

BANACH SYSTEMS, HILBERT PORTS, AND ∞ -PORTS

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1. Introduction

This work is a development in gradual stages of several concepts that may become of some value in the analysis and perhaps eventually in the synthesis of physical systems. A typical concept is the idea of an n -port where $n = \infty$. This idea arises not only as a natural mathematical extension of the n -port but, more importantly for the engineer, as a representation of certain physical systems. For example, consider a modal analysis [1; pp. 21-27] of a microwave transmission system. Each mode can be taken to be the excitation at a port of a black box with a separate port for each mode, and therefore the black box has in general an infinity of ports. In network theory it is common, indeed almost the rule, to assume that all but a finite number of modes can be neglected (see, for example, [2; p. 3]) so that it is sufficient to represent the system by an n -port.

But is it? To do so in every situation makes as much sense as would the replacement of every Fourier series occurring in network theory by a finite sum. Network theorists do not resort to the latter simplification since Fourier series are well understood and quite useable. The situation is very different for the systems considered here. The subject is in its infancy and possesses from the engineering (i.e., synthesis) point of view, very few results. Moreover, from the analysis viewpoint, a variety of mathematical difficulties arise. Indeed, the systems considered herein possess input and output signals which take their instantaneous values in Banach spaces, whereas for an n -port these values occur in n -dimensional euclidean space E_n . It happens disconcertingly often that a readily established fact concerning operators on

E_n -valued functions is very difficult if not impossible to extend to operators on Banach-space-valued function.

Nevertheless, we propose to explore this subject. For the mathematician no justification for doing so is needed. (The mountain is there; let's climb it.) The engineer asks, and indeed should ask, "Is it worth it?" We offer no reply to this question other than the hope that perhaps someday it may be. The fact that the electromagnetic waves within microwave systems, solid-state devices, integrated networks, etc. are better represented by Banach-space-valued functions rather than by E_n -valued functions offers some basis for this hope.

To put all this another way, this paper is motivated by physics, but its content is mathematics.

2. Some Definitions and a Summary.

The first in the order of business is to define the phrases appearing in the title. Throughout this work A and B will denote complex Banach spaces and H a complex Hilbert space. A Banach-space-valued function f is a mapping of some domain, which in this work will always be the real line R , into a Banach space, say, A . Thus, for each fixed $t \in R$, $f(t)$ is a member of A , and, as t varies in R , $f(t)$ may vary in A . Such a function is a typical signal in a variety of physical systems, as for example a microwave-transmission system. Indeed, we can conceive of a physical system (more precisely, a model of a physical system) whose signals are Banach-space-valued functions or even Banach-space-valued distributions. It seems natural to call such a system a "Banach system", and this we shall do.

A Banach system may have many different Banach spaces associated with it. Thus, at one location x in the system the signal representing

a particular physical variable may be an A-valued distribution and at another location y the signal for another physical variable may be a B-valued distribution. Moreover, the system defines an operator that maps the signal at x into the signal at y . In general, it defines many different operators depending on the choices of the locations x and y and of the variables of interest (i.e., electric-field intensity, magnetic-field intensity, etc.) We shall always make this distinction between the model of the physical system and the operators that it generates. The term "Banach system" refers to the model and not to any particular operator.

By a "Hilbert port" we mean the following. Assume that in a given Banach system we have singled out two physical variables u and v that are complementary in the following sense: Both u and v take their values in a Hilbert space H and the real part of their inner product (u, v) represents the instantaneous power entering the Banach system. Then, the Banach system with these two variables so singled out is called a "Hilbert port". When discussing a Hilbert port we in general pay no attention to the other variables within the system. We become interested exclusively in the variables u and v and the three operators $\mathfrak{A}: v \rightarrow u$, $\mathfrak{B}: u \rightarrow v$, and $\mathfrak{C}: v + u \rightarrow v - u$. (In microwave transmission systems, it is customary to choose v as the electric-field intensity on a closed surface containing the system and cutting all the wave guides to the system on transverse planes. Also, u is taken to be the magnetic-field intensity on the same closed surface. Then, \mathfrak{A} is called the admittance operator, \mathfrak{B} the impedance operator, and \mathfrak{C} the scattering operator. Furthermore $v_+ \stackrel{\Delta}{=} v + u$ can be related to the incident electric-field wave and $v_- \stackrel{\Delta}{=} v - u$ to the reflected electric-field wave.)

How about "co-ports"? Assume that the space H for a given Hilbert

port is separable; it will be for all practical systems. Choose an orthonormal basis for H . The separability of H implies that the basis will be countable. (Moreover, a modal analysis of v and u suggests a natural orthonormal basis for H .) Then, v and u can be represented by their sequences of Fourier coefficients $\{v_n\}$ and $\{u_n\}$ respectively. Thus, for each n , we have a pair v_n, u_n of complex-valued functions or distributions on R . We can for the sake of analysis assume that each pair v_n, u_n occurs on a separate port, and thus we are led to a system having a countable infinity of ports. We call such a paper network [3] corresponding to the given Hilbert port an " ∞ -port". In this case the operators \mathcal{R} , \mathcal{S} , and \mathcal{U} mentioned above can be represented by $\infty \times \infty$ matrices, as will be indicated in Sec. 8.

These are the kinds of systems with which we will be concerned in this work. Our primary objective is to develop characterizations and representations for various operators generated by such systems when they satisfy various idealized physical properties such as linearity, time-invariance, and passivity. Our theory depends crucially on the concept of a Banach-space-valued distribution, and therefore we present an introduction to this concept in the next section. Time-varying Banach systems are taken up in Sec. 4 and both a kernel representation and a composition representation for their continuous linear operators are established. To the authors knowledge, this theory does not appear elsewhere in the literature and so we also present proofs. Time-invariant Banach systems are discussed in Sec. 5; in this case the kernel and composition representations become convolution representations. The results of Secs. 4 and 5 apply to both active and passive systems. Next, we restrict our attention to passive Hilbert ports and observe that such systems have frequency-domain descriptions. The frequency-domain character-

izations of their admittance and scattering formulisms are given in Secs. 6 and 7 respectively. The proofs of the results in Secs. 5, 6, and 7 appear elsewhere [4], [5], and so in these sections we merely survey some pertinent results but omit all proofs. In Sec. 8 and its tail we return to detailed arguments. There the $\infty \times \infty$ matrix representation for the admittance operator of an ∞ -port is developed, and the problem of synthesizing an ∞ -port is considered but not resolved.

3. Banach-space-valued Distributions.

The natural language for the theory of continuous linear systems is that provided by distribution theory. It not only simplifies proofs but permits one to establish a number of theorems that simply could not be formulated in terms of conventional functions. Moreover, it allows the consideration of many idealized systems, such as those that differentiate an arbitrary number of times, without requiring that the domain of inputs be excessively restricted. The signals we will be concerned with are Banach-space-valued distributions on the real-time line $-\infty < t < \infty$, and so our first objective is to present a brief introduction to the theory of such distributions. This we do in the present section.

Throughout this work we use the following symbolism. If U and V are two topological linear spaces, the symbol $[U; V]$ denotes the linear space of all continuous linear mappings of U into V . Moreover, if $\phi \in U$ and $f \in [U; V]$, then $f\phi$, $f(\phi)$, and $\langle f, \phi \rangle$ all denote that element of V assigned by f to ϕ . H will always denote a complex Hilbert space, and (\cdot, \cdot) will denote its inner product. On the other hand, both A and B will always represent complex Banach spaces. Furthermore, R^n is n -dimensional euclidean space, $R = R^1$ is the real line, and C the complex plane. C_+ denotes the open right-half of C ; i.e., $C_+ = \{\zeta: \zeta \in C, \text{Re } \zeta > 0\}$.

A function φ mapping an open set $I \subset \mathbb{R}$ into A is said to be strongly continuous or strongly differentiable at a point $t \in I$ if the standard difference expression that defines continuity or the derivative at t converges in the norm topology of A . φ is said to be smooth on I if it possesses strong derivatives of all orders at all points of I . Similarly, if φ maps an open domain $J \subset \mathbb{C}$ into A , φ is said to be analytic on J if, for every point $\zeta \in J$, the standard difference expression that defines the derivative at ζ converges in the norm of A independently of the path in which the increment $\Delta\zeta$ is taken to zero. Derivatives are denoted alternatively by

$$\frac{d}{dx} \varphi = D\varphi = D_x \varphi(x) = \varphi^{(1)} .$$

The support of a continuous function φ mapping \mathbb{R} into A is the smallest closed set outside of which φ is the zero function; it is denoted by $\text{supp } \varphi$. $\|\cdot\|_A$ or simply $\|\cdot\|$ denotes the norm of a Banach space A . If nothing else is explicitly stated, it will be understood that $[A; B]$ carries the usual operator-norm topology (i.e., the uniform topology).

If T and X are either members of \mathbb{R} , $+\infty$, or $-\infty$ with $T < X$, then (T, X) and $[T, X]$ represent open and respectively closed intervals in \mathbb{R} , and similarly for the semi-open and semi-closed intervals $(T, X]$ and $[T, X)$.

Let K be a compact (i.e., closed bounded) interval in \mathbb{R} . $\mathcal{B}_K(A)$ denotes the linear space of all smooth functions φ on \mathbb{R} into A whose supports are contained in K . We assign to $\mathcal{B}_K(A)$ the topology generated by the sequence $\{\gamma_k\}_{k=0}^{\infty}$ of seminorms defined by

$$(3-1) \quad \gamma_k(\varphi) \stackrel{\Delta}{=} \sup_{t \in \mathbb{R}} \|D^k \varphi(t)\|_A \quad \varphi \in \mathcal{B}_K(A) .$$

Thus, a sequence $\{\varphi_\nu\}_{\nu=1}^\infty$ is convergent in $\mathcal{D}_K(A)$ if and only if every $\varphi_\nu \in \mathcal{D}_K(A)$ and there exists a $\varphi \in \mathcal{D}_K(A)$ such that, for every $k = 0, 1, \dots$, $\gamma_k(\varphi_\nu - \varphi) \rightarrow 0$ as $\nu \rightarrow \infty$.

For any nonnegative integer m , $\mathcal{D}_K^m(A)$ is defined similarly except that we impose the conditions on the derivatives $D^k \varphi$ for only $k = 0, 1, \dots, m$.

Next, let K_n denote the compact interval $-n \leq t \leq n$. Set

$$\mathcal{D}(A) = \bigcup_{n=1}^\infty \mathcal{D}_{K_n}(A),$$

and equip $\mathcal{D}(A)$ with the inductive-limit topology [6; vol. 15, pp. 61-62]. This implies that a sequence $\{\varphi_\nu\}_{\nu=1}^\infty$ converges in $\mathcal{D}(A)$ if and only if the entire sequence is contained in some fixed space $\mathcal{D}_{K_n}(A)$ and converges in that space. It is also a fact that a linear mapping f of $\mathcal{D}(A)$ into some locally convex space V (every topological linear space considered in this paper is locally convex) is continuous if and only if its restriction to each $\mathcal{D}_{K_n}(A)$ is sequentially continuous [6; vol. 15, p. 62]. The latter means that the convergence of $\{\varphi_\nu\}$ to zero in $\mathcal{D}_{K_n}(A)$ implies the convergence of $\{f, \varphi_\nu\}$ to zero in V . Moreover, a set Ω is bounded in $\mathcal{D}(A)$ if and only if there exists an n such that $\Omega \subset \mathcal{D}_{K_n}$ and, for every k , γ_k remains bounded on Ω . When A is C , we denote $\mathcal{D}(C)$ simply by \mathcal{D} .

$\mathcal{D}^m(A)$ is defined as the inductive-limit space $\bigcup_{n=1}^\infty \mathcal{D}_{K_n}^m(A)$ and has similar properties.

The linear space of all continuous linear mappings of $\mathcal{D}(A)$ into B is denoted by $[\mathcal{D}(A); B]$. It is a space of vector-valued distributions. We assign to it the so-called "topology of bounded convergence". This is the topology generated by the collection of seminorms $\{\sigma_\Omega\}$ where Ω varies through the bounded sets of $\mathcal{D}(A)$ and

$$(3-2) \quad \sigma_{\Omega}(f) \stackrel{\Delta}{=} \sup_{\varphi \in \Omega} \|\langle f, \varphi \rangle\|_B \quad f \in [\mathcal{D}(A); B].$$

When A and B are both C, we get the customary space $\mathcal{D}' = [\mathcal{D}(C); C]$ of scalar distributions on the real line, and the topology of \mathcal{D}' is then the so-called strong topology.

$[\mathcal{D}^m(A); B]$ and its topology of bounded convergence is defined in just the same way.

We will on occasion use a weaker topology for $[\mathcal{D}(A); B]$, namely, the "topology of pointwise convergence"; this is generated by the collection of seminorms (3-2) where now Ω is restricted to the finite sets in $\mathcal{D}(A)$. It corresponds to the weak topology of \mathcal{D}' . The symbol $[\mathcal{D}(A); B]^w$ will denote the space $[\mathcal{D}(A); B]$ equipped with this weaker topology.

On just one occasion (when we discuss a kernel theorem in Sec. 4) we will need functions and distributions defined on the two-dimensional euclidean space R^2 . In this case, the definitions of $\mathcal{D}_K(A)$, $\mathcal{D}(A)$, and $[\mathcal{D}(A); B]$, as well as $\mathcal{D}_K^m(A)$, $\mathcal{D}^m(A)$, and $[\mathcal{D}^m(A); B]$, are the same as above except that now K_n is a compact interval in R^2 (i.e., $K_n = \{z: z \in R^2, |z| \leq n\}$), k is a nonnegative integer in R^2 , and D^k is a partial differentiation corresponding to k . In the definition of $\mathcal{D}_K^m(A)$, $k = \{k_1, k_2\}$ is restricted to those values for which $0 \leq k_1 + k_2 \leq m$. In this two-dimensional case, we will always denote $\mathcal{D}(A)$ by $\mathcal{D}_{t,x}(A)$ and $\mathcal{D}^m(A)$ by $\mathcal{D}_{t,x}^m(A)$; moreover, we will on occasion replace $\langle \cdot, \cdot \rangle$ by $\langle \cdot, \cdot \rangle_{t,x}$ to emphasize that we are dealing with the two-dimensional case. Henceforth, the symbols \mathcal{D} , $\mathcal{D}(A)$, and $\mathcal{D}^m(A)$ will always signify that their members are defined on $R = R^2$.

We return now to the one-dimensional case. Of importance to us is the space $[\mathcal{D}; [A; B]]$ of continuous linear mappings of $\mathcal{D} = \mathcal{D}(C)$ into

the space $[A; B]$. The topology of $[\mathcal{D}; [A; B]]$ is that generated by $\{\sigma_\Omega\}_\Omega$ with σ_Ω defined by

$$\sigma_\Omega(f) \stackrel{\Delta}{=} \sup_{\varphi \in \Omega} \|\langle f, \varphi \rangle\|_{[A; B]}$$

where now $f \in [\mathcal{D}; [A; B]]$ and Ω traverses the bounded sets in \mathcal{D} . A crucial result for our theory is the following [4; theorem 3-1].

Theorem 3-1: There is a bijection (i.e., a one-to-one onto mapping) from $[\mathcal{D}(A); B]$ onto $[\mathcal{D}; [A; B]]$ defined by

$$(3-3) \quad \langle g, \varphi a \rangle = \langle f, \varphi \rangle a$$

where $\varphi \in \mathcal{D}$, $a \in A$, $g \in [\mathcal{D}(A); B]$, and $f \in [\mathcal{D}; [A; B]]$.

