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SYSTEMS CONTAINING RANDOM PARAMETERS WITH  
SMALL CORRELATION TIMES

by

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# SYSTEMS CONTAINING RANDOM PARAMETERS

## WITH SMALL CORRELATION TIMES

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### Abstract

Systems described by differential equations involving delta function correlated (i.e. white noise) random parameters are discussed. To be physically meaningful, solutions of such equations should be interpreted as the limits of solutions of the corresponding equations with realistic, i.e., finite correlation time random processes. The implications of this is explored here.

We consider first linear systems. Using the Poisson case as the basic process, a "superposition" principle is derived, allowing one to treat any delta function correlated process. Closed sets of ordinary linear differential equations are found for the moments. The treatments is then generalized to the non-linear case.

Finally, we derived conditions under which a delta function correlated process is a valid approximation to one with finite correlation time in a specific stochastic equation; and we show how the appropriate approximation may be found.

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1. Introduction

Recently, increasing attention has been given to systems containing parameters which vary in a random way with time, [Ref.1]. This attention has been motivated, in part, by attempts to analyse, e.g., randomly fluctuating media and control systems, [Ref. 2,3].

Consider a system with  $n$  degrees of freedom,  $x_1, x_2, \dots, x_n$ , governed by the set of equations:

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^n [c_{ij}(t) + r_{ij}(t)] x_j(t) \quad (i=1,2,\dots,n) \quad (1)$$

where the  $c_{ij}(t)$  are known (i.e., deterministic) functions and the  $r_{ij}(t)$  are random functions of time. The  $x_i(t)$  will then constitute an  $n$ -dimensional random process dependent in a complicated way on the functions  $r_{ij}(t)$ . However, this dependence cannot be exhibited in an explicit closed form for other than the first order equation.

This is the central difficulty in treating a system of equations such as (1) and suggests that a general theory encompassing the most varied statistical behavior of the  $r_{ij}(t)$  is not likely to be achieved. It is thus natural to consider special cases.

The case, generally considered in the literature, is the one in which the integrals  $\int r_{ij}(t') dt'$  constitute a Gaussian process with independent increments. This implies, in engineering parlance, that each  $r_{ij}(t)$  is a white noise, i.e. a random process with delta function correlation  $\langle r_{ij}(t) r_{ij}(t') \rangle \approx \delta(t-t')$ .

Of course, the assumption of a process with delta function correlation (abbreviated d.f.c.) is an idealization and leads at once to the problem of giving a precise meaning to a set of equations such as (1). This is because only the integrals of  $r_{ij}(t)$  are functions in the ordinary sense, while the  $r_{ij}$  are so called "singular functions, and for such singular functions the usual existence theorems for differential equations do not apply.

The requirements of physical sense and applicability would seem to dictate the following procedure for giving meaning to (1) with  $r_{ij}$  delta function correlated. Consider a family of processes  $R_{ij}(t)$  depending upon a parameter,  $T_c$ , and whose correlation functions  $\langle R_{ij}(t) R_{ij}(t+\Delta t) \rangle$  are non-vanishing for  $\Delta t$  in a small but finite interval whose length is of the order  $T_c$ . For such processes the functions  $R_{ij}$  will be sufficiently well behaved that the solutions of (1) will exist in the usual sense; denote them by  $X_i(t)$ . Now let  $T_c$  tend to zero, so that the  $R_{ij}$  converge to a delta function correlated process, and define the  $x_i(t)$  as the limit of the corresponding  $X_i(t)$ .

It must be noted that in the theory of "stochastic differential equations" described in the purely mathematical literature, [Ref. 4] meaning has been given to eq. (1), for the Gaussian noise case, in an entirely different way. That theory is, of course, in no sense, "wrong" but rather is not directly applicable to physical situations. Indeed, its apparent origin was not in an attempt to derive the statistical

properties of the solutions of an equation such as (1), but, on the contrary, rather to represent in the simplest form as a stochastic integral, a class of processes whose probability density satisfied certain diffusion equations. It is unfortunate however that the use of the term "stochastic differential equations" has led to confusion in the physical literature, and that a clarification has been made only surprisingly recently (cf. our Ref. 1 and the later elucidation by A.H. Gray and T.K. Caughey, Ref. 5).

In defining the solutions of eq(1) as the limit of the solutions of the corresponding equation with coefficients with finite correlation times, a number of questions immediately arise. For a given family of processes  $R_{ij}$  what is the correct limiting  $r_{ij}$  and how may the statistics of the solutions  $X_i(t)$  be found? One of the pleasant properties of linear equations is that the moments satisfy a system of closed ordinary differential equations. An even more pressing question from the engineering point of view where one generally deals with a process with a specific if short correlation time (rather than a parametrized family) is the following: when is it possible to approximate such a process by a delta function correlated one in attempting to solve equations such as (1)? Moreover, how may this approximation be improved, presumably by some expansion in the correlation time, for which the delta function approximation represents the first order term.

The present paper differs from previous work in its emphasis on the delta function correlated, white noise, case as an

approximation whose validity must be explored; and related to this, by its use of the Poisson rather than the Gaussian as the basic process.\* It is natural to do this, since in the theory of processes with independent increments, it is shown that the most general such process (including the Gaussian) can be approximated to any degree of accuracy by a sum of Poisson processes or by the limit of such a sum; such a "superposition" principle holds when these processes appear as coefficients in differential equations and enables us to treat the case of a general d.f.c. process. The use of Gaussian processes alone in approximating processes with finite correlation times is an unnecessary and severe restriction. Finally, apart from the far greater generality ultimately obtainable, the use of the Poisson process is simpler to visualize since such a process consists only of a series of sharp discrete impulses occurring at random times.

In Section 2, we consider the case of linear differential equations, deriving the equations satisfied by the probability density of the  $x_i(t)$  and their moments. It is shown that the moment equations can be expressed entirely in terms of the moment generating function of the d.f.c. process.

In Section 3, we generalize our results to the non-linear

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\* A somewhat similar approach, not however using the Poisson process as a basis, has been given by R.L. Stratonovich (Ref. 6).

system

$$\frac{dx_1}{dt} = \sum_{j=1}^n f_{ij}(x_1, \dots, x_n, t) r_j(t) \quad (2)$$

In Section 4, we derive conditions under which a delta function correlated process is a permissible approximation to one with finite correlation time in a given differential equation; and also discuss how the appropriate approximation may be found. As an example, the relation between the Gaussian white noise and the Uhlenbeck-Ornstein process is discussed: this relation is of importance in the theory of Brownian motion.

