Syntopogenous Structures and Complete Regularity

By

D. V. Thampuran

An extensive theory of syntopogenous structures has been developed by Császár (1). The purpose of this paper is to study the relationship between syntopogenous structures and complete regularity.

A syntopogenous structure gives rise to two topologies in general and so it is natural to associate a bitopological space with a syntopogenous structure. A syntopogenous structure characterizes a particular type of bitopological space; a bitopological space is completely regular iff it is syntopogenizable. This is similar to the result that a symmetric syntopogenous structure characterizes a particular type of topological space --- the completely regular one.

The bitopological space of a perfect syntopogenous structure has some special properties.

Let E denote a set. The empty subset of E will be denoted by ϕ and for a subset A of E we will write cA for the complement of A. If A contains only one element x we will write cx for cA.

Definition 1. Let $\mathcal{T}$, $\mathcal{T}'$ be two topologies for E. Then the ordered triple $(E, \mathcal{T}, \mathcal{T}')$ is said to be a bitopological space; $\mathcal{T}$ and $\mathcal{T}'$ are called the left and right topologies of this space.

Let $<$ be a topogenous structure on E as defined on page 59 of Császár (1). Denote by $\mathcal{T}$ the family of all subsets T of E such that $x \in T$ implies $cT < cx$; it is then obvious that $\mathcal{T}$ is a topology for E. Take $\mathcal{T}'$ to be the family of all subsets T of E such that $x \in T$ implies $x < T$; then $\mathcal{T}'$ is also a topology for E.

Definition 2. We will call $(E, <, \mathcal{T}, \mathcal{T}')$ the bitopological space, $\mathcal{T}$ the left topology and $\mathcal{T}'$ the right topology of $<$.

But when the context makes the meaning clear we will also denote this space by $(E, <)$ or E.

Denote by k, k' the Kuratowski closure functions respectively for $\mathcal{T}$, $\mathcal{T}'$. Express composition of functions by juxtaposition; thus ck will denote c(kA) for all subsets A of E. Take $i = ckc$, $i' = ck'c$; then i and i' are the interior functions for k and k'. We will also write (E, k, k') for the bitopological space $(E, \mathcal{T}, \mathcal{T}')$. When A contains only a single point x we will write kx for kA and k'x for k'A.

Theorem 1. Let A be a subset of E. Then

$iA = \{x : cA < cx\}$ and $i'A = \{x : x < A\}$.

Proof. Let $B = \{x : cA < cx\}$. Then $iA \subset B \subset A$ and so $iA = B$ if $iB = B$. Let $x \in B$. Then $cA < cx$ and so there is $C \subset B$ such that $cA < C < cx$. Now $y \in cC$ implies $C \subset cy$ and so $cA < cy$; hence $y \in B$ which implies $cC \subset B$. Therefore $cB \subset C$ whence $cB < cx$ and so $iB = B$. The other part of the theorem can be proved similarly.

Corollary. $kA = \{x : A \not< cx\}$ and $k'A = \{x : x \not< cA\}$.

Theorem 2. $A < B$ iff $kA < i'B$.

Proof. Let $A < B$. Then $kA \subset B$ since $x \in kA$ implies $x \in B$ for $x \in cB$ implies $B \subset cx$ and so $A < cx$ which is a contradiction. Also $x \in A$ implies $x < B$ and so $x \in i'B$ from which it follows $A \subset i'B$.

Now $A < B$ implies there is $C \subset B$ such that $A < C < B$. Hence $kA \subset C < B$ and so $kA < B$. Also $A < C \subset i'B$ and so $A < i'B$. Thus $A < B$ implies $kA < B$ which in turn implies $kA < i'B$.

The converse is obvious.

Each topogenous structure $<$ gives rise to a topogenous structure $<'$ defined by $A <' B$ iff $cB < cA$. It is easy to see that the left and right topologies of $<'$ coincide respectively with the right and left topologies of $<$. Hence

$<'$ generates no new topologies.

Definition 3. Let $(E, \mathfrak{T}, \mathfrak{T}')$, $(N, \mathfrak{N}, \mathfrak{N}')$ be two bitopological spaces and f a function from E to N. Then f is said to be continuous iff f is $\mathfrak{T}-\mathfrak{N}$ continuous and $\mathfrak{T}'-\mathfrak{N}'$ continuous.

