

ON THE SOLUTION OF THE BOLTZMANN EQUATION
FOR MAXWELLIAN MOLECULES

Robert M. Ziff

Department of Mechanical Engineering
State University of New York
Stony Brook, New York 11794

G. Stell and P. T. Cummings

Departments of Mechanical Engineering and Chemistry
State University of New York
Stony Brook, New York 11794

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ABSTRACT

Some properties of the Boltzmann equation (BE) for a spatially-uniform system of Maxwellian molecules are considered, including the explicit evaluation of the energy-space transition probability $P(y,z;x)$, the evaluation of the constants λ_n and $\mu_{n,i}$ which enter in the moment equations, the form of the energy BE, the linearization of the moment equations, and a symmetry property of the BE itself.

I. INTRODUCTION

Maxwellian molecules (MM) are particles that have an inverse-fourth power repulsive potential ($\phi = \kappa/r^4$) and a differential scattering cross-section $\sigma(g, \theta)$ that is inversely proportional to the relative velocity g . For a spatially uniform system of such particles, the Boltzmann equation (BE) has been all but solved. Among the many results found, it has been shown that: (i) the moments of the distribution function satisfy an infinite, but closed, set of differential equations;¹⁻⁴ (ii) a formal general solution can be written in the form of an infinite series of Laguerre polynomials;²⁻⁴ (iii) there is a closed-form solution, the celebrated Bobylev-Krook-Wu solution;^{1,5} and (iv) the BE can be transformed into an energy-space scalar equation, with an explicit expression given for the transition probability.^{6,7} Yet to be found are a practical method to construct the solution for a given initial condition, and a method to generate additional closed form solutions, if any such exist.

Much of the work that has been done in this subject has concerned simplified models of the MM, for which $\sigma(g, \theta)$ is still inversely proportional to g , but depends upon θ in a different way than the MM.^{4,8-12} Notable among these models is the pseudo-Maxwellian model, discussed by both Bobylev⁵ and Krook and Wu,¹ in which σ is assumed to be independent of θ . The advantage of these various models is that the form and properties of the solution are similar to those for the MM, while the quantities that enter in the solution can generally be

evaluated explicitly. For the MM, those quantities which are related to integrals of $g\sigma(g,\theta)$ (with respect to θ) cannot be evaluated analytically in closed form because of the nature of $\sigma(g,\theta)$. Another feature of the simplified models is that the corresponding σ are non-singular functions of θ , which makes the manipulation of the kinetic equations easier.

The purpose of this paper is to consider some aspects of the solution to the BE specifically for the true MM, with its singular σ . We present explicit calculations of the σ -dependent parameters that enter in the known results, and present a plot of the energy-space collision frequency. We develop an analytical approximation scheme which gives a rapidly converging series of closed-form expressions for this collision frequency, based upon an analysis of the differential scattering cross section which is given in an Appendix. We also discuss a constraint on the form of the kinetic equation for the MM, an important symmetry property of the BE, and consider the linearization of the moment equation. We begin, in the next section, with a summary of the theoretical results. For a more complete review, the reader is referred to Ref. 13.

II. FORMULAS

We define the energy distribution function $F(x,t)$, related to the usual velocity distribution function $f(v,\tau)$ by $F(x,t) = (K_B T/nm) \times 4\pi v f(v,\tau)$, where K_B = Boltzmann's constant, T = temperature, n = number density (assumed to be uniform), m = mass of a particle, v = the magnitude of the velocity, x is an energy variable $x = mv^2/2K_B T$, and t is a dimensionless time related to the actual time τ by $t = 4\pi n(2\kappa/m)^{1/2} \tau$, where $(2\kappa/m)^{1/2}$ is a constant that multiplies $g\sigma(g,\theta)$ (see Appendix B). We also define the dimensionless collision rate $h(\theta) = (2\kappa/m)^{-1/2} g\sigma(g,\theta)$, which for Maxwellian molecules depends only upon θ and is given in the Appendix.

For the uniform system of Maxwellian molecules, it has been shown:

(i) The normalized moments of F , defined by

$$M_n^*(t) = \frac{\Gamma(n)}{\Gamma(n + 3/2)} \int_0^\infty x^n F(x,t) dx \quad (1)$$

satisfy the closed set of moment equations (written in symmetrical form):¹⁻⁴

$$\frac{dM_n^*}{dt} + \lambda_n M_n^* = \sum_{i=1}^{n-1} \mu_{n,i} M_i^* M_{n-i}^* \quad (2a)$$

where

$$\mu_{n,i} = \frac{n!}{i!(n-i)!} \frac{1}{2} \int_0^\pi \sin^{2i+1} \frac{\theta}{2} \cos^{2n-2i+1} \frac{\theta}{2} \times [h(\theta) + h(\pi-\theta)] d\theta \quad (2b)$$

and

$$\lambda_n = \frac{1}{2} \int_0^\pi \left[1 - \sin^{2n} \frac{\theta}{2} - \cos^{2n} \frac{\theta}{2} \right] \sin \frac{\theta}{2} \cos \frac{\theta}{2} \times [h(\theta) + h(\pi-\theta)] d\theta . \quad (2c)$$

Conservation of mass and energy imply $M_0^* = 1$ and $M_1^* = 1$. Although (2) may be solved successively, a general closed-form solution for all M_n^* has not been found. Some comments about (2c), including expressions for the first seven M_n^* are given in Section VI.

