STATE UNIVERSITY OF NEW YORK AT STONY BROOK

COLLEGE OF ENGINEERING

Report No. 131

ENERGY INEQUALITIES ON A FINITE DIFFERENCE SOLUTION FOR SYMMETRIC HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

by

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MAY, 1969

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ABSTRACT

Energy inequalities are established for the solution of an implicit finite-difference equation approximating the most general linear, first order system of symmetric hyperbolic partial differential equations.

INTRODUCTION

This paper presents some salient properties of a particular implicit finite difference approximation to the solution of the initial-value problem for linear, first-order systems of symmetric hyperbolic partial differential equations with an arbitrary number of independent and dependent variables. The main results of this paper show that the difference equations are unconditionally stable in the discrete L_2 norm. We note that the solution of the difference equations approximates the exact solution of the differential equations in the L_2 norm with a truncation error which is second order in the mesh spacing.

In Section 1 the domain is defined, the finite difference lattice is described, notation is established along lines similar to M. Lees [8], and a number of minor lemmas dealing with finite differences are established for use in later major lemmas and theorems. Most important, here the norm of a vector defined on the lattice is defined as the discrete L_2 norm. This norm is used throughout the paper. Stability and convergence must be understood in the sense of this norm. Continuous analogues of the analysis are given by R. Courant [3].

Section 2 outlines the general properties required of the exact solution of the differential equations and the properties required of the coefficient matrices and the inhomogeneous term. Subsection 2.1 deals with the case of constant coefficient matrices and homogeneous equations. Theorem 1 proves that the finite difference scheme for this special case satisfies, unconditionally, the Von Neumann necessary conditions for stability [9]. In the symmetric case sufficient conditions for unconditional stability are satisfied [10]. By unconditional stabilit we mean that there are no restrictions on the mesh ratios, i.e., Courant-Friedrichs-Lewy conditions are not required [1]. This subsection affords a heuristic basis for conjecturing the unconditional stability in the more general case by considering the coefficients to be "locally constant". Subsection 2.2 begins by establishing the main lemma, which is essentially the discrete analogue of integration by parts for centered differences. The section continues with Theorem 2 which establishes one of the central inequalities, the so-called "energy inequality", which is the heart of any analysis concerning the solution of the partial differential equations by finite difference methods. Theorem 3 is another "energy inequality", but dealing with centered differences of the finite-difference solution. Theorem 2 immediately establishes the unconditional stability of the linear operator L_Δ in the discrete L_2 norm. It is to be noted that Theorems 2 and 3 require no restriction on the size of the domain of solution.

Background readings for this study are also obtained in references [2], [4], [5], and [7]. These papers should contain sufficient secondary references to acquaint the reader with the scope of this work. Especially valuable are references [6] and [8] after which much of this paper is modelled.

We wish to express our appreciation and thanks to Professor H. B. Keller for the encouragement, direction and advice he extended in the course of the research and preparation of this paper.

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In Section 1 the domain is defined, the finite difference lattice is described, notation is established along lines similar to M. Lees [8], and a number of minor lemmas dealing with finite differences are established for use in later major lemmas and theorems. Most important, here the norm of a vector defined on the lattice is defined as the discrete L_2 norm. This norm is used throughout the paper. Stability and convergence must be understood in the sense of this norm. Continuous analogues of the analysis are given by R. Courant [3].

Section 2 outlines the general properties required of the exact solution of the differential equations and the properties required of the coefficient matrices and the inhomogeneous term. Subsection 2.1 deals with the case of constant coefficient matrices and homogeneous equations. Theorem 1 proves that the finite difference scheme for this special case satisfies, unconditionally, the Von Neumann necessary conditions for stability [9]. In the symmetric case sufficient conditions for unconditional stability are satisfied [10]. By unconditional stability we mean that there are no restrictions on the mesh ratios, i.e., Courant-Friedrichs-Lewy conditions are not required [1]. This subsection affords a heuristic basis for conjecturing the unconditional stability in the more general case by considering the coefficients to be "locally constant". Subsection 2.2 begins by establishing the main lemma, which is essentially the discrete analogue of integration by parts for centered differences. The section continues with Theorem 2 which establishes one of the central inequalities, the so-called "energy inequality", which is the heart of any analysis concerning the solution of the partial differential equations by finite difference methods. Theorem 3 is another "energy inequality", but dealing with centered differences of the finite-difference solution. Theorem 2 immediately establishes the unconditional stability of the linear operator L_Δ in the discrete L_2 norm. It is to be noted that Theorems 2 and 3 require no restriction on the size of the domain of solution.

