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REPRESENTATION THEOREMS FOR POSITIVE RATIONAL FUNCTIONS

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Representation Theorems for Positive Rational Functions

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Irving Gerst

Let $f(s)$ be a positive, rational function, so that $f(s)$ is analytic and $\mathrm{Re}\ f(s) \geq 0$, for $\mathrm{Re}\ (s) > 0$. Then the familiar transformation

$$F(s) = \frac{f(s)-1}{f(s)+1} \tag{1}$$

which yields a rational, unimodular bounded function $F(s)$, (i.e. $F(s)$ analytic and $|F(s)| \leq 1$, for $\mathrm{Re}\ (s) > 0$),[1] immediately provides a unique representation for $f(s)$. For, let us write $F(s)$ in the form

$$F(s) = k_0 \frac{P(s)}{Q(s)} , \tag{2}$$

where P and Q are relatively prime, complex, monic polynomials and k_o is a (complex) constant. Then from (1) and (2)

$$f(s) = \frac{Q(s) + k_0P(s)}{Q(s) - k_0P(s)} . \tag{3}$$

Here P,Q and k_o are subject to conditions which characterize $F(s)$ in (2) as a unimodular bounded function. One set of such conditions follows readily from the analyticity of $F(s)$ in $|s| \leq 1$ and the Maximum Modulus Theorem, (cf. [1]), and may be stated as follows: $Q(s)$ must be strictly Hurwitz[2] with $\deg Q \geq \deg P$, and $0 \leq |k_0| \leq \varkappa_m$ where

$$\frac{1}{\varkappa_m} = \max_{s=i\omega} \left|\frac{P(s)}{Q(s)}\right| \quad , \omega \text{ real.} \tag{4}$$

Conversely, it is clear that if these conditions are satisfied, then $f(s)$ given by (3) is a positive function.

It is the purpose of this note to replace the foregoing rather unwieldy condition on k_o in the representation (3), by either of two alterna-

[1] For non-constant $F(s)$, actually $|F(s)| < 1$, for $\mathrm{Re}(s) > 0$, by the Maximum Modulus Theorem; and for rational $F(s)$, it follows easily that $F(s)$ is analytic also on $\mathrm{Re}(s) = 0$, and $|F(s)| \leq 1$ there.

[2] In order to avoid bothersome exceptional cases in the sequel, it is convenient here to define a (complex) polynomial as Hurwitz or strictly Hurwitz respectively, according as it has no zeros in $\mathrm{Re}(s) > 0$ or in $\mathrm{Re}(s) \geq 0$ respectively. Note that a non-zero constant then belongs to both of these categories.

tive conditions which appear to be more tractable. The first of these involves the root locus of a certain polynomial, while the second is the representational form of a theorem on positive functions due to Talbot [2]. We then apply our results to get an alternate proof of another theorem given in [2].

We first prove

Theorem 1. (a) Every non-constant, positive, rational function $f(s)$ has a unique representation of the form (3) where

(i) P and Q are relatively prime, monic polynomials with $\deg P \leq \deg Q$, $\deg Q > 0$, and $k_o \neq 0$;

(ii) Q is strictly Hurwitz;

(iii) for every k such that $0 < |k| \leq |k_o|$, the polynomial $Q - kP$ is Hurwitz.

(b) Conversely, if conditions (i) - (iii) hold, then $f(s)$ given by (3) is a positive function.

Corollary. A positive real, rational function has a unique representation of the form (3) where conditions (i) - (iii) of Theorem 1 hold and in addition: (iv) P and Q are real polynomials and k_o is a real constant. Conversely, if condition (i) - (iv) hold then $f(s)$ given by (3) is positive real.

Proof: (a) In the light of our preceding discussion, we must establish (iii) only.

Suppose $f(s)$ is positive, so that the corresponding $F(s)$ given by (1) and represented as in (2) is unimodular bounded. Then $|F(s)| = 1$ is possible only for such s for which $\text{Re}(s) \leq 0$. Thus, the equation $F(s) = e^{-i\theta}$, for any real θ, has no roots in $\text{Re}(s) > 0$. This implies that $Q - k_o e^{i\theta} P$ is Hurwitz. Since also $\varkappa F(s)/k_o$ is unimodular bounded for $0 < \varkappa \leq |k_o|$, the same conclusion holds for $Q - \varkappa e^{i\theta} P$. Setting $k = \varkappa e^{i\theta}$, we have (iii).

