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LAMINAR FREE CONVECTION ABOVE A LINE SOURCE OF HEAT IN AN AMBIENT SHEAR FIELD

by

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ABSTRACT

The results of Yih, in his paper "Laminar Free Convection above a Line Source of Heat", Ref. 1, have been extended to include a prescribed shear in the horizontal plane. The results show that for an initially linear profile of longitudinal velocity in the horizontal plane in the ambient fluid, the presence of a buoyancy plume flow field causes a diffusion of vorticity towards the vertical plane above the line source, i.e., the transverse gradient of the longitudinal velocity is appreciably amplified towards the vertical plane above the line source, while diminished away from it. The pressure field was also determined.

A few minor corrections of algebraic nature were necessary on Yih's solution so that it is in the corrected form in the present analysis.

The corrected results of Yih are applicable for values of Prandtl number of 5/9 and 2, however, this analysis is restricted to a value of Prandtl number of 5/9.

INTRODUCTION

A line source of heat is placed in an infinite horizontal plane equidistant between two vertical infinite parallel screens as shown in Fig. 1. The screens are in motion in the x-direction with velocities equal in magnitude but opposite in sign U_s and $-U_s$, respectively. The screens will be considered to have the property that they offer no resistance to flow in the direction normal to them, yet they are able to impact shear in the tangential direction. This configuration might be considered a specialization of the more complicated case of the boundary shear field produced by two currents of opposite direction meeting each other. It will further be assumed that the screens offer no resistance to the flow in the vertical direction. This assumption, as will be more clearly seen later, is fairly reasonable if the screens are at moderate distances from the line source, since the buoyance field is restricted to a relatively small region in the neighborhood of the vertical symmetrical plane above the source, for moderate values of the height, z. For larger values of z, the flow may already have become turbulent so that this analysis is inapplicable in any case.

The complete velocity field and buoyancy field are sought.

NOMENCLATURE

G	buoyancy parameter of the plume as defined by $G = \int_{-\infty}^{\infty} w \Delta\gamma dy$
L	distance from line source to either screen
p	local static pressure
p_o	pressure of ambient undisturbed fluid
$\bar{p}$	hydrostatic pressure of ambient undisturbed fluid
p'	$= \frac{\rho L^2}{\mu^2} p$, as defined
$\bar{p}'$	$= \frac{\rho L^2}{\mu^2} \bar{p}$, as defined
p_o'	$= \frac{\rho L^2}{\mu} p_o$, as defined
R	gas constant
T	local temperature
T_o	temperature of ambient undisturbed fluid
ΔT	$= T - T_o$, local temperature increment
u	velocity in the x-direction
u'	$= \frac{u}{U_s}$, as defined
U_s	screen velocity
v	velocity in the y-direction

v' $= \frac{\rho L}{\mu} v$, as defined

w velocity in the z-direction

w' $= \left(\frac{\mu}{GL}\right)^{\frac{1}{2}} w$, as defined

x coordinate parallel to the line heat source

y the vertical coordinate

y' $= \frac{y}{L}$, as defined

z horizontal coordinate perpendicular to the line heat source

z' $= \left(\frac{\mu^3}{\rho^2 L^5 G}\right)^{\frac{1}{2}} z$, as defined

α thermal diffusivity of fluid

γ local specific weight

γ_o specific weight of ambient undisturbed fluid

$\Delta\gamma$ $= \gamma - \gamma_o$, local buoyancy

μ viscosity of fluid

ρ density of fluid

ANALYSIS AND RESULTS

First, we assume that the variations in p, ρ, and T are small. As a consequence of this we can say that the variations in density are small compared to the magnitude of the density itself, hence, we can use the incompressible continuity equation. Also, from the ideal gas equation:

$$p = \gamma RT \quad (1)$$

we can write from the above assumption:

$$\frac{\Delta T}{T_o} = - \frac{\Delta \gamma}{\gamma_o} \tag{2}$$

Further, we assume that variations of the flow variables in the y-direction are large as compared with their variations in the z-direction, hence, we can formulate a boundary layer type problem.

Making the usual boundary layer type assumptions, and assuming the z-pressure variation is essentially hydrostatic, we obtain the equation of continuity and the equations of conservation of linear momentum, respectively,

Cont.: $$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3}$$

x- Mom.: $$v\frac{\partial u}{\partial y} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \tag{4}$$

y-Mom.: $$\frac{\partial p}{\partial y} = \mu\frac{\partial^2 v}{\partial y^2} \tag{5}$$

z-Mom.: $$v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\mu}{\rho}\frac{\partial^2 w}{\partial y^2} - g\frac{\Delta \gamma}{\gamma_o} \tag{6}$$

Making use of Eq. (2), the energy equation becomes:

$$w\frac{\partial \Delta \gamma}{\partial z} + v\frac{\partial \Delta \gamma}{\partial y} = \frac{\partial^2 \Delta \gamma}{\partial y^2} \tag{7}$$

The boundary conditions are:

$$\left.\begin{aligned} y = L;&\quad u = U_s \\ y = -L;&\quad u = -U_s \end{aligned}\right\} \tag{4a}$$

$$\left.\begin{aligned} y = 0;&\quad v = 0, \frac{\partial w}{\partial y} = 0, \quad u = 0 \\ y = \pm\infty;&\quad w = 0, \ \Delta \gamma = 0 \end{aligned}\right\} \tag{5a}$$

(6a)

$$y = \pm\infty; \quad p = \bar{p}(z)$$

Now, observation of Eqs. (3), (6), and (7) with boundary conditions, Eqs. (5a), yields that these form a complete set, i.e., they may be solved independently of Eqs. (4) and (5). Further, Eqs. (4) and (5) are independent of each other.