## 2. The Linear Case

The possible ambiguity in defining solutions of differential equations with delta function correlated coefficients is illustrated at once by the simplest possible example. Consider the equation

$$\frac{dx}{dt} = r(t)x \quad (3)$$

where  $\int r(t)dt$  is a Poisson process so that  $r(t)$  represents a Poisson (shot) noise. Such a noise may be idealized as a series of infinitely short identical impulses occurring at random times. The impulses are each characterized by a "strength"  $S$ , the integral of  $r(t)$  over any single impulse; and the distribution of times at which impulses occur is described by a "rate"  $\lambda$  such that  $\lambda dt$  is the probability of an impulse

occurring in a short time  $dt$  (Ref. 7). Symbolically, a Poisson noise random function may be written as a sum of delta functions

$$r(t) = S \delta(t-t_1) + S \delta(t-t_2) + \dots + S \delta(t-t_n) \quad (4)$$

with  $t_i$  denoting the instant the  $i^{\text{th}}$  impulse (measured from some initial time) occurs.

Let us integrate both sides of eq. (3) over a time interval ( $t_b \leq t \leq t_a$ , say) in which only a single impulse at  $t = t^*$  occurs, and denote by  $x_b$  and  $x_a$  the values  $S$  of  $x$  before and after this impulse. Thus using eq. (4)

$$x_a - x_b = Sx(t^*) \quad (5)$$

But  $x(t^*)$ , the value of  $x$  during an impulse has no precise meaning. In fact, according to eq. (3),  $\frac{dx}{dt}$  during an impulse is arbitrarily large and  $x(t)$  is varying rapidly between  $x_b$  and  $x_a$ . Different theories for eq. (1) will result depending on the value assigned to  $x(t^*)$ .

Suppose, however, that the impulses last a finite time so that eq. (3) can be integrated in the usual way. Thus

$$x_a = x_b \exp \left[ \int_{t_b}^{t_a} r(t) dt \right]$$

Then, as  $x(t)$  tends to a Poisson noise eq. (4) one has the following simple result which is basic to all that follows:

The change caused by an impulse is given by

$$x_a = x_b e^S \quad (6)$$

where  $x_a$  and  $x_b$  are the values of  $x$  before and after the impulse.

We note that this relation between  $x_a$  and  $x_b$  is different than



the one obtained if the value  $x_b$  is used for  $x(t^*)$  for then  $x_a = x_b(1+S)$ . The reader may verify that if this latter expression is used in the arguments leading to the Gaussian noise case (to be given below) one obtains the results found often in the mathematical literature. We emphasize again that that approach is not correct for physical and engineering cases where one deals with processes with finite correlation times, approximating them by delta function correlated processes. Indeed, the use of  $x(t^*)=x_b$  can lead to physically absurd results if one notes that for sufficiently negative S,  $x_a$  and  $x_b$  may have opposite signs which is impossible if say,  $x$  represents an intrinsically positive quantity.

Let us extend (6) to the more general equation

$$\frac{dx}{dt} = r(t) + A \underline{x} \quad (7)$$

where A is a matrix  $[a_{ij}]$   $i, j=1, 2, \dots, n$   $\underline{x}(t)$  is now the vector with components  $x_i(t)$  ( $i=1, 2, \dots, n$ ) and as before, we write  $\underline{x}_b$  for  $\underline{x}(t_b)$ ,  $\underline{x}_a$  for  $\underline{x}(t_a)$ . Then integrating (7) from  $t_b$  to  $t_a$  one finds for the change in x over an impulse

$$\underline{x}_a = e^{\frac{SA}{dt}} \underline{x}_b \quad (8)$$

An equation for the probability density  $p(\underline{x}, t)$  may now be found. Suppose the system has coordinates lying in the volume  $[\underline{x}, \underline{x}+d\underline{x}]$  at the time t, [probability  $p(\underline{x}, t) d\underline{x}$ ]. Then, at the time  $t - dt$  either (a) the coordinates were in the volume  $[\underline{x}, \underline{x}+d\underline{x}]$  and no impulse occurred during the time dt, (the probability of this being  $(1-\lambda dt)p(\underline{x}, t)d\underline{x}$ ) or (b) the

system had coordinates in a volume  $[x_b, x_b + dx_b]$  and an impulse did occur, probability of event (b) being  $\lambda p(\underline{x}_b, t-dt) d\underline{x}_b dt$ . But by equation (8)  $\underline{x}_b = e^{-SA} \underline{x}$ ,  $d\underline{x}_b = |e^{-SA}| d\underline{x}$ , where  $|e^{-SA}|$  is the determinant of the matrix  $e^{-SA}$ . Therefore,

$$p(\underline{x}, t) = (1 - \lambda dt) p(\underline{x}, t-dt) + \lambda dt p(e^{-SA} \underline{x}, t-dt) |e^{-SA}| ;$$

and on passing to the limit, one finds that

The probability density  $p(\underline{x}, t)$  satisfies the equation

$$\frac{\partial p(\underline{x}, t)}{\partial t} = \lambda p(e^{-SA} \underline{x}, t) |e^{-SA}| - \lambda p(\underline{x}, t) \quad (9)$$

For the slightly more general system

$$\frac{d\underline{x}}{dt} = (\underline{C} + r(t) A) \underline{x} \quad (10)$$

where  $\underline{C} = (c_{ij})$  is a deterministic matrix, (9) becomes

$$\frac{\partial p(\underline{x}, t)}{\partial t} = - \sum_{i,j=1}^n c_{ij} \frac{\partial (x_j p(\underline{x}, t))}{\partial x_i} + \lambda p(e^{-SA} \underline{x}, t) |e^{-SA}| - \lambda p(\underline{x}, t) \quad (11)$$

From its derivation, it is clear that equation (11) is valid even if  $\lambda, S, A$  or  $\underline{C}$  are functions of the time.

On the basis of eq. (11) one can readily show that the moments of  $\underline{x}$  satisfy linear differential equations. In particular, expectations  $\langle x_i \rangle$  and covariance  $\langle x_i x_j \rangle$  we have

$$\frac{d\langle x_i \rangle}{dt} = c_{ij} \langle x_j \rangle + \lambda (B_{ij} - 1) \langle x_j \rangle \quad (12)$$

$$\frac{d\langle x_i x_j \rangle}{dt} = \sum_{k=1}^n c_{ik} \langle x_j x_k \rangle + \sum_{k=1}^n c_{jk} \langle x_i x_k \rangle + \lambda \sum_{k,\ell=1}^n B_{ik} B_{j\ell} \langle x_k x_\ell \rangle - \lambda \langle x_i x_j \rangle. \quad (13)$$

where  $B_{ij} = (e^{SA})_{ij}$ .

