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A TIME DOMAIN CHARACTERIZATION OF RATIONAL POSITIVE-REAL MATRICES

by

A. H. ZEMANIAN

First Scientific Report
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Prepared for

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ABSTRACT

The following time-domain characterization of any rational positive-real matrix is established: Let $\underset{\sim}{w}(t)$ be an nxn matrix distribution having its support bounded on the left and let $\underset{\sim}{W}(s)$ be its Laplace transform. The necessary and sufficient conditions for $\underset{\sim}{W}(s)$ to be a rational positive-real matrix are the following.

1. $\underset{\sim}{w}(t) = \underset{\sim}{k}_1 \delta^{(1)}(t) + \underset{\sim}{k}_0 \delta(t) + \underset{\sim}{w}_1(t)$

Here, $\underset{\sim}{k}_1$ is a real symmetric nonnegative-definite constant matrix, $\underset{\sim}{k}_0$ is a real constant matrix, and $\underset{\sim}{w}_1(t)$ is a real matrix distribution whose elements consist of finite linear combinations of terms of the form,

$$u(t)\, t^{\nu} e^{-\gamma t} ,$$

where ν is a nonnegative integer and γ is a complex constant.

2. Let

$$\underset{\sim}{w}_h(t) = \frac{1}{2}\,[\underset{\sim}{w}(t) + \underset{\sim}{w}^T(-t)] .$$

For every nx1 constant vector $\underset{\sim}{y}$ the quantity $\underset{\sim}{y}^{*}\underset{\sim}{w}_h(t)\underset{\sim}{y}$ is a nonnegative-definite distribution.

I INTRODUCTION

Let w(t) denote the distributional inverse Laplace transform of a rational function W(s), where the corresponding region of convergence is taken to be a half-plane extending infinitely to the right. w(t) will be, in general, a distribution because of the possible delta function and its derivatives at the origin. If we assume, in addition, that w(t) is an ordinary function that satisfies certain conditions on its integrability, then it can be shown that a necessary and sufficient condition for W(s) to be positive-real is that the even part $w_h(t)$ of w(t) be a nonnegative-definite function [1,2]. Without this additional assumption, one is forced to use distribution theory or an equivalent theory of generalized functions in order to perform an analagous analysis. Moreover, it turns out in this general case that the nonnegative-definiteness of $w_h(t)$ is necessary but no longer sufficient for W(s) to be positive-real.

The question arises, therefore, as to what additional conditions must be imposed on w(t) in order to insure the positive-reality of W(s). The answer is given in theorem 3 below. A similar result for positive-real matrices is given in theorem 4. In view of the correspondence between positive-reality and the passivity of physically realizable networks, these two theorems give a complete time-domain characterization of lumped linear fixed finite and passive n-ports. This is the objective of this report.

More recently, a time-domain characterization for any positive-real function or matrix, which need not be rational, has been obtained and will be presented in the next report. It contains the results stated in theorems 3 and 4 below, as special cases. The discussion for rational positive-real functions or matrices that is given here is considerably more detailed than the more powerful development of our next report. Indeed, we present here a discussion of what types of distributions are generated at the poles of a rational function. As a result, we obtain a specific relationship between the fact that a positive-real function or matrix must have only simple imaginary poles that satisfy a certain residue condition, and the fact that the unit impulse response function or matrix generates a nonnegative-definite distribution. (See lemmas 1 and 3 below.)

II SOME PRELIMINARY CONSIDERATIONS

This report uses the same notations and definitions that were employed in some previous ones [3,4]. The elements of distribution theory that are needed for an understanding of this work are summarized in the appendix of [3].

The definitions of positive-real functions and matrices that are employed here are the following.

The function $W(s)$ of the complex variable, $s = \sigma + jw$, is said to be positive-real if the following conditions are satisfied in the open right-half s-plane (i.e., for Re $s > 0$).

1. $W(s)$ is analytic.

2. $W(s)$ is real whenever s is real.

3. $\text{Re}\, W(s) \geq 0$.

The nxn matrix $\underset{\sim}{W}(s)$ is said to be positive-real if the following conditions are satisfied in the open right-half s-plane.

1. $\underset{\sim}{W}(s)$ is analytic.

2. $\underset{\sim}{W}(s)$ is real whenever s is real.

3. The hermitian part,

$$\underset{\sim}{W}_h(s) \triangleq \frac{1}{2} [\underset{\sim}{W}(s) + \underset{\sim}{W}^*(s)],$$

of $\underset{\sim}{W}(s)$ is the matrix of a nonnegative-definite hermitian form.

In the sequel we shall say that a matrix is hermitian and nonnegative-definite if it is the matrix of a nonnegative-definite hermitian form.

We shall subsequently make use of the symbolic expression,

$$\int_{-\infty}^{\infty} e^{j\omega t} dG(\omega),$$

where $G(\omega)$ is the sum of a distribution $G_1(\omega)$ of bounded support and a function $G_2(\omega)$ of slow growth that is of bounded variation over every finite interval. It denotes the complex conjugate of the distributional Fourier transform of $G(\omega)$. This is the distribution of slow growth that assigns to each testing function ϕ in D the value,

$$\int_{-\infty}^{\infty} [G_1^{(1)}(\omega) \cdot e^{j\omega t}]\, \phi(t) dt + \lim_{A \to \infty} \int_{-\infty}^{\infty} \phi(t) \int_{-A}^{A} e^{j\omega t} dG_2(\omega) dt .$$

The following representations were established in [3,4].

Theorem 1: The function W(s) is positive-real if and only if its inverse Laplace transform w(t) for the half-plane, Re s > 0, has the symbolic representation,

$$w(t) = A\delta^{(1)}(t) + u(t) \int_{-\infty}^{\infty} (1 + \eta^2) \cos \eta t \, dK(\eta),$$

where A is a real nonnegative constant, K(η) is a real odd non-decreasing bounded function, and u(t) is the unit step function.

Theorem 2: The nxn matrix $\underset{\sim}{W}(s)$ is positive-real if and only if its inverse Laplace transform $\underset{\sim}{w}(t)$ for the half-plane, Re s > 0, has the symbolic representation,

$$\underset{\sim}{w}(t) = \underset{\sim}{Q}\delta(t) + \underset{\sim}{A}\delta^{1}(t) + u(t) \int_{-\infty}^{\infty} (1 + \eta^2) \cos \eta t \, d\underset{\sim}{K}(\eta)$$

$$+ \int_{-\infty}^{\infty} \left(\frac{d^2}{dt^2} - 1\right) [u(t) \sin \eta t] \, d\underset{\sim}{L}(\eta) ,$$

where $\underset{\sim}{Q}$ is a real constant skew-symmetric nxn matrix, $\underset{\sim}{A}$ is a real constant symmetric nonnegative-definite nxn matrix, and the nxn matrices, $\underset{\sim}{K}(\eta)$ and $\underset{\sim}{L}(\eta)$ are real and possess the following properties.