Hence, our plan of solution should be first to solve the system of Eqs. (3), (6), and (7) with boundary conditions Eqs. (5a) for the variables v, w, and $\Delta\gamma$. With v known, we can go to Eq. (4) and solve for u. With Eq. (5) we can readily determine the pressure distribution.

We have thus reduced the solution of what appeared to be a rather complex three-dimensional problem into two two-dimensional problems.

A similarity solution to the system of equations, Eqs. (3), (6), and (7), with boundary conditions Eqs. (5a) was obtained by Yih, Ref. 1, for values of Prandtl number of 5/9 and 2. Using Yih's results for a value of Prandtl number of 5/9, we will proceed to solve Eq. (4) with boundary conditions Eqs. (4a).

Yih's results, in corrected form, are:

$$w = .801\left(\frac{G^2}{\rho\mu}\right)^{1/5} z^{1/5} \operatorname{sech}^2\left[.3654\left(\frac{\rho^2 G}{\mu^3}\right)^{1/5} z^{-2/5}\, y\right] \qquad (8)$$

$$v = -\left(\frac{\mu^2 G}{\rho^3}\right)^{1/5} z^{-2/5}\left\{1.3153 \tanh\left[.3654\left(\frac{\rho^2 G}{\mu^3}\right)^{1/5} z^{-2/5}\, y\right] - .3204\left(\frac{\rho^2 G}{\mu^3}\right)^{1/5} z^{-2/5}\, y \operatorname{sech}^2\left[.3654\left(\frac{\rho^2 G}{\mu^3}\right)^{1/5} z^{-2/5}\, y\right]\right\} \qquad (9)$$

$$\Delta\gamma = -.3420\left(\frac{G^4\rho^3}{\mu^2}\right)^{1/5} z^{-3/5} \operatorname{sech}^2\left[.3654\left(\frac{\rho^2 G}{\mu^3}\right)^{1/5} z^{-2/5}\, y\right] \qquad (10)$$

where $G = \int_{-\infty}^{\infty} w\Delta\gamma\, dy \qquad (11)$

is the buoyancy parameter.

The buoyancy parameter, G. which, as shown by Yih, is independent of z, is a measure of the source strength, and may be shown to be directly proportional to the heat release rate of the source per unit length in the x-direction.

Employing the dimensionless variables shown in "Nomenclature", Eqs. (8)

and (9) become:

$$w' = .801\, z'^{1/5} \operatorname{sech}^2 \left[.3654\, z'^{-2/5}\, y' \right] \tag{12}$$

$$v' = -z'^{-2/5} \left\{ 1.3153 \tanh \left[.3654\, z'^{-2/5}\, y' \right] - .3204\, z'^{-2/5}\, y' \operatorname{sech}^2 \left[.3654\, z'^{-2/5} y' \right] \right\} \tag{13}$$

which are plotted as shown in Figs. 2 and 3 respectively.

Similarly, Eq. (4) becomes

$$v' \frac{\partial u'}{\partial y'} = \frac{\partial^2 u'}{\partial y'^2} \tag{14}$$

and the corresponding boundary conditions from Eqs. (4a)

$$\begin{aligned} y' &= 1; \quad u' = 1 \\ y' &= -1; \quad u' = -1 \\ y' &= 0; \quad u' = 0 \end{aligned} \tag{14a}$$

Eq. (14) may be integrated to yield:

$$\frac{\partial u'}{\partial y'} = f(z') \exp \left[\int_0^{y'} v' dy' \right] \tag{15}$$

where $f(z') = \left(\frac{\partial u'}{\partial y'} \right)_{y' = o}$ which is the value of $\frac{\partial u'}{\partial y'}$ evaluated at $y' = o$.

Substituting into Eq. (15) the expression for v' from Eq. (13), we obtain after performing the indicated integration

$$\frac{\partial u'}{\partial y'} = f(z') \operatorname{sech}^6 \left[.3654\, z'^{-2/5} y' \right] \cdot \exp \left[.8768\, z'^{-2/5} y' \tanh(.3654\, z'^{-2/5} y') \right] \tag{16}$$

To find u', we integrate Eq. (16), and noting that $u'(0,z') = 0$ from the last equation of Eqs. (14a), we have:

$$u' = \frac{z'^{2/5} f(z')}{.3654} I(.3654 z'^{-2/5} y') \tag{17}$$

Where

$$I(\beta) = \int_{0}^{\beta} \operatorname{sech}^6 \phi \cdot \exp\left[2.4\phi \cdot \tanh\phi\right] d\phi$$

The function $f(z')$ can be obtained by applying to Eq. (17) the boundary conditions, $u' = 1$ at $y' = 1$ or $u' = -1$ at $y' = -1$, from the first two equations of Eqs. (14a):

$$f(z') \equiv \left(\frac{\partial u'}{\partial y'}\right)_{y'=0} = \frac{.3654}{z'^{2/5}} \frac{1}{I(.3654z'^{-2/5})} \qquad (18)$$

The integral $I(\beta)$ was evaluated numerically and the results for the dimensionless velocity gradiant $\frac{\partial u'}{\partial y'}$ evaluated at the symmetrical plane $y'=0$, $f(z') \equiv \left(\frac{\partial u'}{\partial y'}\right)_{y'=0}$, from Eq. (18) are plotted in Fig. 4. The results for the dimensionless velocity in the x'-direction, u', from Eq. (17) are plotted in Fig. 5.

