AFCRL-63-391

STATE UNIVERSITY OF NEW YORK
AT STONY BROOK

COLLEGE OF
ENGINEERING

Report No. 13

A TIME DOMAIN CHARACTERIZATION OF POSITIVE-REAL MATRICES

by

A. H. ZEMANIAN

Second Scientific Report
Contract No. AF 19(628)-2981
Project No. 5628
Task No. 562801

AUGUST 16, 1963

Prepared for

Air Force Cambridge Research Laboratories
Office of Aerospace Research
United States Air Force
Bedford, Massachusetts

AFCRL -63-391

A TIME DOMAIN CHARACTERIZATION OF POSITIVE-REAL MATRICES

A. H. ZEMANIAN

State University of New York at Stony Brook

Department of Engineering Analysis

Stony Brook, New York

Second Scientific Report

Contract No. AF 19(628)-2981

Project No. 5628

Task No. 562801

August 16, 1963

Prepared for

Air Force Cambridge Research Laboratories

Office of Aerospace Research

United States Air Force

Bedford, Massachusetts

Requests for additional copies by Agencies of the Department of Defense, their contractors, and other Government agencies should be directed to the:

DEFENSE DOCUMENTATION CENTER (DDC)
CAMERON STATION
ALEXANDRIA, VIRGINIA

Department of Defense contractors must be established for DDC services or have their 'need-to-know' certified by the cognizant military agency of their project or contract.

All other persons and organizations should apply to the:

U.S. DEPARTMENT OF COMMERCE
OFFICE OF TECHNICAL SERVICES
WASHINGTON 25, D. C.

ABSTRACT

The following time-domain characterization of any positive-real matrix is established: Let $\underset{\sim}{w}(t)$ be an $n \times n$ matrix distribution and let $\underset{\sim}{W}(s)$ be its Laplace transform. Necessary and sufficient conditions for $\underset{\sim}{W}(s)$ to be a positive-real matrix are the following.

1. $\underset{\sim}{w}(t) = \underset{\sim}{A}\,\delta^{(1)}(t) + \underset{\sim}{w}_o(t)$

Here, A is a real symmetric nonnegative-definite constant matrix and $\underset{\sim}{w}_o(t)$ is a real matrix distribution of zero order, whose support is contained in $0 \leq t < \infty$.

2. Let

$$\underset{\sim}{w}_h(t) = \tfrac{1}{2}\left[\underset{\sim}{w}(t) + \underset{\sim}{w}^T(-t)\right].$$

For every $n \times 1$ constant vector $\underset{\sim}{y}$ the quantity $\underset{\sim}{y}^* \underset{\sim}{w}_h(t)\, \underset{\sim}{y}$ is a nonnegative-definite distribution.

1. We continue in this report the discussion of time-domain characterization of positive-real matrices $W(s)$ that was presented in a previous report [1]. The present development is considerably briefer and yet more general since the assumption of rationality for $W(s)$ is no longer imposed. Indeed, a complete time-domain characterization of any arbitrary positive-real matrix, whether rational or not, is obtained and the conclusions of our previous report [1] are included as a special case.

Our development is based upon a time-domain representation for the unit impulse response matrix for a single-valued linear time-invariant continuous, causal, and passive n-port, which was presented in a previous paper [2]. An analysis for one-ports that overlaps the present work (i.e., it achieves the necessity of condition 1 of the forthcoming theorem for the case where n=1) has been given by Konig and Meixner [3,4].

We shall continue to use the notations that were employed in [1]. Moreover, we especially call the reader's attention to Sec. II of [1] wherein a summary is given of certain fundamental concepts and theorems that will be used in the subsequent discussion. This summary will not be repeated here. No other material in [1] other than Sec. II is needed for an understanding of this report.

Another concept that we shall make use of here is that of the order of a distribution. The order of a matrix distribution is that least nonnegative integer r for which the $(r + 2)$th - order primitives of the elements in the matrix are all continuous functions.

$\underset{\sim}{Q}\delta(t)$ is of zero order whereas $\underset{\sim}{A}\delta^{(1)}(t)$ is of first order. The third term on the right-hand side of (3) is a symbolic expression for that matrix distribution which assigns to each function ϕ in D the number

$$\int_{-\infty}^{\infty} d\underset{\sim}{K}(\eta)\int_{0}^{\infty}\phi^{(2)}(t)\,\frac{1-\cos\eta t}{\eta^2}(1+\eta^2)\,dt$$

$$= u(t)\int_{-\infty}^{\infty}\cos\eta t\; d\underset{\sim}{K}(\eta)\cdot\phi(t) + u(t)\int_{-\infty}^{\infty}(1-\cos\eta t)\,d\underset{\sim}{K}(\eta)\cdot\phi^{(2)}(t) \qquad (4)$$

(See [2; theorem 4]). Since the elements of

$$u(t)\int_{-\infty}^{\infty}\cos\eta t\; d\,\underset{\sim}{K}(\eta) \qquad (5)$$

are all locally integrable functions of t, (5) is a zero-order matrix distribution. Also, the elements of

$$u(t)\int_{-\infty}^{\infty}(1-\cos\eta t)\; d\underset{\sim}{K}(\eta)$$

are continuous functions everywhere and the second term in (4) defines the second distributional derivative of (6). Therefore, (6) is also a zero-order matrix distribution.