(ii) The formal solution to the BE is given by²⁻⁴

$$F(x,t) = \frac{x^{1/2} e^{-x}}{\Gamma(3/2)} \sum_{n=0}^{\infty} c_n(t) (-1)^n L_n^{(1/2)}(x) \quad (3)$$

where $L_n^{(1/2)}(x)$ is the associated Laguerre polynomial of order 1/2. The $c_n(t)$ satisfy an equation identical to (2), with the M_n^* replaced by the c_n , and with $c_0 = 1$, $c_1 = 0$. The c_n at $t = 0$ can be determined from $F(x,0)$ by the equation

$$c_n(0) = \frac{(-1)^n \Gamma(3/2) n!}{\Gamma(n + 3/2)} \int_0^\infty F(x,0) L_n^{(1/2)}(x) dx . \quad (4)$$

Although (2), (3), and (4) provide a procedure to find a general solution, this procedure is of limited practical use because of the necessity of solving the set of equations (2).

(iii) The closed-form solution of Bobylev⁵ and of Krook and Wu¹ is given by

$$F(x,t) = \frac{x^{1/2} e^{-x/\alpha}}{\Gamma(3/2) \alpha^{5/2}} \left[\frac{5\alpha-3}{2} + x \frac{1-\alpha}{\alpha} \right] \quad (5a)$$

where

$$\alpha = 1 - e^{-\lambda(t-t_0)} \quad (5b)$$

$$\lambda = \lambda_2/2 . \quad (5c)$$

(iv) The energy-space BE satisfied by $F(x,t)$ is given by

$$\frac{dF(x,t)}{dt} = \int_x^\infty d\xi \int_0^\xi dy [F(y,t)F(\xi-y,t)P(y,\xi-y;x) - F(x,t)F(\xi-x,t)P(x,\xi-x;y)] \quad (6)$$

where $P(y,z;x)$ represents the probability that a pair of particles with energies y and z collide and take on energies x and $y+z-x$, in the laboratory frame of reference. The form of the above equation and its relation to other ways of writing an energy-space BE are discussed in Section V. For Maxwellian molecules, $P(y,z;x)$ is given by^{6,7}

$$P(y,z;x) = \frac{1}{\sqrt{yz}} \begin{cases} q\left(\frac{x}{\xi}, \frac{y}{\xi}\right) & 0 < x < y \\ q\left(\frac{y}{\xi}, \frac{x}{\xi}\right) & y < x < z \\ q\left(1 - \frac{x}{\xi}, \frac{y}{\xi}\right) & z < x < (y+z) \end{cases} \quad (7a)$$

for $y < z$, where $\xi = y+z$ and

$$\begin{aligned} q(u,v) &= \frac{1}{4} \int_a^b [h(\theta) + h(\pi-\theta)] d\theta \\ a &= 2[\sin^{-1} \sqrt{v} - \sin^{-1} \sqrt{u}] \\ b &= 2[\sin^{-1} \sqrt{v} + \sin^{-1} \sqrt{u}] . \end{aligned} \quad (7b)$$

Equations (6) and (7) represent a reformulation (and simplification) of the BE for the MM.

III. $P(y,z;x)$

First, consider the transition probability function $P(y,z;x)$, which is determined by (7) and the expression for $h(\theta)$ given in the Appendix. Using numerical techniques to eliminate the parameter s from (A1) - (A2) and to do the integration in (7), we produced the plot of $P(0.3\xi, 0.7\xi, x)$ as a function of x/ξ , given in Fig. 1. This function gives the probability that a MM leaves a collision with energy x/ξ , given that the particles collided with energies 0.3ξ and 0.7ξ . Figure 1 may be compared to the plot given in Fig. 1 of Ref. 12 of the $P(y,z;x)$ for the Bobylev-Krook-Wu pseudo-Maxwellian model (which corresponds to $h = 1$). Figure 1 of Ref. 12 also exhibits the $P(y,z;x)$ for a class of models of general dimensionality d , with $h(\theta)$ proportional to $\sin^{3-d} \theta$, which includes the model of Tjon and Wu⁸ when $d = 2$. The plot of $P(y,z;x)$ for the Maxwellian molecule given here may also be compared to the $P(y,z;x)$ of a class of models considered by Ernst and Hendriks, which are plotted in Fig. 1.4 of Ref. 13. Finally, one may also compare the $P(y,z;x)$ for a three-dimensional system with $h(\theta) = \cos^2 \theta$, whose plot is given in Ref. 6. The most notable difference between the $P(y,z;x)$ of the MM and the $P(y,z;x)$ of these other model systems is that in the case of the MM the $P(y,z;x)$ is singular at $x = y$ and $x = z$, while none of the others are (although some show discontinuities in the derivatives at these points).