Employing the dimensionless variables in "Nomenclature", Eq. (5) may be written:

$$\frac{\partial p'}{\partial y'} = \frac{\partial^2 v'}{\partial y'^2} \qquad (19)$$

which, upon integrating, performed between the limits of $y' = \infty$ and $y' = y'$, gives

$$p' = \frac{\partial v'}{\partial y'} + \bar{p}'(z') \qquad (20)$$

where $\bar{p}'(z')$ represents the contribution of hydrostatic pressure of the ambient fluid. Neglecting the variation in the hydrostatic pressure, we can have the boundary condition:

$$y = \infty; \quad p' = \bar{p}' = p'_0 \qquad (21)$$

Since $\frac{\partial v'}{\partial y'} = 0$ as $y \to \infty$, we have from Eqs. (20) and (21),

$$p' - p_o' = \frac{\partial v'}{\partial y} \tag{22}$$

which, upon substitution of the expression for v' from Eq. (13), gives the difference between the dimensionless local static pressure and the undisturbed ambient pressure

$$p' - p_o' = z'^{-4/5} \operatorname{sech}^2[.3654z'^{-2/5}y'] \cdot \left\{.1602 + .23414z'^{-2/5}y' \tanh(.3654z'^{-2/5}y')\right\} \tag{23}$$

which is plotted in Fig. 6.

DISCUSSIONS

Observation of Fig. 5 clearly shows the effect on the shear field of the buoyancy. Consider a flow initially with zero buoyance, i.e., at $z' = \infty$. We have, essentially, plane Couette flow with a linear x-direction velocity distribution. The addition of buoyance then causes a distortion of the x-direction velocity field, increasing the u' velocity gradient at $y' = 0$ and decreasing it towards $y' = 1$. As the buoyance is increased, we reach a point such that the u-velocity gradient is nearly zero as y' approaches one, but becomes very large as y' approaches zero. This, in essence, signifies that the buoyance causes a diffusion of z-vorticity to the center. The variation of z-vorticity at the centerline, $y' = 0$, with buoyance may be seen firm Fig. 4. Since z' varies inversely as the square root of the buoyance parameter, G, $z' = 0$ corresponds to zero buoyance or infinite height, z. The z-vorticity, then, approaches a constant value, corresponding to plane Couette flow, at $z' = \infty$, and approaches infinity as $z' \to 0$.

These results, however, are not unexpected. In order to explain this, consider a system with plane Couette flow. If a fluid particle is displaced towards the center, its speed will be retarded by the slower moving fluid particles, but they, in turn, will be speeded up, or, the particle will transfer x-momentum to the slower moving particles. In our system, we have a steady flow of these particles moving towards the center, hence, we have a continuous transfer of x-momentum towards the center. Since, as seen from Fig. 3, the v' velocity component decreases rapidly towards the center, there will also be a corresponding decrease in the transfer of x-momentum since the inward flux of the particles carrying this momentum is reduced. The result, then, is that we have a decrease in the transverse gradient of longitudinal velocity in the vicinity of $y'=1$ and an increase near $y'=o$.

The pressure distribution is shown in Fig. 6. As might be expected, the smaller the value of the dimensionless height above the line source, z', the greater the value of the minimum pressure, yet, the smaller is its range of influence, i.e., for small values of z', the more rapidly the pressure approaches the ambient pressure p_0 with y'. It may also be observed that for $y' = \text{const.} \neq o$, the pressure decreases with z', approaches a minimum, and then increases to atmospheric.

One might also observe that the point of minimum pressure moves away from the vertical plane of symmetry, the $x' - z'$ plane, for increasing z'. This may be readily seen from Fig. 7 where the locus of points of minimum pressure is plotted.

ACKNOWLEDGMENT

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REFERENCE

1. Yih, C. S., "Laminar Free Convection due to a Line Source of Heat", Transactions of American Geographical Union, Vol. 33, No. 5, October 1952, pp. 669-672.

