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GENERAL THEORY OF OPTIMAL PROCESSES*

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In the present paper a general version of the maximum principle is formulated and proved. Pontryagin's maximum principle [1] and various extensions [2]–[9] thereof become special cases which can be readily derived from the general version. Of special interest are the following generalizations: (1) discrete systems, (2) systems with multiple merit criteria, (3) restriction of the control function u to a special class of functions, and (4) systems with bounded state variables.

Operative addition and convexity. Let ϵ denote an infinitesimal quantity and $\hat{u}(t)$ a given function defined on $T = \{t: t_1 \leqq t \leqq t_2\}$. A function $u(t)$ is said to vary infinitesimally from $\hat{u}(t)$ if

$$\int_{t_1}^{t_2} \| u(t) - \hat{u}(t) \| \, dt < \epsilon A, \tag{1}$$

where A is a positive constant. Obviously if $u(t)$ is different from $\hat{u}(t)$ for a finite amount, it can be only for an infinitesimal interval of time. In the present paper, one needs only to consider infinitesimal variations of the following form:

$$\begin{aligned} \delta u(t) &\equiv u(t) - \hat{u}(t) = a_i \text{ on } T_i' = \{t: t_i' < t < t_i' + \epsilon\Delta_i\}, \\ \delta u(t) &= \epsilon\, \xi(t) \text{ on } T - T', \qquad T' = \bigcup T_i', \end{aligned} \tag{2}$$

where i may range from 1 to any finite number, $t_i' \in T$, and a_i, Δ_i and $\xi(t)$ are finite numbers and function, respectively.

Operative addition is denoted by $\oplus$ and is defined as follows: let $\delta u_1(t)$ and $\delta u_2(t)$ denote two infinitesimal variations from $\hat{u}(t)$. If the two sets of t_i' have no element in common, the two sets T' are disjoint for sufficiently small ϵ. Then

$$\delta u_1(t) \oplus \delta u_2(t) = \delta u_1(t) + \delta u_2(t). \tag{3}$$

If finite variations occur at the same instant, the variations are rearranged in sequence in $\delta u_1(t) \oplus \delta u_2(t)$. For instance, given

$$\delta u_1(t) = a_k', \quad t_k' \leqq t < t_k' + \epsilon\Delta_k',$$

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$$\delta u_2(t) = a_k'', \quad t_k' \leqq t < t_k' + \epsilon\Delta_k'',$$

then

$$\begin{aligned} \delta u_1(t) \oplus \delta u_2(t) &= a_k', \quad t_k' \leqq t < t_k' + \epsilon\Delta_k', \\ \delta u_1(t) \oplus \delta u_2(t) &= a_k'', \quad t_k' + \epsilon\Delta_k' \leqq t < t_k' + \epsilon(\Delta_k' + \Delta_k''), \end{aligned}$$

A variation from a to b can be considered as a variation from b to a for a negative interval.

A set of functions C is *operatively convex* if it has the following property: given any u and infinitesimal variations δu_1 and δu_2 such that all three functions u, $u + \delta u_1$, and $u + \delta u_2$ belong to C, then

$$u + [h\, \delta u_1 \oplus (1 - h)\delta u_2]$$

belongs to C for all values of h in the interval $0 < h < 1$.

The control problem. The controlled system is described by the following set of differential equations:

$$\frac{dx_i}{dt} \equiv \dot{x}_i = f_i(x, u, t), \qquad i = 1, 2, \cdots, n, \tag{4}$$

where x and u are the state vector and the control vector, respectively, and f is a vector function having continuous and bounded first derivatives in x and being continuous in u. From known existence theorems, given $x(t_1)$ and $u(t)$ on $T = \{t: t_1 \leqq t \leqq t_2\}$, $x(t)$ is completely determined.

The control function $u(t)$ is required to satisfy three conditions:

(a) $u(t)$ belongs to an operatively convex set of functions C on T,

(b) the $x(t)$ resulting from $u(t)$ stays within an allowed region X, $x(t) \in X$ for all $t \in T$,

(c) the path terminates at a point $x(t_2)$, where t_2 may be fixed or arbitrary, $t_2 \leqq T$.