The points $x = y$ and $x = z$ correspond to the two cases where a particle leaves a collision with the same energy that one particle had

coming into the collision. The divergence of $P(y,z;x)$ at these points is a consequence of the infinite cross-section for the Maxwellian molecule. Since all long-range potentials have an infinite classical cross-section,¹⁴ they should all have $P(y,z;x)$'s which are divergent.

We can analyze the divergence of $P(y,z;x)$ by using the expansion of $h(\theta)$ given in (A6). To lowest order we have

$$h(\theta) \approx \frac{\sqrt{3\pi}}{8} \theta^{-5/2} \quad (8)$$

and therefore

$$q(u,v) \approx \frac{1}{16} \sqrt{\pi/3} \left\{ a^{-3/2} - b^{-3/2} + (\pi-a)^{3/2} - (\pi-b)^{3/2} \right\} \quad (9)$$

where a and b are given in (7b). It follows from (9) that $P(y,z;x)$ diverges as $|y-x|^{-3/2}$ at $x = y$, and is not integrable. The implications of this divergence regarding the form of the BE will be discussed in Section V. Successively closer approximations to $q(u,v)$ [and therefore to $P(y,z;x)$] can be found by taking further terms in the expansion (A6).

Figure 2 shows $P(y,z;x)$ for the two limiting cases $y/\xi = 0.5$ (both particles have the same incoming energy) and $y/\xi = 0$ (one particle is stationary). The curve for the latter case also illustrates the function $h(\theta) + h(\pi-\theta)$, since it follows from (7) that¹⁵

$$P(0,z;x) = \frac{1}{2z} [h(2\sin^{-1} \sqrt{x/z}) + h(\pi - 2\sin^{-1} \sqrt{x/z})] . \quad (10)$$

We note that (9) can be thought of as the definition of a new model, obviously very closely related to the MM. The corresponding $P(y,z;x)$

for this model would then be given exactly and in closed form by (7a) and (9). The sequence of functions $h(\theta)$ found by taking successively higher terms in (A6) could then be thought of as a sequence of new models, rapidly converging to the MM. Although the $P(y,z;x)$ of these models can be written in explicit form, the $\mu_{n,i}$ and λ_n apparently cannot.

IV. THE λ_n AND THE $\mu_{n,i}$

Using the expression for $h(\theta)$ given in the Appendix we have also calculated the λ_n and the $\mu_{n,i}$. The results of these calculations are given in Tables I and II, and the techniques used are described in the Appendix.

First note that these numbers are not independent. The λ_n for all odd n can be determined from the λ_n of even n ,¹⁶ for example, by repeated use of the relation

$$2\lambda_n = \sum_{i=2}^{n-1} \binom{n}{i} (-1)^i \lambda_i \quad (11)$$

which follows from (2c). The $\mu_{n,i}$ are related to the λ_n by¹⁷

$$\mu_{n,i} = \frac{1}{2} \binom{n}{i} \sum_{k=i}^n \binom{n-i}{n-k} (-1)^{i+k+1} \lambda_k . \quad (12)$$

Second note that some λ_n have been given previously by Alterman, Frankowski, and Pekeris¹⁸ and also by Cornille and Gervois.¹⁶ The λ_n given here are proportional to the $\lambda_{n,0}$ of Alterman, et al., which

are normalized such that $\lambda_{2,0} = 0.4$. In Alterman, et al., the first eighteen values of $\lambda_{n,0}$, $n = 0, 1, \dots, 18$, are given, and these $\lambda_{n,0}/\lambda_{2,0}$ agree with our λ_n/λ_2 to seven figures (note that only five figures are shown in Table I). The work of Alterman, et al., was done (20 years ago) on the "WEIZAC" computer while ours was done on a pocket calculator. Partly to our advantage was the change of variables described in the Appendix.

Cornille and Gervois¹⁶ give seven values of ϕ_n which are related to $\lambda_2, \lambda_3, \dots, \lambda_{15}$ by $\lambda_2 = \phi_1/12$, $\lambda_3 = \phi_1/8$, $\lambda_4 = \phi_1/6 - \phi_2/60$, etc. Their numbers, which are given to seven significant figures, agree with ours only to three figures.

With the numbers of Table I and Table II, the BE for the MM has been reduced to a set of equations with no undetermined constants. Thus, for example, one can write the first few $M_n(t)$, using the solutions $M_n(t)$ given in Section VI.