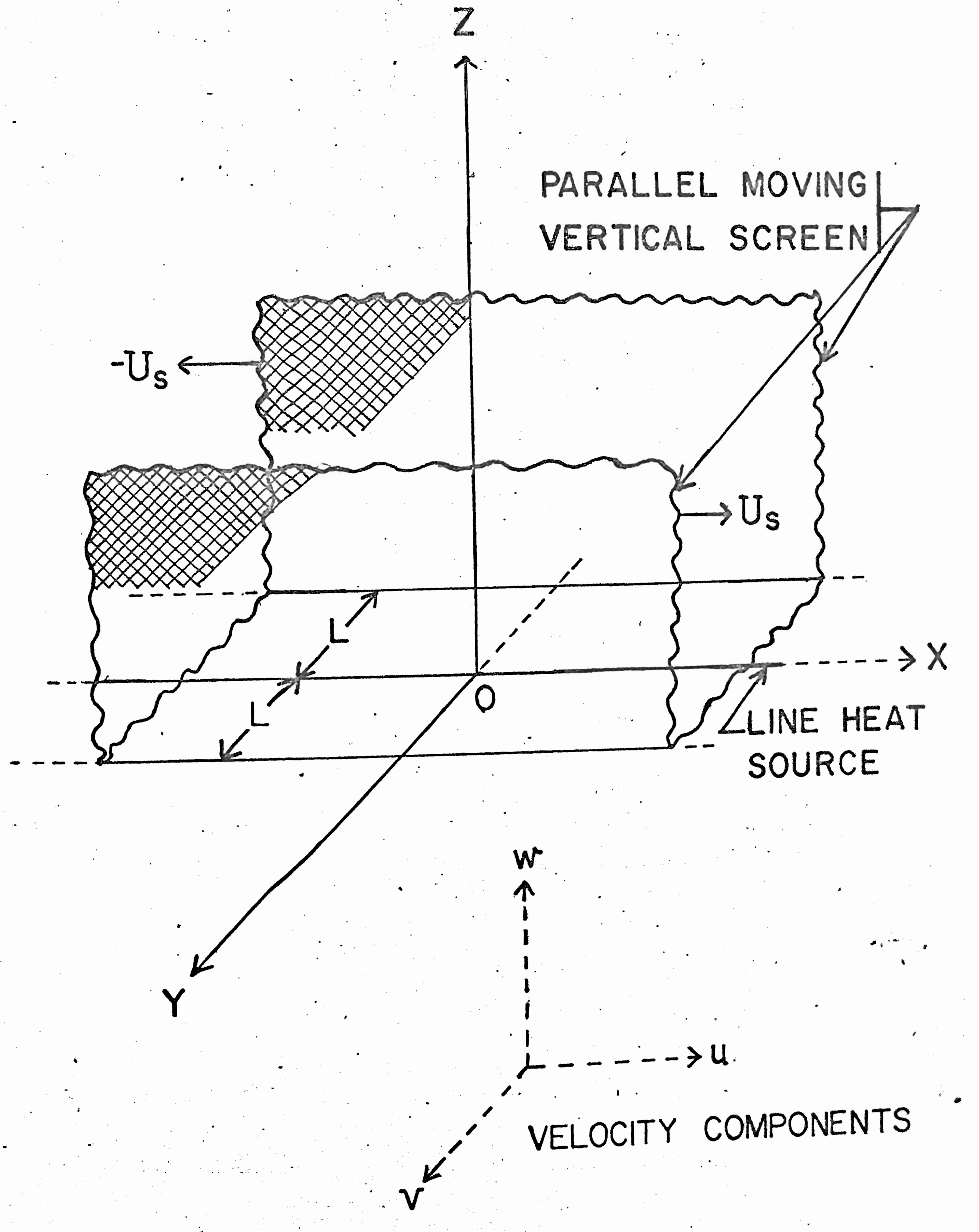

Figure I DEFINATION SKETCH

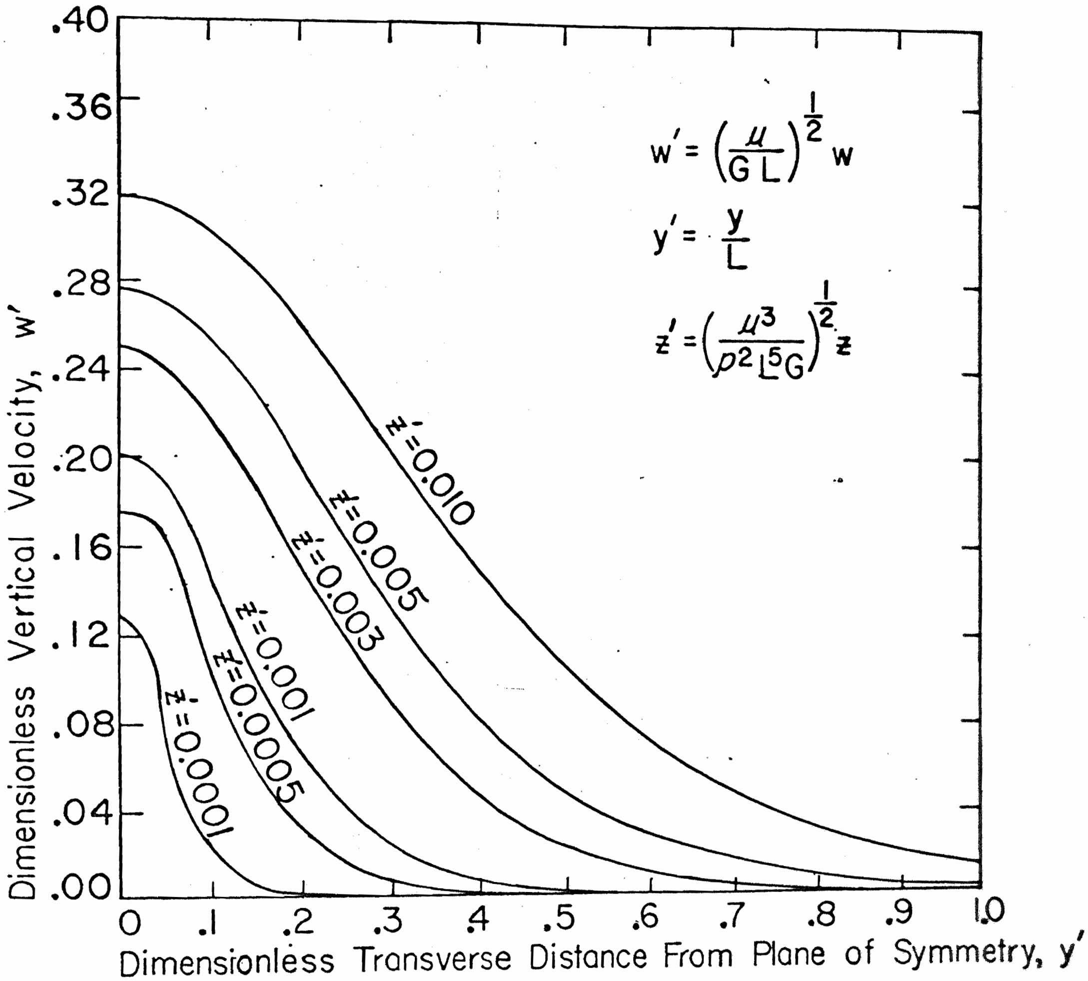

Figure 2 RESULTS OF DISRIBUTION