The fourth term on the right-hand side of (3) is a symbolic expression that assigns to each function ϕ in D the number,

$$\int_{-\infty}^{\infty} d\underset{\sim}{L}(\eta)\left\{\int_{0}^{\infty}\phi^{(1)}(t)\,\frac{1-\cos\eta t}{\eta}\,dt + \int_{0}^{\infty}\phi^{(2)}(t)\,\sin\eta t\,dt\right\}$$

$$= -u(t)\int_{-\infty}^{\infty}\sin\eta t\; d\underset{\sim}{L}(\eta)\cdot\phi(t) + u(t)\int_{-\infty}^{\infty}\sin\eta t\; d\underset{\sim}{L}(\eta)\cdot\phi^{(2)}(t). \qquad (7)$$

(Again, see [2; theorem 4]). Now, the elements of

$$u(t)\int_{-\infty}^{\infty}\sin\eta t\; d\underset{\sim}{L}(\eta) \qquad (8)$$

are all continuous functions for all t and it follows that both terms on the right-hand side of (7) define zero-order matrix distributions. This establishes all of condition 1.

Constructing $w_h(t)$ according to (2) and using the properties of the matrices Q, A, $K(\eta)$, and $L(\eta)$, we see that

$$w_h(t) = \frac{1}{2}\int_{-\infty}^{\infty} e^{j\eta t}(1+\eta^2)\, d\left[K(\eta) + jL(\eta)\right].$$

Therefore,

$$y^* w_h(t) y = \frac{1}{2}\int_{-\infty}^{\infty} e^{j\eta t}(1+\eta^2)\, d\left\{y^*\left[K(\eta) + jL(\eta)\right]y\right\}.$$

Since $y^*\left[K(\eta) - jL(\eta)\right]y$ is a real nondecreasing bounded function of η for all choices of y, $y^*\left[K(\eta) + jL(\eta)\right]y$ must have the same properties. Condition 2 now follows from the Bochner-Schwartz theorem. (See Sec. II of [1] for a statement of this theorem.)

<u>Sufficiency</u>: Assume that $w(t)$ satisfies conditions 1 and 2. Since $y^* w_h(t) y$ is nonnegative-definite, it is also of slow growth [5: Vol. II, p. 132]. Moreover, in forming $w_h(t)$ from $w(t)$, only those terms in the elements of $w(t)$ that are concentrated on the origin can cancel out. These terms are known to be only the delta function and a finite number of derivatives, which are also distributions of slow growth. It follows that $w(t)$ is a matrix distribution of slow growth. Therefore, the Laplace transform $W(s)$ of $w(t)$ exists and is analytic for Re $s > 0$. In addition, since $w(t)$ is real, $W(s)$ is real for real positive s.

The proof will be completed when we show that

$$y^* W_h(s) y \geq 0 \tag{9}$$

for Re $s > 0$ and for every choice of y, where

$$W_h(s) = \frac{1}{2}\left[W(s) + W^*(s)\right]. \tag{10}$$

According to the distributional inverse Laplace transformation, we may relate $\underset{\sim}{w}(t)$ to $\underset{\sim}{W}(c+jw)$ for $c > 0$ through the symbolic expression,

$$e^{-ct}w(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underset{\sim}{W}(c+j\omega)e^{j\omega t}d\omega \tag{11}$$

Since $\underset{\sim}{W}(s)$ is real for real positive s, the reflection principle shows that $\underset{\sim}{W}(c-j\omega) = \overline{\underset{\sim}{W}(c+j\omega)}$. Using this fact, we may write

$$e^{-ct}\underset{\sim}{w}(t) + e^{ct}\underset{\sim}{w}^T(-t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\underset{\sim}{W}_h(c+j\omega)\, e^{j\omega t}d\omega$$

and

$$\underset{\sim}{y}^*\left[e^{-ct}\underset{\sim}{w}(t) + e^{ct}\underset{\sim}{w}^T(-t)\right]\underset{\sim}{y} = \frac{1}{\pi}\int_{-\infty}^{\infty}\underset{\sim}{y}^*\underset{\sim}{W}_h(c+j\omega)\underset{\sim}{y}e^{j\omega t}d\omega \qquad (c > 0) \tag{12}$$

If we can show that the left-hand side of (12) is a nonnegative-definite distribution, then (9) will follow from the Bochner-Schwartz theorem and the fact that $\underset{\sim}{W}(s)$ is analytic for Re $s > 0$.

