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ON THE BEHAVIOR OF PASSIVE [illegible]

IN A TURBULENT FLUID

by

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On the Behavior of Passive Scalars in a Turbulent Fluid

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The formal description of the evolution of n-point scalar correlations in a turbulent field is presented subject only to the requirement of dynamic passivity of the scalar quantity. Detailed examination is carried out for the following situations: Inhomogeneous scalars in homogeneous turbulence, anisotropic scalars in an isotropic turbulent field and the first order (in time) effect of molecular diffusion on an isotropic two point scalar correlation in an isotropic turbulent field. In isotropic turbulence the asymptotic state of an initially anisotropic two point scalar correlation is deduced to be, in the strictest sense, anisotropic although the time scales for effective isotropy are derived. The initial tendency of molecular conduction to enhance the growth of the scalar correlation between distant spatial regions is indicated as is its initial tendency to diminish the correlation between neighboring points. For the case of a point source tag the effect of a correlation between the scalar and the Eulerian field at the tagging point is established for both the initial wake and asymptotic wake.

I Introduction

Transport of a dynamically passive scalar contaminant by turbulence has received a great deal of attention from a variety of authors. Notable among them are the papers of Batchelor[1], Corrsin[2], Roberts[3], and Lumley[4]. Batchelor's interest was principly in the turbulent dispersion of finite clouds and he demonstrated the significance of certain Lagrangian probability density functions, defined on the turbulence field, in determining motions of the mass center of the cloud and the evolution of some measures of the cloud shape.

Corrsin considered the interesting special case, later generalized by O'Brien[5], of the uniform mean scalar gradient in a turbulence with a non-zero mean velocity. Using a "backward Lagrangian analysis" and the statistical properties of single point diffusion he was able to investigate the evolution of the mean profile, the growth of scalar fluctuations from an assumed initial value of zero and the flux of the scalar normal to the mean motion. A further contribution has been the application by Roberts of a statistical approximation technique, due to Kraichnan[6], to the turbulent diffusion problem. He is able to make predictions about

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1. G. K. Batchelor, Proc. Camb. Phil. Soc. 48, 345, 1952.
2. S. Corrsin, J. Appl. Phys. 23, 113, 1952.
3. P. H. Roberts, J. Fluid Mechs. 11, 2, 257, 1961.
4. J. L. Lumley, Ph.D. Thesis, The Johns Hopkins University, 1957.
5. E. E. O'Brien, Phys. Fluids, 5, 6, 656, 1962.
6. R. H. Kraichnan, J. Fluid Mechs. 5, 497, 1959.

the same sort of quantities as Batchelor discussed.

The essential problem in the contaminant convection arena is to obtain more information about the Lagrangian probability functions that Batchelor has shown are important. Lumley[7], among others, has investigated the possibilities of relating Eulerian averages to Lagrangian but despite some progress and a very clear formulation of the difficulties there seems to be no immediate hope of success. Experiments on cloud distortion and convection in laboratory and atmospheric conditions have been used to obtain statistical information about one and two point motions in the spirit of Batchelor. With well defined turbulence single particle motions have been successfully investigated but the experimental difficulties in studying the joint motion of two points are numerous and information about the joint probability density function is meager.

In the present report we explore an alternative approach to the problem. If one postulates an initial statistical distribution of the scalar field instead of a deterministic distribution it will be shown that the subsequent Eulerian n-point moment of the scalar field can be directly related to the initial Eulerian n-point scalar moment and furthermore that the quantity relating these two moments, for the passive scalar with no diffusion, is just that which arises in Batchelor's analysis of cloud shapes. This gives rise to two hopes. In those circumstances in which the velocity field Lagrangian statistics can be considered well established the evolution of Eulerian moments of the scalar field become determinate. Secondly, there may be a means for obtaining some information about Lagrangian statistical quantities of the

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7. J. L. Lumley, J. Math. Phys., 3, 2, 309, 1962.

turbulence in more complicated situations from the relatively well established process of correlation measurements of the scalar field. In this report we limit ourselves to an investigation of the first of these two notions.

It is quite possible to include, formally, molecular diffusion in the statement of the problem. It is well known that the very small time effect of molecular diffusion is as a phenomena uncoupled from the turbulence. To this order there is no increase in difficulty in including molecular effects and some detailed considerations of its effect on scalar spectra can be made. To include molecular diffusion in general we shall see that we need a great deal of Lagrangian statistical information of such a variety that it seems to be unreasonable to expect this formulation to be a practical one for situations in which the molecular contributions are crucial.

.1 A Formulation in the Absence of Molecular Diffusion

Let $p(\underset{\sim}{x}, t)$ define a nondiffusive scalar field which has no dynamical influence on the turbulent field which convects it and let $\underset{\sim}{\chi}(\underset{\sim}{a}, t)$ be the position of particle $\underset{\sim}{a}$ at time t. Suppose further that the probability density of p is known at some time origin $t = 0$ over the whole spatial field.

$$\text{Define} \quad \underset{\sim}{\chi}(\underset{\sim}{a}, 0) = \underset{\sim}{a}$$

then we presume knowledge of the probability density function of $p(\underset{\sim}{a}, 0)$ which will be indicated by the shorthand notation

$$\mathrm{Prob}\left\{ p(\underset{\sim}{a}, 0) \cong p^* \right\}$$

and which is assumed independent of the turbulent field.

