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# AN APPLICATION OF THE MODIFIED NEWTON'S METHOD FOR THE CONSTRUCTION OF PERIODIC SOLUTIONS OF NONLINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

by

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## I Introduction

The question of periodic solutions for nonlinear hyperbolic partial differential equations arises in a natural manner as one tries to consider steady-state dynamical problems in physics and engineering. Up to now much progress has been made in this particular area of research and we shall not give a detailed account of it here except those of direct relevance to our work. Recently, J. B. Keller and L. Ting [1] and M. Millman [2] have presented a general perturbation method for finding the formal periodic solutions of a large class of nonlinear hyperbolic partial differential equations. On the other hand, the existence of periodic solutions of a very special class of nonlinear hyperbolic partial differential equations has been proved by L. Cesari [3] and P. H. Rabinowitz [4].

The main purposes of this paper are to present a method, which not only does produce approximate periodic solutions of a large class of nonlinear hyperbolic partial differential equations but also offers relative simpler criteria for testing the local existence and uniqueness; for estimating the rate of convergence and the truncation error, and to demonstrate how it can be applied in practice by using few concrete examples as those of [1]. To no-one's surprise, our proposed method here is one of the modified Newton's methods in nonlinear functional analysis [5], [6]. This method not only suits our purposes but also sometimes offers few distinct computational advantages over the perturbation method.

In Section II, Newton's method and its modifications are briefly outlined. The well-known Kantorovich's theorem [5], [6] for Newton's methods for nonlinear functional equations in Banach spaces and its variation are stated without proof. Moreover, several relevant known remarks are also given.

A general formulation of the application of the second modified Newton's method to construct approximate periodic solutions of a class of nonlinear hyperbolic partial differential equations which possess time independent solutions is given in Section III. There the sequence of approximate angular frequencies, $\{\omega_n\}$, is obtained as a by-product.

In Sections IV and V, a simple nonlinear scalar wave equation and a equation governing the longitudinal vibration of a string are solved explicitly and compared with the corresponding results in [1]. Finally, a brief discussion on the relative advantages of the second modified Newton's method over the perturbation method is given in the last section.

II Newton's Method and its Variations

Let $B_1$ and $B_2$ be two real Banach spaces, and F a nonlinear operator which maps an open set $\Omega$ of $B_1$ into $B_2$. Let F have a zero and a continuous Fréchet derivative $F'(u)$ for $u \in \Omega$. Starting with an initial approximate solution $u_o \in \Omega$ of

$$F(u) = 0. \tag{2.1}$$

Newton's method is the process of forming the sequence $\{u_n\}$ such that

$$u_{n+1} = u - [F'(u_n)]^{-1} F(u_n), \; n = 0, 1, 2, 3, \dots \tag{2.2}$$

the first modified Newton's method is the process of forming the sequence $\{u_n\}$ such that

$$\tilde{u}_{n+1} = \tilde{u}_n - [F'(\tilde{u}_o)]^{-1} F(\tilde{u}_n), \; n = 0, 1, 2, 3, \dots \tag{2.3}$$

where $\tilde{u}_o = u_o$,

and the second modified Newton's method is the process of forming the sequence $\{\hat{u}_n\}$ such that

$$\hat{u}_{n+1} = \hat{u}_n - \Gamma F(\hat{u}_n), \; n = 0, 1, 2, 3, \dots \tag{2.4}$$

where $\hat{u}_o = u_o$ and $\Gamma$ is an linear operator close to $[F'(\hat{u}_o)]^{-1}$.

By the Kantorovich's theorem and other relevant theorems [5], [6], Newton's method and its variations not only do produce the approximate solution of(2.1), but also yield conclusions on the local existence, uniqueness, rate of convergence and error estimate. Hence these may apply where fixed point principles and implicit functional theorems [5], [7] do not.

Theorem 1 (Kantorovich): Let

(1) the mapping F be defined, as previously, on $\Omega$ and have a continuous second Fréchet derivative in the sphere

$$S \{u \in \Omega: \|u - u_o\| \leq R < \infty\};$$

(2) $F'(u_o)$ map $B_1$ onto $B_2$ and have an inverse $\Gamma_o = [F'(u_o)]^{-1}$;

(3) $\|\Gamma_0 F(u_0)\| \leq J$ ;

(4) $\|\Gamma_0 F''(u)\| \leq K$ , $u \in S$ .