To derive such equations, one need only multiply eq. (11) by the appropriate moment and integrate over  $x$ . The only point to note is that in evaluating such integrals as

$$\int x_i p(e^{-SA} \underline{x}, t) d\underline{x} \quad (14)$$

one makes the substitution  $\underline{y} = e^{-SA} \underline{x}$ .

#### A Superposition Principle for Poisson Noise Processes.

Eq. (11) may be extended to the case where the random part of (7) consists of a sum of an arbitrary number of independent Poisson noises. Such an extension is important for two reasons: first, it enables us to handle equations such as (1), and secondly, to treat the case where  $r(t)$  is an arbitrary delta function correlated process. This is because any such process can be approximated to any degree of accuracy by a sum of Poisson noise processes with suitable strengths and rates, or by a limit of such sums. [Ref. 8].

Consider, for example, the equation.

$$\frac{dx}{dt} = (\underline{C} + r_1(t) \underline{A}_1 + r_2(t) \underline{A}_2) \underline{x} \quad (15)$$

with  $r_1(t)$  and  $r_2(t)$  being independent Poisson noises of

strengths  $S_1$  and  $S_2$  and rates  $\lambda_1$  and  $\lambda_2$  respectively. Noting, that because of this independence the probability of an impulse in both  $r_1(t)$  and  $r_2(t)$  during a short time  $dt$  is  $\lambda_1 \lambda_2 (dt)^2$  and hence may be neglected in the limit  $dt \rightarrow 0$ , one finds that

$$\begin{aligned} \frac{\partial p(\underline{x}, t)}{\partial t} = & - \sum c_{ij} \frac{\partial [x_j p(\underline{x}, t)]}{\partial x_i} + \lambda_1 p(e^{-S_1 A_1} \underline{x}, t) \left| e^{-S_1 A_1} \right| \\ & + \lambda_2 p(e^{-S_2 A_2} \underline{x}, t) \left| e^{-S_2 A_2} \right| - (\lambda_1 + \lambda_2) p(\underline{x}, t) \end{aligned} \quad (16)$$

In general, if the r.h.s. contains an arbitrary number of Poisson noises, one has the following "Superposition Principle":

To obtain the equation for  $\frac{\partial p(\underline{x}, t)}{\partial t}$ , one need only sum the contributions that would be made by each Poisson noise in the absence of the others.

A similar statement holds for the moments. From this, one can immediately obtain the equation satisfied by  $p(\underline{x}, t)$  if  $x_i(t)$  is given by an equation such as (1) with  $r_{ij}(t)$  being a Poisson noise. One simply lets the matrix  $A$  corresponding to the  $r_{ij}(t)$  term in (1) be a matrix containing all zeros except for a 1 in its  $i^{\text{th}}$  row and  $j^{\text{th}}$  column.

If  $r(t)$  in eq. (10) is an arbitrary delta function correlated process, then the equation for  $p(\underline{x}, t)$  can be found by considering the Poisson noises which compose  $r(t)$  and using the superposition principle. However, as far as the moments go, one can show that it is not necessary to find the decomposition of  $r(t)$  into Poisson noises explicitly; indeed,

the moment equations for  $\underline{x}(t)$  can be expressed entirely in terms of the moment generating function of the integral of  $r(t)$

To show this let

$$f(u) = \langle \exp \left[ -u \int_0^1 r(t) dt \right] \rangle$$

and write  $k(u) = \log f(u)$ . Suppose that  $r(t)$  was composed of Poisson noises with strengths  $S_j$  and rates  $\lambda_j$  then

$$k(u) = \sum_j \lambda_j (e^{-uS_j} - 1)$$

Now, on the r.h.s. of eq. (12) the coefficient of  $\langle \underline{x} \rangle$  if written as a matrix would be  $C + \sum_j \lambda_j (e^{S_j A} - E)$  where  $E$  is the identity matrix. But this is just  $C + k(-A)$ . Hence one has

$$\frac{d\langle \underline{x} \rangle}{dt} = [C + k(-A)] \underline{x} \quad (17)$$

Similarly the r.h.s. of (13) may be written in matrix form if one introduces the direct product  $A \times A$  of the matrix  $A$ , and the direct product  $\underline{x} \times \underline{x}$  of the vector  $\underline{x}$  [Ref.10]. Then the coefficient of  $\langle \underline{x} \times \underline{x} \rangle$  in (14) is

$$[C \times E + E \times C + \sum_j \lambda_j (\exp S_j A \times A - E \times E)]$$

(noting that  $\exp [A \times A] = \exp (A) \times \exp (A)$ ), or

$$\frac{d\langle \underline{x} \times \underline{x} \rangle}{dt} = C \times E + E \times C + k(-A \times A) \underline{x} \times \underline{x} \quad (18)$$

And in a similar way one obtains expressions for the higher moments.

Gaussian Noise. Suppose that in equation (15)  $r_1$  and  $r_2$  are shot noises of the same rate  $\lambda$  and with strengths of equal

magnitude but opposite sign. Letting  $\lambda \rightarrow \infty$ , and  $S \rightarrow 0$  in such a way that  $\lambda S^2 \rightarrow \frac{1}{2} \sigma^2$ , the process  $r_1(t) + r_2(t)$  will tend to a Gaussian noise  $g(t)$  with power  $\sigma^2$ :  $\langle g(t) g(t') \rangle = \sigma^2 \delta(t-t')$ . Using this fact, the equation for the probability density of the solutions of the stochastic equations

$$\frac{d\underline{x}}{dt} = [C + A g(t)] \underline{x} \quad (19)$$

may be found directly from (16). Indeed, making the identifications  $\underline{A}_1 = \underline{A}_2 = \underline{A}$ ,  $S_1 = -S_2 = S$ ,  $\lambda_1 = \lambda_2 = \lambda$  gives

$$\begin{aligned} \frac{\partial p(\underline{x}, t)}{\partial t} = & - \sum_{i,j} c_{ij} \frac{\partial [x_j p(\underline{x}, t)]}{\partial x_i} + \lambda p(e^{-\underline{SA}} \underline{x}, t) \left| e^{-\underline{SA}} \right| \\ & + \lambda p(e^{\underline{SA}} \underline{x}, t) \left| e^{\underline{SA}} \right| - 2\lambda p(\underline{x}, t). \end{aligned} \quad (20)$$

