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TRANSIENT FLOW THROUGH POROUS MEDIA

by

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ABSTRACT

Analyses of one-dimensional transient seepage problems are presented. This one-dimensional approach may be considered as an approximation to the two-dimensional problem, i.e., flow from a basin of rectangular cross-section having a large width to depth ratio. Mathematically, the problem involves a free-boundary value problem of potential theory, and is non-linear. Porous media of both infinite and finite depth, and having successive strata of different permeabilities, are considered. Solutions are found corresponding to constant head, constant inflow intensity, and certain combinations of these parameters.

It is also shown that, given the total duration, maximum intensity, and total flow of a storm, it is possible to approximate the inflow intensity to obtain a complete solution of the seepage problem, i.e., the penetration depth and head in the basin as a function of time. It is felt that this result has immediate application in the design of storm water seepage basins.

In all cases, exact solutions are obtained, but where more convenient, a numerical analysis of the governing differential equation is outlined.

INTRODUCTION

This report is concerned with the movement of a fluid in a porous medium in which there is a fluid-gas interface. As is customary, the macroscopic flow equations assume the validity of Darcy's generalized law, with the actual medium replaced by a continuum. For the above to apply the material must be granular, i.e., of the grain size of at least a fine sand. For finer gain soils where capillarity and other molecular forces are significant, the analyses given herein should provide order-of-magnitude estimates for the flow rate and the free-surface location.

The problem involves an unknown boundary location and is non-linear, hence general solutions are not available. For certain applications, approximations and linearizations are sometimes possible. For two-dimensional steady state problems, solutions can occasionally be effected by successive conformal mappings. Recently DeWiest [1], utilizing conformal mapping together with a perturbation procedure, obtained an asymptotic solution for a two-dimensional transient problem. A procedure for extending the range of validity of the solution was presented in [2]. Both methods, however, involve the solution of secular equations of large degree, and the solutions are not valid for early times.

In the present work, certain one-dimensional transient problems are studied for which exact implicit solutions are obtained. Some of these have previously been obtained [3]. However, these have never been exploited for the analysis of seepage problems, the main concern of this study.

Moreover, it turns out that the one-dimensional solutions obtained are limiting cases of two-and-three-dimensional transient problems, so that there are immediate practical applications of this research, namely in the design of storm-water seepage basins and nuclear waste disposal.

With regard to the first of these, seepage basin design, it is necessary that the basin be of sufficient dimensions to accommodate the largest storm runoff anticipated without overflowing. To this end, county design criteria, e.g. Suffolk and Nassau on Long Island, stipulate a basin volume to drainage area ratio. This implies a steady state condition which is actually almost never attained. Many such basins never contain more than a foot or two of water, although the depths provided are twelve to twenty feet. Evidently they are grossly oversized perhaps by an order of magnitude. A more realistic procedure would be one in which the transient nature of the phenomenon is included, for which the following analysis is pertinent.

With regard to the disposal of low level wastes at nuclear power plants, it is important that the seepage paths and velocities of radioactive effluent be predictable. The initial motion of such effluent is primarily one-dimensional since gravity is the generating force. Hence, the analysis presented herein should be of value in this area also. The problem of lateral dispersal, which is actually a two-dimensional transient phenomenon, is being studied.

1. THE BOUNDARY VALUE PROBLEM OF FLOW THROUGH POROUS MEDIA

By use of the kinematical free-surface equation, the equation of continuity for an imcompressible fluid, and the result known as Darcy's law, the differential equation describing the one-dimensional free surface will be derived.

The experimental result formulated by Henri Darcy [5], Darcy's law, expresses a proportionality relationship between the filtration velocity v (also called the specific discharge when denoted by q) and the change in head, $\frac{\partial h}{\partial x_i}$, in the direction of the velocity component. h is defined by $h = \frac{p}{\gamma} + y$, where γ is the specific weight of water and y is the vertical distance from some datum to the point in question. Thus

$$v_{x_i} = -k \frac{\partial h}{\partial_{x_i}} \tag{1}$$

Equation (1) serves to define the constant of proportionality k, called the permeability. The permeability is a function of fluid properties as well as those of the medium.

Following Muskat [6] and Polubarinova-Kochina [3], it is assumed that Darcy's law can be generalized to three dimensions, viz.,

$$\begin{aligned} v_x &= -k \frac{\partial h}{\partial x} \\ v_y &= -k \frac{\partial h}{\partial y} \\ v_z &= -k \frac{\partial h}{\partial z} \end{aligned} \tag{2}$$

for an isotropic medium. In recent works the above assumption has been questioned, since, as may be shown, the above implies that the permeability matrix $[k_{ij}]$ admits of a diagonal form. Although there appears to be no theoretical justification for this assumption, experimental evidence to date seems to indicate the existence of orthogonal principal axes for all samples which have been tested [7].

However, the experimental results represented by equations (2) are incomplete, inasmuch as they do not include the effects of inertia. This effect becomes more pronounced when large granules of porous media are considered, because the large pores enable the fluid passing through them to come under the influence of inertial forces. A separate study indicates that inertial terms are significant only for a very short time, of the order of a second, so that in the following analysis inertial effects are excluded. It is noted, however, that time effects enter via the moving boundary.

The equation of continuity for an incompressible fluid may be written in the form

$$\nabla \cdot \bar{V} = 0 \tag{3}$$

where $\bar{V} = u\underline{i} + v\underline{j} + w\underline{k}$, and $\nabla = \underline{i}\,\frac{\partial}{\partial x} + \underline{j}\,\frac{\partial}{\partial y} + \underline{k}\,\frac{\partial}{\partial z}$.

Expressing equations (2) in the form

$$\bar{v} = k \text{ grad } h = \text{grad } kh$$

and substituting into (3) gives

$$\nabla \cdot (\text{grad } kh) = 0$$

Introducing the "potential" $\phi \equiv kh$,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{4}$$

Hence the quantity kh satisfies Laplace's equation.

Let equation (4) be satisfied in a region, shown in Fig. 1, whose boundaries are fixed. Also, on one boundary segment, let $\phi(P) = g(P)$, and on the other segment, $\frac{\partial \phi}{\partial n} = f(P)$, where P denotes any point on the boundary. This represents a basic problem in potential theory and is a well-defined boundary value problem, i.e., a solution exists and is unique. If one of the boundary segments is a free surface, however, the condition on it is of the form

$$G(\phi, \phi_x, \phi_y) = 0 \tag{5}$$

with the condition on the other segment being either $\phi(P) = g(P)$ or $\frac{\partial\phi}{\partial n} = f(P)$. Condition (5) may not be sufficient to guarantee uniqueness for all such problems. Certain one-dimensional cases will be investigated for which existence and uniqueness have recently been established. [8]

2. THE ONE-DIMENSIONAL TRANSIENT FLOW EQUATION

Consider a semi-infinite porous medium. The standard datum is taken at y = 0, y being measured positively downward, and the surface y = 0 is taken to be horizontal. The applied head H(t) is measured positively upward from this surface, shown in Fig. 2.

The free surface $F(x,y,t) = 0$, which defines the fluid-gas interface can be written in the following form

$$F(x,y,t) = y - \zeta(x,t) = 0 . \tag{6}$$

What is sought is a one-dimensional solution for which the free surface would be horizontal, i.e., $\zeta = \zeta(t)$, so that $F = y - \zeta(t) = 0$.

The kinematical free surface condition is

$$\frac{DF}{Dt} = 0 \tag{7}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$.

Application of (7) to (6) yields the boundary condition where the subscript f denotes the free surface,

$$\frac{\partial F(y,t)}{\partial t} + v_f\frac{\partial F(y,t)}{\partial y} = 0 \tag{8}$$

at $y = \zeta$. Laplace's equation becomes, for $t > 0$ and $0 \leq y \leq \zeta(t)$

$$\frac{\partial^2\phi}{\partial y^2} = 0 \tag{9}$$

the general solution of which is

$$\phi(y) = Ay + B \tag{10}$$

The head, $h(y,t)$, is given by

$$h = \frac{P}{\gamma} - y \tag{11}$$

and at $y = 0$,

$$h = \frac{P}{\gamma} \tag{12}$$

In proceeding to derive the transient differential equation, the applied head, $h(0,t)$, is represented as $H(t)$, a function of time. From the equation of continuity it is seen that because $\bar{V} = v\underline{j}$, so that $v = v(y,t)$ reduces to $v = v(t)$.

Now the solution given by equation (10) is applicable, and boundary condition (12) becomes

$$\phi(0) = kH(t) \tag{13}$$

Since $\phi(0)$ is the value of the coefficient B, (10) becomes

$$\phi(y) = Ay + kH(t) \tag{14}$$

The filtration velocity may be defined in terms of a total discharge Q over a cross-section area A. Thus

$$v = \frac{Q}{A} \tag{15}$$

A relationship between the filtration velocity, given above, and the free surface velocity, given in equation (8), will now be developed.

The porosity of a given porous medium, denoted by ϵ, is defined as the ratio of open space, or voids, V', through which the fluid passes, to the total volume of porous media V containing the voids. Thus

$$\epsilon = \frac{V'}{V} = \frac{A'}{A} \tag{16}$$

where A is the total unit cross-section area, and A' is the area of pores in this total area.

The free surface velocity is, in terms of the total discharge Q and the pore area A',

$$\dot{\varsigma} = \frac{Q}{A'} \tag{17}$$

In view of (16) and (15), the desired relationship between the filtration and free surface velocities is

$$v = \epsilon \dot{\zeta} \tag{18}$$

Alternatively, substitution of (6) and (18) into (8) gives

$$\frac{\partial}{\partial t} [y - \zeta(t)] + \frac{v}{\epsilon} \frac{\partial}{\partial y} [y - \zeta(t)] = 0 ,$$

or

$$v = \epsilon \dot{\zeta} \tag{19}$$

Since v is independent of y, the free surface velocity is equal to a constant times the velocity in the interior.

Now from (11), since at $y = \zeta$, $P = 0$, (using gage pressure),

$$h = - y = - \zeta$$

on the free surface. Hence at $y = \zeta$, (14) becomes

$$- k \zeta = A \zeta + kH(t)$$

and

$$A = \frac{k[\zeta + H(t)]}{\zeta}$$

Substitution into (14) gives, for the velocity potential,

$$\phi = - \frac{k[\zeta + H(t)]}{\zeta} y + kH(t) \tag{20}$$

From (19), $\frac{\partial \phi}{\partial y} = - v = - \epsilon \dot{\zeta}$. (20) becomes, upon differentiation with respect to y,

$$\frac{\partial \phi}{\partial y} = - \frac{k[\zeta + H(t)]}{\zeta}$$

Elimination of $\frac{\partial \phi}{\partial y}$ from these expressions yields

$$\zeta \dot{\zeta} - \frac{k}{\epsilon} \zeta = \frac{k}{\epsilon} H(t) \tag{21}$$

Equation (21) is the one-dimensional differential equation for transient flow through porous media. Rearranging (21),

$$\frac{\zeta \, d\zeta}{\zeta + H(t)} = \frac{k}{\epsilon} dt \tag{22}$$

The physical significance of this one-dimensional approach will now be discussed as a limiting case of a two-dimensional problem.