For the pseudo-Maxwellian model, we have simply $\mu_{n,i} = 1/(n+1)$ and $\lambda_n = (n-1)/(n+1)$.^{1,5} Note that both these λ_n and the λ_n of the MM given in Table II are slowly increasing functions of n . For large n , the λ_n of the pseudo-Maxwellian approaches unity, while for the MM λ_n grows slowly as $n^{1/4}$ -- a result that follows from (2c).

V. THE ENERGY BE

Here we discuss Eq. (6), which is written in a slightly different form than the equation considered previously by one of us:¹²

$$\begin{aligned} \frac{\partial F(x,t)}{\partial t} = & \int_0^\infty dy \int_0^\infty dz F(y,t)F(z,t)P(y,z;x) \\ & - F(x,t) \int_0^\infty dy F(y,t) \int_0^\infty dz P(x,y;z) . \end{aligned} \quad (13)$$

In this way of writing the equation, the interpretation of the two terms as representing gain and loss, respectively, and the interpretation of $P(y,z;x)$ as a transition probability, is perhaps more immediate, but for the MM the two terms on the RHS of (13) are each infinite because of the divergence of $P(y,z;x)$. With both terms written inside the integrations, as in (6), the divergences cancel. Equation (6) is of the form used by Ernst,² who writes $K(x,y;\xi)$ for $P(y,\xi-y;x)$. We can also use the inverse-collision symmetry (or detailed-balance condition),

$$k(x,y;\xi) = k(y,x;\xi) \quad (14)$$

where $k(x,y;\xi) \equiv [y(\xi-y)]^{1/2}P(y,\xi-y;x)$, and rewrite (6) in the form

$$\begin{aligned} \Gamma(3/2)x^{1/2} \frac{\partial \tilde{f}(x,t)}{\partial t} = & \int_0^\infty d\xi \int_0^\xi dy k(x,y;\xi) [\tilde{f}(y,t)\tilde{f}(\xi-y,t) \\ & - \tilde{f}(x,t)\tilde{f}(\xi-x,t)] \end{aligned} \quad (15)$$

where here $\tilde{f}(x,t) \equiv \Gamma(3/2)x^{-1/2}F(x,t)$. [This \tilde{f} is closely related, but not identical, to the $f(v,\tau)$ introduced at the beginning.] Written in this form, the cancellation of the two terms at $x = y$ and $x = \xi - y$ is

obvious. Note that (14) is more nearly in the form of a BE than either (6) or (13) since the four \tilde{f} terms are written in a factor that multiplies the function $k(x,y;\xi)$, which evidently takes the place of the differential scattering cross-section, with the variable ξ taking the place of the scattering angle. In this form, for example, the H-theorem is most readily proven.¹²

VI. THE MOMENT EQUATIONS

Explicit expressions for the first few $M_n^*(t)$ or $c(t)$ have been given by Krook and Wu¹ (for the pseudo-Maxwellian model), Hauge and Praestgaard,¹⁷ and Cornille and Gervois.¹⁶ One way the solutions may be written is in a linearized form, which is possible since the equation (2) for M_n^* depends only linearly upon M_n^* . That is, (2) may be written

$$\frac{db_n}{dt} + \lambda_n b_n = 0 \quad (16)$$

with solution $b_n(t) = b_n(0)e^{-\lambda_n t}$, where

$$\begin{aligned} b_2 &= M_2 - 1 \\ b_3 &= M_3 - 3M_2 + 2 \\ b_4 &= M_4 - 4M_3 + 3M_2^2 \\ b_5 &= M_5 - 5M_4 + 2M_2M_3 + 8M_3 - 6M_2^2 \end{aligned} \quad (17)$$

where we have dropped the asterisks on the M_n . Although this procedure has recently been described by Bobylev,¹⁹ these explicit expressions

of the b_n were not previously given. The corresponding linearization of (2) for the c_n leads to simpler expressions for the b_n , since $c_1 = 0$.

We find

$$\begin{aligned}
 b_2 &= c_2 \\
 b_3 &= c_3 \\
 b_4 &= c_4 + 3c_2^2 \\
 b_5 &= c_5 + 2c_2c_3 .
 \end{aligned} \tag{18}$$

The interesting thing about (17) and (18) is that no λ_n enters the b_n for $n \leq 5$. This does not seem to be the case for higher n ; for example, for b_6 we find

$$\begin{aligned}
 b_6 &= c_6 + \frac{15(\lambda_6 - 4\lambda_4 - 5\lambda_2)}{\lambda_6 - \lambda_2 - \lambda_4} c_2 c_4 + \frac{45(\lambda_6 - 4\lambda_4 + 5\lambda_2)(\lambda_4 - 2\lambda_2)}{(\lambda_6 - \lambda_2 - \lambda_4)(\lambda_6 - 3\lambda_2)} c_2^3 \\
 &\quad + \frac{5(2\lambda_6 - 9\lambda_4 + 12\lambda_2)}{(3\lambda_2 - \lambda_6)} c_3^2 .
 \end{aligned} \tag{19}$$

We note that for the BKW solution, where $c_n = (-1)^{n-1} (n-1) \exp[-n\lambda(t - t_0)]$, it follows that $b_4 = b_5 = 0$, which would be expected since only the λ_2 enters in the solution, (5).