1. $\underset{\sim}{K}(\eta) = - \underset{\sim}{K}(-\eta)$, $\underset{\sim}{L}(\eta) = \underset{\sim}{L}(-\eta)$

2. $\underset{\sim}{K}^T(\eta) = \underset{\sim}{K}(\eta)$, $\underset{\sim}{L}^T(\eta) = - L(\eta)$

3. For every constant nx1 vector $\underset{\sim}{y}$,

   $\underset{\sim}{y}^*[\underset{\sim}{K}(\eta) - jL(\eta)]\underset{\sim}{y}$

is a real nondecreasing bounded function of η .

A distribution f(t) is said to be nonnegative-definite if for every testing function $\phi(t)$ in D the quantity,

$$\int_{-\infty}^{\infty} \overline{\phi(t)}[f(\tau) \cdot \phi(t-\tau)]dt = f(t) \cdot \int_{-\infty}^{\infty} \overline{\phi(t)}\phi(t-\tau)dt \qquad (2\text{-}1)$$

is nonnegative [5; Vol. II, p. 131]. These distributions can also be characterized as follows. (For a proof see [1; Vol. II, pp. 132-133].)

The Bochner-Schwartz theorem: In order for the distribution f(t) to be nonnegative-definite it is necessary and sufficient that symbolically

$$f(t) = \int_{-\infty}^{\infty} e^{j\omega t} dB(\omega) , \qquad (2\text{-}2)$$

where $B(\omega)$ is a real nondecreasing function of slow growth.

As a very simple example, consider the delta function $\delta(t)$. It is nonnegative-definite because

$$\int_{-\infty}^{\infty} \overline{\phi(t)} [\delta(\tau) \cdot \phi(t-\tau)]dt = \int_{-\infty}^{\infty} \overline{\phi(t)}\phi(t)dt \geq 0.$$

Moreover, its Fourier transform is the function that equals one everywhere so that the corresponding inverse Fourier transform assumes the form required by the Bochner-Schwartz theorem.

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega . \qquad (2\text{-}3)$$

Every nonnegative-definite distribution must be of slow growth because the Fourier transformation is a mapping of S' onto S'.

Finally, let us point out certain properties of the function log s and $1/s^n$ $(n = 1,2,3,\ldots)$, which we shall employ later on. First of all, consider the principal branch of log s in the closed right-half s-plane. On the vertical line, $s = c + j\omega$ $(c > 0)$,

$$\log s = \frac{1}{2} \log (c^2 + \omega^2) + j \tan^{-1} \frac{\omega}{c} .$$

As $c \to 0+$, this function converges in D' to the regular distribution,

$$\lim_{c \to 0+} \log (c + j\omega) = \begin{cases} \log |\omega| + \frac{j\pi}{2} & (\omega > 0) \\ \log |\omega| - \frac{j\pi}{2} & (\omega < 0) . \end{cases} \tag{2-4}$$

Since any sequence of distributions that converges in D' can be differentiated to yield a sequence of derivatives, which converge in D' to the derivative of the limit of the original sequence, we may differentiate (2-4) n times to obtain

$$\lim_{c \to 0+} \frac{1}{(c+j\omega)^n} = \frac{j^{n-1}\pi}{(n-1)!} \delta^{(n-1)}(\omega) + F_p \frac{1}{(j\omega)^n} \tag{2-5}$$

Here, the second term on the right-hand side is the singular distribution generated by Hadamard's "finite part" of a divergent integral, as follows:

$$< F_p \frac{1}{(j\omega)^n} , \phi(\omega) > = F_p \int_{-\infty}^{\infty} \frac{\phi(\omega)}{(j\omega)^n} d\omega$$

$$= (-j)^n \lim_{\varepsilon \to 0+} \left\{ (-1)^n \left[\int_{-\infty}^{-\varepsilon} \frac{\phi(\omega)}{|\omega|^n} d\omega - \sum_{\nu=0}^{n-2} \frac{(-1)^\nu \phi^{(\nu)}(0)}{\nu!(n-1-\nu)\varepsilon^{n-1-\nu}} \right.\right.$$

$$\left. + \frac{(-1)^{n-1}\phi^{(n-1)}(0)}{(n-1)!} \log \varepsilon \right]$$

$$\left. + \int_{\varepsilon}^{\infty} \frac{\phi(\omega)}{\omega^n} d\omega - \sum_{\nu=0}^{n-2} \frac{\phi^{(\nu)}(0)}{\nu!(n-1-\nu)\varepsilon^{n-1-\nu}} + \frac{\phi^{(n-1)}(0)}{(n-1)!} \log \varepsilon \right\}$$

$$= (-j)^n \lim_{\varepsilon \to 0+} \left\{ \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{\phi(\omega)}{\omega^n} d\omega - \sum_{\nu=0}^{n-2} [1 + (-1)^{n-\nu}] \frac{\phi^{(\nu)}(0)}{\nu!(n-1-\nu)\varepsilon^{n-1-\nu}} \right\}$$

$$(n = 1,2,3,\ldots)$$

(The summation on the right-hand side is absent when n = 1).

III CERTAIN FOURIER-STIELTJES TRANSFORMS

Our objective in this section is to establish the following lemma.

Lemma 1: Let $W(s)$ be a real rational function of the form,

$$W(s) = k_1 s + k_0 + W_1(s) \ , \qquad (3\text{-}1)$$

where $W_1(s)$ has more poles than zeros and is analytic for $\operatorname{Re} s > 0$. In order for the even part,

$$w_h(t) \triangleq \frac{1}{2} [w(t) + w(-t)],$$

of the unit impulse response $w(t)$ corresponding to $W(s)$ to be representable by the symbolic expression,

$$w_h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} dG_0(\omega), \qquad (3\text{-}2)$$

where $G_0(\omega)$ is a nondecreasing ordinary function, all the imaginary poles of $W(s)$ must be simple and have real positive residues.

If we perform a partial fraction expansion on $W_1(s)$, we convert it into a finite sum of terms, each of which has the form,

$$\frac{a}{(s-\alpha-j\beta)^{\mu}},$$

where a is, in general, a complex constant, α and β are real constants with α nonpositive, and μ is a positive integer. Because $W_1(s)$ has real coefficients, a must be real whenever β is zero. In addition, when β is not zero, there will be another term in the partial fraction expansion whose pole is at $s = \alpha - j\beta$ and whose constant multiplier is $\bar{a}$. Hence, $W_1(s)$ is a finite sum of terms, each of which has either the form,

$$F(s) = \frac{a}{(s - \alpha)^{\mu}} \quad (a \text{ real}), \tag{3-3}$$

or the form,

$$H(s) = \frac{a}{(s - \alpha \; j\beta)^{\mu}} + \frac{\bar{a}}{(s - \alpha + j\beta)^{\mu}} . \tag{3-4}$$

(For definiteness, we shall assume henceforth that $\beta > 0$.) The time function corresponding to (3-3) is

$$f(t) = \frac{a}{(\mu - 1)!} t^{\mu - 1} e^{\alpha t} u(t) \tag{3-5}$$

and that corresponding to (3-4) is

$$h(t) = \frac{2|a|}{(\mu - 1)!} t^{\mu - 1} e^{\alpha t} \cos [\beta t + \arg a] u(t) \tag{3-6}$$

where u(t) is the unit step function

$$u(t) = \begin{cases} 0 & (t < 0) \\ \frac{1}{2} & (t = 0) \\ 1 & (t > 0). \end{cases}$$

Actually, we are interested in the contribution that (3-3) and (3-4) make to the function

$$G_0(\omega) = \int_0^{\omega} \operatorname{Re} W(c + ix)\, dx \quad (c > 0) \tag{3-7}$$

and we wish to determine what happens to these contributions as $c \to 0+$. We also wish to determine the effect of these contributions on

$$\frac{1}{2} [w(t)\, e^{-ct} + w(-t)\, e^{ct}] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} dG_c(\omega) \tag{3-8}$$

in the limit as $c \to 0+$.

