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Regular Spaces and Functional Separation

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A topological space M is said to be completely regular iff A is a closed subset and x is a point not in A imply there is a continuous function f from M to the closed unit interval $[0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for all y in A. The main purpose of this paper is to prove that regular spaces can also be characterized by a similar property.

Weil (3) introduced uniform spaces and generalized the concept of uniform continuity for pseudometric spaces; the topologies of uniform spaces are completely regular. Thampuran (2) has shown that regular spaces can be characterized by a structure which has some similarities to a uniformity and hence the concept of uniform continuity can be generalized to this structure.

Unless otherwise specified the terminology of this paper conforms to that of Kelley (1).

Let M be a set and $\mathcal{T}$ a topology for M. Denote by k the Kuratowski closure function of $\mathcal{T}$ and by (M,k) the topological space. Take $cA = M - A$ for

$A \subset M$. Composition of functions will be denoted by juxtaposition; thus ck will represent c(kA) for $A \subset M$. If A consists of a single point x we will write x for A.

Definition 1. Let M be a topological space. Then M is said to be regular iff A is a closed subset and x a point not in A imply x and A have disjoint neighborhoods.

Definition 2. A set-valued set-function n from the power set, of M, to itself is said to be a neighborhood function for M iff for all subsets A,B of M

1. $n\emptyset = \emptyset$
2. $A \subset nA$ and
3. $nA \subset nB$ if $A \subset B$.

The ordered pair (M,n) is said to be a neighborhood space. In a neighborhood space (M,n), a subset A of M is said to be a neighborhood of a point x iff $x \in cncA$.

Let (M,n) be a neighborhood space and A a subset of M. It is easy to show that x is in nA iff every neighborhood of x intersects A.

Definition 3. Let (M,n), (L,p) be two neighborhood spaces and f a function from M to L. We will say f is continuous at a point x of M iff B is a neighborhood of f(x) implies the inverse of B, under f, is a neighborhood of x; f is said to be continuous iff f is continuous at each point M.

It is easy to show that f is continuous iff $fn \subset pf$.

Let R be the reals. Define a neighborhood function r for the reals as follows. u, v will denote real numbers.

$$1. \quad ru = \begin{cases} (1/3,\infty) & \text{if } 1/2 < u \\ (1/4,\infty) & \text{if } 1/3 < u \leq 1/2 \\ (1/(m+2),\ 1/(m-1)] & \text{if } 1/(m+1) < u \leq 1/m,\ m = 3,4,\ldots \\ (-\infty,0] & \text{if } u \leq 0 \end{cases}$$

$$2. \quad rA = \bigcup \{ru : u \in A\} \quad \text{if } A \subset (-\infty,0] \cup (v,\infty) \text{ for some } 0 < v$$

$$3. \quad rA = \bigcup \{ru : u \in A\} \cup r0 \quad \text{if } (0,1/m) \subset A \text{ for some } m = 1,2,3,\ldots$$

It is obvious that a set A is a neighborhood of a point u iff

1. $ru \subset A$ for u in $(0,\infty)$ and
2. $(-\infty,1/m) \subset A$ for u in $(-\infty,0]$, for some m 1,2,3,...

Lemma. Let (M,k) be a topological space, $S(t) = M$ for $t > 1$ and for $t,u = 1/m, m = 1,2,3,\ldots$ let S(t) be an open set such that $kS(t) \subset S(u)$ when $t < u$. Define a function f from M to the neighborhood space (R,r) by $f(x) = \inf\{t : x \in S(t)\}$ for all x in M. Then f is continuous.

Proof. Let $y \in M$. First consider the case where $f(y)$ is in $(1/(m+1), 1/m]$, $m = 3,4,\ldots$. It is enough to show that the inverse under f of $(1/(m+2), 1/(m-1)]$ is a neighborhood of y. Let $A = \{x : f(x) \leq 1/(m-1)\}$. Then x is in A iff x is in $S(1/(m-1))$ and so $A = S(1/(m-1))$. It is clear that y is in A and so A is a neighborhood of y. Next take $B = \{x : f(x) > 1/(m+2)\}$. Then x in $cS(1/(m+2))$ implies $f(x) > 1/(m+2)$, since $f(x) \leq 1/(m+2)$ would mean $x \in S(1/(m+2))$, and so $cS(1/(m+2)) \subset B$. Now $y \in cS(1/(m+1))$ since $y \in S(1/(m+1))$ would mean $f(y) \leq 1/(m+1)$. We also know $S(1/(m+2)) \subset kS(1/(m+2)) \subset S(1/(m+1))$. Hence it follows $y \in cS(1/(m+1)) \subset ckS(1/(m+2)) \subset cS(1/(m+2)) \subset B$. Therefore B is also a neighborhood of y and so $A \cap B$ is a neighborhood of y. This proves the continuity of f at y.

If $f(y) \leq 0$ then $S(1/m) = \{x : f(x) \leq 1/m\}$ is a neighborhood of y for each $m = 1,2,\ldots$. If $f(y)$ is in $(1/3, 1/2]$ then $\{x : f(x) > 1/4\}$ is a neighborhood of y and if $f(y) > 1/2$ then $\{x : f(x) > 1/3\}$ is a neighborhood of y. Hence f is continuous.

Theorem. A topological space (M,k) is regular iff A a closed subset and x a point, of M, not in A imply there is a continuous function f from (M,k) to (R,r) such that $f(x) = 0$ and f is 1 on A.

Proof. Let the space be regular. Take $S(t) = M$ for $t > 1$ and $S(1) = cA$. Since (M,k) is regular we can define by induction open neighborhoods $S(t)$ of x such that

$kS(t) \subset S(u)$ if $t < u$ for all $t, u = 1/m$, $m = 1,2,3,\ldots$.
Take $f(y) = \inf \{ t : y \in S(t) \}$, for all y in M.

The converse is obvious.

Instead of taking all the reals R, it is clearly equivalent to take $N = 1, 1/2, 1/3, \ldots, 0$ and define r as follows. Let u denote a member of N and A a subset of N.

$$ru = \begin{cases} \{1, \tfrac{1}{2}\} \text{ if } u = 1 \\ \{1/(m+1),\ 1/m,\ 1/(m-1)\} \text{ if } u = 1/m,\ m = 1/2,\ 1/3, \ldots \\ 0 \quad \text{if } u = 0 \end{cases}$$

$rA = \bigcup \{ ru : u \in A \}$ if there is a positive integer m such that v in A implies $v = 0$ or $1/m < v$ and

$rA = \{0\} \cup \{ ru : u \in A \}$ if $\inf \{ u : u \in A \} = 0$

We can also consider the set of all positive integers $1,2,3,\ldots$ together with an entity which is not a positive integer and define r in the obvious way.

References

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3. A. Weil, Sur les espaces à structure uniforme et sur la topologie générale, Act. sci. et ind., 551 Paris (1937).

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