Now,

$$e^{-ct}\underset{\sim}{A}\delta^{(1)} + e^{ct}\underset{\sim}{A}^T\delta^{(1)}(-t) = 2\underset{\sim}{A}c\delta(t)$$

and, consequently, we obtain from the decomposition (1) that

$$e^{-ct}\underset{\sim}{w}(t) + e^{ct}\underset{\sim}{w}^T(-t) = 2\underset{\sim}{A}c\delta(t) + e^{-ct}\underset{\sim}{w}_0(t) + e^{ct}\underset{\sim}{w}_0^T(-t).$$

Moreover, (1) also shows that

$$2\underset{\sim}{w}_h(t) = \underset{\sim}{w}_0(t) + \underset{\sim}{w}_0^T(-t). \tag{13}$$

Since $\underset{\sim}{w}_0(t)$ is of zero order, $\underset{\sim}{w}_0(t)\cdot\phi(t)$ depends only on the values of $\phi(t)$ over $0 \le t < \infty$ and not on the derivatives of $\phi(t)$ [5: Vol. I, P. 93]. It follows that

$$e^{-ct}\underset{\sim}{w}_0(t) = e^{-c|t|}\underset{\sim}{w}_0(t).$$

Therefore,

$$\underset{\sim}{y}^*\left[e^{-ct}\underset{\sim}{w}(t) + e^{ct}\underset{\sim}{w}^T(-t)\right]\underset{\sim}{y} = 2\underset{\sim}{y}^*\underset{\sim}{A}\underset{\sim}{y}c\delta(t) + e^{-c|t|}\underset{\sim}{y}^*\left[\underset{\sim}{w}_0(t) + \underset{\sim}{w}_0^T(-t)\right]\underset{\sim}{y} \tag{14}$$

Now, $\underset{\sim}{y}^*\underset{\sim}{A}\underset{\sim}{y} \geq 0$ and $\delta(t)$ is a nonnegative-definite distribution. Therefore, the first term on the right-hand side of (14) is non-negative-definite for every $\underset{\sim}{y}$ and for $c > 0$.

Also, by condition 2 and (13), $\underset{\sim}{y}^*[\underset{\sim}{w}_0(t)+\underset{\sim}{w}_0^T(-t)]\underset{\sim}{y}$ is nonnegative-definite for every $\underset{\sim}{y}$. It is a fact that $e^{-c|t|}$ $(c > 0)$ is also non-negative-definite. We now invoke a theorem of Schwartz [5: Vol. II, p. 134, theorem XIX] to conclude that the second term on the right-hand side of (14) is nonnegative-definite for every $\underset{\sim}{y}$ and for $c > 0$. Q.E.D.

The principal conclusion of our previous report [1], which characterizes the unit impulse response matrix of any lumped linear fixed finite and passive n-port, now appears as a special case of the above theorem.

Corollary: Let $\underset{\sim}{w}(t)$ be an $n \times n$ matrix distribution and let $\underset{\sim}{W}(s)$ be its Laplace transform. Necessary and sufficient conditions for $\underset{\sim}{W}(s)$ to be positive-real and rational are the following.

1. $\underset{\sim}{w}(t) = \underset{\sim}{A}\,\delta^{(1)}(t) + \underset{\sim}{B}\delta(t) + \underset{\sim}{w}_1(t)$.

Here, $\underset{\sim}{A}$ is a real symmetric nonnegative-definite matrix, $\underset{\sim}{B}$ is a real constant matrix, and $\underset{\sim}{w}_1(t)$ is a real matrix distribution whose elements consist of finite linear combinations of terms of the form,

$$u(t)t^{\nu}e^{-\gamma t}, \tag{15}$$

where ν is a nonnegative integer and γ is a complex constant.

2. For every $n \times 1$ constant vector y and for $w_h(t)$ given by (2) the quantity $y^* w_h(t) y$ is a nonnegative-definite distribution.

When dealing with one-ports our conclusions simplify into the following.

Corollary: Let $w(t)$ be a distribution and let the function $W(s)$ be its Laplace transform. $W(s)$ is positive-real if and only if

1. $w(t) = A\delta^{(1)}(t) + w_0(t)$,

where A is a nonnegative constant and $w_0(t)$ is a real distribution of zero order with support in $0 \leq t < \infty$, and

2. the even part,

$$w_h(t) = \tfrac{1}{2}\left[w(t) + w(-t)\right], \tag{16}$$

of $w(t)$ is a nonnegative-definite distribution.

Corollary: Let $w(t)$ be a distribution and let the function $W(s)$ be its Laplace transform. $W(s)$ is positive-real and rational if and only if

1. $w(t) = A\delta^{(1)}(t) + B\delta(t) + w_1(t)$,

where A and B are real constants with $A \geq 0$ and $w_1(t)$ is a real distribution that consists of a finite linear combination of terms of the form (15), and

2. the even part (16) of $w(t)$ is a nonnegative-definite distribution.