Consider a rectangular channel as illustrated in Fig. 3, along with its steady state free surface. Water is contained in this rectangular channel, with fixed dimensions, at a specified height H_o. The free surface is of necessity a function of the two spatial variables, and approaches vertical asymptotes.

In the two-dimensional transient problem, the free surface is, of course, time dependent. The one-dimensional model has significance since the free stream lines for the two-dimensional steady state extend downward to a vertical asymptote. Hence W, the horizontal distance between the free stream lines at an infinite depth, is finite. Also, the difference W-b is a function of $\frac{H_o}{b}$, and as $b \to \infty$ $(\frac{H_o}{b} \to 0)$, $W \to b$ [9]. Therefore the one-dimensional problem can be considered an approximation to a two-dimensional problem having a large parameter i.e., channel width to depth ratio. It is noted, however, that the problem of flow in a half-plane $y \geq 0$ where

$$H = \begin{cases} H(t) & |x| \leq b/2 \\ 0 & |x| > b/2 \end{cases}$$

is not a one-dimensional one. Again, the one-dimensional approximation is valid for large b/H, and becomes exact as $H/b \to 0$.

3. THE FLOW BALANCE EQUATION

An equation can be found relating the total inflow, the head H(t), and the penetration depth $\zeta(t)$. Such an equation enables a specification of q; hence an expression for H(t), which appears in (22), may be found.

The flow balance equation can be stated as

TOTAL INFLOW = TOTAL FLUID REMAINING ABOVE THE LEVEL $y = 0$

+

TOTAL FLUID PENETRATED TO THE DEPTH $\zeta(t)$.

This is, of course, the equation of continuity in integrated form. Symbolically, the above may be written as

$$\int Qdt = AH(t) + A'\zeta(t) \tag{23}$$

Now by (15), the quantity $\frac{Q}{A}$ is the filtration velocity. Another term sometimes used for this quantity is the specific discharge, defined by the symbol

$$q = \frac{Q}{A} \tag{24}$$

which is the inflow in CFS per unit of horizontal area of basin. Inertion of (24) and (16) into (23) gives the result

$$\int qdt = H(t) + \epsilon\zeta(t) \tag{25}$$

Equation (25) is the flow balance equation.

4. SOME ELEMENTARY SOLUTIONS OF THE TRANSIENT FLOW EQUATION

For the problems to be considered in this section, the initial condition $\zeta(0) = 0$ will be assumed.

(1) CONSTANT HEAD $(H(t) = C)$

Under this assumption, (22) becomes

$$\frac{\zeta d\zeta}{\zeta + C} = \frac{k}{\epsilon} dt \tag{26}$$

The solution being

$$\zeta - C \log \frac{\zeta + C}{\zeta} = \frac{k}{\epsilon} t \tag{27}$$

Equation (27) is plotted, ζ versus log t, in Fig. 4, for a stated range of permeabilities and for a porosity of $\epsilon = 0.4$, and constant head C = 5 ft. The permeabilities listed along with Fig. 4 are for soils that range from coarse sands to tightly packed fine clays.

(2) CONSTANT SPECIFIC DISCHARGE (q = CONSTANT)

Under this assumption, (25) becomes

$$H(t) = qt - \epsilon\zeta(t). \tag{28}$$

Substitution of (43) into (36) yields

$$\frac{\zeta\, d\zeta}{(1-\epsilon)\zeta + qt} = \frac{k}{\epsilon}\, dt$$

Assuming a solution of the form $\zeta(t) = At$, where A is a constant, the following results

$$\zeta(t) = \left\{ \frac{\frac{k}{\epsilon}(1-\epsilon) + \sqrt{k^2\left(\frac{1-\epsilon}{\epsilon}\right)^2 + \frac{4k}{\epsilon} q}}{2} \right\} t \tag{29}$$

(29) shows that ζ is a linear function of time. We note that substitution of $\zeta(t) = At$ into (28) shows that the same form of solution (29) hold for $H(t) = (\text{const.}) \times t$ as for $q = \text{const.}$

(3) $q = 0$, $t \geq 0$, AND A DETERMINATION OF THE TOTAL TIME FOR DRAINAGE

The given condition on q implies a sudden stopping of flow. This means that an initial head of, say, size H_o, remains. Hence (25), for later times, becomes

$$H_o = H(t) + \epsilon\zeta(t),$$

and substitution into (22) gives

$$\frac{\zeta\, d\zeta}{(1-\epsilon)\zeta + H_o} = \frac{k}{\epsilon}\, dt$$

Integration gives the solution

$$\frac{1}{1-\epsilon}\left\{(1-\epsilon)\zeta - H_o \log\left[\frac{H_o + (1-\epsilon)\zeta}{H_o}\right]\right\} = \frac{k}{\epsilon} t \tag{30}$$

Now the basin is completely drained when $H(t) = 0$. Hence (25) becomes

$$\epsilon\zeta = H_o$$

Substituting for ζ into (30) gives the time for complete drainage. Thus

$$t = \frac{\epsilon H_o}{k(1-\epsilon)^2}\left[\frac{1-\epsilon}{\epsilon} + \log\epsilon\right] \tag{31}$$

(4) $H(t) = (\text{CONST.}) \times t = Ct,\ 0 \leq t \leq t_o$; $H(t) = \text{CONST.} = Ct_o,\ t_o < t \leq t_1$;

$q = 0,\ t > t_1$

An application of this problem will be described later on, in the discussion of a "Design Storm."

In view of the discussion at the end of subsection (2) of this section, the specification $H(t) = Ct$ corresponds to a constant q. Also, considering solution (27) along with the flow balance equation, the specification $H(t) = Ct_o$ corresponds to a transcendentally decreasing q. The plot of q versus t is shown in Fig. 5.

For the interval $0 \leq t \leq t_o$, substitution of $H(t) = Ct$ into (22) yields

$$\frac{\zeta d\zeta}{\zeta + Ct} = \frac{k}{\epsilon}\, dt$$

The solution being

$$\zeta(t) = \left\{ \frac{\frac{k}{\epsilon} + \sqrt{\left(\frac{k}{\epsilon}\right)^2 + \frac{4Ck}{\epsilon}}}{2} \right\} t \qquad (32)$$

For the interval $t_o < t \leq t_1$, the solution is obtained as in subsection (1), with the initial condition being (32) at $t = t_o$. Thus

$$\zeta + Ct_o \log\left\{ \frac{\left(\frac{a}{2} + C\right)t_o}{\zeta + Ct_o} \right\} = \frac{k}{\epsilon} t + \frac{b}{2} t_o \qquad (33)$$

where

$$a = \frac{k}{\epsilon} + \sqrt{\left(\frac{k}{\epsilon}\right)^2 + \frac{4Ck}{\epsilon}}, \quad b = -\frac{k}{\epsilon} + \sqrt{\left(\frac{k}{\epsilon}\right)^2 + \frac{4Ck}{\epsilon}}$$

For $t > t_1$, equation (25) becomes

$$H(t_1) + \epsilon\zeta(t_1) = H(t) + \epsilon\zeta(t) \qquad (34)$$

where $\zeta(t_1)$ is the initial condition obtained from (33) as an initial condition at the point t_1. Substitution into (22) yields

$$\frac{\zeta d\zeta}{(1-\epsilon)\zeta + V} = \frac{k}{\epsilon} dt$$

where $V = H(t_1) + \epsilon\zeta(t_1)$

Integration yields the solution

$$\frac{1}{(1-\epsilon)^2}\left\{\left[(1-\epsilon)\zeta - \zeta(t_1)\right] - V \log\left[\frac{V + (1-\epsilon)\zeta}{V + (1-\epsilon)\zeta(t_1)}\right]\right\} \frac{k}{\epsilon}(t-t_1) \qquad (35)$$

By means of equations (32),(33), and (35) one may easily obtain the overall $\zeta(t)$ as a function of $q(t)$ where the latter is given in Fig. 5.

5. SOLUTION OF THE TRANSIENT FLOW EQUATION FOR AN ARBITRARY INITIAL CONDITION ON ζ WITH q = CONSTANT

In the previous section, solutions in the functional form $t = t(\zeta)$ were obtained. For the arbitrary initial condition on the penetration depth, however, both ζ and t are given implicitly. This is shown next.

The arbitrary initial condition is taken to be $\zeta(0) = \zeta_0$, and the initial head to be H_o.

Equation (25) reads, for the given conditions

$$H(t) + \epsilon\zeta(t) = qt + H_o + \epsilon\zeta_o = qt + V$$

Substitution into (22) for $H(t)$ gives

$$\frac{\zeta d\zeta}{(1-\epsilon)\zeta + qt + V} = \frac{k}{\epsilon} dt$$

Letting $u = t + a$ and setting $a = \frac{V}{q}$,

$$\frac{\zeta d\zeta}{(1-\epsilon)\zeta + qu} = \frac{k}{\epsilon} du \qquad (36)$$

(36) can be made separable by means of change of variables given by $\zeta = xu$, whereby the differential equation becomes

$$\frac{du}{u} = -\frac{xdx}{x^2 - \frac{k}{\epsilon}(1-\epsilon)x - \frac{k}{\epsilon}q} \qquad (37)$$

Integration, use of the initial condition, and substitution back for $\zeta(t)$ provides the result

$$\left(\frac{t+a}{a}\right)\left[\frac{\left(\frac{\zeta}{t+a}\right)^2 - b\,\frac{\zeta}{t+a} - \frac{k}{\epsilon}\,q}{\left(\frac{\zeta_o}{a}\right)^2 - b\,\frac{\zeta_o}{a} - \frac{k}{\epsilon}\,q}\right]^{\frac{1}{2}} \left[\frac{\left(\frac{2\zeta}{t+a} - b - d\right)\left(\frac{2\zeta_o}{a} - b + d\right)}{\left(\frac{2\zeta}{t+a} - b + d\right)\left(\frac{\zeta}{a} - b - d\right)}\right]^{\frac{b}{2d}} = 1 \tag{38}$$

where

$$b = \frac{k}{\epsilon}\,(1-\epsilon), \qquad d = \sqrt{\left(\frac{1-\epsilon}{\epsilon}\right)^2 k^2 + \frac{4k}{\epsilon}\,q}$$

This result for an arbitrary initial condition and constant q may also be used to solve a problem for which an arbitrary q(t) curve is given, by approximating the curve in small intervals by lines of constant q. Note that (38) must be rewritten for a non-zero initial instant, i.e., for $\zeta(t_1) = \zeta_1$.