VII. THE TERM $h(\theta) + h(\pi-\theta)$

Finally, we would like to comment on the meaning of the combination $h(\theta) + h(\pi-\theta)$, which occurs in (2b), (2c), and (7c) above. Evidently, for each collision with scattering angle θ the contribution from a

collision with scattering angle $\pi-\theta$ contributes equally. Viewed from outside the collision region, the outgoing particles from these two collisions have the same directions and velocities, but the identity of the two particles are switched, as is shown in Fig. 3. But since the identity of the particles cannot be determined, the two collisions cannot be distinguished. Therefore, the occurrence of $h(\theta)$ always in the combination $h(\theta) + h(\pi-\theta)$ reflects the symmetry with respect to interchange of the outgoing particles.

At first glance, the Boltzmann equation does not appear to contain this symmetry, since only $h(\theta)$ [or $\sigma(g,\theta)$] occurs, and one formally distinguishes between particle 1 and particle 2 in the collision integral. However, because θ is integrated over all values $0 \leq \theta \leq \pi$, and the integrand [besides $h(\theta)$] is symmetric under interchange of θ and $\pi-\theta$, both the θ and $\pi-\theta$ collisions contribute equally and in fact the BE does not distinguish between particles. Although unnecessary, it would be conceptually more accurate to replace $h(\theta)$ by $\frac{1}{2} [h(\theta) + h(\pi-\theta)]$ in the BE to explicitly exhibit this symmetry.

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APPENDIX

The expression for $h(\theta)$ for Maxwellian molecules is given by²⁰

$$h = \frac{1}{4} \frac{1}{\sin\theta} \cdot \left(\frac{1-2s}{s(1-s)} \right)^{1/2} \times [(1-s)K(s) - (1-2s)E(s)]^{-1} \quad (\text{A1})$$

where $s(\theta)$ is determined from

$$\frac{\pi-\theta}{2} = (1-2s)^{1/2} K(s), \quad (\text{A2})$$

with $K(s)$ and $E(s)$ the complete elliptic integrals of the first and second kind, respectively. Equation (A2) follows from the general scattering relation between θ , g , m and the collision parameter b ,

$$\theta = \pi - 2 \int_0^{\eta_0} \frac{d\eta}{[1 - \eta^2 - 2/mg^2 \Phi(b/\eta)]^{1/2}} \quad (\text{A3})$$

for $\Phi = \kappa/r^4$, with s defined by

$$1 - 2s = \left(1 + \frac{1}{x^2} \right)^{-1/2} \\ x = \sqrt{m/2\kappa} \frac{gb^2}{2}. \quad (\text{A4})$$

In (A4), η_0 is the smallest zero of the denominator of the integrand.

Then (A1) follows by virtue of the relation

$$h \equiv \sqrt{m/2\kappa} g\sigma = \sqrt{m/2\kappa} \frac{g}{\sin\theta} \frac{db}{d\theta} = \frac{1}{\sin\theta} \frac{dx}{d\theta} \quad (\text{A5})$$

Using the expressions (A1) - (A2) in a numerical integration is somewhat slow because of the necessity to eliminate s for each integration point.

This problem is eliminated by a change of variables from θ to x defined by (A4) with $s(\theta)$ determined by (A2). By virtue of (A5), we have for any function $f(\theta)$,

$$\int f(\theta) \sin\theta h(\theta) d\theta = \int f(\theta) dx \quad (\text{A6})$$

Now, for each value x of the numerical integration, θ is calculated directly via (A2) and (A4) and f is evaluated at that point. In the integral (7b) for $P(y,z;x)$ the determination of the limits on x require the solution of (A2) for given θ . In this way the curves in Fig. 1 and Fig. 2 were generated. In the integrals for λ_n and $\mu_{n,i}$, in which $0 \leq \theta \leq \pi$, we have simply $0 \leq x \leq \infty$. These integrals were done by Gaussian quadrature, transforming to $y = (x-1)/(x+1)$ so that $-1 < y < 1$. To estimate convergence, we did 16-, 32-, and 64-point integration. For the significance listed in Tables I and II, it turns out that 16-point integration is sufficient.

