Regular Spaces and Relations

By

D. V. Thampuran

The concept of uniform continuity of pseudometric spaces has been generalized in two ways both of which preserve symmetry: the method of Weil (5) through uniform spaces and the method of Efremovich (1,2) and Smirnov (4) through proximity spaces. Both of theses methods yield completely regular spaces. The method involving proximity spaces can be generalized still further, with preservation of symmetry, and this will yield regular topological spaces. This can be accomplished through the use of a symmetric binary relation, having all the properties of a proximity except one which is modified, whose topology is regular; every regular topology is the topology of such a relation. Products of such binary relations are also considered in this paper.

Regarding proximity spaces, the terminology used in this paper is that of Mamuzić (3).

Let E be a set. For a subset A of E we will write $cA = E - A$. If $b \subset 2^E \times 2^E$ and $(A,B) \notin b$ we will also write $(A,B) \in cb$; if $A = \{x\}$ we will write (x,B) for (A,B).

Let $r \subset 2^E \times 2^E$ be such that for all subsets A,B,C,D of E and all x in E

1. $(A,B) \in r$ implies $(B,A) \in r$
2. $(E,\emptyset) \in cr$
3. $(A,B) \in r$, $A \subset C$, $B \subset D$ imply $(C,D) \in r$
4. $(A \cup B,C) \in r$ implies $(A,C) \in r$ or $(B,C) \in r$
5. $A \cap B \neq \emptyset$ implies $(A,B) \in r$
6. $(x,A) \in cr$ implies there is $S \subset E$ such that $(x,S) \in cr$ and $(cS,A) \in cr$.

Definition 1. A binary relation r defined as above is said to be a regular relation for E. The ordered pair (E,r) is said to be a regular space.

We will usually write E for (E,r) if the context makes the meaning clear.

Let $\mathcal{T}$ be the family of all subsets T of E such that x in T implies $(x,cT) \in cr$. It is obvious that $\mathcal{T}$ is a topology for E.

Definition 2. Let (E,r) be a regular space. The topology $\mathcal{T}$ obtained as above is called the topology of r or of (E,r) and $(E,r,\mathcal{T})$ the topological space of r.

When (E,r) is considered as a topological space we will write $(E,r,\mathcal{T})$. Let k be the Kuratowski closure function of the topology $\mathcal{T}$. We will denote $(E,r,\mathcal{T})$ also by (E,r,k) or by E alone if the meaning is clear.

Let k' denote the interior function of k. Then $k' = ckc$.
If $A = \{x\}$ we will write kx for kA.

Theorem 1. $k'A = \{x : (x, cA) \in cr\}$ where $A \subset E$.

Proof. Let $B = \{x : (x, cA) \in cr\}$. Now $k'A \subset B \subset A$. Hence if $k'B = B$ then $k'A = B$. Let $x \in B$. Then $(x, cA) \in cr$ and so $(x, C) \in cr$, $(cC, cA) \in cr$ for some subset C of E. Then $y \in cC$ implies $(y, cA) \in cr$ and so $y \in B$ from which it follows $cC \subset B$. Hence $cB \subset C$ and so $(x, cB) \in cr$ which implies $k'B = B$.

Corollary. $kA = \{x : (x, A) \in r\}$.

It is easy to see that $(A, B) \in cr$ implies $A \cap kB = \emptyset$ and $kA \cap B = \emptyset$. Also $(x, A) \in cr$ implies $(kx, kA) \in cr$ for from $(x, A) \in cr$ we get $(x, C) \in cr$, $(cC, A) \in cr$ for some subset C of E and then $(x, D) \in cr$, $(cD, C) \in cr$ and $(cC, A) \in cr$ for some subset D of E; hence $kx \subset cD$, $kA \subset C$ and so $(kx, kA) \in cr$.

Definition 3. A topology $\mathcal{T}$ for E is said to be regular iff A is a closed subset of E and x is a point, of E, not in A imply x and A have disjoint neighborhoods. A topology $\mathcal{T}$ for E is said to be completely regular iff A is a closed subset of E and x is a point not in A imply there is a continuous function f from E to the closed unit interval $[0,1]$ such that $f(x) = 0$ and $fA = 1$.