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1. PRELIMINARIES

In this paper all finite difference equations, operators and functions are defined on a lattice, L, with mesh width h_ℓ in the coordinate direction $x^{(\ell)}$, $\ell=1, \cdots, N$. L consists of the points of intersection of the coordinate lines,

$$\begin{cases} x^{(\ell)} = ih_\ell \ ; \quad i = 0, \pm 1, \pm 2, \cdots, \text{ and,} \\ t = jk \ ; \quad j = 0, \cdots, J, \text{ where} \end{cases}$$

k is the mesh width in the coordinate direction t.

For functions defined on lattice L we employ the following notation. Where there is no chance of misunderstanding we write x for the vector argument $(x^{(1)}, \cdots, x^{(N)})$. Where no argument of a function is written, the argument (x,t) is understood.

Let $\langle A, B\rangle$ represent the usual scalar product of two vectors A and B: $\sum_{\ell=1}^{N} A^{(\ell)} B^{(\ell)}$. We define:

$$\sum_x A = \sum_{x^{(1)}} \sum_{x^{(2)}} \cdots \sum_{x^{(N)}} A, \text{ and set}$$

$$H = \prod_{\ell=1}^{N} h_\ell .$$

We now define the norm of a vector A defined on L at time line t as:

$$\text{(1.1)} \qquad || A(t) ||^2 = H \sum_x \langle A, A\rangle$$

Let A be any scalar or vector function defined on L. Then one of two properties will be assumed to hold:

(a) A (t) is identically zero outside some bounded region L for any fixed value of t, and the summation, $\sum_x$, extends only over L.

(b) A (t) is periodic in each direction $x^{(\ell)}$, and the summation, $\sum_x$, extends only over one period in each direction, $x^{(\ell)}$.

This assumption will be referred to by the euphemism, "satisfying suitable boundary conditions".

We define the shift operator $T^{\pm h_\ell}$:

(1.2) $$\begin{cases} T^{\pm h_\ell} A = A\,(x^{(1)}, \ldots, x^{(\ell-1)}, x^{(\ell)} \pm h_\ell, x^{(\ell+1)}, \ldots, x^{(N)}, t). \\ T^{\pm k} \quad A = A\,(x, \quad t \pm k) \end{cases}$$

In terms of the shift operator the difference quotients of A are:

(1.3) $A_x(\ell) = h_\ell^{-1}\,[T^{h_\ell} A - A]$, the forward difference quotient;

(1.4) $$\begin{cases} A_{\bar{x}}(\ell) = h_\ell^{-1}\,[A - T^{-h_\ell} A], \text{ the backward difference quotients;} \\ A_{\bar{t}} = k^{-1}\,[A - T^{-k} A] \end{cases}$$

(1.5) $A_{\hat{x}}(\ell) = \frac{1}{2}\,(A_{x}(\ell) + A_{\bar{x}}(\ell))$, the centered difference quotient.

We define the time-average, $\hat{A}$, as:

(1.6) $$\hat{A} = \frac{1}{2}\,[A + T^{-k} A]$$

LEMMA 1. For any vector function, A defined on L,

(1.7) $$\frac{1}{2}\left[\,||\,A(t)\,||^2\right]_{\bar{t}} = H \sum_x < \hat{A}, A_{\bar{t}} >.$$

PROOF: For each component $A^{(\ell)}$ of vector A it follows from (1.4) and (1.6) that:

(1.8) $$A^{(\ell)} A^{(\ell)}_{\bar{t}} = \frac{1}{2} [A^{(\ell)}]^2_{\bar{t}}$$

Summing (1.8) over ℓ we have:

(1.9) $$\langle \hat{A} \quad A_{\bar{t}} \rangle = \frac{1}{2} \langle A, A \rangle_{\bar{t}}.$$

Whence, summing (1.9) with $H \sum_x$ the lemma follows from definition (1.1).