(b) Suppose now that conditions (i) - (iii) hold. Then it follows from (iii) that the equation $\varkappa P/Q = \bar{e}^{i\theta}$ has its roots in $\text{Re}(s) \leq 0$ for all real θ and for all $\varkappa$ such that $0 < \varkappa \leq |k_o|$. We will show that this statement implies that $|k_o| \leq \varkappa_m$ where $\varkappa_m$ is given by (4). Thus k_oP/Q is unimodular bounded, and f(s) given by (3) is positive. Our proof that $|k_o| \leq \varkappa_m$ will be indirect.

Suppose, therefore, that $|k_o| > \varkappa_m$. We will show that this assumption leads to a contradiction.

Consider the unimodular bounded function $F^*(s) = \varkappa_m P/Q$. Let $s = i\omega_o$ be a point at which $|P/Q|$ achieves its maximum on $s = i\omega$, ω real. From (4), $|P(i\omega_o)/Q(i\omega_o)| = 1/\varkappa_m$ so that $F^*(i\omega_o) = e^{i\varphi}$, φ real. We next expand $F^*(s)$ in the neighborhood of $s = i\omega_o$ to get [3]

$$F^*(s) = e^{i\varphi} + \alpha(s-i\omega_o) + \dots \quad . \tag{5}$$

It follows from the unimodular character of $F^*(s)$ that

$$\alpha e^{-i\varphi} < 0 \quad , \tag{6}$$

which also implies that $\alpha \neq 0$. (So as not to interrupt the argument at this point, we defer the proof of (6) until later.)

Write u for $F^*(s)$. Then the inverse of the series in (5) begins as follows:

$$s(u) = i\omega_o + \frac{1}{\alpha}\cdot(u-e^{i\varphi}) + \dots \quad .$$

If u is close enough to $e^{i\varphi}$, the two terms written here will be the dominant terms of the series. Hence for such u

$$\operatorname{sgn} \operatorname{Re}\,[s(u)] = \operatorname{sgn} \operatorname{Re}\left[\frac{u-e^{i\varphi}}{\alpha}\right] \quad .$$

In particular let $u = u_o = \varkappa_m e^{i\varphi}/\varkappa_1$ where $\varkappa_1$ is sufficiently close to $\varkappa_m$ and $\varkappa_m < \varkappa_1 < |k_o|$. Then

$$\operatorname{sgn} \operatorname{Re}\,[s(u_o)] = \operatorname{sgn} \operatorname{Re}\left[\frac{e^{i\varphi}(\varkappa_m-\varkappa_1)}{\alpha\varkappa_1}\right] = +1$$

by (6).

That is to say, the equation $F^*(s) = u_o$ or $\varkappa_1 P/Q = e^{i\varphi}$ has a root in

[3] If $i\omega_o$ is the point at infinity, this expansion, of course, would be in terms of powers of 1/s, but the remainder of the argument goes through unchanged.

the interior of the right half-plane. This result contradicts the italicized statement above.

The argument for the proof of (b) is now complete. The proof of the corollary to Theorem 1 is straightforward and will be left to the reader.

There remains the proof of (6). Denote by $f^*(s)$ the positive function corresponding to $F^*(s)$ via eq. (1). If $e^{i\varphi} \neq 1$, then $f^*(i\omega_o) = (1 + e^{i\varphi})/(1 - e^{i\varphi}) = iy_o$, say, where y_o is real and finite. We have the power series expansion

$$f^*(s) = iy_o + \beta(s-i\omega_o) + \ldots \quad .$$

It is known that here $\beta > 0$.[4]

The constant α in (5) is now determined in terms of β and y_o by substituting the series for $f^*(s)$ in (1). We find

$$\alpha = -\frac{2\beta e^{i\varphi}}{1+y_o^2} \quad ,$$

from which (6) follows immediately.

If $e^{i\varphi} = 1$, then $f^*(s)$ has a pole at $s = i\omega_o$. The argument is again the same but uses the Laurent expansion of $f^*(s)$ at $s = i\omega_o$. We leave the details to the reader.

Remarks: 1. In the representation (3) for f(s) given by Theorem 1(a), it follows in the usual way that the numerator and denominator of the fraction in (3) are relatively prime polynomials.