OF VERTICAL VELOCITY

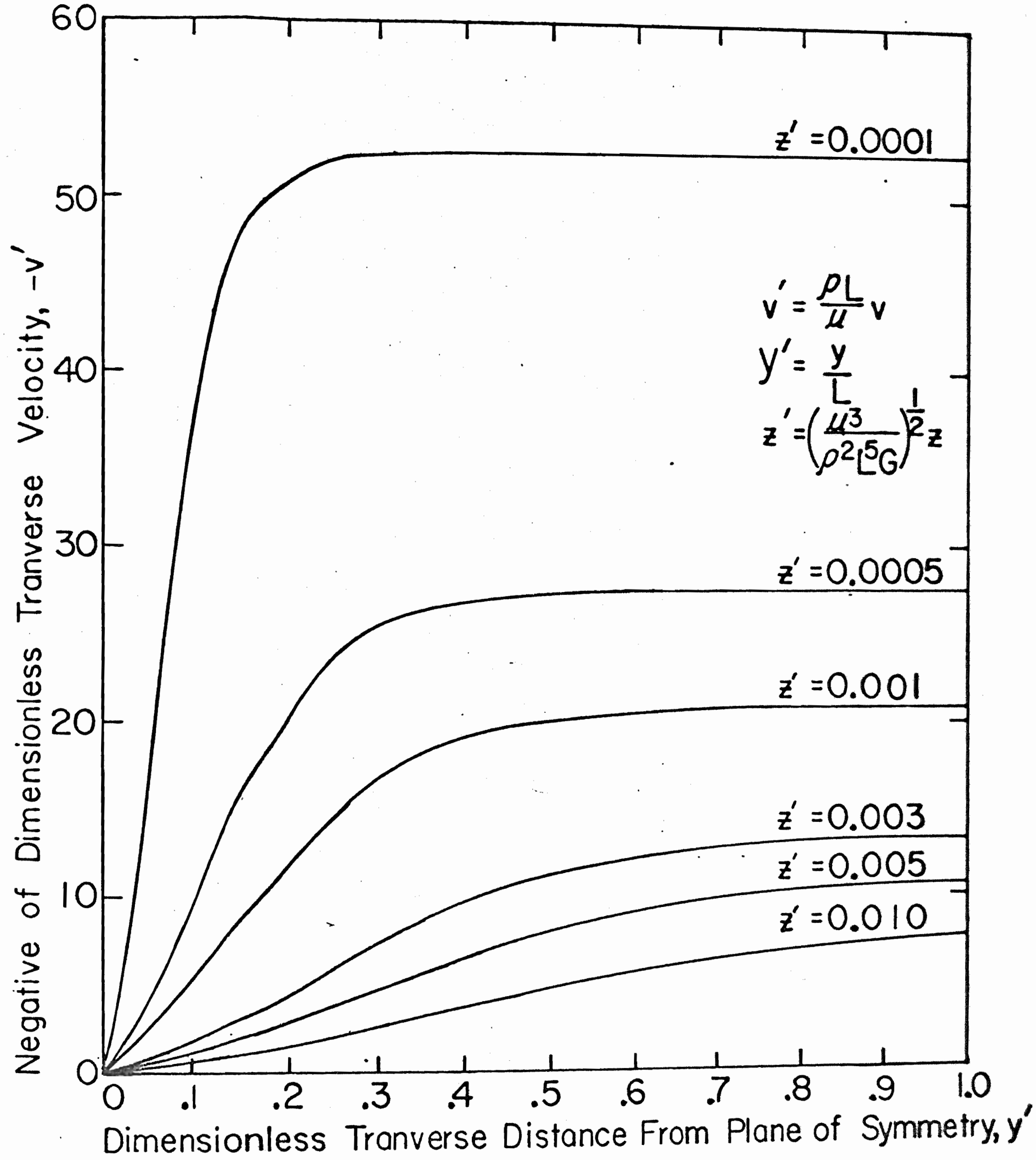

Figure 3 RESULTS OF DISTRIBUTION OF TRANSVERSE VELOCITY

