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MINIMAX SOLUTION OF n+1 INCONSISTENT LINEAR EQUATIONS IN n UNKNOWNS

by

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ABSTRACT. An algorithm for computing the Chebyshev solution of n+1 inconsistent linear equations in n unknowns is given. It makes use of orthogonal triangularization followed by the back-substitution part of the Gaussian elimination.

KEY WORDS AND PHRASES: linear equations, minimax solution, inconsistent equations, Chebyshev solution, orthogonal triangularization.

CR Categories: 5.14

1. Introduction

Let us consider the system of equations

$$Ax = b, \tag{1.1}$$

where A is an (n+1) x n matrix of rank n, x and b are column vectors with n and (n+1) elements respectively. We also assume that the system (1.1) is inconsistent viz., b does not lie in the column space or range $\mathcal{R}(A)$ of A. In otherwords, no x in the n dimensional Euclidean space E^n can be found for which (1.1) is exactly satisfied and as a consequence (1.1) is not a proper equality. If for a given $x \in E^n$, r(x) denotes the residual vector, then (1.1) can be written properly as

$$Ax - b = r(x) . \tag{1.2}$$

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Let $r_i(x)$ denote the i^{th} element of $r(x)$. In this paper we shall give an algorithm for the solution of the following problem:

$$\min_{x} \max_{i} |r_i(x)|, \ i = 1, 2, \ldots, n+1; \ x \in E^n. \tag{1.3}$$

In the general case, when A is m x n, with m > n; b is m x 1 and rank of A = n, a subset of n+1 rows out of the m rows is chosen and the solution of this sub-problem is determined. This is followed by an iterative process which yields the solution of the complete problem. An excellent discussion of this is given by Cheney [1]. In view of this, a numerically stable and relatively simple computational method (which is given below) for the solution of the sub-problem (1.3) may be of some interest.

2. Theoretical Development

Let z_i denote the i^{th} element of a column vector z in E^{n+1}. The i^{th} element, $\operatorname{sgn} z_i$, of the vector sgn z is defined as follows:

$$\operatorname{sgn} z_i = \begin{cases} 1 \text{ if } z_i > 0, \\ 0 \text{ if } z_i = 0, \\ -1 \text{ if } z_i < 0. \end{cases} \tag{2.1}$$

We shall make use of the Theorem given by Cheney [1, p. 41]. In our case it can be restated as follows:

Theorem 2.1. If y is the least square solution of (1.1), then the minimax solution of (1.3) is the exact solution of

$$Ax - b = \epsilon \sigma, \tag{2.2}$$

where $\sigma = \operatorname{sgn} r(y)$ and $\epsilon = \sum_{i=1}^{n+1} r_i^2(y) \, / \sum_{i=1}^{n+1} |r_i(y)|$.

In order to make use of the above Theorem we will need the following for orthogonal triangularization [2,8] of A and a few other results that are related to it. For $k = 1, 2, \ldots, n$, let us define

$$V^{(k+1)} = \theta_k V^{(k)}, \tag{2.3}$$

where $\theta_k = I_{n+1} - \beta_k u^{(k)} u^{(k)^T}$ and $V^{(1)} = A$. If the element in the i^{th} row and the j^{th} column of $V^{(k)}$ is denoted by $v_{ij}^{(k)}$, then the scalar β_k is given by

$$\beta_k = \sigma_k \left(\sigma_k + |v_{kk}^{(k)}|\right)^{-1}, \text{ where } \sigma_k = \left(\sum_{i=k}^{n+1} (v_{ik}^{(k)})^2\right)^{\frac{1}{2}}$$