From a physical point of view we are concerned with two random processes. First the tagging is random and described to some extent by the probability density of $p(\underset{\sim}{a}, 0)$ and secondly we consider $\underset{\sim}{\chi}(\underset{\sim}{a}, t)$ as a random path whose statistics will depend only on the turbulence and not on the initial tag.

It is important to notice here the inherent simplicity of the problem we have set as compared to the vastly more complicated problem of relating Eulerian and Lagrangian velocity field statistics. In the classifaction of Lumley[8], we are concerned with the task of obtaining the statistics of $p(z(t), t)$ on $[0, T]$ where $z(t)$ is a random path with a prescribed distribution statistically independent of p.

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8. J. L. Lumley, Colloques Internationaux Du Centre National de la Recherche Scientifique, No. 108, 17, 1961.

A convenient statement of this fact is, in our notation,

$$\text{Prob}\left\{p(\underset{\sim}{x},t) \cong p^*\right\} = \int_{\underset{\sim}{a}} \text{Prob}\left\{p(\underset{\sim}{a},0) \cong p^*\right\} \text{Prob}\left\{\underset{\sim}{\chi}(\underset{\sim}{a},t) \cong \underset{\sim}{x}\right\} d\underset{\sim}{a} \qquad (2.1)$$

That this is valid can easily be seen as follows: in general we may write

$$\text{Prob}\left\{p(\underset{\sim}{x},t) \cong p^*\right\} = \int_{\underset{\sim}{a}} \text{Prob}\left[\left\{p(\underset{\sim}{a},t) \cong p^*\right\};\left\{\underset{\sim}{\chi}(\underset{\sim}{a},t) \cong \underset{\sim}{x}\right\}\right] d\underset{\sim}{a} \qquad (2.2)$$

But from the nondiffuse assumption

$$\frac{\partial p(\underset{\sim}{a},t)}{\partial t} = 0$$

and

$$\text{Prob}\left\{p(\underset{\sim}{x},t) \cong p^*\right\} = \int_{\underset{\sim}{a}} \text{Prob}\left[\left\{p(\underset{\sim}{a},0) \cong p^*\right\};\left\{\underset{\sim}{\chi}(\underset{\sim}{a},t) \cong \underset{\sim}{x}\right\}\right] d\underset{\sim}{a}$$

from which (2.1) follows in view of the assumptions of dynamic passivity and the statistical independence of the initial tag and the particle path.

The usefulness of the formulation of the problem presented as equation (2.2) is evident from the fact that the first probability density under the integral sign is in fact Eulerian as is the left hand side and we expect to be able to transform this into a relation between experimentally accessible quantities such as Eulerian moments of various orders.

Before preceeding in that manner it is evident that (2.1) can be extended to the analysis of joint distributions. We obtain

$$\text{Prob}\Big[\{p(\underset{\sim}{x}_1,t)\cong p_1^*\}\{p(\underset{\sim}{x}_2,t)\cong p_2^*\} \;---\; \{p(\underset{\sim}{x}_n,t)\cong p_n^*\}\Big]$$

$$= \iiint_{\underset{\sim}{a}_1\,\underset{\sim}{a}_2\,\underset{\sim}{a}_3} -- \int_{\underset{\sim}{a}_n} \text{Prob}\Big[\{p(\underset{\sim}{a}_1,0)\cong p_1^*\}\{p(\underset{\sim}{a}_2,0)\cong p_2^*\}---\{p(\underset{\sim}{a}_n,0)\cong p_n^*\}\Big]. \quad (2.3)$$

$$\text{Prob}\Big[\{\underset{\sim}{\chi}(\underset{\sim}{a}_1,t)\cong \underset{\sim}{x}_1\}\{\underset{\sim}{\chi}(\underset{\sim}{a}_2,t)\cong \underset{\sim}{x}_2\}\cdots\{\underset{\sim}{\chi}(\underset{\sim}{a}_n,t)\cong \underset{\sim}{x}_n\}\Big]$$

$$d\underset{\sim}{a}_1\, d\underset{\sim}{a}_2 \;---\; d\underset{\sim}{a}_n$$

In terms of scalar field moments we can obtain formulations which are suitable for particular situations. For example, to examine the single point properties of scalars we note that if we are interested in the statistics of some suitable function of $p(\underset{\sim}{x},t)$ say $f\big(p(\underset{\sim}{x},t)\big)$ we can, by carrying out the appropriate integration of (2.1), obtain

$$\langle f(p(\underset{\sim}{x},t))\rangle = \int_{\underset{\sim}{a}} \langle f(p(\underset{\sim}{a},0))\rangle \Big\{\text{Prob}\{\underset{\sim}{\chi}(\underset{\sim}{a},t)\cong \underset{\sim}{x}\}\Big\}\, d\underset{\sim}{a} \qquad (2.4)$$