2. In Theorem 1 (b), it is easily seen that the requirement in condition (i) that P and Q be relatively prime can be omitted. However, since Q must be strictly Hurwitz by (ii), any common factor of P and Q must also be strictly Hurwitz.

[4] This result is usually proved for positive real functions in the case $y_o = 0$. It holds as well in the present situation.

3. Since $|k_o P(i\omega_o)/Q(i\omega_o)| = 1$ would lead to the contradiction $|\varkappa_m P(i\omega_o)/Q(i\omega_o)| > 1$ when $|k_o| < \varkappa_m$, it follows in Theorem 1(a) that, actually, $Q - kP$ is <u>strictly</u> Hurwitz for $0 < |k| \leq |k_o|$ except when $|k_o| = \varkappa_m$, in which case there is at least one zero on $Re(s) = 0$ (possibly at ∞) when $|k| = |k_o|$. Thus $\varkappa_m$ may be described, alternatively, as the first value of $|k|$ for which $Q - kP$ has a zero, $s = i\omega_o$, ω_o real, $(0 \leq |\omega_o| \leq \infty)$, as $|k|$ increases monotonically from zero.

In Theorem 1, condition (iii) involves the root locus of $Q - \varkappa e^{i\theta}P$ as $\varkappa$ varies from 0 to $|k_o|$. The question arises as to the conclusion which can be drawn when, in addition to conditions (i) and (ii), it is assumed only that $Q - k_o e^{i\theta}P$ is Hurwitz for all real θ. Then, as in the proof of Theorem 1(b), we have $|F(s)| \neq 1$ for $R(s) > 0$, where $F(s) = k_o P/Q$. By a theorem of Talbot ([2], p. 608), it follows that $\pm f(s)$ is positive, where $f(s)$ corresponds to $F(s)$ by (1) and is represented as in (3). Therefore, in order to arrive at a representation theorem here, we must determine additional conditions which serve to distinguish whether f or $-f$ is positive. We do this by expanding $f(s)$ in (3) at $s = \infty$, and noting that if $f(s)$ is a positive function in the neighborhood of ∞, then this will imply that $f(s)$ must be positive.

Let $Q(s) = s^n + as^{n-1} + \ldots$, $P(s) = s^m + bs^{m-1} + \ldots$. We distinguish three cases.

<u>Case (a)</u>. $m < n$.

Then $f(s) = 1 + \ldots$. It follows that $f(s)$ must be positive, so that no further condition is required here.

Case (b). $m = n$, $k_o \neq 1$.

We have $f(s) = c_o + c_1/s + \dots$, where

$$c_o = \frac{1+k_o}{1-k_o}, \quad c_1 = \frac{2k_o(b-a)}{(1-k_o)^2} .$$

As is known, f(s) will be a positive function in the neighborhood of ∞ if $Re(c_o) > 0$, or if $Re(c_o) = 0$ and $c_1 > 0$. These conditions correspond respectively to $|k_o| < 1$, or $|k_o| = 1$ and $a - b > 0$. (The latter inequality follows since $c_1 = 2(a-b)/|1-e^{i\psi}|^2$ when $k_o = e^{i\psi}$).

Case (c). $m = n$, $k_o = 1$.

Then f(s) has a pole at $s = \infty$, and will be a positive function in the neighborhood of ∞ when $f(s) = c_{-1}s + \dots$, where $c_{-1} = 2/(a-b) > 0$.

We thus arrive at sufficient conditions for f(s) in (3) to be positive. It is clear that these conditions are also necessary.

Summarizing, we have established the following theorem:

Theorem 2. (a) Every non-constant, positive rational function has a unique representation of the form (3) where

(i) P and Q are relatively prime, monic polynomials with deg P $\leq$ deg Q, deg Q > 0, and $k_o \neq 0$;

(ii) Q is strictly Hurwitz;

(iii) for every real θ, $Q - k_o e^{i\theta} P$ is Hurwitz;

(iv) for deg P = deg Q, either $|k_o| < 1$, or $|k_o| = 1$ and $a - b > 0$, where $Q(s) = s^n + as^{n-1} + \dots$, $P(s) = s^n + bs^{n-1} + \dots$.

(b) Conversely, if conditions (i) - (iv) hold, then f(s) given by (3) is positive.