Because of this theorem, we can identify the members of $[\mathcal{D}(A); B]$ with those of $[\mathcal{D}; [A; B]]$, and in place of (3-3) we will write $\langle f, \varphi a \rangle = \langle f, \varphi \rangle a$.

Here are some examples.

Example 3-1: Let F denote a fixed member of $[A; B]$ and let δ denote the customary delta functional. $F\delta$ is defined on any $\theta \in \mathcal{D}(A)$ by

$$(3-4) \quad \langle F\delta, \theta \rangle \stackrel{\Delta}{=} F\theta(0) \in B$$

Clearly, $F\delta \in [\mathcal{D}(A); B]$. On the other hand, we define $F\delta$ on any $\varphi \in \mathcal{D}$ by

$$(3-5) \quad \langle F\delta, \varphi \rangle \stackrel{\Delta}{=} F\varphi(0) \in [A; B]$$

and obtain thereby a member of $[\mathcal{D}; [A; B]]$. That these two definitions accord with (3-3) follows from the fact that, for any $a \in A$, we have $\varphi a \in \mathcal{D}(A)$ so that by (3-4) and (3-5)

$$\langle F\delta, \varphi a \rangle = F\varphi(0)a = \langle F\delta, \varphi \rangle a$$

Thus, $F\delta$ is simultaneously a member of $[\mathcal{D}(A); B]$ and $[\mathcal{D}; [A; B]]$.

Example 3-2: Let h be an $[A; B]$ -valued function on R that is continuous in the norm topology of $[A; B]$. Define a mapping (also denoted by h) on any $\theta \in \mathcal{D}(A)$ by

$$\langle h, \theta \rangle = \int_R h(t) \theta(t) dt \in B .$$

The integral on the right-hand side is the strong limit in B of the corresponding Riemann sums because $h(t) \theta(t)$ is strongly continuous from R into B . That h is a linear mapping follows from the fact that integration is a linear process. The continuity of h from $\mathcal{D}(A)$ into B is implied by the estimate:

$$\|\langle h, \theta \rangle\|_B \leq \int_R \|h(t)\|_{[A; B]} \|\theta(t)\|_A dt \leq \sup_{t \in R} \|\theta(t)\|_A \int_K \|h(t)\|_{[A; B]} dt$$

where K is a compact interval that contains $\text{supp } \theta$. Thus, $h \in [\mathcal{D}(A); B]$.

On the other hand, we can define h as a mapping of \mathcal{D} into $[A; B]$ by

$$\langle h, \varphi \rangle = \int_R h(t) \varphi(t) dt \in [A; B] \quad \varphi \in \mathcal{D},$$

and considerations similar to those above show that $h \in [\mathcal{D}; [A; B]]$.

Thus, the originally given $[A; B]$ -valued function h on R generates both a member of $[\mathcal{D}(A); B]$ and a member of $[\mathcal{D}; [A; B]]$.

We can generate still other members of $[\mathcal{D}(A); B]$ or $[\mathcal{D}; [A; B]]$ by differentiating in a generalized sense the distributions of the preceding examples. Such a differentiation D^p of order p is defined on any f in $[\mathcal{D}(A); B]$ (or in $[\mathcal{D}; [A; B]]$) by

$$(3-6) \quad \langle D^p f, \varphi \rangle = (-1)^p \langle f; D^p \varphi \rangle$$

where φ is an arbitrary member of $\mathcal{D}(A)$ (or respectively of \mathcal{D}). Since D^p is a continuous linear mapping of $\mathcal{D}(A)$ (or \mathcal{D}) into itself and hence

maps bounded sets into bounded sets, it follows that (generalized) differentiation D^p is a continuous linear mapping of $[\mathcal{B}(A); B]$ (or $[\mathcal{B}; [A; B]]$) into itself.

Example 3-3: We can now define a member of $[\mathcal{B}(A); B]$ or of $[\mathcal{B}; [A; B]]$ by applying D^p to the $F\delta$ of example 3-1. Upon applying (3-6), we obtain

$$(3-7) \quad \langle D^p(F\delta), \varphi \rangle = (-1)^p F\varphi^{(p)}(0)$$

where φ is either in $\mathcal{B}(A)$ or respectively in \mathcal{B} . Since $\langle D^p\delta, \varphi \rangle = (-1)^p \varphi^{(p)}(0)$, we see that $D^p(F\delta) = F D^p\delta$.

Example 3-4: Similarly, by applying D^p to the results of example 3-2, we get the member $D^p h$ of $[\mathcal{B}(A); B]$ or of $[\mathcal{B}; [A; B]]$ defined by

$$(3-8) \quad \langle D^p h, \varphi \rangle = (-1)^p \int_R h(t) \varphi^{(p)}(t) dt$$

where again $\varphi \in \mathcal{B}(A)$ or respectively $\varphi \in \mathcal{B}$.

Let us also state the definition of the shifting operator σ_τ ; we shall have need of it later on. For any given $\tau \in R$, σ_τ is defined on any $\varphi \in \mathcal{B}(A)$ by $\sigma_\tau \varphi(t) \stackrel{\Delta}{=} \varphi(t - \tau)$. σ_τ is a continuous linear mapping of $\mathcal{B}(A)$ into itself. Next, σ_τ is defined on any $f \in [\mathcal{B}(A); B]$ by

$$\langle \sigma_\tau f, \varphi \rangle \stackrel{\Delta}{=} \langle f, \sigma_{-\tau} \varphi \rangle,$$

and consequently σ_τ is a continuous linear mapping of $[\mathcal{B}(A); B]$ into itself.

There are several other spaces of Banach-space-valued functions and mappings on them that we shall need. One of these is $\mathcal{E}(A)$, the space of all smooth A -valued functions on R equipped with the topology generated by the collection of seminorms $\{\gamma_{K,k}\}_{K,k}$ where K traverses the compact sets in R , k traverses the nonnegative integers, and

$$(3-9) \quad \gamma_{K,k}(\varphi) \stackrel{\Delta}{=} \sup_{t \in K} \|D^k \varphi(t)\|_A \quad \varphi \in \mathcal{E}(A).$$

Here again, when A is C , we denote $\mathcal{E}(C)$ by simply \mathcal{E} . A sequence $\{\varphi_\nu\}$ converges in $\mathcal{E}(A)$ if and only if there exists a $\varphi \in \mathcal{E}(A)$ such that, for every K and k , $\gamma_{K,k}(\varphi_\nu - \varphi) \rightarrow 0$ as $\nu \rightarrow \infty$. A set in $\mathcal{E}(A)$ is bounded if and only if, for every K and k , $\gamma_{K,k}$ remains bounded on Ω . A linear mapping f of $\mathcal{E}(A)$ into any topological linear space V is continuous if and only if the convergence of $\{\varphi_\nu\}$ to zero in $\mathcal{E}(A)$ implies the convergence of $\{\langle f, \varphi_\nu \rangle\}$ to zero in V .

Let m be a nonnegative integer. We define $\mathcal{E}^m(A)$ as was $\mathcal{E}(A)$ except that conditions on $D^k \varphi$ are imposed only for $k = 0, 1, \dots, m$.

$[\mathcal{E}(A); B]$ denotes the linear space of all continuous linear mappings of $\mathcal{E}(A)$ into B . The distribution $D^p(F\delta)$, when extended onto $\mathcal{E}(A)$ in accordance with (3-7), is a member of $[\mathcal{E}(A); B]$. So too is $D^p h$, as defined by (3-8) with $\varphi \in \mathcal{E}(A)$, so long as $\text{supp } h$ is a compact set. In fact, it can be shown that every member f of $[\mathcal{E}(A); B]$ has a compact support (i.e., there exists some compact set K such that $\langle f, \varphi \rangle = 0$ for every $\varphi \in \mathcal{E}(A)$ with $\text{supp } \varphi$ contained in the complement of K). We assign to $[\mathcal{E}(A); B]$ the topology of bounded convergence, that is, the topology generated by the collection $\{\sigma_\Omega\}_\Omega$ of seminorms defined on any $f \in [\mathcal{E}(A); B]$ by (3-2) where now Ω traverses the bounded sets in $\mathcal{E}(A)$.

We obtain the definition of the space $[\mathcal{E}^m(A); B]$ by replacing $\mathcal{E}(A)$ by $\mathcal{E}^m(A)$ in the preceding paragraph. $[\mathcal{E}^m(A); B]$ is also a space of distributions of bounded support.

Here's another space of importance to us. Let $p \in \mathbb{R}$ be such that $1 \leq p < \infty$. $\mathcal{S}_{L^p}(\mathbb{R})$ is the set of all smooth functions φ on \mathbb{R} such that, for every nonnegative integer k ,

$$(3-10) \quad \alpha_k(\varphi) \stackrel{\Delta}{=} \left[\int_{\mathbb{R}} \|D^k \varphi(t)\|_A^p dt \right]^{1/p} < \infty .$$

We assign to $\mathcal{L}_{L_p}(A)$ the topology generated by $\{\alpha_k\}_{k=0}^{\infty}$. As before, we use the notation $\mathcal{L}_{L_p}(C) = \mathcal{L}_{L_p}$. We can make the same comments concerning convergent sequences, bounded sets, and continuous linear mappings on $\mathcal{L}_{L_p}(A)$ as those made for $\mathcal{E}(A)$ except now $\gamma_{K,k}$ is replaced by α_k .

$[\mathcal{L}_{L_p}(A); B]$ is the topological linear space of all continuous linear mappings of $\mathcal{L}_{L_p}(A)$ into B equipped with the topology of bounded convergence.

We now turn our attention to the spaces $\mathcal{L}(0, \infty; A)$ and $[\mathcal{L}(0, \infty; A); B]$.

A detailed discussion of the scalar version of these spaces, where A and B are both C , is given in [7; chapter 3]. First of all, for any $c \in \mathbb{R}$ and $d \in \mathbb{R}$ we define $\mathcal{L}_{c,d}(A)$ as the space of all smooth functions on \mathbb{R} into A such that, for every nonnegative integer k ,

$$(3-11) \quad \gamma_{c,d,k}(\varphi) \triangleq \sup_{t \in \mathbb{R}} \|\kappa_{c,d}(t) D^k \varphi(t)\|_A < \infty$$

where

$$\kappa_{c,d}(t) \triangleq \begin{cases} e^{ct} & t \geq 0 \\ e^{dt} & t < 0 \end{cases}.$$

The topology of $\mathcal{L}_{c,d}(A)$ is that generated by the collection $\{\gamma_{c,d,k}\}_{k=0}^{\infty}$ of seminorms on $\mathcal{L}_{c,d}(A)$. Once again, the comments concerning convergent sequences, bounded sets, and continuous linear mappings on $\mathcal{L}_{c,d}(A)$ are quite the same as those made for $\mathcal{E}(A)$.

Next, let w be either a member of \mathbb{R} or $-\infty$ and let z be either a member of \mathbb{R} or $+\infty$. Also, let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two monotonic sequences in \mathbb{R} such that $a_n \rightarrow w +$ and $b_n \rightarrow z -$. It follows that, for $m > n$, $\mathcal{L}_{a_n, b_n}(A) \subset \mathcal{L}_{a_m, b_m}(A)$ and the topology of $\mathcal{L}_{a_n, b_n}(A)$ is stronger than that induced on it by $\mathcal{L}_{a_m, b_m}(A)$. We set

$$\mathfrak{L}(w, z; A) = \bigcup_{n=1}^{\infty} \mathfrak{L}_{a_n, b_n}(A)$$

and assign to $\mathfrak{L}(w, z; A)$ the inductive-limit topology. This means among other things that a linear mapping f on $\mathfrak{L}(w, z; A)$ into a locally convex space V is continuous if and only if the restriction of f to each $\mathfrak{L}_{a_n, b_n}(A)$ is continuous. Here again, we set $\mathfrak{L}(w, z; C) = \mathfrak{L}(w, z)$

$[\mathfrak{L}(w, z; A) ; B]$ is the linear space of all continuous linear mappings of $\mathfrak{L}(w, z; A)$ into B . We assign to it the following topology.

Let \mathfrak{C}_n denote the collection of bounded sets in $\mathfrak{L}_{a_n, b_n}(A)$ (i.e., $\Omega \in \mathfrak{C}_n$ if and only if $\Omega \subset \mathfrak{L}_{a_n, b_n}(A)$ and, for each k , the seminorm $\gamma_{a_n, b_n, k}$ remains bounded on \mathfrak{C}_n). Let $\mathfrak{C} = \bigcup_{n=1}^{\infty} \mathfrak{C}_n$. Then, the topology of $[\mathfrak{L}(w, z; A) ; B]$ is that generated by the collection $\{\sigma_{\Omega}\}_{\Omega \in \mathfrak{C}}$ of seminorms where each σ_{Ω} is defined by

$$\sigma_{\Omega}(f) \stackrel{\Delta}{=} \sup_{\varphi \in \Omega} \|\langle f, \varphi \rangle\|_B \quad f \in [\mathfrak{L}(w, z; A) ; B] .$$

If $w < z$ and if $y \in [\mathfrak{L}(w, z; A) ; B]$, we define the Laplace transform $Y = \mathfrak{L}y$ of y by

$$(3-12) \quad Y(\zeta) a \stackrel{\Delta}{=} \langle y(t), e^{-\zeta t} a \rangle \quad w < \operatorname{Re} \zeta < z .$$

where a is any member of A . The right-hand side of (3-12) has a sense because $e^{-\zeta t} a \in \mathfrak{L}(w, z; A)$ for every ζ such that $w < \operatorname{Re} \zeta < z$. It turns out that Y is an $[A; B]$ -valued analytic function on the strip

$$\Omega_y = \{\zeta : w < \operatorname{Re} \zeta < z\} .$$

(See [4; Sec. 5].) Moreover, the Laplace transformation \mathfrak{L} is unique in the following sense: If $Y(\zeta)$ is the zero member of $[A; B]$ on any open subset of Ω_y , then y is the zero member of $[\mathfrak{L}(w, z; A) ; B]$. (See [4; corollary 5-2a].)

A fact that we shall subsequently employ is the following: If $f \in [\mathcal{D}_{L_p}(A) ; B]$ where $1 \leq p < \infty$ and if $\text{supp } f \subset [T, \infty)$ for some $T \in \mathbb{R}$, then $f \in [\mathcal{E}(0, \infty; A) ; B]$ (see [4; lemma 7-2] for the case $p = 1$ and [5; lemma 2-1] for the case $p = 2$) so that $(\mathcal{Q}y)(\zeta)$ exists on at least the right-half plane $C_+ = \{\zeta : 0 < \text{Re } \zeta < \infty\}$.