Now, if $h = J K \leq \frac{1}{2}$ (2.5)

and $R \geq r_0 = [1 - (1 - 2h)^{\frac{1}{2}}] K^{-1}$ (2.6)

the equation (2.1) will have a solution $u^* \in S$ to which both $\{u_n\}$ and $\{\tilde{u}_n\}$ are convergent and $\|u^* - u_0\| \leq r_0$. (2.7)

Furthermore, if for $h < \frac{1}{2}$

and $R < r = [1 + (1 - 2h)^{\frac{1}{2}}] K^{-1}$ (2.8)

or if for $h = \frac{1}{2}$ (2.9)

and $R \leq r$, (2.10)

the solution $u^*$ is unique in S.

The rate of convergence of Newton's method is characterized by

$$\|u^* - u_n\| \leq (2)^{-n} (2h)^{2^n} K^{-1}, \; n = 0, 1, 2, 3, \ldots \quad (2.11)$$

and that of the first modified Newton's method, for $h < \frac{1}{2}$, is characterized by

$$\|u^* - \tilde{u}_n\| \leq [1 - (1-2h)^{\frac{1}{2}}]^{n+1} K^{-1}, \; n = 0, 1, 2, 3, \ldots \quad (2.12)$$

Theorem 2: Let

(1) the mapping F be defined as previously, on $\Omega$ and have a continuous second Fréchet derivative in S;

(2) there exist a linear operator $\Gamma$ such that

$$\|\Gamma F'(u_0) - 1\| \leq \delta < 1;$$

(3) $\|\Gamma F(u_0)\| \leq \hat{J}$;

(4) $\|\Gamma F''(u)\| \leq \hat{K}$, $u \in S$.

Then if $\hat{h} = \dfrac{\hat{K} \hat{J}}{(1-\delta)^2} \leq \frac{1}{2}$ (2.13)

and $R \geq \hat{r}_0 = [1 - (1-2\hat{h})^{\frac{1}{2}}] (1-\delta) \hat{K}^{-1}$ (2.14)

the equation (2.1) will have a solution $u^* \in S$ to which $\{\hat{u}_n\}$ is convergent and it is unique, if for

$\hat{h} < \frac{1}{2}$, $R < \hat{r} = [1 + (1-2\hat{h})^{\frac{1}{2}}]\ (1-\delta)\ \hat{K}^{-1}$ (2.15)

and for $\hat{h} = \frac{1}{2}$, $R \leq \hat{r}$. (2.16)

Moreover, the rate of convergence of the second modified Newton's method, for $\hat{h} < \frac{1}{2}$, is characterized by

$$\|u^* - \hat{u}_n\| \leq \{1 - (1-\delta)\ [1-2\hat{h}]^{\frac{1}{2}}\}^{n+1}\ \hat{K}^{-1},\ n = 0,\ 1,\ 2,\ \ldots \tag{2.17}$$

Remark 1: Newton's method applied to (2.1) is equivalent to the method of successive approximations applied to the equation

$$u = u - [F'(u)]^{-1}\ F(u), \tag{2.18}$$

the first modified Newton's method applied to (2.1) is equivalent to the method of successive approximations applied to the equation $\tilde{u} = \tilde{u} - [F'(\tilde{u}_o)]^{-1}\ F(\tilde{u})$, (2.19)

and the second modified Newton's method applied to (2.1) is equivalent to the method of successive approximations applied to the equation

$$\hat{u} = \hat{u} - \Gamma\ F(\hat{u}). \tag{2.20}$$

The essence of Newton's method and its variations is that instead of solving a given nonlinear equation, one solves a sequence of linearized local approximating equations.

Remark 2: The boundness of $\|\Gamma_o\ F(u_o)\|$ and $\|\Gamma_o\ F''(u)\|$ $(u \in S)$ of theorem 1 and $\|\Gamma\ F(u_o)\|$ and $\|\Gamma\ F''(u)\|$ $(u \in S)$ of theorem 2 is necessary for the existence and the uniqueness of any local solution of (2.1).

Remark 3: The modified Newton's methods often are much easier to use than Newton's method, because in applying the modified methods there is only one linear operator to invert and there are infinitly many linear operators to invert in the original method.

## III Application of Modified Newton's Methods to Obtain Periodic Solutions of a Class of Nonlinear Hyperbolic Partial Differential Equations

Newton's method has been applied to nonlinear elliptic partial differential equations with some success by A. I. Koshelev [8] and D. S. Cohen [9]. In general it is difficult to apply Newton's method or any one of its variations to general nonlinear hyperbolic partial differential equations. However, if only periodic sections of a class of nonlinear hyperbolic partial differential equations are of main concern, Newton's method or its variations may be used sometimes not only to construct periodic solutions but also to provide conditions for their existence and uniqueness. This is because the above problem can be considered as a boundary value problem for hyperbolic equations in which the boundary conditions for space variables is imposed and the requirement of periodicity in time can be interpreted as a boundary condition for the time variable. Here we are looking for classical periodic solutions of a class of nonlinear hyperbolic partial differential equations which possess time independent solutions.