A set $u(t)$ satisfying (a) and (b) is called an *admissible control*. When all three conditions are satisfied, $u(t)$ is called an *allowed control*. The merit of an allowed control is judged by a set of variables y_i, where

$$y_i(t) = \int_{t_1}^{t} g_i(x, u, t)\, dt, \qquad i = 1, 2, \cdots, N. \tag{5}$$

An allowed control A is said to be *inferior to* B if

$$y_i(t_2)|_A \geqq y_i(t_2)|_B, \tag{6}$$

and the inequality sign holds for at least one value of i. An allowed control is said to be noninferior if it is not inferior to any other allowed control in the sense defined above.

The noninferior controls are generalizations of optimal controls for a system with multivalued criteria.

First variations of state and merit variables. Due to the infinitesimal variation in $u(t)$ (see (2)), $x(t)$ and $y(t)$ are different from $\hat{x}(t)$ and $\hat{y}(t)$:

$$x(t) - \hat{x}(t) = \epsilon\Delta x(t) + O(\epsilon), \quad (7)$$
$$y(t) - \hat{y}(t) = \epsilon\Delta y(t) + O(\epsilon),$$

where $\Delta x(t)$ and $\Delta y(t)$ are finite and are called the first variations of $x(t)$ and $y(t)$, respectively.

Let z denote the $(n + N)$-dimensional vector

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

and $h(x, u, t)$ denote the $(n + N)$-dimensional vector function

$$\begin{pmatrix} f \\ g \end{pmatrix}.$$

Equations (4) and (5) can be combined as

$$\dot{z} = h(x, u, t). \quad (8)$$

The first variation in z for $t \notin T'$ is readily shown to be

$$\Delta z \equiv \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \sum_{\text{all } k \text{ with } t_k' < t} A(t, t_k')[h(\hat{x}, u, t_k') - h(\hat{x}, \hat{u}, t_k')]\Delta_k \quad (9)$$
$$+ \int_{t_1}^{t} A(t, t') \frac{\partial h(\hat{x}, \hat{u}, t)}{\partial \hat{u}} \xi(t')\, dt',$$

where $A(t, t')$ is an $(n + N)$-dimensional square matrix satisfying

$$\frac{\partial A(t, t')}{\partial t'} = \frac{\partial h(\hat{x}, \hat{u}, t)}{\partial \hat{z}} A(t, t'), \quad (10)$$

$$\frac{\partial A(t, t')}{\partial t'} = -A(t, t') \frac{\partial h(\hat{x}, \hat{u}, t')}{\partial \hat{z}}, \quad (11)$$

$$A(t, t) = 1,$$

and $\partial h/\partial z$ is an $(n + N)$-dimensional square matrix with

$$\left(\frac{\partial h}{\partial z}\right)_{i,j} = \frac{\partial h_i}{\partial z_j}.$$

Since the vector function h is independent of y, the last N columns of the matrix $\partial h/\partial z$ are identically zero.

The following theorem is obvious from (9).

THEOREM 1. *Let $(\delta u)_1$, $(\delta u)_2$ and $(\delta u)_3$ represent infinitesimal variations about $\hat{u}(t)$ related by*

$$(\delta u)_1 \oplus (\delta u)_2 = (\delta u)_3.$$

Let $(\Delta z)_1$, $(\Delta z)_2$ and $(\Delta z)_3$ denote the first variations in z resulting from $(\delta u)_1$, $(\delta u)_2$ and $(\delta u)_3$, respectively. Then

$$(\Delta z)_1(t) + (\Delta z)_2(t) = (\Delta z)_3(t).$$

COROLLARY. *The set of admissible first variations about any terminal point $z(t_2)$ is convex.*

General theorems on optimal control.

