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NONUNIFORM SEMI-INFINITE GROUNDED GRIDS

A. H. Zemanian

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NONUNIFORM SEMI-INFINITE GROUNDED GRIDS*

A.H.Zemanian

State University of New York at Stony Brook

Abstract: Semi-infinite resistive grounded grids are countably infinite electrical networks that arise from the discretization of the partial differential equation governing the minority-carrier density in a doped semiconductor. If the doping varies with depth from the surface of the semiconductor, the grid's resistances also vary with distance from the inputs to the grid. This nonuniformity prevents the use of the characteristic-resistance method for determining currents and voltages. A computational method for making such a determination is presented herein. It is based upon the theory of infinite continued fractions whose entries are positive operators on a Hilbert space. It is also shown that the solution given by the method is precisely that solution for which the power dissipated in the network is finite. Finally, the method is extended to RLC networks, and this allows the computation of transient responses in semi-infinite grounded grids of positive-real impedances.

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I. Introduction

The purpose of this work is to examine the behavior of a certain class of countably infinite electrical networks. Although individual networks in this class can be highly complex, its prototype is comparatively simple and results from the discretization of a certain partial differential equation relating to semiconductor behavior. Therefore, to motivate our work, we first indicate why the prototype is of interest in the theory of semiconductors.

The partial differential equation that governs the minority-carrier density δ in a doped semiconductor is

$$\nabla^2\delta = \frac{\delta}{\tau D} \tag{1.1}$$

where τ is the minority-carrier lifetime and D is the minority-carrier diffusion constant [9; p. 99]. Ordinarily, the doping concentration, and therefore τ as well, varies with distance from the surface through which the impurities were introduced. (There are of course lateral variations along the surface where the p-n junctions appear, but these variations disappear just below that section.) Because of this, no closed-form solution for (1.1) exists, and computational techniques must be used to get an approximate determination of δ. However, the standard techniques, such as difference methods or finite-element methods, lead to excessively large computer times when the full thickness of the semiconductor wafer is modelled.

An alternative possibility is to assume that one surface of the wafer is at infinity and then make use of the theory of semi-infinite transmission lines. This approach was explored in [17] in the case where the doping does not vary with position. It led to the adaptation of the characteristic-resistance method to semi-infinite grounded grids, the kind of electrical network that arises from the discretization of (1.1). In fact, if the spatial variations for (1.1) are in only two dimensions, we get a square grid of resistances, all having the same value, and also branches connecting the nodes of the grid to a common ground; the resistances of the latter branches represent the quantity τD. This is illustrated in Figure 1, where a and the c_k denote conductances. For constant doping, the c_k are all the same; otherwise, they vary. The h_k are current sources representing the electrical excitation of the semiconductor at its surface. In the case of three spatial dimensions, we get the same configuration except that we now have a cubic grid.

This is the motivation for the problem attacked in this work. We wish to determine the currents and voltages in a semi-infinite grounded grid where the grid's resistances are allowed to vary with distance from its input section. We even allow resistances to vary in a certain restricted fashion for spatial displacements that remain equidistant from the input section. Our analysis is immediately extendable to far more complicated grids than the prototypical square or cubic grids mentioned above. Herein, we allow this generality.

The basis of our computational method is the theory of infinite continued fractions whose elements are positive operators on a Hilbert space. Those operators represent the admittances and impedances of ∞-ports consisting of sections of the grid