LEMMA 2. For vectors A and B defined on L:

(1.10) $$H \sum_x \langle A, B \rangle \leq \frac{1}{2} (|| A ||^2 + || B ||^2).$$

PROOF: By Schwarz's inequality:

$$H \sum_x \langle A, B \rangle \leq H \sum_x [\langle A, A \rangle^{\frac{1}{2}} \langle B, B \rangle^{\frac{1}{2}}].$$

Then by the inequality between the geometric and arithmetic means:

$$H \sum_x \langle A, A \rangle^{\frac{1}{2}} \langle B, B \rangle^{\frac{1}{2}} \leq \frac{H}{2} \sum_x [\langle A, A \rangle + \langle B, B \rangle] = \frac{1}{2} (||A||^2 + ||B||^2),$$

and hence the lemma is proved.

Next, we state three identities which are immediate consequences of definitions (1.2) (1.3) and (1.4). For any two scalar functions, A and B, defined on L:

(1.11) $$T^{h_\ell} A_{\bar{x}(\ell)} = A_{x(\ell)} ,$$

(1.12) $$AB_{x(\ell)} = (AB)_{x(\ell)} - A_{x(\ell)} T^{h_\ell} B,$$

(1.13) $$(AB)_{\bar{x}(\ell)} = T^{-h_\ell} A B_{\bar{x}(\ell)} + BA_{\bar{x}(\ell)}.$$

For any vector or scalar function, A, defined on L and satisfying suitable boundary conditions:

(1.14) $$\sum_{x} T^{h_\ell} A = \sum_{x} A.$$

For any matrix A and vector B defined on L we have:

(1.15) $$\langle AB, B_{x(\ell)} \rangle = \sum_{j=1}^{N} (AB)^{(j)} B^{(j)}_{x(\ell)}$$

$$= \sum_{j=1}^{N} \left\{ \left[(AB)^{(j)} B^{(j)} \right]_{x(\ell)} - (AB)^{(j)}_{x^{\ell}} T^{h_\ell} B^{(j)} \right\},$$

$$= \langle AB, B \rangle_{x(\ell)} - \langle (AB)_{x(\ell)}, T^{h_\ell} B \rangle,$$

By applying (1.12) to each term, $(AB)^{(j)} B^{(j)}_{x(\ell)}$, in (1.15) above.

Also, for any (M x M) matrix, A, with elements a_{ij}, and vector B defined on L, we have:

(1.16) $$(AB)_{\bar{x}(\ell)} = \sum_{j=1}^{M} (a_{ij} B^{(j)})_{\bar{x}(\ell)}$$

$$= \sum_{j=1}^{M} \left[a_{ij} B^{(j)}_{\bar{x}(\ell)} + (a_{ij})_{\bar{x}(\ell)} T^{-h_\ell} B^{(j)} \right]$$

$$= AB_{\bar{x}(\ell)} + A_{\bar{x}(\ell)} (T^{-h_\ell} B),$$

By applying (1.13) to each term, $\left(a_{ij} B^{(j)}\right)_{\bar{x}(\ell)}$, above.

2. THE FINITE DIFFERENCE EQUATION

Let us consider a system of finite difference equations consistent with the system of partial differential equations:

(2.01) $$L(U) \equiv \frac{\partial U}{\partial t} + \sum_{\ell=1}^{N} A^{(\ell)} \frac{\partial U}{\partial_{x(\ell)}} + BU = 0$$

Here U is an M-dimensional vector function of argument (x,t). $A^{(\ell)}$, $\ell=1, \cdots, N$, are each (M x M) symmetric matrices with elements that are functions of (x,t), and B is an (M x M) matrix, also a function of (x,t). Equations (2.01) are subject to initial conditions U (x,0) = g(x), with g(x) a given function.

U satisfies suitable boundary conditions, as described in Section 1 [following (1.1)], above.