We also derived an explicit expansion of $h(\theta)$ in powers of θ . We numerically inverted (A2) to find $s(\theta)$ (to ninth order) and used (A4) to find x as a function of s and then θ . Then applying (A5), we found

$$h(\theta) = \frac{1}{8} \cdot \frac{\sqrt{3\pi}}{\theta^{5/2}} \left[1 + \frac{35}{24} \left(\frac{\theta}{\pi} \right) + \left(\frac{\pi^2}{6} - \frac{35}{384} \right) \left(\frac{\theta}{\pi} \right)^2 + \dots + c_n \left(\frac{\theta}{\pi} \right)^n + \dots \right] \quad (\text{A7})$$

with $c_3 = 0.9114128$, $c_4 = 0.7273464$, $c_5 = 0.3183929$, $c_6 = 0.3518717$, $c_7 = 0.0565650$, and $c_8 = 0.1807578$. The first two terms of (A6) were previously derived by Uhlenbeck and Ford.²⁰ The accuracy of Eq. (A7)

is obviously very high for small θ ; for $0 \leq \theta \leq \pi/2$, (A6) with $n \leq 8$ is accurate within 1×10^{-6} . When $\theta \rightarrow \pi$, however, (A6), $n \leq 8$, is in error by about one percent. The results for the integrals of $h(\theta)$ appearing in Eq. (7) are much more accurate than that, because (i) in the regions (those of large θ) where (A6) is less accurate, its value is very small and thus its contribution to the integrals is small, and (ii), in each integral the combination $h(\theta) + h(\pi-\theta)$ always appears, so that also for $\theta \rightarrow \pi$ the large (and accurate) contribution again dominates. For the combination $h(\theta) + h(\pi-\theta)$, (A6) with $n \leq 8$ is accurate to 1×10^{-4} for all $0 \leq \theta \leq \pi$. This series may be used to evaluate the integrals for λ_n and $\mu_{n,i}$ directly in terms of θ , accurate to within 1×10^{-4} . The integral (7b) for $P(y,z;x)$ may be done explicitly, as shown in (9) when only the first term of (A7) is used.

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TABLE I

i	n							
	2	3	4	5	6	7	8	9
1	0.30844	0.23133	0.21110	0.20305	0.19952	0.19813	0.19786	0.19825
2			0.14600	0.12167	0.11033	0.10395	0.09998	0.09737
3					0.09622	0.08419	0.07724	0.07276
4							0.07183	0.06465

TABLE II

The λ_n

n	λ_n	n	λ_n
2	0.30845	14	1.03618
3	0.46167	15	1.06356
4	0.56823	16	1.08941
5	0.64946	17	1.11391
6	0.71597	18	1.13720
7	0.77259	19	1.15941
8	0.82206	20	1.18066
9	0.86612	25	1.2751
10	0.90594	30	1.3550
11	0.94232	40	1.4863
12	0.97587	50	1.5932
13	1.00704	100	1.9574

FIGURE CAPTIONS

Figure 1. $\xi P(y,z;x)$ as a function of x/ξ , for $y/\xi = 0.3$, where
 $\xi = y+z$.

Figure 2. $\xi P(y,z;x)$ for $x/\xi = 0$ (solid line) and $x/\xi = 0.5$ (dashed
line).

Figure 3. Collisions of two particles with scattering angles of θ
(solid line) and $\pi-\theta$ (dashed line).