When $\alpha < 0$, the contributions of (3-3) and (3-4) to $G_c(\omega)$ are bounded continuous (in fact, infinitely differentiable) functions. Since the $s = j\omega$ axis is in the interior of the region of convergence of the Laplace transforms of (3-5) and (3-6), these contributions continue to possess the same properties when $c = 0$. For the same reason, if these contributions are substituted for $G_c(\omega)$ in (3-8) and the limit $c \longrightarrow 0+$ is taken, the Fourier-Stieltjes transform corresponding to (3-2) will still be a valid expression. In other words, in these simple cases there is no difficulty in interchanging the limit $c \longrightarrow 0+$ with the integration in (3-8).

Next, let us consider the terms corresponding to simple imaginary poles of W(s). In this case, α equals zero and μ equals one. The corresponding contribution of (3-3) to (3-7) is the principal branch of

$$a \tan^{-1}\frac{\omega}{c} \quad (a \text{ real}, c > 0). \qquad (3\text{-}9)$$

Replacing $G_c(\omega)$ in (3-8) by (3-9), we obtain

$$\frac{a}{2} e^{-c|t|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d(a \tan^{-1}\frac{\omega}{c}). \qquad (3\text{-}10)$$

Now, as $c \longrightarrow 0+$, the principal branch of (3-9) becomes the single step function,

$$a\pi[u(\omega) - \tfrac{1}{2}] = \begin{cases} -\frac{a\pi}{2} & (\omega < 0) \\ 0 & (\omega = 0) \\ \frac{a\pi}{2} & (\omega > 0) \end{cases} \qquad (3\text{-}11)$$

Inserting this function into the right-hand side of (3-10), the resulting Stieltjes integral becomes a/2. But, this is precisely the value that the left-hand side of (3-10) becomes as $c \to 0+$. Hence, here too we can interchange the limit $c \to 0+$ and the integration on the right-hand side of (3-10).

Considering again the case where $\alpha = 0$ and $\mu = 1$ and setting $a = a_1 + ja_2$, we find that the contribution of (3-4) to (3-7) is

$$a_1 \left[\tan^{-1} \frac{\omega-\beta}{c} + \tan^{-1} \frac{\omega+\beta}{c}\right] + \frac{a_2}{2} \ln \frac{c^2+(\omega-\beta)^2}{c^2+(\omega+\beta)^2} . \tag{3-12}$$

As $c \to 0+$, the first term in (3-12) becomes the monotonic step function,

$$a_1\pi[u(\omega - \beta) + u(\omega + \beta) - 1] = \begin{cases} -a_1\pi & (\omega < -\beta) \\ \frac{-a_1\pi}{2} & (\omega = -\beta) \\ 0 & (-\beta < \omega < \beta) \\ \frac{a_1\pi}{2} & (\omega = \beta) \\ a_1\pi & (\omega > \beta) \end{cases} \tag{3-13}$$

whereas the second term approaches

$$a_2 \ln \left|\frac{\omega - \beta}{\omega + \beta}\right| . \tag{3-14}$$

Now, (3-14) is a nonmonotonic unbounded function, having logarithmic singularities at $\omega = \pm \beta$. We shall use this fact later.

If we replace $G_0(\omega)$ in (3-2) by the sum of (3-13) and (3-14) and if the resulting Fourier-Stieltjes integral is interpreted in terms of distribution theory, the result will be the even part of (3-6). On the other hand, if $a_2 = 0$, we need not resort to distribution theory to obtain the even part of (3-6); here again, the limit $c \longrightarrow 0+$ and the integration in the Fourier-Stieltjes transform can be interchanged in the classical sense.

Finally, we shall discuss the terms corresponding to the multiple imaginary poles of W(s). For terms of the form a/s^{μ} (a real, $\mu = 2,3,4,\ldots$) we have from (2-5) that

$$\mathrm{Re}\left\{\lim_{c\to 0+} \frac{a}{(c + j\omega)^{\mu}}\right\} = \begin{cases} \dfrac{a(-1)^{\frac{\mu-1}{2}}\pi}{(\mu-1)!}\,\delta^{(\mu-1)}(\omega) & (\mu \text{ odd}) \\ \\ F_p\,\dfrac{a(-1)^{\frac{\mu}{2}}}{\omega^{\mu}} & (\mu \text{ even}). \end{cases} \qquad (3\text{-}15)$$

Therefore, as $c \longrightarrow 0+$ the regular distributions corresponding to

$$\int_0^{\omega} \mathrm{Re}\, \frac{a}{(c + jx)^{\mu}}\, dx$$

approach the generalized functions,

$$a\,\frac{(-1)^{\frac{\mu-1}{2}}\pi}{(\mu-1)!}\,\delta^{(\mu-2)}(\omega) \qquad (\mu \text{ odd}), \qquad (3\text{-}16)$$

and

$$F_p\,\frac{a(-1)^{\frac{\mu}{2}-1}}{(\mu-1)\omega^{\mu-1}} \qquad (\mu \text{ even}). \qquad (3\text{-}16a)$$

Similarly, for terms of the form

$$\frac{a}{(s - j\beta)^n} + \frac{\bar{a}}{(s + j\beta)^n} \qquad (a = a_1 + j\, a_2)$$

The corresponding limits of the regular distributions, generated by

$$\int_0^\omega \mathrm{Re}\left\{\frac{a}{[c + j(x - \beta)]^\mu} + \frac{\bar{a}}{[c + j(x + \beta)]^\mu}\right\} dx$$

as $c \longrightarrow 0+$, are

$$(-1)^{\frac{\mu-1}{2}}\left\{\frac{a_1\pi}{(\mu-1)!}\left[\delta^{(\mu-2)}(\omega-\beta) + \delta^{(\mu-2)}(\omega+\beta)\right] + F_p\frac{a_2}{\mu-1}\left[\frac{1}{(\omega-\beta)^{\mu-1}} - \frac{1}{(\omega+\beta)^{\mu-1}}\right]\right\} \quad (\mu \text{ odd}) \qquad (3\text{-}17)$$

and

$$(-1)^{\frac{\mu}{2}-1}\left\{F_p\frac{a_1}{\mu-1}\left[\frac{1}{(\omega-\beta)^{\mu-1}} + \frac{1}{(\omega+\beta)^{\mu-1}}\right] - \frac{a_2\pi}{(\mu-1)!}\left[\delta^{(n-2)}(\omega-\beta) - \delta^{(n-2)}(\omega+\beta)\right]\right\} \quad (\mu \text{ even}). \qquad (3\text{-}17a)$$

None of the expressions given by (3-16), (3-16a), (3-17), and (3-17a) are ordinary functions and, for all ω (except the singular point, $\omega = \pm\,\beta$), the terms $F_p\, 1/(\omega \pm \beta)^n$ are nonmonotonic and unbounded. In order to obtain the even parts of (3-5) or (3-6) by substituting (3-16), (3-16a) or (3-17), respectively, for $G_0(\omega)$ in (3-2), we must use distribution theory, the classical theory being inadequate in this case.