Let $C_\omega$ be the space of continuous real valued functions of $\tau$ and $\underline{x} = (x_1, x_2, \ldots, x_N)$ defined on the one dimensional torus D: $\{0 \le x_i \le L_i$, $L_i > 0$, $i = 1, 2, \ldots, N$; $2\pi$ periodic in $\tau = \omega t$, where $\omega$ is the unknown real angular frequency$\}$ with boundary $\delta D$ and let $\overline{D} = D \cup \delta D$. $C_\omega(\overline{D})$ is a Banach space with respect to the norm $\|\psi\|_{C_\omega} = \max_{(\underline{x},\tau)\in\overline{D}} |\psi(\underline{x},\tau)|$. Let $C_\omega^\infty$ be the space of continuous infinitely differentiable real valued functions of $\tau$ and $\underline{x}$ defined on D. Then $C_\omega^\infty \subset C_\omega$. Let the Hilbert space $H_\omega$ be the completion of $C_\omega$ with respect to

$$\|\psi\|_{H_\omega} = \left[\int_0^{2\pi}\int_0^{L_1}\cdots\int_0^{L_n}|\psi(\underline{x},\tau)|^2\, d\,\underline{x}\, d\,\tau\right]^{\frac{1}{2}}$$ and its inner product be denoted by $\langle , \rangle$. Then $C_\omega^\infty \subset H_\omega$.

A nonlinear hyperbolic partial differential equation can be written as

$$F[u(\underline{x}, t)] = 0 \tag{3.1}$$

with the solution u subjected to some kind of boundary conditions on $\delta D$. Here we let $u_{-1}(\underline{x})$ be the time independent solution of (3.1). As a consequence of the transformation $\tau = \omega t$. (3.1) becomes

$$F[u(\underline{x}, \tau), \omega] = 0 \tag{3.2}$$

which contains $\omega$ explicitly and $u_{-1}(\underline{x})$ becomes the solution of (3.2) for any value of $\omega$. The selection of the proper method and of the proper initial approximate solution $u_0$ are entirely determined by computational and physical considerations. For simplicity in computation (Remark 3), we shall give up Newton's method here. Since we are mainly interested to study how other solutions bifurcate from $u_{-1}(\underline{x})$ and from what values of $\omega$ solutions split, it is reasonable to try $u_{-1}(\underline{x})$ or some other function which is close to $u_{-1}(\underline{x})$ as the initial approximation. Because $u_{-1}(\underline{x})$ itself is a honest solution of (3.2), trying $u_{-1}(\underline{x})$ as $u_0$ will not lead to any other solution. Hence the next reasonable choice for $u_0$ is the solution of the linearized version of (3.2), i.e.

$$u_0(\underline{x}, \tau) = u_{-1}(\underline{x}) + \Phi(\underline{x}, \tau) \tag{3.3}$$

where $\Phi(\underline{x}, \tau) \in C_\omega^\infty$ is the general nontrivial solution of

$$\{F'[u_{-1}(\underline{x}), \omega_0]\}\Phi = 0 \tag{3.4}$$

satisfying the corresponding homogeneous boundary conditions of (3.2) and $\omega_0'$s are the values of $\omega$ for the existence of $\Phi$. Next we adopt the second modified Newton's method by setting $\Gamma = [F'(u_{-1}, \omega_0)]^{-1}$, for it is much more easier to invert $F'(u_{-1}, \omega_0)$ than $F'(u_0, \omega_0)$. Also the selection of the second modified Newton's method with this particular choice of $\Gamma$ gives us not only a method of successive approximations for constructing u but also for $\omega$.