We now take note of certain relations between some of the above spaces. We have

$$(3-13) \quad \mathcal{D}(A) \subset \mathcal{D}_{L_p}(A) \subset \mathcal{E}(A)$$

and

$$(3-14) \quad \mathcal{D}(A) \subset \mathcal{E}(w, z; A) \subset \mathcal{E}(A) .$$

In both (3-13) and (3-14), any space therein is a dense subspace of any space occurring to the right of it, and the topology of the former space is stronger than the topology induced on it by the latter space. By virtue of these facts, we have

$$(3-15) \quad [\mathcal{E}(A) ; B] \subset [\mathcal{D}_{L_p}(A) ; B] \subset [\mathcal{D}(A) ; B]$$

and

$$(3-16) \quad [\mathcal{E}(A) ; B] \subset [\mathcal{E}(w, z; A) ; B] \subset [\mathcal{D}(A) ; B] .$$

4. Time-varying Banach Systems, a Kernel Theorem, and Composition.

We wish to consider a model of a physical system which determines an operator that maps a class of input signals into a class of output signals. Our purpose is to investigate the relationship between various analytic properties of the operator and certain (idealized) physical properties. In this section the only physical properties we shall impose are single-valuedness, linearity, and continuity. (Throughout this paper,

whenever we specify that an operator is linear on some domain Ω , it will also be understood that it is single-valued on Ω . On occasion we will allow multivalued operators but these will not be called linear.) Thus, time-varying systems are allowed at this point. Active systems are also allowed. The adjective "active" signifies merely that no requirement of passivity, as defined in Secs. 6 and 7 below, is being imposed but may nevertheless be satisfied. Thus, we view passive systems as being special cases of active ones.

The kind of operator we have in mind is one that is generated by a Banach system and therefore maps, say, A -valued distributions into B -valued distributions. We can obtain an analytic representation for the operator by extending Schwartz's kernel theorem [8; p. 531] to Banach-space-valued distributions. This extension is given by theorem 4-1 below. (A proof of theorem 4-1 is provided in a paper by Bogdanowicz [9]. Actually, Bogdanowicz's work restricts the range space B to the complex plane C , but it is not difficult to extend his argument to the more general case considered here.)

To say that \mathfrak{M} is a separately continuous bilinear mapping of $\mathcal{D} \times \mathcal{D}(A)$ into B means the following: \mathfrak{M} maps any ordered pair φ, v of a $\varphi \in \mathcal{D}$ and a $v \in \mathcal{D}(A)$ into a member $\mathfrak{M}(\varphi, v)$ of B , and, in addition, if v (respectively φ) is kept fixed, then \mathfrak{M} is linear and continuous with respect to φ (respectively v).

Theorem 4-1: If \mathfrak{M} is a separately continuous bilinear mapping of $\mathcal{D} \times \mathcal{D}(A)$ into B , then there exists a unique continuous linear mapping $f = f(t, x)$ of $\mathcal{D}_{t,x}(A)$ into B such that

$$(4-1) \quad \mathfrak{M}(\varphi, v) = \langle f(t, x), \varphi(t) v(x) \rangle$$

for every $\varphi \in \mathcal{D}$ and every $v \in \mathcal{D}(A)$.

Next step: We define a composition product $f \bullet v$ on any $f \in [\mathcal{D}_{t,x}(A) ; B]$ and any $v \in \mathcal{B}(A)$ by

$$(4-2) \quad \langle f \bullet v, \varphi \rangle \stackrel{\Delta}{=} \langle f(t, x), \varphi(t) v(x) \rangle \quad \varphi \in \mathcal{D} .$$

We shall refer to the process of forming this product $f \bullet v$ as "composition \bullet " to distinguish it from another such process, called "composition \circ ", which will be defined subsequently. The right-hand side of (4-2) has a sense and determines a member of B because $\varphi(t) v(x) \in \mathcal{D}_{t,x}(A)$. Thus, we can consider the composition \bullet operator $f \bullet : v \mapsto f \bullet v$ as a mapping of $\mathcal{B}(A)$ into the space of mappings of \mathcal{D} into B .

Theorem 4-2: For any given $f \in [\mathcal{D}_{t,x}(A) ; B]$, the composition operator $f \bullet$ is a continuous linear mapping of $\mathcal{B}(A)$ into $[\mathcal{B}; B]$.

Proof: We first observe that $f \bullet v \in [\mathcal{B}; B]$. The linearity of $f \bullet v$ on \mathcal{B} is obvious. Its continuity follows from the fact that, if $\varphi_\nu \rightarrow 0$ in \mathcal{D} , then $\varphi_\nu(t) v(x) \rightarrow 0$ in $\mathcal{D}_{t,x}$ so that $\langle f \bullet v, \varphi_\nu \rangle \rightarrow 0$ in B .

Now consider $f \bullet$. Its linearity on $\mathcal{B}(A)$ is again clear. To show its continuity, let Ω be any bounded set in \mathcal{B} and let $v_\nu \rightarrow 0$ in $\mathcal{B}(A)$. Consequently, $\varphi(t) v_\nu(x) \rightarrow 0$ in $\mathcal{D}_{t,x}(A)$ uniformly for all $\varphi \in \Omega$. Moreover, there exists a compact set $K \subset \mathbb{R}^2$, a nonnegative integer $m \in \mathbb{R}$, and a constant $Q > 0$ such that $\text{supp } \varphi(t) v_\nu(x) \subset K$ for all ν and all $\varphi \in \Omega$, and, in addition,

$$(4-3) \quad \begin{aligned} \sigma_\Omega(f \bullet v_\nu) &= \sup_{\varphi \in \Omega} \|\langle f \bullet v_\nu, \varphi \rangle\|_B \\ &= \sup_{\varphi \in \Omega} \|\langle f(t, x), \varphi(t) v_\nu(x) \rangle\|_B \\ &= \sup_{\varphi \in \Omega} Q \max_{0 \leq |k| \leq m} \sup_{t,x} \|D^k \varphi(t) v_\nu(x)\|_A \end{aligned}$$

where $k = \{k_1, k_2\}$ is a nonnegative integer in \mathbb{R}^2 and $|k| = k_1 + k_2$.

In view of the uniformity of the convergence of $\varphi(t) v_\nu(x)$ with respect to all $\varphi \in \Omega$, the right-hand side of (4-3) tends to zero as $\nu \rightarrow \infty$. Since Ω was arbitrary, we conclude that $f * v_\nu \rightarrow 0$ in $[\mathcal{D}; B]$.

Theorem 4-2 possesses a converse. In order to obtain it, we will need

Lemma 4-1: Let \mathfrak{N} be a continuous linear mapping of $\mathcal{D}(A)$ into $[\mathcal{D}; B]^W$.

Define \mathfrak{M} from \mathfrak{N} by

$$(4-4) \quad \mathfrak{M}(\varphi, v) \stackrel{\Delta}{=} \langle \mathfrak{N}v, \varphi \rangle \quad v \in \mathcal{D}(A), \varphi \in \mathcal{D}.$$

Then, \mathfrak{M} is a uniquely defined separately continuous bilinear mapping of $\mathcal{D} \times \mathcal{D}(A)$ into B .

Proof: Since $\mathfrak{N}v \in [\mathcal{D}; B]$, the right-hand side of (4-4) is a member of B . Thus, \mathfrak{M} maps $\mathcal{D} \times \mathcal{D}(A)$ into B .

Next, fix φ . Let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $v_1 \in \mathcal{D}(A)$, and $v_2 \in \mathcal{D}(A)$. Then, the linearity of \mathfrak{M} with respect to v is established by

$$\begin{aligned} \mathfrak{M}(\varphi, \alpha v_1 + \beta v_2) &= \langle \mathfrak{N}(\alpha v_1 + \beta v_2), \varphi \rangle = \langle \alpha \mathfrak{N}v_1 + \beta \mathfrak{N}v_2, \varphi \rangle \\ &= \alpha \langle \mathfrak{N}v_1, \varphi \rangle + \beta \langle \mathfrak{N}v_2, \varphi \rangle = \alpha \mathfrak{M}(\varphi, v_1) + \beta \mathfrak{M}(\varphi, v_2). \end{aligned}$$

To show the continuity of \mathfrak{M} with respect to v , let $v_\nu \rightarrow 0$ in $\mathcal{D}(A)$. We have that $\mathfrak{M}(v_\nu, \varphi) = \langle \mathfrak{N}v_\nu, \varphi \rangle$, and this tends to zero in B because \mathfrak{N} is continuous from $\mathcal{D}(A)$ into $[\mathcal{D}; B]^W$.

Similar arguments establish the linearity and continuity of \mathfrak{M} with respect to φ when v is held fixed.

We may now combine lemma 4-1, theorem 4-1, and the definition (4-2) to get the aforementioned converse to theorem 4-2.

Theorem 4-3: For every continuous linear mapping \mathfrak{N} of $\mathcal{D}(A)$ into

$[\mathcal{D}; B]^W$, there exists a unique $f \in [\mathcal{D}_{t,x}^0(A); B]$ such that, for all $v \in \mathcal{D}(A)$, $\mathfrak{R}v = f \circ v$ in the sense of equality in $[\mathcal{D}; B]$.

Theorem 4-3 provides an explicit analytic representation for a sufficiently well-behaved operator \mathfrak{R} of a time-varying active Banach system, but it does so for only a very restricted domain for the representation, namely, $\mathcal{D}(A)$. We can construct a composition representation for certain such \mathfrak{R} with wider domains for the representation by appropriately extending to Banach-space-valued distributions the concept of the composition of distributions as developed by Cristescu [10], Cristescu and Marinescu [11], Sabac [12], Wexler [13], Cioranescu [14], Pondelicek [15], and Dolezal [16]. But, before doing so, let us present some examples of composition \circ operators.

Example 4-1: We first present the composition \circ representation of an n^{th} order differential operator hD^n with a variable coefficient $h \in \mathcal{E}^0([A; B])$. That is, h is a strongly continuous $[A; B]$ -valued function on \mathbb{R} . For this purpose, we define the operator $I(t, x) \in [\mathcal{D}_{t,x}^0(A); A]$ by

$$\langle I(t, x), \theta(t, x) \rangle = \int_{\mathbb{R}} \theta(t, t) dt \in A \quad \theta \in \mathcal{D}_{t,x}^0(A) .$$

Consider $f(t, x) = h(t) (-D_x)^n I(t, x)$; it is a member of $[\mathcal{D}_{t,x}^n(A); B] \subset [\mathcal{D}_{t,x}(A); B]$, as can be seen from the equation:

$$\langle f(t, x), \psi(t, x) \rangle = \int_{\mathbb{R}} h(t) [D_x^n \psi(t, x)]_{x=t} dt \quad \psi \in \mathcal{D}_{t,x}^n(A)$$

With this choice of f , we have for any $v \in \mathcal{D}(A)$ and $\phi \in \mathcal{D}$

$$\begin{aligned} \langle f \circ v, \phi \rangle &= \langle f(t, x), \phi(t) v(x) \rangle = \int_{\mathbb{R}} h(t) \phi(t) D^n v(t) dt \\ &= \langle hD^n v, \phi \rangle . \end{aligned}$$

It follows from this expression that the composition * operator $f * v$ is a continuous linear mapping of $\mathcal{B}(A)$ into $[\mathcal{B}; B]$. Thus, we have arrived at the desired representation for hD^n ; namely, in the sense of equality in $[\mathcal{B}; B]$,

$$(4-5) \quad hD^n v = f * v$$

$$v \in \mathcal{B}(A), h \in \mathcal{E}^0([A; B]), f(t, x) = h(t) (-D_x)^n I(t, x) \in [\mathcal{B}_{t,x}^n(A); B].$$

Example 4-2: Here's the composition * representation for the operator $h \sigma_\tau$ where $h \in \mathcal{E}^0([A; B])$ again, $\tau \in \mathbb{R}$, and $\sigma_\tau(x)$ is the shifting operator defined on any $\theta \in \mathcal{B}_{t,x}^0(A)$ by $\sigma_\tau(x) \theta(t, x) \stackrel{\Delta}{=} \theta(t, x - \tau)$ and on any $f \in [\mathcal{B}_{t,x}^0(A); B]$ by

$$\langle \sigma_\tau(x) f(t, x), \theta(t, x) \rangle \stackrel{\Delta}{=} \langle f(t, x), \theta(t, x + \tau) \rangle.$$

Then, $h \sigma_\tau \stackrel{\Delta}{=} h(t) \sigma_\tau(t)$ denotes the operation of first shifting and then multiplying-by- h quantities defined on, say, the t axis.

Let $f(t, x) = h(t) \sigma_{-\tau}(x) I(t, x)$. It is easily seen that $f(t, x) \in [\mathcal{B}_{t,x}^0(A); B] \subset [\mathcal{B}_{t,x}(A); B]$. Then, for all $v \in \mathcal{B}(A)$ and $\phi \in \mathcal{B}$,

$$\begin{aligned} \langle f * v, \phi \rangle &= \langle h(t) \sigma_{-\tau}(x) I(t, x), \phi(t) v(x) \rangle = \langle I(t, x), h(t) \phi(t) v(x - \tau) \rangle \\ &= \int_{\mathbb{R}} h(t) \phi(t) v(t - \tau) dt = \langle h \sigma_\tau v, \phi \rangle \in B. \end{aligned}$$

This equation shows that $f * v$ is a continuous linear mapping of $\mathcal{B}(A)$ into $[\mathcal{B}; B]$. Thus, in the sense of equality in $[\mathcal{B}; B]$, we have

$$(4-6) \quad h \sigma_\tau v = f * v$$

$$v \in \mathcal{B}(A), h \in \mathcal{E}^0([A; B]), f(t, x) = h(t) \sigma_{-\tau}(x) I(t, x) \in [\mathcal{B}_{t,x}^0(A); B].$$

Example 4-3: We now show that convolution is a special case of composition. Let $y \in [\mathcal{D}(A) ; B]$. Define $y(t - x)$ as a member of $[\mathcal{D}_{t,x}(A) ; B]$ by

$$(4-7) \quad \langle y(t - x), \psi(t, x) \rangle_{t,x} \stackrel{\Delta}{=} \langle y(t), \int_{\mathbb{R}} \psi(t + x, x) dx \rangle .$$

Let $f(t, x) = y(t - x)$. Then, for all $v \in \mathcal{D}(A)$ and $\varphi \in \mathcal{D}$,

$$\begin{aligned} \langle f \circ v, \varphi \rangle &= \langle y(t - x), \varphi(t) v(x) \rangle_{t,x} = \langle y(t), \int_{\mathbb{R}} \varphi(t + x) v(x) dx \rangle \\ &= \langle y(t), \langle v(x), \varphi(t + x) \rangle \rangle . \end{aligned}$$

The last expression is the definition of the convolution product $y * v$ applied to φ [4; Sec. 4]. Thus, in the sense of equality in $[\mathcal{D}; B]$, we have

$$(4-8) \quad y * v = f \circ v$$

$$v \in \mathcal{D}(A), y \in [\mathcal{D}(A) ; B], f(t, x) = y(t - x) \in [\mathcal{D}_{t,x}(A) ; B] .$$

We now attack the problem of finding a composition procedure that is explicitly defined for a pair of Banach-space-valued distributions neither of which are in $\mathcal{D}(A)$. (In fact, they may both be singular distributions.) We shall refer to this latter procedure as "composition o" to distinguish it from the previously discussed composition \circ procedure. We first develop some properties of Banach-space-valued distributions that we shall need.

In the following N is a compact interval in \mathbb{R} . Also, $L_{n,N}$ with $n = 1$ or 2 is the customary Banach space of complex Lebesgue-measurable functions f on N with $|f|^n$ Lebesgue integrable on N . The norm for $L_{n,N}$ is $\|\cdot\|_{n,N}$ where

$$\|f\|_{n,N} = \left[\int_N |f(t)|^n dt \right]^{1/n} .$$

By the Schwarz inequality,

$$\|f\|_{1,N} \leq c \|f\|_{2,N}$$

where c denotes the square root of the length of N .

Theorem 4-4: Let the sequence $\{v_j\}_{j=1}^{\infty}$ tend to zero in $[B; A]$, and let N and K be compact intervals in R such that K contains a neighborhood of N . Then, there exists an integer $p \geq 0$ such that the following two conditions are satisfied.