However, it is more convenient in such cases to work directly with (22) and (25) in a finite difference scheme. In finite difference form, (22) becomes

$$\Delta\zeta = \frac{k}{\epsilon}\left[\frac{\zeta + H(t)}{\zeta}\right]\Delta t \tag{39}$$

For the given initial conditions, (25) is written as

$$H(t) + \epsilon\zeta(t) = H_o + \epsilon\zeta_o\, q_j'\, t \tag{40}$$

where q_j' is the average value of q in the subinterval (t_j, t_{j+1}) of the q(t) curve. By use of (39) and (40), a finite difference solution may be constructed, as shown in Fig. 6.

6. DIMENSIONLESS VARIABLES AND THE DIMENSIONLESS FREE-SURFACE VELOCITY

Several of the results previously obtained lend themselves to analysis by means of dimensionless variables. Introducing them to be

$$\xi = \frac{\zeta}{H} \tag{41}$$

and

$$\tau = \frac{k}{\epsilon H} t \tag{42}$$

the differential equation (26) becomes, with H = C

$$\frac{\xi \, d\xi}{\xi + 1} = d\tau \tag{43}$$

The solution is, with $\xi(0) = 0$,

$$\xi - \log(\xi + 1) = \tau \tag{44}$$

A graph of (44) appears in Fig. 7.

Differentiation of (44) with respect to τ yields the dimensionless free surface velocity

$$\frac{d\xi}{d\tau} = \frac{\xi + 1}{\xi} \tag{45}$$

A graph of $\frac{d\xi}{d\tau}$ versus τ , obtained using (45) and (44), is shown in Fig. 8.*

From Fig. 8, it is seen that $\frac{d\xi}{d\tau}$ is infinite at the initial instant. Since this is not physically realizable, it must be concluded that either the analysis is incomplete or that the physical model is incorrect. Recent investigations have shown that the singularity at the origin ($\tau = 0$) is removed when inertial terms are included in the differential equation. However, for times other than the initial instant the inertia effects decay rapidly. The model chosen in this study is, therefore, valid except for the aforementioned limitation [10].

7. A DESIGN STORM FOR GIVEN PERTINENT PARAMETERS

Up to this point, equation (22) has been solved with the aid of

* This discussion follows [3].

(25), for specified $H(t)$ and/or $q(t)$ curves. In this section it will be shown that the $q(t)$ and $H(t)$ curves can be designed to correspond to the following data supplied for a storm: (a) the maximum intensity, q_{max}; (b) the total inflow, $\int q dt \equiv V^*$, and (c) the total duration, t_2 ($q = 0$, $t \geq t_2$).

Consider the curve of a typical storm, shown in Fig. 9. After a short time, the intensity reaches a maximum value and then tapers off gradually. This actual $q(t)$ curve may be approximated by a constant q, q_{max}, in the interval $0 \leq t \leq t_1$, some transcendentally decreasing $q(t)$ [see subsection (4) of section 4] in $t_1 < t \leq t_2$, and $q(t) = 0$ for $t > t_2$. This is shown in Fig. 10. It should be noted that, whereas the response would probably not vary much with any of several different approximations, the possibility of analytical computations depends upon being able to provide a specific type of approximation, hence Fig. 10.

In view of the discussion in section 4, subsection (2) and (4), the $H(t)$ curve may be constructed as shown in Fig. 11.

The following quantities must now be obtained: (1) the depths of penetration in each interval; (2) the expression for the $q(t)$ curve, in the second interval; (3) the complete $H(t)$ curve; and (4) the value of t_1.

By use of (25), equation (22) becomes, in each interval,

in $0 \leq t \leq t_1$:
$$\frac{\zeta d\zeta}{(1-\epsilon)\zeta + q_{max} t} = \frac{k}{\epsilon} dt \qquad (46)$$

in $t_1 < t \leq t_2$
$$\frac{\zeta d\zeta}{\zeta + C} = \frac{k}{\epsilon} dt \qquad (47)$$

in $t > t_2$
$$\frac{\zeta d\zeta}{(1-\epsilon)\zeta + V} = \frac{k}{\epsilon} dt \qquad (48)$$

where $V = H(t_2) + \zeta(t_2) = C + \zeta(t_2)$ with C yet to be determined. Introduction of (41) and (42) into (47), (46), and (48) gives

$$\frac{\xi \, d\xi}{\xi + \frac{q_{max}}{k}\left(\frac{\epsilon}{1-\epsilon}\right)\tau} = (1-\epsilon)d\tau \tag{49}$$

$$\frac{\xi \, d\xi}{\xi + 1} = d\tau \tag{50}$$

$$\frac{\xi \, d\xi}{\xi + \frac{1}{1-\epsilon}[1+\epsilon\,\xi(\tau_2)]} = (1-\epsilon)d\tau \tag{51}$$

The solutions are, with the initial condition $\xi(0) = 0$,

$(0 \leq \tau \leq \tau_1)$

$$\xi(\tau) = \frac{(1-\epsilon) + \sqrt{(1-\epsilon)^2 + \frac{4q_{max}\,\epsilon}{k}}}{2}\,\tau \tag{52}$$

$(\tau_1 \leq \tau \leq \tau_2)$

$$\xi(\tau) + \log\frac{a\tau_1 + 1}{\xi(\tau) + 1} = \tau + \tau_1(a-1) \tag{53}$$

where a is the coefficient of τ in (52),

$(\tau > \tau_2)$

$$\xi(\tau) - \xi(\tau_2) + \frac{1}{1-\epsilon}[1+\epsilon\,\xi(\tau_2)]\log\left\{\frac{\frac{1}{1-\epsilon}[1+\xi(\tau_2)]}{\frac{1}{1-\epsilon}[1+\xi(\tau_2)]+\xi}\right\} \tag{54}$$

$$= (1-\epsilon)(\tau - \tau_2)$$

Equations (52), (53), and (54) specify the $\xi(\tau)$ curve.

In order to solve for q(t) in $t_1 \leq t \leq t_2$, introduce the dimensionless variables (41) and (42) into the flow balance equation (25) to get

$$\epsilon\,\xi(\tau) = \epsilon\,\xi(\tau_1) + \int\frac{\epsilon}{k}\,q(\tau)d\tau$$

Differentiation with respect to τ yields

$$\frac{d\xi}{d\tau} = \frac{q(\tau)}{k}$$

But from (50)

$$\frac{d\xi}{d\tau} = \frac{\xi + 1}{\xi}$$

Hence, in $t_1 < t \leq t_2$

$$q(\tau) = k \frac{\xi + 1}{\xi} \tag{55}$$

Thus, knowing ξ as a function of τ from (53), (τ, as well as C, will subsequently be found) conversion back to dimensional variables will give ζ as a function of t. Now substituting (41) into (55) gives q(t) in terms of ζ. Hence the q(t) curve is obtained.

A flow balance in the interval $0 \leq t \leq t_1$ gives

$$H(t) + \epsilon \zeta(t) = \int q_{max} dt = q_{max} t \tag{56}$$

Also, in $0 \leq t \leq t_1$, $H(t) = \alpha t$, and the dimensional solution is $\zeta(t) = At$, A being the coefficient of t in (29), with $q = q_{max}$. Substituting these results into (56) gives for α,

$$\alpha = q_{max} - \epsilon A \tag{57}$$

or

$$H(t) = (q_{max} - \epsilon A)t \qquad 0 \leq t \leq t_1 \tag{58}$$

In order to find H = C, note that, if the total inflow V^* is given, reference to Fig. 9 yields

$$q_{max} t_1 + \int_{t_1}^{t_2} q(t)dt = V^*$$

Substitution of (42) and division by H(=C) leaves

$$\epsilon \frac{q_{max}}{k} \tau_1 + \int_{\tau_1}^{\tau_2} \frac{\epsilon}{k} q(\tau) d\tau = \frac{V^*}{H} \tag{59}$$

Now a flow balance at t_2 reads

$$\epsilon \zeta(t_2) + H(t_2) = \epsilon \zeta(t_1) + H(t_1) + \int_{t_1}^{t_2} q(t)\,dt$$

Introduction of (41) and (42) into the above yields, since $H(t_2) = H(t_1) = C$,

$$\epsilon\,[\xi(\tau_2) - \xi(\tau_1)] = \int_{\tau_1}^{\tau_2} \frac{\epsilon}{k}\, q(\tau)\,d\tau \qquad (60)$$

Now substitution of (60) for the integral in (59) gives

$$\epsilon\left[\frac{q_{max}}{k}\tau_1 + \xi(\tau_2) - \xi(\tau_1)\right] = \frac{V^*}{H}$$

But, since $H = C$,

$$H(t) = C = \frac{V^*}{\epsilon\left[\frac{q_{max}}{k}\tau_1 + \xi(\tau_2) - \xi(\tau_1)\right]} \qquad (61)$$

Equation (61) gives $H(t)$ for $t_1 < t \leq t_2$. Finally, in $t > t_2$, (25) reads

$$\epsilon \zeta(t) + H(t) = \epsilon \zeta(t_2) + H(t_2)$$

Divide by $H = C = H(t_2)$ to get

$$\epsilon \xi(\tau) + \frac{H(t)}{C} = \epsilon \xi(\tau_2) + 1$$

or

$$H(t) = C\,[\epsilon(\xi(\tau_2) - \xi(\tau)) + 1] \qquad (62)$$

Hence, equations (58), (61), and (62) completely specify the $H(t)$ curve.

The remaining parameter, t_1, is found by using (42). Thus

$$H = C = \alpha t_1 = \alpha\left[\frac{\epsilon H}{k}\tau_1\right] = \alpha\left(\frac{\epsilon C}{k}\right)\tau_1 \qquad (63)$$

From which

$$C = \frac{\alpha \epsilon C}{k}\tau_1$$

or

$$\tau_1 = \frac{k}{\alpha\epsilon} \tag{64}$$

Now the dimensional time t_1 is obtained. From (63)

$$C = \alpha t_1$$

or

$$t_1 = \frac{C}{\alpha} \tag{65}$$

Substitution of (61) and (57) into (65) gives

$$t_1 = \frac{V^*}{\epsilon\,(q\ max - \epsilon A)\ [\frac{q\ max}{k}\ \tau_1 + \xi(\tau_2) - \xi(\tau_1)]} \tag{66}$$

Knowing $\xi(\tau_1)$, $\xi(\tau_2)$ and τ_1, t_1 may be evaluated.

With the expressions for the penetration depth and the specification of the q(t) and H(t) curves, the design problem has been solved.

8. FLOW THROUGH POROUS MEDIA WITH AN IMPERMEABLE BARRIER AT A UNIFORM FINITE DEPTH

All of the preceding discussion has dealt with infinite porous media. In this section, a differential equation that describes the behavior of the free surface $\zeta(t)$ passing through a media of bounded depth will be derived, the boundary being impermeable. This physical problem is illustrated in Fig. 12, the finite depth being labelled ζ^*. Examples of such a model may be found abundantly in nature.

Following the development in section 2, it is seen that Laplace's equation is again applicable, its general solution is given by, in the one-dimensional case, equation (10),

$$\phi = Ay + B$$

Boundary conditions are needed to evaluate A and B.