where $\langle\ \rangle$ indicates an ensemble average. There are several remarks to be made concerning single point statistics as represented by (2.4). If $\langle f(p(\underset{\sim}{a},0))\rangle$ is a homogeneous function, that is, independent of $\underset{\sim}{a}$, then $\langle f(p(\underset{\sim}{x},t))\rangle$ is independent of t no matter what the velocity field properties are. Also if we are interested in the behavior of the mean, $\langle p(\underset{\sim}{x},t)\rangle$, the existence of initial fluctuations about the mean gives no contribution. They seem to have been generally ignored on the basis of being second order effects if the turbulence level is low[9]. Furthermore if we concede detailed knowledge of $\mathrm{Prob}\{\mathbf{X}(\mathbf{q},t)=\mathbf{x}\}$ in certain circumstances, as for example in isotropic turbulence or free shear flows, specific predictions, within the limitations of the non-diffusive assumption, about the evolution of single point scalar functions can be made.

In the following section we apply the single point diffusion theory with statistical initial conditions and without molecular diffusion to the problem of transport of a variable mean gradient tag by a homogeneous turbulence with a uniform mean velocity.

.2 Single Point Properties in the Variable Mean Gradient Problem

The problem of heat transport across a homogeneous stationary turbulent field with a non-zero mean velocity has been examined[2][5] under the assumptions of a known determinate tag at some fixed Eulerian plane and a sufficiently low turbulence level that the displacement of a particle in the direction of the mean flow direction at time t is just Ut where U is the mean velocity of the turbulent field. The latter assumption is simply for convenience and can easily be removed. We retain it here for the advantage it offers of being able to replace the time variable by an Eulerian spatial coordinate - a kind of visual aid. We are interested however in determining what role the statistics of the temperature field with which the particles are tagged plays in

9. H. K. Wiskind, J. Geophys. Res., 67, 8, 3033, 1962.

determining the evolution of the mean profile, the growth of fluctuations and the mean rate of transport of temperature.

It is well to point out that in the usual experimental technique for examining a model such as the above[9] heated grids are placed in the wind stream and the temperature tagging is accomplished by suitable control of grid dimensions and temperatures. There is then the added complication that the initial scalar field statistics will be to some extent correlated with the velocity field. Attempts to include this possibility, except in a purely formal sense, have been largely unsuccessful and we here retain the assumption that the statistics of the scalar tagging are independent of the turbulence. However in section (2.3) we derive some simple properties of point source tag which is correlated with the Eulerian field at the source.

Under the assumptions mentioned above it has already been pointed out in paragraph 2.1 that the mean profile evolution is independent of the initial fluctuations and is a function only of the initial mean profile and the turbulence statistics. Moreover the detailed evolution of certain classes of determinate profile has been discussed elsewhere[5] and is therefore still relevant.

To examine the heat transport properties we notice from the Eulerian energy equation that

$$\frac{\partial}{\partial x_i} \langle u_i T \rangle = - \frac{\partial}{\partial t} \langle T \rangle \qquad (2.5)$$

so that only the mean profile is relevant. And again the details that have been discussed elsewhere(5) remain valid.

Finally the fluctuation equation becomes in the terminology of 2.4

$$\langle T^2(x,y)\rangle = \int_{-\infty}^{+\infty} \langle T^2(0,y_0)\rangle \, \mathrm{Prob}\left\{X_2(y_0, \tfrac{x}{v}) \cong y\right\} dy_0$$

where $\mathrm{Prob}\left\{X_2(y_0, \frac{x}{v}) \cong y\right\}$ is the probability

that a particle with initial y coordinate position y_0 at time zero will be at position y at time $\frac{x}{v}$.

If we define the displacement $X = y - y_0$ then

$$\langle T^2(x,y)\rangle = \int_{-\infty}^{+\infty} \langle T^2(0, y-X)\rangle f(X)\, dX \qquad (2.6)$$

where $f(X)$ is the probability distribution of X and homogeneity has been assumed.

To calculate the behavior of fluctuations we are interested in the quantity

$$\left\langle\left\{T(x,y) - \langle T(x,y)\rangle\right\}^2\right\rangle = \langle T^2(x,y)\rangle - \langle T(x,y)\rangle^2 \qquad (2.7)$$

And evidently only the first term on the right hand side of (2.4) can reflect the role of initial fluctuations

Writing $T(0, y-X) = \langle T(0,y-X)\rangle + T'(0,y-X)$

we obtain

$$\langle T^2(x,y)\rangle = \int_{-\infty}^{+\infty} \left\{ \langle T(0,y-X)\rangle^2 + \langle T'^2(0,y-X)\rangle \right\} f(X)\,dX$$

from which we infer the not too surprising result that as far as mean square fluctuations are concerned the initial mean square fluctuation contribute in the same fashion as does the square of the mean profile.

