SHOCK WAVE SOLUTIONS AT LARGE DISTANCES IN A ONE DIMENSIONAL LATTICE

by

James Tasi

Department of Mechanics, State University of New York
Stony Brook, N.Y. 11790

Report No. 215

February 1972

SUMMARY

A semi-infinite lattice chain, with finite velocity step applied at the first mass point, is analyzed to obtain shock wave solutions which are asymptotically valid at large distances along the chain. For the case of linear nearest neighbor interaction, solutions are given in terms of tabulated functions for all quantities of interest. For the case of nonlinear nearest neighbor interaction, a solution previously obtained by Manvi, Duvall, and Lowell is put in asymptotic form and its limitations are discussed.

NOTATION

Ai	Airy function
α	coefficient of nonlinearity (nondimensional)
β	argument of Airy function for nonlinear case
$J_{2(N-1)}$	Bessel function of first kind and of order $2(N-1)$
Γ	Gamma function
η	argument of Airy function for linear case
p	Laplace transform parameter
S_N	dimensionless displacement from equilibrium of the Nth mass point = (displacement) /(equilibrium distance between mass points)
T	dimensionless time = (time)·(reference frequency), reference frequency is the square root of the spring stiffness divided by the mass of a particle
T_N	arrival time of shock wave for nonlinear solution
u_1	dimensionless velocity step at $N = 1$ = (velocity)/(Newtonian sound speed)
U_s	dimensionless shock velocity from asymptotic solutions. = (shock velocity)/(Newtonian sound speed)
$\overline{U}_s$	= dimensionless average shock velocity
ω	dimensionless frequency = (frequency)/(reference frequency)
X_N	dimensionless position of Nth mass point = (position coordinate)/(equilibrium distance between mass points)

INTRODUCTION

The exact solution for shock waves in a semi-infinite chain of mass points has been obtained by Manvi, Duvall, and Lowell[1] for linear nearest neighbor interaction. The solution for acceleration is given by $S_N''(T) = 2u_1(N-1)T^{-1}J_{2(N-1)}(2T)$ for $N \geq 2$, where u_1 is the velocity step at the first particle and differentiation with respect to time is indicated by a prime. The linear case is extended in Ref. 1 to a nonlinear nearest neighbor interaction case by a time averaging procedure. In both cases, the velocity and displacement at any mass point are prescribed by successive integration of the acceleration with respect to time. However, the integrals are not tabulated for large values of N because of large orders of the Bessel functions involved. It is the purpose of this note to point out that in the case of linear interaction, simple asymptotic solutions exist in terms of tabulated functions for large values of N for all of the quantities of interest (acceleration, particle velocity, displacement and strain). In the case of nonlinear interaction, the solutions of Ref. 1 are put in asymptotic form. The nonlinear asymptotic solutions yield a shock velocity expression for the head of the wave; they also show that integration with respect to time of the analytical solution for acceleration given in Ref. 1 does not yield physically acceptable solutions for particle velocity or displacement.

LINEAR CASE

For large order, Bessel functions are essentially zero until the argument of the function approaches the order, at which time a Bessel function can be represented by an Airy function.[2] Rather than use this property of Bessel functions, however, it is simpler in the linear case to obtain asymptotic solutions directly from a Laplace transform solution of the initial value problem. The dimensionless equation of motion for linear nearest neighbor interaction is[1]

$$S_N''(T) = S_{N+1} - 2S_N \quad S_{N-1} \qquad , N > 1 \tag{1}$$

For zero initial conditions and a velocity step at $N = 1$, the transform solutions for acceleration, velocity, and displacement are

$$(S_N'', S_N', S_N) = \frac{u_1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \left(1, \frac{1}{p}, \frac{1}{p^2}\right)\exp[pT-\lambda(N-1)]dp$$

with $\lambda = 2\sinh^{-1}(\frac{1}{2}p)$. The asymptotic solutions for $N >> 1$ can be carried out in a straightforward manner.[3,4] With $p = i\omega$, $\lambda = i\lambda_1$, and $\lambda_1 \approx \omega + w^3/24$ for low frequencies, the asymptotic solutions are