Now let $v_{n+1} = \hat{u}_{n+1} - \hat{u}_n$, $n = 0, 1, 2, 3, \ldots$ (3.5)

where $\{\hat{u}_n\} \in C_\omega^\infty$. From (2.4) and $\Gamma = [F'(u_{-1}, \omega_0)]^{-1}$, we obtain a sequence of inhomogeneous linear equations.

$$\{F'[u_{-1}\ (\underline{x}),\ \omega_o]\}\ v_n = -\ F[\hat{u}_{n-1}\ (\underline{x},\ \tau),\ \omega_n],\ n = 1,\ 2,\ 3,\ \ldots \tag{3.6}$$

where $v_n (n = 1, 2, 3, \ldots)$ satisfy the proper homogeneous boundary conditions. The subscript "n" is inserted in $\omega$ here to indicate that $\{\omega_n\}$ is considered as the approximating sequence for $\omega$. As a consequence of our particular choice of $\Gamma$, the homogeneous form of (3.6) has nontrivial solutions. The solvability condition of (3.6) in $H_\omega$ [10] is that its right hand side must be orthogonal to all nontrivial solutions of the adjoint homogeneous form of (3.6).

Often $\omega_o$, the values of $\omega$ for which nontrivial solutions of (3.4) exist, are called eigenvalues and form a discrete set, for example

$$\omega_o = \lambda_k. \tag{3.7}$$

If $\lambda_k$ is a simple eigenvalue, there is only one corresponding eigenfunction or nontrivial solution $\varphi_k(\underline{x}, \tau)$ and then

$$\Phi\ (\underline{x},\ \tau) = a\ \varphi_k(\underline{x},\ \tau); \tag{3.8}$$

if $\lambda_k$ is a multiple eigenvalue with multiplicity $\sigma$, there are $\sigma$ different corresponding eigenfunctions $\varphi_{k\eta}(\underline{x}, \tau)$, $\eta = 1, 2, \ldots, \sigma$ and then

$$\Phi\ (\underline{x},\ \tau) = \sum_{\eta=1}^{\sigma} a_\eta\ \varphi_{k\eta}(\underline{x},\ \tau), \tag{3.9}$$

where a's are arbitrary constants.

Let the adjoint of (3.4) have eigenvalues $\lambda_k$ and corresponding eigenfunctions $\varphi_k^*$ or $\varphi_{k\eta}^*$, $\eta = 1, 2, \ldots, \sigma$.

In the first case, the orthogonal condition

$$\langle \varphi_k^*,\ F[\hat{u}_{n-1}\ (\underline{x},\ \tau),\ \omega_n] \rangle = 0,\ n = 1,\ 2,\ 3,\ \ldots \tag{3.10}$$

gives the expression of $\omega_n$ as a function of a, and in the latter case, the orthogonal conditions

$$\langle \varphi_{k\eta}^*,\ F[\hat{u}_{n-1}\ (\underline{x},\ \tau),\ \omega_n] \rangle = 0, \tag{3.11}$$

$$n = 1,\ 2,\ 3,\ \ldots,$$

$$\eta = 1,\ 2,\ \ldots,\ \sigma$$

lead to a system of $\sigma$ nonlinear homogeneous algebraic equations of $\sigma + 1$ unknown $\omega_n$ and $a_\eta$, $\eta = 1, 2, \ldots, \sigma$ which can be hopefully solved for $a_\eta\, a_1^{-1}$, $1 < \eta \leq \sigma$ and $\omega_n$ as a function of $a_\eta\, a_1^{-1}$. If $\sigma \to \infty$, (3.11) will be a system of infinitely many equations whose solution $\{a_\eta\}$ will make $\|u_o\|_{C_\omega} \to \infty$ unless only finite number of $\{a_\eta\}$ are non-zero.

Now let $B_1 = B_2 = C_\omega(\bar{D})$ and F a nonlinear operator which maps an open subspace $C_\omega^\infty(\bar{D})$ of $B_1$ into $B_2$. Let F have a zero and a continuous second Fréchet derivative in the sphere $S\ \{u \in C_\omega^\infty(\bar{D}):\ \|u - u_o\|_{C_\omega} \leq R < \infty\}$. Then we have a theorem for the existence of a solution $u^* \neq u_{-1} \in S$ of (3.2) such that $\|u^* - u_{-1}\|_{C_\omega}$ is small.

Theorem 3: Let there exist a bounded linear operator $\Gamma$

$$\text{such that } \|\Gamma\, F(u_o, \omega_1)\|_{C_\omega} \leq \hat{J}$$

$$\text{and} \qquad \|\Gamma\, F''(u,\omega)\|_{C_\omega} < \infty,\ u \in S.$$

If $\hat{J}$ and $\|F'(u_o, \omega_o) - \Gamma^{-1}\|_{C_\omega}$ are continuous functions of $\|\Phi\|_{C_\omega}$ and both of them equal to zero as $\|\Phi\|_{C_\omega} = 0$, then for $\|\Phi\|_{C_\omega} > 0$ but small enough, $\{\hat{u}_n\}$ converges uniformly to a solution $u^* \in S$ of (3.2) such that $u^* \not\equiv u_{-1}$.