The time function corresponding to $k_1 s$ is $k_1\delta^{(1)}(t)$ and the corresponding contribution to $G_c(\omega)$ is $k_1 ca$, which approaches zero as $c \rightarrow 0+$. This is reflected in the fact that $\delta^{(1)}(t)$

is an odd distribution.

Similarly, the time function for the constant term k_0 in (3-1) is $k_0\delta(t)$. Its contribution to $G_0(\omega)$ is $k_0\omega$; $\delta(t)$ is an even distribution.

Turning now to the proof of lemma 1, let us first observe that for any real rational function W(s) the function,

$$G_0(\omega) = \lim_{c \to 0+} G_c(\omega) = \lim_{c \to 0+} \int_0^{\omega} \operatorname{Re} W(c + jx)\, dx \quad (c > 0),$$

is a finite sum of terms of the forms (3-11), (3-13), (3-14), (3-16), (3-16a), (3-17), and (3-17a), plus possibly $k_0\omega$. Furthermore, we may relate the even part $w_h(t)$ of the unit impulse response to $G_0(\omega)$ through (3-2) if we interpret (3-2) as a symbolic expression for a distribution.

For $G_0(\omega)$ to be a nondecreasing ordinary function, it is clearly necessary that terms of the form (3-14), (3-16), (3-16a), (3-17), and (3-17a) must not appear in $G_0(\omega)$. This means that every imaginary pole of W(s) must be simple and the corresponding residue must be real. In fact, from (3-11), (3-13), and (3-14) we see that these residues must be positive. This establishes lemma 1.

IV FUNCTIONS WHOSE LAPLACE TRANSFORMS ARE POSITIVE-REAL

We shall now develop the properties of w(t) which completely characterize the fact that W(s) is a rational positive-real function. The precise result is as follows.

Theorem 3: Let w(t) be a distribution having its support bounded on the left and let W(s) be its Laplace transform. The necessary and sufficient conditions for W(s) to be a rational positive-real function are the following.

1. $w(t) = k_1 \delta^{(1)}(t) + k_0 \delta(t) + w_1(t)$

Here, $k_1 \geq 0$, k_0 is real, and $w_1(t)$ is a real distribution that consists of a finite linear combination of terms of the form,

$$u(t)\, t^{\mu} e^{-\gamma t} \; , \tag{4-1}$$

where μ is a nonnegative integer and γ is a complex constant.

2. The even part $w_h(t)$ of $w(t)$,

$$w_h(t) \triangleq \frac{1}{2} [w(t) + w(-t)] \; , \tag{4-2}$$

is a nonnegative-definite distribution.

Note that condition 2 above does not suffice to insure the positive-reality of $W(s)$ since $-\delta^{(2)}(t)$ is nonnegative-definite and yet its Laplace transform $-s^2$ is not positive-real. This is in contrast to certain special cases where a classical analysis is possible [1,2].

Proof: Necessity: Assume that $W(s)$ is rational and positive-real. Since $W(s)$ is real for real s in the region of convergence, $W(\bar{s}) = \overline{W(s)}$ there and, consequently, $w(t)$ is real. Furthermore, if $W(s)$ has a pole at $s = \infty$, it must be simple and its residue k_1 must be positive. Expanding $W(s)$ into partial fractions and applying the inverse Laplace transformation, we completely establish condition 1.

Furthermore, $w(t)$ has the form stated in theorem 1. Taking its even part, we obtain

$$w_h(t) = \int_{-\infty}^{\infty} e^{j\eta t} dH(\eta) \; ,$$

where $H(\eta)$ is the real odd nondecreasing function of slow growth given by

$$H(\eta) = \frac{1}{2} \int_0^{\eta} (1 + \chi^2)\, d\, K(\chi) \;.$$

By the Bochner-Schwartz theorem, $w_h(t)$ is a nonnegative-definite distribution.

Sufficiency: Assume that $w(t)$ satisfies conditions 1 and 2. Since $w(t)$ is real, $W(s)$ is real for real s in the region of convergence. Moreover, $W(s)$ is a rational function of the form,

$$W(s) = k_1 s + k_0 + W_1(s),$$

where $W_1(s)$ has more poles than zeros in the finite s-plane. $W_1(s)$ cannot have any poles with positive-real parts. For, if it did, $w(t)$ would contain at least one term of the form (4-1) such that Re > 0. This would mean that $w_h(t)$ could not be of slow growth and, therefore, nonnegative-definite, which would violate condition 2. Thus, the region of convergence contains the open right-half s-plane, which implies the analyticity of $W(s)$ therein.

We can convert the distributional inverse Laplace transform of $W(s)$ into the symbolic equation,

$$\frac{1}{2}\left[w(t)e^{-ct}+w(-t)e^{ct}\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\left[ck_1\omega + k_0\omega + G_{1c}(\omega)\right] \qquad (4\text{-}3)$$

$$(c > 0) \;,$$

where

$$G_{1c}(\omega) = \int_0^{\omega} \text{Re}\, W_1\, (c + j\chi)\, d\chi. \qquad (4\text{-}4)$$

As $c \to 0+$, the left-hand side of (3) converges in D' to $w_h(t)$. Interchanging the limit process $c \to 0+$ with the integration on the right-hand side, we obtain the symbolic equation,

$$w_h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d[k_0\omega + G_{10}(\omega)],$$

where

$$G_{10}(\omega) = \lim_{c \to 0+} G_{1c}(\omega) .$$

By condition 2 and the Bochner-Schwartz theorem, $k_0\omega + G_{10}(\omega)$ is a nondecreasing function. It now follows from lemma 1 that W(s) has simple poles on the s = jω axis with positive residues. Also, at every point on the s = jω axis where $W_1(s)$ is analytic, $G_{10}(\omega)$ is a primitive of Re $W_1(j\omega)$. Thus,

$$k_0 + \text{Re } W_1(j\omega) \geq 0 .$$

Applying the maximum-modulus theorem to k_0 + Re $W_1(s)$ over the right-half s-plane in the usual way, we see that k_0 + Re $W_1(s) \geq 0$ for Re s > 0. Hence, $k_0 + W_1(s)$ is positive-real. Since $k_1 \geq 0$, W(s) is also positive-real. Q.E.D.

