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THE DISTRIBUTIONAL LAPLACE AND MELLIN TRANSFORMATIONS

by

A. H. Zemanian

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Scientific Reports

1. A. H. Zemanian, "A Time-Domain Characterization of Rational Positive-Real Matrices," First Scientific Report, AFCRL-63-390, College of Engineering Tech. Rep. 12, State University of New York at Stony Brook; August 5, 1963.

2. A. H. Zemanian, "A Time-Domain Characterization of Positive-Real Matrices," Second Scientific Report, AFCRL-63-391, College of Engineering Tech. Rep. 13, State University of New York at Stony Brook; August 16, 1963.

3. A. H. Zemanian, "The Time-Domain Synthesis of Distributions," Third Scientific Report, AFCRL-64-191, College of Engineering Tech. Rep. 19, State University of New York at Stony Brook; August 15, 1964.

Papers

1. A. H. Zemanian, "The Time-Domain Synthesis of Distributions," Proceedings of the First Allerton Conference on Circuit Theory, University of Illinois; 1963.

2. A. H. Zemanian, "The Approximation of Distributions by the Impulse Response of RLC Two-ports," Proceedings of the International Conference on Microwaves, Circuit Theory, and Information Theory; Tokyo; September, 1964.

3. A. H. Zemanian, "The Time-Domain Synthesis of Distributions," IEEE Transactions on Circuit Theory, accepted for publication in December, 1964 issue.

ABSTRACT

A new definition for the distributional two-sided Laplace transformation $\mathcal{L}$ is devised as follows. Let t be a one-dimensional real variable. Spaces of testing functions on $-\infty < t < \infty$ are constructed, which contain exponential functions e^{-st}, s being a complex parameter. Their dual spaces turn out to be subspaces for the space $\mathcal{D}'$ of distributions. Then, for any f in such a subspace, $\mathcal{L}f$ is defined as the application of f to e^{-st}

$$\mathcal{L}f = \langle f, e^{-st} \rangle$$

This definition is entirely equivalent to L. Schwartz's definition. Moreover, it simplifies a number of proofs and derivations for the various properties of this transformation. It also provides greater facility in manipulating specific distributional transforms.

The distributional Mellin transform of a distribution on $0 < x < \infty$ is analogously defined as

$$\langle f, x^{s-1} \rangle$$

where x^{s-1} is a member of a certain space of testing functions on $0 < x < \infty$ and f is in the dual space.

These ideas are developed for the case where t and x are n-dimensional.

1. Introduction. This report appears in two parts. In the first we develop an apparently new approach to the n-dimensional two-sided Laplace transformation for distributions. The second part is devoted to a similar analysis for the n-dimensional Mellin transformation of distributions. The basic idea in the one-dimensional case is the following.

A space of testing functions on $-\infty < t < \infty$ is constructed, which contains exponential functions e^{-st}, s being a complex parameter. Its dual space turns out to be a subspace of the space $\mathcal{D}'$ of distributions. Then, the one-dimensional two-sided Laplace transformation $\mathcal{L}$ of a distribution f is simply defined by

$$\mathcal{L} f = \langle f(t) , e^{-st} \rangle . \tag{1-1}$$

Analogously, the one-dimensional Mellin transformation $\mathfrak{M}$ of the distribution f on $0 < x < \infty$ is defined as

$$\mathfrak{M} f = \langle f(x) , x^{s-1} \rangle , \tag{1-2}$$

where x^{s-1} is a member of a certain testing function space and $f(x)$ is in the dual space. These ideas carry over to the n-dimensional case.

The customary definition for the distributional two-sided Laplace transformation is due to L. Schwartz [1]. It defines $\mathcal{L} f$ as a Fourier transformation $\mathcal{F}$.

$$\mathcal{L} f = \mathcal{F}\left\{ e^{-\sigma t} f(t) \right\} \tag{1-3}$$

Here, σ is a real number restricted to those values for which $e^{-\sigma t} f(t)$ is a temperate distribution. The definition proposed here is entirely equivalent to Schwartz's definition in that (1-1) exists in our sense if and only if (1-3) exists in Schwartz's sense.

The use of definition (1-1) simplifies a number of proofs and derivations for various properties of the distributional Laplace transformation. Furthermore, it provides a fairly direct method of introducing the distributional Mellin transformation through a change of variables.

T. Ishihara [2] has adopted the methods of Gelfand and Shilov [3] to extend the Laplace transformation to all distributions. Our definition, being equivalent to Schwartz's, is not as general but can be developed more concisely and manipulated more simply.

A number of other methods have been proposed for assigning a sense to (1-1) but they were designed for the one-sided Laplace transformation and require that the supports of the distributions be bounded on one side. See [4] - [8]. To be sure, one can decompose any distribution into a sum of a distribution with support bounded on the left and a distribution with support bounded on the right. Then, these other techniques can be used to generalize the two-sided Laplace transformation [8]. But, this is awkward and computationally troublesome.

In the second part of this report, the distributional Mellin transformation is defined by applying a sense directly to (1-2). To the author's knowledge, there has been only one other extension of the Mellin transformation to distributions. It is due to Fung Kang [9] and uses the method of Gelfand and Shilov [3] to generalize the one-dimensional Mellin transformation. It too does not assign a sense directly to (1-2). As before, the definition given here is not as general. It remains instead within the framework of distribution theory and leads to a number of simplifications. In addition, our results have been developed for the n-dimensional case.

We shall make use of the following notation. $\mathcal{R}^n$ and $\mathcal{C}^n$ are respectively the real and the complex n-dimensional Euclidean spaces. An integer in $\mathcal{R}^n$ is an element of $\mathcal{R}^n$ whose components are integers. Moreover, we shall always set

$$t = \{t_1, \ldots, t_n\} \in \mathcal{R}^n,$$

$$x = \{x_1, \ldots, x_n\} \in \mathcal{R}^n,$$

and

$$s = \{s_1, \ldots, s_n\} \in \mathcal{C}^n.$$

If f is a function on a subset of $\mathcal{R}^1$, then f(x) shall denote

$$f(x) = \{f(x_1), \ldots , f(x_n)\}$$

If f is a function on a subset of $\mathcal{R}^2$, we set

$$f(x,t) = \{f(x_1,t_1), \ldots , f(x_n,t_n)\}$$

The same notations are used for functions on C^1, C^2, or $\mathcal{R}^1 \times C^1$

For example,

$$\log x = \{ \log x_1 , \ldots , \log x_n \} ,$$

$$xt = \{x_1 t_1 , \ldots , x_n t_n \} ,$$

$$x^s = \{x_1^{s_1} , \ldots , x_n^{s_n} \} ,$$

$$e^{-st} = \{ e^{-s_1 t_1} , \ldots , e^{-s_n t_n} \} .$$

By $[x]$, we mean the product $x_1 x_2 \ldots x_n$. Thus,

$$[e^{-st}] = \exp (-s_1 t_1 - \ldots - s_n t_n)$$

and

$$[x^s] = x_1^{s_1} \ldots x_n^{s_n}$$

The notations, $x \leq t$ and $x < t$, mean $x_\nu \leq t_\nu$ and, respectively, $x_\nu < t_\nu$ $(\nu = 1, 2, \ldots , n)$. k shall always denote a nonnegative integer in $\mathcal{R}^n$. We follow standard procedure in setting $|k| = k_1 + \ldots + k_n$. This should not be confused with the "magnitude" symbol.

D_x^k denotes

$$\frac{\partial^{|k|}}{\partial x_1^{k_1}\,\partial x_2^{k_2}\cdots\cdots\partial x_n^{k_n}}$$

Similarly, D_s^k denotes a partial derivative with respect to the components of $s \in C^n$.

By a smooth function we mean a function that possesses (ordinary) partial derivatives of all orders at all points of its domain. Throughout this paper the principal branch of any multivalued function is always understood.

PART I

THE DISTRIBUTIONAL LAPLACE TRANSFORMATION.

2. The Testing Function Space $\mathcal{L}_{a,b}$

Let t, a, $b \in R^n$ with $a < b$. Also, let t_ν, a_ν, b_ν be arbitrary components of t, a, b, respectively. The function $\kappa_{a_\nu, b_\nu}(t_\nu)$ from R^1 into R^1 is defined to be a positive smooth function such that $a_\nu < b_\nu$ and

$$\kappa_{a_\nu, b_\nu}(t_\nu) = \begin{cases} \exp a_\nu t_\nu & (t_\nu > 1) \\ \exp b_\nu t_\nu & (t_\nu < -1). \end{cases}$$

The precise values for this function on $-1 < t_\nu < 1$ is unimportant but we assume throughout that it is a fixed function. Let $\kappa_{a,b}(t)$ be the positive smooth function from R^n into R^1 given by

$$\kappa_{a,b}(t) = \prod_{\nu=1}^{n} \kappa_{a_\nu, b_\nu}(t_\nu).$$

$\mathcal{L}_{a,b}$ shall denote the space of all smooth functions $\varphi(t)$ from R^n into C^1 such that, for each fixed k,

$$|\kappa_{ab}(t) D_t^k \varphi(t)| < C_k \qquad (-\infty < t < \infty) \qquad (2\text{-}1)$$

where C_k is a constant depending upon k and φ. $\mathcal{L}_{a,b}$ is a linear space over the field C^1. Note that $[e^{-st}]$ is in $\mathcal{L}_{a,b}$ for $a \leqslant \mathrm{Re}\, s \leqslant b$ and $[t^k e^{-st}]$ is in $\mathcal{L}_{a,b}$ for $a < \mathrm{Re}\, s < b$. On the other hand, if at least one of the components of the integer k is positive, neither $[t^k e^{-at}]$ nor $[t^k e^{-bt}]$ is in $\mathcal{L}_{a,b}$.

We assign a topology to $\mathcal{L}_{a,b}$ by making use of the following separating system of seminorms.

$$\gamma_\nu = \gamma_\nu(\varphi) = \max_{0 \le |k| \le \nu} \sup_t \left| \kappa_{a,b}(t) \quad D_t^k \ \varphi(t) \right| \tag{2-2}$$

$$(\nu = 0, 1, 2, \ldots)$$

(Here, it is understood that k traverses all integers in R^n for which $0 \leqslant |k| \leqslant \nu$.) That the γ_ν are truly seminorms follows from the fact that they possess the following properties (Taylor [10] , p. 143). For $\alpha \in C'$, $\varphi \in \mathcal{L}_{a,b}$, and $\psi \in \mathcal{L}_{a,b}$, we have

$$\gamma_\nu(\varphi + \psi) \leqslant \gamma_\nu(\varphi) + \gamma_\nu(\psi) \tag{2-3}$$

and

$$\gamma_\nu(\alpha\varphi) = |\alpha| \, \gamma_\nu(\varphi) \tag{2-4}$$

The γ_ν constitute a separating set of seminorms in the sense that for every $\varphi \not\equiv 0$ in $\mathcal{L}_{a,b}$ there is a γ_ν such that $\gamma_\nu(\varphi) \neq 0$. This is because γ_0 is a norm; that is, it also satisfies the condition,

$$\gamma_0(\varphi) = 0 \iff \varphi = 0. \tag{2-5}$$

These seminorms generate a topology in $\mathcal{L}_{a,b}$ in the following way. A neighborhood of a given $\psi \in \mathcal{L}_{a,b}$ is any subset of $\mathcal{L}_{a,b}$ that contains a set consisting of all $\varphi \in \mathcal{L}_{a,b}$ satisfying

$$\gamma_{\nu_j}(\varphi - \psi) \leq \varepsilon_j \qquad (j = 1, 2, \cdots, N),$$

where the γ_{ν_j} comprise a finite collection of seminorms and the ε_j are positive numbers. The collection of all such neighborhoods (for all $\psi \in \mathcal{L}_{a,b}$) constitutes the topology of $\mathcal{L}_{a,b}$. Note that these neighborhoods satisfy axioms for a topological linear space (Martineau et Treves [11], pp. 1-2). Since our set of seminorms is a seperating one, all neighborhoods of a given ψ in $\mathcal{L}_{a,b}$ have only ψ in common.

A sequence $\{\varphi_\mu\}_{\mu=1}^{\infty}$ (or, more generally, a directed set $\{\varphi_\mu\}_{\mu \to \infty}$) is said to be a Cauchy sequence in $\mathcal{L}_{a,b}$ if every φ_μ is in $\mathcal{L}_{a,b}$ and if for each neighborhood Ω of the zero function here exists an integer N such that, for all μ and ξ greater than N, $\varphi_\mu - \varphi_\xi$ is in Ω. It follows that $\{\varphi_\mu\}_{\mu=1}^{\infty}$ is a Cauchy sequence if and only if for each fixed ν the $\gamma_\nu(\varphi_\mu - \varphi_\xi)$ converges to zero as μ and ξ go to infinity independently. In view of (2-2) and the fact that $\kappa_{ab}(t) > 0$, this means that the sequence of derivatives $\{D_t^k \varphi_\mu(t)\}_{\mu=1}^{\infty}$ converges uniformly over every bounded t-domain. Thus, there exists some limit function φ , which is smooth. Moreover, it satisfies the inequalities (2-1). As a result, φ is in $\mathcal{L}_{a,b}$ and $\{\varphi_\mu\}_{\mu=1}^{\infty}$ converges to it in the topology of $\mathcal{L}_{a,b}$; that is, for every neighborhood Ω of the zero function, there exists an N such that, for all $\mu > N$, $\varphi_\mu - \varphi$ is in Ω. Because this is true for every Cauchy sequence, $\mathcal{L}_{a,b}$ is said to be <u>sequentially</u> <u>complete</u>.

If $\{\varphi_\mu\}_{\mu=1}^{\infty}$ is a Cauchy sequence in $\mathcal{L}_{a,b}$ and has φ as its limit, we shall say that $\{\varphi_\mu\}_{\mu=1}^{\infty}$ <u>converges</u> <u>in</u> $\mathcal{L}_{a,b}$ <u>to</u> φ. (If $\varphi \equiv 0$, we say that the sequence converges to zero.) It follows that a sequence $\{\varphi_\mu\}_{\mu=1}^{\infty}$ converges in $\mathcal{L}_{a,b}$ if and only if each φ_μ is in $\mathcal{L}_{a,b}$ and for each $k \geqslant 0$

$$\{ \kappa_{a,b}(t)\ D_t^k\ \varphi_\mu(t) \}_{\mu=1}^{\infty}$$

converges uniformly on $\mathcal{R}^n$. (The uniformity of the convergence need not hold over all k.)

A subset B of $\mathcal{L}_{a,b}$ is said to be bounded if there exist a set of constants C_ν such that, for all φ in B, $\gamma_\nu(\varphi) \leqslant C_\nu$ $(\nu = 0, 1, \ldots)$.

Two easily established facts concerning $\mathcal{L}_{a,b}$ are the following:

I. Let $\mathcal{D}$ be the space of all smooth functions having compact supports. Then, $\mathcal{D} \subset \mathcal{L}_{a,b}$ for every $a,b \in \mathcal{R}^n$ $(a < b)$.

Moreover, convergence in $\mathcal{D}$ implies convergence in $\mathcal{L}_{a,b}$.

II. If $a \leq c < d \leq b$, then $\mathcal{L}_{c,d} \subset \mathcal{L}_{a,b}$ and convergence in $\mathcal{L}_{c,d}$ implies convergence in $\mathcal{L}_{a,b}$.

We now prove two results that we shall need subsequently. $\mathcal{S}$ shall denote the space of all smooth functions of rapid descent. We assign to it the customary topology. (Schwartz 12, Vol. II, Pp. 89).

Lemma 1: *Let* $a, b, \sigma \in \mathcal{R}^n$ *with* $a \leq \sigma \leq b$. *If* $\psi \in \mathcal{S}$, *then* $[e^{-\sigma t}]\psi \in \mathcal{L}_{a,b}$. *If* $\{\psi_\nu\}_{\nu=1}^{\infty}$ *converges in* $\mathcal{S}$ *to zero, then* $\{[e^{-\sigma t}]\psi_\nu\}_{\nu=1}^{\infty}$ *also converges in* $\mathcal{L}_{a,b}$ *to zero*.

Proof: We prove the last statement, the proof of the preceding one being almost the same. We may write

$$\kappa_{a,b}(t)\, D_t^k \{\psi_\nu(t)[e^{-\sigma t}]\} = \{\kappa_{a,b}(t)[e^{-\sigma t}]\}\left\{\sum_q a_q p_q(\sigma)\, D_t^q\, \psi_\nu(t)\right\}.$$

The summation in the right-hand side is on the finite number of n-dimensional integers q that satisfy $0 \leq q \leq k$. Moreover, the a_q are constants and the $p_q(\sigma)$ are polynomials in the components in σ. The quantity, $\kappa_{a,b}(t)[e^{-\sigma t}]$ is a bounded function for all t. Also, each term in the summation converges uniformly to zero for all t. Thus, the same may be said of the left-hand side. Q. E. D.

Lemma 2: *Let* $a, b, c, d \in \mathcal{R}^n$ *with* $a < c < d < b$. *If* $\varphi \in \mathcal{L}_{c,d}$, *then* $\kappa_{a,b}\varphi \in \mathcal{S}$. *If* $\{\varphi_\nu\}_{\nu=1}^{\infty}$ *converges in* $\mathcal{L}_{c,d}$ *to zero, then* $\{\kappa_{a,b}\varphi_\nu\}_{\nu=1}^{\infty}$ *also converges in* $\mathcal{S}$ *to zero*.

<u>Proof</u> : We again consider just the last statement. Let m and k be arbitrary nonnegative integers in $\mathcal{R}^n$. Then, using the notation described above, we write

$$[t^m] D_t^k \{ \chi_{a,b}(t) \varphi_\nu(t) \} = \sum_q a_q \{ \chi_{c,d}(t) D_t^q \varphi_\nu(t) \} \left\{ \frac{[t^m] D_t^{k-q} \chi_{a,b}(t)}{\chi_{c,d}(t)} \right\}.$$

The quantity in the first set of braces under the summation sign converges to the zero functions uniformly for all t, whereas the quantity in the second set of braces is bounded for all t. Thus, the left-hand side converges to the zero function uniformly for all t. Q. E. D.

We have already noted that $\mathcal{L}_{a,b}$ is a linear space, which means (among other things) that it is closed under the operation of addition and multiplication-by-a-complex-number. We shall list in a moment a number of operations that may be applied to $\mathcal{L}_{a,b}$. But, first some terminology.

