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Abstract

A recursive method for solving for the equilibrium state probabilities of a three tandem queue network with limited buffer space is presented. A set of 4x4 linear equations is solved at each step of the recursion, resulting in large computational savings. Such tandem networks are useful for modeling packets or calls flowing over sequential paths.

1 Introduction

In [2,4] two classes of non-product form, continuous time, queueing networks defined by their state transition diagram (lattice) topology for which it is possible to solve for the equilibrium state probabilities in an exact, decomposed and recursive manner were presented. This is of interest as many practical performance measures are simple functions of the equilibrium state probabilities. The use of recursions in non-product form networks was first studied systematically in [1]. A matrix based interpretation of such recursions is also possible [3].

Among the examples in [2,4] were recursive expressions for two Markovian queues in tandem (series) where one had a buffer size of one and the other had a finite sized buffer. In this letter a network consisting of three tandem Markovian queues operating in continuous time are considered. Two of the queues have a buffer size of one and the other has a finite sized buffer and will be referred to as the multi-buffer queue. Arrivals follow a Poisson process and service times are independent random variables following negative exponential distributions. Such tandem networks are useful for modeling calls or packets flowing over sequential paths.

The state transition diagrams of the networks considered here possess the Type A structure of [2,4]. The state probabilities, relative to a reference state, can be solved for in sets of four states at a time. Thus each step of the recursion procedure involves the solution of a set of four simultaneous linear equations. Assuming that linear equation solution time is proportional to the cube of the number of equations (N), then such a recursive procedure is $\frac{N}{16}^2$ times faster than the direct solution of the entire set of global balance equations, though execution overhead may reduce the magnitude of this speed-up. In the following, recursive solutions are presented for two different orderings of the queues in the three queue tandem network.

2 Source Model

In the first model to be considered the multi-buffer queue is placed immediately after the input and is followed by two servers of buffer size one. This is illustrated in Fig. 1. The state transition diagram appears in Fig. 2. The reference state is state (0,0,0), that is the state with all buffers empty. The states are solved in sets of four starting in the vicinity of the reference state and moving from left to right in the state transition diagram.

Three sets of linear equations are presented below. Each equation is some state's global balance equation. The first set (initial set) is for the first four states near the reference state. The second set (recursive set) is used recursively for four states at a time in the diagram, selected from left to right. The last set (final set) is for four states in the vicinity of the right boundary of the state transition diagram.

Initialization

$$p(0,0,0) = 1.0 \tag{2.1}$$

$$p(0,0,1) = \frac{\lambda}{\beta} p(0,0,0) \tag{2.2}$$

$$p(0,1,0) = \frac{\lambda(\lambda+\beta)}{\alpha\beta} p(0,0,0) \tag{2.3}$$

Initial Set:

$$\begin{bmatrix} -(\lambda+\mu(1)) & \beta & 0 & 0 \\ \mu(1) & 0 & \beta & 0 \\ 0 & -(\lambda+\beta+\mu(1)) & 0 & \alpha \\ 0 & \mu(1) & -(\lambda+\beta) & 0 \end{bmatrix} \begin{bmatrix} p(1,0,0) \\ p(1,0,1) \\ p(0,1,1) \\ p(1,1,0) \end{bmatrix} = \begin{bmatrix} -\lambda p(0,0,0) \\ (\lambda+\alpha)p(0,1,0) \\ -\lambda p(0,0,1) \\ 0 \end{bmatrix} \tag{2.4}$$

Recursive Set i=2,3 ... M-1

$$\begin{bmatrix} -(\lambda+\mu(i)) & \beta & 0 & 0 \\ \mu(i) & 0 & \beta & 0 \\ 0 & -(\lambda+\beta+\mu(i)) & 0 & \alpha \\ 0 & \mu(i) & -(\lambda+\beta) & 0 \end{bmatrix} \begin{bmatrix} p(i,0,0) \\ p(i,0,1) \\ p(i-1,1,1) \\ p(i,1,0) \end{bmatrix} = \tag{2.5}$$

$$\begin{bmatrix} -\lambda p(i-1,0,0) \\ (\lambda+\alpha)p(i-1,1,0) - \lambda p(i-2,1,0) \\ -\lambda p(i-1,0,1) \\ -\lambda p(i-2,1,1) \end{bmatrix}$$