FIG. 1, ZIFF, STELL

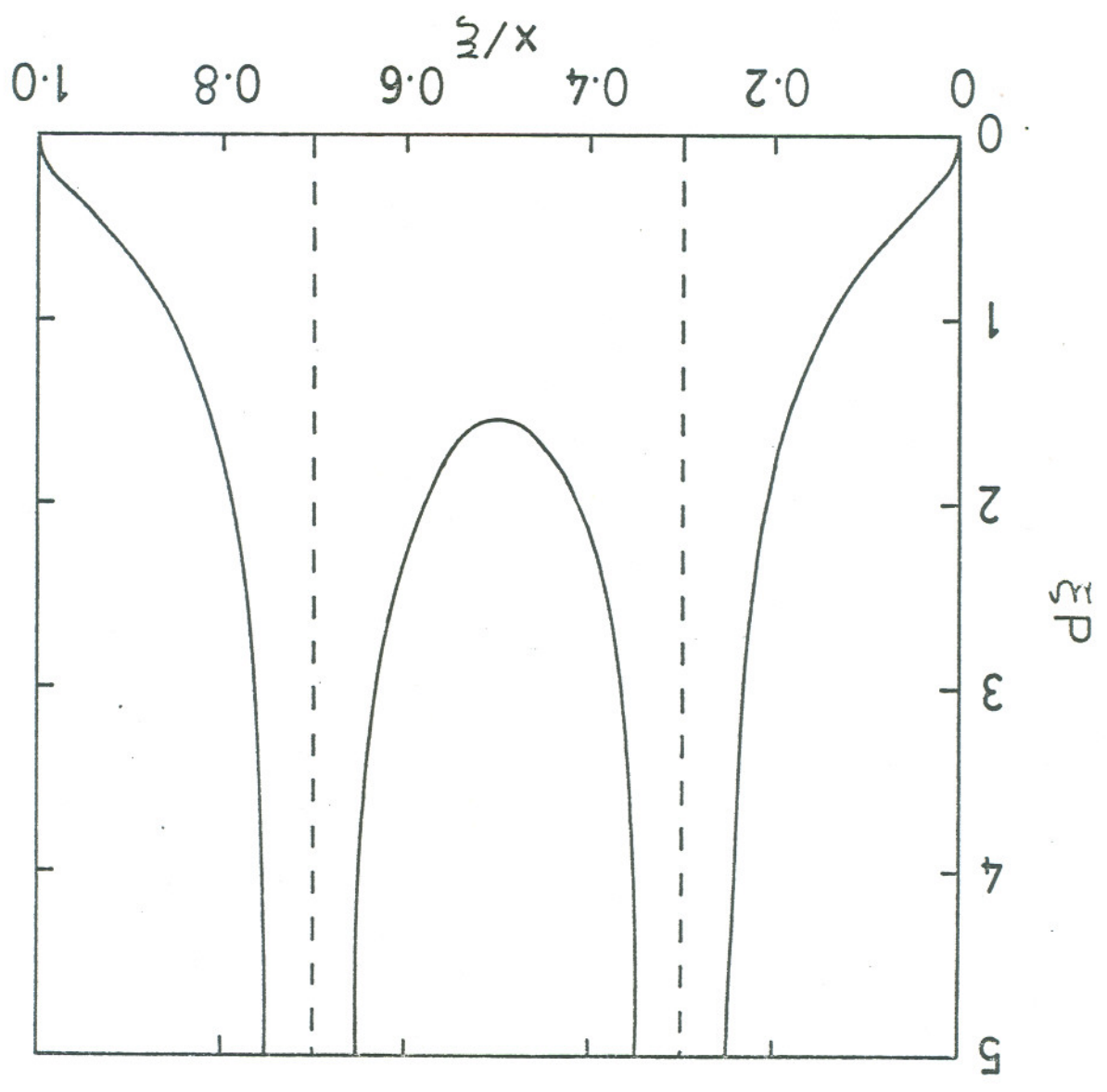
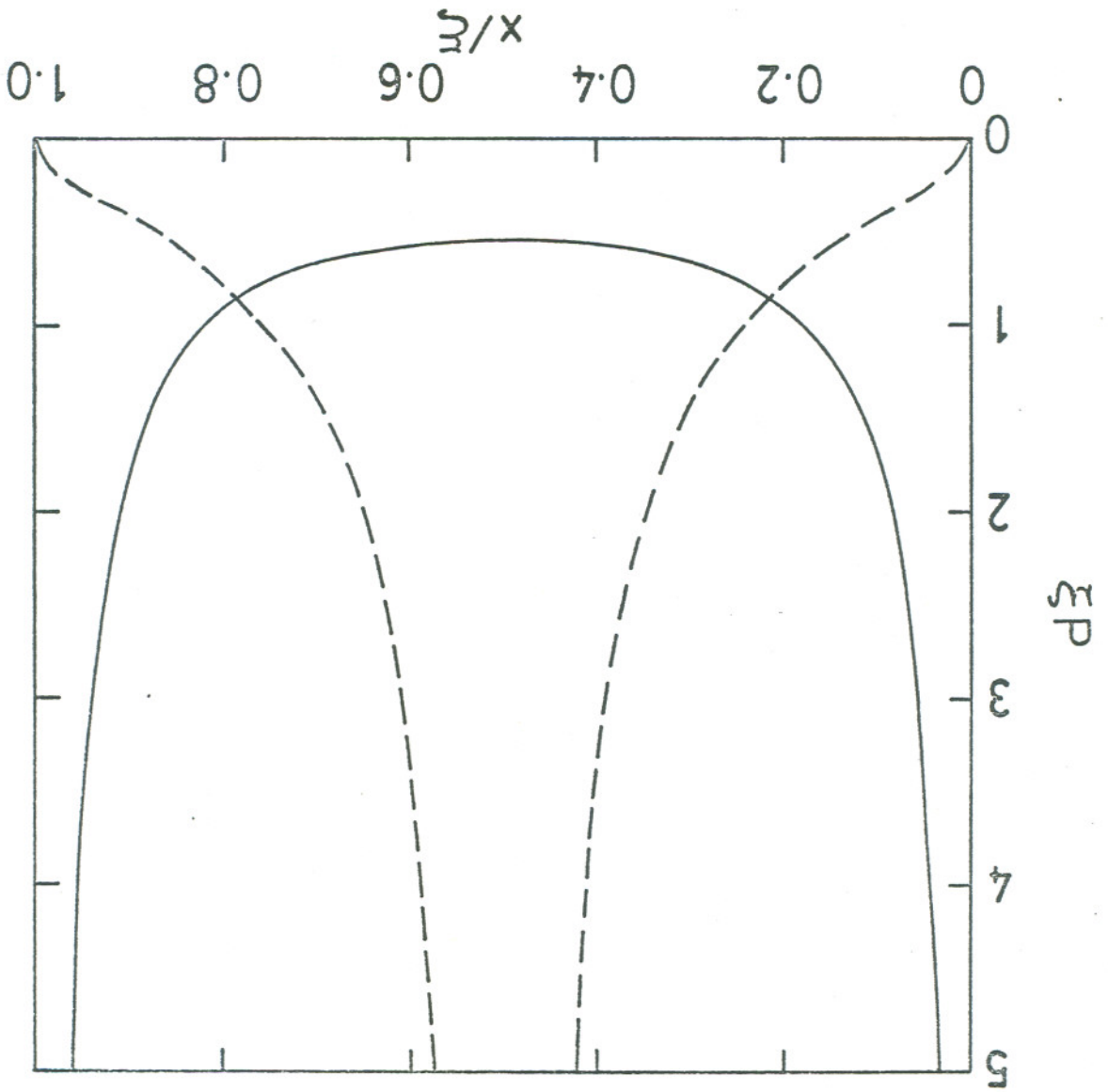
