QUASIUNIFORM SPACES AND FUNCTIONAL SEPARATION

by

D. V. THAMPURAN

It is well known that a uniform space is completely regular and so if A is a closed subset of the space and x is a point, of the space, not in A then there is a continuous function f from the space to the closed unit interval [0,1] such that f maps x into 0 and each point of A into 1; it is also known that a topological space is completely regular iff it is uniformizable. Thampuran (3,4) has shown that a bitopological space is completely regular iff it is quasiuniformizable and that the syntopogenous structures of Császár (1) also characterize completely regular bitopological spaces.

The main purpose of this paper is to show that quasiuniform spaces possess a much stronger functional separation property than that specified by complete regularity. Uniform spaces have a similar property.

Terms not specificlaly defined in this paper have the same meaning as in Kelley (2).

Let M be a set and $\mathcal{U}$ a family of subsets of $M \times M$ such that

(1) each U in $\mathcal{U}$ contains the diagonal

(2) U in $\mathcal{U}$ implies there is V in $\mathcal{U}$ such $VV \subset U$

(3) U,V in $\mathcal{U}$ imply $U \cap V$ is in $\mathcal{U}$

(4) $U \subset V \subset M \times M$ and $U \in \mathcal{U}$ imply $V \in \mathcal{U}$.

Definition 1. $\mathcal{U}$ is said to be a quasiuniformity for M and $(M,\mathcal{U})$ is said to be a quasiuniform space.

Let $\mathcal{T}$ be the family of all subsets T of M such that $x \in T$ implies $\{y: (y,x) \in U\} \subset T$ for some U in $\mathcal{U}$; then $\mathcal{T}$ is a topology for M. Let $\mathcal{T}'$ be the family of all subsets T of M such that x in T implies $\{y: (x,y) \in U\} \subset T$ for some U in $\mathcal{U}$; then $\mathcal{T}'$ is also a topology for M.

Definition 2. $(M,\mathcal{T},\mathcal{T}')$ is said to be the bitopological space of $\mathcal{U}$ and $\mathcal{T},\mathcal{T}'$ are said to be the left and right topologies of $\mathcal{U}$.

Definiton 3. A function d from $M \times M$ to the non-negative reals is said to be a quasimetric for M iff for all x,y,z in M

$$(1) \quad d(x,x) = 0 \text{ and}$$

$$(2) \quad d(x,y) \leq d(x,z) + d(z,y).$$

(M,d) is said to be a quasimetric space. The family $\mathcal{U}$ of all subsets U of $M \times M$ such that $\{(x,y) : d(x,y) < r\} \subset U$ for some $r > 0$ is said to be the quasiuniformity of d. The left and right topologies and the bitopological space of $\mathcal{U}$ are said to be the left and right topologies and the bitopological space of d.

Let m be the quasimetric, for the reals R, defined by $m(x,y) = \max\{y-x, 0\}$.

Definition 4. For a subset A of R, the quasimetric m restricted to A is called the usual quasimetric for A and the bitopological space of the usual quasimetric for A is called the usual bitopological space for A.

Let K denote the closed unit interval $[0,1]$ of the reals. We will also denote by K the usual quasimetric space for K and the usual bitopological space for K.

Definition 5. Let $(M,\mathcal{T},\mathcal{T}')$, $(N,\mathcal{n},\mathcal{n}')$ be two bitopological spaces and f a function from M to N. Then f is said to be continuous iff f is $\mathcal{T}$-$\mathcal{n}$ continuous and $\mathcal{T}'$-$\mathcal{n}'$ continuous.

Definition 6. Let $(M,\mathcal{T},\mathcal{T}')$ be a bitopological space and K the usual bitopological space for the closed unit interval. Then M is said to be completely regular iff

(1) A is a $\mathcal{T}$-closed subset of M and $y \notin A$ imply there is a continuous function f from M to K such that $fA = 0$ and $f(y) = 1$ and

(2) B is a $\mathcal{T}'$-closed subset of M and $x \notin B$ imply there is a continuous function g from M to K such that $g(x) = 0$ and $gB = 1$.

Thampuran (3,4,5) has proved that a bitoplogical space is completely regular iff it is the bitopological space of a quasiuniformity or a syntopogenous structure or a quasiproximity.

The object of this paper is to prove a stronger functional separation, for a quasiuniform space, from which complete regularity will follow.

