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TURBULENT NATURAL CONVECTION PLUME ABOVE A FINITE CIRCULAR SOURCE OF MASS, MOMENTUM AND BUOYANCY

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SUMMARY

An investigation is made of the turbulent plume generated by steady release of finite mass, momentum and buoyancy fluxes from a circular source of finite size situated in a semi-infinite region of ambient fluid. By the use of the lateral entertainment assumption, the analysis shows that the behavior of such a plume can be characterized by a source Froude number. The results are summarized in the comprehensive quadrature solutions of two separate ranges of the source Froude number.

INTRODUCTION

In most of the previous successful work on the theoretical investigation of the turbulent convection plume, the method used is the lateral entrainment assumption on the ambient otherwise stationary air first introduced by Taylor [1]. The assumption states that the time-mean velocity of entrained surrounding air at a certain level is proportional to a certain characteristic time-mean velocity of the turbulent natural convection plume at the same level.

Morton, Taylor and Turner [2] investigated the simple two-dimentional case of the plume from an idealized mathematical line source of infinitesimal physical size, infinite buoyancy intensity and zero mass and momentum fluxes. Their results check very well with the experimental findings of Rouse, Yih and Humphreys [3] for plumes above a very small gas flame and a line of very small gas flames designed to simulate the idealized point and line sources. However, for these two cases of different geometry, the entrainment proportionality coefficient takes on different values.

Morton [4, 5] investigated the axisymmetrical case of the plume from an axisymmetrical source of finite mass, momentum and buoyancy fluxes. His results depend on the solution for the case of plume from a ficticious point source of finite buoyancy and momentum fluxes but zero mass flux at a lower level. Furthermore, in his analysis, no description has been made about the physical size of the source and its possible influence on the behavior of the plume.

Lee and Emmons [6] investigated theoretically the two-dimensional case of the plume from a finite-size strip source of finite mass, momentum and buoyancy fluxes. Their results bring out the significance of a source Froude number F. A quadrature solution was obtained for each of two separate ranges of the Froude number, $F > 1$ or $F < 1$. In neither of these cases can the finite-size strip source be accurately represented by an equivalent mathematical line source at a lower level. Only the special case, $F = 1$, can be so represented and its solutions, with a shift of reference coordinates, check with the line source solutions obtained by Morton, Taylor and Turner [2].

Lee and Emmons [6] also investigated experimentally the behavior of the plume of hot gases above a diffusion flame of liquid fuel burned in a long finite-size channel burner. Their measurements check closely with the results of their theoretical investigation of the two-dimensional case of the plume from a finite-size source of finite mass momentum and buoyancy fluxes [6]. Among other interesting findings, a comparison between their theoretical and experimental results shows that the location of the finite-size strip source for the theoretical analysis can be assumed to be at the level of the average flame height. And furthermore, the configurations of the finite-size strip source for the plume calculation can be computed from the configurations of the diffusion flame in the finite-size channel burner. These findings further justify the attempt to split the convection plume part out of the total diffusion fire problem.

In all established experimental data, Gaussian profile velocity and temperature distribution have been found to exist in both axisymmetrical and two-dimensional cases. For the axisymmetrical case, measurements of Rouse, Yih and Humphreys [3] give a value for the entrainment proportionality coefficient which agrees closely with the values evaluated from experimental results on axisymmetrical turbulent jet mixings given by Squire [7], Kuethe [8] and Ruden [9] as indicated by Morton [5]. For the two-dimensional case, measurements of Rouse, Yih and Humphreys [3] and Lee and Emmons [6] both give the same value for the entrainment proportionality coefficient which checks very well with a value computed from velocity measurements of a two-dimensional turbulent free jet by Reichard [10].

In the present treatment, use will be made of the lateral entrainment assumption in the investigation of the behavior of the turbulent plume issuing from a finite circular source of finite fluxes of mass, momentum and buoyancy.