All operations discussed in the report are understood to be single-valued. An operation (or mapping) $\mathfrak{N}$ from a sequentially complete topological linear space $\mathcal{a}$ into another such space $\mathcal{B}$ is said to be linear if for every scalar α and every two elements $\varphi, \psi \in \mathcal{a}$ one has

$$\mathfrak{N}(\varphi + \psi) = \mathfrak{N}\varphi + \mathfrak{N}\psi \in \mathcal{B}, \quad \mathfrak{N}(\alpha\varphi) = \alpha \mathfrak{N}\varphi \in \mathcal{B}$$

$\mathfrak{N}$ is said to be continuous if for each sequence $\{\varphi_\nu\}_{\nu=1}^\infty$ that converges in the topology of $\mathcal{a}$ to the limit φ one has that $\{\mathfrak{N}\varphi_\nu\}_{\nu=1}^\infty$ converges in the topology of the space $\mathcal{B}$ to $\mathfrak{N}\varphi$. It is a fact that a linear operation is continuous if it is continuous at the origin. $\mathcal{N}$ is said to be a topological isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ if it is one-to-one and if $\mathcal{N}$ and its inverse are both continuous linear operations. (Henceforth, when we say "isomorphism" we shall mean "topological isomorphism.")

Now, for the list of operations. We identify each operation either as an isomorphism or as a continuous linear operation. The proofs of these properties are given in Appendix A.

1. For $\tau \in \mathcal{R}^n$, $\varphi(t) \longrightarrow \varphi(t-\tau)$ is an isomorphism from $\mathcal{L}_{a,b}$ onto $\mathcal{L}_{a,b}$.

2. $\varphi(t) \longrightarrow \varphi(-t)$ is an isomorphism from $\mathcal{L}_{a,b}$ onto $\mathcal{L}_{-b,-a}$

3. For $\tau \in \mathcal{R}^n$ with $\tau > 0$, $\varphi(t) \longrightarrow \varphi(\tau t)$ is an isomorphism from $\mathcal{L}_{a,b}$ onto $\mathcal{L}_{\tau a,\tau b}$. (Here, no component of τ is allowed to be zero.)

4. Let $p(t)$ be a polynomial in the components of t. Then, $\varphi \longrightarrow p\varphi$ is a continuous linear mapping of $\mathcal{L}_{c,d}$ into $\mathcal{L}_{a,b}$ for every a and b such that $a < c < d < b$.

5. For $\alpha \in C^n$ and $y = \text{Re}\,\alpha$, $\varphi(t) \rightarrow [e^{-\alpha t}]\,\varphi(t)$ is an isomorphism from $\mathcal{L}_{a,b}$ onto $\mathcal{L}_{a+y,b+y}$.

6. Let $\lambda(t)$ be a smooth function from $\mathcal{R}^n$ into $\mathcal{R}^1$ such that it and all its derivatives are bounded functions. $\varphi \longrightarrow \lambda\varphi$ is a continuous linear mapping of $\mathcal{L}_{a,b}$ into $\mathcal{L}_{a,b}$.

7. $\varphi \longrightarrow D_t^{k}\varphi$ is a continuous linear mapping of $\mathcal{L}_{a,b}$ into $\mathcal{L}_{a,b}$.

3. The Dual Space $\mathcal{L}'_{a,b}$.

A functional on a space $\mathcal{A}$ is a mapping of $\mathcal{A}$ into C^1. Thus, it is a special case of an operator and we define its linearity and continuity as we did for operators.

$\mathcal{L}'_{a,b}$ shall denote the space of all continuous linear functionals on $\mathcal{L}_{a,b}$. It is called the dual of $\mathcal{L}_{a,b}$. The complex number that an $f \in \mathcal{L}'_{a,b}$ assigns to a $\varphi \in \mathcal{L}_{a,b}$ is denoted by $\langle f, \varphi \rangle = \langle f(t), \varphi(t) \rangle$ (We write f(t) to indicate the independent variable of the testing functions for f.) Two numbers f and g, of $\mathcal{L}'_{a,b}$ are said to be equal if $\langle f, \varphi \rangle = \langle g, \varphi \rangle$ for every $\varphi \in \mathcal{L}_{a,b}$. Addition and multiplication-by-a-complex-number are defined by

$$\langle f + g, \varphi \rangle = \langle f, \varphi \rangle + \langle g, \varphi \rangle$$

$$\langle \alpha f, \varphi \rangle = \langle f, \alpha \varphi \rangle$$

where φ traverses $\mathcal{L}_{a,b}$. Clearly, $\mathcal{L}'_{a,b}$ is closed under these operations. With these operations, $\mathcal{L}'_{a,b}$ becomes a linear space. Some other pertinent facts are the following.

I. $\mathcal{L}'_{a,b}$ is a subspace of $\mathcal{D}'$, the space of distributions. This follows from note I of Sec. 2 . We may, therefore, use all the definitions, properties, and operations that are applicable to distributions.

II. The space $\mathcal{E}'$ of distributions of bounded support is a subspace of $\mathcal{L}'_{a,b}$. This is because the space $\mathcal{E}'$ of smooth functions contains $\mathcal{L}_{a,b}$ and because convergence in $\mathcal{L}_{a,b}$ implies convergence in $\mathcal{E}$. (See Schwartz 12, Vol.1, p. 88.)

III. If $a \leq c < d \leq b$, then $\mathcal{L}'_{a,b} \subset \mathcal{L}'_{c,d}$. This follows from Note II of Sec. 2.

IV. If $f \in \mathcal{L}'_{c,d}$ and if $f(t) = 0$ whenever any component of t is less than a given number, then $f \in \mathcal{L}'_{c,b}$ for every $b > d$. If $f \in \mathcal{L}'_{c,d}$ and if $f(t) = 0$ whenever any component of t is greater than a given number, then $f \in \mathcal{L}'_{a,d}$ for every $a < c$.

V. If f is a locally integrable function such that $f / \kappa_{a,b}$ is absolutely integrable over $\mathcal{R}^n$, then the regular distribution f is in $\mathcal{L}'_{a,b}$.

A (weak topology) is generated in $\mathcal{L}'_{a,b}$ through the following system of seminorms. Each φ in $\mathcal{L}_{a,b}$ produces a seminorms $\rho_\varphi(f)$ on $\mathcal{L}'_{a,b}$ through the expression,

$$\rho_\varphi(f) \triangleq |\langle f, \varphi \rangle| \qquad (f \in \mathcal{L}'_{a,b}). \tag{3-1}$$

Note that the axioms for a seminorm are satisfied since

$$\rho_\varphi(\alpha f) = |\langle \alpha f, \varphi \rangle| = |\alpha| |\langle f, \varphi \rangle| = |\alpha| \rho_\varphi(f)$$

and

$$\rho_\varphi(f_1 + f_2) = |\langle f_1 + f_2, \varphi \rangle| \leq |\langle f_1, \varphi \rangle| + |\langle f_2, \varphi \rangle| = \rho_\varphi(f_1) + \rho_\varphi(f_2).$$

The ρ_φ comprise a separating set of seminorms since, if $f \neq 0$, there is some norm φ in $\mathcal{D}$ for which $\langle f, \varphi \rangle \neq 0$.

Let g be a given distribution in $\mathcal{L}'_{a,b}$. A neighborhood of g is any subset of $\mathcal{L}'_{a,b}$ that contains a set consisting of all $f \in \mathcal{L}'_{a,b}$ such that

$$\rho_{\varphi_j}(f - g) \leq \varepsilon_j \qquad (j = 1, 2, \ldots, N),$$

where the ρ_{φ_j} comprise a finite collection of seminorms and the ε_j are positive numbers. The (weak) topology T_w of $\mathcal{L}'_{a,b}$ is the set of all neighborhoods in $\mathcal{L}'_{a,b}$ generated by the $\rho_\varphi(f)$. Such neighborhoods will be called T_w-neighborhoods.

A sequence $\{f_\nu\}_{\nu=1}^{\infty}$ of distributions in $\mathcal{L}'_{a,b}$ (or, more generally, a directed set $\{f_\nu\}_{\nu\to\infty}$) is called a Cauchy sequence (or, respectively, a Cauchy directed set) with respect to the topology T_w if, for each T_w-neighborhood Ξ of the zero distribution, there exists an integer N such that, for all ν and μ greater than N, $f_\nu - f_\mu$ is in Ξ. Thus, $\{f_\nu\}_{\nu=1}^{\infty}$ is a Cauchy sequence if and only if for every φ the corresponding seminorm $\rho_\varphi(f_\nu - f_\mu)$ converges to zero as ν and μ tend to infinity separately. This is the same as requiring that for each φ the numerical sequence $\{\langle f_\nu, \varphi \rangle\}_{\nu=1}^{\infty}$ converges. The limits of all such numerical sequences define a functional on $\mathcal{L}_{a,b}$. It is a fact that this functional is linear and continuous on $\mathcal{L}_{a,b}$ and is therefore, in $\mathcal{L}'_{a,b}$. (This can be established by adapting M. S. Brodsku's proof of the sequential completeness of $\mathcal{D}'$. The details are given in Appendix C.) Thus, every Cauchy sequence with respect to T_w possesses a limit, which is also in $\mathcal{L}'_{a,b}$, In other words, $\mathcal{L}'_{a,b}$ is sequentially complete with respect to T_w. If $\{f_\nu\}_{\nu=1}^{\infty}$ is a Cauchy sequence in $\mathcal{L}'_{a,b}$ with this topology, we shall say that $\{f_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{L}'_{a,b}$ to f. (We simply say "converges to zero" if f is the zero distribution.)

A subset B' of $\mathcal{L}'_{a,b}$ is said to be bounded (with respect to T_w) if, for every φ in $\mathcal{L}_{a,b}$,

$$\sup_{f \in B'} \rho_\varphi(f)$$

exists (i. e., is a finite number.)

A strong topology can also be generated in $\mathcal{L}'_{a,b}$ by using the seminorms $\rho_B(f)$, which are defined as follows. For each bounded set B in $\mathcal{L}_{a,b}$,

$$\rho_B(f) = \sup_{\varphi \in B} |\langle f, \varphi \rangle| \qquad (f \in \mathcal{L}'_{a,b}). \tag{3-2}$$

Note that every $\rho_\varphi(f)$ is also a $\rho_B(f)$. When B is a finite set, $\rho_B(f)$ is obviously a finite number. This is still true even when B is an infinite set. Indeed, let us assume the opposite. Then, there exists a sequence $\{\varphi_\mu\}_{\mu=1}^{\infty}$ of elements in B such that $\gamma_\nu(\varphi_\mu) < C_\nu$ $(\nu = 0, 1, 2, \ldots)$, the C_ν being independent of μ, and such that $|\langle f, \varphi_\mu\rangle| \longrightarrow \infty$ as $\mu \longrightarrow \infty$. Therefore, we can choose a subsequence $\{\varphi_\mu'\}_{\mu=1}^{\infty}$ from $\{\varphi_\mu\}$ such that $|\langle f, \varphi_\mu'\rangle| > \mu$. Set $\psi_\mu = \varphi_\mu'/\mu$. Then,

$$|\langle f, \psi_\mu \rangle| > 1 \tag{3-3}$$

Moreover, $\{\psi_\mu\}_{\mu=1}^{\infty}$ converges in $\mathcal{L}_{a,b}$ to zero because for each ν

$$\gamma_\nu(\psi_\mu) = \frac{1}{\mu}\gamma_\nu(\varphi_\mu') = \frac{C_\nu}{\mu} \longrightarrow 0$$

as $\mu \longrightarrow \infty$. Since f is a continuous functional on $\mathcal{L}_{a,b}$, $\{\langle f, \psi_\mu\rangle\}_{\mu=1}^{\infty}$ converges to zero. But, this contradicts (3-3), which proves that $\rho_B(f)$ must be finite.

We can construct a system of neighborhoods in $\mathcal{L}_{a,b}'$, by using the seminorms (3-2) in precisely the same way as was done before using the seminorms (3-1). The resulting collection of neighborhoods constitutes the strong topology T_s of $\mathcal{L}_{a,b}'$. Obviously, the weak topology T_w is a subset of the strong topology T_s.

Similarly, we can define Cauchy sequences with respect to the strong topology T_s as we did for the weak topology T_w. A neighborhood with respect to T_w is clearly a neighborhood with respect to T_s. Therefore, if a sequence is a Cauchy sequence with respect to T_s, it is certainly a Cauchy sequence with respect to T_w. By the sequential completeness of $\mathcal{L}'_{a,b}$ with respect to T_w, it follows that $\mathcal{L}'_{a,b}$ is sequentially complete with respect to T_s.

A subset B' of $\mathcal{L}'_{a,b}$ is said to be bounded with respect to T_s if, for every bounded set B in $\mathcal{L}_{a,b}$,

$$\sup_{f \in B'} \rho_B(f)$$

is a finite number.

We shall not make use of the strong topology in our subsequent discussion. Whenever we speak of "convergence in $\mathcal{L}'_{a,b}$," it will be understood that this is with respect to the weak topology T_w.

We now relate the space $\mathcal{L}'_{a,b}$ to the space $\mathcal{S}'$. First note that $[e^{-\sigma t}]\, f \in \mathcal{S}'$ <u>for</u> $a \leq \sigma \leq b$ <u>if and only if</u> $f / \kappa_{a,b} \in \mathcal{S}'$. Indeed, assume $f / \kappa_{a,b} \in \mathcal{S}'$. We may write

$$[e^{-\sigma t}]\, f(t) = \left\{ \kappa_{a,b}(t)[e^{-\sigma t}] \right\} \left\{ \frac{f(t)}{\kappa_{a,b}(t)} \right\} ,$$

where the quantity in the first set of braces is a bounded function. Thus, $[e^{-\sigma t}]\, f(t)$ is in $\mathcal{S}'$. To prove the converse, we set

$$\frac{1}{\chi_{a,b}(t)} = \prod_{\nu=1}^{n} \{ \theta_{a_\nu}(t_\nu) + \theta_{b_\nu}(t_\nu) \}$$

Here, θ_{a_ν} is a smooth function with a support bounded on the left and equals $\exp(-a_\nu t_\nu)$ for $t_\nu > 1$. Also, θ_{b_ν} is a smooth function with a support bounded on the right and equals $\exp(-b_\nu t_\nu)$ for $t_\nu < -1$. So, $f / \chi_{a,b}$ is a finite sum of terms, a typical term being

$$f\,\theta_{a_1}\theta_{b_2}\cdots\theta_{a_n} = \{ f \exp(-a_1t_1 - b_2t_2 - \cdots - a_nt_n)\} \left\{ \frac{\theta_{a_1}\theta_{b_2}\cdots\cdots\theta_{a_n}}{\exp(-a_1t_1 - b_2t_2 - \cdots - a_nt_n)} \right\}$$

The function in the second set of braces is smooth and bounded over $\mathbb{R}^n$. Moreover, under the assumption that $[e^{-\sigma t}]f(t)$ is in $\mathcal{S}'$ for $a \leq \sigma \leq b$, the distribution in the first set of braces is in $\mathcal{S}'$. Thus, $f / \kappa_{a,b}$ is also in $\mathcal{S}'$. This completely establishes our original assertion.

<u>Theorem 3-1</u> : <u>If</u> $f \in \mathcal{L}'_{a,b}$, <u>then</u> $[e^{-\sigma t}]\, f \in \mathcal{S}'$ <u>for</u> $a \leq \sigma \leq b$ (<u>or, equvalently</u> ; $f / \chi_{a,b} \in \mathcal{S}'$).

<u>Proof</u> : By lemma 1, $[e^{-\sigma t}]\,\psi \in \mathcal{L}_{a,b}$ whenever $\psi \in \mathcal{S}$. Then, $[e^{-\sigma t}]\,f$ is defined as a functional on $\mathcal{S}$ through the equation,

$$\langle [e^{-\sigma t}]f, \psi \rangle = \langle f, [e^{-\sigma t}]\psi \rangle .$$

Clearly, $[e^{-\sigma t}]f$ is a linear functional on $\mathcal{S}$. Moreover, by the last statement of lemma 1, if $\{\psi_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{S}$ to zero, then

$$\langle [e^{-\sigma t}]f, \psi_\nu \rangle = \langle f, [e^{-\sigma t}]\psi_\nu \rangle \longrightarrow 0$$

as $\nu \to \infty$. Thus, $[e^{-\sigma t}]f$ is a continuous functional on $\mathcal{S}$. Q. E. D.

Theorem 3-2 : *If* $[e^{-\sigma t}]f \in \mathcal{S}'$ *for* $a \leq \sigma \leq b$ (*or, equivalently, if* $f/\kappa_{a,b} \in \mathcal{S}'$), *then* $f \in \mathcal{L}'_{c,d}$ *for every* c *and* d *such that* $a < c < d < b$.

Proof : By lemma 2, $\kappa_{a,b}\varphi \in \mathcal{S}$ whenever $\varphi \in \mathcal{L}_{c,d}$. We define f as a functional on $\mathcal{L}_{c,d}$ by

$$\langle f, \varphi \rangle = \langle \frac{f}{\kappa_{a,b}}, \kappa_{a,b}\varphi \rangle .$$

f is clearly a linear functional on $\mathcal{L}_{c,d}$. That it is a continuous functional on $\mathcal{L}_{c,d}$ follows from the second statement of lemma 2. Q. E. D.

4. Some Operations on $\mathcal{L}'_{a,b}$.

Since an arbitrary f in $\mathcal{L}'_{a,b}$ is a distribution, we may perform any operation on f that is applicable to distributions. However, the resulting distribution may not be in $\mathcal{L}'_{a,b}$. We have already mentioned that the space $\mathcal{L}'_{a,b}$ is closed under the operations of addition and multiplication-by-a-complex-number and that $\mathcal{L}'_{a,b}$ is a linear space. We shall now describe some other operations of interest to us. But first, some rather general remarks are in order.

Let $\mathfrak{N}$ be a continuous linear mapping of $\mathcal{L}_{c,d}$ into $\mathcal{L}_{a,b}$. (Here, we do not place any restrictions on the real points, a, b, c, and d other than $a < b$ and $c < d$.) We define the adjoint operator $\mathfrak{N}'$ (acting on $\mathcal{L}'_{a,b}$) by

$$\langle \mathfrak{N}' f, \varphi \rangle = \langle f, \mathfrak{N}\varphi \rangle, \tag{4-1}$$

where $f \in \mathcal{L}'_{a,b}$ and φ traverses all of $\mathcal{L}_{c,d}$. Thus, $\mathfrak{N}\varphi$ is in $\mathcal{L}_{a,b}$ and the right-hand side has a sense. This equation defines $\mathfrak{N}' f$ as a functional on $\mathcal{L}_{c,d}$.