Proof: Since $\|\Gamma\, F'(u_o, \omega_o) - I\|_{C_\omega} \leq \|\Gamma\|_{C_\omega}\, \|F'(u_o, \omega_o) - \Gamma^{-1}\|_{C_\omega}$, from hypothesis we can make the right hand side smaller than one by choosing $\|\Phi\|_{C_\omega} > 0$ (or $\{a_\eta\}$) but small enough. Hence the condition (2) of theorem is satisfied. Next, if $\|\Gamma\, F''(u, \omega)\|_{C_\omega}$ is bounded, we can always take a positive real number M large enough such that $\|\Gamma\, F''(u, \omega)\|_{C_\omega} < M$ $(\hat{K} \leq M)$ for $u \in S$.

Upon rewriting the existence conditions (2.13) and (2.14) of theorem 2, we obtain the inequality for the existence of solutions,

$$\hat{J} < \text{Min.}\ \{\tfrac{1}{2}\,(1-\delta)^2\, M^{-1},\ R\,[(1-\delta) - \tfrac{1}{2}\,R\,M]\}.$$

From the hypothesis, $\hat{J}$ can be made small enough to satisfy the above inequality by choosing $\|\Phi\|_{C_\omega} > 0$ but small enough. Because of the norm in $C_\omega$, $\{\hat{u}_n\}$ converges uniformly to a solution $u^* \in S$ of (3.2). By our particular way of constructing the approximate solution, it is obvious that $u^* \neq u_{-1}$ unless $\|\Phi\|_{C_\omega} = 0$. Q.E.D.

Remark 4: The uniform convergence of $\{\hat{u}_n\}$ implies the convergence of $\{\omega_n\}$.

IV A Nonlinear Scalar Wave Equation

As a simple example in application of the second modified Newton's method, we shall consider the following two-dimensional nonlinear scalar wave equation,

$$u_{tt} - u_{xx} - f(u) = 0 \qquad 0 \leq x \leq \pi \qquad -\infty < t < \infty \tag{4.1}$$

where f(u) is a infinitely Fréchet differentiable real functional such that

$$f(u) = \sum_{\nu=1}^{\infty} \frac{1}{\nu!} f^{(\nu)}(0)\, u^{\nu} . \tag{4.2}$$

We seek a solution of (4.1) satisfying the simple boundary condition

$$u(0, t) = u(\pi, t) = 0 \tag{4.3}$$

and the periodicity condition

$$u(x, t + 2\pi\, \omega^{-1}) = u(x, t) , \tag{4.4}$$

where $\omega$ is the unknown angular frequency of the solution. For convenience, we introduce a new time scale $\tau = \omega t$. Under this transformation, (4.1), (4.3) and (4.4) become

$$F(u, \omega) = \omega^2 u_{\tau\tau} - u_{xx} - f(u) = 0, \tag{4.5}$$

$$u(0, \tau) = u(\pi, \tau) = 0 \tag{4.6}$$

and

$$u(x, \tau + 2\pi) = u(x, \tau) . \tag{4.7}$$

The time independent solution of this problem is

$$u_{-1}(x) = 0 . \tag{4.8}$$

If we restrict the domain of F to be $C_\omega^\infty(\bar{D})$, then F maps $C_\omega^k$ into $C_\omega^{k-2}$ for any $k \geq 2$ and has infinitely many continuous Fréchet derivatives in $C_\omega^\infty(\bar{D})$. The first two of them are

$$F'(u, \omega) = \omega^2 \frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial x^2} - f'(u) \tag{4.9}$$

and $$F''(u) = - f''(u) \; . \tag{4.10}$$

Hence from (3.3), (3.4), (3.6), (4.8) and (4.9), we obtain

$$\omega_o^2 \frac{\partial^2 v_n}{\partial\tau^2} - \frac{\partial^2 v_n}{\partial x^2} - f'(0)v_n = - \omega_n^2 \frac{\partial^2 \hat{u}_{n-1}}{\partial\tau^2} + \frac{\partial^2 \hat{u}_{n-1}}{\partial x^2} + f(\hat{u}_{n-1}) \equiv - g(\hat{u}_{n-1}, \omega_n), \tag{4.11}$$

$$v_n(0, \tau) = v_n(\pi, \tau) = 0 \tag{4.12}$$

and $$v_n(x, \tau + 2\pi) = v_n(x, \tau), \quad n = 1, 2, 3, \ldots , \tag{4.13}$$