Now from definition (11), at $y = 0^-$ (i.e., approaching $y = 0$ from y negative),

$$h = \frac{P(0^-,t)}{\gamma}$$

Since $\phi = kh$,

$$\phi(0) = k\,\frac{P(0^-,t)}{\gamma} = B$$

For the ensuing development, as will be seen, the absolute pressure rather than the gage pressure must be used. Thus, at $y = 0^-$,

$$P(0^-,t) = P_{gage} + P_{at} = \gamma H(t) + P_{at} \tag{67}$$

Hence

$$B = k\,[H(t) + \frac{P_{at}}{\gamma}] \tag{68}$$

Now at $y = \zeta$,

$$h = \frac{P(\zeta\,,t)}{\gamma} - \zeta$$

Thus, since $\phi = kh$, use of (10) and (68) at $y = \zeta$ gives

$$k[\frac{P(\zeta\,,t)}{\gamma} - \zeta\,] = A\zeta + k[H(t) + \frac{P_{at}}{\gamma}] \tag{69}$$

To evaluate A it is necessary that $P(\zeta,t)$ be known. Unlike the problems previously studied, the pressure at the interface is not equal to a constant; the thermodynamics of the air-compression process must be taken into consideration.

If it is assumed that the air trapped between the free surface and the impenetrable barrier is compressed in a quasi-steady process, and that the thermodynamic behavior of air may be approximated by an ideal gas, the equation

$$PV = nRT \tag{70}$$

may be used. P is the pressure, V the volume, n the number of moles of gas, R the universal gas constant, and T the temperature. Further, if the process is assumed isothermal, the pressure and volume at the initial instant when $y = 0$ ($\zeta = 0$) may be related to that at the instant when $y = \zeta$. Thus, from (70),

$$P(0^+,t)V(0^+,t) = P(\zeta,t)V(\zeta,t) \tag{71}$$

Note that the value of $P(0^+,t)$ in (71) is just the pressure of the enclosed mass of air, here taken as atmospheric, whereas in (67) $P(0^-,t)$ represents the pressure exerted by the air and water <u>above</u> the surface. Naturally, for free surface motion, there may be a discontinuity in pressure at the level $y = 0$, $t < 0$.

Now if a unit area is considered, $V(0^+,t) = \zeta^*$, and $V(\zeta,t) = \zeta^* - \zeta$. Hence (71) becomes

$$P_{at}\zeta^* = P(\zeta,t)(\zeta^* - \zeta)$$

Solving for $P(\zeta,t)$ gives

$$P(\zeta,t) = \frac{P_{at}\zeta^*}{\zeta^* - \zeta} \tag{72}$$

Substitution of (72) into (69) yields, upon solution for A,

$$A = \frac{k}{\zeta}\left[\frac{P_{at}}{\gamma}\,\frac{\zeta^*}{\zeta^* - \zeta} - \zeta - H(t) - \frac{P_{at}}{\gamma}\right] \tag{73}$$

Substitution of (73) and (68) into (10) gives the result

$$\phi = \frac{k}{\zeta}\left[\frac{P_{at}}{\gamma}\,\frac{\zeta^*}{\zeta^* - \zeta} - \zeta - H(t) - \frac{P_{at}}{\gamma}\right]y + k\left[H(t) + \frac{P_{at}}{\gamma}\right] \tag{74}$$

Now by (19)

$$\frac{\partial\phi}{\partial y} = -v = -\epsilon\dot{\zeta}$$

Hence, substituting for $\frac{\partial\phi}{\partial y}$, obtained from (74), gives

$$-\epsilon\dot{\zeta} = \frac{k}{\zeta}\left[\frac{P_{at}}{\gamma}\left(\frac{\zeta^*}{\zeta^* - \zeta}\right) - \zeta - H(t) - \frac{P_{at}}{\gamma}\right]$$

Some rearrangement yields the result

$$\frac{(\zeta - \zeta^*)\zeta\, d\zeta}{\zeta^2 - \left[\zeta^* - H(t) - \frac{P_{at}}{\gamma}\right]\zeta - \zeta^* H(t)} = \frac{k}{\epsilon}\, dt \tag{75}$$

Equation (75) is the differential equation for the free surface with the presence of a barrier; it shall be called the barrier flow equation.

9. SOLUTION OF THE DIFFERENTIAL EQUATION FOR BARRIER FLOW WITH H(t) = CONSTANT

For H(t) = Constant = C, equation (75) becomes

$$\frac{(\zeta - \zeta^*)\,\zeta\, d\zeta}{\zeta^2 - \left[\zeta^* - C - \frac{P_{at}}{\gamma}\right]\zeta - C\zeta^*} = \frac{k}{\epsilon}\, dt \tag{76}$$

Integration, using $\zeta(0) = 0$, gives

$$\zeta + \frac{1}{r_2 - r_1}\left[\left\{(\alpha - \zeta^*)r_2 + \beta\right\}\log\left|\frac{\zeta - r_2}{-r_2}\right| - \left\{(\alpha - \zeta^*)r_1 + \beta\right\}\log\left|\frac{\zeta - r_1}{r_1}\right|\right] = \frac{k}{\epsilon}\, t \tag{77}$$

where

$$\alpha = \zeta^* - C - \frac{P_{at}}{\gamma}, \quad \beta = C\zeta^*, \quad r_1, r_2 = \frac{\alpha \pm \sqrt{\alpha^2 + 4\beta}}{2}$$

Because $\alpha^2 + 4\beta > 0$, r_1 and r_2 are real.

Equation (77) is the solution to the barrier flow equation for H(t)= CONST. It is seen that as $\zeta \rightarrow r_1$, $t \rightarrow +\infty$, since the second log term, with the minus sign goes to $+\infty$. It is also noted that the value of r_1 is always less than that of r_2, for if ζ would traverse the value r_2 on its way to r_1, the first term would approach $-\infty$, which is physically impossible. Thus the value $\zeta = r_1$, say ζ_1, gives the value for the maximum penetration depth.

By substitution of the dimensionless variables $\xi = \frac{\zeta}{\zeta^*}$ and $\tau = \frac{k}{\epsilon H}$ into (76) and integration, there results the solution

$$\xi + \frac{1}{R_2 - R_1}\left[\left\{\left(\frac{\alpha}{\zeta^*} - 1\right) R_2 + \frac{\beta}{\zeta^{*2}}\right\} \log\left|\frac{\xi - R_2}{- R_2}\right| - \left\{\left(\frac{\alpha}{\zeta^*} - 1\right) R_1 + \frac{\beta}{\zeta^{*2}}\right\} \log\left|\frac{\xi - R_1}{- R_1}\right|\right] = \tau \tag{78}$$

where $R_1 = \frac{r_1}{\zeta^*}$ and $R_2 = \frac{r_2}{\zeta^*}$. Also, using the argument of the previous paragraph, R_1 is the maximum value of the dimensionless penetration depth. A plot of (78), using several values of ζ^*, is shown in Fig. 13. For a proper comparison to the ζ/τ plot shown in Fig. 7, the dimensionless variables used are those given by (41) and (42). These are easily obtained from the present dimensionless variables by multiplication by ζ^*/H. In Fig. 13, the asymptotic behavior of $\xi(\tau \to \infty)$ is shown, along with the comparison to $\zeta^* = \infty$.