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\* The easiest way to show this is to prove that the characteristic function of  $\int_0^t [r_1(t') + r_2(t')] dt'$  tends to that of  $\int_0^t g(t') dt'$  for any  $t$ , as  $S \rightarrow 0$ ,  $\lambda \rightarrow \infty$ ,  $\lambda^2 S \rightarrow \frac{1}{2} \sigma^2$ . But this follows on noting that  $\langle \exp i \delta \int_0^t [r_1(t') + r_2(t')] dt' \rangle = \exp [\lambda t (e^{i\delta S} - 1) + \lambda t (e^{i\delta S} - 1)]$ , and  $\langle \exp i \delta \int_0^t g(t') dt' \rangle = \exp [ -\frac{1}{2} \delta^2 \sigma^2 t ]$ . Physically, Gaussian noise arises in this way so this procedure seems entirely natural.

Expanding to terms of order  $S^2$ ,

$$\begin{aligned}
 p(e^{\underline{SA}} \underline{x}, t) \quad |e^{\underline{SA}}| &= p\left(\left[1 + S\underline{A} + S^2 \frac{\underline{A}^2}{2}\right] \underline{x}, t\right) \left[1 + S\underline{\Gamma} + \frac{S^2}{2} \underline{\Gamma}^2\right] \\
 &= p + S[\underline{\Gamma}p + \sum_{i,j} \frac{\partial p}{\partial x_i} (a_{ij} x_j)] \\
 &+ \frac{1}{2} S^2 \left[ \sum_{\substack{i,j, \\ k,l}} \frac{\partial^2 p}{\partial x_i \partial x_j} a_{ik} x_k a_{jl} x_l + 2 \sum_{i,j} \frac{\partial p}{\partial x_i} a_{ij} x_j \underline{\Gamma} \right. \\
 &\quad \left. p\underline{\Gamma}^2 \right] \tag{21}
 \end{aligned}$$

where  $\underline{\Gamma} = \text{Trace } \underline{A} = \sum_i a_{ii}$ , and  $p = p(\underline{x}, t)$ .

Performing a similar expansion with  $p(e^{-\underline{SA}} \underline{x}, t) |e^{-\underline{SA}}|$  substituting in (20) allowing  $\lambda S^2 \rightarrow \frac{1}{2} \sigma^2$ , one finds that

$$\begin{aligned}
 \frac{\partial p(\underline{x}, t)}{\partial t} &= - \sum_{i,j} c_{ij} \frac{\partial [x_j p]}{\partial x_i} + \frac{1}{2} \sigma^2 \left[ - \sum_{i,j} \frac{\partial (a_{ij} x_j p)}{\partial x_i} + \right. \\
 &\quad \left. \sum_{\substack{i,j, \\ k,l}} \frac{\partial (a_{ik} a_{jl} x_k x_l p)}{\partial x_i \partial x_j} \right] \tag{22}
 \end{aligned}$$

The "Superposition" principle used for Poisson noise extends at once to the Gaussian noise case. If, e.g.,  $g_1(t)$

and  $g_2(t)$  are independent Gaussian noises with zero means and powers  $\sigma_1^2$  and  $\sigma_2^2$  respectively, then to obtain the equation for the probability density of  $\underline{x}$  where

$$\frac{d\underline{x}}{dt} = [\underline{C} + g_1(t) \underline{A}_1 + g_2(t) \underline{A}_2] \underline{x} \quad (23)$$

one need only write each  $g_i(t)$  ( $i=1,2$ ) as the limit of a sum of shot noises with rates  $\lambda_i$  and of strengths  $S_i$  and  $-S_i$  such that  $\lambda_i S_i^2 \rightarrow \frac{\sigma_i^2}{2}$ . Using the superposition procedure for the Poisson noise, the differential equation for  $p(x,t)$  will follow in precisely the same way as (22) from (20).

The case where the  $g_i(t)$  are mutually correlated can be readily reduced to the above. Suppose the system for  $\underline{x}$  is

$$\frac{d\underline{x}}{dt} = [\underline{C} + \sum_{i=1}^n g_i(t) \underline{A}_i] \underline{x} \quad (24)$$

where

$$\langle g_i(t) g_j(t') \rangle = \rho_{ij} \delta(t-t'), \quad (\rho_{ij} = \sigma_i^2 \text{ for } i=j)$$

Let  $\underline{g}(t)$  denote the vector  $[g_i(t), i=1,2,\dots,n]$  and  $\underline{\rho}$  the matrix  $(\rho_{ij})$ . Then there exists an orthogonal matrix  $\underline{H}$  and a diagonal matrix  $D$  such that  $\underline{H} \underline{\rho} \underline{H}^{-1} = D$ . This follows from the symmetry of  $\rho$ . But then the random processes  $h_i(t)$  defined by  $\underline{h}(t) = \underline{H} \underline{g}(t)$  ( $\underline{h}(t)$  is the vector  $[h_i(t), i=1,2,\dots,n]$ ) will be mutually uncorrelated. Hence, (24) may be written as

$$\frac{d\underline{x}}{dt} = [\underline{C} + \sum_{i=1}^n h_i(t) \underline{A}_i^*] \underline{x} \quad (25)$$



where  $A_i^* = \sum_{j=1}^n h_{ij} A_j$  and the  $h_i(t)$ , being mutually uncorrelated and Gaussian are mutually independent.

### 3. Non-Linear Systems

Consider now the non-linear system of equations

$$\frac{dx_i}{dt} = \sum_{j=1}^n f_{ij}(x_1, \dots, x_n) r_j(t) \quad (26)$$

where the  $f_{ij}$ , apart from being at least twice differentiable, may be arbitrary functions of the variables  $x_i$ . To determine the equation satisfied by  $p(\underline{x}, t)$ , we proceed as in the linear case. Thus, suppose first (26) has the simpler form

$$\frac{dx_i}{dt} = r(t) f_i(x_1, \dots, x_n) \quad (27)$$

where  $r(t)$  is a shot noise of strength  $S$  and impulse rate  $\lambda$ .