ANALYSIS

It is assumed that turbulent flow is fully developed in the plume and that local density variations are small compared to some reference density in the flow field. As direct consequences, turbulent transfer mechanisms dominate the mixing process and the significant role played by the density variations due to temperature variations is the vertical acceleration caused by the associated buoyant force. It is also assumed that the flow field of the plume is confined to a relatively narrow vertical region. Therefore, the usual boundary layer type assumptions can be made in the analysis.

Let the steady turbulent convection plume be symmetrical with respect to a vertical axis x with its origin 0 located at the level of the source producing the plume. The local time-mean component of velocity in the vertical direction x is denoted by $u(x, r)$, and that in the horizontal radial direction r is denoted by $v(x, r)$.

As first suggested by Taylor [1] and later adopted by Morton, Taylor and Turner [2], Morton [4, 5] and Lee and Emmons [6], the rate of entrainment of the ambient fluid into the convection plume is taken as proportional to the vertical velocity on the axis of the plume, thus

$$v(x, \infty) = -\alpha u(x, 0) \tag{1}$$

where α is the entrainment coefficient. The negative sign on the right hand side of Eqn. (1) appears as a result of the convention that the entrainment velocity v, pointing opposite to the direction of the r-coordinate, should be negative.

Consider an axisymmetric plume generated from a finite source in an incompressible environment. The plume will be assumed to have Gaussian profiles of time-mean vertical velocity and time-mean buoyancy. Therefore

$$u(x, r) = u(x) \exp\left(-r^2/b^2\right) \tag{2}$$

where $j(x) = u(x, 0)$ and $b = b(x)$ is a horizontal length-scale characterizing the velocity profile, and

$$\Delta\gamma(x, r) = \Delta\gamma(x) \exp\left(-r^2/\lambda^2 b^2\right) \tag{3}$$

where $\Delta\gamma(x, r) = g\,\Delta\rho = g(\rho_1 - \rho)$ is the local buoyancy and $\Delta\gamma(x) = \Delta\gamma(x, 0)$. ρ is the local density and ρ_1 is the density of the undisturbed ambient fluid. g is the usual gravitational acceleration. λb is a length-scale characterizing the associated buoyancy profile. λ as well as α is assumed to be a universal constant.

The finite source will be characterized by $b(0) = b_o$, $u(0) = u_o$ and $\Delta\gamma(0) = \Delta\gamma_o$. Therefore, with the assumptions expressed in Equations (1), (2) and (3) and the assumption of small density variation, we can integrate with respect to r the equations of continuity, conservation of momentum and conservation of energy to obtain the governing equations

$$\frac{d}{dx}\left[ub^2\right] = 2\alpha bu \tag{4}$$

$$\frac{d}{dx}\left[b^2u^2\right] = 2\lambda^2 g b^2 \Delta\gamma/\gamma_1 \tag{5}$$

$$\frac{d}{dx}\left[b^2 u \Delta\gamma\right] = 0 \tag{6}$$

subject to the boundary conditions

$$b = b_o, \quad u = u_o, \quad \Delta\gamma = \Delta\gamma_o \qquad at \quad x = 0 \tag{7}$$

Introducing the transformations

$$x' = 2\alpha x/b_o, \qquad b' = b/b_o$$

$$p' = (\Delta\gamma/\gamma_1)/(\Delta\gamma_o/\gamma_1), \quad u' = \left[(\gamma_1/\Delta\gamma_o)\alpha/g b_o\right]^{1/2} u/\lambda \tag{8}$$

we have, from Equations (4), (5) and (6),

$$\frac{d}{dx'}\left[u'b'^2\right] = b'u' \tag{9}$$

$$\frac{d}{dx'}\left[b'^2u'^2\right] = b'^2 p' \tag{10}$$

$$\frac{d}{dx'}\left[b'^2 u' p'\right] = 0 \tag{11}$$

subject to the boundary conditions

$$b' = 1, \quad u' = F, \quad p' = 1 \qquad at \quad x' = 0 \tag{12}$$

where $F = \left[(\gamma_1/\Delta\gamma_o)\alpha/g b_o\right]^{1/2} u_o/\lambda$ is a source Froude number.