and the n+1 components of the column vector $u^{(k)}$ are

$$u_i^{(k)} = 0,\ i < k;\ u_i^{(k)} = v_{ik}^{(k)},\ i > k.$$

$$u_k^{(k)} = \operatorname{sgn}(v_{kk}^{(k)})\ (\sigma_k + |v_{kk}^{(k)}|),$$

$u^{(k)^T}$ is the transpose of $u^{(k)}$ and I_{n+1} is the identity matrix of order n+1. Thus θ_k is a symmetric orthogonal matrix of order n+1, which transforms all the elements $v_{ik}^{(k)}$, $i > k$ in equation (2.3) to zero [2,8]. Since the columns of A are linearly independent, we have

$$Q\,A = \begin{pmatrix} R \\ 0 \end{pmatrix}, \tag{2.4}$$

where $Q = \theta_n \cdots \theta_2 \theta_1$ and R is an upper triangular non-singular matrix of order n. If Q_n denotes the first n rows and q the last row of Q and Q^T its transpose, then the fact that Q is orthogonal gives

$$I_{n+1} = QQ^T = \begin{pmatrix} Q_n \\ q \end{pmatrix} (Q_n^T, q^T) = \begin{pmatrix} Q_n Q_n^T & Q_n q^T \\ q\,Q_n^T & q\,q^T \end{pmatrix} \Rightarrow Q_n Q_n^T = I_n \text{ and } qq^T = 1, \tag{2.5}$$

and
$$I_n = Q^TQ = (Q_n^T, q^T)\begin{pmatrix} Q_n \\ q \end{pmatrix} = Q_n^TQ_n + q^Tq \,. \tag{2.6}$$

Now, from (2.4) we have
$$A = Q^T\begin{pmatrix} R \\ 0 \end{pmatrix} = Q_n^T R \,. \tag{2.7}$$

To find the least square solution of (1.1) we shall need the generalized inverse of A [3,4]. For a given A the generalized inverse is defined as the unique matrix A^+ satisfying each of the following equations:

$$A\,A^+A = A,\ A^+\,A\,A^+ = A^+,\ (AA^+)^* = AA^+ \text{ and } (A^+A)^* = A^+A, \tag{2.8}$$

where * denotes the conjugate transpose of the relevant matrix. For real matrices, * implies the usual transpose. Now, in view of (2.7), (2.8) and (2.5), it is easy to verify by direct substitution, that
$$A^+ = R^{-1}\,Q_n \,. \tag{2.9}$$

Let the orthogonal matrix Q also operate on b viz.,

$$Q\,b = \begin{pmatrix} f \\ \lambda \end{pmatrix} \Rightarrow \begin{pmatrix} Q_n \\ q \end{pmatrix} b = \begin{pmatrix} f \\ \lambda \end{pmatrix} \Rightarrow Q_n b = f \text{ and } q\,b = \lambda, \tag{2.10}$$

where f is an n x 1 vector and λ is a scalar. We can now state and prove the following:

Theorem 2.2. The solution of (1.3) is given by

$$x = R^{-1}f - \frac{\lambda}{q\sigma}R^{-1}Q_n\sigma \,, \tag{2.11}$$

where $\sigma = -\,\mathrm{sgn}(\lambda q^T)$.

Proof: The least square solution of (1.1) is [4]

$$y = A^+b + (I_n - A^+A)\,\gamma \tag{2.12}$$

where γ is an arbitrary vector in E^n. Therefore, the least square residual is

$$r(y) = Ay - b = AA^+b - b + (A - AA^+A)\,\gamma = -\,(I_{n+1} - AA^+)\,b, \tag{2.13}$$

since in view of (2.8), $AA^+A = A$. On using (2.7) and (2.9) in (2.13) and simplifying the result by keeping (2.6) and (2.10) in view, we have

$$r(y) = -\,(I_{n+1} - Q_n^T R\, R^{-1}\, Q_n)\, b = -\,(I_{n+1} - Q_n^T Q_n)\, b = -\,q^T q b = -\,\lambda q^T, \tag{2.14}$$

and σ which was defined in Theorem 2.1 is now given by,

$$\sigma = \operatorname{sgn} r(y) = -\operatorname{sgn}(\lambda q^T). \tag{2.15}$$