Actually, $\mathfrak{N}' f$ <u>is a member of</u> $\mathcal{L}'_{c,d}$. Indeed, let $\alpha, \beta \in C'$ and $\varphi, \psi \in \mathcal{L}_{c,d}$. Then,

$$\langle \mathfrak{N}'f, \alpha\varphi + \beta\psi \rangle = \langle f, \mathfrak{N}(\alpha\varphi + \beta\psi) \rangle = \langle f, \alpha\mathfrak{N}\varphi + \beta\mathfrak{N}\psi \rangle$$
$$= \alpha\langle f, \mathfrak{N}\varphi \rangle + \beta\langle f, \mathfrak{N}\psi \rangle = \alpha\langle \mathfrak{N}'f, \varphi \rangle + \beta\langle \mathfrak{N}'f, \psi \rangle,$$

which shows that $\mathfrak{N}' f$ is a linear functional on $\mathcal{L}_{c,d}$. Moreover, let $\{\varphi_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{L}_{c,d}$ to zero. Then, as $\nu \longrightarrow \infty$,

$$\langle \mathfrak{N}' f, \varphi_\nu \rangle = \langle f, \mathfrak{N}\varphi_\nu \rangle \longrightarrow 0$$

and this shows that $\mathfrak{N}' f$ is a continuous functional on $\mathcal{L}_{c,d}$.

Thus, $\mathfrak{N}'f$ is truly in $\mathcal{L}'_{c,d}$.

Furthermore, the fact that $\mathfrak{N}$ is a continuous linear mapping of $\mathcal{L}_{c,d}$ into $\mathcal{L}_{a,b}$ implies that $\mathfrak{N}'$ is a continuous linear mapping of $\mathcal{L}'_{a,b}$ into $\mathcal{L}'_{c,d}$. To show the linearity of $\mathfrak{N}'$, let $\varphi \in \mathcal{L}_{c,d}$, $\alpha, \beta \in C'$, and $f, g \in \mathcal{L}'_{a,b}$. Then,

$$\langle \mathfrak{N}'(\alpha f + \beta g), \varphi \rangle = \langle \alpha f + \beta g, \mathfrak{N}\varphi \rangle = \alpha \langle f, \mathfrak{N}\varphi \rangle + \beta \langle g, \mathfrak{N}\varphi \rangle$$
$$= \alpha \langle \mathfrak{N}'f, \varphi \rangle + \beta \langle \mathfrak{N}'g, \varphi \rangle = \langle \alpha \mathfrak{N}'f + \beta \mathfrak{N}'g, \varphi \rangle.$$

To show its continuity, let $\{f_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{L}'_{a,b}$ to zero. Then, as $\nu \longrightarrow \infty$,

$$\langle \mathfrak{N}'f_\nu, \varphi \rangle = \langle f_\nu, \mathfrak{N}\varphi \rangle \longrightarrow 0.$$

Similarly, if $\mathfrak{N}$ is an isomorphism from $\mathcal{L}_{c,d}$ onto $\mathcal{L}_{a,b}$, then $\mathfrak{N}'$ is an isomorphism from $\mathcal{L}'_{a,b}$ onto $\mathcal{L}'_{c,d}$. Indeed, by the definition of an isomorphism, $\mathfrak{N}$ is a continuous linear mapping of $\mathcal{L}_{c,d}$ onto $\mathcal{L}_{a,b}$ and there exists a unique inverse operator $\mathfrak{N}^{-1}$ for $\mathfrak{N}$, which is a continuous linear mapping of $\mathcal{L}_{a,b}$ onto $\mathcal{L}_{c,d}$. The adjoint $(\mathfrak{N}^{-1})'$ to $\mathfrak{N}^{-1}$ is defined by

$$\langle (\mathfrak{N}^{-1})'g, \psi \rangle = \langle g, \mathfrak{N}^{-1}\psi \rangle, \qquad (4\text{-}2)$$

where $\psi \in \mathcal{L}_{a,b}$ and $g \in \mathcal{L}'_{c,d}$. From our preceding results, $(\mathfrak{N}^{-1})'g \in \mathcal{L}'_{a,b}$ and $(\mathfrak{N}^{-1})'$ is a continuous linear mapping of $\mathcal{L}'_{c,d}$ onto $\mathcal{L}'_{a,b}$. To complete the proof, we have to show that $(\mathfrak{N}^{-1})'$ is the inverse of $\mathfrak{N}'$. Let $\psi = \mathfrak{N}\varphi$ or, equivalently, $\varphi = \mathfrak{N}^{-1}\psi$ and let $g = \mathfrak{N}' f$. Then, from (4-1) and (4-2), we have

$$\langle f, \psi \rangle = \langle \mathfrak{N}' f, \varphi \rangle = \langle g, \varphi \rangle = \langle (\mathfrak{N}^{-1})' g, \psi \rangle ,$$

which shows that $f = (\mathfrak{N}^{-1})' g$. Thus, $(\mathfrak{N}^{-1})'$ is truly the inverse of $\mathfrak{N}'$. This completes the proof.

We now define a number of operations on $\mathcal{L}'_{a,b}$ as the adjoint operations of those given in Sec. 2. Our definitions conform with those that apply to distributions in $\mathcal{D}'$ and testing functions in $\mathcal{D}$. We assume throughout the following list that $f \in \mathcal{L}'_{a,b}$ and $\mathfrak{N}\varphi \in \mathcal{L}_{a,b}$ where $\mathfrak{N}$ denotes the particular operation on φ under consideration.

1. For $\tau \in \mathcal{R}^n$, $f(t) \longrightarrow f(t-\tau)$ is defined by

$$\langle f(t-\tau), \varphi(t) \rangle = \langle f(t), \varphi(t+\tau) \rangle$$

It is an isomorphism from $\mathcal{L}'_{a,b}$ onto $\mathcal{L}'_{a,b}$.

2. $f(t) \longrightarrow f(-t)$ is defined by

$$\langle f(-t), \varphi(t) \rangle = \langle f(t), \varphi(-t) \rangle$$

It is an isomorphism from $\mathcal{L}'_{a,b}$ onto $\mathcal{L}'_{-b,-a}$.

3. For $\tau \in \mathcal{R}^n$ with $\tau > 0$, $f(t) \longrightarrow f(\tau t)$ is defined by

$$\langle f(\tau t), \varphi(t) \rangle = \langle f(t), [\tau^{-1}] \varphi(t/\tau) \rangle$$

It is an isomorphism from $\mathcal{L}'_{a,b}$ onto $\mathcal{L}'_{\tau a,\tau b}$.

4. Let ρ be a polynomial in the components of t. $f \longrightarrow \rho f$ is defined by

$$\langle \rho f, \varphi \rangle = \langle f, \rho \varphi \rangle$$

It is a continuous linear mapping of $\mathcal{L}'_{a,b}$ into $\mathcal{L}'_{c,d}$, where $a < c < d < b$.

5. For $\alpha \in C^n$ and $y = \operatorname{Re}\alpha$, $f \longrightarrow [e^{-\alpha t}]f$ is defined by

$$\langle [e^{-\alpha t}]\ f, \varphi \rangle = \langle f, [e^{-\alpha t}]\varphi \rangle$$

It is an isomorphism from $\mathcal{L}'_{a,b}$ onto $\mathcal{L}'_{a-y,b-y}$.

6. Let λ be a smooth function from R^n into R^1 such that it and all its derivatives are bounded functions. $f \longrightarrow \lambda f$ is defined by

$$\langle \lambda f, \varphi \rangle = \langle f, \lambda \varphi \rangle$$

It is a continuous linear mapping of $\mathcal{L}'_{a,b}$ into $\mathcal{L}'_{a,b}$.

7. $f \longrightarrow D^k f$ is defined by

$$\langle D^k f, \varphi \rangle = \langle f, (-1)^{|k|} D^k \varphi \rangle.$$

It is a continuous linear mapping of $\mathcal{L}'_{a,b}$ into $\mathcal{L}'_{a,b}$.

5. A Boundedness Property for Distributions in $\mathcal{L}'_{a,b}$.

Theorem 5-1 : For each $f \in \mathcal{L}'_{a,b}$ there exist a nonnegative integer $r \in R^1$ and a positive constant $C \in R^1$ such that, for all φ in $\mathcal{L}_{a,b}$,

$$|\langle f, \varphi \rangle| \leq C\, \gamma_r(\varphi) \qquad (5\text{-}1)$$

Proof : Assume that no such relation (5-1) holds. Then, for each nonnegative integer ν, there is a $\varphi_\nu \in \mathcal{L}_{a,b}$ such that

$$|\langle f, \varphi_\nu \rangle| > \nu \gamma_\nu(\varphi) \qquad (5\text{-}2)$$

Let $\psi_\nu \in \mathcal{L}_{a,b}$ be defined by

$$\psi_\nu = \frac{\varphi_\nu}{\nu \gamma_\nu(\varphi_\nu)} .$$

From the definition (2-2), we clearly have that $0 \leqslant \gamma_0 \leqslant \gamma_1 \leqslant \gamma_2 \cdots$.

Thus, for $\mu \leqslant \nu$,

$$\gamma_\mu(\psi_\nu) \leqslant \gamma_\nu(\psi_\nu) = \frac{\gamma_\nu(\varphi_\nu)}{\nu \gamma_\nu(\varphi_\nu)} = \frac{1}{\nu} .$$

Hence, for each fixed μ , $\{\gamma_\mu(\psi_\nu)\}_{\nu=1}^\infty$ converges to zero and consequently, $\{\psi_\nu\}_{\nu=1}^\infty$ converges in $\mathcal{L}_{a,b}$ to zero. Since f is a continuous functional on $\mathcal{L}_{a,b}$,

$$\langle f, \psi_\nu \rangle \longrightarrow 0 \qquad (5\text{-}3)$$

as $\nu \longrightarrow \infty$.

On the other hand, (5-2) implies that

$$|\langle f, \psi_\nu \rangle| > 1 .$$

This contradicts (5-3). Q. E. D.

6. The Distributional Two-sided Laplace Transformation $\mathcal{L}$.

We turn now to the (apparently new) definition of the Laplace transformation $\mathcal{L}$. We shall say that a distribution f is $\mathcal{L}$-transformable if there exist some a, b $\in \mathcal{R}^n$ (a < b) such that $f \in \mathcal{L}'_{a,b}$. In this case the Laplace transform $\mathcal{L} f$ of f is defined as that function F(s) from a subset of C^n

into C^1, given by

$$\mathcal{L} f = F(s) = \langle f(t), [e^{-st}] \rangle$$

We shall also speak of the tube of existence Ω_f, which is a set in C^n defined as follows. A point $s \in C^n$ is in Ω_f if and only if there exist two points $a, b \in R^n$ $(a < b)$, which depend in general on s, such that $a < \text{Re } s < b$ and $f \in \mathcal{L}'_{a,b}$. The right-hand side of (6-1) will have a sense as the application of $f \in \mathcal{L}'_{a,b}$ on $[e^{-st}] \in \mathcal{L}_{a,b}$ if $s \in \Omega_f$. Hence, the name, "tube of existence." Note that by this definition Ω_f is an open set in R^n. In certain cases the right-hand side of (6-1) will have a sense on the boundary of Ω_f, but we will never include this boundary as part of Ω_f.

Whenever we write $\mathcal{L} f$ it shall be understood that f is $\mathcal{L}$-transformable and that $\mathcal{L} f$ exists in the aforementioned sense. Ω_f shall always designate the tube of existence.

<u>Theorem 6-1</u> : <u>Ω_f is a convex set</u>.

<u>Proof</u> : Assume that $f \in \mathcal{L}'_{a,x}$ and also $f \in \mathcal{L}'_{y,b}$, where $a < x < y < b$. We shall show that $f \in \mathcal{L}'_{a,b}$. By the definition of Ω_f, this will prove that Ω_f is a convex set.

Let $\lambda(t_\nu)$ be a smooth function from R^1 into R^1 such that it and all its derivatives are bounded functions on R^n. Also, assume that $\lambda(t_\nu) = 1$ for $t_\nu > 1$ and $\lambda(t_\nu) = 0$ for

$t_\nu < -1$. Set

$$\lambda(t) = \prod_{\nu=1}^{n} \lambda(t_\nu)$$

and $$f = \lambda f + (1-\lambda)f.$$

Since $f \in \mathcal{L}'_{a,x}$ it follows from item 6 of Sec. 4 that $\lambda f \in \mathcal{L}'_{a,x}$ Hence, by note IV of Sec. 3, $\lambda f \in \mathcal{L}'_{a,b}$. A similar argument shows that $(1-\lambda)f \in \mathcal{L}'_{a,b}$. Q. E. D.

The Laplace transformation is a linear operation in the following sense. If $\mathcal{L} f = F(s)$ for $s \in \Omega_f$ and $\mathcal{L} g = G(s)$ for $s \in \Omega_g$ and if $\Omega_f \cap \Omega_g$ is nonvoid, then for $\alpha, \beta \in C^1$ we have

$$\mathcal{L}(\alpha f + \beta g) = \alpha F(s) + \beta G(s)$$

for at least all $s \in \Omega_f \cap \Omega_g$.

The classical two-sided Laplace transformation is based on the integral,

$$F(s) = \int_{\mathbb{R}^n} f(t)\,[e^{-st}]\,dt.$$

If $f(t)$ is a locally integrable function such that, for all real σ in some open convex subset Ξ of $\mathbb{R}^n$, $[e^{-\sigma t}]\, f(t)$ is absolutely integrable on $\mathbb{R}^n$, then this integral certainly converges for all s such that $\operatorname{Re} s \in \Xi$. In this case the regular distribution corresponding to $f(t)$ is certainly in $\mathcal{L}'_{a,b}$ for each $a, b \in \Xi$ $(a < b)$ and the above integral can be interpreted as the application of the regular distribution f to the testing function $[e^{-st}] \in \mathcal{L}_{a,b}$ $(a \leq \operatorname{Re} s \leq b)$. Thus, the classical Laplace transform of a function f(t) that satisfies the aforementioned conditions is a special case of our distributional Laplace transform.

As in the classical case, F(s) is an analytic function within the tube of existence. To prove this we shall need

Lemma 3 : Let $s \in C^n$, $a, b \in R^n$, and $a < \operatorname{Re} s < b$. With s being fixed, let Δs_ν be an increment in the ν th component of s such that $|\Delta s_\nu| < \gamma$ and $a_\nu < \operatorname{Re} s_\nu - \gamma < \operatorname{Re} s_\nu + \gamma < b_\nu$. Finally, for $\Delta s_\nu \neq 0$, set

$$\varphi_{\Delta s_\nu}(t) = \frac{\exp(-\Delta s_\nu t_\nu) - 1}{\Delta s_\nu} [e^{-st}]$$

Then, as $|\Delta s_\nu| \longrightarrow 0$, $\varphi_{\Delta s_\nu}(t)$ converges in $\mathcal{L}_{a,b}$ to $-t_\nu [e^{-st}]$

Proof : Let

$$\begin{aligned}\psi_{\Delta s_\nu}(t) &= \varphi_{\Delta s_\nu}(t) + t_\nu [e^{-st}] \\ &= [e^{-st}] \left\{ \frac{\exp(-\Delta s_\nu t_\nu) - 1}{\Delta s_\nu} + t_\nu \right\} \quad (\Delta s_\nu \neq 0)\end{aligned}$$

The derivative $D_t^k \psi_{\Delta s_\nu}(t)$ is a finite linear combination of terms of the forms,

$$M_{|k|}(s) [e^{-st}] \left\{ \frac{\exp(-\Delta s_\nu t_\nu) - 1}{\Delta s_\nu} + t_\nu \right\}, \tag{6-2}$$

$$M_{|k|-1}(s) [e^{-st}] \{ 1 - \exp(-\Delta s_\nu t_\nu) \}, \tag{6-3}$$

and

$$M_{|k|-\mu}(s)\left[e^{-st}\right](-\Delta s_\nu)^{\mu-1}\exp(-\Delta s_\nu t_\nu)$$

$$(2 \leqslant \mu \leqslant |k|), \qquad (6\text{-}4)$$

where $M_\nu(s)$ designates some monomial in the components of s, of degree ν. As $|t| \longrightarrow \infty$, $\kappa_{a,b}(t)D_t^k \psi_{\Delta s_\nu}(t)$ tends to zero uniformly for $|\Delta s_\nu| < r$. In other words, given an $\varepsilon > 0$, there exists a $T > 0$ such that, for all $|t| > T$ and for all $|\Delta s_\nu| < r$, we have that

$$\left|\kappa_{a,b}(t)\, D_t^k\, \psi_{\Delta s_\nu}(t)\right| < \varepsilon$$

Fix T in this way. Also as $|\Delta s_\nu| \longrightarrow 0$, all the terms (6-2), (6-3), and (6-4) can be made less than ε in magnitude over the domain, $|t| \leqslant T$. All this shows that $\psi_{\Delta s_\nu}(t)$ converges in $\mathcal{L}_{a,b}$ to the zero function as $|\Delta s_\nu| \longrightarrow 0$. Q. E. D.

<u>Theorem 6-3(The Analyticity Theorem)</u> : <u>If</u> $\mathcal{L} f = F(s)$ <u>for</u> $s \in \Omega_f$, <u>then</u> $F(s)$ <u>is</u> <u>analytic</u> <u>on</u> Ω_f <u>and</u>

$$\frac{\partial F}{\partial s_\nu} = \left\langle -t_\nu f(t),\, \left[e^{-st}\right]\right\rangle, \quad (s \in \Omega_f). \qquad (6\text{-}5)$$

<u>Proof</u> : Let a, b be real points in Ω_f and restrict s and Δs_ν as in lemma 3. Then, by the linearity of f,

$$\frac{1}{\Delta s_\nu}\left\{F(s_1, \ldots, s_\nu + \Delta s_\nu, \ldots, s_n) - F(s_1, \ldots, s_\nu, \ldots, s_n)\right\} = \left\langle f(t), \varphi_{\Delta s_\nu}(t)\right\rangle.$$

In view of lemma 3, as $|\Delta s_\nu| \longrightarrow 0$ the right-hand side converges to

$$\langle f(t), -t_\nu [e^{-st}] \rangle ,$$

which is equal to the right-hand side of (6-5). Since s can be chosen as any point in Ω_f by choosing a and b appropriately, the proof is complete.