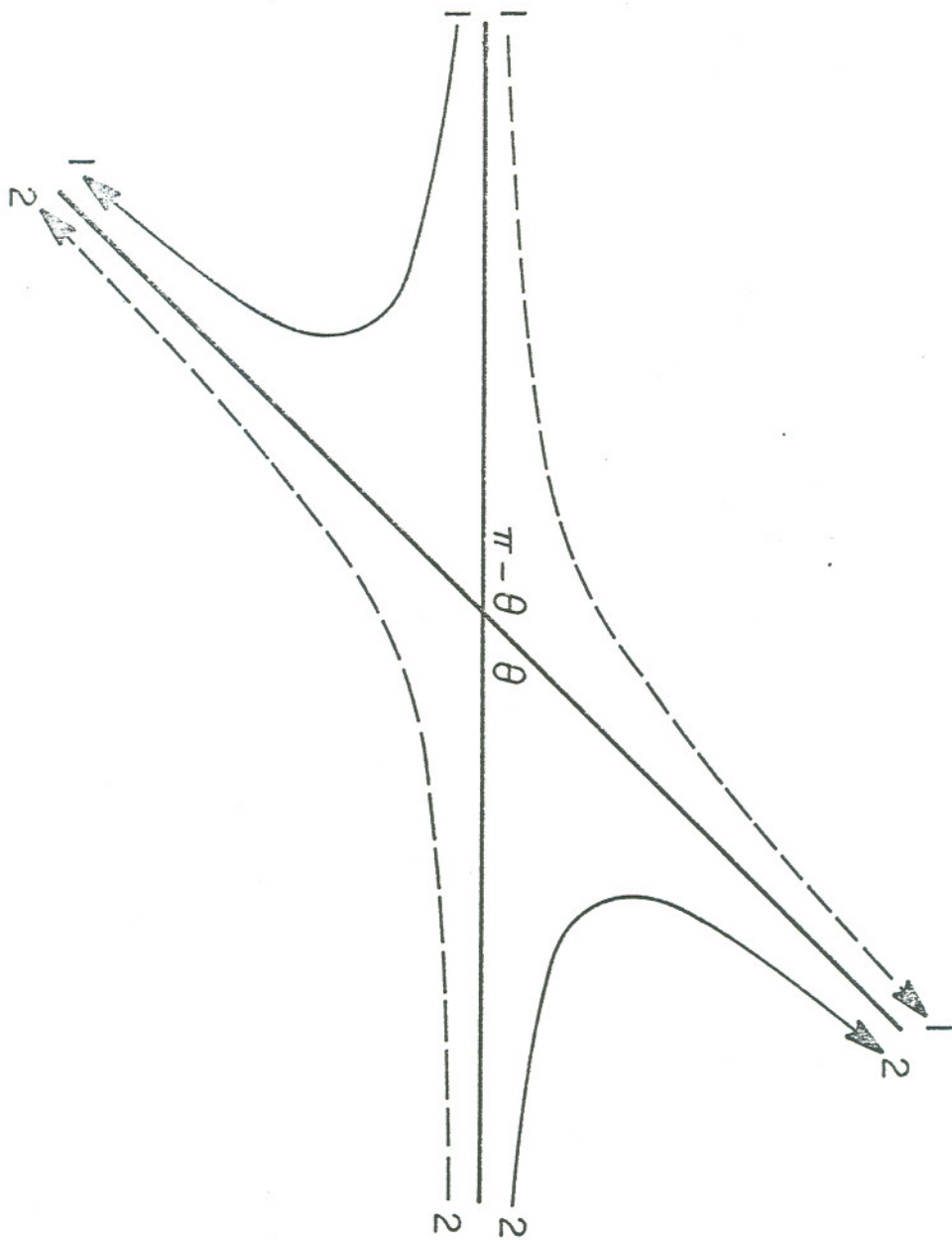


Fig. 2, Ziff, Steele,
CUMMINGS





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