Also, from Theorem 2.1, (2.14) and (2.5) it follows that

$$\epsilon = \frac{\Sigma\, r_i^2(y)}{\Sigma |r_i(y)|} = \frac{[r(y)]^T[r(y)]}{[r(y)^T[\sigma]} = \frac{\lambda q \lambda q^T}{-\lambda q \sigma} = -\frac{\lambda}{q\sigma}\ . \tag{2.16}$$

In view of Theorem 2.1, the solution of (1.3) is the exact solution of (2.2). But the general solution of (2.2) is [4]

$$x = A^+ b + \epsilon A^+ \sigma + (I_n - A^+ A)\,\gamma\ . \tag{2.17}$$

Now, from (2.7), (2.9) and (2.5) we have

$$I_n - A^+ A = I_n - R^{-1} Q_n Q_n^T R = I_n - R^{-1} R = 0_n\ ,$$

where 0_n is the null matrix of order n. Therefore, (2.17) becomes

$$x = A^+ b + \epsilon A^+ \sigma\ . \tag{2.18}$$

This is the exact solution of (2.2). This can be seen by computing the corresponding residual which in view of (2.16) will turn out to be zero. Using (2.9) and (2.16) in (2.18), we have

$$x = R^{-1} Q_n b - \frac{\lambda}{q\sigma} R^{-1} Q_n \sigma$$

$$= R^{-1} f - \frac{\lambda}{q\sigma} R^{-1} Q_n \sigma\ , \text{ using (2.10).}$$

3. The Algorithm

1. Use orthogonal triangularization (2.3) to compute R, f and λ as follows:

$$Q(A,b) = \begin{pmatrix} R & f \\ 0 & \lambda \end{pmatrix} . \tag{3.1}$$

If the Haar condition (every subset of n rows of A is a non-singular matrix) is not satisfied then in the course of the above reduction, a row other than the last row may become zero. In this case, one row permutation will be required to bring the zero row to be the last one.

2. Compute $R^{-1} f$, $R^{-1} Q_n$ by performing the back substitution (backward course) part of the Gaussian elemination [8, p. 200] as follows:

$$\begin{pmatrix} R & f & Q_n \\ 0 & \lambda & q \end{pmatrix} \longrightarrow \begin{pmatrix} I_n & R^{-1} f & R^{-1} Q_n \\ 0 & \lambda & q \end{pmatrix} . \tag{3.2}$$

In other words, we transform the upper triangular matrix R by elementary row operations to I_n and perform the same row operations on f and Q_n.

3. Using the results obtained in (3.2), compute $\sigma = -\operatorname{sgn}(\lambda q^T)$ and $x = R^{-1} f - \frac{\lambda}{q\sigma} R^{-1} Q_n \sigma$ (see (2.15) and (2.11)).

We conclude this paper with a few remarks regarding the algorithm described above. In view of (2.14), (2.2) and (2.16), if needed the least square residual $r(y) = -\lambda q^T$ and the Chebyshev residual $\epsilon\sigma = -\frac{\lambda}{q\sigma}\sigma$ can be easily evaluated from (3.2). The principal advantages of using the orthogonal symmetric matrices in the reduction (3.1) are: (i) stability is unconditionally guaranteed [8, p. 245], (ii) during the reduction no dangerous growth of elements can take place [8, p. 245] and (iii) if A is sparse, then considerable savings in storage and computational effort can be made [5,6]. The "high accuracy in the inversion of triangular matrices" is well known [7]. Therefore in the reduction (3.2) we expect low round - off errors in $R^{-1} f$ and $R^{-1} Q_n$, because

R is upper triangular with reasonable sized non-zero diagonal elements $(-\operatorname{sgn} v_{kk}^{k})\,\sigma_k$; $k = 1, 2, \ldots, n$, [2,8].

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