<u>Corollary 6-3a</u> : <u>Under the hypothesis of theorem</u> 6-3,

$$D_s^k F = \langle (-1)^{|k|} [t^k] f(t), [e^{-st}] \rangle \qquad (s \in \Omega_f) \qquad (6\text{-}6)$$

Our next objective is to relate our Laplace transformation $\mathcal{L}$ to Schwartz's distributional Fourier transformation $\mathcal{F}$ (Schwartz 2 , Vol. II). $\mathcal{S}_t'$ denotes the space of temperate distributions on $\mathbb{R}^n$, where t is the independent variable for the corresponding testing functions. Let $\omega \in \mathbb{R}^n$. $\mathcal{S}_{t,\omega}$ shall be the space of testing functions $\psi(t,\omega)$ of rapid descent defined on $\mathbb{R}^n \times \mathbb{R}^n$.

<u>Lemma 4</u> : <u>If</u> $g \in \mathcal{S}_t'$ <u>and</u> $\psi(t,\omega) \in \mathcal{S}_{t,\omega}$, <u>then</u>

$$\langle g(t), \int_{\mathbb{R}^n} \psi(t,\omega)\, d\omega \rangle = \int_{\mathbb{R}^n} \langle g(t), \psi(t,\omega) \rangle \, d\omega$$

<u>Proof</u> : In the following $1(\omega)$ shall designate the function of ω that equals 1 everywhere. The direct product $g(t) \times 1(\omega)$ is a temperate distribution over $\mathbb{R}^n \times \mathbb{R}^n$ (Schwartz [10] , Vol. II, pp. 99). By the commutativity of the direct product, we may write

$$\langle g(t), \int_{R^n} \psi(t,\omega)d\omega \rangle = \langle g(t), \langle 1(\omega), \psi(t,\omega)\rangle\rangle$$

$$\langle 1(\omega), \langle g(t), \psi(t,\omega)\rangle\rangle = \int_{R^n} g(t), \psi(t,\omega)\, d\omega.$$

Q. E. D.

<u>Theorem 6-4</u> : <u>If</u> $\mathcal{L} f = F(s)$ <u>for</u> $s \in \Omega_f$, <u>then</u>

$$\mathcal{L} f = F(\sigma+i\omega) = \mathcal{F}\{[e^{-\sigma t}]\ f\} \qquad (\sigma \in \Omega_f), \qquad (6\text{-}7)$$

<u>where</u> ω <u>is taken to be the independent variable for the Fourier transform</u>.

<u>Proof</u> : Let a, b, and σ be real points in Ω_f with $a < \sigma < b$. Consequently, $f \in \mathcal{L}'_{a,b}$. According to theorem 3-1, $[e^{-\sigma t}]\ f(t) \in \mathcal{S}'_t$. Since the Fourier transformation is an isomorphism from $\mathcal{S}'_t$ onto itself, $\mathcal{F}\{[e^{-\sigma t}]\ f\} \in \mathcal{S}'_\omega$. Thus,

$$\langle \mathcal{F}\{[e^{-\sigma t}]\ f\}, \varphi(\omega) \rangle = \langle [e^{-\sigma t}]\ f(t), \int_{R^n} \varphi(\omega)\, [e^{-i\omega t}]\, d\omega \rangle$$

$$= \langle \frac{f(t)}{K_{a,b}(t)}, \int_{R^n} K_{a,b}(t)\, [e^{-(\sigma+i\omega)t}]\ \varphi(\omega)\, d\omega \rangle.$$

The integrand inside the right-hand side is a testing function in $\mathcal{S}_{t,\omega}$. Therefore, by lemma 4 the last expression equals

$$\int_{R^n} \langle \frac{f(t)}{K_{a,b}(t)}, K_{a,b}(t)[e^{-(\sigma+i\omega)t}]\varphi(\omega)\rangle\, d\omega = \langle\langle f(t), [e^{-st}]\rangle, \varphi(\omega)\rangle$$

$$= \langle F(\sigma+i\omega), \varphi(\omega) \rangle$$

(Note that theorem 5-1 implies that $F(\sigma+i\omega) \in \mathcal{S}'_\omega$.)

Since σ can be any real point in Ω_f, we have proved (6-7).

Theorem 6-5 (The Uniqueness Theorem) : If $\mathcal{L} f = F(s)$ for $s \in \Omega_f$ and $\mathcal{L} g = G(s)$ for $s \in \Omega_g$, if $\Omega_f \cap \Omega_g$ is nonvoid, and if $F(s) = G(s)$ for $s \in \Omega_f \cap \Omega_g$, then $f = g$.

Proof : Let σ be a fixed real point in $\Omega_f \cap \Omega_g$. Assume for the moment that $F(s) = 0$ on $\Omega_f \cap \Omega_g$. Then, by theorem 6-4,

$$\mathcal{F}\{[e^{-\sigma t}]\, f\} = F(\sigma + i\omega) = 0.$$

Since the Fourier transformation is an isomorphism from $\mathcal{S}'$ onto itself,

$$[e^{-\sigma t}]\, f = 0$$

or, equivalently, for every $\varphi \in \mathcal{D}$,

$$\langle [e^{-\sigma t}]\, f(t), \varphi(t) \rangle = \langle f(t), [e^{-\sigma t}]\, \varphi(t) \rangle = 0.$$

But, $[e^{-\sigma t}]\varphi$ traverses all of $\mathcal{D}$ as φ traverses all of $\mathcal{D}$. Hence, $f = 0$.

Now, assume that $F(s) = G(s) \neq 0$ on $\Omega_f \cap \Omega_g$. By the linearity of the Fourier transformation,

$$\mathcal{F}\{[e^{-\sigma t}]\,(f - g)\} = F(s) - G(s) = 0.$$

By our preceding result, $f - g = 0$. Q. E. D.

Theorem 6-6 (The Continuity Theorem) : *If* $\{f_\nu\}_{\nu=1}^{\infty}$ *converges in* $\mathcal{L}'_{a,b}$ *to* f *for some* a, b $\in \mathbb{R}^n$ (a < b), *and if* $\mathcal{L} f_\nu = F_\nu(s)$, *then* $\mathcal{L} f = F(s)$ *exists for at least* $a \leq \operatorname{Re} s \leq b$, *and* $\{F_\nu(s)\}_{\nu=1}^{\infty}$ *converges pointwise in the tube* $a \leq \operatorname{Re} s \leq b$ *to* F(s).

Proof : Since $[e^{-st}]$ is in $\mathcal{L}_{a,b}$ for each s satisfying $a \leq \operatorname{Re} s \leq b$, this theorem follows from the definition of convergence in $\mathcal{L}'_{a,b}$ and the fact that $\mathcal{L}'_{a,b}$ is sequentially complete.

7. Some Operation-transform Formulas for the Laplace Transformation.

We now list some operation-transform formulas. Each one is a direct consequence of some operation listed in Sec. 4 except for the first one which is a restatement of (6-6). We assume here that $\mathcal{L} f(t) = F(s)$ for $s \in \Omega_f$ and that $\tau \in \mathbb{R}^n$, $\alpha \in C^n$. Each formula represents a continuous (linear) operation $\mathcal{L}\mathfrak{N}$ in the sense that, if $\{f_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{L}'_{a,b}$ to f, then $\{\mathcal{L}\mathfrak{N} f_\nu\}_{\nu=1}^{\infty}$ converge pointwise to $\mathcal{L}\mathfrak{N} f$ in the appropriately transformed tube.

$$\mathcal{L}\{[t^k] f(t)\} = (-1)^{|k|} D_s^k F(s) \qquad (s \in \Omega_f) \qquad (6\text{-}6)$$

$$\mathcal{L} D_t^k f(t) = [s^k] F(s) \qquad (s \in \Omega_f) \qquad (7\text{-}1)$$

$$\mathcal{L} f(t-\tau) = [e^{-s\tau}] F(s) \qquad (s \in \Omega_f) \qquad (7\text{-}2)$$

$$\mathcal{L}\{[e^{-\alpha t}]\, f\} = F(s+\alpha) \qquad (s+\alpha \in \Omega_f) \tag{7-3}$$

$$\mathcal{L}\, f(-t) = F(-s) \qquad (-s \in \Omega_f) \tag{7-4}$$

$$\mathcal{L}\, f(\tau t) = [\tau^{-1}]\, F(\, s/\tau\,) \qquad (\, s/\tau \in \Omega_f) \tag{7-5}$$

8. The Inversion of the Distributional Laplace Transformation.

In this section we establish necessary and sufficient conditions in order for a function $F(s)$ to be a Laplace transform. The sufficiency proof of the following theorem provides a method for inverting the distributional Laplace transform.

Theorem 8-1 : A necessary and sufficient condition for a function $F(s)$ to be the Laplace transform of a distribution f is that there be a tube $a \leqslant \operatorname{Re} s \leqslant b$ $(a < b)$ on which $F(s)$ is analytic and bounded according to

$$|F(s)| \leqslant \mathcal{P}(|s|), \tag{8-1}$$

where $\mathcal{P}(|s|)$ is a polynomial in $|s|$.

Proof : Necessity : If $\mathcal{L} f = F(s)$, then by definition there exists a tube $a \leqslant \operatorname{Re} s \leqslant b$ $(a < b)$ inside the tube of existence Ω_f for which $f \in \mathcal{L}'_{a,b}$. For $a \leqslant \operatorname{Re} s \leqslant b$, $[e^{-st}] \in \mathcal{L}_{a,b}$. Moreover,

$$\sup_t \left| \kappa_{a,b}(t)\, D_t^k [e^{-st}] \right| = [s^k] \sup_t \left| \kappa_{a,b}(t) [e^{-st}] \right| = [s^k] K$$

where $K \in \mathbb{R}^1$ is a constant. So, by the boundedness property (theorem 5-1), there is a constant $C \in \mathbb{R}^1$ and a nonnegative integer $r \in \mathbb{R}^1$ such that

$$F(s) \leqslant \left| \langle f(t), [e^{-st}] \rangle \right| \leqslant C\, \gamma_r([e^{-st}])$$

$$= C\, K \max_{0 \leqslant |k| \leqslant r} [s^k] \leqslant P(|s|).$$

Sufficiency : We shall make use of the following classical fact. If $|G(s)| \leqslant K/|s|^{n+1}$ for $a \leqslant \operatorname{Re} s \leqslant b$ and if

$$g(t) = \frac{1}{(2\pi i)^n} \int_{c_1 - i\infty}^{c_1 + i\infty} \cdots \int_{c_n - i\infty}^{c_n + i\infty} F(s_1, \cdots, s_n) \exp(-s_1 t_1, \cdots - s_n t_n)\, ds_1 \cdots ds_n \qquad (8\text{-}8)$$

$(a < c < b)$,

then, $g(t)$ is a continuous function of $t \in \mathbb{R}^n$ and $\mathcal{L} g = G(s)$ for at least $a < \operatorname{Re} s < b$. (The continuity of $g(t)$ follows from the facts that the integrand in (8-2) is a continuous function of $(s,t) \in C^n \times \mathbb{R}^n$ and that the integral converges uniformly for all $t \in \mathbb{R}^n$. That $\mathcal{L} g = G(s)$ for $a < \operatorname{Re} s < b$ is a consequence of the inversion formula for the classical Fourier transformation ; see Bochner [13], pp. 244 - 245, where we use the fact that on $a \leqslant \operatorname{Re} s \leqslant b$ all partial derivatives up to the second order are also bounded by a polynomial of the same degree as $P(|s|)$, in view of Cauchy's integral formula.)

Now, set $G(s) = [s^{-k}]F(s)$, where k is a nonnegative integer in $\mathbb{R}^n$. Since $|F(s)| \leq \mathcal{P}(|s|)$, we can make $|G(s)| \leq K/|s|^{n+1}$ (K being a constant) for $a \leq \operatorname{Re} s \leq b$ by choosing the components of k large enough. Thus, $F(s) = [s^{k}]G(s)$ and by (7-1) we have

$$f = D^{k} g$$

and $\mathcal{L} f = F(s)$ for at least $a < \operatorname{Re} s < b$. Q. E. D.

If we are given an F(s) that satisfies (8-1), a possible means of obtaining its inverse Laplace transform is first to construct a G(s) as above, then to evaluate its inverse Laplace transform by using (8-2) or perhaps some table of classical Laplace transforms, and finally to differentiate according to (8-3). Of course, in practical cases this procedure may be very difficult to perform.

9. Convolution.

Before stating the definition of the convolution of distributions, we shall establish some facts about the function,

$$\psi(t) = \langle g(\tau), \varphi(t+\tau) \rangle, \tag{9-1}$$

where $g \in \mathcal{L}'_{a,b}$ and $\varphi \in \mathcal{L}_{a,b}$.

Lemma 5 :

$$D_t^k \psi(t) = \langle g(\tau), D_t^k \varphi(t+\tau) \rangle \tag{9-2}$$

Proof : Let t be fixed, let $\Delta t = \{0,\cdots,0,\Delta t_\nu,0,\cdots,0\}$ and consider the function,

$$\theta(\Delta t_\nu, \tau) = K_{a,b}(\tau)\left[\frac{\varphi(t+\tau+\Delta t) - \varphi(t+\tau)}{\Delta t_\nu} - \frac{\partial}{\partial t_\nu}\varphi(t+\tau)\right]$$

$(\Delta t_\nu \neq 0)$.

Clearly, $\theta(\Delta t_\nu, \tau)$ is a smooth function of τ for each fixed Δt_ν. Assuming t and τ are fixed and using Taylor's formula with exact remainder for Δt_ν as the single independent variable, we may write

$$\varphi(t+\tau+\Delta t) = \varphi(t+\tau) + \Delta t_\nu \frac{\partial\varphi(t+\tau)}{\partial t_\nu} + \frac{1}{2}\int_0^{\Delta t_\nu} \frac{\partial^2\varphi(t+\tau+y)}{\partial y_\nu^2}(\Delta t_\nu - y_\nu)\,dy_\nu$$

where $y = \{0,\cdots,0,y_\nu,0,\cdots,0\}$

So,

$$\left|\theta(\Delta t_\nu,\tau)\right| = \left| \frac{K_{a,b}(\tau)}{2\Delta t_\nu}\int_0^{\Delta t_\nu} \frac{\partial^2\varphi(t+\tau+y)}{\partial y_\nu^2}(\Delta t_\nu - y_\nu)\,dy_\nu \right|$$

Moreover,

$$\left| K_{a,b}(\tau) \sup_{|y_\nu|\leq|\Delta t_\nu|} \frac{\partial^2\varphi(t+\tau+y)}{\partial y_\nu^2} \right|$$

K being some constant with respect to τ. Hence,

$$\left|\theta(\Delta t_\nu,\tau)\right| \leq \frac{K}{2\Delta t_\nu}\left|\int_0^{\Delta t_\nu}(\Delta t_\nu - y_\nu)\,dy_\nu\right| = \frac{K|\Delta t_\nu|}{4}.$$

Thus, as $|\Delta t_\nu| \longrightarrow 0$, $\theta(\Delta t_\nu, \tau)$ tends to the zero function uniformly for all τ.

Next, consider

$$\kappa_{a,b}(\tau)\left\{\frac{D_\tau^k\varphi(t+\tau+\Delta t)-D_\tau^k\varphi(t+\tau)}{\Delta t_\nu}-\frac{\partial}{\partial t_\nu}D_\tau^k\varphi(t+\tau)\right\},$$

where $\Delta t_\nu \neq 0$. The same argument as above shows that as $\Delta t_\nu \longrightarrow 0$ this also converges to the zero function uniformly for all τ. Thus,

$$\frac{\varphi(t+\tau+\Delta t)-\varphi(t+\tau)}{\Delta t_\nu}$$

converges to $\partial\varphi(t+\tau)/\partial t_\nu$ in $\mathcal{L}_{a,b}$.

The continuity and linearity of $g \in \mathcal{L}'_{a,b}$ now shows that

$$\begin{aligned}\frac{\partial\psi}{\partial t_\nu} &= \lim_{\Delta t_\nu\to 0}\frac{\psi(t+\Delta t)-\psi(t)}{\Delta t_\nu}\\ &= \lim_{\Delta t_\nu\to 0}\left\langle g(\tau), \frac{\varphi(t+\tau+\Delta t)-\varphi(t+\tau)}{\Delta t_\nu}\right\rangle\\ &= \left\langle g(\tau), \frac{\partial}{\partial t_\nu}\varphi(t+\tau)\right\rangle.\end{aligned}$$

Repeatedly applying this last result, we get (9-2). Q.E.D.

<u>Lemma 6</u> : <u>$\psi(t)$ is a member of</u> $\mathcal{L}_{a,b}$.

<u>Proof</u> : In view of (9-2), $\psi(t)$ is smooth. Thus, we need merely show that it is bounded according to (2-1). Using (9-1) and the boundedness property (5-1), we may write, for each fixed t,

$$\left|\kappa_{a,b}(t)\,D_t^k\psi(t)\right| \leq \left|\kappa_{a,b}(t)\,C\gamma_r\{D_t^k\varphi(t+\tau)\}\right| \qquad (9\text{-}3)$$

Next, consider

$$\kappa_{a,b}(\tau)\left\{\frac{D_\tau^k\varphi(t+\tau+\Delta t)-D_\tau^k\varphi(t+\tau)}{\Delta t_\nu}-\frac{\partial}{\partial t_\nu}D_\tau^k\varphi(t+\tau)\right\},$$

where $\Delta t_\nu \neq 0$. The same argument as above shows that as $\Delta t_\nu \longrightarrow 0$ this also converges to the zero function uniformly for all τ. Thus,

$$\frac{\varphi(t+\tau+\Delta t)-\varphi(t+\tau)}{\Delta t_\nu}$$

converges to $\partial\varphi(t+\tau)/\partial t_\nu$ in $\mathcal{L}_{a,b}$.

The continuity and linearity of $g \in \mathcal{L}'_{a,b}$ now shows that

$$\begin{aligned}\frac{\partial\psi}{\partial t_\nu} &= \lim_{\Delta t_\nu\to 0}\frac{\psi(t+\Delta t)-\psi(t)}{\Delta t_\nu}\\ &= \lim_{\Delta t_\nu\to 0}\left\langle g(\tau), \frac{\varphi(t+\tau+\Delta t)-\varphi(t+\tau)}{\Delta t_\nu}\right\rangle\\ &= \left\langle g(\tau), \frac{\partial}{\partial t_\nu}\varphi(t+\tau)\right\rangle.\end{aligned}$$

Repeatedly applying this last result, we get (9-2). Q.E.D.

<u>Lemma 6</u> : $\psi(t)$ <u>is</u> <u>a</u> <u>member</u> <u>of</u> $\mathcal{L}_{a,b}$.