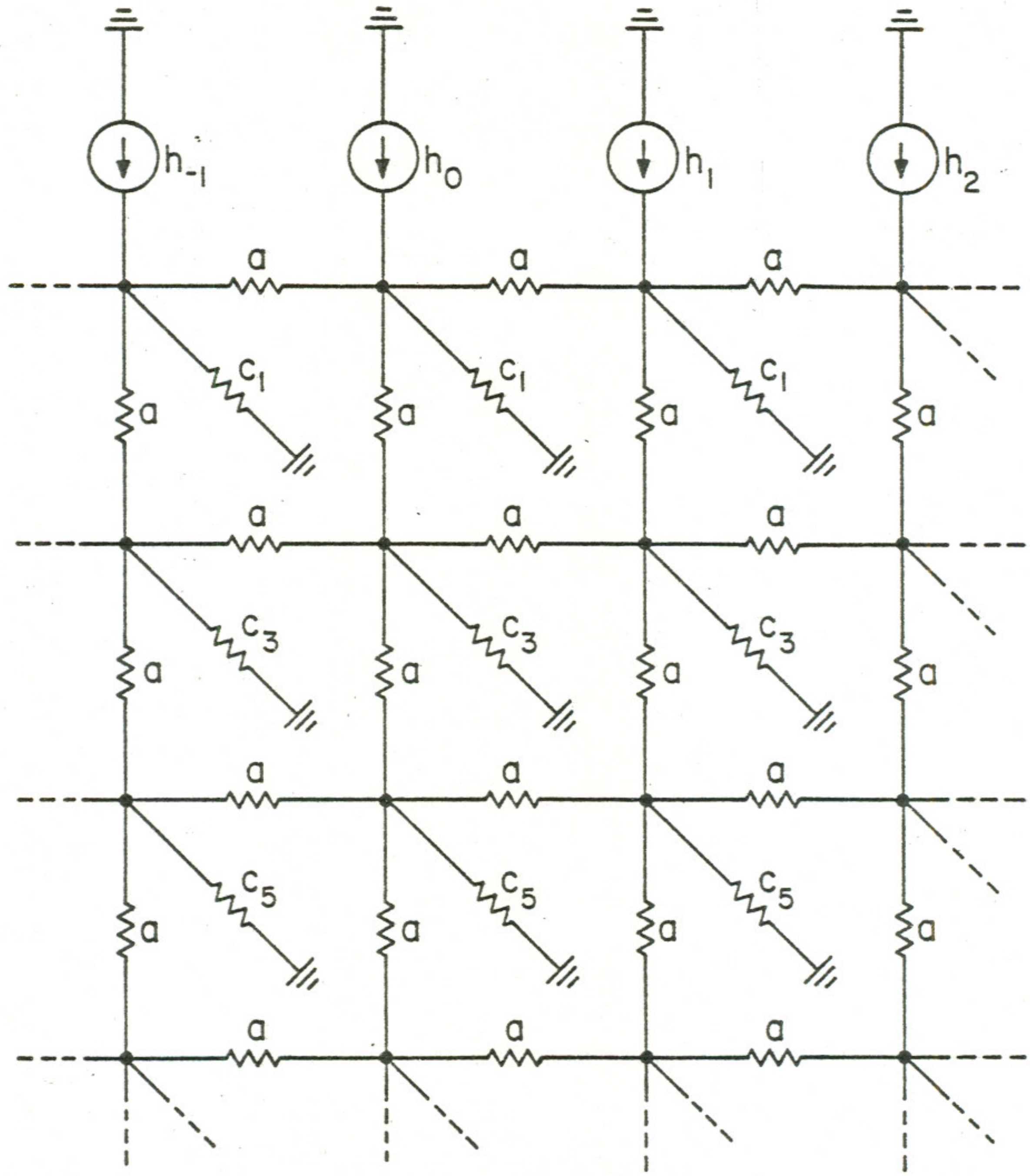

Figure 1.

lying parallel to the input section. The ∞-ports are connected together to make a semi-infinite ladder whose input impedance is the aforementioned continued fraction. Our analysis of the ladder network yields that unique set of voltages and currents for which the total power dissipated in the network is finite. By using the theory of Laurent operators, we also obtain a computational procedure for calculating the currents and voltages in the original grid.

All of this is extendable to grounded grids whose branches are positive-real impedances. We end this paper by indicating how the transient responses of such impedance grids can be computed. The solution we now obtain is characterized by a finite-power condition applied this time to points on the real positive axis of the complex-frequency domain.

Before proceeding, let us explain some of the notation we will be using. If H is a Hilbert space, [H; H] denotes the Banach space of bounded linear operators that map H into H. By an "operator" we will always mean a member of [H; H] for some H. The symbol 1 is used to denote either the number one, or a function whose range is the singleton {1}, or the identity operator in [H; H]. Which meaning that symbol has in a particular case will either be stated or will be clear from the context in which it is used. If **A** is an operator, W(**A**) denotes the numerical range of **A**:

$$W(\mathbf{A}) = \{(\mathbf{A}x, x) : x \in H, \|x\| = 1\}$$

where (α, β) is the inner product of the elements α and β in H.