V THE FOURIER-STIELTJES TRANSFORMS OF CERTAIN MATRICES

We now turn to the extension of these results to matrices. We shall subsequently make use of the sufficiency part of the following known result.

Lemma 2: Let $\underset{\sim}{W}(s)$ be an n x n matrix whose elements are rational functions. The necessary and sufficient conditions for $\underset{\sim}{W}(s)$ to be positive-real are the following.

1. $\underset{\sim}{W}(s)$ is analytic for Re s > 0 and real for s real.
2. Every imaginary pole (including the possible one at s = ∞) is simple so that $\underset{\sim}{W}(s)$ is a finite sum of terms of the form,

$$\underset{\sim}{W}(s) = \underset{\sim}{k}_1 s + \underset{\sim}{k}_0 + \frac{\underset{\sim}{a}_0}{s} + \sum_{\nu=1}^{p} \frac{\underset{\sim}{a}_\nu}{s - j\beta_\nu} + \frac{\bar{\underset{\sim}{a}}_\nu}{s + j\beta_\nu} + \underset{\sim}{W}_a(s), \tag{5-1}$$

where the β_ν are real and positive and $\underset{\sim}{W}_a(s)$ is analytic for Re s ≥ 0 and has a zero at $s = \infty$. Furthermore, k_1 and a_0 are nonnegative-definite real symmetric matrices and the a_ν ($\nu = 1,2,\ldots,p$) are non-negative-definite hermitian matrices.

3. For $s = j\omega$ the hermitian part of $k_0 + W_a(s)$, given by

$$\frac{1}{2}\,[k_0 + k_0^T + W_a(j\omega) + W_a^*(j\omega)]\ ,$$

is a nonnegative-definite hermitian matrix.

Proof: Necessity: Let $W(s)$ be positive-real. The first condition holds by the definition of positive-real matrices.

If $W(s)$ has a pole of multiplicity μ at $s = \infty$, the following asymptotic expression holds as $s \to \infty$.

$$W(s) \sim k_\mu s^\mu$$

Here, the matrix of coefficients k_μ is real since $W(s)$ is real for s real. The hermitian part $W_h(s)$ of $W(s)$ is given by

$$W_h(s) = \frac{1}{2}\,[W(s) + W^*(s)]$$

Letting y be any $n \times 1$ constant vector, we have that as $s \to \infty$ the hermitian form $y^* W_h(s)\, y$ is asymptotic to

$$\frac{1}{2}\,[y^* k_\mu y(\sigma + j\omega)^\mu + y^* k_\mu^T\, y\ (\sigma - j\omega)^\mu] \tag{5-2}$$

Since $y^* k_\mu^T y$ is the complex conjugate of $y^* k_\mu y$, (5-2) is equal to

$$\mathrm{Re}\ [y^* k_\mu y\ (\sigma + j\omega)^\mu] \tag{5-3}$$

and this expression cannot be nonnegative for Re $s > 0$ in every neighborhood of $s = \infty$ and for every y if $\mu > 1$. On the other hand, if $\mu = 1$, (5-3) is nonnegative for Re $s > 0$ in every neighborhood of $s = \infty$ and for every y if and only if k_μ is a nonnegative-definite real symmetric matrix. Therefore, for $y^* W_h(s) y$ to be a nonnegative-definite hermitian form for Re $s > 0$, it is necessary that any pole of $W(s)$ at $s = \infty$ be simple and that the corresponding matrix of residues k_1 be a nonnegative-definite real symmetric form.

If $W(s)$ has a pole at the origin, we can come to a similar conclusion by constructing essentially the same argument in the vicinity of $s = 0$.

Now, let us assume that $W(s)$ has a pair of imaginary poles at $s = \pm j\beta_\nu$ $(\beta_\nu > 0)$ of multiplicity μ. The contribution of the pole at $s = j\beta_\nu$ to $y^* W_h(s) y$ is

$$\frac{y^* a_\nu y}{2(s - j\beta_\nu)^\mu} + \frac{\overline{y^* a_\nu^* y}}{2(s - j\beta_\nu)^\mu}, \qquad (5\text{-}4)$$

and, as $s \rightarrow j\beta_\nu$, $u^* W_h(s) u$ is asymptotic to this expression. Since $y^* a^* y$ is the complex conjugate of $y^* a y$, (5-4) simplifies into

$$\mathrm{Re}\,\frac{y^* a_\nu y}{(s - j\beta_\nu)^\mu} \qquad (5\text{-}5)$$

As before, this expression is nonnegative for $\mathrm{Re}\, s > 0$ in every neighborhood of $s = j\beta_\nu$ and for every y if and only if $\mu = 1$ and a_ν is a nonnegative-definite hermitian form. The same result holds for the pole at $s = -j\beta_\nu$.

Since $W(s)$ is a rational, the conditions developed so far show that it can be expanded into a finite sum of terms of the form (5-1). Thus, condition 2 is completely established.

To verify the last condition, let us first note that we have from

$$y^* W_h(s) y \geq 0 \qquad (\mathrm{Re}\ s > 0) \qquad (5\text{-}6)$$

by continuity that, at all points of the $s = j\omega$ axis, except possibly at the imaginary poles,

$$y^* W_h(j\omega) y \geq 0 \quad . \tag{5-7}$$

Referring to (5-5) and using the fact that $y^* a_\nu y$ is real and positive, we see that the contribution of the partial fraction for any simple pole at $s = j\beta_\nu$ to $W_h(j\omega)$ is zero. By the same argument, this assertion holds for any simple pole that may occur at either $s = 0$, $s = -j\beta_\nu$, or $s = \infty$. Because of these results, it follows that, whenever $W(j\omega)$ is analytic,

$$y^* W_h(j\omega) y = \frac{y^*}{2} [k_0 + k_0^T + W_a(j\omega) + W_a^*(j\omega)] y \geq 0 \quad .$$

This result can be extended to all imaginary values of s because of the continuity of $W_a(j\omega)$.

<u>Sufficiency:</u> Because of condition 1, we need merely show that (5-6) holds. Let y be any given constant $n \times 1$ vector and consider a closed path consisting of a portion of the $s = j\omega$ axis and a semicircle in the right-half s-plane of radius r and center at the origin. Since $W_a(s)$ is rational and $W_a(\infty)$ is zero, the quantity,

$$\left| y^* W_{ah}(s) y \right| \doteq \left| \frac{y^*}{2} [W_a(s) + W_a^*(s)] y \right| ,$$

uniformly approaches zero on the semicircle as $r \longrightarrow \infty$. Furthermore, since $W_a(s)$ is analytic for $\text{Re}\ s \geq 0$,

$$\frac{y^*}{2} [k_0 + k_0^T + W_a(s) + W_a(s)] \ y \tag{5-8}$$

is a finite linear combination of harmonic functions for $\text{Re}\ s \geq 0$. By condition 3 it is nonnegative on the $s = j\omega$ axis and approaches the nonnegative quantity,

$$\frac{y^*}{2} (k_0 + k_0^T) y^* \ ,$$

uniformly on the stated semicircle as $r \to \infty$. Consequently, we may apply the maximum-modulus theorem to (5-8) to conclude that (5-8) is nonnegative for Re $s > 0$.