Definition 7. Let $(M, \mathcal{U})$, $(N, \mathcal{V})$ be two quasiuniform spaces and f a function from M to N. We will say f is uniformly continuous relative to $\mathcal{U}$, $\mathcal{V}$ iff V in $\mathcal{V}$ implies there is U in $\mathcal{U}$ such that $(f(x), f(y)) \in V$ for all $(x, y) \in U$. If f is a function from a quasimetric space (M, d) to a quasimetric space (N, e) then f is said to be uniformly continuous iff f is uniformly continuous relative to the quasiuniformities of d and e.

Let (M, d) be a quasimetric space and A a subset of M. For x in M write $D(A, x) = \inf \{d(y, x): y \in A\}$ and $D(x, A) = \inf \{d(x, y): y \in A\}$.

Theorem 1. Let (M, d) be a quasimetric space, $\mathcal{U}$ the quasiuniformity of d and $\mathcal{U}'$ the family of all inverses of members of $\mathcal{U}$. Denote by $\mathcal{V}$ the quasiuniformity of the usual quasimetric for the reals R. Let A,B be subsets of M. Take $f(x) = D(A, x)$ and $g(x) = D(x, B)$ for x in M. Then

(1) f is uniformly continuous relative to $\mathcal{U}$, $\mathcal{V}$ and

(2) g is uniformly continuous relative to $\mathcal{U}'$, $\mathcal{V}$.

Proof.

(1) The family of all sets of the form $V(r) = \{(u, v): v - u < r, u, v \in R\}$, $r > 0$ is a base for $\mathcal{V}$. For $r > 0$ let $U(r) = \{(x,y): d(x,y) < r, x, y \in M\}$. Now $d(z, y) \leq d(z, x) + d(x, y)$ for all x, y, z in M. Taking infima with respect to $z \in A$ we get $D(A, y) \leq D(A, x) + d(x, y)$. Hence $(x, y) \in U(r)$ implies $f(y) - f(x) \leq d(x, y) < r$ and so $(f(x), f(y)) \in V(r)$. Therefore f is uniformly continuous.

(2) Proof for g is similar.

From this it is easy to prove that a quasimetric space is normal, i.e., $\mathcal{T}$, $\mathcal{T}'$ are the left and right topologies of a quasimetric and A, B are disjoint $\mathcal{T}$-closed and $\mathcal{T}'$-closed subsets imply there are disjoint sets C, D such that $C \in \mathcal{T}$, $D \in \mathcal{T}'$, $A \subset D$ and $B \subset C$. First we observe that a uniformly continuous function is continuous and hence it easily follows that $f - g$ is a continuous function from the quasimetric space to the reals. Also, the $\mathcal{T}$-closure of a subset S of M is $\{x: D(S, x) = 0\}$ and the $\mathcal{T}'$-closure of S is $\{x: D(x, S) = 0\}$. Consider the function h from M to the reals defined by $h(x) = f(x) - g(x)$ where $f(x) = D(A, x)$ and $g(x) = D(x, B)$. Take $C = \{x: h(x) > 0\}$ and $D = \{x: h(x) < 0\}$.

Lemma 1. Let (M, d) be a quasimetric space, f a uniformly continuous function from M to R and $k > 0$. Then kf is uniformly continuous.

Lemma 2. Let d be a quasimetric for M and $k > 0$. Then e defined by $e(x, y) = k\, d(x, y)$ for all x, y in M is a quasimetric for M and the quasi-uniformities of d and e are the same.

Proof. The result follows since $\{(x,y): e(x,y) < kr\} = \{(x,y): d(x,y) < r\}$ for each $r > 0$.

Lemma 3. Let d be a quasimetric for M. Take $e(x, y) = \min\{1, d(x,y)\}$ for all x, y in M. Then e is a quasimetric for M and the quasiuniformities of d and e coincide.