The symbol (α, β) will also be used to denote an open interval between the real numbers α and β; once again, which meaning (α, β) has in particular cases will be either clear or specified. The symbols $[\alpha, \beta]$, $[\alpha, \beta)$, and $(\alpha, \beta]$ denote closed and semiclosed intervals with the endpoints α and β.

II. Semi-infinite Grounded Grids

The type of grounded grid we shall examine is indicated symbolically in Figure 2. We have a sequence of infinite networks, which for the sake of illustration we indicate as being contained in a sequence of hypothetical boxes. We number those boxes by $k = 1, 3, 5, \ldots$. We have shown only three nodes in each box, but it is understood that each box contains an infinity of them. The following is assumed.

Rule I. Every node is connected to a ground node through a positive conductance whose value c_k is the same for all the nodes in a particular box. The c_k can vary from box to box, that is, as k varies.

The nodes of a given box are connected together by conductances, which we have not shown in Figure 1 so as not to clutter up the diagram. We assume that the graph of these interconnections within each box is isomorphic (in a graph-theoretical sense) to a uniform structure S_k, which we specify in Rule II. S_k need not be the same for every box. Let n be a positive integer (possibly greater than three) and let R^n denote real Euclidean n-space. The lattice points of R^n are the n-tuples $p = (p_1, \ldots, p_n)$ where each p_i is an integer.

Rule II. The nodes of each S_k occur at all the lattice

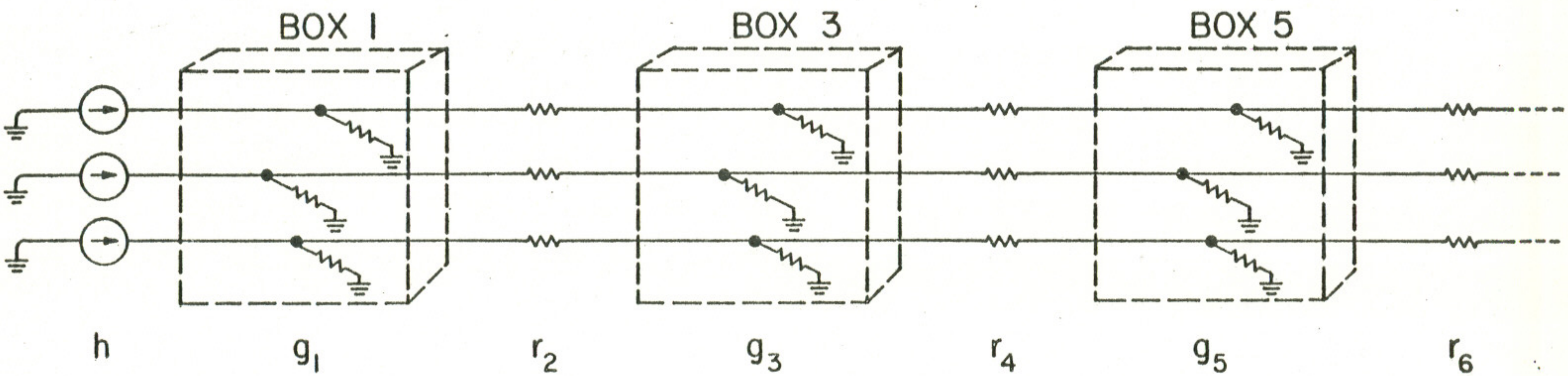

Figure 2.

points p of R^n; n is the same for every S_k. We number the nodes by their lattice numbers p. The degrees of the nodes of a particular S_k are finite and all the same, but those degrees can vary as k varies. Every branch of S_k is a positive conductance. Moreover, in a given S_k, if node p is connected to node q through a branch of conductance a, then every node j is connected to node j+q-p through a branch with the same conductance a.

Thus, when n = 1 and S_k is connected, S_k is simply a series connection of conductances a that extends to infinity in both directions. When n = 2, an infinity of possibilities arises. One of them is shown in Figure 3, wherein a_1, a_2, and a_3 denote conductance values. Still more variety in possible configurations for the S_k arise as n increases beyond 2. Rule II implies that all the branches of S_k can be partitioned into a finite number of classes such that two branches are in the same class if and only if they are parallel, that is, if and only if the difference between the incident-node numbers of one branch is equal to or the negative of the difference in the incident-node numbers of the other branch. We denote these classes of branches in S_k by $\Gamma_{k\mu}$, where $\mu = 1, \ldots, j_k$. The single conductance value for all the branches in a given $\Gamma_{k\mu}$ is denoted by $a_{k\mu}$.