Secondly, the contributions of the imaginary poles (including the possible one at $s = \infty$) to $\underset{\sim}{y}^* \underset{\sim}{W}_h(s) \underset{\sim}{y}$ are also nonnegative as the following relations show. Since $\underset{\sim}{a}_0$ and $\underset{\sim}{k}_1$ are nonnegative-definite real symmetric matrices,

$$\frac{\underset{\sim}{y}^*}{2}\left[\frac{\underset{\sim}{a}_0}{s} + \frac{\underset{\sim}{a}_0'}{\bar{s}}\right]\underset{\sim}{y} = \frac{\sigma}{|s|^2}\, \underset{\sim}{y}^* \underset{\sim}{a}_0 \underset{\sim}{y} \geq 0$$

$$\frac{\underset{\sim}{y}^*}{2}\left[\underset{\sim}{k}_1 s + \underset{\sim}{k}_1^T \bar{s}\right]\underset{\sim}{y} = \sigma\, \underset{\sim}{y}^* \underset{\sim}{k}_1 \underset{\sim}{y} \geq 0 .$$

Also, since $\underset{\sim}{a}_\nu$ is a nonnegative-definite hermitian matrix,

$$\frac{\underset{\sim}{y}^*}{2}\left[\frac{\underset{\sim}{a}_\nu}{s - j\beta_\nu} + \frac{\underset{\sim}{a}_\nu^*}{\bar{s} - j\bar{\beta}_\nu}\right]\underset{\sim}{y} = \frac{\sigma}{|s - j\beta|^2}\, \underset{\sim}{y}^* \underset{\sim}{a}_\nu \underset{\sim}{y} \geq 0,$$

whereas a similar expression holds for the simple pole at $s = j\beta_\nu$.

Thus, we have shown that every contribution to $\underset{\sim}{y}^* \underset{\sim}{W}_h(s) \underset{\sim}{y}$ is nonnegative for Re $s > 0$. Q.E.D.

We shall now develop an extension of lemma 1 for application to matrices. Following the notation of Sec. II, we shall let $\underset{\sim}{w}(t)$ denote an n x n matrix of unit impulse responses that are all zero for $t < 0$. The quantity $\underset{\sim}{w}_h(t)$ is defined by

$$\underset{\sim}{w}_h(t) \triangleq \frac{1}{2}\left[\underset{\sim}{w}(t) + \underset{\sim}{w}^T(-t)\right]$$

As usual, $\underset{\sim}{W}(s)$ denotes the Laplace transform of $\underset{\sim}{w}(t)$ and

$$\underset{\sim}{W}_h(s) \triangleq \frac{1}{2}\left[\underset{\sim}{W}(s) + \underset{\sim}{W}^*(s)\right]$$

If the point $s = c$ is taken large enough to be in the region of convergence of the Laplace transform $\underset{\sim}{W}(s)$ and if $\underset{\sim}{y}$ is any n x 1 constant vector, we may symbolically write

$$\frac{y^*}{2}\left[\underset{\sim}{w}(t)e^{-ct} + \underset{\sim}{w}^T(-t)e^{ct}\right]\underset{\sim}{y} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\,dG_c(\omega), \qquad (5\text{-}9)$$

where $G_c(\omega)$ is any primitive of $\underset{\sim}{y}^*\underset{\sim}{W}_h(c + j\omega)\underset{\sim}{y}$.

If $\underset{\sim}{W}(s)$ has a simple pole at the origin, its partial fraction expansion will have a term of the form $\underset{\sim}{a}_0/s$, where $\underset{\sim}{a}_0$ is a matrix of real constants. The corresponding contribution to $\underset{\sim}{W}_h(s)$ is

$$\frac{(\underset{\sim}{a}_0 + \underset{\sim}{a}_0^T)}{2(\sigma^2 + \omega^2)} - \frac{(\underset{\sim}{a}_0 - \underset{\sim}{a}_0^T)\,j\omega}{2(\sigma^2 + \omega^2)} \,. \qquad (5\text{-}10)$$

As $\sigma \rightarrow 0+$, this converges in D' to

$$\frac{\pi}{2}(\underset{\sim}{a}_0 + \underset{\sim}{a}_0^T)\delta(\omega) + \frac{1}{2}(\underset{\sim}{a}_0 - \underset{\sim}{a}_0^T)F_p\,\frac{1}{j\omega} \qquad (5\text{-}11)$$

This can be ascertained from (2-5).

If W(s) has a pole of multiplicity μ at the origin, the term

$$\frac{\underset{\sim}{a}_0}{s^{\mu}}$$

will appear in the partial fraction expansion of $\underset{\sim}{W}(s)$. From (2-5), its contribution to $\underset{\sim}{W}_h(s)$ as $\sigma \rightarrow 0+$ is found to be

$$\lim_{\sigma \rightarrow 0+} \frac{1}{2}\left[\frac{\underset{\sim}{a}_0}{s^{\mu}} + \frac{\underset{\sim}{a}_0^T}{\bar{s}^{\mu}}\right] = \frac{1}{2}\left\{\underset{\sim}{a}_0\left[\frac{j^{\mu-1}\pi}{(\mu-1)!}\delta^{(\mu-1)}(\omega) + F_p\,\frac{1}{(j\omega)^{\mu}}\right]\right.$$

$$\left. + \underset{\sim}{a}_0^T\left[\frac{(-j)^{\mu-1}\pi}{(\mu-1)!}\delta^{(\mu-1)}(\omega) + F_p\,\frac{1}{(-j\omega)^{\mu}}\right]\right\}$$

$$= \frac{1}{2}\,\frac{j^{\mu-1}\pi}{(\mu-1)!}\left[\underset{\sim}{a}_0 + (-1)^{\mu-1}\underset{\sim}{a}_0^T\right]\delta^{(\mu-1)}(\omega) + \frac{1}{2}(\underset{\sim}{a}_0 + (-1)^{\mu}\underset{\sim}{a}_0^T)F_p\,\frac{1}{(j\omega)^{\mu}} \qquad (5\text{-}12)$$

Similarly, if $\underset{\sim}{W}(s)$ has a pair of single imaginary poles at $s = \pm j\beta_\nu$ ($\beta_\nu > 0$), then the following expression will appear in its partial fraction expansion.

$$\frac{\underset{\sim}{a}_\nu}{s-j\beta_\nu} + \frac{\overline{\underset{\sim}{a}_\nu}}{s+j\beta_\nu} \tag{5-13}$$

where $\underset{\sim}{a}_\nu$ is an n x n matrix of constants. The contribution of (5-13) to $\underset{\sim}{W}_h(s)$ is

$$\frac{1}{2}\left[\frac{\underset{\sim}{a}_\nu}{s-j\beta_\nu} + \frac{\bar{\underset{\sim}{a}}_\nu}{s+j\beta_\nu} + \frac{\underset{\sim}{a}_\nu^*}{s-j\beta_\nu} + \frac{\underset{\sim}{a}_\nu^T}{s+j\beta_\nu}\right] \tag{5-14}$$