The finite difference equations corresponding to (2.01), that we wish to consider are:

$$(2.02)\quad L_\Delta (u) \equiv u_{\bar{t}} + \sum_{\ell=1}^{N} A^{(\ell)} \hat{u}_{\hat{x}^{(\ell)}} + B \hat{u} = 0,$$

subject to initial conditions $u(x,0) = g(x)$. The arguments of (each element of) $A^{(\ell)}$ and B are $(x, t-\frac{k}{2})$, to center the scheme properly. We further require that each element of the matrices $A^{(\ell)}$, satisfy a Lipschitz condition with respect to each of its arguments $x^{(\ell)}$, i.e., there exist non-negative numbers, $\beta_1, \cdots, \beta_N$ such that for any x_1, x_2 in L:

$$|A^{(\ell)}(x_1,t) - A^{(\ell)}(x_2,t)| \leq \beta_\ell \, |x_1-x_2| \; , \; \ell=1, \cdots, N.$$

The matrix norm used here is the natural norm induced by the inner product. Defined in Section 1, i.e., if A is a matrix and B are test vectors:

$$|A| \equiv \underset{|B|=1}{\text{L.U.B.}} |AB| \equiv \underset{\langle B,B\rangle=1}{\text{L.U.B.}} \langle AB, AB\rangle^{\frac{1}{2}}.$$

We require also that the matrix B have a bound β_o such that $||B|| \leq \beta_o$.

2.1 VON NEUMANN STABILITY ANALYSIS

As a motivating analysis for the succeeding sections we now show that $L_\Delta(u)$, as defined in (2.02) with B=0 and $A^{(\ell)}$ matrices with constant coefficients is unconditionally stable.

THEOREM 1: Given the finite difference equations:

(2.10) $u_{\bar{t}} + \sum_{\ell=1}^{N} A^{(\ell)} \hat{u}_{\hat{x}^{(\ell)}} = 0$ satisfying suitable boundary conditions, with $A^{(\ell)}$ being constant matrices, such that (2.01) is hyperbolic, then the equations satisfy the Von Neumann necessary conditions for stability. If the matrices $A^{(\ell)}$ are also symmetric then (2.10) satisfies a sufficient condition for stability.

PROOF: Let a fundamental solution of (2.10) be:

(2.11) $$u(x,tn) = V^{(n)} e^{i<\xi,x>}, \quad <\xi,\xi> = 1, \quad i = \sqrt{-1} \ .$$

We then compute:

(2.12) $$u_{\hat{x}^{(\ell)}}(x,tn) = i\, h_\ell^{-1} \text{ SIN } (\xi_\ell h_\ell)\; u\,(x,tn).$$

Substituting (2.11) and (2.12) into (2.10) we arrive, after cancelling $e^{i<\xi,x>}$, at:

$$(V^{(n)} - V^{(n-1)} + i \sum_{\ell=1}^{N} \eta_\ell A^{(\ell)} \left(V^{(n)} + V^{(n-1)}\right) = 0,$$

where $\eta_\ell = \frac{k}{2h_\ell}$ SIN $(\xi_\ell h_\ell)$. And so we have,

$$\left[I + i \sum_{\ell=1}^{N} \eta_\ell A^{(\ell)}\right] V^{(n)} = \left[I - i \sum_{\ell=1}^{N} \eta_\ell A^{(\ell)}\right] V^{(n-1)}.$$

Thus, the amplification matrix is:

$$G(\eta) = \left[I + i \sum_{\ell=1}^{N} \eta_\ell A^{(\ell)}\right]^{-1} \left[I - i \sum_{\ell=1}^{N} \eta_\ell A^{(\ell)}\right] .$$

Now if we let the eigen-values of $\sum_{\ell=1}^{N} \eta_\ell A^{(\ell)}$ be α_j and remember from the hyperbolicity condition that the α_j are real, then we see that the eigen-values, λ_j of $G(\eta)$ are:

$$\lambda_j = \frac{1 - i\alpha_j}{1 + i\alpha_j} .$$

We note that $|\lambda_j| = 1$, and hence the Von Neumann necessary condition for stability is satisfied. If in addition we have symmetry, i.e., $A^{(\ell)} = A^{(\ell)*}$ for each ℓ, then $G\,G^* = I$ for all η and, a fortiori, G is normal. This is a sufficient condition for Von Neumann stability [10]. This completes the proof.