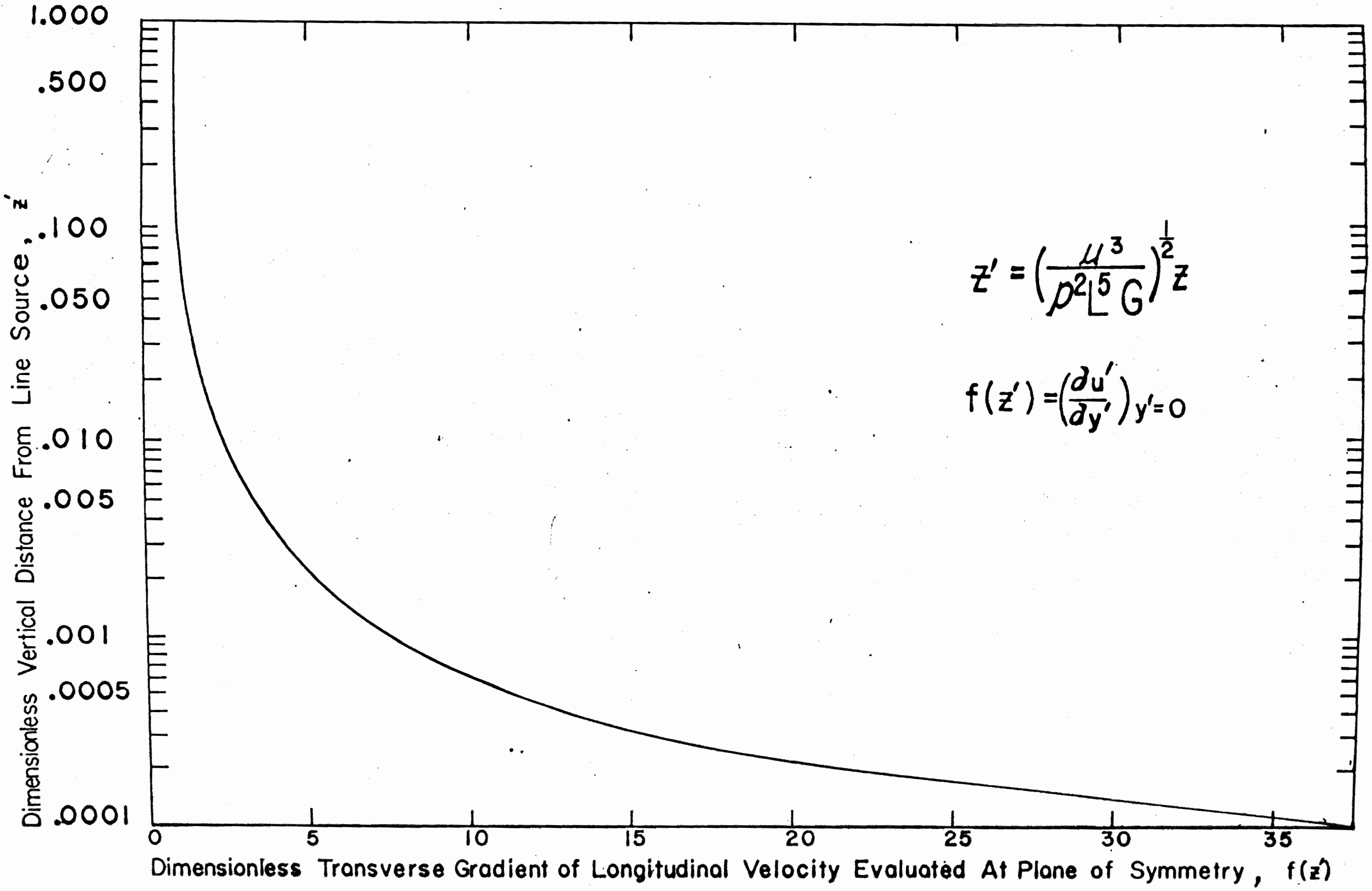

Figure 4 RESULTS OF TRANSVERSE GRADIENT OF LONGITUDINAL VELOCITY EVALUATED AT PLANE OF SYMMETRY