Consider a single impulse and denote by  $\underline{x}_b$  and  $\underline{x}_a$ , the values of  $\underline{x}$  just before and after this impulse. It is crucial to the theory that the change in  $\underline{x}$  over an impulse depend on  $r(t)$  only through the integral  $\int r(t) dt$ . Otherwise, in passing to the limit of a shot noise from a non-singular random process, the limit of the solutions of (27) in the latter case will not depend only on the properties of the limiting shot noise. However, one has the result that

The solutions of eq. (27) depend on  $r(t)$  only through the integral  $\int r(t) dt$ . In particular,  $\underline{x}_a$  depends only on  $\underline{x}_b$  and  $S$ .

In fact, on dividing the  $i^{\text{th}}$  ( $i=2,3,\dots$ ) of eqs. (27) by the equation for  $x_1$ , we have

$$\begin{aligned}(x_1)_b^2 &= -2S [(x_1)_a / (x_2)_a] + (x_1)_a^2, \\(x_2)_b^2 &= -2S [(x_2)_a / (x_1)_a] + (x_2)_a^2.\end{aligned}\tag{32}$$

From the expression for  $\underline{x}_b$  in terms of  $\underline{x}_a$  and  $S$ , the equation for  $p(\underline{x}, t)$  follows by a generalization of the argument leading to (14). One finds that

$p(\underline{x}, t)$  satisfies the equation

$$\frac{\partial p(\underline{x}, t)}{\partial t} = \lambda p[\underline{x}_b(\underline{x}, S), t] \left| \frac{\partial \underline{x}_b}{\partial \underline{x}} \right| - \lambda p(\underline{x}, t)\tag{33}$$

where  $\underline{x}$  stands for  $\underline{x}_a$ ,  $\left| \frac{\partial \underline{x}_b}{\partial \underline{x}} \right|$  is the determinant with elements  $\frac{\partial (x_i)_b}{\partial x_j}$ , and the notation  $p[\underline{x}_b(\underline{x}, S), t]$  indicates that in the  $i^{\text{th}}$  coordinate position  $p(\underline{x}, t)$  is to be evaluated at  $(x_i)_b$  (regarded as a function of  $\underline{x}$  and  $S$ ). To verify eq. (33), it need be only noted that if the system is in the volume element  $[\underline{x}, \underline{x} + d\underline{x}]$  at time  $t$ , either no impulse occurred in the previous time interval  $dt$ , or an impulse did occur, and the system was in the element  $[\underline{x}_b, \underline{x}_b + d\underline{x}_b]$ , where  $\underline{x}_b = \underline{x}_b(\underline{x}, S)$  and  $d\underline{x}_b = \left| \frac{\partial \underline{x}_b}{\partial \underline{x}} \right| d\underline{x}$ . The superposition principle used in the linear case when  $r(t)$  is a sum of shot noises is valid here as well without any change, so that no further comment is necessary.

#### Expansions in powers of $S$

In contrast to the linear case, the equations for the moments do not form a closed system. For example, if (33) is multiplied by and integrated, one has on the left an expression

involving not only the first order moments but generally higher orders as well. The expedient often used in non-linear equations is however, applicable here: namely an expansion in powers of a small parameter. Let S then be taken to be small. Regarding  $(\underline{x})_b$  as a function of S, we seek an expansion of the form

$$(\underline{x})_b = (\underline{x})_a + \left. \frac{\partial(\underline{x})_b}{\partial S} \right|_{S=0} S + \frac{1}{2} \left. \frac{\partial^2(\underline{x})_b}{\partial S^2} \right|_{S=0} S^2 + \dots, \quad (34)$$

noting that  $(\underline{x})_b$  reduces to  $(\underline{x})_a$  when  $S=0$

Differentiating (34) with respect to S gives

$$\frac{-\partial(x_1)_b}{\partial S} \frac{1}{\tilde{f}_1 [(x_1)_b, (x_1)_a, (x_i)_a]} = 1 \quad (35)$$

and in general

$$\frac{\partial(x_i)_b}{\partial S} = -f_i [(x_1)_b, (x_2)_b, \dots, (x_n)_b]; \quad (36)$$

differentiating once more

$$\frac{\partial^2(x_i)_b}{\partial S^2} = \sum_j \frac{-\partial f_i}{\partial(x_j)_b} \frac{\partial(x_j)_b}{\partial S} = \sum_j \frac{\partial f_i}{\partial(x_j)_b} f_j \quad (37)$$

To find the values of these derivatives at  $S=0$ , we need only replace in the r.h.s. of (36) and (37) the arguments  $(x_i)_b$  by  $(x_i)_a$ . Using (34), it follows that the determinant

$\left| \frac{\partial \underline{x}_b}{\partial \underline{x}} \right|_{0(S^2)}$  (writing  $\underline{x}$  for  $(\underline{x})_a$ ) has the expansion correct to terms

$$\left| \frac{\partial \underline{x}_b}{\partial \underline{x}} \right| = 1 - S \sum_i f_{i,i} + \frac{S^2}{2} \sum_{i,j} [f_{i,ij} f_j^{f_{i,i}} + f_{i,i} f_{j,j}] \quad (38)$$

where  $f_{i,j} = \frac{\partial f_i}{\partial x_j}$ ,  $f_{i,jk} = \frac{\partial^2 f_i}{\partial x_j \partial x_k}$

Finally from (33) and (38) we find the approximate equation for  $p = p(\underline{x}, t)$

$$\begin{aligned} \frac{\partial p}{\partial t} &= \lambda S \left[ \frac{\partial p}{\partial x_i} f_i + p f_{i,i} \right] \\ \frac{\lambda S^2}{2} \sum_{i,j} &\left[ \frac{\partial p}{\partial x_i} f_{i,j} f_j + 2 \frac{\partial p}{\partial x_i} f_i f_{j,j} + \frac{\partial^2 p}{\partial x_i \partial x_j} f_i f_j \right. \\ &\left. + p(f_{i,ij} f_j + f_{i,i} f_{j,j}) \right] + O(S^3) \end{aligned} \quad (39)$$

Now regard  $p(\underline{x}, t)$  as a function of  $S$  with the series expansion

$$p(\underline{x}, t) = p_0(\underline{x}, t) + S p_1(\underline{x}, t) + \dots$$

Then on substituting into eq. (39) and equating coefficients of  $S$ , one derives a set of recursion relations for the  $p_k(\underline{x}, t)$ , the first such equation being

$$\frac{\partial p_1}{\partial t} = S \sum_i \left[ \frac{\partial p_0}{\partial x_i} \right] f_i + p_0 f_{i,i}$$