THEOREM 2. *Given fixed points $x(t_1)$ and $x(t_2)$, and letting X be the x-space (unbounded), a necessary condition for a control and path pair $\hat{u}(t)$, $\hat{x}(t)$ to be a noninferior control is that there exists a set of vector functions $\hat{\psi}(t)$ and $H(\hat{\psi}, \hat{x}, \hat{u}, t)$ satisfying*

$$H(\hat{\psi}, \hat{x}, \hat{u}, t) \equiv \sum_{i=1}^{n} \hat{\psi}_i(t) f_i(\hat{x}, \hat{u}, t) - \sum_{k=1}^{N} c_k g_k(\hat{x}, \hat{u}, t), \quad (12)$$

$$\partial_u \int_{t_1}^{t_2} H(\hat{\psi}, \hat{x}, \hat{u}, t)\, dt \leqq 0, \quad (13)$$

and

$$\frac{d\hat{\psi}_i(t)}{dt} = -\frac{\partial H}{\partial \hat{x}_i}, \qquad i = 1, 2, \cdots, n, \quad (14)$$

where

$$C_k \geqq 0, \quad (15)$$

and the equality sign in (15) cannot hold for all values of k; ∂_u is the first variation of the subsequent integral due to an infinitesimal variation of $u(t)$, with $\hat{\psi}$, $\hat{x}$ and t considered as fixed.

Note that there is no restriction on the infinitesimal variation δu except that $\hat{u} + \delta u$ belongs to C. In this and later theorems ∂_u is interpreted in the same way. From the definition of ∂_u, (9) can be written as

$$\Delta z(t) = \partial_u \int_{t_1}^{t} A(t, t')h(\hat{x}, \hat{u}, t')\, dt'.$$

Before proving Theorem 2 some geometrical notions in z-space will be established. A point p in z-space is called *accessible* if there exists an admissible $u(t)$ which brings the system from $z(t_1)$ to p at some time t_2. The set of all accessible points at fixed t_2 and at $t_2 \leqq T$ are denoted by $\Omega(t_2)$ and Ω, respectively.

The set of allowed terminal points is an N-dimensional plane, P, at $x = x(t_2)$. The intersection of Ω with P is denoted by I.

Lemma 1. *The terminal point $\hat{z}(t_2)$ of a noninferior control is a boundary point of I and Ω.*

Lemma 2. *Let Ω_ϵ denote the set $\hat{z}(t_2) + \epsilon\Delta z$ for all admissible first variations Δz. Then $\hat{z}(t_2)$ is a boundary point of Ω_ϵ.*

Lemma 2 follows intuitively from Lemma 1. It has also been proven rigorously by previous authors [1, pp. 86–106].

Lemma 3. *There exists a vector l such that*

$$\sum_{i=1}^{n+N} l_i(\Delta z)_i \leqq 0 \tag{16}$$

for all allowed first variations Δz,

$$l_i \leqq 0, \quad i = n+1, n+2, \cdots, n+N, \tag{17}$$

and the equality sign in (17) cannot hold for all values of i.

Proof. Let P_s denote a section in P which satisfies

$$z_i - \hat{z}_i(t_2) \leqq 0, \quad i = n+1, n+2, \cdots, n+N.$$

Since $\hat{z}(t_2)$ is the terminal point of a noninferior control, Ω_ϵ and P_s do not have interior points in common. Furthermore Ω_ϵ is convex because of the Corollary of Theorem 1. There exists a support plane S which separates Ω_ϵ and P_s. Let l be the normal to S. Points on the Ω_ϵ side of S are represented by (16) and all the points on P_s satisfy

$$\sum_{j=n+1}^{n+N} l_j[z_j - \hat{z}_j(t_2)] \geqq 0. \tag{18}$$

By choosing

$$z_j - \hat{z}_j < 0 \quad \text{for } j = i,$$

$$z_j - \hat{z}_j = 0 \quad \text{for all } j \neq i,$$

(18) leads to (17). Because S can coincide at most with one boundary plane of S, the final assertion of the lemma is obtained.

Proof of Theorem 2. Let l' represent the row vector $(l_1, l_2, \cdots, l_{n+N})$. Inequality (16) can be written as

$$l'\Delta z(t_2) \leqq 0. \tag{19}$$

Since X is the entire x-space, the class of admissible controls is identical with C. From (9),

$$\Delta z(t_2) = \partial_u \int_{t_1}^{t_2} A(t_2, t)h(\hat{x}, \hat{u}, t)\, dt. \tag{20}$$

Substituting (20) into (19) gives

$$\partial_u \int_{t_1}^{t_2} l'A(t_2, t)h(\hat{x}, \hat{u}, t)\, dt \leqq 0. \tag{21}$$

Let $\hat{\psi}'(t)$ denote the row vector $l'A(t_2, t)$, and let $H(\psi, x, u, t)$ be defined as

$$H(\psi, x, u, t) \equiv \psi'(t)\, h(x, u, t). \tag{22}$$

Inequality (21) is identical with (13). Multiplying (11) on the left by l' gives

$$\frac{d\hat{\psi}'(t)}{dt} = -\hat{\psi}' \frac{\partial h}{\partial z}. \tag{23}$$

The first n components of (23) give (14). The last N components of (23) give

$$\psi_i(t) = \text{const.} = l_i, \qquad i = n+1, \cdots, n+N.$$

Let $C_k \equiv -l_{n+k}$. Equation (22) is then identical with (12).