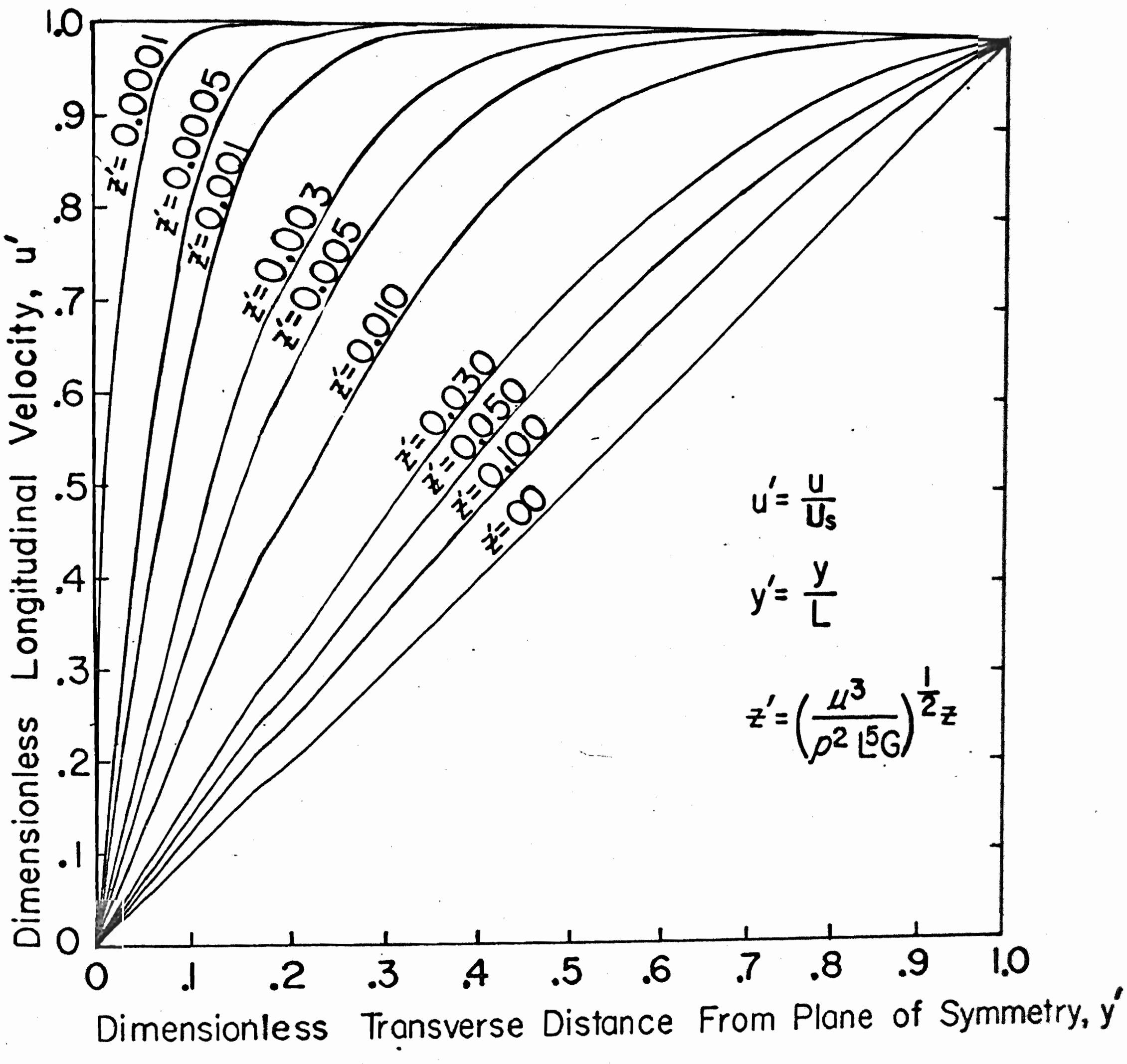

Figure 5 RESULTS OF DISTRIBUTION OF LONGITUDINAL VELOCITY

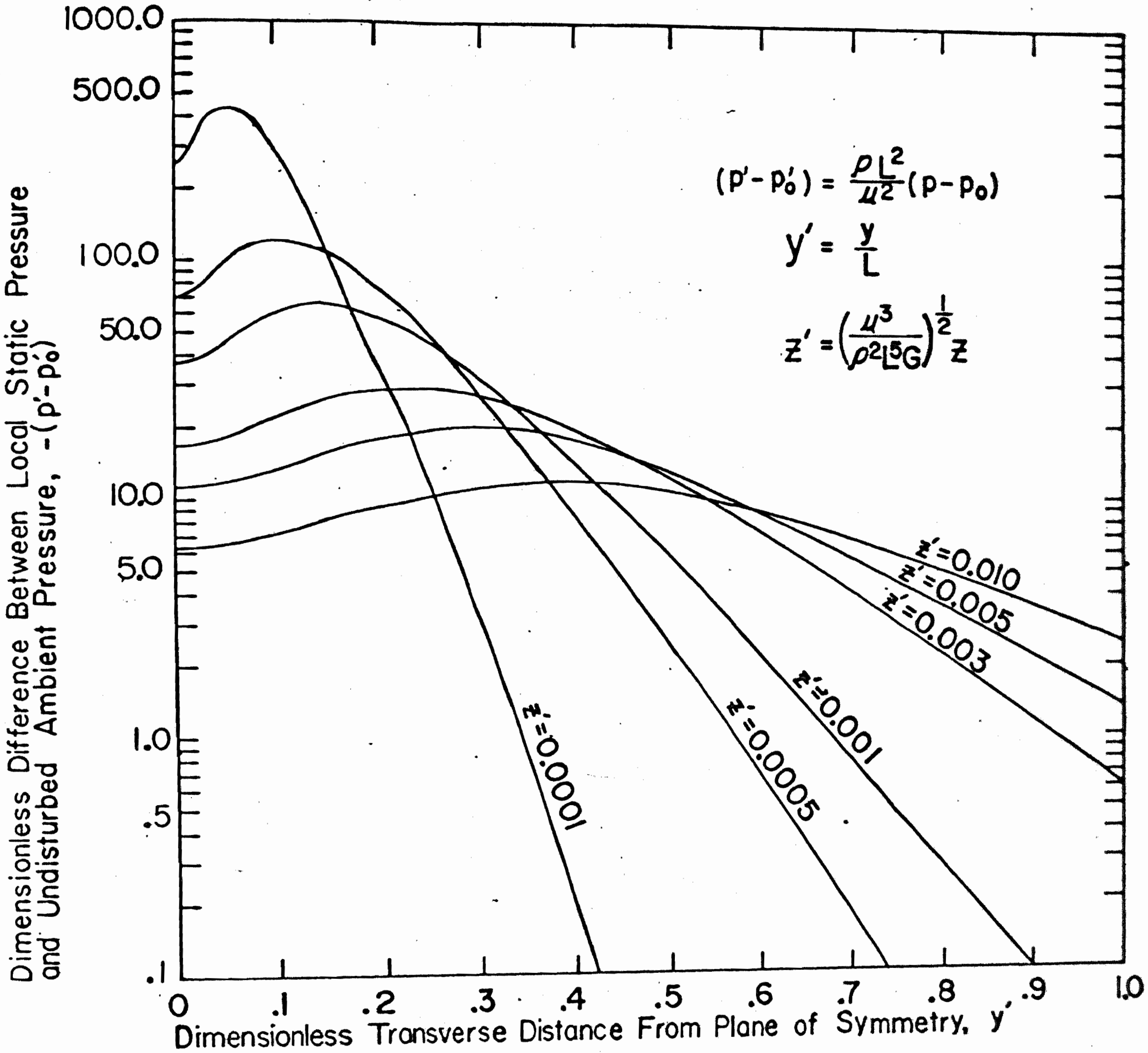