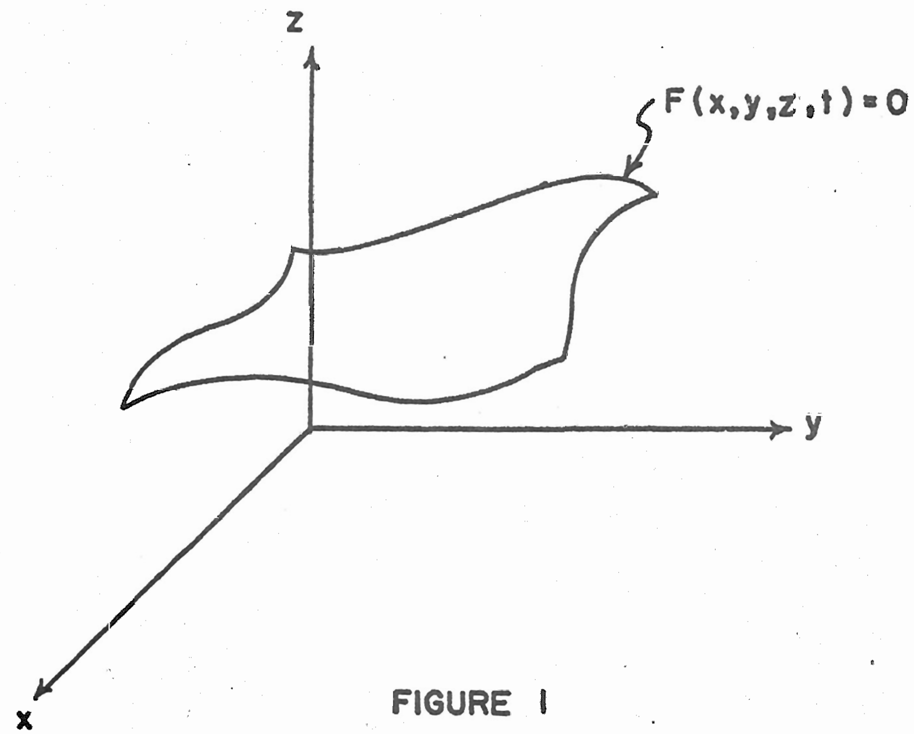


FIGURE 1

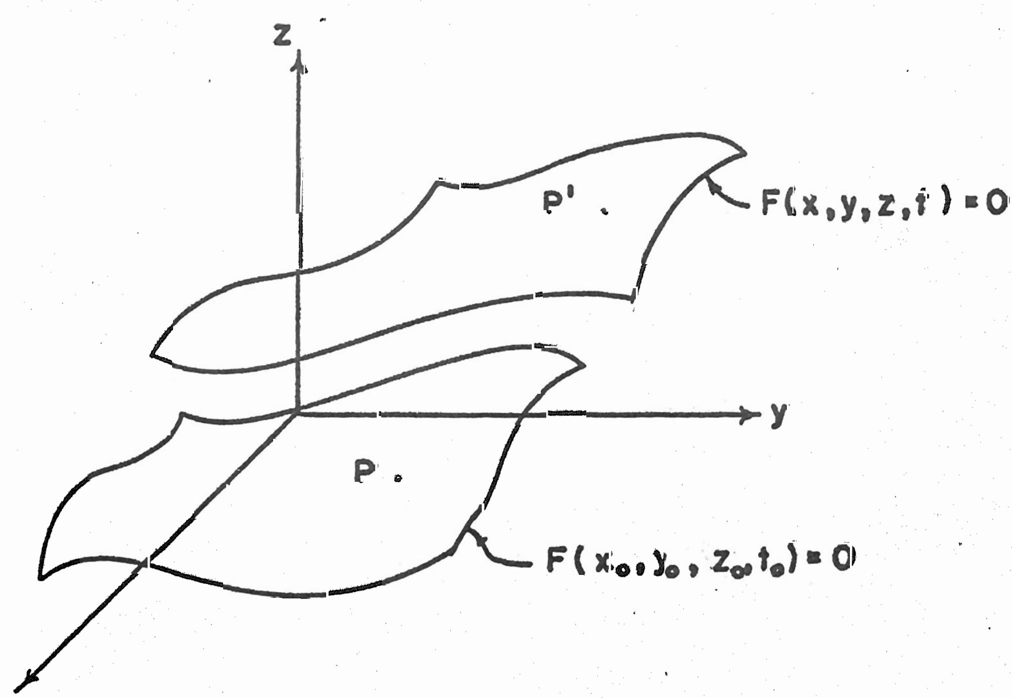


FIGURE 2

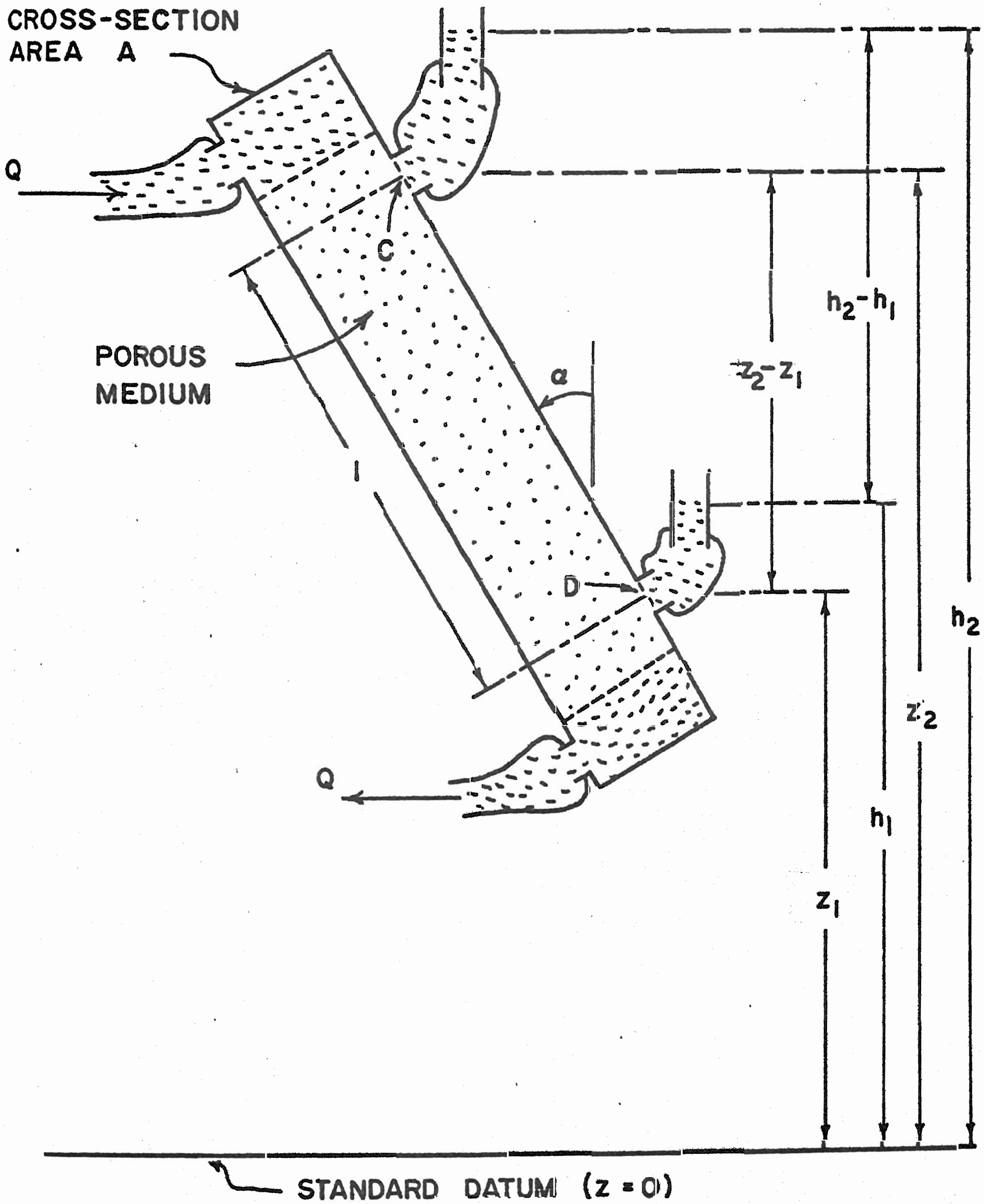

CROSS-SECTION
AREA A
Q
C
POROUS
MEDIUM
l
α
D
Q
$h_2 - h_1$
$z_2 - z_1$
h_2
z_2
h_1
z_1
STANDARD DATUM (z = 0)