Again the previously deduced behavior of fluctuations remains valid if the contribution of the initial mean profile is incorporated. In the circumstance that

$$\frac{\langle T'^2(0,y_0)\rangle}{\langle T(0,y_0)\rangle^2} \ll 1 \qquad \text{for all } y_0$$

the initial fluctuations can be ignored. Furthermore since the asymptotic result has been shown to be spacial uniformity of the fluctuations for X large it will be sufficient in some circumstances to require only the averages over y_0 to satisfy the requirement

$$\langle T'^2(0,y_0)\rangle_{av.} \ll \langle T(0,y_0)\rangle^2_{av.}$$

Further extensions of this problem, for example the consideration of n-point correlations are evident, but we will not pursue the matter further.

3.3 A Point Source with a Dependence of the Scalar Tag on the Velocity Field

The simplest situation in which to investigate the effect of some correlation between the value of the scalar with which a particle is tagged and the particle velocity at the point of tagging is that of a point source. We avoid all other complexities by assuming that any fluctuation in the scalar about its mean is due only to fluctuations in the velocity field.

The appropriate formulation with which to replace equation (2.2) is easily seen to be

$$\mathrm{Prob}\{p(\underset{\sim}{x},t) \cong p^*\} = \mathrm{Prob}\{p(0,0) \cong p^* \mid \underset{\sim}{X}(0,t) \cong \underset{\sim}{x}\}\,\mathrm{Prob}\{\underset{\sim}{X}(0,t) = \underset{\sim}{x}\} \tag{2.8}$$

where the first term on the right hand side is just the probability that the particle at the origin at zero time will be tagged with a value p^* given that the subsequent position of the same particle at time t is $\underset{\sim}{x}$.

Let us restrict our interest to the mean square scalar field

$$\langle p^2(\underset{\sim}{x},t)\rangle = \mathrm{Prob}\{\underset{\sim}{X}(0,t) \cong \underset{\sim}{x}\} \int_{-\infty}^{+\infty} p^{*2}\,\mathrm{Prob}\{p(0,0) \cong p^* \mid \underset{\sim}{X}(0,t) = \underset{\sim}{x}\}\,dp^* \tag{2.9}$$

The classical result implied by Taylor's[10] work concerns situations in which the probability density under the integral sign is independent of the particle path and one obtains

$$\langle p^2(\underset{\sim}{x},t)\rangle = \langle p^2(0,0)\rangle\,\mathrm{Prob}\{\underset{\sim}{X}(0,t) \cong \underset{\sim}{x}\} \tag{2.10}$$

To this we add that equation (2.10) is evidently the asymptotic form of (2.9) even with dependence of p on the particle path, since for times very much greater than the Lagrangian time scale

10. G. I. Taylor, Proc. London Math. Soc., 4, 1921.

it is unreasonable to expect

$$\mathrm{Prob}\left\{p(0,0) \cong p^* \mid \underset{\sim}{\chi}(0,t) \cong \underset{\sim}{x}\right\}$$

to be significantly dependent on the instantaneous particle position.

For short times we follow Taylor and assign the Eulerian velocity $\underset{\sim}{u}(0,0)$ to the particle so that (2.9) becomes

$$\langle p^2(\underset{\sim}{x},t)\rangle = \mathrm{Prob}\left\{\underset{\sim}{\chi}(0,t) \cong \underset{\sim}{x}\right\} \int_{-\infty}^{+\infty} p^{*2}\, \mathrm{Prob}\left\{p(0,0) \cong p^* \mid \underset{\sim}{u}(0,0) = \frac{\underset{\sim}{x}}{t}\right\} dp^* \quad (2.11)$$

Knowledge of the joint probability density that occurs in (2.11) is then sufficient to establish the behavior of scalar fluctuations subsequently. As a particular class of phenomena suppose we know $p(0,0)$ as a definite function of $\underset{\sim}{u}(0,0)$ only; say

$$p(0,0) = f(\underset{\sim}{u}(0,0))$$

then

$$\mathrm{Prob}\left\{p(0,0) \cong p^* \mid \underset{\sim}{u}(0,0) = \frac{\underset{\sim}{x}}{t}\right\} = \delta\left(f\left(\frac{\underset{\sim}{x}}{t} - p^*\right)\right)$$

and (2.11) can be seen to become

$$\langle p^2(\underset{\sim}{x},t)\rangle = \mathrm{Prob}\left\{\underset{\sim}{\chi}(0,t) \cong \underset{\sim}{x}\right\} f^2\left(\frac{\underset{\sim}{x}}{t}\right) \quad (2.12)$$

As a specific example consider the case of temperature tagging by a constant temperature point. With increased velocity one can expect a decrease in the actual tagged temperature of the air.

Let us for simplicity consider a flow with a mean velocity U in the x-direction and idealize the tagging physics such that $p(0,0)$ depends only on the magnitude of the y velocity component v.

Suppose $$p(0,0) = A - \frac{B}{U}|v(0,0)|$$

where A, B, are positive constants.