Referring to Figure 2, we impose still another condition.

Rule III. The nodes of box k are connected to the nodes of box k+2 in the following fashion. A node of box k is adjacent to a node of box k+2 if and only if the nodes have the same lattice number. Moreover, the branches connecting two consecutive boxes are all purely resistive and have the same resistance value, but that value is allowed to change as k changes. Furthermore,

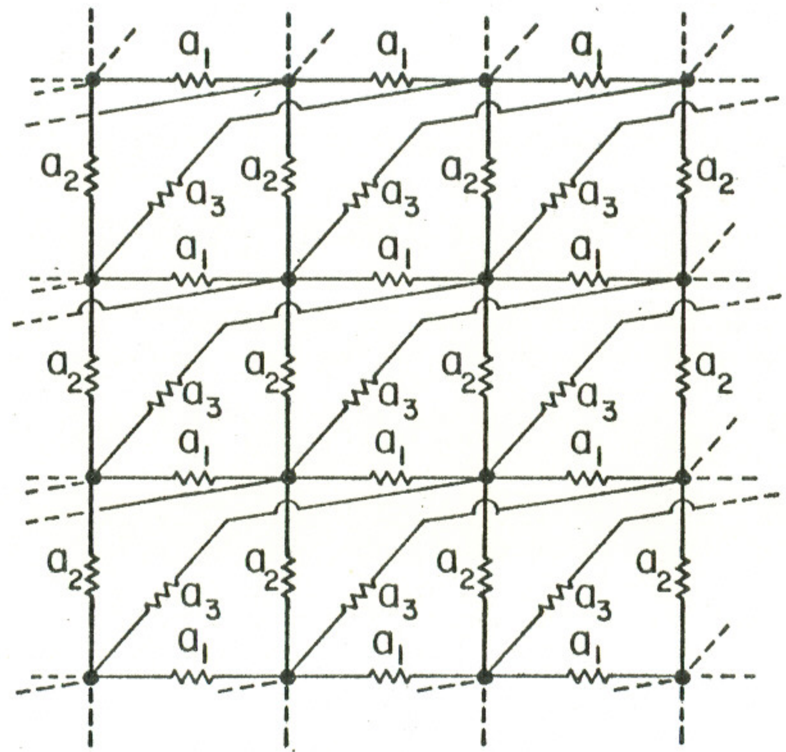

Figure 3.

current generators are connected from ground to the nodes of the first box; these current generators are not in general the same.

Note that, under the three rules, Figure 1 is a special case of Figure 2.

III. Existence and Uniqueness of Solutions

We wish to examine the solution of the countably infinite electrical networks satisfying the above three rules. By a solution we mean a set of branch currents and branch voltages that satisfy Kirchhoff's node and loop laws and Ohm's law. However, such networks have in general on infinity of solutions [14]. This is because power can be injected into the network from infinity. On the other hand, practical considerations (i.e., there is really no such thing as an infinite network - the idea is simply a mathematical convenience) dictate that the "natural" solutions are those that obtain their power only from the sources within the network. But, a particular infinite network may even have an infinity of natural solutions; see [15]. In this section, we shall impose conditions on our network that insure the existence of one and only one natural solution.

For a subsequent purpose, we shall allow our branch conductances to be operators on a certain Hilbert space. In particular, let H_r be any real Hilbert space. $\underline{l}_2(H_r)$ will denote the real Hilbert space of vectors

$$x = [x_1, x_2, x_3, \dots]^T$$

where every element x_m is a member of H_r, the superscript T denotes matrix transpose, and

$$\|x\| = \left[\sum_{m=1}^{\infty} \|x_m\|^2\right]^{\frac{1}{2}} < \infty.$$

The inner product of two members a and b of $\underline{l}_2(H_r)$ is

$$(x, y) = \sum_{m=1}^{\infty} (x_m, y_m).$$

Here, $\|x_m\|$ and (x_m, y_m) are of course the norm and inner product in H_r.

Another set of conditions we will employ are the following.