FIGURE 3

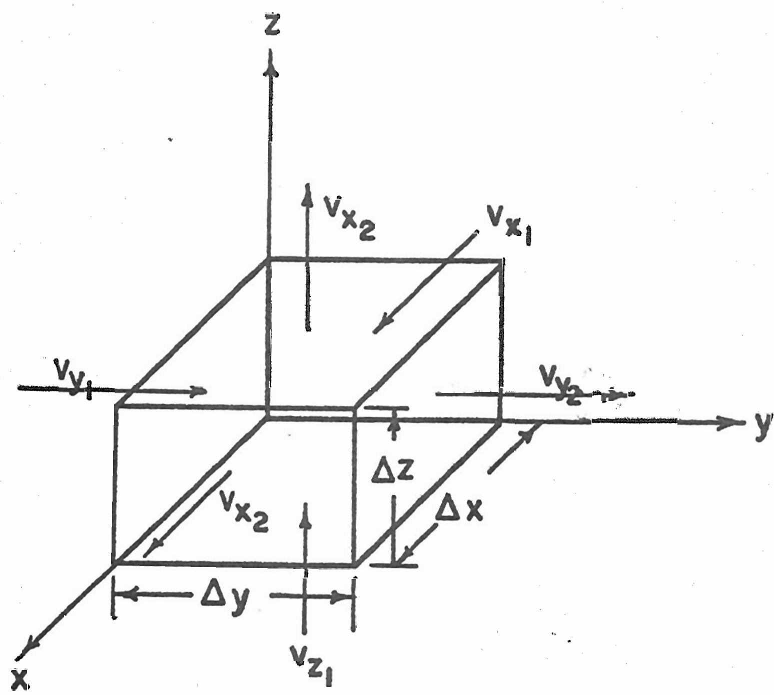


FIGURE 4

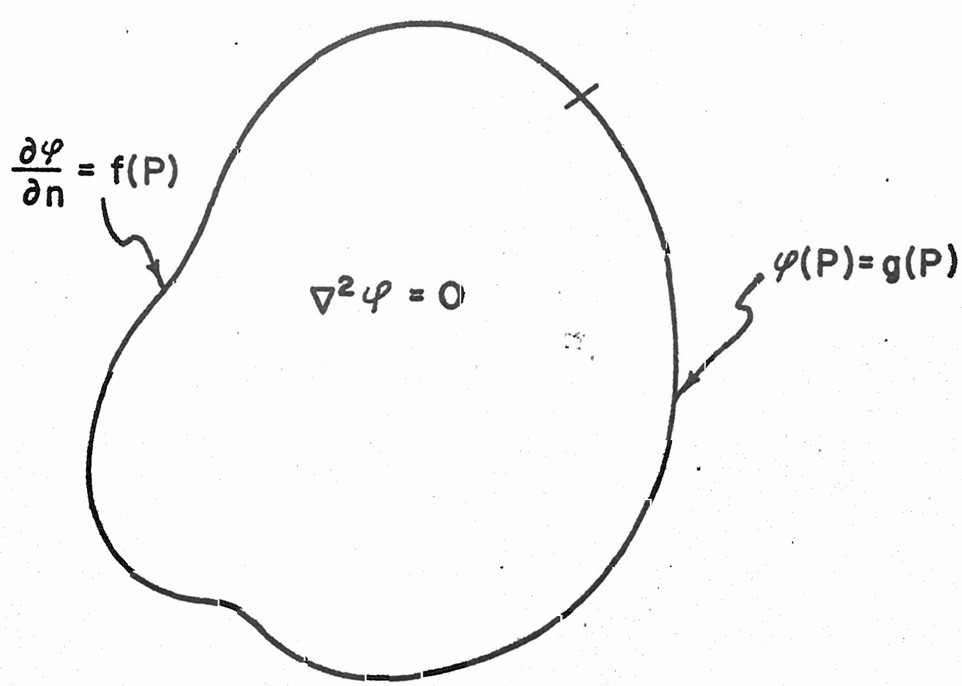


FIGURE 5

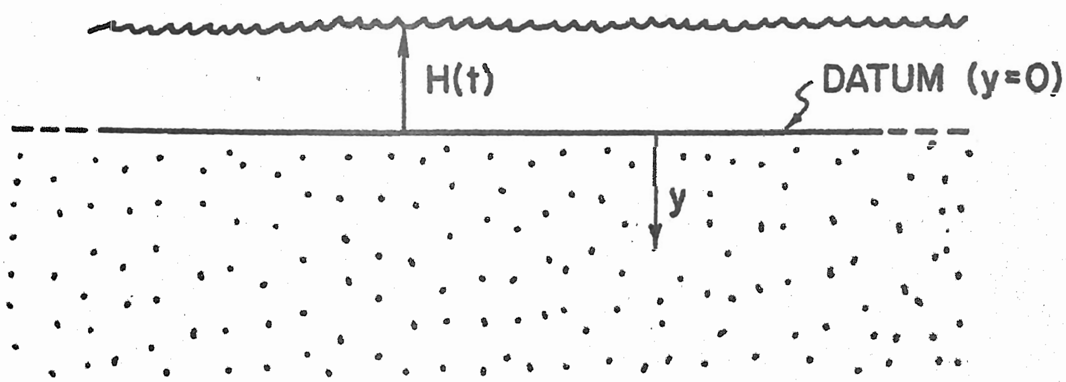


FIGURE 6

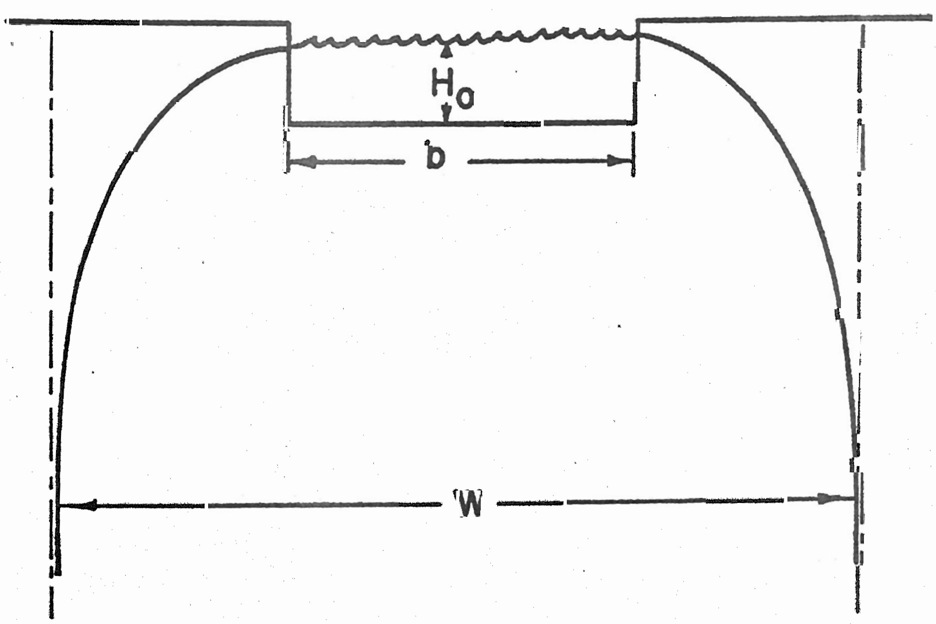
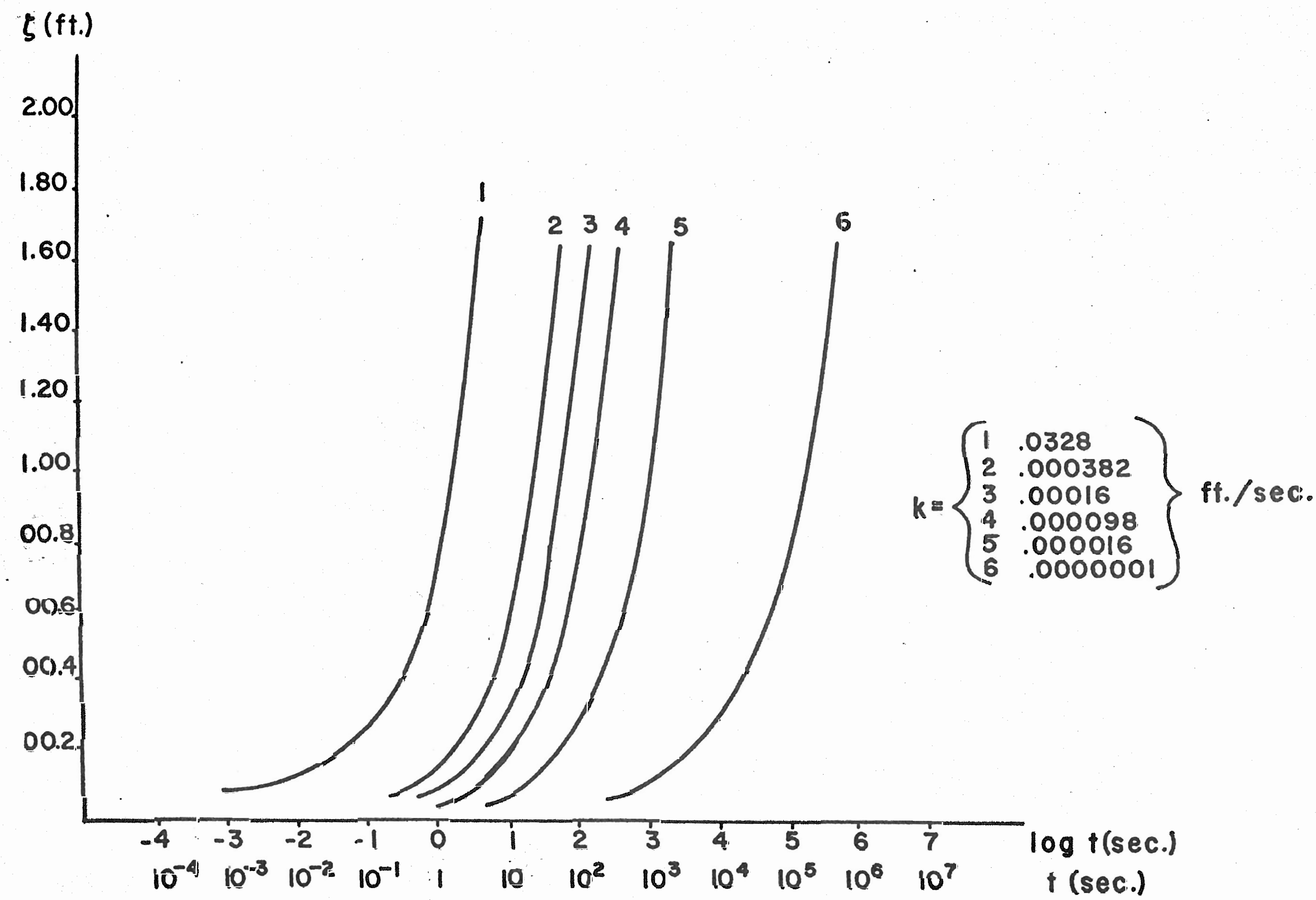


FIGURE 7



ζ (ft.)
2.00
1.80
1.60
1.40
1.20
1.00
00.8
00.6
00.4
00.2
1
2
3
4
5
6
k =
1 .0328
2 .000382
3 .00016
4 .000098
5 .000016
6 .0000001
ft./sec.
-4 -3 -2 -1 0 1 2 3 4 5 6 7
log t(sec.)
10^-4 10^-3 10^-2 10^-1 1 10 10^2 10^3 10^4 10^5 10^6 10^7
t (sec.)

FIGURE 8

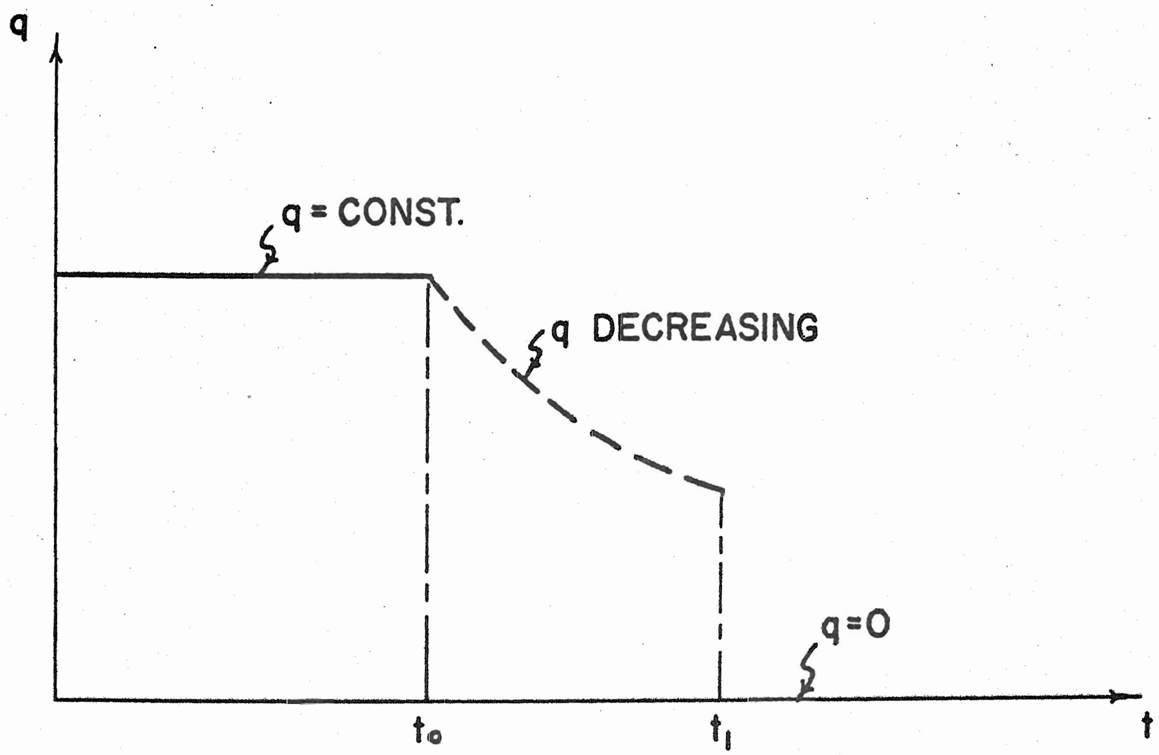


FIGURE 9

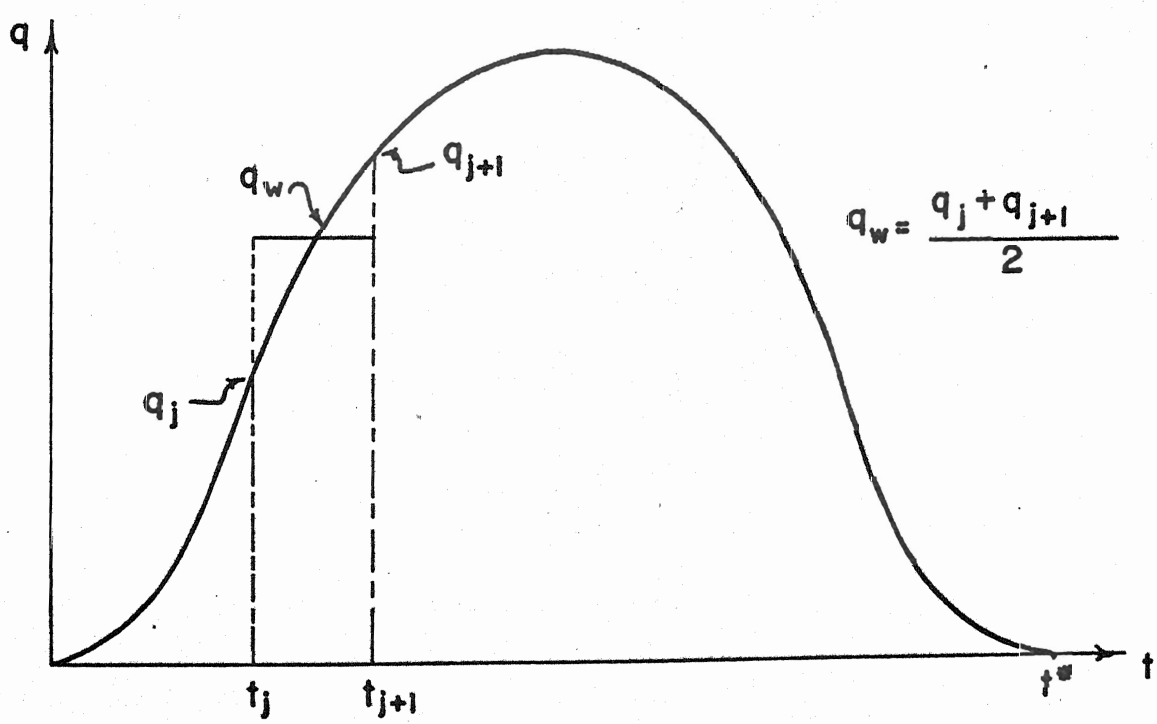


FIGURE 10

t	Δt	$\Delta\zeta$	q	ζ	$H(t)$
0	Δt_1	$\Delta\zeta_1$	q_1'	ζ_0	H_0
t_1	Δt_2	$\Delta\zeta_2$	q_2'	$\zeta_0+\Delta\zeta_1=\zeta_1$	H_1
t_2	Δt_3	$\Delta\zeta_3$	q_3'	$\zeta_1+\Delta\zeta_2=\zeta_2$	H_2
t_3	Δt_4	$\Delta\zeta_4$	q_4'	$\zeta_2+\Delta\zeta_3=\zeta_3$	H_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t_i	Δt_{i+1}	$\Delta\zeta_{i+1}$	q'_{i+1}	$\zeta_{i-1}+\Delta\zeta_1=\zeta_i$	H_i