2.2 THE ENERGY INEQUALITIES

LEMMA 3. (Main Lemma). For any symmetric matrix A, defined on L and any vector B defined on L and satisfying suitable boundary conditions:

(2.20) $$2H \sum_x \langle B, A\, B_{\hat{x}(\ell)} \rangle = -H \sum_x \langle T^{-h_\ell} B, A_{\bar{x}(\ell)} B \rangle.$$

PROOF: We have,

$$\langle B, A\, B_{x(\ell)} \rangle = \langle A\, B, B_{x(\ell)} \rangle, \text{ since } A \text{ is symmetric;}$$

$$= \langle B, A\, B \rangle_{x(\ell)} - \langle (AB)_{x(\ell)}, T^{h_\ell} B \rangle, \text{ by (1.15);}$$

$$= \langle B\ A\ B \rangle_{x(\ell)} - T^{h_\ell} \langle (AB)_{\bar{x}(\ell)}, B \rangle, \text{ by (1.11);}$$

$$= \langle B, AB \rangle_{x(\ell)} - T^{h_\ell} \langle AB_{\bar{x}(\ell)}, B \rangle$$

$$- T^{h_\ell} A_{\bar{x}(\ell)} (T^{-h_\ell} B) , B \rangle, \text{ by (1.16).}$$

From the symmetry of A it follows that $A_{\bar{x}}$ is symmetric, and hence:

(2.21) $$\langle B, AB_{x(\ell)} \rangle = \langle B, AB \rangle_{x(\ell)} - T^{h_\ell} \langle B, AB_{\bar{x}(\ell)} \rangle - T^{h_\ell} \langle T^{-h_\ell} B, A_{\bar{x}(\ell)} B \rangle.$$

Now, applying the summation, $H \sum_x$ to (2.21) and using (1.14) and the fact that $H \sum_x \langle B, AB \rangle_{x(\ell)} = 0$, since B satisfies suitable boundary conditions; we have:

(2.22) $$H \sum_x \left[\langle B, AB_{x(\ell)} \rangle + \langle B, AB_{\bar{x}(\ell)} \rangle \right] = -H \sum_x \langle T^{-h_\ell} B, A_{\bar{x}(\ell)} B \rangle.$$

But the term on the left side of (2.22) is $2H \sum_x \langle B, AB_{\hat{x}(\ell)} \rangle$, whence our Lemma is proved.

THEOREM 2: (Energy Inequality 1)

Let u be a vector defined on L and satisfying suitable boundary conditions. Then there exists constants C_0 and C_1, depending only on T and the bounds $\beta_o, \cdots, \beta_N$ such that, for sufficiently small k:

$$||u(T)||^2 \leq C_1 \left[||u(0)||^2 + \frac{2k}{2-k\ C_o} \sum_{t=k}^{T} ||\ L_\Delta\ [u(t)]||^2 \right].$$

PROOF: We have (see 2.02):

$$L_\Delta\ [u(t)] \equiv u_{\bar{t}} + \sum_{\ell=1}^{N} A^{(\ell)}\ \hat{u}_{\hat{x}(\ell)} + B\hat{u}.$$

By taking the inner product of each side of this equation with $\hat{u}$ and summing with $kH \sum_x$, we have:

$$(2.23)\quad kH \sum_x \left\langle \hat{u},\ L_\Delta(u) \right\rangle = kH \sum_x \left[\left\langle \hat{u}, u_{\bar{t}} \right\rangle + \sum_{\ell=1}^{N} \left\langle \hat{u},\ A^{(\ell)} \hat{u}_{\hat{x}(\ell)} \right\rangle + \left\langle \hat{u},\ B\hat{u} \right\rangle \right]$$

But from Lemma 2, with $A \to \hat{u}$, $B \to L_\Delta\ (u)$, we have:

$$(2.24)\quad kH \sum_x \left\langle \hat{u},\ L_\Delta(u) \right\rangle \leq \frac{k}{2} \left[||\hat{u}(t)||^2 + ||L_\Delta[u(t)]||^2 \right].$$

From Lemma 1, with $A \to u$, we have:

$$(2.25)\quad kH \sum_x \left\langle \hat{u},\ u_{\bar{t}} \right\rangle = \frac{1}{2} \left[||u(T)||^2 - ||u(T-k)||^2 \right].$$