Final Set

$$\begin{bmatrix} -\mu(M) & \beta & 0 & 0 \\ \mu(M) & 0 & \beta & 0 \\ 0 & -(\beta+\mu(M)) & 0 & \alpha \\ 0 & \mu(M) & -(\lambda+\beta) & 0 \end{bmatrix} \begin{bmatrix} p(M,0,0) \\ P(M,0,1) \\ p(M-1,1,1) \\ p(M,1,0) \end{bmatrix} = \tag{2.6}$$

$$\begin{bmatrix} -\lambda p(M-1,0,0) \\ (\lambda+\alpha)p(M-1,1,0) - \lambda p(M-2,1,0) \\ -\lambda p(M-1,0,1) \\ -\lambda p(M-2,1,1) \end{bmatrix}$$

and

$$p(M,1,1) = \frac{\lambda}{\beta} p(M-1,1,1) \tag{2.7}$$

Here M is the maximum size of the multi-buffer queue. Naturally the solutions produced by this recursion must be scaled so that the sum of the state probabilities equals one.

These equations were found to agree to 12 significant digits with the "complete" solution found by solving all the global balance equations simultaneously.

Scaling problems are possible with the use of this set of equations. To see why this happens, consider the multi-buffer queue by itself. Treating it as a state independent M/M/1 queueing model means that the equilibrium probability at the ith state is equal to λ/μ times the equilibrium probability of at the (i-1)st state. Here λ is the arrival rate and μ is the service rate. Thus, for instance, the probability that there are a hundred customers in the buffer is $(\lambda/\mu)^{100}$ times the probability that the buffer is empty. This can naturally lead to scaling problems.

However because the marginal probability of the number of customers in the multi-buffer queue is either monotonically increasing or decreasing in the number of multi-buffer queue customers and because this is the same order in which the previous equations are solved the necessary scaling can be performed automatically. Basically a check is performed at each iteration to see if the state probabilities are approaching the upper limit of the machine register size. If this is so, all state probabilities calculated so far are scaled downward by a common factor. This technique was used successfully for the previous equations over a wide range of parameter values.

The great advantage of the use of these equations is their speed. This is illustrated in the following table, The results were computed on a VAX 11/780 where the IMSL routine LEQT2F was used as the linear equation solver.

# of States	Buffer Size	Complete (sec)	Recursive (sec)
24	5	.5	.3
44	10	1.9	.4
84	20	10.7	.8
124	30	34.8	1.2
164	40	76.1	1.7
204	50	150.5	2.0
244	60	248.8	2.4
284	70	381.9	2.8
324	80	569.5	3.2
404	100	1056.6	4.1

3 Destination Model

In this model arrivals proceed sequentially through two single buffer servers and then enter the multi-buffer queue (Fig. 3). The state transition diagram appears in Fig. 4. Because of the state transition diagram topology it is now necessary to let the reference state be state (0,1,M).

Again, three sets of linear equations are presented. The first set (initial set) is for four states near the reference state. The second set (recursive set) is used recursively for four states at a time in the diagram, selected from right to left. The last set (final set) is for four states in the vicinity of the left boundary of the state transition diagram.

Initialization

$$p(0,1,M) = 1.0 \tag{3.1}$$

$$p(1,0,M) = \frac{\lambda + \mu(M)}{\alpha} p(0,1,M) \tag{3.2}$$

$$p(1,1,M) = \frac{\lambda}{\mu(M)} p(0,1,M) \tag{3.3}$$

Initial Set

$$\begin{bmatrix} -(\lambda+\mu(M)) & \beta & 0 & 0 \\ \lambda & 0 & \beta & 0 \\ 0 & -(\lambda+\beta+\mu(M-1)) & 0 & \alpha \\ 0 & \lambda & -(\beta+\mu(M-1)) & 0 \end{bmatrix} \begin{bmatrix} p(0,0,M) \\ p(0,1,M-1) \\ p(1,1,M-1) \\ p(1,0,M-1) \end{bmatrix} = \tag{3.4}$$

$$\begin{bmatrix} 0 \\ (\alpha+\mu(M))p(1,0,M) \\ -\mu(M)p(0,1,M) \\ -\mu(M)p(1,1,M) \end{bmatrix}$$

Recursive Set i=M-1,M-2, ... 3,2.