$$S_N''(T) \sim 2u_1(N-1)^{-1/3}\, Ai(-\eta) \tag{2}$$

$$S_N'(T) \sim u_1\left[\frac{1}{3} + \int_0^{\eta} Ai(-\xi)d\xi\right] \tag{3}$$

$$S_N(T) \sim u_1\left\{\frac{1}{3}(T-(N-1)) + \frac{1}{2}(N-1)^{1/3}\left[3^{1/6}\Gamma(\tfrac{2}{3})/(2\pi) + \int_0^{\eta}\int_0^{\eta'} Ai(-\xi)d\xi d\eta'\right]\right\} \tag{4}$$

in which stationary phase contributions are ignored. In equations (2)-(4)

$$\eta = 2\,[T-(N-1)](N-1)^{-1/3} \tag{5}$$

and $Ai(-\eta)$ is the Airy function. The strain between two mass points N, N-1 is

$$\text{strain} = S_N(T) - S_{N-1}(T)$$

$$= \frac{u_1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} - \frac{1}{2}[1 + \frac{1}{p}(4+p^2)^{1/2}]\exp[pT-\lambda(N-1)]dp$$

$$\sim -u_1 \left\{ \frac{1}{3} + \int_0^\eta Ai(-\xi)d\xi + (N-1)^{-1/3}Ai(-\eta) + \frac{1}{2}(N-1)^{-2/3} d\, Ai(-\eta)/d\eta \right\} \tag{6}$$

The asymptotic expression in Eq. (6) is valid near the head of the wave. Stationary phase contributions could also be included, but they yield terms of order $(N-1)^{-4}$ and are only important behind the head of the wave. The wiry function, its derivative and its integrals that appear in equations (2)–(6) are tabulated in Ref, 5 and 6.

The particle velocity from equation (3) is shown in Fig. 1 for three widely separated particles. The existence of a precursor prior to the arrival of the head of the pulse is due to the frequency dispersion characteristics of the lattice system. The displacement-time relationship given by equation (4) yields the position–time curves shown in Fig. 2 for several neighboring particles. To calculate the shock speed, let $T_{(N)}$ be the time at which the disturbance associated with a given value of η arrives at particle N. From equation (5)

$$T_{(N)} = \frac{1}{2}(N-1)^{1/3}\,\eta + (N-1)$$

For particle N + 1, the disturbance associated with the same value of η occurs at time $T_{(N+1)}$. The speed U_s with which the disturbance travels from particle N to N + 1 is

$$U_s = [(N + 1) - N]/(T_{(N+1)} - T_{(N)})$$

$$\sim 1/\{1 + \eta/[6(N-1)^{2/3}]\} \qquad , N-1 \gg 1$$

or

$$U_s \sim 1 - \eta/[6(N-1)^{2/3}] \qquad , N-1 \gg |\eta| \tag{7}$$

The dispersion of the shock in the neighborhood of the head of the wave, given explicitly by equation (7), is also indicated by the curves in Fig. 1. For η negative $(T < N-1)$, the shock travels at a faster speed than the sound speed; for η positive $(T > N-1)$ the shock travels at a slower speed than the sound speed, with the difference in shock speed in the neighborhood of the head of the wave becoming vanishingly small at increasingly large distances in the lattice. The decrease of the slope of the shock [illegible] with [illegible] has been observed by Tsai and Beckett[7] in their numerical studies of shock waves.

NONLINEAR CASE

For nonlinear nearest neighbor interaction force of the type

$$F_{N,N+1} = -(S_{N+1} - S_N) + \alpha\,(S_{N+1} - S_N)^2,$$

where a is the coefficient of parabolic interaction, the equation of motion is

$$S_N''(T) = (S_{N+1} - 2S_N + S_{N-1})\,[1 - \alpha\,(S_{N+1} - S_{N-1})] \tag{8}$$