<u>Proof</u> : In view of (9-2), $\psi(t)$ is smooth. Thus, we need merely show that it is bounded according to (2-1). Using (9-1) and the boundedness property (5-1), we may write, for each fixed t,

$$\left|\kappa_{a,b}(t)\, D_t^k\,\psi(t)\right| \leq \left|\kappa_{a,b}(t)\; C\gamma_r\left\{D_t^k\,\varphi(t+\tau)\right\}\right| \tag{9-3}$$

$$= C \max_{0 \le |j| \le r} \sup_{\tau} \left| \kappa_{a,b}(t)\, \kappa_{a,b}(\tau)\, D_\tau^j\, D_t^k\, \varphi(t+\tau) \right|$$

$$\le C \max_{0 \le |j| \le r} \sup_{\tau} \left| \frac{\kappa_{a,b}(t)\, \kappa_{a,b}(\tau)}{\kappa_{a,b}(t+\tau)}\; C_{j+k} \right|$$

Here, the C_{j+k} denote the constants given in (2-1). Moreover, as a function of $(t,\tau) \in R^n \times R^n$,

$$\frac{\kappa_{a,b}(t)\, \kappa_{a,b}(\tau)}{\kappa_{a,b}(t+\tau)} \tag{9-4}$$

is clearly a bounded function over every bounded domain. Even more is true. It is bounded over all of $R^n \times R^n$. To see this, we may assume that t, τ, a, and b are in R^1 since (9-4) is simply a product of such one-dimensional factors. We then consider the following possible ways that (t,τ) may approach infinity. (i) For $t > 1$ and $\tau > 1$, (9-4) equals one. (ii) For $t > 2$ and $|\tau| < 1$, (9-4) remains bounded. (iii) If $t \longrightarrow \infty$ and $\tau \longrightarrow -\infty$ such that $t + \tau$ remains bounded, then (9-4) approaches zero since $a < b$. (iv) If $t \longrightarrow \infty$ and $\tau \longrightarrow -\infty$ such that $t + \tau \longrightarrow \infty$, then (9-4) again approaches zero since $a < b$. Because of the symmetry in the form of (9-4), we can make similar statements for the corresponding four cases when $t \longrightarrow -\infty$. We can now conclude that (9-4) is bounded for all (t,τ). Thus, the right-hand side of (9-3) is a finite number, which proves that $\psi(t)$ satisfies (2-1). Q. E. D.

Lemma 7 : Let $\{\varphi_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{L}_{a,b}$ to zero and let

$$\psi_\nu(t) = \langle g(\tau), \varphi_\nu(t+\tau) \rangle,$$

where $g \in \mathcal{L}'_{a,b}$. Then, $\{\psi_\nu\}_{\nu=1}^{\infty}$ also converges in $\mathcal{L}_{a,b}$ to zero.

The proof of this lemma is almost identical to the preceding proof. In this case we rely on the fact that for each fixed j and k and for $\nu \longrightarrow \infty$ the functions,

$$K_{a,b}(t+\tau)\, D_\tau^{j}\, D_t^{k}\, \varphi_\nu(t+\tau),$$

converge to the zero function uniformly for all (t,τ).

We turn now to the definition of the convolution of two $\mathcal{L}$-transformable distributions, f and g. Let $\mathcal{L}f$ and $\mathcal{L}g$ exist for $s \in \Omega_f$ and $s \in \Omega_g$, respectively, and let $\Omega_f \cap \Omega_g$ be nonvoid. Let a and b be arbitrary real points in $\Omega_f \cap \Omega_g$ with $a < b$. We define the convolution $f*g$ as a functional on $\mathcal{L}_{a,b}$ by

$$\langle f*g, \varphi \rangle = \langle f(t), \langle g(\tau), \varphi(t+\tau)\rangle\rangle \quad (\varphi \in \mathcal{L}_{a,b}) \qquad (9\text{-}5)$$

The right-hand side has a sense since $f \in \mathcal{L}'_{a,b}$ and, according to lemma 6, $\langle g(\tau), \varphi(t+\tau)\rangle \in \mathcal{L}_{a,b}$. Actually, $f*g$ is a distribution in $\mathcal{L}'_{a,b}$. Indeed, its linearity as a functional on $\mathcal{L}_{a,b}$ is obvious. Moreover, by lemma 7 and the continuity of f as a functional on $\mathcal{L}_{a,b}$, we have that, if $\{\varphi_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{L}_{a,b}$ to the zero function, then

$$\langle f*g, \varphi_\nu \rangle = \langle f, \psi_\nu \rangle \longrightarrow 0$$

as $\nu \longrightarrow \infty$. Thus, $f*g$ is a continuous linear functional on $\mathcal{L}_{a,b}$ as was asserted.

The convolution of distributions can be defined even when the distributions are not $\mathcal{L}$-transformable (see Zemanian [8], chapter 5) but we do not do this here.

Convolution is a linear process in the sense that, if $\alpha, \beta \in \mathbb{C}^1$, if f, g, and h are distributions whose Laplace transforms exist for $s \in \Omega_f$, $s \in \Omega_g$, and $s \in \Omega_h$, respectively, and if $\Omega_f \cap \Omega_g \cap \Omega_h$ is nonvoid, then

$$f*(\alpha g + \beta h) = \alpha f*g + \beta f*h$$

and

$$(\alpha f + \beta g)*h = \alpha f*h + \beta g*h$$

Under certain restrictions on f and g, our distributional convolution $f*g$ is equivalent to the classical convolution. In particular, assume that f and g are locally integrable functions from $\mathbb{R}^n$ into $\mathbb{C}^1$ and that $f/\varkappa_{a,b}$ and $g/\varkappa_{a,b}$ are absolutely integrable on $\mathbb{R}^n$ for some $a, b \in \mathbb{R}^n$ $(a < b)$. Then, for $\varphi \in \mathcal{L}_{a,b}$ we may write

$$\langle f*g, \varphi \rangle = \langle f(t), \langle g(\tau), \varphi(t+\tau) \rangle \rangle$$

$$= \int_{R^n} dt \int_{R^n} f(t) g(\tau)\, \varphi(t+\tau)\, d\tau. \qquad (9\text{-}6)$$

The last integrand is locally integrable as a function of (t,τ). It is also absolutely integrable over the (t,τ) Euclidean space. (Indeed,

$$\left| f(t) g(\tau) \varphi(t+\tau) \right| = \left| \frac{f(t) g(\tau)}{\kappa_{a,b}(t)\, \kappa_{a,b}(\tau)} \right| \left| \frac{\kappa_{a,b}(t)\, \kappa_{a,b}(\tau)}{\kappa_{a,b}(t+\tau)} \right| \left| \kappa_{a,b}(t+\tau) \varphi(t+\tau) \right|.$$

The first factor on the right-hand side is integrable on the (t,τ) space whereas the second and third factors are bounded and continuous on this space.) Applying the change of variable $x = t$ and $y = t + \tau$ and noting that the Jacobian determinant equals one, we convert (9-6) into

$$\langle f*g, \varphi \rangle = \int_{R^n} dy \int_{R^n} f(x) g(y-x) \varphi(y) dx = \langle \int_{R^n} f(x) g(y-x) dx, \varphi(y) \rangle.$$

Thus, we may write

$$f*g(y) = \int_{R^n} f(x) g(y-x)\, dx, \qquad (9\text{-}7)$$

where the right-hand side is a locally integrable function in $\mathcal{L}'_{a,b}$. Expression (9-7) is the classical form for a convolution.

The Laplace transformation converts convolution into multiplication, as follows.

Theorem 9-1 : Let $\mathcal{L} f = F(s)$ for $s \in \Omega_f$ and $\mathcal{L} g = G(s)$ for $s \in \Omega_g$. Also, assume that $\Omega_f \cap \Omega_g$ is nonvoid. Then,

$$\mathcal{L}(f*g) = F(s)G(s) \qquad (s \in \Omega_f \cap \Omega_g). \tag{9-8}$$

Proof : In the definition of $f*g$, the points a and b $(a < b)$ can be chosen arbitrarily in $\Omega_f \cap \Omega_g$. Therefore, $\mathcal{L}(f*g)$ exists for $s \in \Omega_f \cap \Omega_g$. Hence,

$$\begin{aligned}\mathcal{L}(f*g) &= \langle f(t), \langle g(\tau), [e^{-s(t+\tau)}] \rangle\rangle \\ &= \langle f(t), [e^{-st}] \rangle \cdot \langle g(\tau), [e^{-s\tau}] \rangle \\ &= F(s)\,G(s)\end{aligned}$$

Q. E. D.

We can use this result to prove the commutativity and associativity of our convolution.

Theorem 9-2 : Under the hypothesis of theorem 9-1,

$$f*g = g*f \qquad \text{(commutativity)}. \tag{9-9}$$

If, in addition, $\mathcal{L} h = H(s)$ for $s \in \Omega_h$ and $\Omega_f \cap \Omega_g \cap \Omega_h$ is nonvoid, then

$$f*(g*h) = (f*g)*h \qquad \text{(associativity)}. \tag{9-10}$$

Proof : From theorem 9-1, we have

$$\mathcal{L}(f*g) = F(s)G(s) = G(s)F(s) = \mathcal{L}(g*f) \qquad (s \in \Omega_f \cap \Omega_g)$$

Equation (9-9) now follows from the uniqueness of the Laplace transformation.

Similarly,

$$\mathcal{L}\{f*(g*h)\} = F(s)\,\mathcal{L}(g*h) = F(s)\,G(s)\,H(s)$$

$$= \{\mathcal{L}(f*g)\}H(s) = \mathcal{L}\{(f*g)*h\} \qquad (s \in \Omega_f \cap \Omega_g \cap \Omega_h\ )$$

The uniqueness theorem now implies (9-10). Q. E. D.

We can now conclude that $\mathcal{L}'_{a,b}$ is a commutative algebra of convolution having the n-dimensional delta functional $\delta(t)$ as its unit element. It does not have any divisors of zero. Indeed, if $f*g = 0$, then $\mathcal{L}(f*g) = F(s)\,G(s) = 0$ for $s \in \Omega_f \cap \Omega_g$. By the analyticity of F and G, either F(s) or G(s) (or both) equals zero on $\Omega_f \cap \Omega_g$. By the uniqueness theorem, either f or g (or both) is the zero distribution.

Convolution is a continuous operation in the following way.

<u>Theorem 9-3</u> : <u>Let</u> $\{f_\nu\}_{\nu=1}^{\infty}$ <u>converge</u> <u>in</u> $\mathcal{L}'_{a,b}$ <u>to</u> f <u>and</u> <u>let</u> $g \in \mathcal{L}'_{a,b}$. <u>Then,</u> $\{f_\nu * g\}_{\nu=1}^{\infty}$ <u>converges</u> <u>in</u> $\mathcal{L}'_{a,b}$ <u>to</u> $f*g$.

<u>Proof</u> : Let $\varphi \in \mathcal{L}_{a,b}$. In view of (9-5) and lemma 6, we may write,

$$\langle f_\nu * g, \varphi \rangle = \langle f_\nu, \psi \rangle \longrightarrow \langle f, \psi \rangle = \langle f*g, \varphi \rangle \quad . \quad \text{Q. E. D.}$$

Equation (9-9) now follows from the uniqueness of the Laplace transformation.

Similarly,

$$\mathcal{L}\{f*(g*h)\} = F(s)\,\mathcal{L}(g*h) = F(s)\ G(s)\ H(s)$$

$$= \{\mathcal{L}(f*g)\}H(s) = \mathcal{L}\{(f*g)*h\} \qquad (s \in \Omega_f \cap \Omega_g \cap \Omega_h \quad)$$

The uniqueness theorem now implies (9-10). Q. E. D.

We can now conclude that $\mathcal{L}'_{a,b}$ is a commutative algebra of convolution having the n-dimensional delta functional $\delta(t)$ as its unit element. It does not have any divisors of zero. Indeed, if $f*g = 0$, then $\mathcal{L}(f*g) = F(s)\ G(s) = 0$ for $s \in \Omega_f \cap \Omega_g$. By the analyticity of F and G, either F(s) or G(s) (or both) equals zero on $\Omega_f \cap \Omega_g$. By the uniqueness theorem, either f or g (or both) is the zero distribution.

Convolution is a continuous operation in the following way.

<u>Theorem 9-3</u> : <u>Let</u> $\{f_\nu\}_{\nu=1}^{\infty}$ <u>converge</u> <u>in</u> $\mathcal{L}'_{a,b}$ <u>to</u> f <u>and</u> <u>let</u> $g \in \mathcal{L}'_{a,b}$. <u>Then,</u> $\{f_\nu * g\}_{\nu=1}^{\infty}$ <u>converges</u> <u>in</u> $\mathcal{L}'_{a,b}$ <u>to</u> $f*g$.

<u>Proof</u> : Let $\varphi \in \mathcal{L}_{a,b}$. In view of (9-5) and lemma 6, we may write,

$$\langle f_\nu * g, \varphi \rangle = \langle f_\nu , \psi \rangle \longrightarrow \langle f, \psi \rangle = \langle f*g, \varphi \rangle \quad . \quad \text{Q. E. D.}$$

PART II

THE DISTRIBUTIONAL MELLIN TRANSFORMATION.

10. The Testing Function Space $\mathcal{M}_{a,b}$

$\mathcal{R}^n_+$ shall denote the open domain $0 < x < \infty$. The change of variable, $x = e^{-t}$, $t = -\log x$, will be of importance to us. As t traverses $\mathcal{R}^n$, x traverses $\mathcal{R}^n_+$. We set $\mathfrak{Z}_{a,b}(x) = K_{a,b}(-\log x)$. Thus, $\mathfrak{Z}_{a,b}(x)$ is a smooth positive function defined on $\mathcal{R}^n_+$. Moreover,

$$\mathfrak{Z}_{a,b}(x) = \prod_{\nu=1}^{n} \mathfrak{Z}_{a_\nu b_\nu}(x_\nu),$$

where

$$\mathfrak{Z}_{a_\nu b_\nu}(x_\nu) = \begin{cases} x_\nu^{-a_\nu} & (0 < x_\nu < e^{-1}) \\ x_\nu^{-b_\nu} & (e < x_\nu < \infty). \end{cases}$$

$\mathcal{M}_{a,b}$ is the space of all smooth functions $\theta(x)$ defined on $\mathcal{R}^n_+$ with values in $\mathcal{C}'$, which satisfy the following set of inequalities. For each fixed k,

$$\left|\mathfrak{Z}_{a,b}(x)\left[x^{k+1}\right] D_x^k \theta(x)\right| \leq K_k \qquad (0<x<\infty), \qquad (10\text{-}1)$$

where K_k denote constants that depend only upon the choices of k and θ.

$\mathcal{M}_{a,b}$ is a linear space over the field $\mathcal{C}'$. Any smooth function whose support is contained in $\mathcal{R}^n_+$ is in $\mathcal{M}_{a,b}$.

Other menbers of $\mathcal{M}_{a,b}$ are $[x^{s-1}]$ for $a \leqslant \operatorname{Re} s \leqslant b$ and $[(\log x)^k x^{b-1}]$ for $a < \operatorname{Re} s < b$. However, $[(\log x)^k x^{a-1}]$ and $[(\log x)^k x^{b-1}]$ are not in $\mathcal{M}_{a,b}$ if at least one of the components of k is positive.

We assign to $\mathcal{M}_{a,b}$ the topology generated by the semi-norms,

$$\lambda_\nu = \lambda_\nu(\theta) = \max_{0 \leqslant |k| \leqslant \nu} \sup_x \left| \zeta_{a,b}(x) \, [x^{k+1}] \quad D_x^k \theta(x) \right| \tag{10-2}$$

A sequence $\{\theta_\nu\}_{\nu=1}^{\infty}$ is a Cauchy sequence in $\mathcal{M}_{a,b}$ if and only if each $\theta_\nu \in \mathcal{M}_{a,b}$ and for each fixed k the functions,

$$\zeta_{a,b}(x) \, [x^{k+1}] \quad D_x^k \quad \theta_\nu(x),$$

converges uniformly on $\mathbb{R}_+^n$ as $\nu \longrightarrow \infty$. It follows that the limit function $\theta(x)$ for this sequence is also in $\mathcal{M}_{a,b}$; that is, $\mathcal{M}_{a,b}$ is sequentially complete. We shall refer to this type of convergence as "convergence in $\mathcal{M}_{a,b}$." If the limit is the zero function, we say "convergence in $\mathcal{M}_{a,b}$ to zero."

If a sequence $\{\theta_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{D}$ to θ such that the supports of all the θ_ν are contained in a fixed finite closed subset of $\mathbb{R}_+^n$, then $\{\theta_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{M}_{a,b}$ to θ for every a, $b \in \mathbb{R}^n$ $(a < b)$. Also, if $a \leqslant c < d \leqslant b$, then $\mathcal{M}_{c,d} \subset \mathcal{M}_{a,b}$ and convergence in $\mathcal{M}_{c,d}$ implies convergence in $\mathcal{M}_{a,b}$.

Theorem 10-1 : The mapping,

$$\theta(x) \longrightarrow [e^{-t}]\,\theta(e^{-t}) = \varphi(t), \tag{10-3}$$

is an isomorphism from $\mathcal{M}_{a,b}$ onto $\mathcal{L}_{a,b}$. The inverse mapping is given by

$$\varphi(t) \longrightarrow [x^{-1}]\,\varphi(-\log x) = \theta(x) \tag{10-4}$$

Proof : That the mappings (10-3) and (10-4) are inverses of each other is obvious. Now, assume that $\theta(x) \in \mathcal{M}_{a,b}$ The derivative,

$$D_t^k\{[e^{-t}]\;\theta(e^{-t})\},$$

is equal to a finite sum of terms, a typical term being

$$a_p\,[x^{p+1}]\;D_x^p\,\theta(x),$$

where $x = e^{-t}$, $0 \leq p \leq k$, and a_p is a constant. Thus,

$$\left| \kappa_{a,b}(t)\; D_t^k\{[e^{-t}]\theta(e^{-t})\} \right| = \zeta_{a,b}(x) \sum_p a_p[x^{p+1}]\;\; D_x^p\;\theta(x)$$

$$\leq \sum a_p K_p\,,$$

where the K_p are the constants indicated in (10-1). Thus, $\varphi(t) \in \mathcal{L}_{a,b}$.

A similar argument shows that, if $\{\theta_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{M}_{a,b}$ to zero and if $\varphi_\nu(t) = [e^{-t}]\,\theta_\nu(e^{-t})$, then $\{\varphi_\nu\}_{\nu=1}^{\infty}$

converges in $\mathcal{L}_{a,b}$ to zero. The mapping (10-3) is clearly linear. Thus, we have shown that it is a continuous linear mapping from $\mathcal{M}_{a,b}$ into $\mathcal{L}_{a,b}$.