Theorem 3. *Given fixed points $x(t_1)$ and $x(t_2)$, and letting X be an n-dimensional smooth region in x-space, a necessary condition for a control and path pair $\hat{u}(t)$, $\hat{x}(t)$ to be a noninferior control is that there exist $\hat{\psi}(t)$, $\zeta(\hat{x}, t)$ and $H(\hat{\psi}, \hat{x}, \hat{u}, t)$ satisfying* (12), (13), (15), *and the following:*

$$\frac{d\hat{\psi}_i}{dt} = -\frac{\partial H}{\partial \hat{x}_i} \zeta(\hat{x}, t)\eta_i(\hat{x}), \tag{24}$$

where

$$\zeta(\hat{x}, t) \begin{cases} = 0 \text{ if } \hat{x} \text{ is an interior point of } X, \\ \geqq 0 \text{ if } \hat{x} \text{ is on the boundary of } X, \end{cases} \tag{25}$$

and η is the normal to X at $\hat{x}$ pointing away from X.

Proof. The proof of Theorem 3 is identical with that of Theorem 2 up to (19). Inequality (19) holds only for infinitesimal admissible variations of u. Condition (2) defining admissible variations can be written as

$$\sum_{i=1}^{n} \eta_i(\hat{x})\Delta x_i(t) \leqq 0 \quad \text{on} \quad \Gamma', \tag{26}$$

where Γ' is the part of the path $\hat{x}(t)$ lying on the boundary of X. Thus (19) is replaced by itself together with the side condition (26). Let η' represent the $(n + N)$-component row vector with the η_i as its first n components and 0 for the remaining N components. Inequality (26) is identical with

$$\eta'(\hat{x})\Delta z(t) \leqq 0 \quad \text{on} \quad \Gamma'. \tag{27}$$

From a well-known result in variation calculus, (19) together with the side condition (27) is equivalent to the existence of a $\zeta(t)$ such that

$$l'\Delta z(t_2) - \int_\Gamma \zeta(t)\eta'(\hat{x})\Delta z(t)\,dt \leqq 0. \tag{28}$$

The integral is taken over periods of time in which x lies on the boundary of X.

LEMMA 4. *A necessary condition for $\hat{u}(t)$ and $\hat{x}(t)$ to be a noninferior control and path pair is that there exists a $\zeta(t)$ such that (28) is satisfied by all first variations $\Delta(z)$ resulting from $\delta u(t)$ with $\hat{u} + \delta u$ belonging to C.*

Let $\zeta(\hat{x}, t) = \zeta(t)$ when $\hat{x}(t)$ is a boundary point, and equal zero when $\hat{x}(t)$ is an interior point. From (9),

$$\begin{aligned}\int_{\Gamma'} \zeta(t)\eta'(\hat{x})\Delta z(t)\,dt &= \int_{t_1}^{t_2} \zeta(\hat{x}, t)\eta'(\hat{x})\partial_u \int_{t_1}^{t} A(t, t')h(\hat{x}, \hat{u}, t')dt'\,dt \\ &= \partial_u \int_{t_1}^{t_2}\int_{t_1}^{t} \zeta(\hat{x}, t)\eta'(\hat{x})A(t, t')h(\hat{x}, \hat{u}, t')\,dt'\,dt.\end{aligned}$$

Changing the order of integration but integrating over the same area gives

$$\int_{t_1}^{t_2}\int_{t_1}^{t} \cdots\,dt'\,dt = \int_{t_1}^{t_2}\int_{t}^{t_2} \cdots\,dt\,dt'.$$