Proof. Obviously $e(x, x) = 0$ for each x in M. To prove the triangle inequality for e it is enough to prove that if a, b, c are nonnegative numbers such that $a \leq b + c$ then $\min\{1, a\} \leq \min\{1, b\} + \min\{1, c\}$. If either b or c is ≥ 1 then the result is immediate since $\min\{1, a\}$ is ≤ 1. If both b and c are less than 1 then $\min\{1, a\} \leq a \leq b + c$ completes the proof of the triangle inequality. Hence e is a quasimetric for M. Now the family of all sets of the form $U(r) = \{(x, y): d(x, y) < r < 1\}$ for $r > 0$ is a base for the quasiuniformity of d and the family of all sets of the form $V(r)=\{(x,y): e(x, y) < r < 1\}$ for $r > 0$ is a base for the quasiuniformity of e. It is obvious that $(x, y) \in U(r)$ implies $e(x, y) < r$ and so $(x, y) \in V(r)$; hence $U(r) \subset V(r)$. Also, if $(x, y) \in V(r)$ then $d(x, y) < r$ as otherwise we get a contradiction and so $V(r) \subset U(r)$.

By combining Lemmas 2 and 3 we get the following result.

Lemma 4. Let (M, d) be a quasimetric space and $k > 0$. Define e by $e(x, y) = \min\{1, d(x, y)/k\}$ for all x, y in M. Then e is a quasimetric for M and the quasiuniformities of d and e are identical.

Let (M, d) be a quasimetric space and A, B subsets of M. Write $D(A, B) = \inf\{d(x, y): x \in A, y \in B\}$.

Theorem 2. Let (M, d) be a quasimetric space and A, B subsets of M such that $D(A, B) > 0$. Then there is a uniformly continuous function f from M to K such that $fA = 0$ and $fB = 1$.

Proof. Let $D(A, B) = k$. Then $k > 0$. Take $e(x, y) = \min\{1, d(x,y)/k\}$ for all x, y in M. Define $E(A, u) = \inf\{e(x, u): x \in A\}$ for all u in M. Then $E(A, u) = 0$ for all u in A. If $x \in A$ and $y \in B$ then $d(x, y) \geq k$ and so $d(x, y)/k \geq 1$; hence $e(x, y) = 1$. Therefore $E(A, u) = 1$ for all u in B. For u in M take $f(u) = E(A, u)$. Then f is a uniformly continuous function from

M to K such that $fA = 0$ and $fB = 1$.

Let $U \subset M \times M$ and $A \subset M$. Then we will write $AU = \{y:(x,y) \in U$ for some $x \in A\}$.

Theorem 3. Let $(M, \mathcal{U})$ be a quasiuniform space and A, B subsets of M such that $AU \cap B = \emptyset$ for some $U \in \mathcal{U}$. There is then a uniformly continuous function f from M to K such that $fA = 0$ and $fB = 1$.

Proof. Thampuran (3) has proved that $\mathcal{U}$ has a gage G. There is then a quasimetric d in G such that $D(A, B) > 0$. Hence there is a uniformly continuous function f from (M, d) to K such that $fA = 0$ and $fB = 1$. Since d is in the gage of $\mathcal{U}$ this implies f is uniformly continuous when f is considered as a function from $(M, \mathcal{U})$ to K.

Corollary. The bitopological space of a quasiuniform space is completely regular.

Proof. A uniformly continuous function is continuous. Let $(M, \mathcal{T}, \mathcal{T}')$ be the bitopological space of $\mathcal{U}$. Then A is $\mathcal{T}$-closed and $y \notin A$ imply $AU \cap \{y\} = \emptyset$ for some $U \in \mathcal{U}$. Also B is $\mathcal{T}'$-closed and $x \notin B$ imply $\{x\}V \cap B = \emptyset$ for some $V \in \mathcal{U}$.

This corollary thus proves what had independently been proved before.

The procedure used in this paper can also be employed to prove the analagous result for uniform spaces. If d is a pseudometric for M then $D(A, x) = f(x)$ is a uniformly continuous function — first we prove this. Next, we prove that if $(M, \mathcal{U})$ is a uniform space and A, B subsets of M such that $AU \cap B = \emptyset$ for some U in $\mathcal{U}$ then there is a uniformly continuous function f from M to K such that $fA = 0$ and $fB = 1$. It follows immediately that a uniform space is completely regular.

References

1. Á. Császár, Foundations of General Topology, New York (1963).

2. J. L. Kelley, General Topology, Princeton (1968).

3. D. V. Thampuran, Bitopological Spaces and Quasiuniformities (to appear).

4. ______________, Syntopogenous Structures and Complete Regularity (to appear).

5. ______________, Quasiproximity Spaces (to appear).