General Topology and Convergence

By

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Kelley (1) has characterized topological spaces by convergence of nets. Nets are defined in terms of a directed set. It is shown in this paper that this property of direction is not needed in the characterization of topological spaces by convergence. The only place where the property of nets depending on direction is used is in the proof of the additivity of the Kuratowski closure function but this can be proved by using the directed property of the neighborhood system of a point.

When a topological concept such as Hausdorffness or compactness imposes extra conditions on the space then the medium used for convergence has to have more structure too; for Hausdorff spaces, for instance, pair-wise intersection property or its equivalent is needed as additional condition.

Neighborhoods, open sets, closed sets, closures etc., can be characterized without any extra conditions; the same is true of continuity.

Unless otherwise specified, the terms used in this paper have the same meaning as in Kelley (2).

Let M be a set. For a subset B of M let cB denote the complement of B with respect to M. Let h be a Kuratowski closure function for M; denote by (M,h) the topological space determined by h. Composition of functions will be denoted by juxtaposition; thus chB will mean c(hB). It is obvious that a subset B of M is a neighborhood of a point x iff x is in chcB.

Definition 1. A nonempty family b of nonempty subsets of M is said to be a generalized filterbase. We will say b is in a subset B of M iff every member of b is a subset of B. A generalized filterbase b is said to be finer than a generalized filterbase d iff every member of d contains a member of b. A generalized filterbase b is said to be eventually in a set B iff some member of b is a subset of B. A generalized filterbase b is said to converge to a point x iff b is eventually in each neighborhood of x.

Theorem 1. $x \in hB$ iff some generalized filterbase in B converges to x.

It is obvious that if a generalized filterbase b converges to x and d is a generalized filterbase finer than b then d also converges to x.

Theorem 2. Let the generalized filterbase b in $A \cup B$ converge to the point x. There is then a generalized filterbase d such that d is either in A or in B and d converges to x.

Proof. Every neighborhood of x intersects $A \cup B$ and so A intersects every neighborhood of x or B intersects every neighborhood of x.

Theorem 3. Let d be a generalized filterbase and x a point in M such that d does not converge to x. There is then a subset X of M such that there is in X a generalized filterbase e finer than d and that no generalized filterbase in X converges to x.

Proof. Since d does not converge to x there is a neighborhood B of x such that cB intersects every member of d. Take X = cB. Then the family of all intersections of members of d with cB is a generalized filterbase e, in X, which is finer than d. It is clear that no generalized filterbase in X converges to x.

Theorem 4. Let B be a subset of M such that no generalized filterbase in B converges to a point x in M. There then exist two sets X, Y whose union is M such that d is a generalized filterbase in X implies d does not converge to x and e is a generalized filterbase in B, y is a point in Y imply e does not converge to y.

Proof. Obviously x is in chB. Take X = hB, Y = chB.

Definition 2. R is said to be a convergence relation iff R is a subset of the Cartesian product of the set of all generalized filters in M and the set M such that

(i) if the generalized filterbase b consists of the one-poin set $\{x\}$ alone then $(b,x) \in R$

(ii) if the generalized filterbase b is finer than d and $(d,x) \in R$ then $(b,x) \in R$

(iii) if $(d,x) \in R$ for a generalized filterbase d in $A \cup B$ then there is a generalized filterbase b such that b is in A or b is in B and $(b,x) \in R$

(iv) if $(d,x) \notin R$ for a generalized filterbase d then there is a subset X of M such that there is in X a generalized filterbase finer than d and $(b,x) \notin R$ for all generalized filterbases b in X.

(v) if b is a generalized filterbase in a subset B of M implies $(b,x) \notin R$ then there exist two subsets X, Y of M such that $X \cup Y = M$, $(d,x) \notin R$ for all generalized filterbases d in X and $(e,y) \notin R$ for all generalized filterbases e in B and for all points y in Y.

It has already been shown that convergence of generalized filterbases in (M,h) satisfies all these conditions.

Theorem 5. Let R be a convergence relation. Define the set-valued set-function h as follows: for a subset B, of M, we will denote by hB the set of all x such that $(b,x) \in R$ for some generalized filterbase b in B. Then (M,h) is a topological space and $(b,x) \in R$ iff b converges to x in the topological space (M,h).