Integration of Equation (11) gives

$$b'^2 u' p' = F \tag{13}$$

and then Equation (10) becomes

$$\frac{d}{dx'}\left[b'^2 u'^2\right] = F/u' \tag{14}$$

Introduce the mass and the momentum fluxes

$$N = u' b'^2, \quad M = u'^2 b'^2 \tag{15}$$

From Equations (9), (14) and (12), we have

$$\frac{dN}{dx'} = M^{1/2} \tag{16}$$

$$\frac{dM}{dx'} = FN/M \tag{17}$$

subject to the boundary conditions

$$n = F, \quad M = F^2 \quad \text{at } x' = 0 \tag{18}$$

Equations (16) and (17) give

$$(5/4) FN^2 = M^{5/2} + c \tag{19}$$

where $c = F^3 (5/4 - F^2)$.

By the use of Equation (19), we obtain from Equation (16)

$$x' = \int_F^N \left[(5/4) FN^2 - c\right]^{-1/5} dN \tag{20}$$

It is convenient to consider the following three separate cases:

I. $c = 0$; $F = (5)^{1/2}/2$

II. $c < 0$; $F > (5)^{1/2}/2$

III. $c > 0$; $F < (5)^{1/2}/2$

Case I. $F = (5)^{1/2}/2$, the neutral source.

For this case, Equations (19) and (20) reduce to

$$(5)^{3/2}(8)^{-1}N^2 = M^{5/2} \tag{21}$$

and

$$x' = (5/3)\left[(2)^{3/5}(5)^{-3/10}N^{3/5} - 1\right] \tag{22}$$

which in turn with Equations (13) and (15) give

velocity $$u' = (5)^{1/2}(2)^{-1}\left[(3/5)x' + 1\right]^{-1/3}$$

plume half-width $$b' = (3/5)x' + 1 \tag{23}$$

buoyancy $$p' = \left[(3/5)x' + 1\right]^{-5/3}$$

the plots of which are shown in Figure 1.

In terms of the conserved buoyancy flux across any horizontal cross-section

$$\begin{aligned} Q &= \int_0^\infty 2\pi r g u(x,r)\Delta\gamma(x,r)/\gamma_1\, dr \\ &= \pi g^{3/2}\alpha^{-1/2}\lambda^3(\lambda^2+1)^{-1}F(\Delta\gamma_0/\gamma_1)^{3/2}b_0^{5/2} \end{aligned} \tag{24}$$

Equations (23) take the more familiar forms

$$u = (5)^{1/3}(2)^{-2/3}\pi^{-1/3}\alpha^{-1/3}(\lambda^2+1)^{1/3}Q^{1/3}\left[(6/5)\alpha x + b_0\right]^{-1/3}$$

$$b = (6/5)\alpha x + b_0 \tag{25}$$

$$\gamma_1/\Delta\gamma = (5)^{1/3}(2)^{-2/3}\pi^{2/3}\alpha^{-1/3}g\lambda^2(\lambda^2+1)^{-2/3}Q^{-2/3}\left[(6/5)\alpha x + b_0\right]^{5/3}$$

which in fact are the solutions for the convection plume issuing from a point source located at a distance $(5/6\alpha)b_0$ below the finite source.

Case II. $F < (5)^{1/2}/2$, the restrained source.

Let

$$N = \beta v \tag{26}$$

with $\beta = F\left[1 - (4/5)F^2\right]^{1/2}$.

Equation (20) then becomes

$$x' = 2(5)^{-1/2}(5/4 - F^2)^{3/10}F^{2/5}\int_{v_0}^{v}(v^2-1)^{-1/5}dv \tag{27}$$

where

$$v = NF^{-1}\left[1-(4/5)F^2\right]^{-1/2}$$
$$v_0 = \left[1-(4/5)F^2\right]^{-1/2} \tag{28}$$

If we further define

$$I_1(v) = \int_1^{v}(v^2-1)^{-1/5}dv \tag{29}$$

then Equation (27) becomes

$$x' = 2(5)^{-1/2}(5/4 - F^2)^{3/10} F^{2/5} \left[I_1(\nu) - I_1(\nu_0) \right]. \tag{30}$$