Interchanging the notations t and t', one finally obtains

$$\begin{aligned}&\int_{\Gamma'} \zeta(t)\eta'(\hat{x})\Delta z(t)\,dt \\ &\quad = \partial_u \int_{t_1}^{t_2}\int_{t}^{t_2} \zeta(\hat{x}, t')\eta'(\hat{x}(t'))A(t', t)h(\hat{x}, \hat{u}, t)\,dt'\,dt.\end{aligned} \tag{29}$$

Substituting (20) and (29) into (28) gives

$$\partial_u \int_{t_1}^{t_2} \left[l'A(t_2, t) - \int_{t}^{t_2} \zeta(\hat{x}, t')\eta'(\hat{x})A(t', t)\,dt' \right] h(\hat{x}, \hat{u}, t)\,dt \leqq 0. \tag{30}$$

Let $\hat{\psi}'(t)$ be defined as

$$\hat{\psi}'(t) \equiv l'A(t_2, t) - \int_{t}^{t_2} \zeta(\hat{x}, t')\eta'(\hat{x})A(t', t)\,dt', \tag{31}$$

and let $H(\psi, x, u, t)$ be defined the same way as in (22); then (30) is identical with (13). From (31),

$$\begin{aligned}\frac{d\hat{\psi}'(t)}{dt} &= l'\,\frac{\partial A(t_2, t)}{\partial t} + \zeta(\hat{x}, t)\eta'(\hat{x}) - \int_{t}^{t_2} \zeta(\hat{x}, t')\eta'(\hat{x})\,\frac{\partial A(t', t)}{\partial t}\,dt' \\ &= -\psi'\,\frac{\partial h(\hat{x}, \hat{u}, t)}{d\hat{z}} + \zeta(\hat{x}, t)\eta'(\hat{x}).\end{aligned} \tag{32}$$

The first n components of (32) give (24). The last N components give

$$\psi_{n+k}(t) = -C_k, \quad k = 1, 2, \cdots, N, \tag{33}$$

and (12).

The condition (25) is proved in a previous paper for a less general problem (10). It also follows from the intuitive notion that $y_i(t_2)$ can be reduced if the path $x(t)$ is allowed an excursion beyond X.

The following theorem is the well-known transversality condition, and will be stated without a proof.

THEOREM 4. *Let $u(t)$, $t_1 \leqq t \leqq t_2$, be an admissible control which transfers the phase point from some position $x(t_1) \in S_1$ to the position $x(t_2) \in S_2$, where S_1 and S_2 are smooth regions of points satisfying the following equations and inequalities:*

$$\begin{aligned} S_1:\quad & F_i(x) = 0, \quad i = 1, 2, \cdots, k \leqq n, \\ & G_i(x) \leqq 0, \quad i = 1, 2, \cdots, m, \\ S_2:\quad & F_i'(x) = 0, \quad i = 1, 2, \cdots, k' \leqq n, \\ & G_i'(x) \leqq 0, \quad i = 1, 2, \cdots, m'. \end{aligned}$$

In order that $u(t)$ and $z(t)$ yield the solution of the noninferior problem with variable endpoints, it is necessary that there exists a nonzero continuous vector function $\psi(t)$ which satisfies the conditions of Theorem 3 and, in addition, the transversality condition at both endpoints of the trajectory $z(t)$,

$$\psi_i(t_1) = \sum_{j=1}^{k} a_j \frac{\partial F_j}{\partial x_i} + \sum_{j=1}^{m} b_j \frac{\partial G_j}{\partial x_i}, \tag{34}$$

$$\psi_i(t_2) = \sum_{j=1}^{k'} a_j' \frac{\partial F_j'}{\partial x_i} - \sum_{j=1}^{m'} b_j' \frac{\partial G_j'}{\partial x_i}, \tag{35}$$

where a_j and a_j' are arbitrary constants, and b_j and b_j' are nonnegative constants such that

$$\begin{aligned} b_j &= 0 \quad \text{if } G_j(x) < 0 \text{ or if } x(t_1) \text{ is an interior point of } S_1, \\ b_j &\geqq 0 \quad \text{if } G_j(x) = 0 \text{ and the equation actually defines the boundary of } S_1 \text{ at } x(t_1), \end{aligned} \tag{36}$$

and similar conditions hold for b_j'. In (34), (35) and (36) the values of the functions and partial derivatives are evaluated at the corresponding endpoints.