$$\begin{bmatrix} -(\lambda+\mu(i)) & \beta & 0 & 0 \\ \lambda & 0 & \beta & 0 \\ 0 & -(\lambda+\beta+\mu(i-1)) & 0 & \alpha \\ 0 & \lambda & -(\beta+\mu(i-1)) & 0 \end{bmatrix} \begin{bmatrix} p(0,0,i) \\ p(0,1,i-1) \\ p(1,1,i-1) \\ p(1,0,i-1) \end{bmatrix} = \tag{3.5}$$

$$\begin{bmatrix} -\mu(i+1)p(0,0,i+1) \\ (\alpha+\mu(i))p(1,0,i)-\mu(i+1)p(1,0,i+1) \\ -\mu(i)p(0,1,i) \\ -\mu(i)p(1,1,i) \end{bmatrix}$$

Final Set

$$\begin{bmatrix} -(\lambda+\mu(1)) & \beta & 0 & 0 \\ \lambda & 0 & \beta & 0 \\ 0 & -(\lambda+\beta) & 0 & \alpha \\ 0 & \lambda & -\beta & 0 \end{bmatrix} \begin{bmatrix} p(0,0,1) \\ p(0,1,0) \\ p(1,1,0) \\ p(1,0,0) \end{bmatrix} = \tag{3.6}$$

$$\begin{bmatrix} -\mu(2)p(0,0,2) \\ (\alpha+\mu(1))p(1,0,1)-\mu(2)p(1,0,2) \\ -\mu(1)p(0,1,1) \\ -\mu(1)p(1,1,1) \end{bmatrix}$$

and

$$p(0,0,0) = \frac{\mu(1)}{\lambda} p(0,0,1) \tag{3.7}$$

Again, M is the size of the multi-buffer. The results of these equations also agree to 12 significant places with the results of a complete solution found by solving all the global balance equations simultaneously. The same sort of automatic scaling discussed for the source model was also used successfully for this model. Computationally, these equations are as efficient as those for the source model.

4 Conclusion

Whether or not such recursions can be found for more general tandem models is an open question. However it is interesting to note that if the multi-buffer queue is put in the central position (preceded and followed by single buffer servers) then the topology of the resulting state transition diagram appears to preclude a recursive solution. Fortran implementations of the recursive source and destination model equations are available from the author at tom@sbee.sunysb.edu.

5 Acknowledgements

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[4] I.Y. Wang and T.G. Robertazzi, "Recursive Computation of Steady State Probabilities of Non-Product Form Queueing Networks Associated with Computer Network Models", *IEEE Transactions on Communications*, Vol. 38, No. 1, Jan. 1990, pp. 115-117.

Figure Captions:

Fig. 1: Source Model Queueing Schematic

Fig. 2: Source Model State Transition Diagram (M=4)

Fig. 3: Destination Model Queueing Schematic

Fig. 4: Destination Model State Transition Diagram (M=4)