Now, assume that $\varphi(t) \in \mathcal{L}_{a,b}$. Some computation shows that

$$D_x^k \{[x^{-1}] \varphi(-\log x)\} = [x^{-k-1}] \sum_p b_p D_t^p \varphi(t),$$

where $x = e^{-t}$, the b_p are constants, and the summation extends over all integers $p \in \mathcal{R}^n$ that satisfy $0 \leq p \leq k$. Thus,

$$\left| \mathfrak{Z}_{a,b}(x) [x^{k+1}] D_x^k \theta(x) \right| = \left| \kappa_{a,b}(t) \sum_p b_p D_t^p \varphi(t) \right| \leq \sum_p b_p C_p ,$$

where the C_p are the constants given in (2-1). Consequently, $\theta(x) \in \mathcal{M}_{a,b}$. The same reasoning shows that (10-4) is a continuous mapping. Its linearity is clear.

Thus, we see that (10-3) is a one-to-one continuous linear mapping of $\mathcal{L}_{a,b}$ onto $\mathcal{M}_{a,b}$ and that its inverse mapping has the same properties from $\mathcal{M}_{a,b}$ onto $\mathcal{L}_{a,b}$. Q. E. D.

We now list a number of operations that may be applied to $\mathcal{M}_{a,b}$. The proof of the properties associated with each operation is given in Appendix B.

1. For $y \in \mathcal{R}^n$ and $y > 0$, $\theta(x) \longrightarrow \theta(yx)$ is an isomorphism from $\mathcal{M}_{a,b}$ onto $\mathcal{M}_{a,b}$.

2. $\theta(x) \longrightarrow \theta(x^{-1})$ is an isomorphism from $\mathcal{M}_{a,b}$ onto $\mathcal{M}_{2-b,2-a}$.

3. For $y \in \mathbb{R}^n$ and $y > 0$, $\theta(x) \longrightarrow \theta(x^y)$ is an isomorphism from $\mathcal{M}_{a,b}$ onto $\mathcal{M}_{1+ya-y,1+yb-y}$.

4. For $y \in \mathbb{R}^n$ and $y < 0$, $\theta(x) \longrightarrow \theta(x^y)$ is an isomorphism from $\mathcal{M}_{a,b}$ onto $\mathcal{M}_{1+yb-y,1+ya-y}$.

5. Let $\mathcal{P}(\log x)$ be a polynomial in the components of $\log x$. Then, $\theta \longrightarrow \mathcal{P}(\log x)\theta$ is a continuous linear mapping of $\mathcal{M}_{c,d}$ into $\mathcal{M}_{a,b}$ for every a and b such that $a < c < d < b$.

6. If $\alpha \in \mathbb{C}^n$ and $y = \mathrm{Re}\,\alpha$, then $\theta \longrightarrow [x^\alpha]\,\theta$ is an isomorphism from $\mathcal{M}_{a,b}$ onto $\mathcal{M}_{a+y,b+y}$.

7. $\theta \longrightarrow D_x^k\theta$ is a continuous linear mapping of $\mathcal{M}_{a,b}$ into $\mathcal{M}_{a-k,b-k}$.

8. $\theta \longrightarrow [x^k]\,D_x^k\theta$ is a continuous linear mapping of $\mathcal{M}_{a,b}$ into $\mathcal{M}_{a,b}$.

9. Let $[x\,D_x]^k\theta$ denote

$$x_\xi \frac{\partial}{\partial x_\xi}(\ \dots\ (x_\mu \frac{\partial}{\partial x_\mu}(x_\nu \frac{\partial}{\partial x_\nu}\theta)\ )\ \dots\ ),$$

where the number of times $x_\nu \frac{\partial}{\partial x_\nu}$ appears in the expression is k. (The order in which these terms appear is unimportant in view of the smoothness of $\theta \in \mathcal{M}_{a,b}$.) Then, $\theta \longrightarrow [x\,D_x]^k\theta$ is a continuous linear mapping of $\mathcal{M}_{a,b}$ into $\mathcal{M}_{a,b}$.

10. $\theta \longrightarrow D_x^k([x^k]\,\theta)$ is a continuous linear mapping of $\mathcal{M}_{a,b}$ into $\mathcal{M}_{a,b}$.

11. Let $[D_x x]^k \theta$ denote

$$\frac{\partial}{\partial x_\xi}(x_\xi \ldots \frac{\partial}{\partial x_\mu}(x_\mu \frac{\partial}{\partial x_\nu}(x_\nu \theta)) \ldots),$$

where the number of times $\frac{\partial}{\partial x_\nu} x_\nu$ appears is k_ν. Then, $\theta \longrightarrow [D_x x]^k \theta$ is a continuous linear mapping of $\mathcal{M}_{a,b}$ into $\mathcal{M}_{a,b}$.

11. <u>The Dual Space $\mathcal{M}'_{a,b}$.</u>

$\mathcal{M}'_{a,b}$ is the dual space of $\mathcal{M}_{a,b}$. In other words, $f \in \mathcal{M}'_{a,b}$ if and only if f is a continuous linear functional on $\mathcal{M}_{a,b}$ Equality, addition, and multiplication-by-a-complex number are defined in the usual way. $\mathcal{M}'_{a,b}$ is a linear space over C'. $\langle f, \theta \rangle = \langle f(x), \theta(x) \rangle$ is the number that $f \in \mathcal{M}'_{a,b}$ assigns to $\theta \in \mathcal{M}_{a,b}$.

If the support of the distribution f is contained in a bounded closed subset of $\mathcal{R}^n_+$, then f is in $\mathcal{M}'_{a,b}$ for every $a, b \in \mathcal{R}^n$ with $a < b$. On the other hand, every member of $\mathcal{M}'_{a,b}$ is a distribution on $\mathcal{R}^n_+$. Since convergence in $\mathcal{M}_{c,d}$ implies convergence in $\mathcal{M}_{a,b}$ for $a \leq c < d \leq b$, it follows that $\mathcal{M}'_{a,b} \subset \mathcal{M}'_{c,d}$.

We define a (weak) topology for $\mathcal{M}'_{a,b}$ by using the following seminorms. For each θ in $\mathcal{M}_{a,b}$, we define a seminorm $\xi_\theta(f)$ on $\mathcal{M}'_{a,b}$ by

$$\xi_\theta(f) = |\langle f, \theta \rangle| \qquad (f \in \mathcal{M}'_{a,b}) \qquad (11\text{-}1)$$

It follows that a sequence $\{f_\nu\}_{\nu=1}^{\infty}$ $(f_\nu \in \mathcal{M}'_{a,b})$ is a Cauchy sequence in $\mathcal{M}'_{a,b}$ if and only if for every $\theta \in \mathcal{M}_{a,b}$ the numerical sequence $\{\langle f_\nu, \theta \rangle\}_{\nu=1}^{\infty}$ converges. We shall refer to this type of convergence as "convergence in $\mathcal{M}'_{a,b}$." ("Zero" in $\mathcal{M}'_{a,b}$ means the zero distribution on $\mathbb{R}^n_+$.) One can show that $\mathcal{M}'_{a,b}$ is sequentially complete by using the same proof as that given in Appendix C for $\mathcal{L}'_{a,b}$.

A strong topology can also be defined on $\mathcal{M}'_{a,b}$ as was done for $\mathcal{L}'_{a,b}$, but we shall make no use of it.

We can relate the elements of $\mathcal{M}'_{a,b}$ to those in $\mathcal{L}'_{a,b}$ by formally applying the standard change-of-variable formula for distribution, which is the following. Let $x = u(t)$ and $t = v(x)$, where u and v are smooth functions mapping an open simply-connected domain in the t-space onto an open simply-connected domain Ξ in the x-space in a one-to-one fashion. Also, assume that the Jacobian

$$J(t) = \frac{\partial(x_1, \ldots, x_n)}{\partial(t_1, \ldots, t_n)}$$

does not equal zero on Ω. Then, for every $\theta \in \mathcal{D}$ with compact support (with respect to Ξ) and for every distribution f defined on Ξ, we have

$$\langle f(x), \theta(x) \rangle = \langle f(u(t)), |J(t)|\, \theta(u(t)) \rangle$$

As usual, we set $x = e^{-t}$, $t = -\log x$, $\varphi(t)=[e^{-t}]\theta(e^{-t})$, and $\theta(x) = [x^{-1}]\varphi(-\log x)$. In view of theorem 10-1, we can associate to each $f(x) \in \mathcal{M}'_{a,b}$ a distribution $f(e^{-t}) \in \mathcal{L}'_{a,b}$ by

$$\langle f(e^{-t}), \varphi(t) \rangle = \langle f(x), \theta(x) \rangle \tag{11-2}$$

Conversely, if $g(t) \in \mathcal{L}'_{a,b}$, then $g(-\log x) \in \mathcal{M}'_{a,b}$ given by

$$\langle g(-\log x), \theta(x) \rangle = \langle g(t), \varphi(t) \rangle \tag{11-3}$$

Clearly, we may state

Theorem 11-1 : The mapping $f(x) \to f(e^{-t})$, defined by (11-2), is an isomorphism from $\mathcal{M}'_{a,b}$ onto $\mathcal{L}'_{a,b}$. The inverse mapping is defined by (11-3).

12. Some Operations on $\mathcal{M}'_{a,b}$.

We now list some operations that may be applied to $\mathcal{M}'_{a,b}$ and their characteristics. The definitions given below conform with the formulas one would obtain by either applying theorem 11-1 to the operations listed in Sec. 4 or by formally using the change-of-variable formula to construct the adjoint operations corresponding to those listed in Sec. 10. We assume here that $f \in \mathcal{M}'_{a,b}$ and $\mathfrak{N}\theta \in \mathcal{M}_{a,b}$, where $\mathfrak{N}$ is the particular operation on θ under consideration.

1. For $y \in \mathbb{R}^n$ and $y > 0$, $f(x) \longrightarrow f(yx)$ is defined by

$$\langle f(yx), \theta(x) \rangle = \langle f(x), [y^{-1}]\, \theta(x / y) \rangle$$

It is an isomorphism from $\mathcal{M}'_{a,b}$ onto $\mathcal{M}'_{a,b}$.

2. $f(x) \longrightarrow f(x^{-1})$ is defined by

$$\langle f(x^{-1}), \theta(x) \rangle = \langle f(x), [x^{-2}]\, \theta(x^{-1}) \rangle$$

It is an isomorphism from $\mathcal{M}'_{a,b}$ onto $\mathcal{M}'_{-b,-a}$.

3. For $y \in \mathbb{R}^n$ and $y > 0$, $f(x) \longrightarrow f(x^y)$ is defined by

$$\langle f(x^y), \theta(x) \rangle = \langle f(x), [y^{-1} x^{\frac{1-y}{y}}]\, \theta(x^{1/y}) \rangle.$$

It is an isomorphism from $\mathcal{M}'_{a,b}$ onto $\mathcal{M}'_{ya,yb}$.

4. For $y \in \mathbb{R}^n$ and $y < 0$, $f(x) \longrightarrow f(x^y)$ is defined by

$$\langle f(x^y), \theta(x) \rangle = \langle f(x), |[y^{-1}]|\, [x^{\frac{1-y}{y}}]\, \theta(x^{1/y}) \rangle.$$

It is an isomorphism from $\mathcal{M}'_{a,b}$ onto $\mathcal{M}'_{yb,ya}$.

5. Let $\rho(\log x)$ be a polynomial in the components of $\log x$. Then, $f \longrightarrow \rho(\log x) f$ is defined by

$$\langle \rho(\log x) f, \theta \rangle = \langle f, \rho(\log x)\, \theta \rangle.$$

It is a continuous linear mapping of $\mathcal{M}'_{a,b}$ into $\mathcal{M}'_{c,d}$ for every c and d such that $a < c < d < b$.

6. For $\alpha \in C^n$ and $y = \mathrm{Re}\,\alpha$, $f \longrightarrow [x^\alpha]\, f$ is defined by

$$\langle [x^\alpha]\, f, \theta \rangle = \langle f, [x^\alpha]\, \theta \rangle$$

It is an isomorphism from $\mathcal{M}'_{a,b}$ onto $\mathcal{M}'_{a-y,\, b-y}$.

7. $f \longrightarrow D_x^k f$ is defined by

$$\langle D_x^k f, \theta \rangle = \langle f, (-1)^{|k|} D_x^k\, \theta \rangle$$

It is a continuous linear mapping of $\mathcal{M}'_{a,b}$ into $\mathcal{M}'_{a+k,\, b+k}$.

8. $f \longrightarrow [x^k]\, D_x^k f$ is defined by

$$\langle [x^k] D_x^k f, \theta \rangle = \langle f, (-1)^{|k|} D_x^k ([x^k]\, \theta)$$

It is a continuous linear mapping of $\mathcal{M}'_{a,b}$ into $\mathcal{M}'_{a,b}$.

9. $f \longrightarrow [x\, D_x]^k f$ is defined by

$$\langle [x\, D_x]^k f, \theta \rangle = \langle f, (-1)^{|k|} [D_x\, x]^k\, \theta \rangle$$

It is a continuous linear mapping of $\mathcal{M}'_{a,b}$ into $\mathcal{M}'_{a,b}$.

10. $f \longrightarrow D_x^k ([x^k]\, f)$ is defined by

$$\langle D_x^k ([x^k] f), \theta \rangle = \langle f, (-1)^{|k|} [x^k] D_x^k\, f \rangle.$$

It is a continuous linear mapping of $\mathcal{M}'_{a,b}$ into $\mathcal{M}'_{a,b}$.

11. $f \longrightarrow [D_x\, x]^k f$ is defined by

$$\langle [D_x\, x]^k f, \theta \rangle = \langle f, (-1)^{|k|} [x\, D_x]^k\, \theta \rangle.$$

It is a continuous linear mapping of $\mathfrak{M}'_{a,b}$ into $\mathfrak{M}'_{a,b}$.

13. A Boundedness Property for Distributions in $\mathfrak{M}'_{a,b}$.

By using the same arguments as those employed in Sec. 5, we can prove

Theorem 13-1 : *For each* $f \in \mathfrak{M}'_{a,b}$ *there exist a non-negative integer* $r \in R^1$ *and a positive constant* $C \in R^1$ *such that for all* $\theta \in \mathfrak{M}_{a,b}$,

$$|\langle f, \theta \rangle| \leq C \lambda_r(\theta).$$

14. The Distributional Mellin Transformation $\mathfrak{M}$.

A distribution f defined on R^n_+ will be called $\mathfrak{M}$-transformable if $f \in \mathfrak{M}'_{a,b}$ for some a, b $\in R^n$ such that a < b. Its Mellin transform $\mathfrak{M}f$ is a function F(s) from a subset of C^n into C^1 defined by

$$\mathfrak{M}f = F(s) = \langle f(x), [x^{s-1}] \rangle \qquad (14\text{-}1)$$

The right-hand side has a sense since $[x^{s-1}] \in \mathfrak{M}_{a,b}$ for $a \leq \mathrm{Re}\, s \leq b$. The interior of the set of all s for which (14-1) has a sense in this way is called the tube of existence and is denoted by Ω_f. Note that as s traverses Ω_f the choices of a and b, for which $f \in \mathfrak{M}'_{a,b}$, may have to be altered.

Let us set $\varphi(t) = [e^{-st}]$ and $\theta(x) = [x^{-1}]\varphi(-\log x) = [x^{s-1}]$. Invoking theorem 11-1, we can state

Theorem 14-1 : The distribution $f(x)$ is $\mathfrak{M}$-transformable if and only if $f(e^{-t})$ is $\mathcal{L}$-transformable. In this case,

$$\mathfrak{M} f(x) = F(s) = \mathcal{L} f(e^{-t}) \qquad (s \in \Omega_f).$$

It follows that various properties of the Laplace transformation can be carried over to the Mellin transformation. For example,

(i) The tube of existence Ω_f for a Mellin transform is a convex set.

(ii) The Mellin transformation is linear in the same way as is the Laplace transformation.

(iii) If $f(x)$ is a locally integrable function defined on $\mathbb{R}^n_+$ with values in C^1 such that, for all Re $s = \sigma$ in some open convex subset Ξ of $\mathbb{R}^n$, $[x^{\sigma-1}] f(x)$ is absolutely integrable on $\mathbb{R}^n_+$, then the classical Mellin transform,

$$\mathfrak{M} f(x) = \int_{\mathbb{R}^n_+} f(x)\,[x^{s-1}]\,dx,$$

equals the distributional Mellin transform for $a \leqslant$ Re $s \leqslant b$ whenever $a, b \in \Xi$.

Moreover, we can convert some theorems on the Laplace transformation into the following.

<u>Theorem 14-2 (The Analyticity Theorem)</u> : <u>If</u> $\mathfrak{M}f = F(s)$ <u>for</u> $s \in \Omega_f$, <u>then</u> $F(s)$ <u>is analytic in</u> Ω_f <u>and</u>

$$D_s^k F = \langle [(\log x)^k]\, f(x),\ [x^{s-1}] \rangle \quad (s \in \Omega_f) \quad (14\text{-}2)$$

<u>Theorem 14-3 (The Uniqueness Theorem)</u> : <u>If</u> $\mathfrak{M}f = F(s)$ <u>for</u> $s \in \Omega_f$ <u>and</u> $\mathfrak{M}g = G(s)$ <u>for</u> $s \in \Omega_g$, <u>if</u> $\Omega_f \cap \Omega_g$ <u>is nonvoid</u>, <u>and if</u> $F(s) = G(s)$ <u>for</u> $s \in \Omega_f \cap \Omega_g$, <u>then</u> $f = g$ <u>on</u> $\mathcal{R}^n_+$.

In addition to theorem 14-1, let us invoke the fact that $\{f_\nu(x)\}_{\nu=1}^{\infty}$ converges in $\mathfrak{M}'_{a,b}$ to $f(x)$ if and only if $\{f_\nu(e^{-t})\}_{\nu=1}^{\infty}$ converges in $\mathcal{L}'_{a,b}$ to $f(e^{-t})$, as is asserted by theorem 11-1. Then, theorem 6-6 becomes

<u>Theorem 14-4 (The Continuity Theorem)</u> : <u>If</u> $\{f_\nu\}_{\nu=1}^{\infty}$ <u>converges in</u> $\mathfrak{M}'_{a,b}$ <u>to</u> f <u>for some</u> $a,\ b \in \mathcal{R}^n$ $(a < b)$ <u>and if</u> $\mathfrak{M} f_\nu = F_\nu(s)$, <u>then</u> $\mathfrak{M}f = F(s)$ <u>exists for at least</u> $a \leqslant \mathrm{Re}\ s \leqslant b$, <u>and</u> $\{F_\nu(s)\}_{\nu=1}^{\infty}$ <u>converges pointwise in the tube</u> $a \leqslant \mathrm{Re}\ s \leqslant b$ <u>to</u> $F(s)$.