Now for each value of ℓ, $\ell=1, \cdots, N$ we apply Lemma 3 with, $A \to A^{(\ell)}$, $B \to \hat{u}$ and find:

$$(2.26)\quad kH \sum_x \sum_{\ell=1}^{N} \left\langle \hat{u},\ A^{(\ell)} \hat{u}_{\hat{x}(\ell)} \right\rangle = -\frac{kH}{2} \sum_x \sum_{\ell=1}^{N} \left\langle T^{-h_\ell}\hat{u},\ A^{(\ell)}_{\bar{x}(\ell)} \hat{u} \right\rangle.$$

Applying Lemma 2 to the right hand side of (2.26) with,

$A \to T^{-h_\ell}\hat{u}$ and $B \to A^{(\ell)}_{\bar{x}(\ell)}\ \hat{u}$:

$$(2.27)\quad kH \sum_{x} \sum_{\ell=1}^{N} \hat{u},\ A^{(\ell)}\hat{u}_{\hat{x}(\ell)} \leq \frac{k}{4} N ||\hat{u}(t)||^2 + \sum_{\ell=1}^{N} ||A_{\bar{x}}^{(\ell)}(\ell)\ \hat{u}||^2 .$$

Again, applying Lemma 2 with $A \to \hat{u}$, $B \to B\hat{u}$ we have:

$$(2.28)\quad kH \sum_{x} \langle \hat{u}, B\hat{u} \rangle \leq \frac{k}{2} [||\hat{u}(t)||^2 + ||B\hat{u}(t)||^2].$$

We have the estimates, which follow either by hypothesis or as an immediate consequence of the Lipschitz continuity of the matrices $A^{(\ell)}$.

$$||A_{\bar{x}(\ell)}^{(\ell)}\ \hat{u}\ || \leq ||A_{\bar{x}(\ell)}^{(\ell)}||\ ||\ ||\hat{u}|| \leq \beta_\ell\ ||\hat{u}||, \text{ and}$$

$$||B\ \hat{u}|| \leq ||B||\ \ ||\hat{u}|| \leq \beta_0 ||\hat{u}||.$$

Whence, applying (2.24), (2.25), (2.27) and (2.28) to equation (2.23), we obtain:

$$(2.29)\quad ||u(t)||^2 - ||u(t-k)||^2 \leq kC_o\ ||\hat{u}(t)||^2 + k||L_\Delta[u(t)]||^2,$$

with $C_o = 2 + \beta_o + \frac{1}{2}(N + \sum_{\ell=1}^{N} \beta_\ell)$.

From Schwarz's inequality we have:

$$||\hat{u}(t)||^2 \leq \frac{1}{2}(||u(t)||^2 + ||u(t-k)||^2).$$

Applying this to (2.29):

$$(2.210)\quad ||u(T)||^2 \leq \left(1 + \frac{2k\ C_o}{2-k\ C_o}\right)||u(T-k)||^2 + \frac{2k}{2-k\ C_o}\ ||L_\Delta u(T)||^2,$$

and putting $C_2 = \dfrac{2\ C_o}{2-k\ C_o}$, (2.210) implies in a familiar manner that:

$$||u(T)||^2 \leq C_1\ [||u(0)||^2 + \frac{2k}{2-k\ C_0} \sum_{t=k}^{T} ||L_\Delta\ [u(t)]||^2], \text{ where}$$

$C_1 = \text{EXP}\ [C_2\ (T-k)]$ and $k < \dfrac{2}{C_0}$. This proves Theorem 2.

Theorem 2 immediately implies that the implicit operator L_Δ is unconditionally stable in the mean square norm.