Proof. It is clear that $h\emptyset = \emptyset$, $A \subset hA$ and $hA \subset hB$ for $A \subset B \subset M$. It then follows from condition (iii) of Definition that $h(A \cup B) = hA \cup hB$. We will now prove h is idempotent. Let A be a subset of M and let $x \in chA$. Then b is a generalized filterbase in A implies $(b,x) \notin R$. There are then sets X, Y such that $X \cup Y = M$, $(d,x) \notin R$ for all generalized filterbases d in X and $(e,y) \notin R$ for all generalized filterbases e in A and for all points y in Y. Hence $Y \subset chA$, $x \in chA$ and $cX \subset Y \subset chA$. Therefore $hA \subset X$ and so $hhA \subset hX$ since h is isotone. It then follows $x \in chX \subset chhA$ from which we get $chA \subset chhA$. This implies $hhA \subset hA$ and so $hh = h$ since h is isotone. Therefore (M,h) is a topological space.

Next, let the generalized filterbase d converge to x in (M,h) but let $(d,x) \notin R$. There is then a subset X of M such that there is in X a generalized filterbase e finer than d and $(b,x) \notin R$ for all generalized filterbases b in X. Then e converges to x and so $x \in hX$. But $x \notin hX$ since $(b,x) \notin R$ for all generalized filterbases b in X.

Finally let $(d,x) \in R$ for a generalized filterbase d but let d not converge to x in (M,h). It then follows from Theorem 3 that there is a subset X of M such that there is in X a generalized filterbase e finer than d and that no generalized filterbase in X converges to x. By this last condition $x \notin hX$. But $(e,x) \in R$ since e is finer than d and hence $x \in hX$.

Theorem 6. x is an accumulation point of a subset A of M iff there is a generalized filterbase in $A - \{x\}$ converging to x.

Proof. If x is an accumulation point of A then every neighborhood of x intersects $A - \{x\}$. If a generalized filterbase in $A - \{x\}$ converges to x then every neighborhood of x intersects $A - \{x\}$.

The next result follows from Theorem 1.

Theorem 7. A set B is closed iff no generalized filterbase in B converges to a point of cB.

Theorem 8. A set B is a neighborhood of a point x iff every generalized filterbase converging to x is eventually in B.

Proof. The necessity part is evident. Let B be not a neighborhood of x. Then every neighborhood of x intersects cB and the family of all these intersections is a generalized filterbase b, in cB, converging to x. But b is not eventually in B and this proves sufficiency.

Corollary. A set B is open iff every generalized filterbase converging to a point of B is eventually in B.

It is easily seen that a topological space is T_1 iff for each point x the generalized filterbase consisting of the single member $\{x\}$ converges only to x.

Definition 3. A generalized filterbase b is said to have the pair-wise intersection property iff every pair of members of b has a nonempty intersection.

Theorem 9. A topological space is Hausdorff iff every generalized filterbase with the pair-wise intersection property converges to no more than one point.

Proof. Necessity is obvious. To prove sufficiency let the space be not Hausdorff. Then there are two distinct points x,y such that every neighborhood of x intersects every neighborho of y and the family of all these intersections is a generalized filterbase, with the pair-wise intersection property, converging to both x and y.

Definition 4. A point x is said to be a cluster point of a generalized filterbase b iff b is not eventually in the complement of any neighborhood of x.

The following results are then easy to prove:

(i) if x is a cluster point of a generalized filterbase d finer than b then x is a cluster point of b

(ii) if a generalized filterbase b has the pair-wise intersection property and converges to x then x is a cluster point of b.

(iii) if a generalized filterbase d with the pair-wise intersection property converges to x and is finer than b then x is a cluster point of b

(iv) if x is a cluster point of b then there is a generalized filterbase d converging to x and finer than b

(v) if B is a closed subset of M then every cluster point, of each generalized filterbase in B, is a point of B

(vi) if every cluster point of each generalized filterbase, in B, with the pair-wise intersection property belongs to B then B is closed

(vii) a point x is a cluster point of a generalized filterbase b iff x is in the closure of each member of b.

For a first countable space the usual results can be proved in terms of generalized filterbases.

Regarding continuity we have the following result. Let f be a function mapping the topological space M into the topological space M'. For a generalized filterbase b in M let fb denote the family of all images of members of b under f. Thus fb is a generalized filterbase in M'. Then f is continuous iff b is a generalized filterbase in M converging to x implies the generalized filterbase fb converges to f(x). The proof follows the usual lines.

Next consider compactness. It is easy to see that a space is compact iff each generalized filterbase b, such that the family of all closures of members of b has the finite intersection property, has a cluster point. So we do not get anything essentially new in this case.

It has thus been shown that a topological space as well as open sets, closed sets, closures etc., can be characterized by convergence of generalized filterbases which have little structure. It is obvious how to generalize nets to achieve the same purpose.

References

1. Kelley, J. L., Convergence in Topology, Duke Math. J. 17 (1950) 277 - 283

2. Kelley, J. L., General Topology, Princeton (1968)