Then $$\langle p^2(y,t)\rangle = \left(A - \frac{B}{U}\frac{|y|}{t}\right)^2 \mathrm{Prob}\left\{\chi_2(0,t) \cong y\right\} \tag{2.13}$$

If we further assume a Gaussian Eulerian velocity field we obtain[11]

$$\left\langle p^2\left(y,\frac{x}{U}\right)\right\rangle = \left(A - \frac{Bv'}{U}\frac{|y|}{\langle y^2\rangle^{1/2}}\right)^2 \frac{1}{(2\pi)^{1/2}\langle y^2\rangle^{1/2}} \exp\left[-\frac{y^2}{2\langle y^2\rangle}\right] \tag{2.14}$$

where v' is the root mean square velocity in the y direction and $\langle y^2\rangle = v'^2 t^2$ for t small. From which we can conclude that the negative correlation between y velocity and temperature fluctuations produces a wake that is slightly more peaked than a Gaussian. In this sense the wake can be said to be narrowed under the conditions examined here. Quite evidently the wake becomes broader at its base if a positive scalar fluctuation is produced by a y velocity fluctuation.

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11. G. K. Batchelor and A. A. Townsend, "Surveys in Mechanics", Cambridge University Press, New York, 1956, p. 358.

Asymptotic Isotropy of Anisotropic Scalars in Isotropic Turbulence

It has been common practice[12][13] to investigate jointly isotropic scalar and turbulent fields but there seems to be no clear basis for estimating how or if the isotropy of the turbulence tends to produce isotropy of the scalar field it convects. There is an intuitive feeling motivated by the apparently successful predictions of Kolmogoroff's similarity theory[14] that the small scale structure probably tends to isotropy more rapidly than the large scale, and there seems to be a general expectation that in isotropic turbulence asymptotic scalar isotropy is to be expected of all scales. In the present formulation these expectations, if interpreted correctly, are shown to be very plausible. A time scale for the approach to effective isotropy of the large scale motions is deduced and it is argued that this is the longest time scale that will arise in determining the approach to isotropy.

For convenience we will consider two point correlations and examine anisotropy in terms of the dependence of the scalar correlation on the orientation of the vector separation between the two points. The generalization to n-point correlations is immediate and does not seem to introduce any significant new features.

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12. E. E. O'Brien and G. C. Francis, J. Fluid Mech., 13, 3, 369, 1962.

13. S. Corrsin, J. Aero. Sci., 18, 6, 417, 1951.

14. A. N. Kolmogoroff, C. R. Acad. Sci., U.R.S.S. 30, 301, 1941.

From (2.3)

$$\text{Prob}\left\{\left(p(\underset{\sim}{x}_1,t)\cong p_1^*\right)\left(p(\underset{\sim}{x}_2,t)\cong p_2^*\right)\right\}$$

$$= \int_{\underset{\sim}{a}_1}\int_{\underset{\sim}{a}_2} \text{Prob}\left\{\left(p(\underset{\sim}{a}_1,0)\cong p_1^*\right)\left(p(\underset{\sim}{a}_2,0)\cong p_2^*\right)\right\} \text{Prob}\left\{\left(\underset{\sim}{X}(\underset{\sim}{a}_1,t)\cong \underset{\sim}{x}_1\right)\left(\underset{\sim}{X}(\underset{\sim}{a}_2,t)\cong \underset{\sim}{x}_2\right)\right\} d\underset{\sim}{a}_1\, d\underset{\sim}{a}_2$$

and hence $\langle p(\underset{\sim}{x}_1,t)\, p(\underset{\sim}{x}_2,t)\rangle$

$$= \int_{\underset{\sim}{a}_1}\int_{\underset{\sim}{a}_2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p_1^* p_2^* \,\text{Prob}\left\{\left(p(\underset{\sim}{a}_1,0)\cong p^*\right)\left(p(\underset{\sim}{a}_2,0)\cong p_2^*\right)\right\} \text{Prob}\left\{\left(\underset{\sim}{X}(\underset{\sim}{a}_1,t)=\underset{\sim}{x}_1\right)\left(\underset{\sim}{X}(\underset{\sim}{a}_2,t)=\underset{\sim}{x}_2\right)\right\} dp_1^*\, dp_2^*\, d\underset{\sim}{a}_1\, d\underset{\sim}{a}_2 . \tag{2.15}$$

Assuming homogeneity of both fields and denoting

$$\langle p(\underset{\sim}{x}_1,t)\, p(\underset{\sim}{x}_2,t)\rangle = f(\underset{\sim}{r},t) \quad ; \quad \underset{\sim}{r} = \underset{\sim}{x}_2 - \underset{\sim}{x}_1$$

we can integrate (2.15) to obtain

$$f(\underset{\sim}{r},t) = \int_{\underset{\sim}{r}_0} f(\underset{\sim}{r}_0,0)\, q(\underset{\sim}{r},t/\underset{\sim}{r}_0,0)\, d\underset{\sim}{r}_0 \tag{2.16}$$

The turbulent field two particle displacement probability $q(\underset{\sim}{r},t|\underset{\sim}{r}_0,0)$ which appears in (2.16) is the probability that two particles separated by $\underset{\sim}{r}_0$ at time zero will be separated by $\underset{\sim}{r}$ at time t. The notation is here consistent with that of Batchelor[1] who has carefully summarized the situations in which some information about $q(\underset{\sim}{r},t|\underset{\sim}{r}_0,0)$ or related quantities[15] may be deduced.