The approximations used to obtain an analytical solution for the nonlinear equations are described in Ref. 1. It is assumed that the displacement and velocity are zero prior to the arrival time T_N of the shock wave at the Nth mass point. According to Ref. 1, T_N is calculated by

$$T_N = (N-1)/\bar{U}_s \tag{9}$$

where an average shock velocity, $\bar{U}_s$, with which the wave travels is computed from

$$\bar{U}_s = [1 + \alpha(u_1/\bar{U}_s)]^{\frac{1}{2}} \tag{10}$$

The solutions for particle acceleration and velocity are[1]

$$\text{acceleration} = \begin{cases} S_{N,1}''(T) = 0 & , T < T_N \\ S_{N,2}''(T) = 2u_1(N-1)T^{-1}J_{2(N-1)}(2BT) & , T > T_N \end{cases} \tag{11}$$

$$\text{velocity} = \begin{cases} S_{N,1}'(T) = 0 & , T < T_N \\ S_{N,2}'(T) = 2u_1(N-1)\int_{T_N}^{T} \frac{1}{y}\, J_{2(N-1)}(2By)\,dy & , T > T_N \end{cases} \quad (12$$

where

$$B^2 = 1 + 2\alpha(u_1/\bar{U}_s)$$

The corresponding asymptotic expressions for $S_{N,2}''$ and $S_{N,2}'$ are

$$S_{N,2}'' \sim 2u_1 B(N-1)^{-1/3}\,\mathrm{Ai}(-\beta) \tag{11'}$$

$$S_{N,2}' \sim u_1 \int_{\beta_N}^{\beta} Ai(-\xi)d\xi \qquad (12')$$

where

$$\beta = 2[BT - (N-1)](N-1)^{-1/3}$$

$$\beta_N = \beta(T = T_N)$$

Before showing the consequences of the assumptions on which equations (11) and (12) are based, it should be noted that the solution of Ref. 1 for acceleration and velocity imply a different expression for shock velocity at the head of the wave, U_s, than the average shock velocity $\overline{U}_s$ used in computing T_N from (9). Following the reasoning that led to Eqn. (7) for the linear case, the shock velocity from the asymptotic solution for the nonlinear case is

$$U_s \sim B + 0\ (\beta/(N-1)^{2/3}) \qquad (13)$$

A comparison of $U_s \sim B$ with $\overline{U}_s$ given by (10) can be made by expanding both expressions in powers of αu_1. The results are

$$U_s \sim B \approx 1 + \alpha u_1 - (\alpha u_1)^2 + 0(\alpha u_1)^3 \qquad (14)$$

$$\overline{U}_s \approx 1 + \frac{1}{2}\alpha u_1 - \frac{3}{8}(\alpha u_1)^2 + 0(\alpha u_1)^3 \qquad (15)$$

From (14) and (15), U_s is greater than $\overline{U}_s$, with the two shock velocities differing in their first order dependency on αu_1. Therefore, the shock velocity implied by the wave solution is not consistent with the shock velocity used in obtaining the solution.

With regard to computing particle velocity, if $T_N = (N-1)/U_s = (N-1)/B$ is used, then $\beta_N = 0$ and

$$S_{N,2}' \sim u_1 \int_0^\beta Ai(-\xi)d\xi \tag{12''}$$

For the linear case, the maximum value of particle velocity (from equation (3)) is $1.27u_1$, and far behind the head of the pulse the solution approaches the applied step velocity u_1, as it should. For the nonlinear solution given by equation (12"), the maximum value of particle velocity is $0.94u_1$, and far behind the head of the pulse the solution approaches the value $(2/3)u_1$, which is physically unacceptable.* Similarly, computing displacement by integrating $S_{N,2}'(T)$, with zero value for displacement when $T < T_N$, also yields a physically unacceptable solution. The problem with the nonlinear analysis of Ref. 1 lies in the assumption that the particle displacement and velocity are zero prior to the arrival time T_N of the shook wave. Hence, there is no acceleration prior to T_N, at which time a discontinuity arises in the acceleration. As is seen in Figs. 1 and 2, the existence of nonzero values of velocity and displacement prior to the arrival of the head of the pulse are fundamental characteristics of the lattice chain and should not be ignored.

*It should be noted that if $\overline{U}_s$, rather than U_s, ware used to determine T_N, then the particle velocities would be still lower.