In the absence of a deterministic part,  $p_0(\underline{x}, t)$  is just the initial value of the probability density, namely  $p(\underline{x}, 0)$  and it follows readily that the  $p_k(\underline{x}, t)$  will be polynomials in  $t$  of order  $n$  with coefficients depending on  $p_0(\underline{x})$  and its first  $k$  derivatives. On the other hand if a deterministic part did occur in eq. (27), i.e. if a term  $c_i(\underline{x})$  say was added to the r.h.s. of eq. (27), then (33) and (39) would be modified only by the presence of an additional term  $-\sum \frac{\partial(c_i p)}{\partial x_i}$  in their

right hand sides.  $p_0(x,t)$  then would not be simply  $p(x,0)$  but rather  $p[\underline{x}(\underline{x}_0, t), 0]$  where  $\underline{x}(\underline{x}_0, t)$  is the solution of the differential equations

$$\frac{dx_i}{dt} = c_i(\underline{x}) \quad [i=1,2,\dots,n]$$

with initial conditions  $\underline{x}(t=0) = \underline{x}_0$ .

The Gaussian Case. By means of (39) we can write down the equation for  $p(x,t)$  when  $r(t)$  is a Gaussian noise with mean zero and power  $\sigma^2$ . Proceeding exactly as before, i.e., regarding a Gaussian noise as the sum of two Poisson processes of strengths  $S$  and  $-S$ , and rate  $\lambda$  such that  $\lambda S^2 \rightarrow \sigma^2/2$  as  $\lambda \rightarrow \infty$ ,  $S \rightarrow 0$ , one obtains

$$\frac{\partial p}{\partial t} = \frac{\sigma^2}{2} \left[ - \sum_{i,j} \frac{\partial(p f_{i,j} f_j)}{\partial x_i} + \sum_{i,j} \frac{\partial^2(p f_i f_j)}{\partial x_i \partial x_j} \right] \quad (40)$$

Let us compare eq. (40) with the familiar Fokker-Planck equation. [Ref. 9]. Suppose  $x_i(t)$  were a Markoff process, such that  $\langle \Delta x_i \rangle = \langle x_i(t+\Delta t) - x_i(t) \rangle$  and  $\langle \Delta x_i \Delta x_j \rangle = \langle [x_i(t+\Delta t) - x_i(t)][x_j(t+\Delta t) - x_j(t)] \rangle$  are both of order  $\Delta t$  for small  $\Delta t$ , while higher order moments are of lower order, then  $p = p(\underline{x}, t)$  would satisfy the equation

$$\frac{\partial p}{\partial t} \Delta t + o(\Delta t) = - \sum_{i=1}^n \frac{\partial}{\partial x_i} (p \langle \Delta x_i \rangle) + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (p \langle \Delta x_i \Delta x_j \rangle) \quad (41)$$

On division by  $\Delta t$  and passage to the limit  $\Delta t=0$  this is the same as (40) when  $\langle \Delta x_i \rangle = \frac{1}{2} \sigma^2 \sum_{j=1}^n f_{i,j} f_j \Delta t$ , and

$$\langle \Delta x_i \Delta x_j \rangle = \sigma^2 f_i f_j \Delta t.$$

The interesting point is the non-zero value of  $\langle \Delta x_i \rangle$  even though the first moment of  $g(t)$  and its integral vanishes. This is because over the interval  $\Delta t$  the values of  $x_1, \dots, x_n$  and hence  $f_i(x_1, \dots, x_n)$  varies according to the values assumed by  $g(t')$  and hence the correlation  $\langle g(t') f_i \rangle$  (and therefore  $\langle \Delta x_i \rangle$ ) is in general non-zero. Ultimately, if we consider the linear case, the non-vanishing of  $\langle \Delta x_i \rangle$  is a consequence of the fact that an impulse of strength  $S$  and one of strength  $-S$  do not average out, to second order, as may be seen by expanding  $e^S + e^{-S}$ .

#### 4. Approximation to the Finite Correlation Time Case

In previous sections, we have considered differential equations with d.f.c. coefficients, interpreting such equations as the limit of the same differential equations but having coefficients that are random processes correlated over a finite time. Our purpose here is to determine when the d.f.c. idealization is a valid approximation to a physically realistic case, and how this approximation is to be made.

Suppose that the differential equation is

$$\frac{dx}{dt} = \sum_{i=1}^m A_i r_i(t) \underline{x} \quad (42)$$

where the  $A_i$  are constant matrices and the  $r_i(t)$  are stationary independent random processes\* whose sample functions are continuous (except perhaps for a finite number of jump discontinuities)

In general a closed solution of eq(42) is not obtainable but an iterative solution is possible. Writing

$$D(t) = \sum_{i=1}^m A_i r_i(t)$$

---

\* Note that the presence of a linear deterministic part can be taken into account by letting one of the  $r_i(t)$  be a constant

$$\underline{x}(t+\tau) = \underline{x}(t) + \left[ \int_t^{t+\tau} \underline{D}(t') dt' + \int_t^{t+\tau} \int_t^{t'} \underline{D}(t') \underline{D}(t'') dt' dt'' \right. \quad (43)$$

$$\left. + \dots \int_t^{t+\tau} \dots \int_t^{t^{(n-1)}} \underline{D}(t') \underline{D}(t'') \dots \underline{D}(t^{(n)}) dt' dt'' \dots dt^{(n)} + \dots \right] \underline{x}(t)$$

which gives  $\underline{x}$  at time  $t+\tau$  in terms of  $\underline{x}$  at some earlier time  $t$ .

We now introduce two assumptions which allow the expression (43) to be greatly simplified, and which in fact form the basis of the approximation by d.f.c. processes. They both involve the time increment  $\tau$ : namely, we suppose it possible to choose  $\tau$  such that

$$\int_t^{t+\tau} r_i(t') dt' \quad \text{(A) The integrals}$$

are statistically independent of the history of the random processes  $r_i$  before the time  $t$ .

(B) The relative change in  $\underline{x}$  the time  $\tau$  is small.

Generally, it is possible to meet assumption (A) by taking  $\tau$  sufficiently large; and it is possible to meet assumption (B) by taking  $\tau$  sufficiently small. The key point is that we are requiring a  $\tau$  such that both (A) and (B) are satisfied.