Turbulent isotropy is equivalent to demanding that

$$q(\underset{\sim}{r},t|\underset{\sim}{r}_0,0) = q(r,r_0,\underset{\sim}{r}\cdot\underset{\sim}{r}_0,t)$$

where r and r_0 are the magnitudes of $\underset{\sim}{r}$ and $\underset{\sim}{r}_0$ respectively.

15. G. K. Batchelor, Quart. J. Roy. Met. Soc. 76, 133, 1950.

Since $f(\underset{\sim}{r}, t)$ as given by (2.8) is a linear function of $f(\underset{\sim}{r_0}, 0)$ it is appropriate to investigate the condition

$$f(\underset{\sim}{r_0}, 0) = \alpha(\underset{\sim}{\beta})\, \delta(\underset{\sim}{\beta} - \underset{\sim}{r_0})$$

where $\delta(\underset{\sim}{\beta} - \underset{\sim}{r_0})$ is the Dirac delta function.

Then $$f_{\underset{\sim}{\beta}}(\underset{\sim}{r}, t) = \alpha(\underset{\sim}{\beta})\, g(\underset{\sim}{r}, t \mid \underset{\sim}{\beta}, 0)$$

or $$f_{\underset{\sim}{\beta}}(\underset{\sim}{r}, t) = \alpha(\underset{\sim}{\beta})\, g(r, \beta, \underset{\sim}{r} \cdot \underset{\sim}{\beta}, t) \qquad (2.17)$$

Isotropy of $f_{\underset{\sim}{\beta}}(\underset{\sim}{r}, t)$ is the requirement that it can be written $F_{\beta}(r, t)$ and it is evident from (2.17) that the dependence of g on $\underset{\sim}{r} \cdot \underset{\sim}{\beta}$ is the source of anisotropy of the scalar field.

There are several regions of $|\underset{\sim}{\beta}|$ to be considered. If $|\underset{\sim}{\beta}|$ is small compared to the Kolmogoroff microscale then the particles can be considered to be undergoing a uniform straining motion over the time scale characteristic of the small eddies. If $|\underset{\sim}{\beta}|$ is of the order of the small eddies an immensely complicated situation arises in which the material line joining the two particles becomes significantly convoluted during relative motion of the particles in the time scale of the appropriate Fourier components of the velocity field. Finally when the initial separation $|\underset{\sim}{\beta}|$ is larger than say the integral length scale of the turbulence the particles may be able to be considered as wandering separately and the problem becomes one of following the independent statistics of two fluid points. The latter operation can be satisfactorily accomplished since asymptotic single point behavior in isotropic turbulence is reasonably well documented.

From (2.16) we have

$$f(\underset{\sim}{r},t) = \int_{\underset{\sim}{r}_0} f(\underset{\sim}{r}_0,0)\, q(\underset{\sim}{r},t\,|\,\underset{\sim}{r}_0,0)\, d\underset{\sim}{r}_0 .$$

Suppose we consider $f(\underset{\sim}{r}_0,0)$ defined over two regions of $\underset{\sim}{r}_0$:

$$f(\underset{\sim}{r}_0,0) = f_\eta(\underset{\sim}{r}_0,0) \quad ; \; 0 < |\underset{\sim}{r}_0| \le \eta$$

and

$$f(\underset{\sim}{r}_0,0) = f_M(\underset{\sim}{r}_0,0) \quad ; \; \eta < |\underset{\sim}{r}_0| ,$$

where η is meant to represent a neighborhood around the origin of the order of the Kolmogoroff microscale.

For all pairs of points described by $\eta < |\underset{\sim}{r}_0|$ we expect an assumption of independent wandering after finite time to be legitimate whereas for nearer neighbors, that is those within the Kolmogoroff microscale, jointly dependent motion is assumed to exist for times which may approach infinity. In particular coincident points and infinitesimally separated particles effectively remain bound together statistically for all time.

Then we may write

$$f(\underset{\sim}{r},t) = \int_{\eta} f_\eta(\underset{\sim}{r}_0,0)\, q(\underset{\sim}{r},t\,|\,\underset{\sim}{r}_0,0)\, d\underset{\sim}{r}_0 + \int_M f_M(\underset{\sim}{r}_0,0)\, q(\underset{\sim}{r},t\,|\,\underset{\sim}{r}_0,0)\, d\underset{\sim}{r}_0 . \quad (2.18)$$

For the second integral the assumption of separate wanderings for τ large implies[1]

$$q(\underset{\sim}{r},\tau \mid \underset{\sim}{r_0},0) = \int_{\chi} Q(\underset{\sim}{\chi},t;0)\, Q(\underset{\sim}{\chi}+\underset{\sim}{r}-\underset{\sim}{r_0},t;0)\, d\underset{\sim}{\chi} \tag{2.19}$$

where $\underset{\sim}{\chi}$ is the vector displacement of either of the two particles and $Q(\underset{\sim}{\chi},t;0)$ is the single particle displacement probability density.