THEOREM 5. *Consider a control problem satisfying the following conditions:*

$$f(x, u, t) = A(t)x + B(t)u + f(t), \tag{37}$$

$$g(x, u, t) = p(x, t) + q(u, t), \tag{38}$$

where $A(t)$ and $B(t)$ are matrices, $f(t)$, $p(x, t)$, and $q(u, t)$ are vector functions, $p(x, t)$ is convex in x, and $q(u, t)$ is convex in u,
X, S_1, S_2, and the class C are convex.

If for an allowed control and path pair, $\hat{u}(t)$ and $\hat{x}(t)$, a set of functions, $H(\hat{\psi}, \hat{x}, \hat{u}, t)$, $\hat{\psi}(t)$, $\zeta(\hat{x}, t)$, and $C_i > 0$, $i = 1, 2, \cdots, N$, can be found such that (12), (13), (24), (25), and the transversality condition are satisfied, then $\hat{u}(t)$ is a noninferior control among all admissible controls which transfer the phase point from a point on S_1 at t_1 to a point on S_2 at t_2.

Proof. Let C' denote the row vector $(C_1, C_2, \cdots, C_N)$. From (12), (37), and (38),

$$H(\psi, x, u, t) = \psi' A(t)x + \psi' B(t)u + \psi' f(t) - C'p(x, t) - C'q(u, t) \tag{39}$$

From (24),

$$\frac{d}{dt}\hat{\psi}' = -\hat{\psi}' A(t) + \frac{\partial C'p(\hat{x}, t)}{\partial \hat{x}} + \zeta(\hat{x}, t)\eta'(\hat{x}). \tag{40}$$

Consider any other allowed control and path pair $u(t)$, $x(t)$ which satisfy the same terminal conditions. Evaluate the following total derivative:

$$\begin{aligned} \frac{d}{dt}[\hat{\psi}'(\hat{x} - x)] &= \hat{\psi}' B(t)(\hat{u} - u) \\ &\quad + \frac{\partial C'p(\hat{x}, t)}{\partial \hat{x}}(\hat{x} - x) + \zeta(\hat{x}, t)\eta'(\hat{x})(\hat{x} - x). \end{aligned} \tag{41}$$

Subtracting $C'p(\hat{x}, t) + C'q(\hat{u}, t) - C'p(x, t) - C'q(u, t)$ from both sides of (41) and integrating from t_1 to t_2 give

$$\begin{aligned} &\hat{\psi}'(\hat{x} - x)\big|_{t_1}^{t_2} - C'[\hat{y}(t_2) - y(t_2)] \\ &= \int_{t_1}^{t_2} \left\{\hat{\psi}' B(t)(\hat{u} - u) - \frac{\partial C'q(\hat{u}, t)}{\partial \hat{u}}(\hat{u} - u)\right] dt \\ &\quad + \int_{t_1}^{t_2} \left[\frac{\partial C'q(\hat{u}, t)}{\partial \hat{u}}(\hat{u} - u) - C'q(\hat{u}, t) + C'q(u, t)\right] dt \\ &\quad + \int_{t_1}^{t_2} \left[\frac{\partial C'p(\hat{x}, t)}{\partial \hat{x}}(\hat{x} - x) - C'p(\hat{x}, t) + C'p(x, t)\right] dt \\ &\quad + \int_{t_1}^{t_2} \zeta(\hat{x}, t)\eta'(\hat{x})(\hat{x} - x)\, dt. \end{aligned} \tag{42}$$

On the right-hand side of (42), the first integral is nonnegative because of (13) and the convexity of the class C. The second and third integrals are nonnegative because of the convexity of the functions $p(x, t)$ and $q(u, t)$, and $C_i > 0$, $i = 1, 2, \cdots, N$. The last integral is nonnegative because of (25) and the convexity of X. On the left-hand side of (42),

$$\hat{\psi}'(\hat{x} - x)\big|_{t_1}^{t_2}$$

is nonpositive because of the transversality condition and convexity of S_1 and S_2. Therefore

$$C'[\hat{y}(t_2) - y(t_2)] \leqq 0.$$

Examples of application of the theorems to discrete systems and other special cases are given in a companion paper.

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