Definition 4. Let $(E, \mathfrak{T}, \mathfrak{T}')$ be a bitopological space and F a subset of E. Denote by $\mathfrak{N}, \mathfrak{N}'$ the relativizations respectively of $\mathfrak{T}, \mathfrak{T}'$ to N. Then $(N, \mathfrak{N}, \mathfrak{N}')$ is said to be a subspace of $(E, \mathfrak{T}, \mathfrak{T}')$.

Let R be the set of all real numbers. Define a quasimetric m for R as follows: for all real x,y,

$$m'(x,y) = \begin{cases} y - x, & x \leq' y \\ 0, & y <' x \end{cases}$$

where $<'$ denotes the usual order for the reals.
For subsets A,B of R write $M(A,B) = \inf\{m'(x,y) : x \in A, y \in B\}$. Define $<^*$ for R by $A <^* B$ iff $M(A, cB) > 0$. Then $<^*$ is a topogenous structure on R.

Definition 5. We will call $<^*$ the usual topogenous structure on R and the bitopological space of $<^*$ the usual bitopological space for R. If A is a subset of R then the subspace for A is said to be the usual bitopological space for A.

Denote by (R, r, r') the usual bitopological space for R. Let I denote the closed unit interval $[0,1]$ of the reals. We will also denote by I the usual bitopological space for I.

Definition 6. A bitopological space $(E, \mathcal{T}, \mathcal{T}')$ is said to be completely regular iff

(i) A is $\mathcal{T}$-closed and y in cA imply there is a continuous function f from E to I such that $fA = 0$ and $f(y) = 1$ and

(ii) B is $\mathcal{T}'$-closed and $x \in cB$ imply there is a continuous function g from E to I such that $g(x) = 0$ and $gB = 1$.

For x,y in R let $xR = \{ y : x <' y \}$ and $Rx = \{ y : y <' x \}$ where $<'$ is the usual order relation for the reals. Then the set of all xR for x in R is a base for the left topology of R and the set of all Rx for x in R is a base for the right topology of R.

$<'$ will denote the usual order relation for the reals in Lemmas 1, 2 and Theorem 3. Lemma 1 is well known.

Lemma 1. For each t in a dense subset D of the positive reals let S(t) be a subset of E such that

(i) $S(t) \subset S(u)$ if $t <' u$ and

(ii) $\bigcup \{ S(t) : t \in D \} = E$.

For x in E take $f(x) = \inf \{ t : x \in S(t) \}$. Then

$\{x : f(x) <' u\} = \bigcup \{ S(t) : t \in D \text{ and } t <' u\}$ and $\{x : f(x) \leq' u\} = \bigcap \{ S(t) : t \in D \text{ and } u <' t\}$ for every real u.

Lemma 2. Let (E, k, k') be a bitopological space. For each t in a dense subset D of the positive reals let S(t) be a subset of E such that

(i) $i'S(t) = S(t)$

(ii) $kS(t) \subset S(u)$ if $t <' u$ and

(iii) $\bigcup \{ S(t) : t \in D \} = E$.

Then the function f from E to R defined by $f(x) = \inf \{ t : x \in S(t)\}$ is continuous.

Proof. For a real u the set $f^{-1} Ru = \{ x : f(x) <' u\}$ is the union of i'-open sets and so is i'-open. Hence f is $k' - r'$ continuous.

Next, for a real u, the set $f^{-1} uR = \{ x : u <' f(x)\}$ and so $cf^{-1} uR = \{ x : f(x) \leq' u \} = \bigcap \{ S(t) : t \in D, u <' t\} = A$, say. Now $A \subset \bigcap \{ kS(t) : t \in D, u <' t\}$. Also $\bigcap \{ kS(t) : t \in D, u <' t\} \subset A$ since $t \in D$, $u <' t$ imply there is v in D such that $u <' v <' t$ and so $kS(v) \subset S(t)$. Hence A is the intersection of k-closed sets and so is k-closed. Therefore $icA = cA$ and this implies f is k-r continuous.

Theorem 3. Let $(E, <, k, k')$ be a bitopological space and let $A < B$. There is then a continuous function f from

E to I such that $fA = 0$ and $fcB = 1$.