Thus $q(\underset{\sim}{r},t \mid \underset{\sim}{r_0},0)$ can be written $q(\underset{\sim}{r}-\underset{\sim}{r_0},t)$ and the last term in (2.10) reads

$$\int_M f_M(\underset{\sim}{r_0},0)\; q(\underset{\sim}{r}-\underset{\sim}{r_0},t)\, d\underset{\sim}{r_0}$$

Furthermore from (2.11) $q(\underset{\sim}{r}-\underset{\sim}{r_0},t)$ is a convolution $Q(\underset{\sim}{r_0}-\underset{\sim}{r},t) * Q(\underset{\sim}{r_0}-\underset{\sim}{r},t)$ with $Q(\underset{\sim}{\chi},t)$ almost certainly asymptotically Gaussian.

In fact defining

$$f_M(\underset{\sim}{r_0},0) = 0 \qquad |r_0| \leqq \eta$$

and taking the Fourier transform of the final term of (2.18) we obtain

$$\mathcal{F}_M(\underset{\sim}{k},t) = \mathcal{F}_M(\underset{\sim}{k},0)\, \Pi(\underset{\sim}{k},t) \tag{2.20}$$

where $\mathcal{F}_M(\underset{\sim}{k})$, $\Pi(\underset{\sim}{k})$ are the transforms of $f_M(\underset{\sim}{r})$ and $q(\underset{\sim}{r})$.

The remarkable feature of (2.20) is the permanence of anisotropy in $\mathcal{F}_M(\underset{\sim}{k},t)$ due to initial anisotropy in $\mathcal{F}_M(\underset{\sim}{k},0)$ and independent of isotropy of the velocity field quantity $\Pi(\underset{\sim}{k},t)$.

It is to be noted however that this is the kind of permanent anisotropy typical of diffusion problems. For example Bachelor has shown[1] a similar consequence for asymptotic probability density of marked fluid at a point after a finite cloud of marked particles is released.

It is just the sort of anisotropy that remains for infinite times when thermal diffusion from a non-isotropic heated region takes place in an isotropic media. One might better ask for a measure, in terms of the initial cloud geometry, of the time until the cloud appears spherical when viewed from a suitably defined distance.

For our purposes we would in this sense expect approximate isotropy when the mean square motion of a particle is very much greater than the initial separation between the most widely separated correlated points. If the scalar correlation extends a distance L initially where $L >> \eta$ then effective isotropy at least for the larger scale motions might be expected when $\langle X^2 \rangle >> L^2$, where $\langle X^2 \rangle$ is the mean square particle displacement in some direction. Asymptotically $\langle X^2 \rangle = 2\langle u^2 \rangle T t$ where T is a Lagrangian time scale. Hence as an estimate for effective large scale isotropy we obtain

$$t >> \frac{L^2}{\langle u^2 \rangle T}$$

The small scale motion in homogeneous turbulence has been the subject of numerous investigations. Batchelor[16] in his work on material lines has made use of the model of two particles initially so close that they may be considered to experience a uniform rate of strain over the whole time history of interest. This of course

16. G. K. Batchelor, Proc. Roy. Soc. A 213, 349, 1952.

is relevant to the situation $|\underset{\sim}{r}_0| \ll \eta$ and we may seek from his results some estimate of the time to effective isotropy of near neighbors. In fact he introduces an assumption which requires fairly rapid loss of influence of the initial orientation of $\underset{\sim}{r}_0$ for $\underset{\sim}{r}_0$ small enough. By fairly rapid we mean in a time less than a characteristic time of elongation of the material line spanning the separation $\underset{\sim}{r}_0$.

Following Batchelor if $\left\{\langle|\underset{\sim}{r}|\rangle^{-1} \frac{d\langle|\underset{\sim}{r}|\rangle}{d\tau}\right\}^{-1}$ is defined to be such a time scale it can by dimensional reasoning be related to $\left(\frac{\nu}{\epsilon}\right)^{1/2}$ the Kolmogoroff time scale characteristic of the small eddies. ν is the kinematic viscosity and ϵ the dissipation rate.

It is reasonable to expect then that isotropy of the regions of the scalar spectrum associated with the small scale motion of the turbulence will become rapidly isotropic in a time scale . Such a result agrees nicely with the expectations of the universal similarity theory and in that sense is not unexpected. It may also be remarked that even in nonisotropic turbulence for high enough Reynolds numbers small scale isotropy is predicted by Kolmogorovian reasoning and thus the above analysis has wider validity than that done on the larger scale motions which are unlikely to exhibit isotropy in any practical situation.

3 Formulation With Molecular Diffusion

The difficulties of including diffusion phenomena in a Lagrangian framework are widely recognized. A formal statement of the evolution of a diffusing scalar p in terms of the field variables $\underset{\sim}{a}$ and t can easily be shown to be described by

$$\frac{\partial p(\underset{\sim}{a},t)}{\partial t} - k \frac{\partial a_j}{\partial x_i} \frac{\partial}{\partial a_j} \left\{ \frac{\partial a_k}{\partial x_i} \frac{\partial p(\underset{\sim}{a},t)}{\partial a_k} \right\} = 0 \qquad (3.1)$$

where $\underset{\sim}{X}(\underset{\sim}{a},t)$ is the Lagrangian position field for which an inverse is assumed and which allows an Eulerian description of the initial position of a particle.