Assumption (A), which in practice can be met only to a certain degree of approximation, may be expressed in a more transparent form if the notion of the correlation time  $T_c$  is used. Consider the process  $r_i$ , for example, at two instants  $t$



and  $t + \Delta t$ . Then the correlation time is such that  $r_i(t)$  and  $r_i(t + \Delta t)$  are very nearly independent for  $\Delta t > T_c$ . A useful measure of  $T$  is given by

$$T_c = \frac{\int_0^{\infty} [r_i(t)r_i(t+\Delta t) - \langle r_i(t) \rangle^2] dt}{\langle r_i^2(t) \rangle - \langle r_i(t) \rangle^2} \quad (44)$$

i.e., the time integral of the correlation coefficient over the variance. If  $\tau \gg T_c$  the predominant contribution to the integrals of  $r_i$  will come from those  $t'$  for which  $t' - t \gg T_c$ . Therefore (A) will hold when

(A') The time increment  $\tau$  is such that  $\tau \gg T_c^*$

where  $T_c^*$  is the maximum of the correlation times of the processes  $r_i(t)$ .

We introduce now the moments

$$M_{i,k} = \left\langle \left[ \int_{t+\tau}^t r_i(t') dt' \right]^k \right\rangle \quad (45)$$

and the quantities

$$\mu_{i,k} = M_{i,k} / \tau \quad (46)$$

The  $\mu_{i,k}$  will then be certain functions of  $\tau$  which are independent of the history of the processes  $r_i(t')$  before the time  $t$ , provided that  $\tau \gg T_c$ .

Consider a class of processes  $r_i(t)$  containing the correlation time  $T_c$  as a parameter, and imagine that this parameter approaches zero. Then letting both  $T_c/\tau$ , and  $\tau \rightarrow 0$ , the  $\mu_{i,k}$  will tend to certain (possibly zero) constants and

these will define a limiting d.f.c. process for  $r_i(t)$ . Thus, if  $\mu_{i,k} \rightarrow \lambda_i S_i^k$  where  $S_i$  is some positive number, then  $r_i(t)$  approaches a shot noise with strength  $S_i$  and frequency  $\lambda_i$ ; or if  $\mu_{i,k} \rightarrow \sigma_i^2$  for  $k=2$ , and 0 for  $k > 2$ , the limit will be a Gaussian with power  $\sigma^2$

An additional condition will be needed here: namely that the  $k$ -fold correlations  $\langle r(t')r(t'')\dots \rangle$  do not change sign over the interval  $[t, t+\tau]$ . Since  $\tau$  is small this condition is a weak one. It can be removed if we use instead of the  $M_{i,k}$  the integrals of the absolute values of the  $k$ -fold correlations. We now prove that, subject to the above condition,

To terms of the first order in  $\tau$  the moments of  $\underline{x}(t+\tau) - \underline{x}(t)$  are identical with those of the expression

$$\sum_{i=1}^m \left[ \exp \left( A_i \int_t^{t+\tau} r_i(t') dt' \right) - E \right] \underline{x}(t) \quad (47)$$

The theorem is clear if the matrices  $A_i$  mutually commute for then (43) is equal to

$$\left[ \exp \sum_{i=1}^m A_i \int_t^{t+\tau} r_i(t') dt' \right] \quad (48)$$

and to order  $\tau$ , the moments of (47) are the same as those of (48),

In general, let us consider the difference  $\delta(\tau)$  between the r.h.s. of (43) and (47). To prove that the moments of  $\delta(\tau)$  are of order  $\tau^2$ , we note that an upper bound for

these moments may be obtained by the following procedure:  
 Replace  $\underline{x}(t)$  by  $\underline{x}_{\max} \underline{e}$  where  $\underline{x}_{\max} = \max_j |x_j(t)|$  and  $\underline{e}$  is a vector having unity for all its components; and similarly replace the matrices  $A_i$  by  $n a_{i,\max} E$ . Finally, in calculating moments of the resulting expression

$$\left\{ \exp \left[ \sum_{i=1}^m n a_{i,\max} \int_t^{t+\tau} r_i(t') dt' \right] - E \right. \quad (49)$$

$$\left. - \exp \left[ \sum_{i=1}^m \left[ n a_{i,\max} \int_t^{t+\tau} r_i(t') dt' - E \right] \right] \right\} \underline{x}_{\max} \underline{e}$$

use the absolute values of the quantities  $M_{i,k}^{12}$ . But from (46), in the resulting moments (and hence in those of  $\underline{\Delta}(\tau)$ ) only terms of order  $\tau^2$  or higher enter.

Indeed one observes that terms of order  $\tau$  can only arise on taking moments of expressions involving but a single one of the  $r_i(t)$ ; but no such terms can appear in (49), for on expanding the exponentials only products of different  $r_i$  enter.

Thus, for sufficiently small  $\tau$  (assuming always that  $\tau \gg T_c$ ) the statistics of  $\underline{x}(t+\tau) - \underline{x}(t)$  as determined from (43), reduce to those of (47). The form of (43) leads to two conclusions. Namely, that the dependence of  $\underline{x}(t+\tau) - \underline{x}(t)$  on the random terms  $r_i(t)$  is additive, i.e., the change in  $\underline{x}(t)$  is just the sum of the changes that would be caused by each term in the absence of the others; and secondly, the dependence on each  $r_i(t)$  is only through their integrals. But

these integrals are completely characterized by the  $\mu_{i,k}$

The introduction of the d.f.c. approximation can then be effected by finding a d.f.c. process whose integral over the interval  $[t, t+\tau]$  has the same statistics as that of  $r_i(t')$ . Since any such process can be written as a sum of Poisson noises, we need only determine a set of constants  $\lambda_{ij} > 0$  ( $j=1,2,\dots$ ) and  $S_{ij}$  such that

$$\mu_{ik} = \sum_j \lambda_{ij} S_{ij}^k. \quad (50)$$

and regard the  $\lambda_{ij}$  and  $S_{ij}$  as being the rates and strengths of shot noises  $r_{ij}(t)$ . From section 2, we note that to determine the moments of  $\underline{x}(t)$ , the  $\lambda_{ij}$  and  $S_{ij}$  need not be found explicitly. The moments of  $\underline{x}(t)$  may be expressed entirely in terms of  $\mu_{ik}$ .