(i) For each j there exists a strongly continuous A -valued function g_j on R such that $\text{supp } g_j \subset K$ and $v_j = D^p g_j$ in the sense of equality in $[A_N; A]$.

(ii) The sequence $\{g_j(x)\}$ tends to zero in A uniformly for all $x \in R$.

Proof: By hypothesis, $v_j \rightarrow 0$ in $[A_N; A]$. By [4; lemma 3-1], there exists a constant $M > 0$ and an integer $r \geq 0$ such that, for all $\phi \in B_N$,

$$(4-9) \quad \sup_j \|\langle v_j, \phi \rangle\|_A \leq M \max_{0 \leq k \leq r} \sup_{t \in R} |D^k \phi(t)| .$$

Next, for each derivative $D^k \phi$, we may write

$$D^k \phi = \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{r-k}} D^{r+1} \phi(t_{r+1-k}) dt_{r+1-k} .$$

Let $T = \min N$ and $\tau = \max N$. We get

$$\begin{aligned} \sup_{t \in R} |D^k \phi(t)| &\leq (\tau - T)^{r-k} \sup_{t \in R} \int_T^t |D^{r+1} \phi(x)| dx \\ &= (\tau - T)^{r-k} \int_N |D^{r+1} \phi(x)| dx . \end{aligned}$$

We see from (4-9) that there exists a constant $M_1 > 0$ such that

$$(4-10) \quad \sup_j \|\langle v_j, \phi \rangle\|_A \leq M_1 \int_N |D^{r+1} \phi(x)| dx = M_1 \|D^{r+1} \phi\|_{1,N}.$$

Next, consider the following linear subspace of \mathcal{L}_N :

$$\Delta = \{\psi : \psi = D^{r+1} \phi, \phi \in \mathcal{L}_N\}$$

Let U_j be the linear mapping of Δ into A defined by

$$(4-11) \quad \langle U_j, \psi \rangle \stackrel{\Delta}{=} \langle v_j, \phi \rangle.$$

This defines U_j uniquely because, if $\psi \equiv 0$ and $\phi \in \mathcal{L}_N$, then $\phi \equiv 0$ so that $\langle U_j, 0 \rangle = \langle v_j, 0 \rangle = 0$. Moreover, U_j is continuous when Δ is supplied the topology induced by $L_{2,N}$ because by (4-10)

$$(4-12) \quad \|\langle U_j, \psi \rangle\|_A = \|\langle v_j, \phi \rangle\|_A \leq M_1 \|\psi\|_{1,N} \leq M_1 c \|\psi\|_{2,N}.$$

We extend U_j onto the closure $\bar{\Delta}$ of Δ in $L_{2,N}$ by continuity. The extended mapping, which we also denote by U_j , is continuous and linear on $\bar{\Delta}$ and satisfies

$$\|\langle U_j, \psi \rangle\|_A \leq M_1 c \|\psi\|_{2,N}$$

for all $\psi \in \bar{\Delta}$. Furthermore, let $C\Delta$ be the orthogonal complement of Δ in $L_{2,N}$, and define $\langle U_j, \psi \rangle = 0$ on every $\psi \in C\Delta$. Finally, define U_j on all of $L_{2,N}$ by linearity. Thus, we have arrived at a uniquely defined continuous linear mapping U_j of $L_{2,N}$ into A .

We now invoke a result [17; p. 259, theorem 1] which asserts that there exists an A -valued measure m_j defined on the Borel subsets of N such that

$$(4-13) \quad \langle U_j, \psi \rangle = \int_N \psi \, dm_j$$

for each $\psi \in L_{2,N}$. In addition, (4-13) tends to zero as $j \rightarrow \infty$. Indeed, $\psi = \psi_1 + \psi_2$ where $\psi_1 \in \bar{\Delta}$ and $\psi_2 \in G\Delta$. Thus,

$$\int_N \psi \, dm_j = \langle U_j, \psi \rangle = \langle U_j, \psi_1 + \psi_2 \rangle = \langle U_j, \psi_1 \rangle = \langle U_j, \psi_1 - \theta \rangle + \langle U_j, \theta \rangle$$

for any $\theta \in \Delta$. Hence,

$$\left\| \int_N \psi \, dm_j \right\|_A \leq M_1 c \|\psi_1 - \theta\|_{2,N} + \|\langle U_j, \theta \rangle\|_A.$$

Given any $\epsilon > 0$, we can choose $\theta \in \Delta$ such that

$$M_1 c \|\psi_1 - \theta\|_{2,N} < \frac{\epsilon}{2}.$$

By the hypothesis on v_j and (4-11), $\|\langle U_j, \theta \rangle\|_A$ is also less than $\epsilon/2$ for all sufficiently large j . Thus, (4-13) truly tends to zero as $j \rightarrow \infty$.

Moreover, we also have that, for every $\psi \in L_{2,N}$,

$$(4-14) \quad \left\| \int_N \psi \, dm_j \right\|_A = \|\langle U_j, \psi_1 \rangle\| \leq M_1 c \|\psi_1\|_{2,N} \leq M_1 c \|\psi\|_{2,N}.$$

Next, consider the function

$$K_x(t) = \begin{cases} 1 & \text{for } t \leq x, t \in N \\ 0 & \text{otherwise,} \end{cases}$$

where x is any member of R . As before, we set $T = \min N$ and define an A -valued function f_j on R by

$$f_j(x) = \int_T^x dm_j = \int_N K_x \, dm_j \quad x \in R$$

Since the restriction of $K_x(t)$ to N is a member of $L_{2,N}$ for each $x \in R$,

$f_j(x) \rightarrow 0$ as $j \rightarrow \infty$. Moreover, $\{f_j\}_j$ is a strongly equicontinuous set because by (4-14)

$$\begin{aligned} \|f_j(x) - f_j(y)\|_A &= \left\| \int_N (K_x - K_y) \, dm_j \right\|_A \leq M_1 c \|K_x - K_y\|_{2,N} \\ &= M_1 c \sqrt{|y - x|} \quad . \end{aligned}$$

Consequently, $\{f_j(x)\}_j$ tends to 0 in A uniformly for all $x \in \mathbb{R}$.

Finally, we have from (4-11), (4-13), and an integration by parts that, for every $\phi \in \mathcal{D}_N$,

$$\langle v_j, \phi \rangle = \langle U_j, D^{r+1} \phi \rangle = \int_N D^{r+1} \phi \, dm_j = - \int_N f_j D^{r+2} \phi \, dt \quad .$$

Let $g_j = (-1)^{r+1} \alpha f_j$ where $\alpha \in \mathcal{D}$ is such that $\alpha \equiv 1$ on a neighborhood of N and $\alpha \equiv 0$ outside K . Then, for all $\phi \in \mathcal{D}_N$,

$$\begin{aligned} \langle v_j, \phi \rangle &= \langle -f_j, \alpha D^{r+2} \phi \rangle = \langle (-1)^{r+1} D^{r+2} (\alpha f_j), \phi \rangle \\ &= \langle D^{r+2} g_j, \phi \rangle \quad . \end{aligned}$$

This completes the proof of theorem 4-4.

We note in passing that the foregoing proof can be modified to eliminate the integral representation in the right-hand side of (4-13). One need merely work with the left-hand side of (4-13) and define $f_j(x)$ as $\langle U_j, K_x \rangle$. But then, this will require the use of the concepts of the primitive of U_j and its differentiation.

Theorem 4-5: If the sequence $\{v_j\}_{j=1}^{\infty}$ converges in $[\mathcal{E}; A]$ to zero, then there exist a compact interval N , an integer $p \geq 0$, and strongly continuous A -valued functions $h_{k,j}$ on \mathbb{R} such that, in the sense of equality in $[\mathcal{E}; A]$,

$$v_j = \sum_{k=0}^p D^k h_{k,j}$$

where $\text{supp } h_{k,j} \subset N$ for all k and j and, for each fixed k , $h_{k,j} \rightarrow 0$ in A uniformly for all $x \in R$.

Proof: That $\{v_j\}$ converges in $[\mathcal{E}; A]$ implies that there exists a compact interval $G \subset R$ such that $\text{supp } v_j \subset G$ for all j [18; chapter 1, pp.62-63]. Let N be a compact interval in R containing a neighborhood of G . Choose $\lambda \in \mathcal{D}$ such that $\lambda \equiv 1$ on a neighborhood of G and $\lambda \equiv 0$ outside N . Then, for any $\phi \in \mathcal{E}$, we have that $\lambda\phi \in \mathcal{D}_N$ and $\langle v_j, \phi \rangle = \langle v_j, \lambda\phi \rangle$. By theorem 4-4 and the fact that convergence in $[\mathcal{E}; A]$ implies convergence in $[\mathcal{D}; A]$,

$$\begin{aligned} \langle v_j, \phi \rangle &= \langle v_j, \lambda\phi \rangle = \langle D^p g_j, \lambda\phi \rangle = (-1)^p \langle g_j, D^p(\lambda\phi) \rangle \\ &= (-1)^p \langle g_j, \sum_{k=0}^p \binom{p}{k} (D^{p-k}\lambda) (D^k\phi) \rangle = \langle \sum_{k=0}^p D^k h_{k,j}, \phi \rangle \end{aligned}$$

where

$$h_{k,j} = (-1)^{p+k} \binom{p}{k} (D^{p-k}\lambda) g_j .$$

The functions $h_{k,j}$ possess the properties stated in the theorem.

Lemma 4-2: Any given $v \in [\mathcal{E}; A]$ generates a unique $\hat{v} \in [\mathcal{E}([A; B]); B]$ by means of the definition:

$$(4-15) \quad \langle \hat{v}, J\theta \rangle = J \langle v, \theta \rangle \quad J \in [A; B], \theta \in \mathcal{E} .$$

Moreover, the mapping $v \rightarrow \hat{v}$ is a sequentially continuous linear injection of $[\mathcal{E}; A]$ into $[\mathcal{E}([A; B]); B]$.

Proof: Let P denote the set of all elements in $\mathcal{E}([A; B])$ of the form $J\theta$. Equation (4-15) uniquely defines from the given $v \in [\mathcal{E}; A]$

a mapping g_v of P into B .

Next, as a consequence of theorem 4-5, we have that v is a member of $\mathcal{E}; A$ if and only if there exists an integer $p \geq 0$, a compact interval N , and a finite set $\{h_k\}_{k=0}^p$ of strongly continuous A -valued functions h_k on R with $\text{supp } h_k \subset N$ such that

$$v = \sum_{k=0}^p D^k h_k$$

in the sense of equality in $\mathcal{E}; A$. We define a linear mapping \hat{v} of $\mathcal{E}([A; B])$ into B by

$$(4-16) \quad \langle \hat{v}, \psi \rangle = \left\langle \sum_{k=0}^p D^k h_k, \psi \right\rangle = \sum_{k=0}^p (-1)^k \int_N h_k D^k \psi dt \in B$$

$$\psi \in \mathcal{E}([A; B]).$$

(For any $J \in [A; B]$ and any $a \in A$, we denote the application of J to a by either Ja or aJ . Thus, $h_k D^k \psi$ denotes a B -valued function on R , which also happens to be strongly continuous.) \hat{v} is continuous because

$$\begin{aligned} \|\langle \hat{v}, \psi \rangle\|_B &\leq \sum_{k=0}^p \int_N \|h_k\|_A \|D^k \psi\|_{[A; B]} dt \\ &\leq \sum_{k=0}^p \int_N \|h_k\|_A dt \sup_{t \in N} \|D^k \psi(t)\|_{[A; B]} \end{aligned}$$

Thus, $\hat{v} \in [\mathcal{E}([A; B]); B]$.

We now observe that the substitution of $J\theta$ for ψ in (4-16) yields (4-15). Thus, \hat{v} coincides with g_v on P . Moreover, P is total in $\mathcal{E}([A; B])$; that is, the span of P is dense in $\mathcal{E}([A; B])$. Indeed, $Q \stackrel{\Delta}{=} \{J\chi : J \in [A; B], \chi \in \mathcal{A}\}$ is total in $\mathcal{A}([A; B])$ according to [4; lemma 6-1]. But, $\mathcal{A}([A; B])$ is dense in $\mathcal{E}([A; B])$ and the topology of

$\mathcal{B}([A; B])$ is stronger than that induced on it by $\mathcal{E}([A; B])$. Hence, Q is total in $\mathcal{E}([A; B])$. Since $Q \subset P$, so too is P . This implies that \hat{v} is uniquely determined by its restriction to P . This restriction is g_v , which is uniquely determined by v , as was noted above. This proves the first sentence of the lemma.

Turning to the second sentence, we first show that $v \rightarrow \hat{v}$ is an injection (i.e., a one-to-one mapping). Assume that v_1 and v_2 are both members of $[\mathcal{E}; A]$ and that $\langle v_1, \theta \rangle \neq \langle v_2, \theta \rangle$ for some $\theta \in \mathcal{E}$. There exists at least one $J \in [A; B]$ such that $J\langle v_1, \theta \rangle \neq J\langle v_2, \theta \rangle$. (Indeed, by the Hahn-Banach theorem [8; p. 187, corollary 2], there exists a continuous linear functional F on A such that $F\langle v_1, \theta \rangle \neq F\langle v_2, \theta \rangle$. Now, set $J = bF$ where b is any member of B other than the zero member.) Next, let \hat{v}_1 and \hat{v}_2 be the members of $[\mathcal{E}([A; B]); B]$ defined by v_1 and v_2 respectively in accordance with (4-15). Therefore, $\langle \hat{v}_1, J\theta \rangle \neq \langle \hat{v}_2, J\theta \rangle$. So truly, $v \rightarrow \hat{v}$ is an injection.

That $v \rightarrow \hat{v}$ is a linear mapping is clear. Its sequential continuity follows directly from (4-16) and theorem 4-5. Lemma 4-2 is now established.

The concept of "composition \circ " employs the idea of a distribution y_x depending on a parameter $x \in R$. For our purpose, we will impose upon y_x the

Conditions G:

G1. For each fixed $x \in R$, y_x is a member of $[\mathcal{D}; [A; B]]$. Thus, for any given $\phi \in \mathcal{D}$, the equation:

$$(4-17) \quad \psi(x) \stackrel{\Delta}{=} \langle y_x, \phi \rangle$$

defines an $[A; B]$ -valued function ψ on R .

G2. $\phi \mapsto \psi$ is a mapping of \mathcal{D} into $\mathcal{E}([A; B])$.

The next lemma is due to Dolezal [16].

Lemma 4-3: Assume that y_x satisfies conditions G. Then, the mapping $\varphi \mapsto \psi$ of \mathcal{D} into $\mathcal{E}([A; B])$ is linear and continuous.

We now define the composition o product $v \circ y_x$ of any $v \in [\mathcal{E}; A]$ with y_x by

$$(4-18) \quad \langle v \circ y_x, \omega \rangle \stackrel{\Delta}{=} \langle \hat{v}, \psi \rangle \stackrel{\Delta}{=} \langle \hat{v}(x), \langle y_x(t), \varphi(t) \rangle \rangle \quad \omega \in \mathcal{D}$$

where $\hat{v} \in [\mathcal{E}([A; B]); B]$ is defined by (4-15) or equivalently by (4-16)

At times, we will denote the operator $v \mapsto v \circ y_x$ by $\circ y_x$.

Theorem 4-6: Let $v \in [\mathcal{E}; A]$ and let y satisfy conditions G.