FIGURE 11

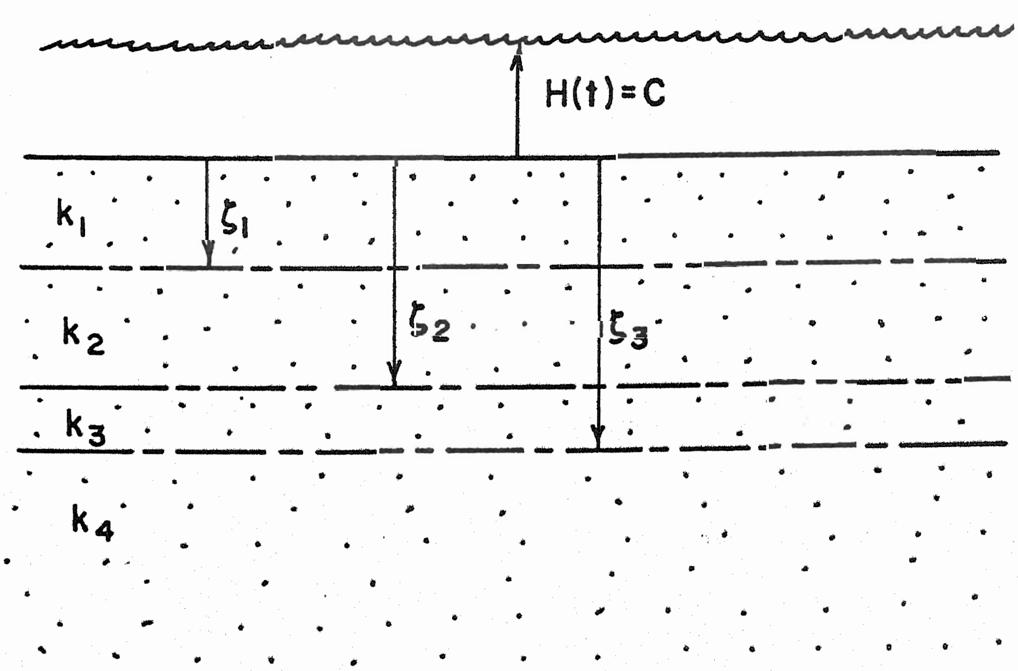


FIGURE 12

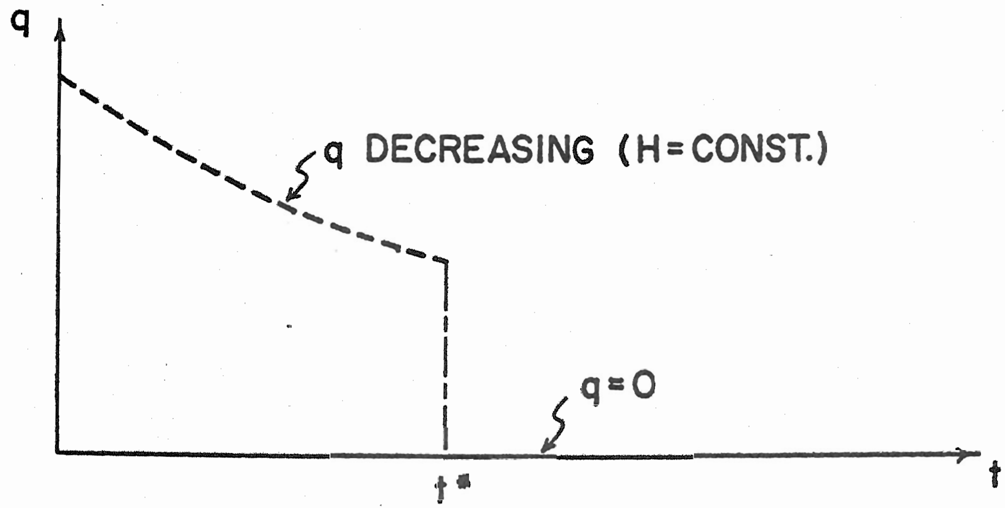


FIGURE 13

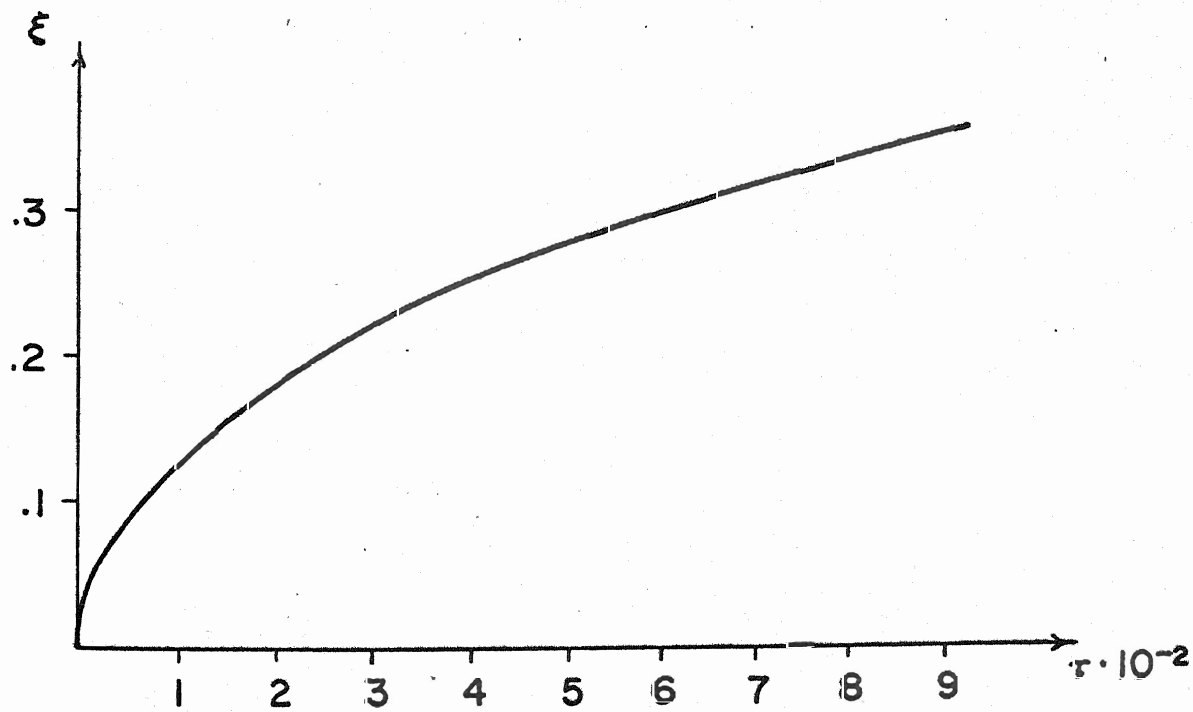


FIGURE 14

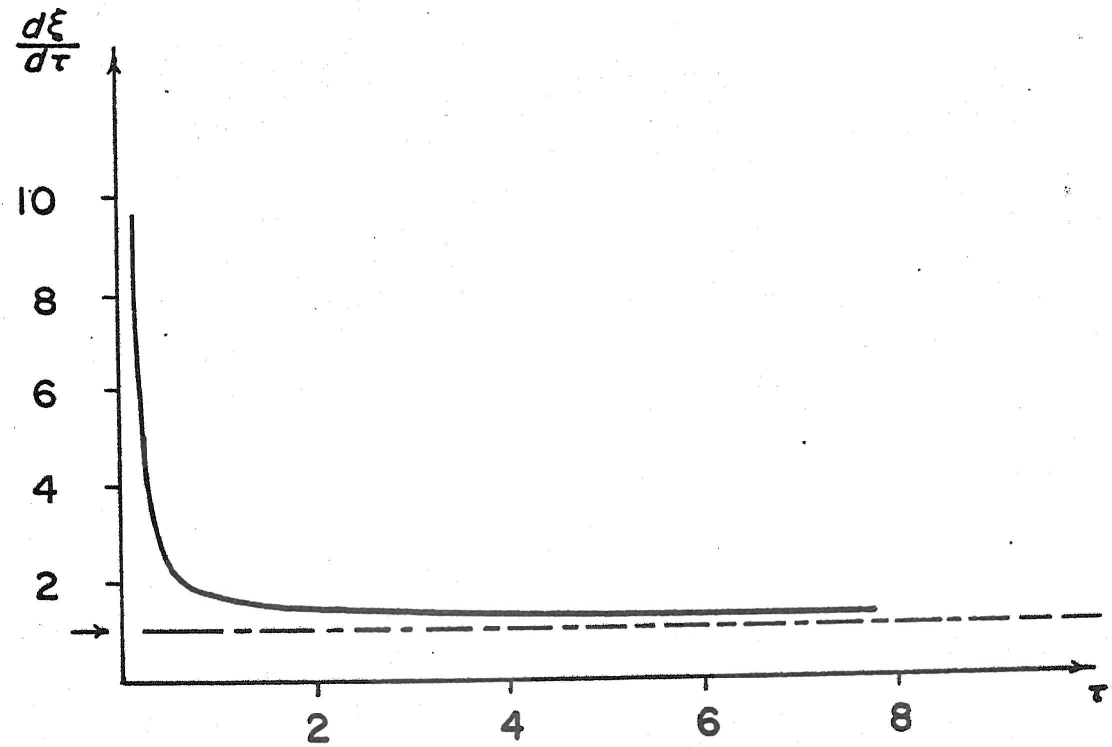


FIGURE 15

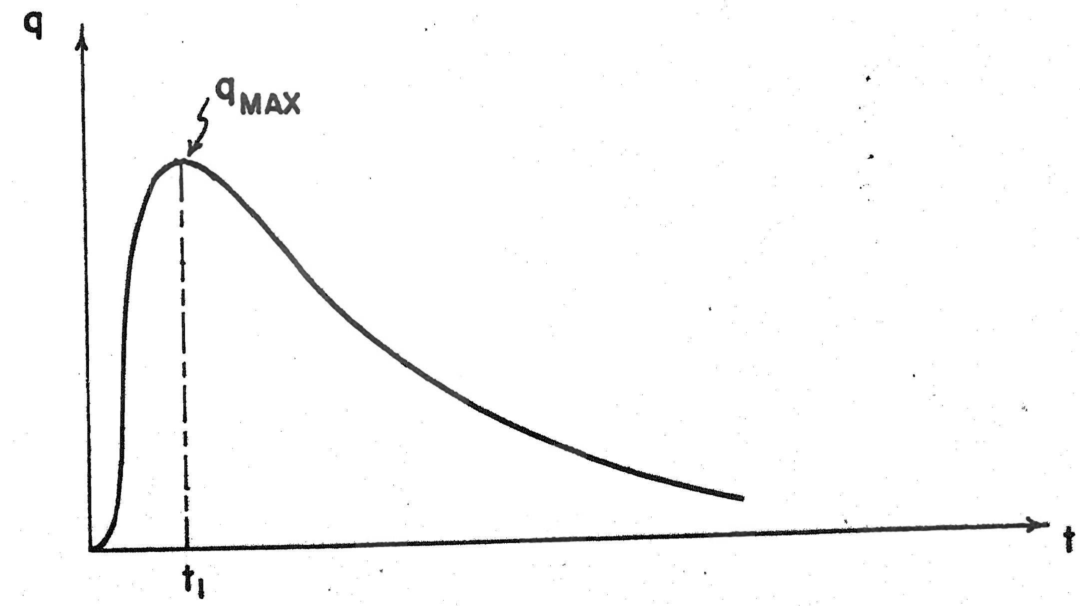