Conditions A. The currents and voltages of the network are members of H_r. Each branch is a parallel connection of a (possibly zero) current source $h \in H_r$ and a conductance g which is a positive invertible operator mapping H_r into H_r. There are no other current sources and no voltage sources. (Actually, voltage sources can be incorporated by making a Thevenin-to-Norton transformation.) The numerical ranges of all the conductance operators are uniformly contained in a fixed compact subinterval of the open half-axis $(0, \infty)$. The current sources (with any appropriate indexing) comprise a vector in $\underline{l}_2(H_r)$.

In [17; Theorem 2.2] we proved the following theorem. It was established by modifying the circle of ideas concerning infinite electrical networks first introduced by Flanders [3].

Theorem 3.1. Let N be a connected infinite electrical network which is locally finite except possibly for one ground node; the ground node may be of infinite degree. Assume N satisfies Conditions A. Then, there exist a unique vector v $\underline{l}_2(H_r)$ of branch voltages and a unique vector i $\underline{l}_2(H_r)$ of branch currents such that Kirchhoff's node and loop laws and Ohm's law are satisfied.

(When the ground node has infinite degree, it is not required to satisfy Kirchhoff's node law, that law being an assertion only about nodes of finite degree [12; p. 275].)

This theorem may be applied to any network satisfying Rules I through III, where now H_r is the real line, so long as the current-source values at the left-hand side of Figure 2 are quadratically summable and all conductance values are contained in a compact subinterval of the open half-axis $(0, \infty)$.

IV. ∞-ports and Laurent Operators

The network in any box of Figure 2 can be viewed as a grounded ∞-port, where the two terminals of each port are the ground node and one of the nodes within the box. In order to make use of Theorem 3.1., we shall restrict the voltage and current vectors of these ∞-ports to Hilbert's coordinate space $\underline{l}_{2r} = \underline{l}_2(R^1)$ but will alter the indexing of the components of any vector in $\underline{l}_{2r}$ to conform with Rule II. Let N^n denote the set of lattice points in R^n; that is, each member of N^n is an ordered n-tuple $p = (p_1, \ldots, p_n)$ whose entries are integers. A member of $\underline{l}_{2r}$ will now be an n-dimensional array $\{a_p : p \in N^n\}$ of real numbers a_p such that

$$\sum_{p \in N^n} a_p^{\,2} < \infty.$$

Thus, the inner product of two members $a = \{a_p\}$ and $b = \{b_p\}$ in $\underline{l}_{2r}$ is the n-tuple infinite series

$$(a, b) = \sum_{p \in N^n} a_p b_p.$$

A bounded linear mapping F of $\underline{l}_{2r}$ into $\underline{l}_{2r}$ has a matrix-like representation, but it should be borne in mind that its matrix $[F_{p,q}]$, where p, q $\in N^n$, is a 2n-dimensional array of real numbers. Thus, if $y = Fx$, where $y = (y_1, \ldots, y_n) \in \underline{l}_{2r}$ and $x = (x_1, \ldots, x_n) \in \underline{l}_{2r}$, then

$$y_p = \sum_{q \in N^n} F_{p,q} x_q .$$

Of course, not all 2n-dimensional arrays of real numbers will represent bounded linear mappings of $\underline{l}_{2r}$ into $\underline{l}_{2r}$ [5; p. 126], but those we encounter below will do so.

Now consider the kth box of Figure 2. As an ∞-port, it has a linear conductance operator whose matrix representation can be determined by making a nodal analysis. g_k has the structure of a Laurent matrix [1]; that is, upon letting $(g_k)_{p,q}$ denote the p, q entry of the matrix representation for g_k, we have for every p, q, m $\in N^n$

$$(4.1) \qquad (g_k)_{p,q} = (g_k)_{p+m,q+m}$$

This is an immediate consequence of Rules I and II.

Moreover, g_k truly is a bounded mapping of $\underline{l}_{2r}$ into $\underline{l}_{2r}$. Indeed, for $x = \{x_q\} \in \underline{l}_{2r}$, we may write the following, where every summation is understood to be over N^n.

$$\|g_k x\|^2 = \sum_p \left| \sum_q (g_k)_{p,q} x_q \right|^2$$

By virtue of Rule II, for each fixed p one finds only a finite number, say, ν of nonzero $(g_k)_{p,q}$ as q traverses N^n. Moreover, in view of (4.1), the same values appear whatever be p; those values merely shift their indices as p changes. Let M be a bound on those values. By applying Schwarz's inequality to the inner summation of the last expression and taking into account all the zero values of $(g_k)_{p,q}$, we get

$$\|g_k x\|^2 \leq M^2 \nu^2 \sum_m |x_m|^2 = M^2 \nu^2 \|x\|^2 .$$

This verifies our assertion.