We again see from (2-5) that, as $\sigma \to 0+$, (5-14) converges in D' to

$$\frac{\pi}{2}\left[(\underset{\sim}{a}_\nu + \underset{\sim}{a}_\nu^*)\,\delta(\omega-\beta_\nu) + (\bar{\underset{\sim}{a}}_\nu + \underset{\sim}{a}_\nu^T)\,\delta(\omega+\beta_\nu)\right]$$
$$+ \frac{1}{2j}\left[(\underset{\sim}{a}_\nu - \underset{\sim}{a}_\nu^*)\,Fp\,\frac{1}{\omega-\beta_\nu} + (\bar{\underset{\sim}{a}}_\nu - \underset{\sim}{a}_\nu^T)\,Fp\,\frac{1}{\omega+\beta_\nu}\right] \tag{5-15}$$

Assume, more generally, that these poles have a multiplicity of μ so that the terms,

$$\frac{\underset{\sim}{a}_\nu}{(s-j\beta_\nu)^\mu} + \frac{\underset{\sim}{a}_\nu}{(s+j\beta_\nu)^\mu},$$

appear in the partial fraction expansion of $\underset{\sim}{W}(s)$. Here, $\underset{\sim}{a}_\nu$ denotes a matrix of constants. Their contribution to $\lim_{\sigma \to 0+} \underset{\sim}{W}_h(\sigma+j\omega)$ is

$$\frac{j^{\mu-1}\pi}{(\mu-1)!\,2}\Big\{[\underset{\sim}{a}_\nu + (-1)^{\mu-1}\underset{\sim}{a}_\nu^*]\,\delta^{(\mu-1)}(\omega-\beta_\nu)$$
$$+ [\bar{\underset{\sim}{a}}_\nu + (-1)^{\mu-1}\underset{\sim}{a}_\nu^T]\,\delta^{(\mu-1)}(\omega+\beta_\nu)\Big\} \tag{5-16}$$
$$+ \frac{1}{2}[\underset{\sim}{a}_\nu + (-1)^\mu \underset{\sim}{a}_\nu^*]\,Fp\,\frac{1}{(j\omega-j\beta_\nu)^\mu} + \frac{1}{2}[\bar{\underset{\sim}{a}} + (-1)^\mu \underset{\sim}{a}^T]\,Fp\,\frac{1}{(j\omega+j\beta_\nu)^\mu}$$

The desired extension of lemma 1 is given by

Lemma 3: Let $\underset{\sim}{W}(s)$ be an n x n matrix whose elements are rational functions that are real for s real. Also, let $\underset{\sim}{W}(s)$ be of the form,

$$\underset{\sim}{W}(s) = \underset{\sim}{k}_1 s + \underset{\sim}{k}_0 + \underset{\sim}{W}_1(s), \tag{5-17}$$

where $\underset{\sim}{k}_1$ is a real symmetric constant matrix, $\underset{\sim}{k}_0$ is a real constant matrix, and $\underset{\sim}{W}_1(s)$ is analytic for Re s > 0 and has more poles than zeros in the finite s-plane. In order for $\underset{\sim}{y}^* \underset{\sim}{w}_h(t) \underset{\sim}{y}$ to be representable by the symbolic expression,

$$\underset{\sim}{y}^* \underset{\sim}{w}_h(t) \underset{\sim}{y} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} dG_0(\omega), \tag{5-18}$$

for every n x 1 constant vector $\underset{\sim}{y}$, where $G_0(\omega)$ is a nondecreasing ordinary function, all imaginary poles of W(s) must be simple and the matrix of residues at each such pole must be a nonnegative-definite hermitian matrix.

Proof: The proof of this lemma is almost the same as that of lemma 1.

Let

$$G_0 = \lim_{c \to 0+} G_c(\omega)$$

where again G_c is any primitive of $\underset{\sim}{y}^* \underset{\sim}{W}_h(c + j\omega) \underset{\sim}{y}$ $(c > 0)$.

The partial fractions corresponding to poles of $\underset{\sim}{W}(s)$ that lie in the open left-half s-plane contribute terms to $G_0(\omega)$ and to $G_c(\omega)$ that are continuous bounded functions and for these terms we may exchange the limit $c \to 0+$ with the integration in (5-9) to obtain (5-18) in the classical sense.

To do the same thing for the imaginary poles of $\underset{\sim}{W}(s)$, we shall resort to distribution theory (as we must if these poles are multiple ones or, in case they are simple, their residue matrices do not satisfy the conditions stated in the lemma). If $\underset{\sim}{W}(s)$ has a simple pole at the origin, its contribution to $\underset{\sim}{W}_h(s)$ is given by (5-10). Forming a hermitian form out of (5-10), then constructing a primitive (any primitive will do), and finally letting $\sigma \to 0+$, we find that the contribution to $G_0(\omega)$ is

$$\frac{\pi}{2} \underset{\sim}{y}^*(\underset{\sim}{a}_0 + \underset{\sim}{a}_0^T) \underset{\sim}{y} \left[u(\omega) - \frac{1}{2}\right] + \frac{1}{2j} \underset{\sim}{y}^*(\underset{\sim}{a}_0 - \underset{\sim}{a}_0^T) \underset{\sim}{y} \log |\omega|$$

If $\underset{\sim}{a}_0$ is not symmetric, this pole cannot contribute to $G_0(\omega)$ a monotonic function. On the other hand, if $\underset{\sim}{a}_0$ is symmetric, this pole contributes to $G_0(\omega)$ the bounded function,

$$\frac{\pi}{2} \underset{\sim}{y}^*(\underset{\sim}{a}_0 + \underset{\sim}{a}_0^T) \underset{\sim}{y} \left[u(\omega) - \frac{1}{2}\right]$$

and this expression is nondecreasing if $\underset{\sim}{a}_0$ is nonnegative-definite.

The contribution of the multiple pole $\underset{\sim}{a}_0/s^\mu$ to $G_0(\omega)$ is computed in the same way. That is, we construct a hermitian form out of (5-12) and then take one of its primitives. This yields for $\mu = 2, 3, 4, \ldots$

$$\frac{j^{\mu-1}\pi}{(\mu-1)!\,2} \underset{\sim}{y}^* \left[\underset{\sim}{a}_0 + (-1)^{\mu-1}\underset{\sim}{a}_0^T\right] \underset{\sim}{y}\, \delta^{(\mu-2)}(\omega)$$

$$- \frac{1}{2j(\mu-1)} \underset{\sim}{y}^* \left[\underset{\sim}{a}_0 + (-1)^{\mu}\underset{\sim}{a}_0^T\right] \underset{\sim}{y}\, \mathrm{Fp} \frac{1}{(j\omega)^{\mu-1}} \quad .$$

So, when $\mu > 1$, this contribution does not correspond to a nondecreasing ordinary function.