Numerical evaluation of Equation (29) with the use of Equations (13), (15), (19) and (26) give the solutions

velocity $\quad u' = (5)^{1/2}(2)^{-1} F^{1/5} (5/4 - F^2)^{-1/10} (\nu^2 - 1)^{2/5} \nu^{-1}$

plume half-width $\quad b' = 2(5)^{-1/2} F^{2/5} (5/4 - F^2)^{3/10} (\nu^2 - 1)^{-1/5} \nu \qquad (31)$

buoyancy $\quad p' = (5)^{1/2}(2)^{-1}(5/4 - F^2)^{-1/2} \nu^{-1}$

the graphical representations as shown in Figure 2. The actual source can be imagined located at a distance

$$x_0'' = (5)^{1/2}(2)^{-1}(5/4 - F^2)^{-3/10} F^{-2/5} x_0' = I_1\left[\left\{1 - (4/5)F^2\right\}^{-1/2}\right] \tag{32}$$

to the right of the origin and the convection plume characteristics plotted in the region to the right of the location of the actual source in the figure. It is seen that the results for the case of a restrained source approach asymptotically those for the case of a neutral source when the value of F approaches $(5)^{1/2}/2$.

Case III. $F > (5)^{1/2}/2$, the impelled source.

Let

$$N = \beta \nu \tag{33}$$

with $$\beta = F\left[(4/5)F^2 - 1\right]^{1/2}.$$

Equation (20) then becomes

$$x' = 2(5)^{-1/2}(F^2 - 5/4)^{3/10} F^{2/5} \int_{\nu_0}^{\nu} (\nu^2 + 1)^{-1/5} d\nu \qquad (34)$$

where

$$\nu = NF^{-1}\left[(4/5)F^2 - 1\right]^{-1/2}$$

$$\nu_0 = \left[(4/5)F^2 - 1\right]^{-1/2}. \qquad (35)$$

If we further define

$$I_2(\nu) = \int_0^{\nu} (\nu^2 + 1)^{-1/5} d\nu \qquad (36)$$

then Equation (34) becomes

$$x' = 2(5)^{-1/2}(F^2 - 5/4)^{3/10} F^{2/5}\left[I_2(\nu) - I_2(\nu_0)\right]. \qquad (37)$$

Numerical evaluation of Equation (36) with the use of Equations (13), (15), (19) and (33) gives the solutions

velocity $$u' = (5)^{1/2}(2)^{-1} F^{1/5} (F^2 - 5/4)^{-1/10} (\nu^2 + 1)^{2/5} \nu^{-1}$$

plume half-width $$b' = 2(5)^{-1/2} F^{2/5} (F^2 - 5/4)^{3/10} (\nu^2 + 1)^{-1/5} \nu \qquad (38)$$

buoyancy $$p' = (5)^{1/2}(2)^{-1}(F^2 - 5/4)^{-1/2} \nu^{-1}$$

the graphical representations as shown in Figure 3. The actual source can be imagined located at a distance

$$x_0'' = (5)^{1/2}(2)^{-1}\left(F^2 - 5/4\right)^{-3/10} F^{-2/5} x_0' = I_2\left[\left\{(4/5)F^2 - 1\right\}^{-1/2}\right] \tag{39}$$

to the right of the origin and the convection plume characteristics plotted in the region to the right of the location of the actual source in the figure. It is seen that the results for the case of an impelled source approach asymptotically those for the case of a neutral source when the value of F approaches $(5)^{1/2}/2$.

CONCLUSION

By the use of the assumptions of lateral entrainment and Gaussian velocity and temperature profiles, a quadrature solution has been obtained for an axisymmetrical turbulent natural convection plume above a finite circular source of finite mass, momentum and buoyancy fluxes situated in a semi-finite region of ambient fluid. The behavior of such a plume has been found to be described solely by a characteristic source Froude number F. $F < (5)^{1/2}/2$ is associated with cases with restrained source and $F > (5)^{1/2}/2$ is associated with cases with an impelled source. The situation of $F = (5)^{1/2}/2$ is associated with the limiting case with a neutral source approached asymptotically from both sides by the cases associated with the separated ranges of the source Froude number. The solution for the case with a neutral source is identified as that for an axisymmetrical turbulent plume generated by a point source of infinite buoyancy flux and zero mass and momentum fluxes and physical size situated at a lower level than the finite source.