Then, $v \mapsto v \circ y_x$ is a sequentially continuous linear mapping of $[\mathcal{E}; A]$ into $[\mathcal{D}; B]$.

Proof: That $v \circ y_x \in [\mathcal{D}; B]$ follows immediately from the definition (4-18) and the following two facts: $\varphi \mapsto \psi$ is a continuous linear mapping of \mathcal{D} into $\mathcal{E}([A; B])$ according to lemma 4-3. $\psi \mapsto \langle \hat{v}, \psi \rangle$ is a continuous linear mapping of $\mathcal{E}([A; B])$ into B according to lemma 4-2.

It is clear that $v \mapsto v \circ y_x$ is a linear mapping. To verify its sequential continuity, let Ω be an arbitrarily chosen bounded set in \mathcal{D} , let $\{v_j\}_{j=1}^{\infty}$ tend to zero in $[\mathcal{E}; A]$, and define \hat{v}_j from v_j as in lemma 4-2. By lemma 4-3, ψ traverses a bounded set Λ in $\mathcal{E}([A; B])$ when φ traverses Ω . So, for the corresponding seminorm σ_{Ω} on $[\mathcal{D}; B]$, we have

$$\sigma_{\Omega}(v_j \circ y_x) \stackrel{\Delta}{=} \sup_{\varphi \in \Omega} \|\langle v_j \circ y_x, \varphi \rangle\|_B = \sup_{\psi \in \Lambda} \|\langle \hat{v}_j, \psi \rangle\|_B$$

By theorem 4-5 and (4-16),

$$\begin{aligned} \sigma_{\Omega}(v_j \circ y_x) &= \sup_{\psi \in \Lambda} \left\| \sum_{k=0}^p (-1)^k \int_N \psi^{(k)}(t) h_{k,j}(t) dt \right\|_B \\ &\leq \sum_{k=0}^p \sup_{t \in N} \|h_{k,j}(t)\|_A \sup_{\psi \in \Lambda} \int_N \|\psi^{(k)}(t)\|_{[A; B]} dt, \end{aligned}$$

and the right-hand side tends to zero as $j \rightarrow \infty$. This shows that $v_j \circ y_x$ tends to zero in $[\mathcal{D}; B]$ and completes the proof.

In much the same way, a variation of theorem 4-6 can be established. In this case we assume that $v \in [\mathcal{D}; A]$ and that y_x satisfies the following two conditions:

Condition G':

G1'. y_x satisfies condition A1.

G2'. $\varphi \mapsto \psi$ is a mapping of \mathcal{D} into $\mathcal{D}([A; B])$.

Dolezal [16] has shown that conditions G' imply that the mapping $\varphi \mapsto \psi$ of \mathcal{D} into $\mathcal{D}([A; B])$ is linear and continuous. Furthermore, by modifying the proof of lemma 4-2 (we now use theorem 4-4 instead of 4-5), we can also show that any given $v \in [\mathcal{D}; A]$ defines a unique $\hat{v} \in [\mathcal{D}([A; B]); B]$ via the equation (4-15), where now $\theta \in \mathcal{D}$ and, in addition, $v \mapsto \hat{v}$ is a sequentially continuous linear injection of $[\mathcal{D}; A]$ into $[\mathcal{D}([A; B]); B]$. Once again, we define $v \circ y_x$ by (4-18). Then, an argument almost identical to the proof of theorem 4-6 establishes

Theorem 4-7: Let $v \in [\mathcal{D}; A]$ and let y_x satisfy conditions G'. Then $v \mapsto v \circ y_x$ is a sequentially continuous linear mapping of $[\mathcal{D}; A]$ into $[\mathcal{D}; B]$.

We can relate composition of mappings to composition of mappings in the following way. Let there be given a composition of mapping $\mathfrak{M}: v \mapsto v \circ y_x$ where $v \in [\mathcal{E}; A]$ and y_x satisfies conditions G. Then, the restriction of \mathfrak{M} to $\mathcal{D}(A)$ is a continuous linear mapping \mathfrak{N} of $\mathcal{D}(A)$ into $[\mathcal{D}; B]^W$ given

by $v \mapsto f \circ v$, $v \in \mathcal{A}(A)$, where $f \in [\mathcal{D}_{t,x}(A) ; B]$ is uniquely determined. Indeed, $\mathcal{A}(A) \subset [\mathcal{E} ; A]$, and the topology of $\mathcal{A}(A)$ is stronger than that induced on it by $[\mathcal{E} ; A]$. On the other hand, the topology of $[\mathcal{D} ; B]$ is stronger than that of $[\mathcal{D} ; B]^W$. Hence, by theorem 4-6, the restriction \mathfrak{M} of \mathfrak{M} to $\mathcal{A}(A)$ is a sequentially continuous linear mapping of $\mathcal{A}(A)$ into $[\mathcal{D} ; B]^W$. It is even continuous since $\mathcal{A}(A)$ is the inductive limit of Fréchet spaces. Thus, we may invoke theorem 4-3 to conclude that $\mathfrak{M} = f \circ$ on $\mathcal{A}(A)$, where $f \in [\mathcal{D}_{t,x}(A) ; B]$ is uniquely determined.

It is also true that the mapping $\mathfrak{M} \mapsto \mathfrak{M}$ is injective because $\mathcal{A}(A)$ is dense in $[\mathcal{E} ; A]$. We shall see later on (theorem 4-8) that any given continuous linear mapping \mathfrak{M} of $[\mathcal{E} ; A]$ into $[\mathcal{D} ; B]$ uniquely determines a y_x satisfying conditions G such that $\mathfrak{M}v = v \circ y_x$ for every $v \in [\mathcal{E} ; A]$. It follows from these two facts that, if $f \in [\mathcal{D}_{t,x}(A) ; B]$ is given and if $f \circ$ can be extended into a continuous linear mapping \mathfrak{M} of $[\mathcal{E} ; A]$ into $[\mathcal{D} ; B]$, then there exists a unique y_x satisfying conditions G such that $f \circ v = v \circ y_x$ for all $v \in \mathcal{A}(A)$.

As examples we now develop the composition \circ operators corresponding to the composition \circ operators presented in examples 4-1, 4-2, and 4-3.

Example 4-1a: For $x \in \mathbb{R}$ and n a positive integer, we set

$$(4-19) \quad y_x(t) = h(t) D_t^n \delta_x(t)$$

where now we assume that $h \in \mathcal{C}([A ; B])$ in contrast to the less restrictive assumption made in example 4-1. (The symbol δ_x denotes the shifted delta functional; that is, $\langle \delta_x, \varphi \rangle = \varphi(x)$.) For any $\varphi \in \mathcal{D}$

$$\langle y_x, \varphi \rangle = \langle h D^n \delta_x, \varphi \rangle = (-D_x)^n [h(x) \varphi(x)] \in \mathcal{A}([A ; B]) .$$

Thus, y_x satisfies both conditions G and G'. So, by theorem 4-7, $v \circ y_x \in [\mathcal{D}; B]$ for any $v \in [\mathcal{D}; A]$. In particular, for $\varphi \in \mathcal{D}$, we have

$$\begin{aligned} \langle v \circ y_x, \varphi \rangle &= \langle \hat{v}(x), \langle h(t) D_t^n \delta_x(t), \varphi(t) \rangle \rangle \\ &= \langle \hat{v}(x), (-D_x)^n [h(x) \varphi(x)] \rangle \in B. \end{aligned}$$

By virtue of the representation $v = D_k^p g$, $g \in \mathcal{D}_k^0(A)$, from which \hat{v} is defined via the analogue to (4-16), the right-hand side is equal to

$$\int_N g(x) (-D_x)^{n+p} [h(x) \varphi(x)] dx = \langle h D^n v, \varphi \rangle .$$

Thus, in the sense of equality in $[\mathcal{D}; B]$,

$$(4-20) \quad v \circ y_x = h D^n v .$$

Compare this to (4-5).

Example 4-2a: Let $\tau \in \mathbb{R}$, let σ_τ be the shifting operator as before, let $h \in \mathcal{C}([A; B])$, and set

$$(4-21) \quad y_x = h \sigma_\tau \delta_x = h \delta_{x+\tau} .$$

Here too, y_x satisfies conditions G and G'. For any $v \in [\mathcal{D}; A]$ and $\varphi \in \mathcal{D}$,

$$\begin{aligned} \langle v \circ y_x, \varphi \rangle &= \langle \hat{v}(x), \langle h(t) \delta_{x+\tau}(t), \varphi(t) \rangle \rangle \\ &= \langle \hat{v}(x), h(x + \tau) \varphi(x + \tau) \rangle \in B . \end{aligned}$$

As in the previous example, the right-hand side can be shown to be equal to

$$\langle h(x) v(x - \tau), \varphi(x) \rangle = \langle h \sigma_\tau v, \varphi \rangle .$$

Thus, in the sense of equality in $[\mathcal{D}; B]$,

$$(4-22) \quad v \circ y_x = h \sigma_T v .$$

Example 4-3a: We now turn to convolution once again. Let $y \in [\mathcal{D}(A); B]$ and $v \in [\mathcal{E}; A]$. By theorem 3-1, y is also a member of $[\mathcal{D}; [A; B]]$. For each $x \in \mathbb{R}$, we define y_x as either a member of $[\mathcal{D}(A); B]$ or $[\mathcal{D}; [A; B]]$ by $y_x = \sigma_x y$. So, for any $\varphi \in \mathcal{D}$,

$$(4-23) \quad \langle v \circ y_x, \varphi \rangle = \langle \hat{v}(x), \langle y_x(t), \varphi(t) \rangle \rangle = \langle \hat{v}(x), \langle y(t), \varphi(t+x) \rangle \rangle .$$

Here, the right-hand side has the sense of the application of $\hat{v} \in [\mathcal{E}([A; B]); B]$ to $\langle y(t), \varphi(t+x) \rangle \in \mathcal{E}([A; B])$. (See [4; theorems 3-1 and 4-3].) We now employ the representation of \hat{v} as a sum of derivatives of strongly continuous A -valued functions on \mathbb{R} of compact support (see theorem 4-5) and the representation of y on \mathcal{D}_K for any given compact interval K as the derivative of an $[A; B]$ -valued function on \mathbb{R} that is continuous in the norm topology of $[A; B]$ (see [4; theorem 3-1 and equations (3.8) and (3.9)]). This allows us to invoke Fubini's theorem and then to rewrite (4-23) as follows, where now $y \in [\mathcal{D}(A); B]$.

$$\langle v \circ y_x, \varphi \rangle = \langle y(t), \langle v(x), \varphi(t+x) \rangle \rangle = \langle y * v, \varphi \rangle$$

Thus, in the sense of equality in $[\mathcal{D}; B]$, $v \circ y_x = y * v$.

Theorem 4-3 states that every continuous linear mapping of $\mathcal{D}(A)$ into $[\mathcal{D}; B]^W$ has a composition \circ representation. On the other hand, the examples of this section show that at least in three particular cases a composition \circ operator has a corresponding composition \circ operator. A natural conjecture therefore is that every continuous linear mapping of $[\mathcal{E}; A]$ into $[\mathcal{D}; B]$ has a composition \circ representation. Theorem 4-8 below, which is a partial converse to theorem 4-6, states that this is in-

deed the case. Its proof makes use of

Lemma 4-4: Every $v \in [\mathcal{E}; A]$ is the limit in $[\mathcal{E}; A]$ of a sequence $\{v_\nu\}_{\nu=1}^\infty$ such that each v_ν is a finite sum of the form:

$$v_\nu(t) = \sum_{\mu=1}^{m_\nu} a_{\nu,\mu} D_X^r \delta_X(t) \Big|_{x=\tau_{\nu,\mu}}$$

where $a_{\nu,\mu} \in A$, $\tau_{\nu,\mu} \in \mathbb{R}$, and r is a nonnegative integer not depending on μ or ν .

The proof of this lemma mimics that of lemma 2 in [19; Sec. 5.8]. In this case, we use theorem 4-5 to write $v = (-D)^{r+2} h$ where $h \in \mathcal{E}^0(A)$. (Here, h need not have a compact support.) As in [19; p. 145], we then set up the functions $h_\nu \in \mathcal{E}^0(A)$, all of which coincide with h outside some compact interval I containing $\text{supp } v$. This allows us to write, for any $\varphi \in \mathcal{E}$,

$$\langle v - (-D)^{r+2} h_\nu, \varphi \rangle = \int_I (h - h_\nu) D^{r+2} \varphi dt.$$

Lemma 4-4 is established by estimating a bound on this integral.

Theorem 4-8: Let \mathfrak{M} be a continuous linear mapping of $[\mathcal{E}; A]$ into $[\mathcal{D}; B]$. Then, there exists a unique γ_X satisfying conditions G such that for every $v \in [\mathcal{E}; A]$, $\mathfrak{M} v = v \circ \gamma_X$ in the sense of equality in $[\mathcal{D}; B]$.

Proof: Define an operator \mathfrak{M} from \mathfrak{M} by

$$(4-24) \quad \langle \mathfrak{M}g, \varphi \rangle \triangleq \langle \mathfrak{M}(g a), \varphi \rangle$$

where $g \in [\mathcal{E}; C]$, $a \in A$, and $\varphi \in \mathcal{D}$. By [4; theorem 3-2], \mathfrak{M} is a continuous linear mapping of $[\mathcal{E}; C]$ into $[\mathcal{D}; [A; B]]$. Moreover, the equation

$\langle a \mathfrak{M}g, \varphi \rangle = \langle \mathfrak{M}g, \varphi \rangle$ defines $a \mathfrak{M}g$ as a member of $[\mathcal{D}; B]$ (see 4: Sec. 3).

Therefore, in the sense of equality in $[\mathcal{D}; B]$,

$$(4-25) \quad \mathfrak{M}g = \mathfrak{N}(ga) .$$

For each $x \in R$, set $y_x = \mathfrak{M}\delta_x \in [\mathcal{D}; [A; B]]$. We now establish two facts:

(i) For any given $\varphi \in \mathcal{D}$, $\langle y_x, \varphi \rangle$ as a function of x is a member of $\mathcal{E}([A; B])$ so that y_x satisfies conditions G. (ii) Moreover, for any nonnegative integer k ,

$$(4-26) \quad D_x^k y_x = \mathfrak{M} D_x^k \delta_x$$

in the sense of equality in $[\mathcal{D}; [A; B]]$. ($D_x^k y_x$ denotes the k^{th} -order parametric derivative of y_x defined by $\langle D_x^k y_x, \varphi \rangle \triangleq D_x^k \langle y_x, \varphi \rangle$, $\varphi \in \mathcal{D}$.) We use an inductive argument. First note that (4-26) is true for $k = 0$ by definition. Next, fix x and choose any $\Delta x \in R$, $\Delta x \neq 0$. Assuming that (4-26) is true for some k , we may write

$$(4-27) \quad \frac{1}{\Delta x} (D_{x+\Delta x}^k y_{x+\Delta x} - D_x^k y_x) = \mathfrak{M} \frac{1}{\Delta x} [D_{x+\Delta x}^k \delta_{x+\Delta x} - D_x^k \delta_x] .$$

The quantity in the right-hand side upon which \mathfrak{M} operates converges in $[\mathcal{E}; C]$ to $D_x^{k+1} \delta_x$. Therefore, (4-27) converges in $[\mathcal{D}; [A; B]]$ to

$$D_x^{k+1} y_x = \mathfrak{M} D_x^{k+1} \delta_x ,$$

and, in addition, $\langle y_x, \varphi \rangle$ is a smooth $[A; B]$ -valued function on R by the definition of parametric differentiation. The statements (i) and (ii) are hereby established.