Thus, for small  $\tau$ , eq. (47) may be written as

$$\underline{x}(t+\tau) - \underline{x}(t) = \left[ \sum_{i=1}^m \exp\left(A_i \sum_j \int_t^{t+\tau} r_{ij}(t') dt'\right) - E \right] \underline{x}(t) \quad (51)$$

Consider the change  $\frac{\partial p(\underline{x}, t)}{\partial t} \tau$  of the probability density  $p(\underline{x}, t)$  over the time  $\tau$ . To first order in  $\tau$  this change will be the sum of those caused by each of the individual terms in (51). But

$\exp\left[A_1 \sum_j \int_t^{t+\tau} r_{1j}(t') dt'\right] \underline{x}(t)$ , e.g., is (according to the definition used in this paper), just the solution of the stochastic

differential equation

$$\frac{d\underline{x}}{dt} = \underline{A}_1 \left( \sum_j r_{1j}(t) \right) \underline{x} \quad (52)$$

Referring to sec. 2, the contribution of the  $\underline{A}_1$  term to

$$\frac{\partial p(\underline{x}, t)}{\partial t} \tau \quad \text{is}$$

$$\sum_j \lambda_{ij} p \left[ \left( e^{-S_{1j} \underline{A}_1} \right) \underline{x}, t \right] \left| e^{-S_{1j} \underline{A}_1} \right| - p(\underline{x}, t) \quad (53)$$

Hence,

$$\frac{\partial p(\underline{x}, t)}{\partial t} = \sum_{i=1}^m \left[ \sum_j \lambda_{ij} \left[ p \left( e^{-S_{ij} \underline{A}_i} \underline{x}, t \right) \left| e^{-S_{ij} \underline{A}_i} \right| - p(\underline{x}, t) \right] \right] \quad (54)$$

Eq. (54) gives an expression for the change in  $p(\underline{x}, t)$  during the interval between  $t$  and  $t+\tau$ . The argument leading to (53) may be repeated over the interval  $t+\tau$  to  $t+2\tau$  and continuing this way, one sees that eq. (54) gives an expression for

$\frac{\partial p(\underline{x}, t)}{\partial t}$  which is valid for all time  $t$  apart from an error

that tends to zero with  $\tau$ .

Let  $M_i^*$  be a quantity typifying the order of magnitude of  $\left| \frac{M_{i,k}}{k} \right|^{1/k}$  and  $a_i$  the elements of the matrices  $\underline{A}_i$ . The

introduction of  $M_i^*$  is motivated by noting that

$$\left\langle \exp \underline{A}_i \int_t^{t+\tau} r_i(t') dt' - E \right\rangle = \sum_{k=1}^{\infty} \underline{A}_i^k \frac{M_{i,k}}{k!} \quad (55)$$

Replacing  $M_{i,k}/k$  by  $M_i^{*k}$  and  $A_i^k$  by  $a_i^k$ , the expression (55) will be of order  $\frac{a_i M_i^*}{1 - a_i M_i^*}$  and this will be small if  $a_i M_i^*$  is. \* Moreover on considering  $\delta(\tau)$  as above, one shows that  $\delta(\tau)$  will be of the second order in the small quantities  $a_i M_i^*$ . The degree to which the inequalities  $a_i M_i^* \ll 1$  hold measures the accuracy of the d.f.c. approximation. \*\*

Example 1. Rectangular Pulse. Consider the approximation of a random process consisting of a sequence of random rectangular pulses of height  $h$  and duration  $l$  occurring at a Poisson rate  $\lambda$  by a shot noise process of strength  $S = lh$  and the same rate  $\lambda$ . Assuming that  $1/\lambda \gg l$ , the correlation time may be taken to be simply  $l$ . \*\*\* Choosing  $\tau$  such that  $1/\lambda \gg \tau \gg l$ , with high probability only one pulse will occur during the time  $\tau$ , so that  $M_k = \lambda \tau S^k$ . Writing  $M^* = \max_k S \left(\frac{\lambda \tau}{k}\right)^{1/k} \approx S / \log(1/\lambda \tau)$  the inequalities then become

$$\lambda \tau e^{aS} \ll 1, \quad (56)$$

\* It should be noted that in most cases of interest  $A_i$  will contain only one or two non-zero elements so that  $a_i^k$  not  $n_i^k a_i^k$  is a closer estimate to the order of  $A_i^k$ .

\*\* It is clear that the results remain unchanged provided that the statistics of  $r_i(t)$  vary by a small fraction of themselves during the  $i$  time  $\tau$ .

\*\*\* We omit the subscript  $i$  here and below.

Example 2. Uhlenbeck-Ornstein Process. A more

significant, indeed classic, example is the replacement of an Uhlenbeck-Ornstein process by a Gaussian white noise.\*

The Uhlenbeck-Ornstein process can be described as a correlated Gaussian process of mean zero and variance  $\sigma_0^2$  with the correlation function being  $\langle r(t)r(t') \rangle = \sigma_0^2 e^{-\beta|t-t'|}$ ;  $1/\beta$  may be regarded as the correlation time. The integral

$\int_t^{t+\tau} r(t') dt'$  has mean  $M_1 = \frac{r(t)}{\beta} (1 - e^{-\beta\tau})$  and second moment

$$M_2 = \frac{2\sigma_0^2}{\beta} + \frac{r(t)^2}{\beta^2} (1 - e^{-\beta\tau})^2 + \frac{\sigma_0^2}{\beta^2} (-3 + 4e^{-\beta\tau} - e^{-2\beta\tau}) \quad (57)$$

Taking  $\tau$  long compared with the correlation time

$\frac{1}{\beta}$ ,  $M_1 \rightarrow 0$ ,  $M_2 \rightarrow \frac{2\sigma_0^2}{\beta} \tau$ , and since  $\int_t^{t+\tau} r(t') dt'$  has a Gaussian distribution,  $M_k \rightarrow 0$  (k odd),  $M_k \rightarrow \frac{(2k-1)!}{2^k k!} \left(\frac{2\sigma_0^2}{\beta} \tau\right)^{k/2}$  (k even). We may take

$M_k = \left(\frac{2\sigma_0^2}{\beta} \tau\right)^{k/2}$  and hence if  $a \left(\frac{2\sigma_0^2}{\beta} \tau\right)^{1/2} \ll 1$  the Gaussian white noise with power  $\sigma^2 = \frac{2\sigma_0^2}{\beta}$  is a valid approximation to the Uhlenbeck-Ornstein process in our stochastic differential equation.

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\* The Gaussian white noise limit here is sometimes referred to as the Einstein Brownian motion process, c.f. Ref. 9.

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