REFERENCES

1. R. Manvi, G.E. Duvall, and S.C. Lowell, Int. J. Mech. Sci. **11**, 1 (1969).
2. G.M. Watson, A Treatise on the Theory of Bessel Functions, p. 249, Cambridge University Press, Cambridge, England, (1962), 2nd ed.
3. M.V. Cerrillo, "An Elementary Introduction to the Theory of the Saddle-Point Method of Integration", Technical Report No. 55:2a, Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass., 1950.
4. O.E. Jones and F.R. Norwood, J. Appl. Mech. **34**, 718 (1967).
5. J.C.P. Miller, "The Airy Integral", British Assoc. Adv. Sci. Mathematical Tables, Part-vol. B, Cambridge Univ. Press, Cambridge, England (1946).
6. E.E. Osborne, "Integrals of Airy Functions", National Bureau of Standards Applied Mathematics Series 52 (1958).
7. D. H. Tsai and C. W. Rcckatt, J. Geophys. Res. **71**, 2601 (1966)

ACKNOWLEDGMENT

This study developed out of work started by the author while employed for a summer at the Sandia Corporation.

FIGURE LEGEND

Fig. 1   Particle velocity for three particles, linear case. Dashed line indicates the solution for the continuum approximation.

Fig. 2   Position-time relationship for three neighboring particles, linear case with $u_1 = 0.2$