In the same way, it can be seen from (5-15) that a pair of simple poles at $s = \pm j\beta_\nu$ $(\beta_\nu > 0)$ contributes a nondecreasing ordinary function to $G_0(\omega)$ if and only if the corresponding matrix of residues $\underset{\sim}{a}_\nu$ is a nonnegative-definite hermitian form. Furthermore, (5-16) indicates that a pair of poles of multiplicity μ $(\mu > 1)$ at $s = \pm j\beta_\nu$ cannot contribute a nondecreasing function to $G_0(\omega)$.

In the vicinity of each imaginary pole of $\underset{\sim}{W}(s)$, the contributions of all the other terms in the partial fraction expansion of $\underset{\sim}{W}(s)$ to $G_0(\omega)$ are bounded ordinary functions, whereas this pole contributes singularities of the form, $\delta^{(\mu-2)}$ and/or $1/\omega^{\mu-1}$ if it does not fulfill the conditions stated in this lemma. Hence, for $G_0(\omega)$ to be a nondecreasing ordinary function these conditions must be fulfilled. The proof is completed by noting that we may interchange the limit process $c \to 0+$ with the symbolic integration in (5-9) to obtain (5-18). Q.E.D.

VI MATRICES WHOSE LAPLACE TRANSFORMS ARE POSITIVE-REAL

The t-domain characterization of a rational positive-real matrix is as follows:

Theorem 4: Let $\underset{\sim}{w}(t)$ be an n x n matrix distribution having its support bounded on the left and let $\underset{\sim}{W}(s)$ be its Laplace Transform. The necessary and sufficient conditions for $\underset{\sim}{W}(s)$ to be a rational positive-real matrix are the following:

1. $\underset{\sim}{w}(t) = \underset{\sim}{k}_1 \, \delta^{(1)}(t) + \underset{\sim}{k}_0 \, \delta(t) + \underset{\sim}{w}_1(t)$

Here, $\underset{\sim}{k}_1$ is a real symmetric nonnegative-definite constant matrix, $\underset{\sim}{k}_0$ is a real constant matrix, and $\underset{\sim}{w}_1(t)$ is a real matrix distribution whose elements consist of finite linear combinations of terms of the form

$$u(t)t^{\nu} e^{-\gamma t} ,$$

where ν is a nonnegative integer and γ is a complex constant.

2. Let

$$\underset{\sim}{w}_h(t) = \frac{1}{2} [\underset{\sim}{w}(t) + \underset{\sim}{w}^T(-t)] . \qquad (6\text{-}1)$$

For every n x 1 constant vector $\underset{\sim}{y}$ the quantity $\underset{\sim}{y}^* \underset{\sim}{w}_h(t) \, \underset{\sim}{y}$ is a non-negative-definite distribution.

Proof: Necessity: Assume that $\underset{\sim}{W}(s)$ is a rational positive-real matrix. Since $\underset{\sim}{W}(s)$ is real for s real and positive, the reflection principle [6;p. 155] shows that $\underset{\sim}{W}(\bar{s}) = \overline{\underset{\sim}{W}(s)}$ for at least Re s > 0. Consequently, $\underset{\sim}{w}(t)$ is real. The rest of condition 1, including the properties of $\underset{\sim}{k}_1$, follows from lemma 2, a partial fraction expansion of $\underset{\sim}{W}(s)$, and an application of the inverse Laplace transformation.

To verify condition 2, substitute the representation for $\underset{\sim}{w}(t)$, given in theorem 2, into (6-1). Using the properties of the various matrices in this representation, we may simplify the resulting expression for $\underset{\sim}{w}_h(t)$ into

$$\underset{\sim}{w}_h(t) = \frac{1}{2} \int_{-\infty}^{\infty} e^{j\eta t}(1 + \eta^2) \, d \, [\underset{\sim}{K}(\eta) + j \, \underset{\sim}{L}(\eta)] .$$

Hence,

$$\underset{\sim}{y}^* \underset{\sim}{w}_h(t) \, \underset{\sim}{y} = \frac{1}{2} \int_{-\infty}^{\infty} e^{j\eta t}(1 + \eta^2) d\{\underset{\sim}{y}^*[\underset{\sim}{K}(\eta) + j\underset{\sim}{L}(\eta)]\underset{\sim}{y}\} .$$

Since $\underset{\sim}{y}^{*}[\underset{\sim}{K}(\eta) - j\underset{\sim}{L}(\eta)]\underset{\sim}{y}$ is a real nondecreasing bounded function of η for <u>every</u> choice of $\underset{\sim}{y}$, $\underset{\sim}{y}^{*}[\underset{\sim}{K}(\eta) + j\underset{\sim}{L}(\eta)]\underset{\sim}{y}$, y must have the same properties. Condition 2 now follows from the Bochner-Schwartz theorem.

<u>Sufficiency:</u> Assuming that conditions 1 and 2 hold, we see first of all that $\underset{\sim}{W}(s)$ is real for real s in the region of convergence since $\underset{\sim}{w}(t)$ is real. Also, $\underset{\sim}{W}(s)$ is rational and is of the form,

$$\underset{\sim}{W}(s) = \underset{\sim}{k}_1 s + \underset{\sim}{k}_0 + \underset{\sim}{W}_1(s),$$

where $\underset{\sim}{W}_1(s)$ has more poles than zeros in the finite s-plane. All the finite poles of $\underset{\sim}{W}(s)$ must have nonpositive real parts because, otherwise, $\underset{\sim}{y}^{*}\underset{\sim}{w}_h(t)\underset{\sim}{y}$ would grow exponentially as $t \longrightarrow \infty$ for at least one $\underset{\sim}{y}$. This would mean that $\underset{\sim}{y}^{*}\underset{\sim}{w}_h(t)\underset{\sim}{y}$ is not nonnegative-definite in violation of condition 2. We conclude, therefore, that $\underset{\sim}{W}(s)$ is analytic for Re $s > 0$.

Now, by condition (2) and the Bochner-Schwartz theorem, $\underset{\sim}{y}^{*}\underset{\sim}{w}_h(t)\underset{\sim}{y}$ can be represented symbolically by (5-18), where $G_0(\omega)$ is a nondecreasing function. Hence, by lemma 3 every imaginary pole of $\underset{\sim}{W}(s)$ is simple and each corresponding residue matrix is hermitian and nonnegative-definite. Also, using the notation of lemma 2, we can decompose $G_0(\omega)$ into a step function due to the finite imaginary poles of $\underset{\sim}{W}(s)$ and an infinitely differentiable function that is a primitive of

$$\frac{\underset{\sim}{y}^{*}}{2}[\underset{\sim}{k}_0 + \underset{\sim}{k}_0^{T} + \underset{\sim}{W}_a(j\omega) + \underset{\sim}{W}_a^{*}(j\omega)]\underset{\sim}{y}\ . \qquad (6\text{-}2)$$

Since $G_0(\omega)$ is nondecreasing, (6-2) is nonnegative for every choice of $\underset{\sim}{y}$. We now conclude from lemma 2 that $\underset{\sim}{W}(s)$ is positive-real. Q.E.D.

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