Remarks: As in the case of Theorem 1, when k_o, P and Q are real, we get a representation theorem for a positive real, rational function. Also, Remarks 1 and 2 following Theorem 1 apply here as well. Finally, in both Theorems 1(b) and 2(b), it can be shown that condition (ii) is superfluous, as it follows from the remaining conditions in each theorem.

In [2], Talbot has proved, among other results, the following theorem.

Theorem 3. Let the positive, rational function $f(s)$ be written in the form $f(s) = N(s)/D(s)$ where N and D are relatively prime polynomials and deg $D > 0$. Then also N'/D' is a positive function.[5]

We would like to indicate another proof of this theorem based upon Theorems 1 and 2, in which the desired result is evident almost by inspection. We assume as known the following theorem: The derivative of a non-constant, Hurwitz or strictly Hurwitz polynomial respectively, is Hurwitz or strictly Hurwitz respectively (Cf. [2]). This result follows as a special case of the Lucas theorem [3], which states that the zeros of the derivative of a polynomial, $p(s)$, are contained in the convex hull of the zeros of $p(s)$.

Proof of Theorem 3: Apply Theorem 1(a) to represent $f(s)$ in the form (3) where conditions (i) - (iii) hold. Let R and S denote the numerator and denominator respectively, of the fraction in (3). From $f(s) = N/D = R/S$ and Remark 1, it follows that $N = \alpha R$, $D = \alpha S$ where α is a constant. Thus $N'/D' = R'/S'$.

Let $Q(s) = s^n + as^{n-1} + \ldots$, $P(s) = s^m + bs^{m-1} + \ldots$. We have $m \leq n$ by (i). Suppose $n \geq 2$. Then differentiating in (3), we get

$$\frac{R'}{S'} = \frac{Q_1 + k_1 P_1}{Q_1 - k_1 P_1}, \qquad (7)$$

where we have set $Q_1 = Q'/n$, $P_1 = P'/m$, $k_1 = mk_o/n$. The right member of (7) is again of the same form as the right member of (3). It remains to verify that k_1, P_1, and Q_1 satisfy conditions (i) - (iii) of Theorem 1(b).

It is clear that condition (i) holds in the form as modified by Remark 2, after Theorem 1, except for the case $P_1 \equiv 0$ when the theorem follows trivially. Condition (ii) follows since Q_1 is the derivative of the non-

5 Here the primes denote differentiation with respect to s.

constant, strictly Hurwitz polynomial Q/n. Finally, as $|k_1| \leq |k_0|$, the polynomial $Q_1 - kP_1$ which is the derivative of the Hurwitz polynomial $(Q - kP)/n$, will be Hurwitz for $0 < |k| \leq |k_1|$, except, possibly, if $Q - kP$ is a constant. In the latter case, we must have $k = 1$, $m = n$, and $a = b$ (since $n \geq 2$). The last two conditions, by (iv) of Theorem 2(a) applied to R/S, imply that $|k_0| < 1$, which is incompatible with $k = 1$. It follows that R'/S' is a positive function for $n \geq 2$.

When $n = 1$, the result holds trivially, since $f(s)$ is then of the form $c(s+a)/(s+b)$ or $c/(s+b)$ where $Re(c) \geq 0$.

Footnotes

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(2) In order to avoid bothersome exceptional cases in the sequel, it is convenient here to define a (complex) polynomial as Hurwitz or strictly Hurwitz respectively, according as it has no zeros in $Re(s) > 0$ or in $Re(s) \geq 0$ respectively. Note that a non-zero constant then belongs to both of these categories.

(3) If $i\omega_0$ is the point at infinity, this expansion, of course, would be in terms of powers of 1/s, but the remainder of the argument goes through unchanged.

(4) This result is usually proved for positive real functions in the case $y_0 = 0$. It holds as well in the present situation.

(5) Here the primes denote differentiation with respect to s.

References

1. A. Talbot, "A new method of synthesis of reactance networks," Proc. IEE, vol. 101, pp. 73-90; February 1954.

2. A. Talbot, "Some theorems on positive functions," IEEE Trans. on Circuit Theory (Correspondence), vol. CT-12, pp. 607-608, December 1965.

3. M. Marden, The Geometry of the Zeros of a Polynomial in a Complex Variable, New York: American Mathematical Society, 1949., Chap. 2.