A Laurent operator is a member of $[\underline{1}_{2r};\ \underline{1}_{2r}]$ that satisfies (4.1). We have proven that g_k is a Laurent operator.

Moreover, we can show that each g_k is positive and invertible by examining its numerical range. For any $x \in \underline{1}_{2r}$,

$$(4.2) \qquad (g_k x,\ x) = \sum_p \Big[\sum_q (g_k)_{p,q} x_q \Big] x_p$$

By using the aforementioned properties of $(g_k)_{p,q}$, it is not difficult to see that the right-hand side converges absolutely and therefore can be rearranged. According to Rule I, the branch connecting node p to ground has conductance $c_k > 0$. It therefore introduces the term $c_k x_p^2$ into the summation in (4.2). Now, consider any branch that is not incident to the ground node. Assume that it connects node p to node q and that its conductance is $a > 0$. That branch introduces the following terms into the summation (4.2).

$$a x_p^2 - 2a x_p x_q + a x_q^2 = a(x_p - x_q)^2 \geq 0.$$

Now, we can partition all the branches that are not incident to ground (that is, all the branches in S_k) into a finite number of classes $\Gamma_{k\mu}$, where $\mu = 1, 2, \dots, j_k$, as was explained in Section II. The branches of any class all have the same conductance, say, $a_{k\mu}$. Thus, (4.2) can be rearranged into the following expression.

$$(4.3) \qquad (g_k x, x) = c_k \sum_p x_p^2 + \sum_{\mu=1}^{j_k} a_{k\mu} \sum_{b \in \Gamma_{k\mu}} (x_{p_b} - x_{q_b})^2$$

where p_b and q_b are the indices of the nodes incident to the branch b in class $\Gamma_{k\mu}$. By Rules I and IV, $c_k > 0$ and $a_{k\mu} > 0$. Hence,

$$(g_k x, x) \geq c_k \|x\|^2 .$$

This proves that g_k is positive and invertible.

Actually, the branches connecting two consecutive boxes also comprise an ∞-port. We take the two nodes of each such branch as one of the ports of the ∞-port, and we number those ports in the same way as the nodes to which they connect. According to Rule III, all those branches have the same positive resistance b_k (k is now even). Therefore, the ∞-port has the resistance operator $r_k = b_k 1$, where 1 is the identity operator on $\underline{l}_{2r}$; that is, the element $(r_k)_{p,q}$, where $p, q \in N^n$, of r_k's matrix representation is

$$(r_k)_{p,q} = \begin{cases} b_k & \text{for } p=q \\ 0 & \text{for } p \neq q \end{cases}$$

Thus, each r_k is a positive invertible Laurent operator too.

V. A Ladder Network of Operators

Because of the grounded nature of the g_k ∞-ports (k odd) and the disconnected form of the r_k ∞-ports (k even), we can connect them into the infinite ladder network of Figure 4 without violating the port conditions. We shall analyze the network of Figure 2 by a two-step procedure consisting of an analysis of Figure 4, in which the individual ∞-port currents and ∞-port voltages are vectors in $\underline{l}_{2r}$, followed by a determination of the interior branch currents and voltages of each ∞-port to get the branch currents and voltages of Figure 2. To do so, we shall impose two further assumptions on Figure 2.

Rule IV. (i) The vector h of current-source values h_p, $p \in N^n$, at the input of Figure 2 is a member of $\underline{l}_{2r}$.

(ii) There exist two real numbers α and γ with $0 < \alpha < \gamma < \infty$ such that the conductances-to-ground c_k satisfy $\alpha \leq c_k \leq \gamma$ for all odd k, the conductances $a_{k\mu}$ of the branches inside each box that are not incident to the ground node satisfy

$$\sum_{\mu=1}^{j_k} a_{k\mu} \leq \gamma$$

for all odd k, and the resistances b_k of the branches between the boxes satisfy $\alpha \leq b_k \leq \gamma$ for all even k.