where $$u_o(x, \tau) = (a \sin p\, \tau + b \cos p\, \tau) \cdot \sin q\, x \tag{4.14}$$

and $$\omega_o^2 = p^{-2}[q^2 - f'(0)], \tag{4.15}$$

where $(a, b)$ is a pair of arbitrary constants and $(p, q)$ is a pair of integers with the condition $q^2 \geq f'(0)$ for $\omega_o$ to be real. For simplicity we may choose the origin of $\tau$ - axis such that

$$u_o(x, \tau) = a \sin p\, \tau \sin q\, x = \Phi \; . \tag{4.16}$$

Since (4.11), (4.12) and (4.13) form a self-adjoint system and $\omega_o$ is a simple eigenvalue, the orthogonal condition (3.10) becomes

$$\omega_n^2 = \langle \Phi, \frac{\partial^2 \hat{u}_{n-1}}{\partial\tau^2} \rangle^{-1} \langle \Phi, \frac{\partial^2 \hat{u}_{n-1}}{\partial x^2} + f(\hat{u}_{n-1}) \rangle \qquad n = 1, 2, 3, \ldots \tag{4.17}$$

By the method of generalized Green's function, the solutions of (4.11), (4.12) and (4.13) are

$$\begin{aligned} v_n(x, \tau) &= - \Gamma\, g(\hat{u}_{n-1}, \omega_n) \\ &= \frac{1}{\pi^2} \sum_{m=1}^{\infty} \frac{\int_0^{2\pi}\int_0^{\pi} \sin m x\, g(\hat{u}_{n-1}, \omega_n)\, dx d\tau}{f'(0) - m^2} \sin m x \\ &+ \frac{2}{\pi^2} \sum_{\substack{\ell=1 \\ \ell \neq p}}^{\infty} \sum_{\substack{m=1 \\ m \neq q}}^{\infty} \frac{\left(\int_0^{2\pi}\int_0^{\pi} \sin \ell\tau \sin mx\, g(\hat{u}_{n-1}, \omega_n)\, dx d\tau\right) \sin \ell\tau \sin mx + \left(\int_0^{2\pi}\int_0^{\pi} \cos \ell\tau \sin mx\, g(\hat{u}_{n-1}, \omega_n)\, dx d\tau\right) \cos \ell\tau \sin mx}{f'(0) - m^2 + \omega_o^2 \ell^2} \\ &+ \zeta_n u_o , \qquad n = 1, 2, 3, \ldots , \end{aligned} \tag{4.18}$$

where $\zeta_n$ are arbitrary constants.

In particular, the first correction to $\omega_o^2$ can be obtained by simply using (4.16) and (4.17). Hence

$$\omega_1^2 - \omega_o^2 = - p^{-2}\left[\frac{3}{32} f^{(3)}(0)a^2 + \frac{2}{\pi^2} \sum_{\nu=5,7,9,\ldots} \frac{1}{\nu!} f^{(\nu)}(0)a^{\nu-1} \int_0^{2\pi}\int_0^{\pi}(\sin p\tau \sin qx)^{\nu+1} dx d\tau\right] \tag{4.19}$$

of which the first term agrees exactly with the second correction to $\omega_o^2$ of the perturbation method of J. B. Keller and L. Ting [1] under the assumption of $f^{(2)}(0) = 0$.

To show the convergence of $\{\hat{u}_n\}$, we invoke Theorem 3 of which conditions can be verified easily. Since

$$\|\Gamma\ F(u_o,\ \omega_1)\|_{C_\omega} = \|v_1\|_{C_\omega} \text{ and } \|F'(u_o,\ \omega_o) - \Gamma^{-1}\|_{C_\omega} = \|\sum_{\nu=1}^{\infty} \frac{1}{\nu!} f^{(\nu+1)}(0)u_o^{\nu}\|_{C_\omega},$$

from (4.16) and (4.18) they are continuous functions of $\|\Phi\|_{C_\omega} = a$ and equal to zero as $\|\Phi\|_{C_\omega} = 0$. Also

$$\|\Gamma F''(u,\omega)\|_{C_\omega} \leq \|\Gamma\|_{C_\omega}\|f''(u)\|_{C_\omega} < \infty,\ u \in C_\omega^\infty .$$

Hence by Theorem 3, $\{\hat{u}_n\}$ converges uniformly for small enough $a > 0$.