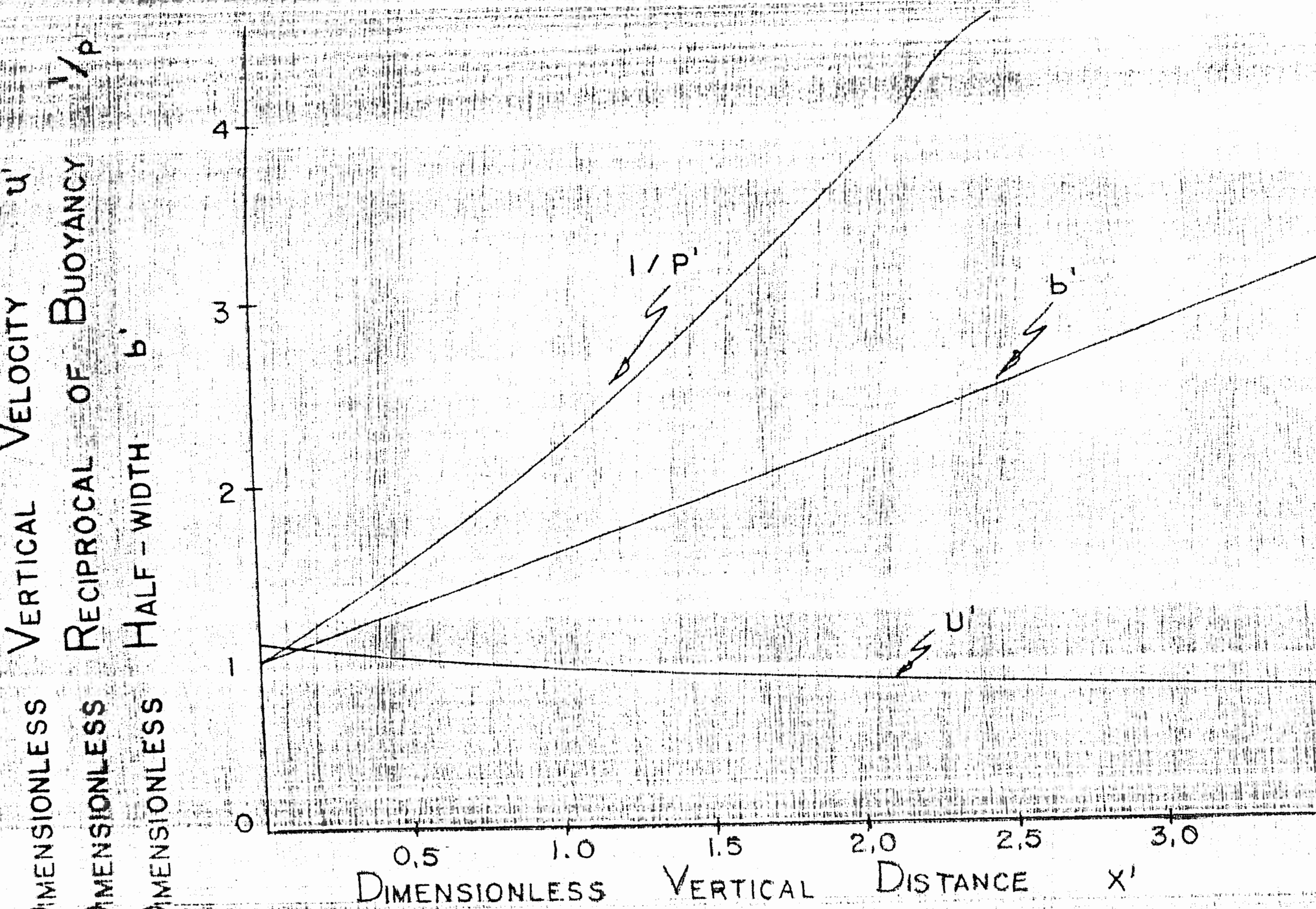

FIG. 1. SOLUTION FOR A PLUME ISSUING FROM A NEUTRAL SOURCE