Theorem 2. The topology of a regular relation is regular.

Proof. Let r be a regular relation for E and (E,r,k) the topological space of r. Let A = kA and x ε cA. Then (x,A) ε cr and so (x,B) ε cr, (cB,A) ε cr for some subset B of E. Hence there is a subset C of E such that (x,C) ε cr and (cC,B) ε cr. Evidently x ε ckC and A $\subset$ k'B. Now ckC and k'B are disjoint open sets since they are subsets respectively of cC and B.

Definition 4. Let (E,r,k) be a regular space. A set B is said to be a r-neighborhood of a set A iff (A,cB) ε cr.

A r-neighborhood is obviously a neighborhood.

If (x,A) ε cr then x and A have disjoint r-neighborhoods, because (x,A) ε cr implies there are subsets B,C of E such that (x,C) ε cr, (cC,B) ε cr, (cB,A) ε cr and then cC and B are disjoint r-neighborhoods respectively of x and A.

Theorem 3. If A is a r-neighborhood of x then there is an open r-neighborhood B of x and a closed r-neighborhood C of x such that C $\subset$ B $\subset$ A.

Proof We are given (x,cA) ε cr. Hence (x,kcA) ε cr and so $B = ckcA$ is an open r-neighborhood of x. Now $(x,cB) \in cr$ and so $(x,D) \in cr$, $(cD, cB) \in cr$ for some subset D. Then $C = kcD$ is such that $(x,cC) \in cr$ and so C is a closed r-neighborhood of x. We also know that kcD and cB are disjoint and so kcD is a subset of B. Hence $C \subset B \subset A$.

Let $B \supset\!\!\!\supset A$ denote B is a r-neighborhood of A; if $A = \{x\}$ we will write $B \supset\!\!\!\supset x$. The following properties of r-neighborhoods are easy to prove:

1. $B \supset\!\!\!\supset A$ implies $cA \supset\!\!\!\supset cB$.
2. $B \supset\!\!\!\supset A$ implies $B \supset kA$.
3. $B \supset\!\!\!\supset A \supset C$ or $B \supset A \supset\!\!\!\supset C$ implies $B \supset\!\!\!\supset C$.
4. $B_i \supset\!\!\!\supset A_i$, $i = 1,2,\ldots,n$ imply $\bigcup B_i \supset\!\!\!\supset \bigcup A_i$ and $\bigcap B_i \supset\!\!\!\supset \bigcap A_i$.
5. $B \supset\!\!\!\supset x$ implies $B \supset\!\!\!\supset A \supset\!\!\!\supset x$ for some A.

It is clear that uniform spaces and proximity spaces are regular spaces. A regular space (E,r,k) is Hausdorff iff it is T_1.

Theorem 4. Let A be a subset of a regular space (E,r,k). Then kA is the intersection of all the r-neighborhoods of A.

Proof. Let B be the intersection of all the r-neighborhoods of A; then kA is a subset of B. Suppose $x \in B - kA$. This implies there is a r-neighborhood C of A such that x is not in C. But B is a subset of C and hence x is in C which is a contradiction.

Corollary. The interior of A is the union of all the sets for which A is a r-neighborhood.

Definition 5. A subset S of a topological space E is said to be compact iff every open cover of S has a finite subcover. A subspace S of E is compact iff S as a subset of E is compact.

Theorem 5. Let A, B be subsets of (E,r,k) such that A is compact and A and kB are disjoint. Then $(A,kB) \in cr$.

Proof. x in A implies $(x,kB) \in cr$ and so there is C such that $(x,C), (cC, kB) \in cr$. Then ckC is an open set containing x; the family of all such ckC for x in A is an open cover of A and so has a finite subcover $D_1, \ldots, D_n$, say. Let $D = D_1 \cup \ldots \cup D_n$. Now $(D_i, kB) \in cr$ for each $i = 1, \ldots, n$ and so $(D,kB) \in cr$. Hence $(A,kB) \in cr$.

Corollary. A, B are disjoint closed subsets of a compact (E,r,k) imply $(A,B) \in cr$.

Theorem 6. Let $\mathcal{T}$ be a regular topology for E. Then $\mathcal{T}$ is the topology of a regular relation r for E.