## V Another Simple Example:

Now we consider another simple example,

$$\rho(x)u_{tt} - T(u_x)u_{xx} = 0, \qquad \begin{matrix} 0 \leq x \leq L \\ -\infty < t < \infty \end{matrix} \tag{5.1}$$

where $T(u_x)$ is a infinitely Fréchet differentiable real function of $u_x$ such that

$$T(u_x) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} T^{(\nu)}(\beta)(u_x-\beta)^{\nu} . \tag{5.2}$$

We seek a solution of (5.1) satisfying the simple boundary condition

$$u(0, t) = 0, \qquad u(L, t) = L \tag{5.3}$$

and the periodicity condition

$$u(x, t + 2\pi\omega^{-1}) = u(x, t) \tag{5.4}$$

with $\omega$ being the unknown angular frequency of the solution. Under the transformation $\tau = \omega t$, (5.1), (5.3) and (5.4) become

$$F(u, \omega) = \omega^2\rho(x)u_{\tau\tau} - T(u_x)u_{xx} = 0, \tag{5.5}$$

$$u(0, \tau) = 0 , \quad u(L, \tau) = L \tag{5.6}$$

and

$$u(x, \tau + 2\pi) = u(x, \tau) . \tag{5.7}$$

In this case, the time independent solution is

$$u_{-1}(x) = x . \tag{5.8}$$

Let the domain of F be $C^{\infty}_{\omega}(\bar{D})$. Then F has infinitely many continuous Fréchet derivatives in $C^{\infty}_{\omega}(\bar{D})$ and the first two are

$$F'(u, \omega) = \omega^2\rho \frac{\partial^2}{\partial\tau^2} - T(u_x) \frac{\partial^2}{\partial x^2} - T'(u_x) \frac{\partial^2 u}{\partial x^2} \frac{\partial}{\partial x} \tag{5.9}$$

and

$$F''(u, \omega) = -2T'(u_x) \frac{\partial^3}{\partial x^3} - T''(u_x) \frac{\partial^2 u}{\partial x^2} \frac{\partial}{\partial x} . \tag{5.10}$$

From (3.3), (3.4), (3.6), (5.8) and (5.9), we obtain

$$\omega_o^2\rho(x) \frac{\partial^2 v_n}{\partial\tau^2} - T(1) \frac{\partial^2 v_n}{\partial x^2} = -\omega_n^2\rho(x) \frac{\partial^2 \hat{u}_{n-1}}{\partial\tau^2} + T(\hat{u}_{n-1x}) \frac{\partial^2 \hat{u}_{n-1}}{\partial x^2} \equiv - g(\hat{u}_{n-1}, \omega_n) , \tag{5.11}$$

$$v_n(0, \tau) = v_n(L, \tau) = 0 , \tag{5.12}$$

and $\quad v_n(x, \tau + 2\pi) = v_n(x, \tau) , \quad n = 1, 2, 3, \ldots \qquad (5.13)$

with

$$u_o(x, \tau) = x + (a \sin p\tau) X_q(x) = x + \Phi \tag{5.14}$$

and $\quad \omega_o^2 = p^{-2}\lambda_q , \qquad (5.15)$

where $\lambda_q$ is a simple eigenvalue of the problem

$$X''(x) + \frac{\rho(x)}{T(1)} \lambda X(x) = 0, \quad X(0) = X(L) = 0 \tag{5.16}$$

such that

$$p^{-2}\lambda_q \neq j^{-2}\lambda_i \tag{5.17}$$

for any $i \neq q$ and any integers j and p;

$X_q(x)$ is the corresponding eigenfunction.

From the orghogonal condition (3.10), we get

$$\omega_n^2 = \langle \Phi, \rho(x) \frac{\partial^2 \hat{u}_{n-1}}{\partial \tau^2} \rangle^{-1} \langle \Phi, T(\hat{u}_{n-1_x}) \frac{\partial^2 \hat{u}_{n-1}}{\partial x^2} \rangle, \quad n = 1, 2, 3, \cdots\cdots \tag{5.18}$$

and by the method of generalized Green's function, we obtain

$$v_n(x, \tau) = -\Gamma\, g(\hat{u}_{n-1}, \omega_n)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^L G(x, x', 0)\, g(\hat{u}_{n-1}, \omega_n)\, dx d\tau$$

$$+ \frac{1}{\pi} \sum_{\ell=1}^{\infty} \Big[ \Big( \int_0^{2\pi} \int_0^L G(x, x', \ell)\, g[\hat{u}_{n-1}(x', \tau'), \omega_n] \sin \ell\tau' dx' d\tau' \Big) \sin \ell\tau$$