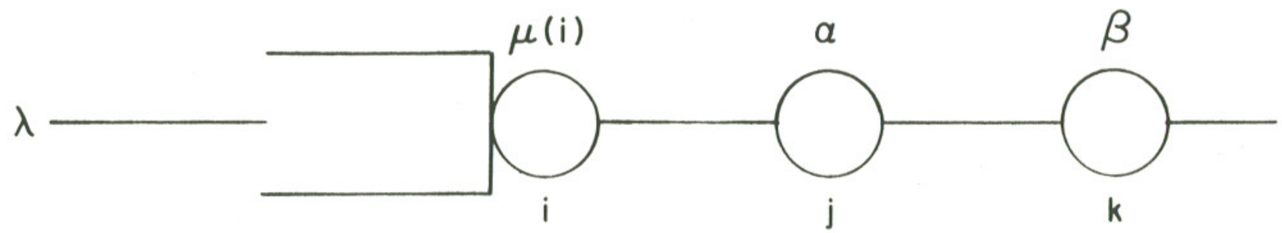

Figure 1

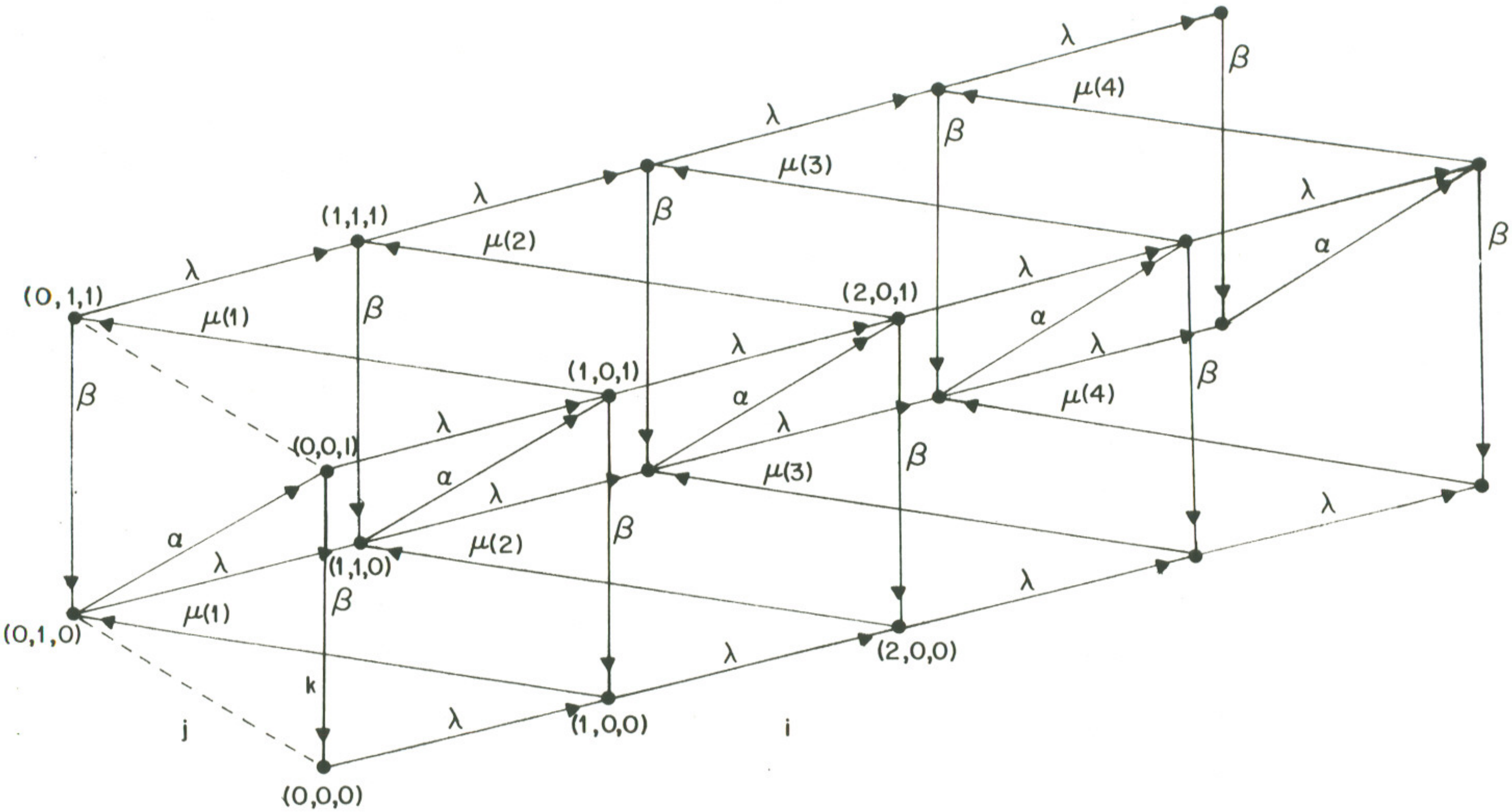

Figure 2