2.3 THEOREM 3: (Energy Inequality 2)

Let u be a vector defined on L and satisfying suitable boundary conditions. Then there exists constants C_4 and C_5 depending only on T, N and the bounds $\beta_0, \cdots \beta_N$ such that, for sufficiently small k:

$$\sum_{\ell=1}^{N} ||u_{\hat{x}(\ell)}(T)||^2 \leq C_5 \left\{ \sum_{\ell=1}^{N} ||u_{\hat{x}(\ell)}(0)||^2 + \frac{k}{1-C_4 k} \sum_{t=k}^{T} \sum_{\ell=1}^{N} \left[||[L_\Delta(u(t))]_{\hat{x}(\ell)}||^2 + \beta_0^2 ||u(T)||^2 \right] \right\}.$$

PROOF: We have,

$$[L_\Delta(u(t))]_{\hat{x}(j)} = u_{\hat{x}(j)\bar{t}} + \sum_{\ell=1}^{N} \left(A^{(\ell)} \hat{u}_{\hat{x}(\ell)} \right)_{\hat{x}(j)} + (B\hat{u})_{\hat{x}(j)}.$$

Applying (1.12) and (1.13) to the second and third terms on the right,

$$(2.30)\ [L_\Delta(u(t))]_{\hat{x}(j)} = u_{\hat{x}(j)\bar{t}} + \sum_{\ell=1}^{N} \Big[A^{(\ell)} \hat{u}_{\hat{x}(j)\hat{x}(\ell)} + \frac{1}{2}\left(A^{(\ell)}_{\bar{x}(j)} T^{-h_j} \hat{u}_{\hat{x}(\ell)} \right) + A^{(\ell)}_{x(j)} T^{h_j} \hat{u}_{\hat{x}(\ell)} \Big] + B\hat{u}_{\hat{x}(j)} + \frac{1}{2}\left(B_{\bar{x}(j)} T^{-h_j} \hat{u} + B_{x(j)} T^{h_j} \hat{u} \right).$$

Now we take the inner product of (2.30) with $\hat{u}_{\hat{x}(j)}$ and sum with $kH \sum_{x}$; and using Lemmas 1 and 2 on the result, we have:

$$(2.31)\quad ||u_{\hat{x}(j)}(t)||^2 \le ||u_{\hat{x}(j)}(t-k)||^2 + \frac{k}{2}\,(2N+3+\sum_{\ell=0}^{N}\beta_\ell{}^2)$$

$$(||u_{\hat{x}(j)}(t)||^2 + ||u_{\hat{x}(j)}(t-k)||^2) + \beta_j{}^2\,\frac{k}{2}\sum_{\ell=1}^{N}$$

$$(||u_{\hat{x}(\ell)}(t)||^2 + ||u_{\hat{x}(\ell)}(t-k)||^2) + k\,\beta_0||\hat{u}(t)||^2$$

$$+\,k\,||[L_\Delta\left(u(t)\right)]_{\hat{x}(j)}||^2.$$

We now sum with respect to j from 1 to N, obtaining from 2.31, with

$$C_4 = \frac{1}{2}\,(2N+3+b_0{}^2) + \sum_{\ell=1}^{N}\beta_\ell{}^2:$$

$$(2.32)\quad \sum_{j=1}^{N}||u_{\hat{x}(j)}(t)||^2 \le \left(1+\frac{k\,C_4}{1-k\,C_4}\right)\sum_{j=1}^{N}||u_{\hat{x}(j)}(t-k)||^2 +$$

$$\frac{k}{1-C_4k}\left[\sum_{j=1}^{N}\left\{||[L_\Delta\,u(t)]_{\hat{x}(j)}||^2\right\} + N\,\beta_0{}^2||u(t)||^2\right]$$

The conclusion follows in a familiar manner with

$$C_5 = \mathrm{EXP}\left[\frac{C_4}{1-C_4\,k}\,(T-k)\right],\ \text{requiring } k < \frac{1}{C_4}.$$

This proves Theorem 3.

An immediate consequence of Theorem 3 is, that, if the initial conditions on u are Lipschitz continuous with constant D, and the function C (x,t), defined on L is also Lipschitz continuous with constant D_2, then for the equation:

$L_\Delta(u) + C = 0$, $u_{\hat{x}(j)}(T)$ is bounded in norm with:

$$\sum_{j=1}^{N}||u_{\hat{x}(j)}(T)||^2 \le N\,C_5\left[D_1{}^2 + \frac{D_2{}^2(T-k)}{1-C_4\,k} + \frac{k\,\beta_0{}^2}{1-C_4\,k}\sum_{t=k}^{T}||\hat{u}(t)||^2\right].$$

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