Proof. Let k be the Kuratowski closure function of $\mathcal{T}$. For subsets A, B of E write $(A,B) \in r$ iff kA, kB are not disjoint.

Definition 6. Let F be a subset of a regular space (E,r). For subsets A,B of F write (A,B) ε s iff (A,B) ε r. Then (F,s) is called a subspace of (E,r).

Definition 7. A regular space (E,r,k) is called a r-extension of a regular space (F,s,h) iff F is dense in E and (F,s) is a subspace of (E,r).

A subspace (F,s) of a regular space (E,r) is obviously regular; also the topology of s is the relativization of the topology of r to F. A regular space (F,s,h) does not always seem to be dense in a compact regular space (E,r,k) for then the topology of h would be completely regular, being the relativization of the completely regular topology of k and so all regular spaces do not have compact r-extensions.

Definition 8. Let (E,r), (F,s) be two regular spaces and f a function from E to F. Then f is said to be a r-mapping or r-function iff A,B $\subset$ E and (A,B) ε r imply (fA, fB) ε s. A r-function f is called a r-homeomorphism iff f is one to one and both f and its inverse are r-functions. Two regular spaces are said to be r-homeomorphic iff there is a r-homeomorphism between them.

Lemma. Let f be a function from a regular space

(E,r,k) to a regular space (F,s,h). Then f is continuous iff $(x,A) \in r$ imply $(f(x), fA) \in s$ for x in E and $A \subset E$.

Theorem 7. Every r-function from (E,r,k) to (F,s,h) is continuous.

The converse of Theorem 7 is obviously not valid. It is easy to see that a function f from (E,r) to (F,s) is a r-function iff $A,B \subset F$, and $(A,B) \in cs$ imply $(f^{-1}A, f^{-1}B) \in cr$. It is also clear that if f is a function from a proximity space (E,δ) to a proximity space (E', δ') then f is a δ-function iff f is a r-function; this shows that r-functions are generalizations of δ-functions and that regular spaces have symmetry and form a class of spaces, more general than proximity spaces, which admit of such generalizations.

Let (E,k), (F,h) be regular topological spaces. Define regular relations r,s for E,F as follows: $A,B \subset E$ imply $(A,B) \in r$ iff $kA \cap kB \neq \emptyset$ and $C,D \subset F$ imply $(C,D) \in s$ iff $hC \cap hD \neq \emptyset$. Then a function f from (E,r,k) to (F,s,h) is a r-function iff f is continuous.

Let s be the intersection of a family of regular relations for E. Then s has all the properties of a regular relation except perhaps the fourth condition in the definition of a regular relation. Similarly the union of a family of regular relations for E will have all the properties of a regular relation except perhaps the sixth condition in the definition of a regular relation.

Definition 9. Let A be a set. Then a finite family $A_1, \ldots, A_m$, of subsets of A and whose union is A, is said to be a partition of A.

Definition 10. Let r, s be two regular relations for a set E. Then r is said to be finer than s (or s is coarser than r) iff $(A,B) \varepsilon r$ implies $(A,B) \varepsilon s$.

Theorem 8. Let s be the intersection of a family F of regular relations for a set E. Denote by u the union of all the regular relations finer than each member of F. Define r as follows: if $A,B \subset E$ then $(A,B) \varepsilon r$ iff $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$ are partitions of A and B imply $(A_i, B_j) \varepsilon s$ for some $i = 1, \ldots, m$ and some $j = 1, \ldots, n$. Then $r = u$ and r is the coarsest regular relation finer than each member of R.

Definition 11. Let F be a family of regular relations for E. Then the coarsest regular relation, finer than each member of F, is said to be generated by F.

Definition 12. A topology $\mathcal{T}$ for E is said to be finer than a topology $\mathcal{S}$ for E (or $\mathcal{S}$ is coarser than $\mathcal{T}$) iff $\mathcal{S}$ is a subfamily of $\mathcal{T}$.

Let r be the regular relation generated by a family F of regular relations for a set E. Then the topology of r is the coarsest topology finer than that of each member of F.