Figure 6 RESULTS OF DISTRIBUTION OF STATIC PRESSURE

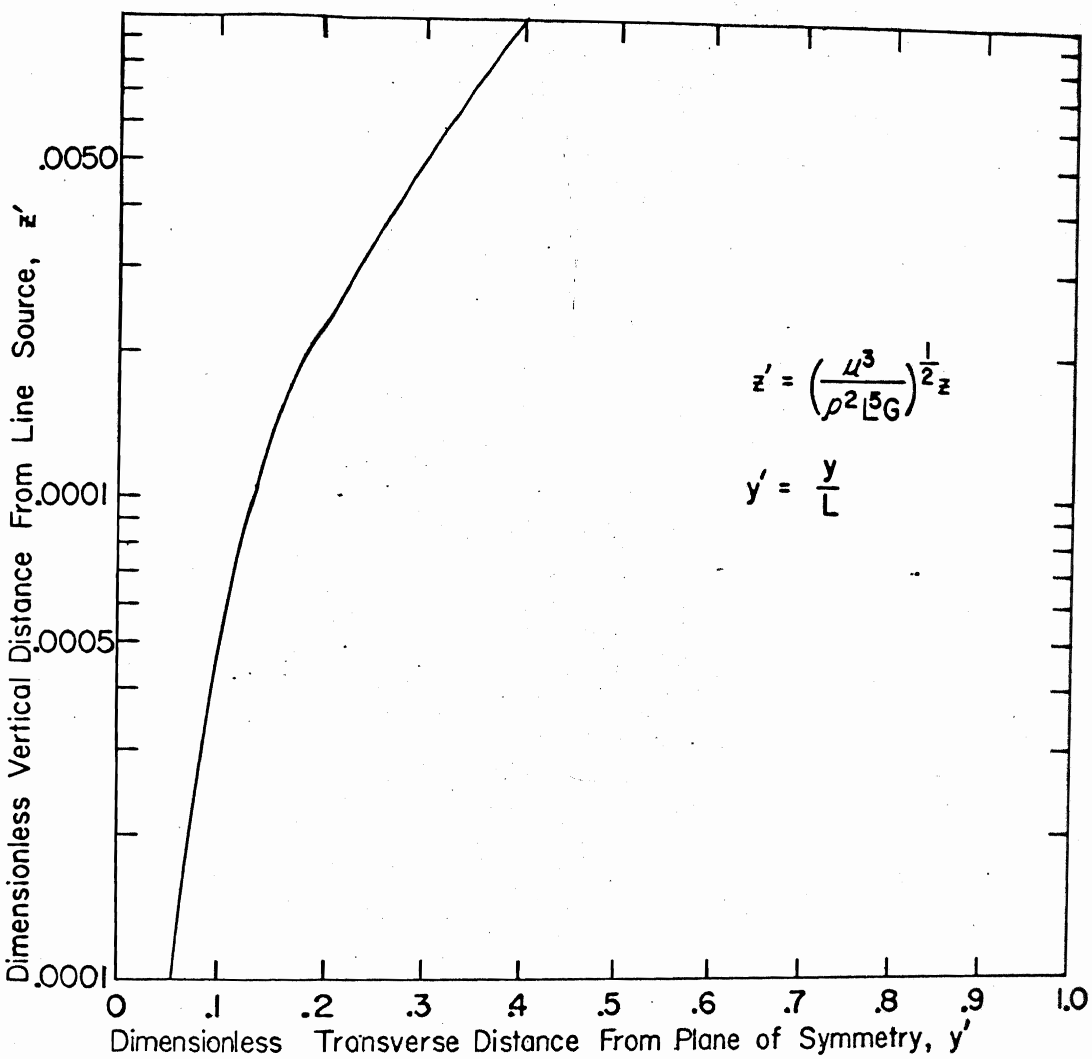

Figure 7 RESULTS OF LOCATION OF POINT OF MINIMUM PRESSURE