We now show that Rule IV(ii) insures that the numerical ranges of all the operators g_k and r_k are uniformly bounded according to

$$(5.1)\qquad \begin{aligned} W(g_k) &\subset [\alpha, \beta], \text{ k odd} \\ W(r_k) &\subset [\alpha, \beta], \text{ k even} \end{aligned}$$

where $0 < \alpha < \beta < \infty$. Since in this case we will also have

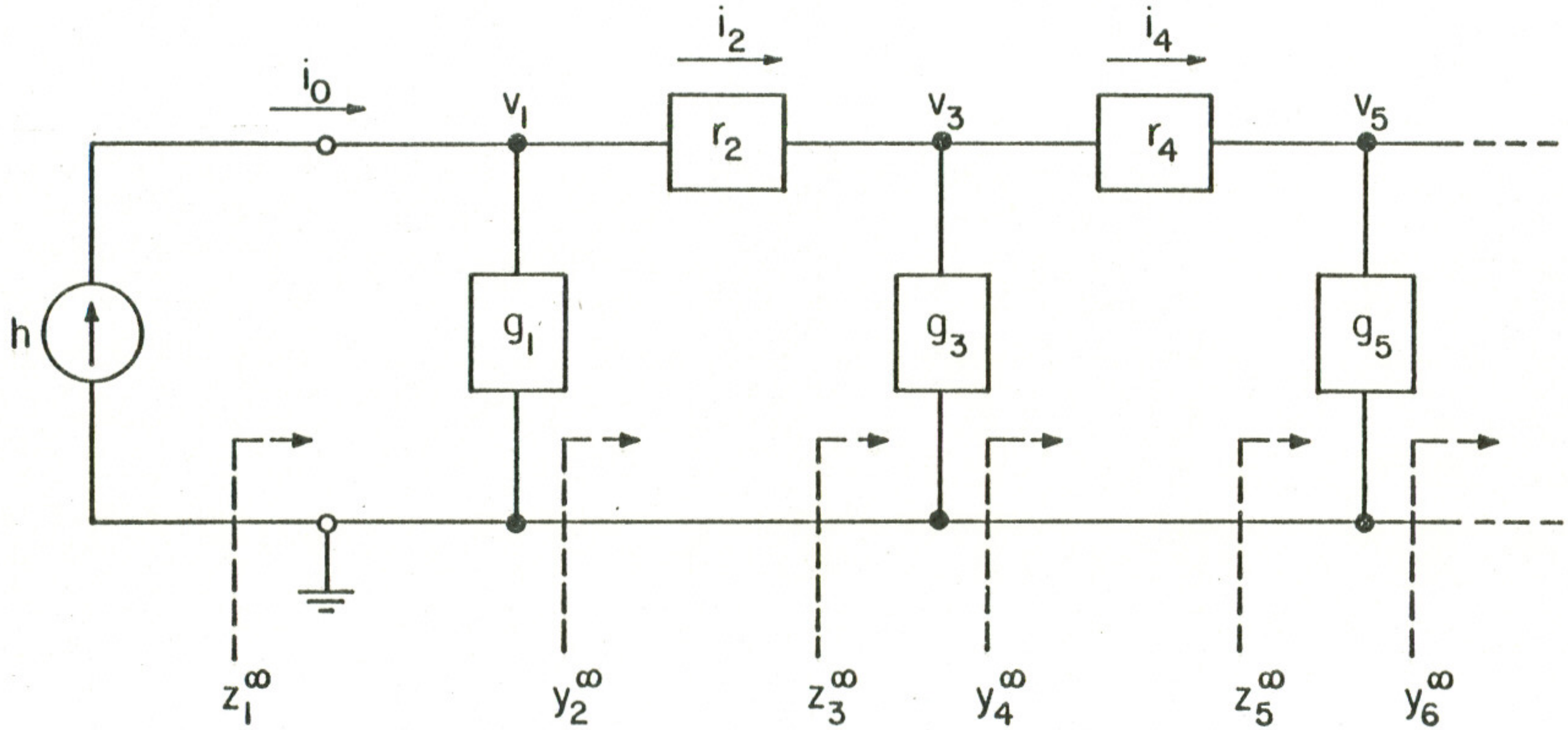

Figure 4.