We now employ the sequence $\{v_\nu\}$ indicated in lemma 4-4. By the linearity of \mathfrak{N} and equations (4-25) and (4-26), we may write

$$(4-29) \quad \mathfrak{N} v_\nu = \sum_{\mu=1}^{m_\nu} a_{\nu,\mu} D_x^r y_x \Big|_x = \tau_{\nu,\mu}$$

By lemma 4-4 and the continuity of \mathfrak{N} , the left-hand side of (4-29) converges in $[\mathcal{D}; B]$ to $\mathfrak{N}v$. On the other hand, the application of the right-hand side of (4-29) to any $\varphi \in \mathcal{D}$ yields

$$\sum_{\mu=1}^{m_\nu} a_{\nu,\mu} D_X^r \langle y_X, \varphi \rangle \Big|_X = \tau_{\nu,\mu}$$

where $\langle y_X, \varphi \rangle \in \mathcal{E}([A; B])$ as was noted above. The last quantity is equal to

$$\begin{aligned} & \sum_{\mu=1}^{m_\nu} a_{\nu,\mu} \langle D_{\tau_{\nu,\mu}}^r \delta_{\tau_{\nu,\mu}}(x), \langle y_X(t), \varphi(t) \rangle \rangle \\ & = \langle v_\nu \circ y_X, \varphi \rangle . \end{aligned}$$

But, this tends to $v \circ y_X$ since $f \mapsto f \circ y_X$ is continuous on $[\mathcal{E}; A]$. Thus, theorem 4-8 is proven.

Finally, we define the concept of causality and point out how it affects the composition \circ and composition \circ representations of operators generated by time-varying Banach systems.

Definition 4-1: Let \mathfrak{N} be an operator mapping a set $\mathcal{X} \subset [\mathcal{D}; A]$ into $[\mathcal{D}; B]$. \mathfrak{N} is said to be causal on \mathcal{X} if, for every $t_0 \in \mathbb{R}$, we have that $(\mathfrak{N}v_1)(t) = (\mathfrak{N}v_2)(t)$ on $-\infty < t < t_0$ whenever $v_1 \in \mathcal{X}$, $v_2 \in \mathcal{X}$, and $v_1(t) = v_2(t)$ on $-\infty < t < t_0$.

The following theorems are established in the same way as in the scalar case [20].

Theorem 4-9: Let $f \in [\mathcal{D}_{t,x}(A); B]$. The operator $f \circ$ is causal on $\mathcal{B}(A)$ if and only if $\text{supp } f$ is contained in the half plane $\{t, x: t \geq x\}$

Theorem 4-10: Let y_X satisfy conditions G. The operator $\circ y_X$ is causal on $[\mathcal{E}; A]$ if and only if, for each fixed $x \in \mathbb{R}$, $\text{supp } y_X$ is contained in the semi-infinite line $[x, \infty)$.

5. Time-invariant Banach Systems and Convolution.

A time-invariant Banach system is of course one whose components do not vary with time. One can define such a system in a mathematical way by saying that every operator generated by the system commutes with the shifting operator σ_τ whatever be the choice of $\tau \in \mathbb{R}$. (We shall also call every such operator time-invariant.) In this case the composition \circ and composition \circ operators become convolution operators. The latter is defined as follows [4; Sec. 4].

Let y be a fixed member of $[\mathcal{B}(A); B]$. Then, the convolution product $y * v$ of y and any $v \in [\mathcal{E}; A]$ is defined by

$$(5-1) \quad \langle y * v, \varphi \rangle \stackrel{\Delta}{=} \langle y(t), \langle v(x), \varphi(t+x) \rangle \rangle \quad \varphi \in \mathcal{D}$$

The right-hand side has a sense because $\psi(t) \stackrel{\Delta}{=} \langle v(x), \varphi(t+x) \rangle \in \mathcal{B}(A)$. This also defines the convolution operator $v \mapsto y * v$, which we denote by $y *$. This operator is a continuous linear mapping of $[\mathcal{E}; A]$ into $[\mathcal{B}; B]$ (see [4; theorem 4-1]). Moreover, it is time-invariant, which means, as was noted above, that it commutes with the shifting operator σ_τ for every $\tau \in \mathbb{R}$ [4; proposition 4-1]. In addition, $y *$ is a continuous linear mapping of $\mathcal{B}(A)$ into $\mathcal{E}(B)$, and in this case we have that

$$(5-2) \quad (y * v)(t) = \langle y(x), v(t-x) \rangle \quad v \in \mathcal{B}(A)$$

in the sense of equality in $[\mathcal{B}; B]$ (see [4; theorem 4-3]). Similar results hold for several other spaces of Banach-space-valued distributions.

Conversely, every continuous linear mapping of $\mathcal{B}(A)$ into $[\mathcal{B}; B]$ that commutes with the shifting operator σ_τ for every $\tau \in \mathbb{R}$ has a convolution representation [4; theorem 6-1]. In particular, we have

Theorem 5-1: \mathcal{R} is a continuous linear time-invariant mapping of $\mathcal{B}(A)$ into $[\mathcal{B}; B]$ if and only if there exists a $y \in [\mathcal{B}(A); B]$ such that

$\mathfrak{R} = y * \text{ on } \mathcal{D}(A)$ (i.e., in the sense of equality in $[\mathcal{D}; B]$, $\mathfrak{R}v = y * v$ for all $v \in \mathcal{D}(A)$). y is uniquely determined by \mathfrak{R} , and conversely.

(This theorem can be refined by replacing $\mathcal{D}(A)$ by the space $\mathcal{D} \otimes A$ which is the span of all elements of the form ϕa where $\phi \in \mathcal{D}$ and $a \in A$ [4].)

In view of theorem 4-3, 4-8, and 5-1, we see that every convolution operator is a special case of a composition \circ operator as well as of a composition \circ operator. This was also observed in examples 4-3 and 4-3a. By theorem 3-1 and the analysis of example 4-3a, we see that the composition \circ operator $\circ y_x$ corresponding to any given convolution operator $y *$ is obtained simply by setting $y_x = \sigma_x y$. Thus, theorem 4-10 immediately yields

Theorem 5-2: Let $y \in [\mathcal{D}(A); B]$. Then, the convolution operator $y *$ is causal on $[\mathcal{E}; A]$ if and only if $\text{supp } y \subset [0, \infty)$.

A causality criterion for $y *$ can be stated in terms of the Laplace transform $\mathcal{L}y$ of y if $\mathcal{L}y$ happens to exist in the sense stated at the end of Sec. 3 [4; theorem 6-2 and proposition 6-3].

Theorem 5-3: Assume that $y \in [\mathcal{L}(w, z; A); B]$ for some w and z . Necessary and sufficient conditions for $y *$ to be causal on $[\mathcal{E}; A]$ (and, in fact, on $[\mathcal{L}(w, z); A]$ are that $z = \infty$ and, on some half plane $\{\zeta : \text{Re } \zeta \geq a, a \in \mathbb{R}\}$, we have

$$\|(\mathcal{L}y)(\zeta)\|_{[A; B]} \leq P(|\zeta|)$$

where P is a polynomial.

6. Hilbert Ports and Passivity: An Admittance Formulism.

As was explained in Sec. 2, the concept of a Hilbert port arises when two physical variables v and u in a system take their values in a complex Hilbert space H and, in addition, are complementary in the sense that their inner product $(u, v) = (u(t), v(t))$ represents the instantaneous complex power entering the system. If this power is Lebesgue integrable on the interval $(-\infty, x)$, then the integral

$$(6-1) \quad \operatorname{Re} \int_{-\infty}^x (u, v) dt$$

represents the total energy entering the system during the time interval $-\infty < t < x$. This allows us to define the passivity of the admittance operator $\mathfrak{M}: v \mapsto u$.

Definition 6-1: Let $\mathcal{W}(H)$ be a set of H -valued functions on \mathbb{R} contained in the domain of an operator \mathfrak{M} . \mathfrak{M} is said to be a passive mapping on $\mathcal{W}(H)$ if, for every $v \in \mathcal{W}(H)$, for $u = \mathfrak{M}v$, and for every finite real number x , we have that $(u(t), v(t))$ is Lebesgue integrable on $-\infty < t < x$ and the integral (6-1) is nonnegative.

When \mathfrak{M} is the admittance operator of a Hilbert port and is passive, we shall also call the Hilbert port passive.

If \mathfrak{M} is a convolution operator $y *$, $y \in [\mathcal{D}(H); H]$, then it turns out that, for every $v \in \mathcal{B}(H)$, $u = y * v$ is a member of $\mathcal{E}(H)$ and that $(u, v) \in \mathcal{B}$ [4, theorem 4-3 and lemma 7-1]. Thus, (6-1) certainly exists for every $x \in \mathbb{R}$, and we may establish the passivity of \mathfrak{M} on $\mathcal{B}(H)$ merely by checking the nonnegativity of (6-1).

Every passive convolution operator $y *$ has a frequency-domain description; namely, its impulse response y possesses a Laplace transform Y which is positive*. The last word is defined as follows.

Definition 6-2: Given a complex Hilbert space H , a function Y of

the complex variable ζ is called a positive $*$ mapping of H into H (or simply positive $*$) if, on the half plane $C_+ = \{\zeta : \text{Re } \zeta > 0\}$, Y is an $[H; H]$ -valued analytic function such that $\text{Re}(Y(\zeta) a, a) \geq 0$ for every $a \in H$.

The principal theorem in the admittance formalism of passive Hilbert ports is the following [4].

Theorem 6-1: Assume that \mathfrak{R} is a continuous linear time-invariant passive mapping of $\mathcal{B}(H)$ into $[\mathcal{B}; H]$. Then, \mathfrak{R} has a convolution representation $\mathfrak{R} = y *$ where $y \in [L_1(H); H]$ and $\text{supp } y \subset [0, \infty)$. Moreover, y possesses a Laplace transform Y on at least C_+ which is positive $*$.

Conversely, assume that Y is positive $*$. Then, there exists a unique convolution operator $\mathfrak{R} = y *$ such that $y \in [L_1(H); H]$, $\text{supp } y \subset [0, \infty)$, and $\mathcal{B}y = Y$ on C_+ . Moreover, \mathfrak{R} is a continuous linear time-invariant passive mapping of $\mathcal{B}(H)$ into $[\mathcal{B}; H]$.

Note that the fact that $\text{supp } y \subset [0, \infty)$ implies that $y *$ is also causal on $[\mathcal{E}; H]$ as was indicated in theorem 5-2.

To introduce the concept of positive $*$ -reality, we must first consider real operators in $[H; H]$, and this in turn requires that we assign to H a somewhat more complicated structure. In particular, we shall now assume that the complex Hilbert space H is generated from a real Hilbert space H_r through complexification [21; Sec. 2.1]. This implies among other things that $H_r \subset H$. Then, an $F \in [H; H]$ is called real if $F \in [H_r; H_r]$.

Definition 6-3: Given H and H_r as stated, a function Y of the complex variable ζ is called positive $*$ -real if it is a positive $*$ mapping of H into H and, for each real positive number σ , $Y(\sigma)$ is real.

Corollary 6-1a: Theorem 6-1 remains valid when H is replaced by H_r , "positive $*$ " by "positive $*$ -real", and \mathcal{B} by the space $\mathcal{B}(R)$ of real-valued functions in \mathcal{B} .

(Note: The spaces $\mathcal{B}(H_R)$, $\mathcal{B}_{L_1}(H_R)$, and $[\mathcal{B}(R); H_R]$, their topologies, and the properties of passivity, time-invariance, etc. are defined just as they are in the complex case.)

7. Hilbert Ports and Passivity: A Scattering Formulism.

Let us consider once again a Hilbert port and its variables v , u which determine its admittance operator $\mathfrak{M}: v \rightarrow u$. A scattering formulism for the Hilbert port is generated by working with the variables:

$$(7-1) \quad v_+ \stackrel{\Delta}{=} v + u, \quad v_- \stackrel{\Delta}{=} v - u.$$

We call v_+ the incident wave and v_- the reflected wave. The mapping $\mathfrak{B}: v_+ \rightarrow v_-$ is the scattering operator of the Hilbert port. In terms of v_+ and v_- , the energy integral (6-1) becomes

$$(7-2) \quad \int_{-\infty}^x [(v_+, v_+) - (v_-, v_-)] dt.$$

Definition 7-1: Let $\mathcal{W}(H)$ be a set of H -valued functions on R contained in the domain of an operator \mathfrak{B} . \mathfrak{B} is said to be scatter-passive on $\mathcal{W}(H)$ if, for every $x \in R$, for every $v_+ \in \mathcal{W}(H)$, and for $v_- = \mathfrak{B}v_+$, we have that (v_+, v_+) and (v_-, v_-) are both Lebesgue integrable on $(-\infty, x)$ and (7-2) is nonnegative. \mathfrak{B} is said to be scatter-passive-at-infinity on $\mathcal{W}(H)$ if, for every $v_+ \in \mathcal{W}(H)$ and for $v_- = \mathfrak{B}v_+$, we have that (v_+, v_+) and (v_-, v_-) are both Lebesgue integrable on R and

$$(7-3) \quad \int_{-\infty}^{\infty} [(v_+, v_+) - (v_-, v_-)] dt$$

is nonnegative.

If \mathfrak{B} is a linear mapping of $\mathcal{B}(H)$ into $[\mathcal{B}; H]$ and is scatter-passive-at-infinity on $\mathcal{B}(H)$, then \mathfrak{B} is also continuous from $\mathcal{B}(H)$ into $[\mathcal{B}; H]$ (see [5; Sec. 3]).

There is an interesting relationship between scatter-passivity and the two properties of causality and scatter-passivity-at-infinity. It was discovered by Wohlers and Beltrami for the scalar case [22]. Its extension to Hilbert ports, which is stated in the next theorem, is established in [5; Sec. 3].

Theorem 7-1: Assume that \mathfrak{B} is a linear time-invariant mapping of $\mathcal{B}(H)$ into $[\mathcal{D}; H]$. Then, \mathfrak{B} is a scatter-passive on $\mathcal{B}(H)$ if and only if \mathfrak{B} is causal and scatter-passive-at-infinity on $\mathcal{B}(H)$.

For the frequency-domain description of our scattering formalism, we will need

Definition 7-2: Given a complex Hilbert space H , a function S of the complex variable ζ is said to be a bounded* mapping of H into H (or simply bounded*) if, on the half plane $C_+ \stackrel{\Delta}{=} \{\zeta : \text{Re } \zeta > 0\}$, S is an $[H; H]$ -valued analytic function such that $\|S(\zeta)\|_{[H; H]} \leq 1$.

A description for the scattering formalism of a passive Hilbert port [5; theorems 4-2 and 5-1] is given by

Theorem 7-2: Assume that \mathfrak{B} is a linear time-invariant causal scatter-passive-at-infinity mapping of $\mathcal{B}(H)$ into $[\mathcal{D}; H]$. Then, \mathfrak{B} has a convolution representation $\mathfrak{B} = s *$, where $s \in [\mathcal{D}_{L_2}(H); H]$ and $\text{supp } s \subset [0, \infty)$. Moreover, s possesses a Laplace transform S on at least C_+ which is bounded*.

Conversely, assume that S is bounded*. Then, there exists a unique convolution operator $\mathfrak{B} = s *$ such that $s \in [\mathcal{D}_{L_2}(H); H]$, $\text{supp } s \subset [0, \infty)$, and $\mathcal{B}s = S$ on C_+ . Moreover, \mathfrak{B} is a continuous linear time-invariant causal scatter-passive-at-infinity mapping of $\mathcal{B}(H)$ into $[\mathcal{D}; H]$.