FIGURE 16

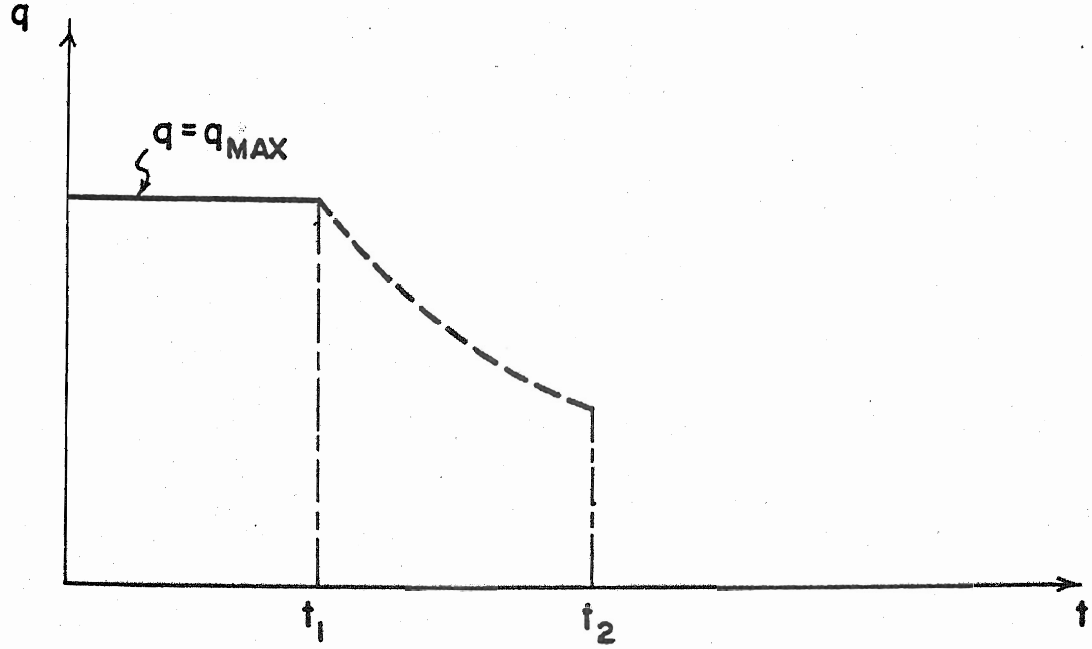


FIGURE 17

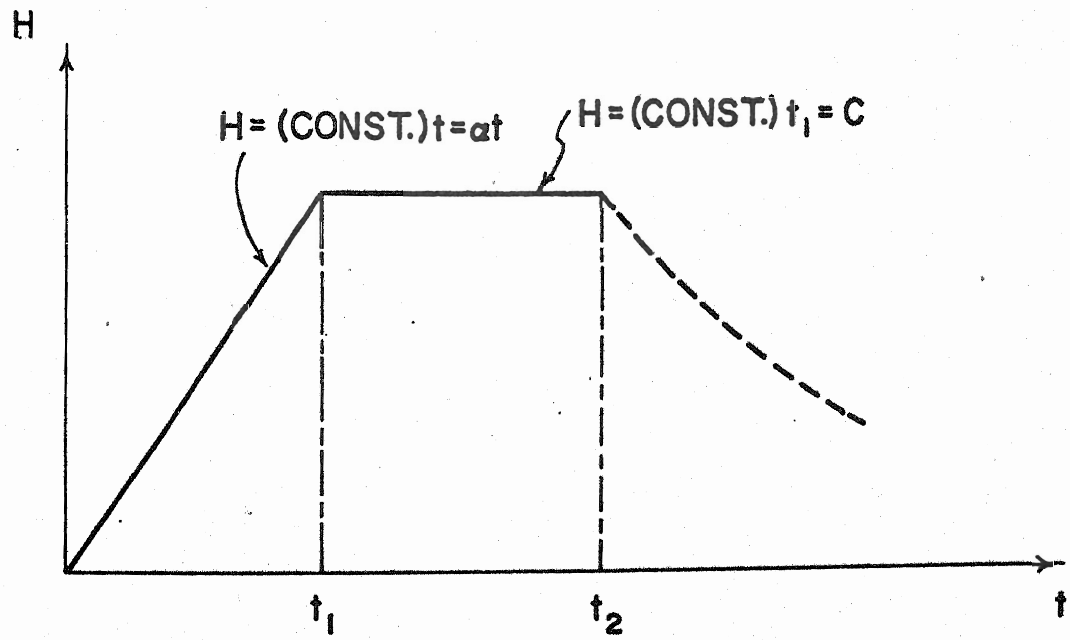


FIGURE 18

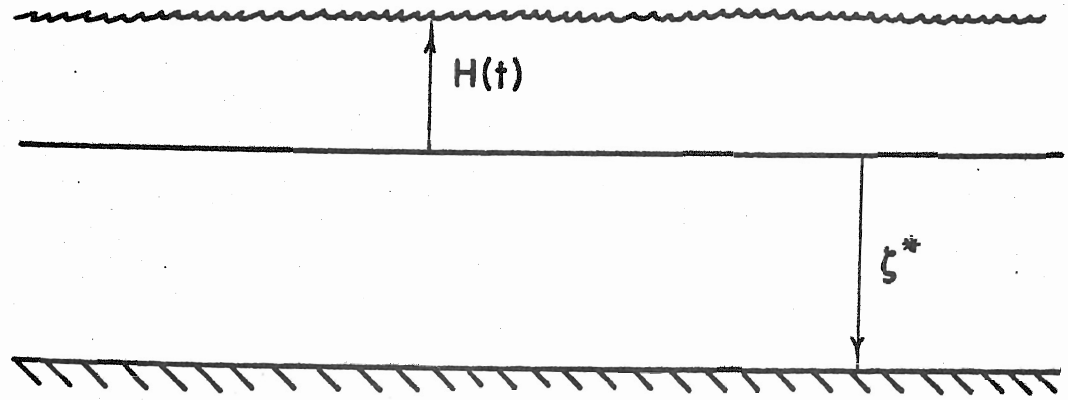


FIGURE 19

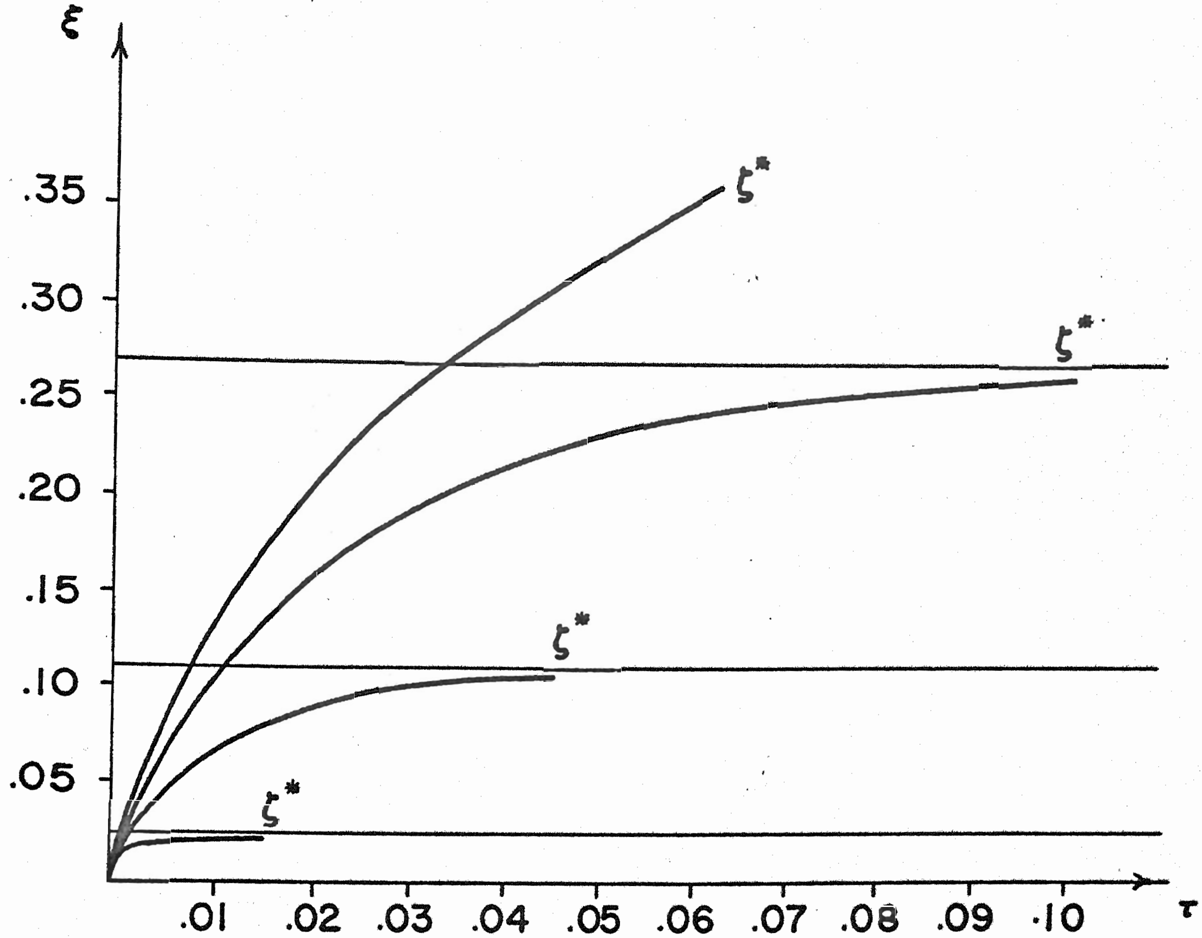


FIGURE 20

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