That is

$$\underset{\sim}{a} = \underset{\sim}{a}(\underset{\sim}{x},t) \quad ; \quad \underset{\sim}{a} = \underset{\sim}{a}(\underset{\sim}{x},0)$$

An expansion of (3.1) in a power series in time indicates that

$$p(\underset{\sim}{a},t) = p(\underset{\sim}{a},0) + k \frac{\partial a_j}{\partial x_i} \frac{\partial}{\partial a_j} \left\{ \frac{\partial a_k}{\partial x_i} \frac{\partial p(\underset{\sim}{a},t)}{\partial a_k} \right\} \Bigg|_{t=0} t + O(t^2)$$

Now

$$\frac{\partial a_j}{\partial x_i} \Bigg|_{t=0} = \delta_{ji}$$

and we obtain

$$p(\underset{\sim}{a},t) = p(\underset{\sim}{a},0) + kt \frac{\partial^2 p(\underset{\sim}{a},0)}{\partial a_k \partial a_k} + O(t^2) \qquad (3.2)$$

An explicit calculation of the term $O(t^2)$ indicates a significant complexity in the coefficients. The difficulty for our purposes is that the coefficients, unlike those in the term of $O(t)$, are dependent upon the statistics of the Lagrangian velocity field. It may be argued that we are already seriously compromised by the necessity of having to prescribe Lagrangian velocity field information; however to include terms of high order in time we will need to prescribe joint probability densities for the Lagrangian displacement and all the higher order spatial derivatives of the particle path at the initial instant.

If we consider (3.2) of finite order Equation (3.1) can be interpreted as a linear stochastic operation.

$$L\, p(\underset{\sim}{a}, 0) = p(\underset{\sim}{a}, t)$$

with $p(\underset{\sim}{a}, 0)$ given statistically and independent of the velocity field but with the coefficients of L stocastic and not statistically independent of the path.

Formally then we can state

$$\mathrm{Prob}\left\{p(\underset{\sim}{x},t) \cong p^{*}\right\} = \int_{\underset{\sim}{a}} \mathrm{Prob}\left\{p(\underset{\sim}{a},t) = p^{*}\, ; \underset{\sim}{\chi}(\underset{\sim}{a},t) \cong \underset{\sim}{x}\right\} d\underset{\sim}{a} \qquad (2.1)$$

as

$$\mathrm{Prob}\left\{p(\underset{\sim}{x},t) \cong p^{*}\right\} = \int_{\underset{\sim}{a}} \mathrm{Prob}\left\{L\, p(\underset{\sim}{a},0) = p^{*}\, ; \underset{\sim}{\chi}(\underset{\sim}{a},t) \cong \underset{\sim}{x}\right\} d\underset{\sim}{a} \qquad (3.3)$$

A certain amount of progress has been made in the problem of handling stochastic linear operators[17] but both the complexity of the

17. G. Adomian, Revs. Mod. Phys. 35, 1, 185, 1963.

operator L for terms of $O(t^2)$ or higher and its statistical coupling with the Lagrangian displacement field are discouraging.

The situation for terms of $O(t)$ is not so depressing. From (3.2) we can note that the operator L is now entirely defined on the scalar field initial statistics and we are required therefore to prescribe not only the initial statistics of the pertinent stocastic measure (mean, 2 point correlation, etc.) but also certain second order derivatives of it.

For example assuming homogeneity of both the turbulence and the scalar field, no dependence of the initial tagging on the turbulence, and including molecular diffusion only to $O(t)$, we find for the two point correlation

$$f(\underset{\sim}{r},t) = \int_{\underset{\sim}{r}_0} \left[(1 + 2kt\nabla^2) f(\underset{\sim}{r}_0, 0) \right] g(\underset{\sim}{r}, t \mid \underset{\sim}{r}_0, 0)\, d\underset{\sim}{r}_0$$

or

$$f(\underset{\sim}{r},t) = f_T(\underset{\sim}{r},t) + 2kt \int_{\underset{\sim}{r}_0} \nabla^2 f(\underset{\sim}{r}_0, 0)\, g(\underset{\sim}{r}, t \mid \underset{\sim}{r}_0, 0)\, d\underset{\sim}{r}_0$$

where the first term is the scalar evolution without molecular diffusion and in the second term

$$\nabla^2 f(\underset{\sim}{r}_0, 0) = \frac{\partial^2}{\partial r_{0_j} \partial r_{0_j}} f(\underset{\sim}{r}_0, 0)$$

As a general expectation $\nabla^2 f(\underset{\sim}{r}_0, 0)$ will be negative in the region near the origin and positive for larger scale $|\underset{\sim}{r}_0|$ so that, noting $g(\underset{\sim}{r}, t \mid \underset{\sim}{r}_0, 0)$ for all $\underset{\sim}{r}$ and $\underset{\sim}{r}_0$ and for small times the contribution to a fixed $\underset{\sim}{r}$ will be primarily from regions of $\underset{\sim}{r}_0$ approximately equal to $\underset{\sim}{r}$, the initial effect of molecular diffusion will be to accentuate the action of turbulent mixing in the sense that the correlation at small scale motions will be decreased and that at larger scales will be accentuated.

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