Proof. Let D be the set of all numbers of the form $p2^{-q}$ where p and q are positive integers. Take $S(t) = E$ for t in D and $1 <' t$, take $S(1) = B$ and take $S(0)$ to be an i'-open set such that $A < S(0) < B$. For t in D and $0 <' t <' 1$ take t in the form $t = (2m+1)2^{-n}$ and choose, inductively on n, $S(t)$ to be an i'-open set such that $S(2m2^{-n}) < S(t) < S((2m+2)2^{-n})$. Such choice is possible since $<$ is a topogenous structure. Take $f(x) = \inf\{t : x \in S(t)\}$. Then f is continuous. Also $fA = 0$ and $fcB = 1$.

Corollary. $A < B$ implies there is a continuous function f from E to I such that $fkA = 0$ and $fk'cB = 1$.

Corollary. The bitopological space $(E, <, k, k)$ is completely regular.

Let S be a syntopogenous structure for E. Define $\mathcal{T}$ to be the family of all subsets T of E such that x in T implies $cT <' cx$ for some $<'$ in S. Then $\mathcal{T}$ is a topology for E. Similarly the family $\mathcal{T}'$, of all subsets T of E such that x in T implies $x <' T$ for some $<'$ in S, is also a topology for E.

Definition 7. We will say $(E, S, \mathcal{T}, \mathcal{T}')$ is the bitopological space of S, $\mathcal{T}$ is the left topology of S and $\mathcal{T}'$ is the right topology of S.

Given a syntopogenous structure S on E define a binary relation $<$ by A $<$ B iff A $<'$ B for some $<'$ in S. Then $<$ is a topogenous structure on E and the left and right topologies of $<$ coincide respectively with the left and right topologies of S. Hence a syntopogenous space (E,S) is completely regular. Also A $<'$ B for some $<'$ in S implies there is a continuous function f from E to I such that fA = 0 and fcB = 1.

Definition 8. A bitopological space (E, $\mathcal{T}$, $\mathcal{T}'$) is said to be syntopogenizable (or topogenizable) iff there is a syntopogenous structure (or topogenous structure) on E whose bitopological space is (E, $\mathcal{T}$, $\mathcal{T}'$).

Thampuran (3) has proved that a completely regular bitopological space is quasiuniformizable. From a quasiuniformity $\mathcal{U}$ we can get a syntopogenous structure S --- in the same way as a symmetric syntopogenous structure can be obtained from a uniformity --- such that $\mathcal{U}$ and S have the same bitopological space.

It is clear that a bitopological space is topogenizable iff it is syntopogenizable. We now have the result:

Theorem 4. A bitopological space is completely regular iff it is topogenizable.

It is obvious that a subspace of a completely regular space is completely regular. Thampuran (2) has proved that a product of completely regular spaces is completely regular.

Definition 9. A bitopological space (E,k,k') is said to be regular iff

(i) $A = kA$ and $y \in cA$ imply there are sets $X = iX$, $X' = i'X'$ such that $A \subset X'$ and $y \in X$ and $X \cap X' = \emptyset$ and

(ii) $B = k'B$ and $x \in cB$ imply there are sets $Y = iY$, $Y' = i'Y'$ such that $x \in Y'$ and $B \subset Y$ and $Y \cap Y' = \emptyset$.

A completely regular space is evidently regular; hence a syntopogenous space is regular. It is clear that a subspace of a regular space is regular. A product of regular spaces has been shown to be regular by Thampuran (2).

Theorem 5. Let $<$ be a perfect topogenous structure on E. Then

(i) $A \not< B$ iff $A \cap k'cB \neq \emptyset$ and

(ii) $kA = \bigcup \{ kx : x \in A \}$.

Proof.

(i) $A \not< B$ implies there is x in A such that $x \not< B$ and so x is in $k'cB$. Conversely, if there is x in A such that x is in $k'cB$ then $x \not< B$ and hence $A \not< B$.

(ii) $x \in kA$ iff $A \not< cx$ and this holds iff there is y in A such that $y \not< cx$ or $x \in ky$.

We also have the following result for a perfect topogenous structure $<$ on E. If $<'$ is also a topogenous structure on E such that both $<$ and $<'$ have the same bitopological space (E,k,k') then $<$ is finer than $<'$, $A < cx$ implies $A <' cx$ and $x < cA$ implies $x <' cA$. But if $<'$ is also perfect then $< = <'$. These follow easily from Theorem 5 and from Corollary to Theorem 1.

References

1. Á. Császár, Foundations of general topology, New York 1963
2. D. V. Thampuran, Bitopological spaces and complete regularity (to appear).
3. D. V. Thampuran, Bitopological spaces and quasiuniformities (to appear).