Finally, theorem 8-1 is converted into

<u>Theorem 14-5</u> : <u>A necessary and sufficient condition for a function</u> $F(s)$ <u>to be the Mellin transform of a distribution</u> $f(x)$ <u>is that there be a tube</u> $a \leqslant \mathrm{Re}\ s \leqslant b$ $(a < b)$

on which $F(s)$ is analytic and bounded according to

$$|F(s)| \leq \rho(|s|),$$

where $\rho(|s|)$ is a polynomial in $|s|$.

15. Some Operation-transform Formulas for the Mellin Transformation.

In the following list we assume that $\mathfrak{M} f(x) = F(s)$ for $s \in \Omega_f$. Also, y is a fixed point in $\mathbb{R}^n$ and α is a fixed point in $\mathbb{C}^n$. The formulas given here can be obtained from the operations listed in Sec. 12, except for the first one, which we stated in theorem 14-2. Each formula represents a continuous linear operation in the same sense as that stated in Sec. 7.

$$\mathfrak{M}\left\{[(\log x)^k]\, f(x)\right\} = D_s^k F(s) \qquad (s \in \Omega_f) \qquad (14\text{-}2)$$

$$\mathfrak{M} f(yx) = [y^{-s}] F(s) \qquad (y > 0,\ s \in \Omega_f) \qquad (15\text{-}1)$$

$$\mathfrak{M} f(x^{-1}) = F(-s) \qquad (-s \in \Omega_f) \qquad (15\text{-}2)$$

$$\mathfrak{M} f(x^y) = \left|[y^{-1}]\right| F(s/y) \qquad (y \neq 0,\ \tfrac{s}{y} \in \Omega_f) \qquad (15\text{-}3)$$

Here, $y \neq 0$ means that every component of y is not zero.

$$\mathfrak{M}\left\{[x^\alpha]\, f(x)\right\} = F(s+\alpha) \qquad (s + \mathrm{Re}\,\alpha \in \Omega_f) \qquad (15\text{-}4)$$

$$\mathfrak{M} D_x^k f = (-1)^{|k|} [s-k]_k F(s-k) \qquad (s-k \in \Omega_f) \qquad (15\text{-}5)$$

Here,

$$[\alpha]_k = \prod_{\nu=1}^{n} (\alpha_\nu)_{k_\nu}$$

where

$$(\alpha_\nu)_{k_\nu} = \begin{cases} 1 & (k_\nu = 0) \\ \alpha_\nu(\alpha_\nu+1)\cdots(\alpha_\nu+k_\nu-1) & (k_\nu = 1,2,\cdots) \end{cases}$$

$$\mathfrak{M}\{[x^k]\, D_x^k\, f\} = (-1)^{|k|}[s]_k\, F(s) \qquad (s \in \Omega_f). \qquad (15\text{-}6)$$

$$\mathfrak{M}\{[xD_x]^k\, f\} = (-1)^{|k|}[s^k]\, F(s) \qquad (s \in \Omega_f). \qquad (15\text{-}7)$$

$$\mathfrak{M}\{D_x^k\,([x^k]\, f)\} = (-1)^{|k|}[s-k]_k\, F(s) \qquad (s \in \Omega_f) \qquad (15\text{-}8)$$

$$\mathfrak{M}\{[D_x\, x]^k\, f\} = (-1)^{|k|}[(s-1)^k]\, F(s) \qquad (s \in \Omega_f) \qquad (15\text{-}9)$$

16. Mellin-type Convolutions.

There are two types of convolutions (which we shall call "Mellin-type") that are readily analyzed by means of the Mellin transformation. If f and g are sufficiently well-behaved functions these convolutions are given by

$$f(x) \vee g(x) = \int_{R_+^n} f(y)g(\tfrac{x}{y})\,[y^{-1}]dy \qquad (16\text{-}1)$$

and

$$f(x) \wedge g(x) = \int_{R_+^n} f(y)g(xy)\,dy. \qquad (16\text{-}2)$$

We can generalize these formulas to the case where f and g are distributions as follows. Let $\mathfrak{M}f$ exist for $s \in \Omega_f$ and $\mathfrak{M}g$ exist for $s \in \Omega_g$. Assume that $\Omega_f \cap \Omega_g$ is nonvoid. Finally, let $a, b \in \mathbb{R}^n$ $(a < b)$ be restricted to $\Omega_f \cap \Omega_g$ but otherwise arbitrary. Then, we define $f \vee g$ as a functional on $\mathfrak{M}_{a,b}$ by

$$\langle f \vee g, \theta \rangle = \langle f(x), \langle g(y), \theta(xy) \rangle\rangle \qquad (\theta \in \mathfrak{M}_{a,b}) \quad (16\text{-}3)$$

We can show that the right-hand side has a sense by demonstrating that $\langle g(y), \theta(xy) \rangle \in \mathfrak{M}_{a,b}$. Indeed; by theorem 11-1, $g(e^{-\tau})$ is in $\mathcal{L}'_{a,b}$, and by lemma 6

$$\langle g(e^{-\tau}), \varphi(t+\tau) \rangle \quad (16\text{-}4)$$

is in $\mathcal{L}_{a,b}$ if $\varphi \in \mathcal{L}_{a,b}$. Setting $t = -\log x$, $\tau = -\log y$, and $\theta(x) = [x^{-1}]\, \varphi(-\log x)$, we invoke theorem 10-1 and 11-1 to convert (16-4) into the following member of $\mathfrak{M}_{a,b}$.

$$[x^{-1}] \langle g(y), [y^{-1}] \varphi(-\log x - \log y) \rangle$$

$$= \langle g(y), \theta(xy) \rangle$$

Furthermore, by the same change of variables we see that $f(e^{-t}) * g(e^{-t}) \in \mathcal{L}'_{a,b}$ and that

$$\langle f(e^{-t}) * (e^{-t}), \varphi(t) \rangle = \langle f(e^{-t}), \langle g(e^{-\tau}), \varphi(t+\tau) \rangle\rangle$$

$$= \langle f(x), [x^{-1}] \langle g(y), [y^{-1}] \varphi(-\log x - \log y) \rangle\rangle \qquad (16\text{-}5)$$

$$= \langle f(x), \langle g(y), \theta(xy) \rangle\rangle = \langle f(x) \vee g(x), \theta(x) \rangle .$$

In view of theorem 11-1, we can conclude that $f \vee g \in \mathcal{M}'_{a,b}$.

Henceforth, whenever we write $f \vee g$, it is understood that f and g are $\mathcal{M}$-transformable, that $\Omega_f \cap \Omega_g$ is nonvoid, and that $f \vee g \in \mathcal{M}'_{a,b}$ is defined by (16-3) for every $a, b \in \mathbb{R}^n$ $(a < b)$ that lie in $\Omega_f \cap \Omega_g$.

The definition (16-3) contains the classical definition (16-1) as a special case. Indeed, in addition to our previous assumptions, assume that f and g are locally integrable functions from $\mathbb{R}^n_+$ into C^1 and that $f/\mathfrak{Z}_{a,b}$ and $g/\mathfrak{Z}_{a,b}$ are absolutely integrable on $\mathbb{R}^n_+$ for some $a, b \in \mathbb{R}^n$ $(a < b)$. Then, for $\theta \in \mathcal{M}_{a,b}$

$$\langle f \vee g, \theta \rangle = \langle f(x), \langle g(y), \theta(xy) \rangle\rangle$$

$$= \int_{\mathbb{R}^n_+} dx \int_{\mathbb{R}^n_+} f(x)\, g(y)\, \theta(xy)\, dy \qquad (16\text{-}6)$$

Let us write

$$f(x)g(y)\theta(xy) = \left\{ \frac{f(x)\, g(y)}{\mathfrak{Z}_{a,b}(x)\, \mathfrak{Z}_{a,b}(y)} \right\} \left\{ \frac{\mathfrak{Z}_{a,b}(x)\, \mathfrak{Z}_{a,b}(y)}{\mathfrak{Z}_{a,b}(xy)} \right\} \left\{ \mathfrak{Z}_{a,b}(xy)\, \theta(xy) \right\} (16\text{-}7)$$

The second and third factors on the right-hand side are bounded continuous functions on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$. Hence, the integrand in (16-6) is locally integrable and absolutely integrable on $\mathbb{R}_+^n \times \mathbb{R}_+^n$. Fubini's theorem allows us to write the repeated integral (16-6) as a double integral. Moreover, making the change of variables, $u = xy$, $v = x$, and noting that the Jacobian determinant equals $[v^{-1}]$, we obtain

$$\int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} f(v)\, g\left(\frac{u}{v}\right) \theta(u)\, [v^{-1}]\, du\, dv.$$

By Fubini's theorem again, we finally get

$$\langle f \vee g, \theta \rangle = \left\langle \int_{\mathbb{R}_+^n} f(v)\, g\left(\frac{u}{v}\right) [v^{-1}]\, dv,\ \theta(u) \right\rangle$$

which is what we had to show.

The Mellin transformation converts our first Mellin-type convolution into multiplication, as follows.

Theorem 16-1: *If $\Omega_f \cap \Omega_g$ is nonvoid, then*

$$\mathfrak{M}(f \vee g) = F(s)\, G(s) \qquad (s \in \Omega_f \cap \Omega_g). \qquad (16\text{-}8)$$

Proof: $f \vee g \in \mathfrak{M}'_{a,b}$ for every pair of real points $a, b \in \Omega_f \cap \Omega_g$ $(a < b)$. Furthermore, for $a \leqslant \operatorname{Re} s \leqslant b$,

$$\begin{aligned} \mathfrak{M}(f \vee g) &= \langle f(x), \langle g(y), [(xy)^{s-1}] \rangle\rangle \\ &= \langle f(x), [x^{s-1}] \rangle \ \langle g(y), [y^{s-1}] \rangle \\ &= F(s)\, G(s). \end{aligned}$$

Since a and b can be any real points in $\Omega_f \cap \Omega_g$, this result holds for all $s \in \Omega_f \cap \Omega_g$. Q. E. D.

We can use theorem 16-1 to establish the commutativity and associativity of the convolution operation defined by (16-3).

<u>Theorem 16-2</u>: <u>If $\Omega_f \cap \Omega_g$ is nonvoid, then</u>

$$f \vee g = g \vee f \qquad (\underline{\text{commutativity}}) \qquad (16\text{-}9)$$

<u>If $\Omega_f \cap \Omega_g \cap \Omega_h$ is nonvoid, then</u>

$$f \vee (g \vee h) = (f \vee g) \vee h \qquad (\underline{\text{associativity}}) \qquad (16\text{-}10)$$

<u>Proof</u>:

$$\mathfrak{M}(f \vee g) = F(s)\, G(s) = G(s)\, F(s) = \mathfrak{M}(g \vee f)$$

$$(s \in \Omega_f \cap \Omega_g)$$

Consequently, (16-9) follows from the uniqueness theorem. Similary,

$$\mathfrak{M}\{f \vee (g \vee h)\} = F(s)\,\{G(s)\, H(s)\} = \{F(s)\, G(s)\}\, H(s)$$

$$= \mathfrak{M}\{(f \vee g) \vee h\}$$

$$(s \in \Omega_f \cap \Omega_g \cap \Omega_h),$$

which establishes (16-10).

This convolution operation is linear. That is, if $\alpha, \beta \in C'$ and if $\Omega_f \cap \Omega_g \cap \Omega_h$ is nonvoid, then

$$f \vee (\alpha g + \beta h) = \alpha f \vee g + \beta f \vee h.$$

It now follows that $\mathfrak{M}'_{a,b}$ is a commutative algebra where the first Mellin-type convolution takes the place of multiplication. It has a unit element, the n-dimensional delta functional concentrated at $x = 1$. Theorems 14-3 and 16-1 can be used to show that this algebra has no divisors of zero.

Furthermore, this Mellin convolution is continuous in the following sense.

__Theorem 16-3__: __If__ $\{f_\nu(x)\}_{\nu=1}^{\infty}$ __converges in__ $\mathfrak{M}'_{a,b}$ __to__ f __and if__ $g \in \mathfrak{M}'_{a,b}$, __then, in the sense of convergence in__ $\mathfrak{M}'_{a,b}$, __we have__

$$\lim_{\nu\to\infty} g \vee f_\nu = \lim_{\nu\to\infty} f_\nu \vee g = f \vee g. \qquad (16\text{-}11)$$

__Proof__: By theorem 11-1, $\{f_\nu(e^{-t})\}_{\nu=1}^{\infty}$ converges in $\mathcal{L}'_{a,b}$ to $f(e^{-t})$. Let $\theta(x) \in \mathfrak{M}_{a,b}$ and $\varphi(t) = [e^{-t}]\,\theta(e^{-t}) \in \mathcal{L}_{a,b}$. Then, by (16-5) and theorem 9-3, we have

$$\langle f_\nu(x) \vee g(x), \theta(x)\rangle = \langle f_\nu(e^{-t}) * g(e^{-t}), \varphi(t)\rangle$$

$$\longrightarrow \langle f(e^{-t}) * g(e^{-t}), \varphi(t)\rangle = \langle f(x) \vee g(x), \theta(x)\rangle$$

The commutativity property (16-9) establishes the other part of (16-11). Q. E. D.

Let us turn now to the generalization of the second Mellin-type convolution (16-2). We shall use the notation,

$$\hat{f}(x) = [x^{-1}]\, f(x^{-1})$$

By operations 2 and 6 of Sec. 12, $f(x) \to \hat{f}(x)$ is an isomorphism from $\mathfrak{M}'_{1-b,1-a}$ onto $\mathfrak{M}'_{a,b}$. Thus, $\mathfrak{M} f = F(s)$ for $s \in \Omega_f$ if and only if $\mathfrak{M} \hat{f} = F(1-s)$ for all s such that $1-s \in \Omega_f$. We denote the tube of existence for $\mathfrak{M} \hat{f}$ by $\hat{\Omega}_f$. Also, assume that g is $\mathfrak{M}$-transformable and that $\hat{\Omega}_f \cap \Omega_g$ is nonvoid. Thus, we define $f \wedge g$ as a distribution in $\mathfrak{M}'_{a,b}$ where a, b $(a < b)$ are any real points in $\hat{\Omega}_f \cap \Omega_g$, by

$$\langle f \wedge g, \theta \rangle = \langle \hat{f} \vee g, \theta \rangle \qquad (\theta \in \mathfrak{M}_{a,b}) \tag{16-12}$$

Thus,

$$\langle f \wedge g, \theta \rangle = \langle [x^{-1}] f(x^{-1}), \langle g(y), \theta(xy) \rangle\rangle$$

$$= \langle f(x), \langle g(y), [x^{-1}] \theta(y/x) \rangle\rangle \tag{16-13}$$

Consequently, an alternative definition of our second Mellin-type convolution is the following. If $f \in \mathfrak{M}'_{1-b,1-a}$ and $g \in \mathfrak{M}'_{a,b}$ then $f \wedge g$ is defined as a distribution in $\mathfrak{M}'_{a,b}$ by (16-13).

Moreover, from (16-5) and (16-12) we have that, for $\varphi(t) = [e^{-t}] \theta(e^{-t}) \in \mathcal{L}_{a,b}$

$$\langle f \wedge g, \theta \rangle = \langle \{[e^{t}] f(e^{t})\} * g(e^{-t}), \varphi(t) \rangle. \tag{16-14}$$

That the second Mellin-type convolution for distributions contains the classical convolution (16-2) as a special case can be shown as follows. In addition to our previous assumptions, assume that f and g are locally integrable functions and that $f/\mathfrak{Z}_{1-b,1-a}$ and $g/\mathfrak{Z}_{a,b}$ are absolutely integrable on R^n_+. Thus, $\hat{f}/\mathfrak{Z}_{a,b}$ is also absolutely integrable on R^n_+, and from what we have shown previously we may write

$$f \wedge g = \hat{f} \vee g = \int_{R^n_+} \hat{f}(y)\; g\left(\frac{x}{y}\right)[y^{-1}]\,dy$$
$$= \int_{R^n_+} f\left(\frac{1}{y}\right) g\left(\frac{x}{y}\right)[y^{-2}]\,dy.$$

Employing the change of varible, $u = y^{-1}$, we obtain (16-2), which is what we set out to do.

Combining (16-12) and theorem 16-1, we obtain

<u>Theorem</u> 16-4: <u>If</u> $\hat{\Omega}_f \cap \Omega_g$ <u>is</u> <u>nonvoid</u>, <u>then</u>

$$\mathfrak{M}(f \wedge g) = F(1-s)\, G(s) \qquad (s \in \hat{\Omega}_f \cap \Omega_g) \qquad (16\text{-}15)$$

The present convolution is clearly a linear operation in the following way. Let $\alpha, \beta \in C^1$. If $\hat{\Omega}_f \cap \Omega_g \cap \Omega_h$ is nonvoid, then

$$f \wedge (\alpha g + \beta h) = \alpha f \wedge g + \beta f \wedge h.$$

If $\hat{\Omega}_f \cap \Omega_g \cap \Omega_h$ is nonvoid, then

$$(\alpha f + \beta g) \wedge h = \alpha f \wedge h + \beta g \wedge h.$$

Continuity also holds.

Theorem 16-5: *If* $\{f_\nu\}_{\nu=1}^{\infty}$ *converges in* $\mathfrak{M}'_{1-b,1-a}$ *to* f *and if* $g \in \mathfrak{M}'_{a,b}$, *then* $\{f_\nu \wedge g\}_{\nu=1}^{\infty}$ *converges in* $\mathfrak{M}'_{a,b}$ *to* $f \wedge g$. *If* $f \in \mathfrak{M}'_{1-b,1-a}$ *and if* $\{g_\nu\}_{\nu=1}^{\infty}$ *converges in* $\mathfrak{M}'_{a,b}$ *to* g, *then* $\{f \wedge g_\nu\}_{\nu=1}^{\infty}$ *converges in* $\mathfrak{M}'_{a,b}$ *to* $f \wedge g$.