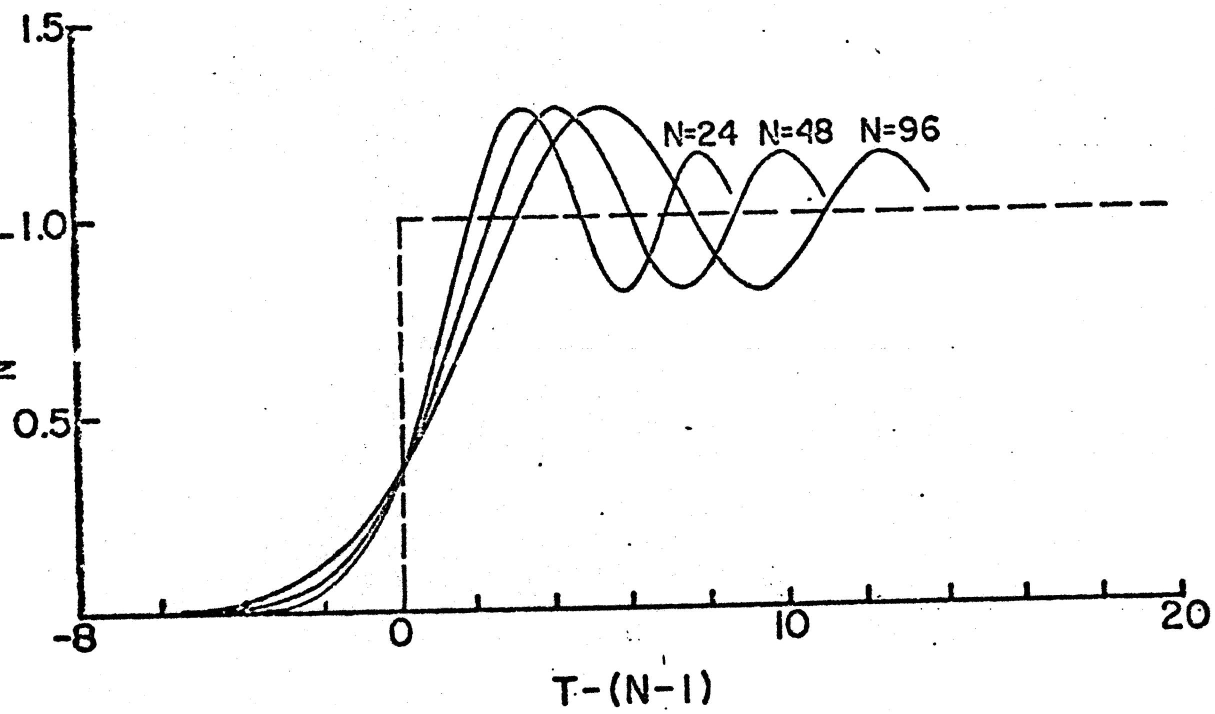

FIG. 1

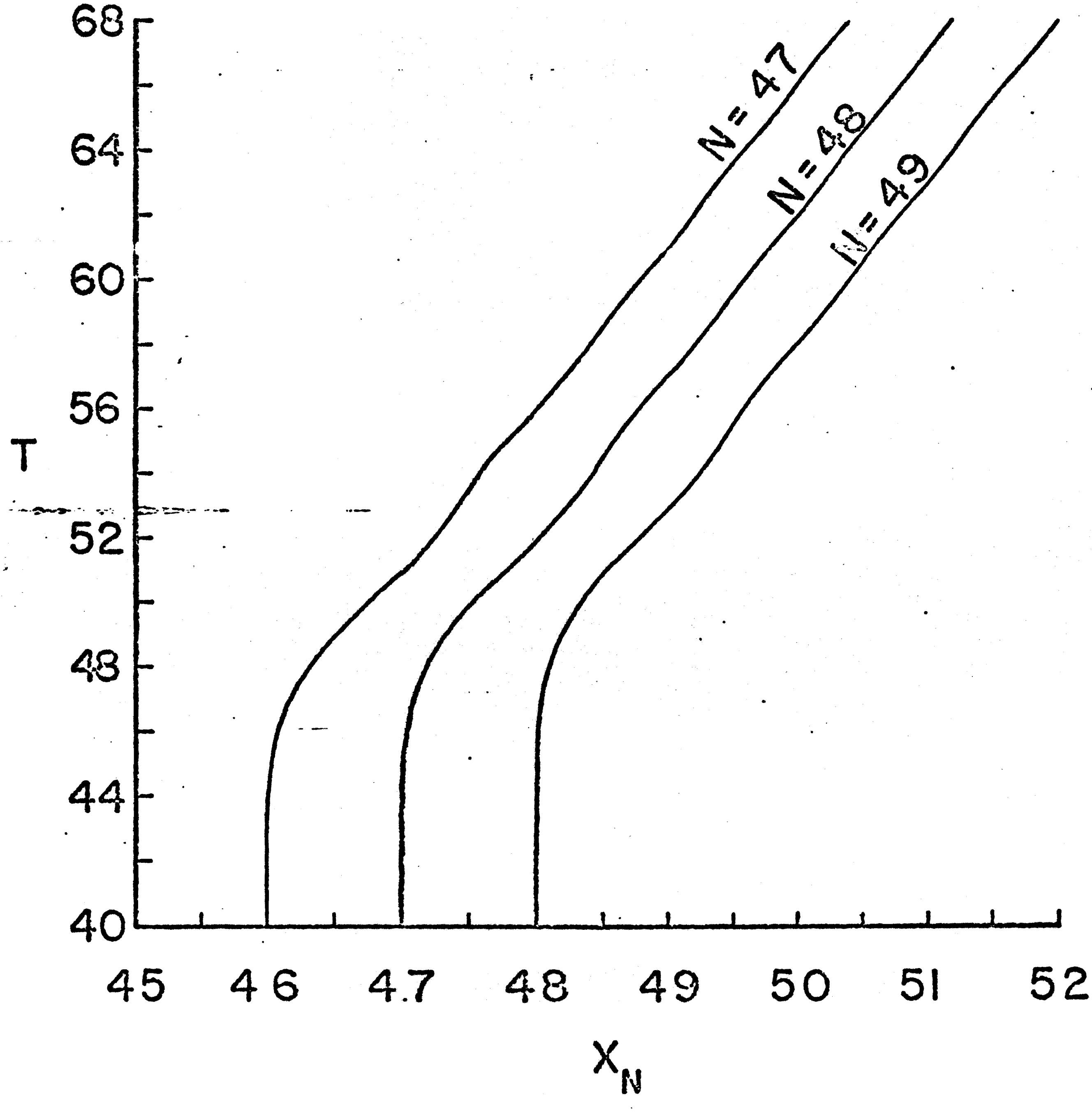

FIG. 2