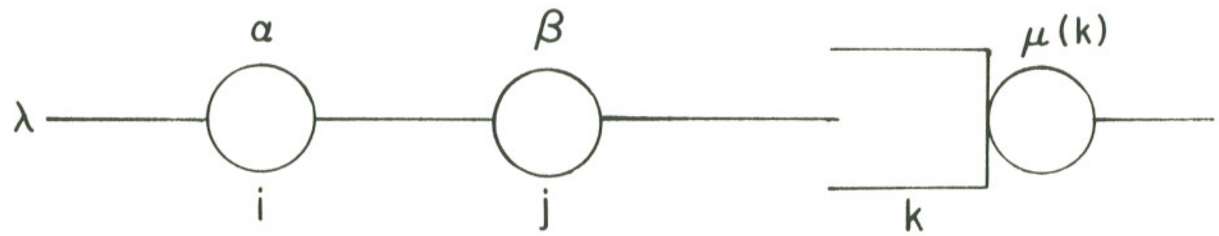

Figure 3

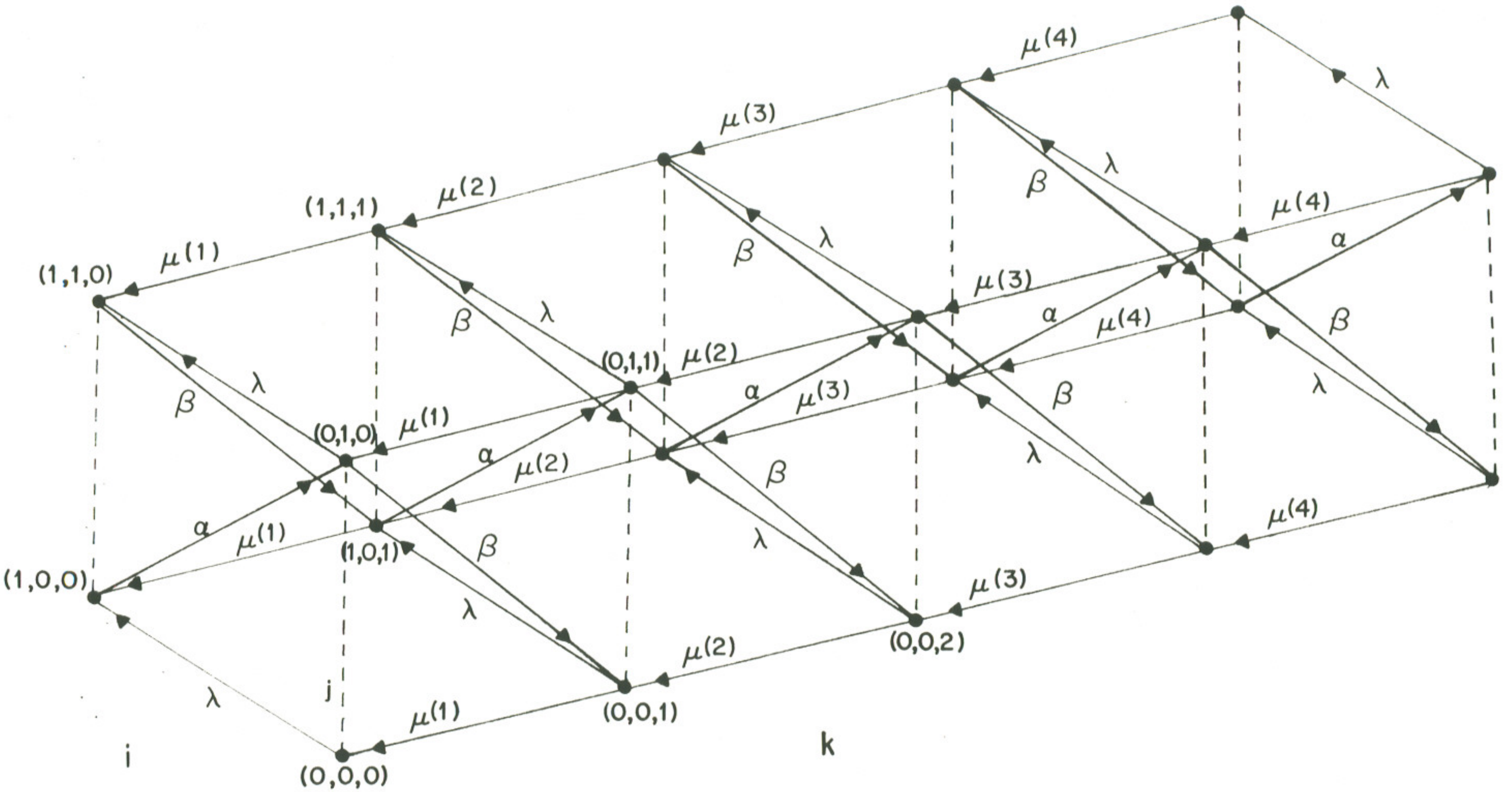

Figure 4