Proof: $\{f_\nu\}_{\nu=1}^{\infty}$ converges in $\mathfrak{M}'_{1-b,1-a}$ to f if and only if $\{\hat{f}_\nu\}_{\nu=1}^{\infty}$ converges in $\mathfrak{M}'_{a,b}$ to $\hat{f}$. Invoking theorem 16-3, we see that in the sense of convergence in $\mathfrak{M}'_{a,b}$

$$f_\nu \wedge g = \hat{f}_\nu \vee g \longrightarrow \hat{f} \vee g = f \wedge g$$

The second statement can be proved in a similar way.

Our last theorem compiles a number of associative and commutative formulas. As usual, $\alpha \in C^1$.

Theorem 16-6: *If* $\Omega_f \cap \Omega_g$ *is nonvoid, then*

$$f \vee g = g \vee f, \qquad (16\text{-}9)$$

$$(f \vee g)^\wedge = \hat{f} \vee \hat{g}, \qquad (16\text{-}16)$$

$$[x^\alpha](f \vee g) = ([x^\alpha]f) \vee ([x^\alpha]g). \qquad (16\text{-}17)$$

If $\Omega_f \cap \Omega_g \cap \Omega_h$ *is nonvoid, then*

$$f \vee (g \vee h) = (f \vee g) \vee h. \qquad (16\text{-}10)$$

If $\hat{\Omega}_f \cap \Omega_g$ *is nonvoid, then*

$$f \wedge g = \hat{f} \vee g, \qquad (16\text{-}12)$$

$$f \wedge g = \hat{g} \wedge \hat{f} \tag{16-18}$$

$$[x^{\alpha}](f \wedge g) = ([x^{-\alpha}]f) \wedge ([x^{\alpha}]g), \tag{16-19}$$

$$(f \wedge g)^{\wedge} = \hat{f} \wedge \hat{g} \tag{16-20}$$

If $\hat{\Omega}_f \cap \hat{\Omega}_g \cap \hat{\Omega}_h$ is nonvoid, then

$$f \wedge (g \wedge h) = (\hat{f} \wedge g) \wedge h \tag{16-21}$$

Proof: Equations (16-9) and (16-10) have already been established, whereas (16-12) is the definition of $f \wedge g$. The other equations can be proved by applying the Mellin transformation and invoking its uniqueness theorem. Thus, for (16-16), we have

$$\mathcal{M}(f \vee g)^{\wedge} = F(1-s)\, G(1-s) = \mathcal{M}(\hat{f} \vee \hat{g}).$$

For (16-17),

$$\mathcal{M}\left\{[x^{\alpha}](f \vee g)\right\} = F(s+\alpha)\, G(s+\alpha) = \mathcal{M}\left\{([x^{\alpha}]f) \vee ([x^{\alpha}]g)\right\}$$

For (16-18),

$$\mathcal{M}(f \wedge g) = F(1-s)\, G(s) = G(s)\, F(1-s) = \mathcal{M}(\hat{g} \wedge \hat{f}).$$

For (16-19),

$$\begin{aligned} \mathcal{M}\left\{[x^{\alpha}](f \wedge g)\right\} &= F(1-s-\alpha)\, G(s+\alpha) \\ &= \mathcal{M}\left\{([x^{-\alpha}]f) \wedge ([x^{\alpha}]g)\right\} \end{aligned}$$

For (16-20),

$$\mathfrak{M}(f \wedge g)^{\wedge} = F(s)\, G(1-s) = \mathfrak{M}(\hat{f} \wedge \hat{g}).$$

Finally, for (16-21),

$$\mathfrak{M}\{f \wedge (g \wedge h)\} = F(1-s)\, G(1-s)\, H(s)$$

$$= \mathfrak{M}\{(\hat{f} \wedge g) \wedge h\}.$$

Q. E. D.

Appendix A.

Here, we shall briefly indicate how the properties of the operations listed in Sec. 2 can be established. We employ the same numbering for the operations as we did in that section.

1. This statement is clear since shifting does not affect the behavior of the function as t tends to infinity.

2. Note that $\kappa_{-b,-a}(t) = \eta(t) \quad \kappa_{a,b}(-t)$, where $\eta(t)$ is bounded on R^n. Then, setting $\tau = -t$, we have

$$\left|\kappa_{-b,-a}(\tau)\ D_\tau^k\ \varphi(-\tau)\right| = \left|\eta(\tau)\ \kappa_{a,b}(-\tau)\ D_\tau^k\ \varphi(-\tau)\right|$$

$$= \left|\eta(-t)\ \ \kappa_{a,b}(t)\ D_t^k\ \varphi(t)\right| = C_k \sup \eta(-t).$$

This inequality also implies the continuity of the mapping. That it is linear and one-to-one from $\mathcal{L}_{a,b}$ onto $\mathcal{L}_{-b,-a}$ is obvious, as are now the corresponding assertions for the inverse mapping.

3. We have that $\kappa_{\tau a,\tau b}(t) = \eta(t)\kappa_{a,b}(\tau t)$, where $\eta(t)$ is a bounded function. Then,

$$\left|\kappa_{\tau a,\tau b}(t)\ D_t^k\varphi(\tau t)\right| = \left|\eta(t)\kappa_{a,b}(\tau t)[\tau^k]D_{\tau t}^k\varphi(\tau t)\right| \leq C_k[\tau^k] \sup \eta(t)$$

The rest of the assertion follows easily.

4. The derivative $D^k(\rho\varphi)$ is a finite linear combination of terms of the form, $D^{k-j}\rho\ D^j\varphi$, where $0 \leq j \leq k$. Also,

$$\left|\kappa_{a,b}\, D^{k-j}\rho\ D^j\varphi\right| = \left|\frac{\kappa_{a,b}}{\kappa_{c,d}} D^{k-j}\rho\right| \ \left|\kappa_{c,d}\, D^j\varphi\right|$$

Both factors on the right-hand side are bounded on $\mathbb{R}^n$. Thus, $\rho\varphi \in \mathcal{L}_{a,b}$. This result also implies the continuity of the mapping; its linearity is obvious.

5. Note that $\kappa_{a+\sigma,b+\sigma}(t) = \eta(t)\,\kappa_{a,b}(t)\,[e^{\sigma t}]$, where $\eta(t)$ is a bounded function on $\mathbb{R}^n$. Also, $D_t^k\{[e^{-st}]\,\varphi(t)\}$ is a finite linear combination of terms of the form,

$$[s^{k-j}\,e^{-st}]\,D_t^j\;\varphi(t),$$

where $0 \leqslant j \leqslant k$. Hence, with B_j denoting constants, we may write

$$\left|\kappa_{a+\sigma,b+\sigma}(t)\,D_t^k\{[e^{-st}]\;\;\varphi(t)\}\right|$$

$$\leq \left|\eta(t)\right| \sum_j \left|\kappa_{a,b}(t)B_j[s^{k-j}]\;\;D_t^k\;\varphi(t)\right|$$

The right-hand side is bounded on $\mathbb{R}^n$. The rest follows easily.

6. This statement is obvious.

7. So is this one.

Appendix B

In this appendix we shall justify the various statements concerning the operations on $\mathcal{M}_{a,b}$ listed in Sec. 10. Once again, we use the same numbering here.

In this discussion we shall at times make use of the following fact. If the operator $\mathcal{N}$ is an isomorphism from a space $\mathcal{A}$ onto a space $\mathcal{B}$ and if the operator $\mathcal{D}$ is an isomorphism from $\mathcal{B}$ onto a space $\mathcal{G}$, then $\mathcal{D}\mathcal{N}$ (the application of $\mathcal{N}$ and then the application of $\mathcal{D}$) is an isomorphism from $\mathcal{A}$ onto $\mathcal{G}$. Also, when we write

$$\theta(x) \in \mathcal{A} \Longleftrightarrow \varphi(t) \in \mathcal{B}$$

we shall mean (in addition to the usual meaning of $\Longleftrightarrow$) that the mapping $\theta(x) \longrightarrow \varphi(t)$ is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Similarly, by writing

$$\theta(x) \in \mathcal{A} \Longrightarrow \varphi(t) \in \mathcal{B}$$

we shall also mean that $\theta(x) \rightarrow \varphi(t)$ is a continuous linear operation from $\mathcal{A}$ into $\mathcal{B}$.

1. We have that $\zeta_{a,b}(x) = \eta(x)\, \zeta_{a,b}(yx)$ where $\eta(x)$ is a bounded function on $0 < x < \infty$. Then,

$$\left| \zeta_{a,b}(x)\, [x^{k+1}]\, D_x^{k}\, \theta(yx) \right|$$

$$= \left| [y^{-1}]\, \eta(x)\, \zeta_{a,b}(yx)\, [(yx)^{k+1}]\, D_{yx}^{k}\, \theta(yx) \right|$$

$$\leq K_k [y^{-1}] \sup_{0<x<\infty} \eta(x),$$

so that $\theta(yx)$ is also in $\mathcal{M}_{a,b}$. That the indicated operation is an isomorphism now follows easily.

2. Using theorem 10-1 and some of the operations listed in Sec. 2, we may write

$$\theta(x) \in \mathcal{M}_{a,b} \Longleftrightarrow [e^{-t}]\,\theta(e^{-t}) \in \mathcal{L}_{a,b} \Longleftrightarrow [e^{t}]\,\theta(e^{t}) \in \mathcal{L}_{-b,-a}$$

$$\Longleftrightarrow [e^{-t}]\,\theta(e^{t}) \in \mathcal{L}_{2-b,2-a} \Longleftrightarrow \theta\left(\frac{1}{x}\right) \in \mathcal{M}_{2-b,2-a}.$$

3. $$\theta(x) \in \mathcal{M}_{a,b} \Longleftrightarrow [e^{-t}]\,\theta(e^{-t}) \in \mathcal{L}_{a,b} \Longleftrightarrow [e^{-\gamma t}]\,\theta(e^{-\gamma t}) \in \mathcal{L}_{\gamma a,\gamma b}.$$

$$\Longleftrightarrow [e^{(\gamma-1)t}]\,[e^{-\gamma t}]\,\theta(e^{-\gamma t}) = [e^{-t}]\,\theta(e^{-\gamma t}) \in \mathcal{L}_{1+\gamma a-\gamma,\,1+\gamma b-\gamma}$$

$$\Longleftrightarrow \theta(x^{\gamma}) \in \mathcal{M}_{1+\gamma a-\gamma,\,1+\gamma b-\gamma}.$$

4. Combine the second and third operations.

5. $$\theta(x) \in \mathcal{M}_{c,d} \Longleftrightarrow [e^{-t}]\,\theta(e^{-t}) \in \mathcal{L}_{c,d} \Rightarrow p(-t)\,[e^{-t}]\,\theta(e^{-t}) \in \mathcal{L}_{a,b}$$

$$\Longleftrightarrow p(\log x)\,\theta(x) \in \mathcal{M}_{a,b}.$$

6. $$\theta(x) \in \mathcal{M}_{a,b} \Longleftrightarrow [e^{-t}]\,\theta(e^{-t}) \in \mathcal{L}_{a,b} \Longleftrightarrow [e^{-\alpha t}]\,[e^{-t}]\,\theta(e^{-t}) \in \mathcal{L}_{a+\gamma,b+\gamma}$$

$$\Longleftrightarrow [x^{\alpha}]\,\theta(x) \in \mathcal{M}_{a+\gamma,\,b+\gamma}.$$

7. Note that

$$[x^{-k}]\,\mathfrak{Z}_{a-k,b-k}(x) = \eta(x)\,\mathfrak{Z}_{a,b}(x),$$

where $\eta(x)$ is a bounded function on R^n_+. Then,

$$\left| \mathfrak{Z}_{a-k,b-k}(x)[x^{p+1}]\, D_x^{p}\, D_x^{k}\ \theta(x) \right|$$

$$= \left| \eta(x)\, \mathfrak{Z}_{a,b}(x)\, [x^{p+k+1}]\, D_x^{p+k}\ \theta(x) \right|$$

$$\leq K_{p+k} \sup_{0<x<\infty} \eta(x)$$

Thus, $\theta \in \mathcal{M}_{a-k,b-k}$. The continuity of D_x^{k} follows from this result. Its linearity is obvious.

8. Combine the sixth and seventh operations.

9. This follows by induction from the eigth operation.

10. Combine the sixth and seventh operations.

11. This follows by induction from the previous operation.

Appendix C

Proof of the Completeness of $\mathcal{L}'_{a,b}$

We shall show that $\mathcal{L}'_{a,b}$ is sequentially complete with respect to its weak topology. This will be done by adapting the proof of the sequential completeness of $\mathcal{D}'$ due to M. S. Brodskiĭ.

Let $\{f_\nu\}_{\nu=1}^{\infty}$ be a Cauchy sequence in $\mathcal{L}'_{a,b}$; that is, every $f_\nu \in \mathcal{L}'_{a,b}$ and for each $\varphi \in \mathcal{L}_{a,b}$ the numerical sequence $\{\langle f_\nu , \varphi\rangle\}_{\nu=1}^{\infty}$ converges. Let f denote the limit functional on $\mathcal{L}_{a,b}$ generated by $\{f_\nu\}_{\nu=1}^{\infty}$. f is clearly linear on $\mathcal{L}_{a,b}$. Our problem is to show that it is continuous.

Assume the opposite. That is, assume there exists a sequence that converges in $\mathcal{L}_{a,b}$ to zero (i.e., to the zero function) such that the corresponding sequence of numbers assigned by f does not converge. We can choose a subsequence $\{\varphi_\nu\}_{\nu=1}^{\infty}$ from the original sequence such that

$$|\langle f , \varphi_\nu\rangle| \geqslant c > 0 \qquad (\nu = 1, 2, \ldots) \quad \text{(C-1)}$$

(where c is a fixed positive number) and such that

$$|\kappa_{a,b}(t)\, D^k \varphi_\nu(t)| \leq 4^{-\nu} \qquad (|k| = 0, 1, \ldots \nu). \quad \text{(C-2)}$$

Now, let $\psi_\nu = 2^\nu \varphi_\nu$. By (C-2), $\{\psi_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{L}_{a,b}$ to zero. By (C-1), $\{|\langle f , \psi_\nu\rangle|\}_{\nu=1}^{\infty}$ diverges to $+\infty$.

Now, we choose a subsequence $\{\psi_\nu'\}_{\nu=1}^{\infty}$ from $\{\psi_\nu\}_{\nu=1}^{\infty}$ and a subsequence $\{f_\nu'\}_{\nu=1}^{\infty}$ from $\{f_\nu\}_{\nu=1}^{\infty}$ as follows. First, choose ψ_1' such that $|\langle f, \psi_1'\rangle| > 1$. Since, for every $\psi \in \mathcal{L}_{a,b}$, $\langle f_\nu, \psi\rangle \longrightarrow \langle f, \psi\rangle$ as $\nu \longrightarrow \infty$, f_1' can be chosen such that $|\langle f_1', \psi_1'\rangle| > 1$.

Assuming that the first $\nu-1$ elements of these subsequences have been chosen, choose ψ_ν' such that

$$|\langle f_j', \psi_\nu'\rangle| < 2^{j-\nu} \qquad (j = 1, \ldots, \nu-1) \tag{C-3}$$

and

$$|\langle f, \psi_\nu'\rangle| > \sum_{\mu=1}^{\nu-1} |\langle f, \psi_\mu'\rangle| + \nu \tag{C-4}$$

(C-3) can be satisfied since $\{\psi_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{L}_{a,b}$ to zero and, for each fixed f_j', $\langle f_j', \psi_\nu\rangle \longrightarrow 0$ as $\nu \longrightarrow \infty$. (C-4) can be satisfied since $\langle f, \psi_\nu\rangle \longrightarrow \infty$ as $\nu \longrightarrow \infty$.

Because $\langle f_\nu, \varphi\rangle \longrightarrow \langle f, \varphi\rangle$ for every $\varphi \in \mathcal{L}_{a,b}$, f_ν' can be chosen such that

$$|\langle f_\nu', \psi_\nu'\rangle| > \sum_{\mu=1}^{\nu-1} |\langle f_\nu', \psi_\mu'\rangle| + \nu . \tag{C-5}$$

Next, we wish to show that

$$\psi = \sum_{\nu=1}^{\infty} \psi_\nu' \tag{C-6}$$

is a member of $\mathcal{L}_{a,b}$. Consider the remainder term,

$$R_m = \sum_{\nu=m}^{\infty} \psi_\nu'$$

For $m \geqslant |k|$,

$$\left| K_{a,b}(t) \sum_{\nu=m}^{\infty} D^{k} \psi_{\nu}'(t) \right| \leq \sum_{\nu=m}^{\infty} \left| K_{a,b}(t)\ D^{k} \psi_{\nu}(t) \right| \leq \sum_{\nu=m}^{\infty} 2^{-\nu} \qquad (C\text{-}7)$$

As $m \longrightarrow \infty$, the right-hand side converges to zero. Thus, over every bounded domain, (C-6) converges uniformly as well as the series of derivatives for any given partial derivative. Thus, we may differentiate (C-6) under the summation sign. (C-7) also shows that the series (C-6) converges in $\mathcal{L}_{a,b}$ to ψ. By the completeness of $\mathcal{L}_{a,b}$, we conclude that $\psi \in \mathcal{L}_{a,b}$.

Finally, we may write

$$\langle f_{\nu}', \psi \rangle = \sum_{\mu=1}^{\nu-1} \langle f_{\nu}', \psi_{\mu}' \rangle + \langle f_{\nu}', \psi_{\nu}' \rangle + \sum_{\mu=\nu+1}^{\infty} \langle f_{\nu}', \psi_{\mu}' \rangle. \qquad (C\text{-}8)$$

By (C-3)

$$\left| \sum_{\mu=\nu+1}^{\infty} \langle f_{\nu}', \psi_{\mu}' \rangle \right| \leq \sum_{\mu=\nu+1}^{\infty} 2^{\nu-\mu} = 1. \qquad (C\text{-}9)$$

By (C-5), (C-8), and (C-9),

$$\left| \langle f_{\nu}', \psi \rangle \right| \geq \left| \langle f_{\nu}', \psi_{\nu}' \rangle \right| - \left| \sum_{\mu=1}^{\nu-1} \langle f_{\nu}', \psi_{\mu}' \rangle \right| - \left| \sum_{\mu=\nu+1}^{\infty} \langle f_{\nu}', \psi_{\mu}' \rangle \right|$$

$$> \nu - 1.$$

Therefore, as $\nu \longrightarrow \infty$, $\left| \langle f_{\nu}', \psi \rangle \right| \longrightarrow \infty$. This contradicts the hypothesis that $\{ f_{\nu} \}_{\nu=1}^{\infty}$ is a Cauchy sequence in $\mathcal{L}_{a,b}$. Our proof is complete.

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