$$+ \Big( \int_0^{2\pi} \int_0^L G(x, x', \ell)\, g[\hat{u}_{n-1}(x', \tau'), \omega_n] \cos \ell\tau' dx' d\tau' \Big) \cos \ell\tau \Big]$$

$$+ \zeta_n \Phi, \qquad n = 1, 2, 3, \ldots \tag{5.19}$$

where $G(x, x', \alpha)$ satisfies

$$\frac{\partial^2 G(x, x', \alpha)}{\partial x^2} + \frac{\omega_0^2 \rho(x) \alpha^2}{T(1)} G(x, x', \alpha) = - \frac{1}{T(1)} \delta(x - x'), \tag{5.20}$$

$$G(0, x', \alpha) = G(L, x', \alpha) = 0.$$

In particular, the first correction to $\omega_0^2$ due to the nonlinearity is

$$\omega_1^2 - \omega_0^2 = \frac{\lambda_q \sum_{\nu=2,4,\ldots\ldots}^{\infty} \frac{a^\nu}{\nu!} T^{(\nu)}(1) \int_0^L \rho(x) [X_q(x)]^2 [X_q'(x)]^\nu dx \int_0^{2\pi} \sin^{\nu+2} p\tau d\tau}{\pi p^2 T(1) \int_0^L \rho(x) [X_q(x)]^2 dx} \tag{5.21}$$

which, in general, is nonzero for at least one $T^{(\nu)}(1) \neq 0$, $\nu = 2, 4, \ldots$ .

The convergence of $\{\hat{u}\}$ can be easily shown also. Since

$$\|\Gamma F(u_o, \omega_1)\|_{C_\omega} = \|v_1\|_{C_\omega} \text{ and}$$

$$\|F'(u_o, \omega_o) - \Gamma^{-1}\|_{C_\omega} = \| \sum_{\nu=1}^{\infty} \frac{1}{\nu!} T^{(\nu)}(1)(\Phi_x)^{\nu} \frac{\partial^2}{\partial x^2} + T'(1 + \Phi_x)\Phi_{xx} \frac{\partial}{\partial x} \|_{C_\omega} ,$$

by (5.19) they are continuous functions of $\|\Phi\|_{C_\omega} \propto a$ and equal to zero as $\|\Phi\|_{C_\omega}=0$.

Furthermore,

$$\|\Gamma F''(u, \omega)\|_{C_\omega} \leq \|\Gamma\|_{C_\omega} \|2T'(u_x) \frac{\partial^3}{\partial x^3} + T''(u_x) \frac{\partial^2 u}{\partial x^2} \frac{\partial}{\partial x}\|_{C_\omega} < \infty, \; u \in C_\omega^\infty .$$

Hence by Theorem 3, $\{\hat{u}_n\}$ converges uniformly for $a > 0$ small enough.

## VI Discussion

In this section we shall make a brief comparison between the perturbation technique of J. B. Keller and L. Ting [1] and our method described in previous sections. Since there is no exact periodic solutions available and rates of convergence are hard to come by, it is very difficult to discuss the relative advantage of these two methods for constructing the approximate periodic solutions. However in computing $\omega^2$, the perturbation method always gives zero contribution to $\omega_1^2-\omega_0^2$ or $\dot{\omega}^2$ in [1], the first correction to $\omega_0^2$, while the second modified Newton's method often yields nonzero contribution. From the mechanism of these two methods, the knowledge of $u_0$ or $\dot{u}$, the non-trivial solution of the linearized homogeneous equation, is needed to calculate $\omega_1^2-\omega_0^2$ or $\dot{\omega}^2$ respectively; the knowledge of $u_0$ and $\hat{u}_1$ or $\dot{u}$ and $\ddot{u}$ is needed to calculate $\omega_2^2-\omega_0^2$ or $\ddot{\omega}^2$ respectively. Because a lot of tedious computation is needed to obtain $\hat{u}$. or $\ddot{u}$ and other higher order approximate solutions in general, we may conclude that the second modified Newton's method has often a distinct computational advantage over the perturbation method. Furthermore, it is rather easy to show the convergence of the approximate solutions of the second modified Newton's method, while to show the convergence of the perturbation series is often difficult. It is to be noted that our method applies equally well to a system of inhomogeneous nonlinear hyperbolic partial differential equations.

It should be mentioned that J. Moser [11] has outlined a similar technique for the construction of solutions of nonlinear differential equations. It essentially consists of constructing a sequence of approximate solutions by using Newton's method and proving the convergence by invoking Nash implicit functional theorem [7].

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