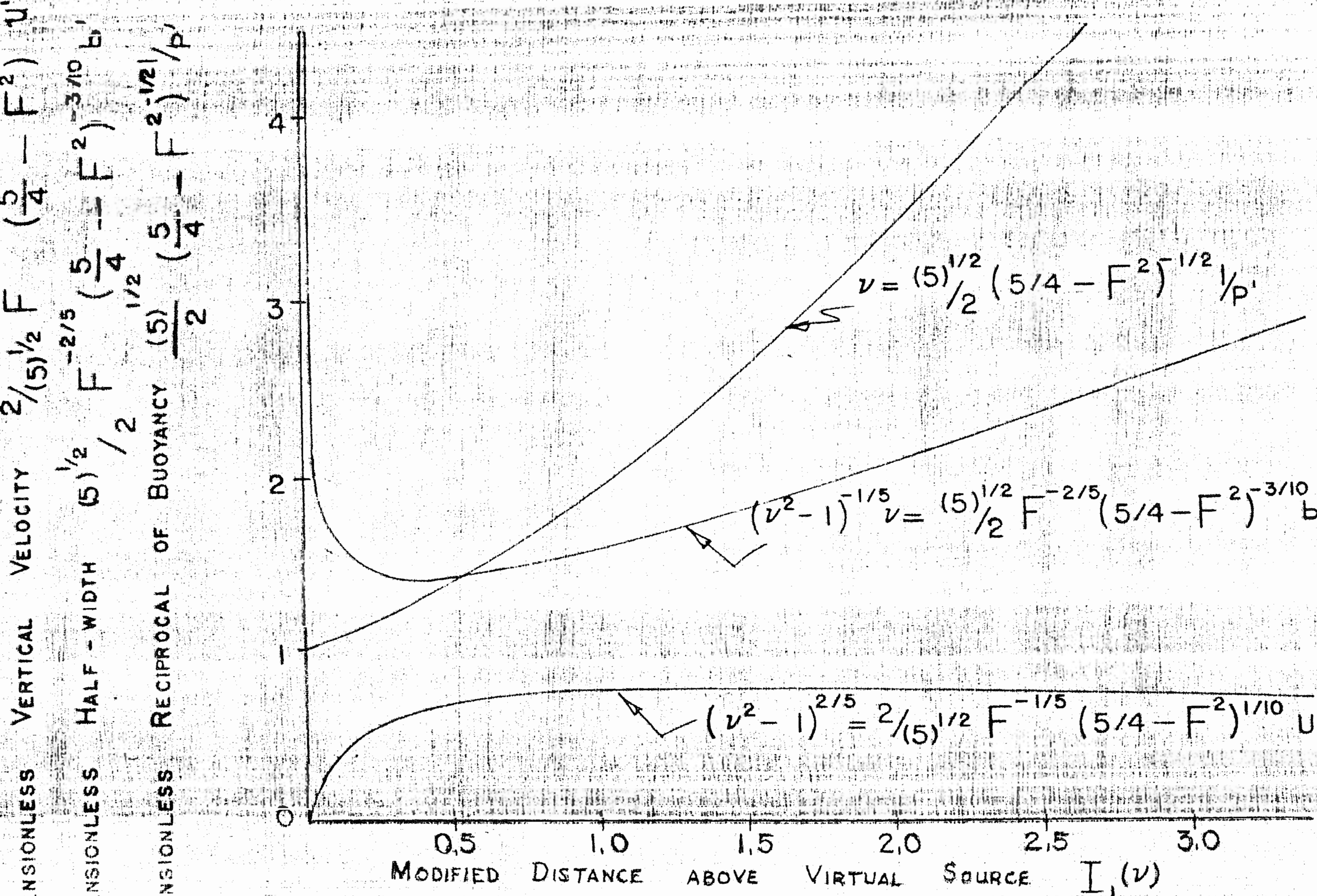

FIG 2. Solution for a plume issuing from a restrained source

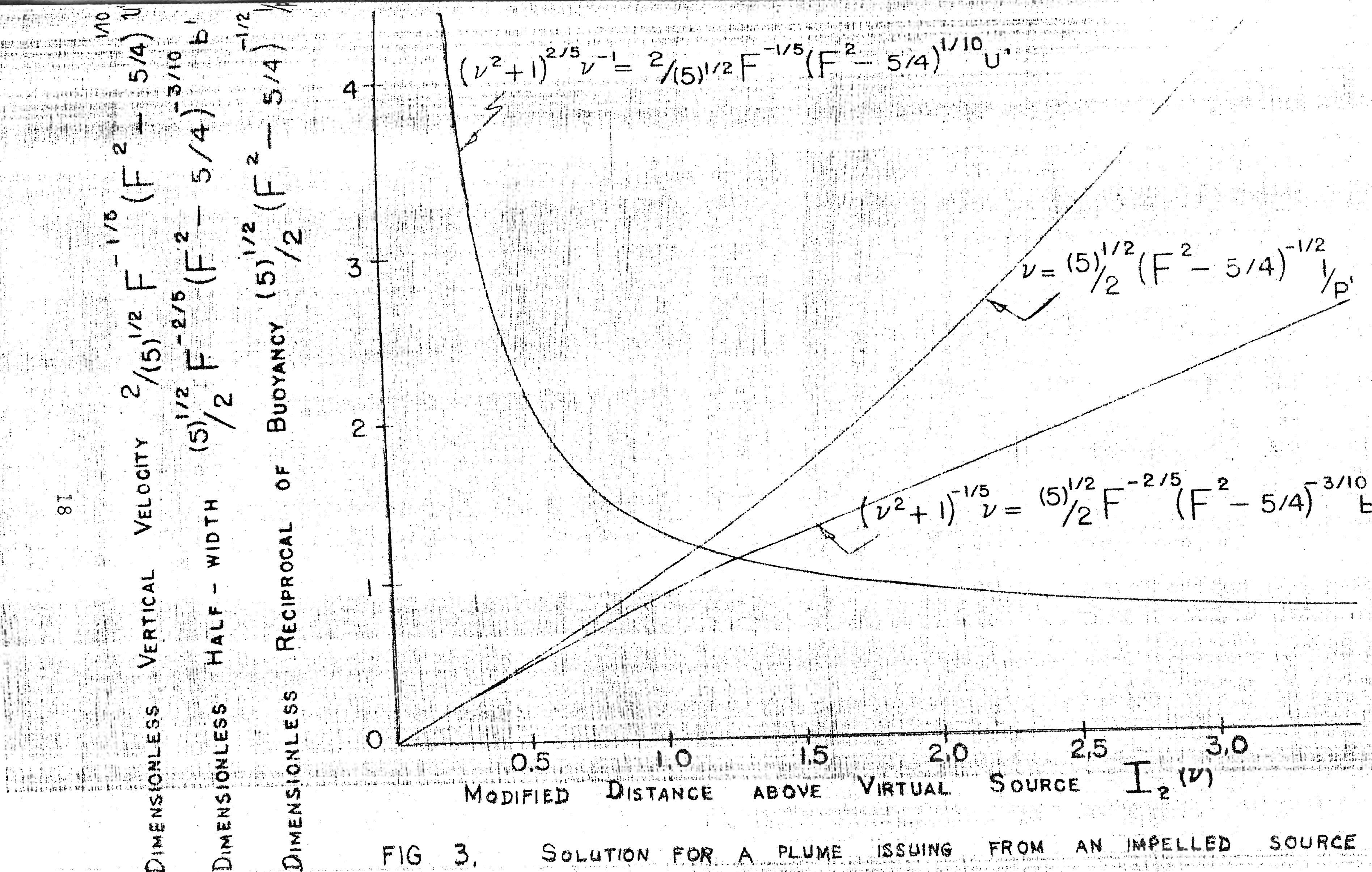

FIG 3. Solution for a plume issuing from an impelled source

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