Now, for bounded*-reality [5; corollaries 4-2a and 5-1a]. Assume once again that H is generated from a real Hilbert space H_r through complexification.

Definition 7-3: Given H and H_r , as stated, a function S of the complex variable ζ is called bounded*-real if it is a bounded* mapping of H into H and, for each real positive number σ , $S(\sigma)$ is real.

Corollary 7-2a: Theorem 7-1 remains valid when H is replaced by H_r , "bounded*" by "bounded*-real", and \mathcal{B} by $\mathcal{B}(R)$.

We end this section by stating the connection between the admittance and scattering formalisms. Given any Hilbert port, whose operators need not satisfy any assumptions of linearity, continuity, etc., we see immediately from (7-1) that the admittance operator $\mathfrak{A}: v \mapsto u$ uniquely determines and is uniquely determined by the scattering operator $\mathfrak{B}: v_+ \mapsto v_-$. In this case, either one or both of these operators may be multivalued. However, when the aforementioned assumptions are imposed, we get the following theorem (see [5; theorem 6-1 and 6-2]).

Theorem 7-3: If the admittance operator $\mathfrak{A}: v \rightarrow u$ of a Hilbert port is a continuous linear time-invariant passive mapping of $\mathcal{B}(H)$ into $[\mathcal{B}; H]$, then its scattering operator $\mathfrak{B}: v_+ \rightarrow v_-$ is a linear time-invariant causal scatter-passive-at-infinity mapping of $\mathcal{B}(H)$ into $[\mathcal{B}; H]$

The converse statement is also true if the scattering transform $S \stackrel{\Delta}{=} \mathfrak{B} s$ corresponding to $\mathfrak{B} = s^*$ is such that, for every $\zeta \in C_+$, $(I + S)^{-1}$ exists. Here, I denotes the identity operator on H .

8. ∞ -ports.

Henceforth, we assume that the complex Hilbert space H is separable. This allows us to exploit the isomorphism between any such Hilbert space and the space ℓ_2 in order to introduce an infinite-dimensional extension to the concept of an n -port. We also assume throughout the rest of this paper that an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ has been chosen in H . (When analyzing systems such as microwave-transmission networks, it is natural to fix upon the orthonormal basis generated by a modal analysis.)

Lemma 8-1: Let $v \in \mathcal{D}(H)$. Then, in the sense of convergence in $\mathcal{D}(H)$,
 (8-1)
$$v = \sum_{k=1}^{\infty} (v, e_k) e_k$$

Proof: Since, for any nonnegative integer p and any fixed t , $D^p v(t) \in H$, we have that

$$D^p v = \sum_{k=1}^{\infty} (D^p v, e_k) e_k.$$

By Parseval's equation,

$$\left\| \sum_{k=m+1}^{\infty} (D^p v, e_k) e_k \right\|_H^2 = \sum_{k=m+1}^{\infty} |(D^p v, e_k)|^2.$$

As $m \rightarrow \infty$, the right-hand side tends to zero monotonically at every point of R . By Dini's theorem [23, p. 117], it therefore tends to zero uniformly on every compact interval in R . But, $D^p v$ has a compact support. Therefore,

$$\sum_{k=1}^m (v, e_k) e_k$$

tends to v in $\mathcal{D}(H)$.

Lemma 8-2: Let $u \in [\mathcal{D}; H]$. Then, in the sense of convergence in $[\mathcal{D}; H]$,
 (8-2)
$$u = \sum_{k=1}^{\infty} (u, e_k) e_k.$$

Moreover, the series is unique; that is, if two such series converge in $[\mathcal{D}; H]$ to the same limit, they must have the same coefficients.

Proof: We first note that, for each k , $(u, e_k) \in [\mathcal{D}; \mathbb{C}]$ according to [4; Sec. 3] so that $(u, e_k)e_k \in [\mathcal{D}; \mathbb{H}]$. Moreover, for each $\varphi \in \mathcal{D}$, we have that $\langle u, \varphi \rangle \in \mathbb{H}$ and

$$\langle \langle u, \varphi \rangle, e_k \rangle e_k = \langle (u, e_k), \varphi \rangle e_k = \langle (u, e_k)e_k, \varphi \rangle$$

again according to [4; Sec. 3]. Hence, we may set up the orthonormal series expansion:

$$(8-3) \quad \langle u, \varphi \rangle = \sum_{k=1}^{\infty} \langle \langle u, \varphi \rangle, e_k \rangle e_k = \sum_{k=1}^{\infty} \langle (u, e_k)e_k, \varphi \rangle.$$

We wish to show that this series converges uniformly with respect to all φ in any given bounded set Ω in \mathcal{D} . Since \mathcal{D} is a Montel space [8; p. 357], the closure $\bar{\Omega}$ of Ω is a compact set in \mathcal{D} . So, we need merely establish the uniformity of the convergence on any compact set Λ in \mathcal{D} .

Consider the function F_m on \mathcal{D} into \mathbb{R} defined by

$$F_m(\varphi) \triangleq \left\| \langle u - \sum_{k=1}^{m-1} (u, e_k)e_k, \varphi \rangle \right\|^2 = \left\| \sum_{k=m}^{\infty} \langle (u, e_k)e_k, \varphi \rangle \right\|^2$$

F_m is continuous on \mathcal{D} . Moreover, by Parseval's equation,

$$F_m(\varphi) = \sum_{k=m}^{\infty} |\langle \langle u, \varphi \rangle, e_k \rangle|^2,$$

and therefore, for each $\varphi \in \mathcal{D}$, $F_m(\varphi)$ tends monotonically to zero as $m \rightarrow \infty$.

By the standard argument, we can conclude that $F_m(\varphi)$ tends to zero uniformly on any compact set Λ in \mathcal{D} . This in turn implies that the series in (8-3) converges uniformly on Λ .

The uniqueness of the expansion follows from the fact that, for any $a \in H$, the mapping $u \mapsto (u, a)$ is a continuous linear mapping of $[\mathcal{B}; H]$ into $[\mathcal{B}; C]$ (see [4; Sec. 3] again). Indeed, we need merely set $a = e_p$ and then apply this mapping to both sides of

$$u = \sum_{k=1}^{\infty} b_k e_k,$$

where $b_k \in [\mathcal{B}; C]$, doing this term by term on the right-hand side, to get $b_p = (u, e_p)$.

Lemma 8-3: Let \mathfrak{M} be a continuous linear mapping of $\mathcal{B}(H)$ into $[\mathcal{B}; H]$.

Then, for every choice of the positive integers j and k , there exists an

$f_{j,k} \in [\mathcal{B}_{t,x}; C]$ such that, for all $\varphi \in \mathcal{B}$,

$$(8-4) \quad \mathfrak{M}\varphi e_k = \sum_{j=1}^{\infty} (f_{j,k} \circ \varphi) e_j$$

where the series converges in $[\mathcal{B}; H]$.

Proof: Since $\mathfrak{M}\varphi e_k \in [\mathcal{B}; H]$, we can expand it according to lemma 8-2 to obtain

$$(8-5) \quad \mathfrak{M}\varphi e_k = \sum_{j=1}^{\infty} (\mathfrak{M}_{j,k}\varphi) e_j$$

where

$$\mathfrak{M}_{j,k}\varphi \triangleq (\mathfrak{M}\varphi e_k, e_j).$$

$\mathfrak{M}_{j,k}$ maps \mathcal{B} into $[\mathcal{B}; C]$ by [4; Sec. 3].

In addition, $\mathfrak{M}_{j,k}$ is both linear and continuous. Indeed, its linearity being clear, consider its continuity. Assume that $\varphi_v \rightarrow 0$ in \mathcal{B} as $v \rightarrow \infty$. Then, $\varphi_v e_k \rightarrow 0$ in $\mathcal{B}(H)$, and therefore $\mathfrak{M}(\varphi_v e_k) \rightarrow 0$ $[\mathcal{B}; H]$. We may also write for any $\psi \in \mathcal{B}$

$$(8-6) \quad \begin{aligned} |\langle \mathfrak{M}_{j,k}\varphi_v, \psi \rangle| &= |\langle (\mathfrak{M}\varphi_v e_k), e_j \rangle, \psi| \\ &= |\langle (\mathfrak{M}\varphi_v e_k, \psi), e_j \rangle| \leq \| \langle \mathfrak{M}\varphi_v e_k, \psi \rangle \| \end{aligned}$$

because $\|e_j\| = 1$. Since $\mathfrak{N}\varphi_\nu e_k \rightarrow 0$ in $[\mathcal{D}; H]$ as $\nu \rightarrow \infty$, the right-hand side of (8-6) tends to zero uniformly for all ψ in any bounded set in \mathcal{D} .

This proves the asserted continuity of $\mathfrak{N}_{j,k}$.

We may now invoke theorem 4-3 with $A = B = C$ to conclude that $\mathfrak{N}_{j,k}\varphi = f_{j,k} \circ \varphi$. (Here, $f_{j,k}$ is defined as in (8-7) below.) Inserting this result into (8-5), we complete the proof.

We are at last ready to establish for the mapping \mathfrak{N} a composition \circ representation that is amenable to an co-port interpretation.

Theorem 8-1: If \mathfrak{N} is a continuous linear mapping of $\mathcal{D}(H)$ into $[D; H]$, then there exists a collection $\{f_{j,k}\}_{j,k}$ of distributions in $[\mathcal{D}_{t,x}, C]$ defined by

$$(8-7) \quad \langle f_{j,k}(t,x), \psi(t)\varphi(x) \rangle \stackrel{\Delta}{=} \langle \mathfrak{N}_{j,k}\varphi, \psi \rangle \stackrel{\Delta}{=} \langle (\mathfrak{N}\varphi e_k, e_j), \psi \rangle$$

$$\varphi \in \mathcal{D}; \psi \in \mathcal{D}; j = 1, 2, \dots; k = 1, 2, \dots$$

such that, for any $v \in \mathcal{D}(H)$,

$$(8-8) \quad \mathfrak{N}v = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} [f_{j,k} \circ (v, e_k)] e_j = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} [f_{j,k} \circ (v, e_k)] e_j$$

where the series converge in $[D; H]$.

Proof: We may apply \mathfrak{N} term by term to the series indicated in lemma 8-1 to get

$$\mathfrak{N}v = \sum_{k=1}^{\infty} \mathfrak{N}(v, e_k) e_k.$$

Upon replacing φ by (v, e_k) in (8-4) and invoking lemma 8-3, we can rewrite the last equation as

$$(8-9) \quad \mathfrak{N}v = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} [f_{j,k} \circ (v, e_k)] e_j,$$

which is the first series in (8-8). Here, it is understood that we sum first on j and then on k to obtain in both cases a limit in $[D; H]$. To show that this order of summation can be reversed, we expand $\mathfrak{N}v \in [D; H]$ into a series according to lemma 8-2 to get

$$(8-10) \quad \mathfrak{M}v = \sum_{j=1}^{\infty} (\mathfrak{M}v, e_j) e_j .$$

Upon applying the operator $u \mapsto (u, e_p)$ to (8-9) as in the proof of lemma 8-2, we obtain

$$(\mathfrak{M}v, e_p) = \sum_{k=1}^{\infty} [f_{p,k} * (v, e_k)] .$$

Then, substituting this result into (8-10) with p replaced by j , we arrive at the second series in (8-8).

We can interpret the representation (8-8) in terms of an ∞ -port.

Think of a black box to whose interior we have access only through a collection of electrical ports which are countably infinite in number. Number these ports 1, 2, 3, Then, given the \mathfrak{M} of theorem 8-1 and any $v \in \mathcal{B}(H)$, set $u = \mathfrak{M}v$. Also, for each positive integer n , assume that the voltage impressed on the j^{th} port is $v_j = (v, e_j) \in \mathcal{B}$ so that the corresponding current is

$$u_j = \sum_{k=1}^{\infty} f_{j,k} * v_k \in [\mathcal{B}; \mathcal{C}] .$$

The operator that maps the vector $\{v_k\}_{k=1}^{\infty}$ into the vector $\{u_j\}_{j=1}^{\infty}$ can be represented by an $\infty \times \infty$ matrix $[f_{j,k}]$. In applying this matrix to any $\{v_k\}$ to get $\{u_j\}$, we follow the customary rule for the multiplication of matrices. Thus, the matrix equation corresponding to the composition $*$ representation (8-8) is

$$(8-11) \quad \{u_j\} = [f_{j,k}] * \{v_k\} .$$

It represents the behavior at the ports of a time-varying ∞ -port corresponding to the Hilbert port whose admittance operator is the \mathfrak{M} of theorem 8-1 and to the given choice of the orthonormal basis $\{e_k\}$.

We now take up the case where \mathfrak{M} commutes with the shifting operator and develops the matrix representation of a time-invariant ω -port. We first note that each $\mathfrak{M}_{j,k}$ in (8-7) also commutes with the shifting operator. Indeed, for any $\tau \in \mathbb{R}$, any $\psi \in \mathcal{D}$ and any $f \in [\mathcal{D}; \mathbb{C}]$, we have that

$$\begin{aligned} \langle \sigma_\tau(f, e_j), \psi \rangle &= \langle (f, e_j), \sigma_{-\tau}\psi \rangle = \langle (f, \sigma_{-\tau}\psi), e_j \rangle \\ &= \langle (\sigma_\tau f, \psi), e_j \rangle = \langle (\sigma_\tau f, e_j), \psi \rangle . \end{aligned}$$

Hence, in the sense of equality in $[\mathcal{D}; \mathbb{C}]$, we have for any $\varphi \in \mathcal{D}$.

$\sigma_\tau \mathfrak{M}_{j,k} \varphi = \sigma_\tau (\mathfrak{M} \varphi e_k, e_j) = (\sigma_\tau \mathfrak{M} \varphi e_k, e_j) = (\mathfrak{M} \sigma_\tau \varphi e_k, e_j) = \mathfrak{M}_{j,k} \sigma_\tau \varphi$, as was asserted.

By virtue of theorem 5-1 with $A = B = \mathbb{C}$, the composition \ast representations in theorem 8-1 become convolution representations, and we have

Corollary 8-1a: If \mathfrak{M} is a continuous linear time-invariant mapping of $\mathcal{D}(\mathbb{H})$ into $[\mathcal{D}; \mathbb{H}]$, then there exists a collection $\{y_{j,k}\}$ of distributions in $[\mathcal{D}; \mathbb{C}]$ defined by

$$(8-12) \quad y_{j,k} \ast \varphi \stackrel{\Delta}{=} \mathfrak{M}_{j,k} \varphi \stackrel{\Delta}{=} (\mathfrak{M} \varphi e_k, e_j) = (y \ast \varphi e_k, e_j)$$

$$\varphi \in \mathcal{D}; j = 1, 2, \dots; k = 1, 2, \dots$$

such that, for any $v \in \mathcal{D}(\mathbb{H})$,

$$(8-13) \quad \mathfrak{M} v = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} [y_{j,k} \ast (v, e_k)] e_j$$

where the series converges in $[\mathcal{D}; \mathbb{H}]$ and the order of summation can be reversed.

The matrix representation for the admittance equation of a time-invariant ω -port corresponding to the chosen $\{e_k\}$ and the Hilbert port whose admittance operator is the \mathfrak{M} of this corollary, is

$$(8-14) \quad \{u_j\} = [y_{j,k}] \ast \{v_k\} .$$

Here again, we use the customary rules for matrix multiplication.