Definition 13. A family R of binary relations $b \subset 2^E \times 2^E$ is said to be a regular family for E iff

1. each member of F satisfies the first five conditions in the definition of a regular relation
2. s,t are in F imply there is u in F such that u is finer than s and t
3. s in F and $(x,A) \in cs$ imply there is t in F and a subset C of E such that $(x,C) \in ct$ and $(cC,A) \in ct$.

The ordered pair (E,R) will also be called a regular space.

A regular relation is a regular family containing only one member.

Definition 14. Let R be a regular family for E. Then the family $\mathcal{T}$, of all subsets T of E such that x in T implies $(x, cT) \in cs$ for some s in R,(which is a topology for E) is said to be the topology of R and we will denote the resulting topological space by (E,R, $\mathcal{T}$) or simply by E.

Definition 15. Let R be a regular family for E. Then r, defined by (A,B) is in r iff (A,B) is in each member of R, (which is a regular relation) is called the regular relation generated by R.

The topology of a regular family R coincides with the topology of the regular relation generated by R.

Let F be a family of regular relations for a set E. For each finite subfamily G of F there is a coarsest regular relation g finer than each member of G; let R be the family of all such g for each finite subfamily G of F. Then R is a regular family. Also R and F generate the same regular relation.

Definition 16. Let R, S be two regular families for E. Then R is said to be finer than S (or S is coarser than R) iff for each s in S there is r in R such that $r \subset s$. We will say R and S are equivalent iff each is finer than the other.

Definition 17. Let (E,R), (E',R') be regular spaces and f a function from E to E'. Then f is said to be (R,R')-continuous iff for each r' in R' there is r in R such that (A,B) is in r implies (fA, fB) is in r'.

Let (E,R) (E',R') be regular spaces and f a function from E to E. Denote by r,r' the regular relations generated by R, R'. Then f from (E, r) to (E', r') is a r-function if f is (R, R')-continuous but the converse is not necessarily true.

Let E be a set, (E', R') a regular space and f a function from E to E'. For each r' in R' write $(A, B) \varepsilon r$ for subsets A, B of E iff $(fA, fB) \varepsilon r'$; let R be the family of all such r. Then R is a regular family for E and f is (R, R')-continuous. Also if S is a regular family for E such that f is (S, R')-continuous then S is finer than R. Write $f^{-1} R' = R$.

Let E be a set and R_i a regular family, for E, for each i in an index set I. Denote by S the union of R_i for i in I. For a finite subfamily T of S define $(A, B) \varepsilon t$ iff $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$ are partitions of A and B imply (A_a, B_b) is in each member of T for some $a = 1, \ldots, m$ and some $b = 1, \ldots, n$; let R be the family of all such t for each finite subfamily T of S. Then R is a regular family for E finer than each R_i, $i \varepsilon I$ and R is coarser than each regular family for E which is finer than each R_i; in this sense we can say R is the coarsest regular family finer than each R_i. Obviously R is unique up to equivalence.

Next, let E be a set and f_i a function from E to a regular space (E_i, R_i') for each i in an index set I. Let $R_i = f_i^{-1} R_i'$. Denote by R the coarsest regular family finer than each R_i for i in I. Then R is the coarsest regular family for E such that each f_i is (R,R_i)-continuous.

Definition 18. Let (E_i, R_i) be a family of regular spaces for each i in an index set I. Denote by E the Cartesian product of E_i for i in I. Let R be the coarsest regular family for E such that projection into the i-th coordinate space is (R,R_i)-continuous for each i in I. Then R is said to be the product of the regular families R_i for i in I and (E,R) is said to be the product regular space.

It is clear that a product R is unique up to equivalence. It is easy to see that the topology of the product regular family is the product of the topologies of the regular families R_i for i in I.

Let (E,R), (F,S) and (G,T) be regular spaces. Let f be a (R,S)-continuous function from E to F and g a (S,T)-continuous function from F to G. Then the composition gf is a (R,T)-continuous function from E to G. The next theorem is easy to prove.

Theorem 9. Let f be a function from a regular space (F,S) to a product (E,R) of regular spaces. Then f is (S,R)-continuous iff the composition $p_i f$ is (S,R_i)-continuous for each i in I (using the notation of Definition 18) and this property determines the equivalence class of R.

Regular spaces do possess a property of functional separation similar to that of completely regular spaces and is proved in another of my papers.

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