Finally, we turn to the case where \mathcal{T} is not only continuous, linear, and time-invariant as a mapping of $\mathcal{D}(\mathcal{H})$ into $[\mathcal{D}; \mathcal{H}]$ but also passive on $\mathcal{D}(\mathcal{H})$. As was stated in theorem 6-1, this implies that $y \in [\mathcal{D}_{L_1}(\mathcal{H}); \mathcal{H}]$ and that $\text{supp } y \subset [0, \infty)$. Let us show that these conditions on y imply that, for each j and k , $y_{j,k} \in [\mathcal{D}_{L_1}; \mathbb{C}]$ and $\text{supp } y_{j,k} \subset [0, \infty)$.

Indeed, first consider $y * (e_k \delta)$. As before, we drop the parentheses in this expression and simply write $y * e_k \delta$. By the standard definition of convolution, we have for any $\phi \in \mathcal{D}_{L_1}$ the expression:

$$\begin{aligned} \langle y * e_k \delta, \phi \rangle &\stackrel{\Delta}{=} \langle y(t), \langle e_k \delta(x), \phi(t+x) \rangle \rangle \\ (8-15) \quad &= \langle y(t), e_k \phi(t) \rangle = \langle y, \phi \rangle e_k \end{aligned}$$

In the right-hand side, y now denotes a member of $[\mathcal{D}_{L_1}; [\mathcal{H}; \mathcal{H}]]$. Thus, $y * e_k \delta$ is a mapping of \mathcal{D}_{L_1} into \mathcal{H} . Moreover, this mapping is both linear and continuous, as is easily seen from the right-hand side of (8-15). Hence, $y * e_k \delta \in [\mathcal{D}_{L_1}; \mathcal{H}]$.

Next, consider $y_{j,k} = (y * e_k \delta, e_j)$. We get this expression from (8-12) by letting ϕ converge in $[\mathcal{E}; \mathbb{C}]$ to δ and invoking the continuity of the operators $y_{j,k} *$ and $y *$ on $[\mathcal{E}; \mathbb{C}]$ and $[\mathcal{E}; \mathcal{H}]$ respectively. So, again for any $\phi \in \mathcal{D}_{L_1}$, we have from [4; Sec. 3] that

$$(8-16) \quad \langle y_{j,k}, \phi \rangle = \langle (y * e_k \delta, e_j), \phi \rangle = \langle (y * e_k \delta, \phi), e_j \rangle.$$

This shows that $y_{j,k}$ maps \mathcal{D}_{L_1} into \mathbb{C} in a continuous linear fashion.

This is, $y_{j,k} \in [\mathcal{D}_{L_1}; \mathbb{C}]$.

As for the support of $y_{j,k}$, let $\phi \in \mathcal{D}$ and rewrite (8-12) as a regularization process [4; theorem 4-3].

$$(8-17) \quad \langle y_{j,k}(x), \phi(t-x) \rangle = \langle (y(x), \phi(t-x) e_k), e_j \rangle$$

Given any $\psi \in \mathcal{D}$ with $\text{supp } \psi \subset (-\infty, 0)$, we can choose $t \in \mathbb{R}$ and $\varphi \in \mathcal{D}$ such that $\varphi(t - x) = \psi(x)$ for all $x \in \mathbb{R}$. Since $\text{supp } y \subset [0, \infty)$, it follows immediately that the right-hand side of (8-17) is equal to zero. Thus, $\text{supp } y_{j,k} \subset [0, \infty)$.

We noted in Sec. 3 that these properties of y and $y_{j,k}$ imply that they have Laplace transforms Y and $Y_{j,k}$ respectively on at least the half-plane $C_+ \stackrel{\Delta}{=} \{\zeta : \text{Re } \zeta > 0\}$. We can relate $Y_{j,k}$ to Y as follows. For any $\zeta \in C_+$,

$$\begin{aligned} Y_{j,k}(\zeta) &= \langle y_{j,k}, e^{-\zeta t} \rangle = \langle (y * e_k \delta, e_j), e^{-\zeta t} \rangle \\ &= (\langle y * e_k \delta, e^{-\zeta t} \rangle, e_j) . \end{aligned}$$

The last equality holds because $y * e_k \delta \in [\mathcal{L}_{c,d}; H]$ for any $c \in \mathbb{R}$ with $c > 0$, and any $d \in \mathbb{R}$ and moreover $\text{supp } y * e_k \delta \subset [0, \infty)$. (See again [4; Sec. 3].) Thus,

$$Y_{j,k}(\zeta) = (\langle y(t), \langle e_k \delta(x), e^{-\zeta(t+x)} \rangle \rangle, e_j) = (\langle y(t), e_k e^{-\zeta t} \rangle, e_j)$$

or

$$(8-18) \quad Y_{j,k}(\zeta) = (Y(\zeta) e_k, e_j) \quad \zeta \in C_+ .$$

In the following l_2 represents the standard Hilbert space of all sequences $\alpha = \{\alpha_k\}$ of complex numbers for which the norm

$$\|\alpha\| = \left[\sum_{k=1}^{\infty} |\alpha_k|^2 \right]^{1/2}$$

exists. We know from theorem 6-1 that, under our stated assumptions on \mathfrak{A} , $Y(\zeta) \in [H; H]$ for each fixed $\zeta \in C_+$. Having fixed upon the orthonormal basis $\{e_k\}$ in H , we construct an $\infty \times \infty$ matrix $[Y_{j,k}(\zeta)]$ that represents the operator in $[l_2; l_2]$ corresponding to $Y(\zeta) \in [H; H]$ under the isomorphism existing between H and l_2 . The elements of this matrix are

given precisely by (8-18). (See [24; Sec. 3.1].) Moreover, if $v \in [A; H]$ has a Laplace transform V whose strip of definition contains the chosen $\zeta \in C_+$, then the Laplace transform of the equation (8-14) yields

$$(8-19) \quad \{U_j(\zeta)\} = [Y_{j,k}(\zeta)] \{V_k(\zeta)\}$$

where $\{U_j\}$ and $\{V_j\}$ denote the componentwise transformations of $\{u_j\}$ and $\{v_k\}$ respectively and, for the given ζ , $\{U_j(\zeta)\} \in \ell_2$ and $\{V_k(\zeta)\} \in \ell_2$.

We conclude this section by relating the positivity* of Y to the positivity* of $[Y_{j,k}]$.

Theorem 8-2: Y is a positive* mapping of H into H if and only if $[Y_{j,k}]$, as defined by (8-18), is a positive* mapping of ℓ_2 into ℓ_2 .
Thus, Y and $[Y_{j,k}]$ are positive* if and only if \mathcal{M} is a continuous linear time-invariant passive mapping of $\mathcal{A}(H)$ into $[A; H]$.

Proof: We have already noted that, for any fixed $\zeta \in C_+$, $Y(\zeta) \in [H; H]$ if and only if $[Y_{j,k}(\zeta)] \in [\ell_2; \ell_2]$.

Now let a and b be arbitrary members of H and let $\{a_k\}$ and $\{b_k\}$ be the corresponding members of ℓ_2 (i.e., $\{a_k\}$ is the sequence of Fourier coefficients of a with respect to $\{e_k\}$). Then, we have that

$$(8-20) \quad (Ya, b) = ([Y_{j,k}] \{a_k\}, \{b_j\}).$$

Thus, Y is weakly analytic on C_+ if and only if $[Y_{j,k}]$ is weakly analytic on C_+ . But, weak analyticity is equivalent to analyticity in the norm topology [25; p. 93].

We check the nonnegativity condition and thereby complete the proof of the first sentence of this theorem by setting $a = b$ in (8-20) and then taking real parts. The second sentence follows immediately from theorem 6-1.

Finally, assume once again that H is obtained from a real separable

Hilbert space H_r through complexification and assume that the orthonormal basis $\{e_k\}$ is contained in H_r . Through the isomorphism between H and ℓ_2 we have that H_r corresponds to the subspace $\ell_{2,r}$ of ℓ_2 , where $\ell_{2,r}$ consists of all sequences of real numbers in ℓ_2 . Thus, for any $\sigma > 0$, $[Y_{j,k}(\sigma)]$ maps $\ell_{2,r}$ into $\ell_{2,r}$ if and only if $Y(\sigma)$ maps H_r into H_r . This allows us to state

Corollary 8-2a: The first sentence of theorem 8-2 remains true if "positive $*$ " is replaced by "positive $*$ -real." In this case, for every $\sigma > 0$ and every j and k , $Y_{j,k}(\sigma)$ is a real number. In addition, the second sentence of theorem 8-2 remains true if "positive $*$ " is replaced by "positive $*$ -real", H by H_r and \mathcal{B} by $\mathcal{B}(R)$.

We have treated only the admittance formulism of the ∞ -port. An analysis of the scattering formulism can also be made, but, since it is quite similar to the foregoing, we omit it.

8 $\frac{1}{2}$. A First Thrust at the Synthesis of an ∞ -port.

Given a positive $*$ -real mapping $[Y_{j,k}]$ of ℓ_2 into ℓ_2 , can one synthesize an ∞ -port to realize it? Here are a few thoughts on the subject.

First of all, let it be said that we are trying to synthesize a "paper network" [3], which is a perfectly legitimate mathematical idea that can only be approximated by a physical system. (So too is the ideal one-ohm resistor.)

Assume once again that H is the complexification of a real separable Hilbert space H_r and that $\{e_k\} \subset H_r$. Let $[Y_{j,k}]_n$ be the $\infty \times \infty$ matrix obtained from $[Y_{j,k}]$ by replacing each element $Y_{j,k}$ in $[Y_{j,k}]$ by 0 if either $j > n$ or $k > n$. This corresponds to the following alteration of the ∞ -port whose admittance matrix is $[Y_{j,k}]$: For every port beyond the n^{th} , disconnect the wires to the port terminals and short them together.

The resulting system which is in fact an n-port, possesses

$[Y_{j,k}]_n$ as its admittance matrix. $[Y_{j,k}]_n$ can be identified with the $n \times n$ matrix obtained by dropping all rows and columns in $[Y_{j,k}]$ beyond the n^{th} row and n^{th} column, and the latter is a positive-real $n \times n$ matrix in the usual sense. Indeed, let $a = \{a_k\}$ be any member of ℓ_2 such that $a_k = 0$ if $k > n$. Then, for $\zeta \in C_+$,

$$\operatorname{Re} \sum_{j=1}^n \sum_{k=1}^n Y_{j,k}(\zeta) \bar{a}_j a_k = \operatorname{Re} ([Y_{j,k}] \{a_k\}, \{a_j\}) \geq 0$$

by virtue of the positivity* of $[Y_{j,k}]$. The analyticity of each $Y_{j,k}(\zeta)$ on C_+ and the reality of $Y_{j,k}(\sigma)$ for $\sigma > 0$ is equally clear.

If we now assume in addition that every $Y_{j,k}$ is a rational function of ζ , then we can apply known synthesis procedures [26] to realize $[Y_{j,k}]_n$ as an ∞ -port whose first n ports connect to a lumped passive network and whose ports beyond the n^{th} all have broken terminal wires. Thus, we have synthesized in this way a sequence $\{[Y_{j,k}]_n\}_{n=1}^{\infty}$ of ∞ -ports.

$\{[Y_{j,k}]_n\}_{n=1}^{\infty}$ is an approximating sequence for the originally given $[Y_{j,k}]$ in the sense that, for each fixed $\zeta \in C_+$, $[Y_{j,k}(\zeta)]_n \rightarrow [Y_{j,k}(\zeta)]$ in the weak topology of $[\ell_2; \ell_2]$. To show this, let $a = \{a_k\}$ and $b = \{b_k\}$ be arbitrary members of ℓ_2 . Then,

$$\begin{aligned} & |([Y_{j,k}(\zeta)] a, b) - ([Y_{j,k}(\zeta)]_n a, b)|^2 \\ &= \left| \left(\sum_{j=n+1}^{\infty} \sum_{k=1}^n + \sum_{j=1}^{\infty} \sum_{k=n+1}^{\infty} \right) Y_{j,k}(\zeta) a_k \bar{b}_j \right|^2 . \end{aligned}$$

By applying the Schwarz inequality to the summations on j , we bound the last expression by

$$\sum_{j=n+1}^{\infty} |b_j|^2 \sum_{j=n+1}^{\infty} \left| \sum_{k=1}^n Y_{j,k}(\zeta) a_k \right|^2 + \sum_{j=1}^{\infty} |b_j|^2 \sum_{j=1}^{\infty} \left| \sum_{k=n+1}^{\infty} Y_{j,k}(\zeta) a_k \right|^2 .$$

We now invoke a result [24; Sec. 3.1], which states that $[Y_{j,k}(\zeta)] \in [l_2; l_2]$ if and only if there exists a constant $M > 0$ such that, for every pair p, q of positive integers and every choice of the complex numbers a_1, \dots, a_p ,

$$\sum_{j=1}^q \left| \sum_{k=1}^p Y_{j,k}(\zeta) a_k \right|^2 \leq M^2 \sum_{k=1}^p |a_k|^2$$

Under the assumptions that $[Y_{j,k}(\zeta)] \in [l_2; l_2]$ and $\{a_k\} \in l_2$, we may take $p \rightarrow \infty$ and/or $q \rightarrow \infty$ and still obtain a valid inequality. In view of this fact, we have that

$$\begin{aligned} |([Y_{j,k}(\zeta)] a, b) - ([Y_{j,k}(\zeta)]_n a, b)|^2 &\leq M^2 \sum_{j=n+1}^{\infty} |b_j|^2 \sum_{k=1}^n |a_k|^2 \\ &+ M^2 \sum_{j=1}^{\infty} |b_j|^2 \sum_{k=n+1}^{\infty} |a_k|^2. \end{aligned}$$

The right-hand side tends to zero as $n \rightarrow \infty$, which establishes our assertion.

If it happens that, for the chosen $\zeta \in C_+$,

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |Y_{j,k}(\zeta)|^2 < \infty,$$

then, the convergence of $[Y_{j,k}(\zeta)]_n$ to $[Y_{j,k}(\zeta)]$ occurs in the norm topology of $[l_2; l_2]$. Indeed, if $b = [Y_{j,k}(\zeta)] a$, $a \in l_2$, then by the Schwarz inequality,

$$\begin{aligned} \|b\|^2 &= \sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} Y_{j,k}(\zeta) a_k \right|^2 \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |Y_{j,k}(\zeta)|^2 \sum_{k=1}^{\infty} |a_k|^2 \\ &= \|a\|^2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |Y_{j,k}(\zeta)|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|[Y_{j,k}(\zeta)] - [Y_{j,k}(\zeta)]_n\| &= \sup_{\|a\|=1} \| \{ [Y_{j,k}(\zeta)] - [Y_{j,k}(\zeta)]_n \} a \| \\ &\leq \left[\sum_{j=n+1}^{\infty} \sum_{k=1}^n |Y_{j,k}(\zeta)|^2 + \sum_{j=1}^{\infty} \sum_{k=n+1}^{\infty} |Y_{j,k}(\zeta)|^2 \right]^{1/2}. \end{aligned}$$

The right-hand side tends to zero as $n \rightarrow \infty$.

In summary, we have not obtained a synthesis of the given positive *-real $[Y_{j,k}]$ but instead have constructed an approximating sequence $\{[Y_{j,k}]_n\}$ whose members have n-port realizations whenever the $[Y_{j,k}]_n$ are rational.

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