$W(r_k^{-1}) \subset [\beta^{-1}, \alpha^{-1}]$, the assertion in Condition A concerning the numerical ranges will be satisfied.

For the g_k we can argue from (4.3) as follows. Since

$$(x_{p_b} - x_{q_b})^2 \leq 2x_{p_b}^2 + 2x_{q_b}^2,$$

we have

$$(5.2) \qquad (g_k x, x) \leq c_k \|x\|^2 + \sum_{\mu=1}^{j_k} a_{k\mu} \sum_{b \in \Gamma_{k\mu}} (2x_{p_b}^2 + 2x_{q_b}^2).$$

But, by Rule II, all the node voltages are traversed by x_{p_b} and by x_{q_b} as b traverses $\Gamma_{k\mu}$. Therefore, the right-hand side of (5.2) is equal to

$$c_k \|x\|^2 + 4\|x\|^2 \sum_{\mu=1}^{j_k} a_{k\mu}$$

So, our assertion for the g_k follows when we set $\alpha = c_k$ and $\beta = 5\gamma$ and then invoke Rule IV(ii).

Since g_k is a strictly positive operator, this result on its numerical range also implies [4; p. 62], [6; p. 145] that for all k

$$(5.3) \qquad \|g_k\| \leq \beta , \quad \|g_k^{-1}\| \leq \alpha^{-1}.$$

The same conclusions for the r_k follow immediately from Rule IV(ii) since $r_k = b_k 1$, where now 1 denotes the identity operator on $\underline{1}_{2r}$.

Now, refer to Figure 4 again. The next thing we want to show is that the driving-point impedances z_k^{∞}, where k is odd, and the driving-point admittances y_k^{∞}, where k is even, exist and are positive invertible Laurent operators on $\underline{1}_{2r}$. For $n > k$, we let z_k^n and y_k^n be the corresponding driving-point impedances and admittances when the ladder network is terminated at its nth element. That is, we set

$$f_j = \begin{cases} g_j , & j = 1, 3, 5, \ldots \\ r_j , & j = 2, 4, 6, \ldots \end{cases}$$

and

$$(5.4) \qquad f_k^n = \begin{cases} z_k^n , & k = 1, 3, 5, \ldots , \quad n > k \\ y_k^n , & k = 2, 4, 6, \ldots , \quad n > k \end{cases}$$

Then, for $n < \infty$, f_k^n is given by the finite continued fraction

$$(5.5) \qquad f_k^n = \frac{1}{f_k} + \frac{1}{f_{k+1}} + \cdots + \frac{1}{f_n}$$

The inverse of a positive, invertible, Laurent operator and the sum of two such operators are again positive, invertible, and Laurent. Therefore, every f_k^n also has these properties.

Now, Laurent operators commute [1]. This fact coupled with the fact that the numerical ranges of all the g_j and r_j are all contained in the interval $[\alpha, \beta]$, where $\alpha > 0$, allows us to invoke a theorem of Fair [2] to conclude that, as $n \to \infty$, (5.5) converges in the uniform operator topology. Its f_k^∞ is the driving-point impedance z_k^∞ or the driving-point admittance y_k^∞ depending on whether k is odd or even.

For $n > k+2$

$$f_k^n = \cfrac{1}{f_k + \cfrac{1}{f_{k+1} + f_{k+2}^n}}$$

Since $W(f_k)$ and $W(f_{k+1})$ are both contained in $[\alpha, \beta]$, where $0 < \alpha < \beta < \infty$, and since f_{k+2}^n is a positive operator, we may invoke the spectral mapping theorem to write the following set inclusions, where the right-hand sides denote closed or half-closed intervals.

$$W(f_{k+1} + f^{n}_{k+2}) \subset [\alpha, \infty)$$

$$W\left(\frac{1}{f_{k+1} + f^{n}_{k+2}}\right) \subset (0, \tfrac{1}{\alpha}]$$

$$W\left(f_k + \frac{1}{f_{k+1} + f^{n}_{k+2}}\right) \subset \left[\alpha, \beta + \frac{1}{\alpha}\right]$$

$$W(f^{n}_{k}) \subset \left[\frac{1}{\beta + \frac{1}{\alpha}}, \frac{1}{\alpha}\right]$$

Since, for every $x \in \underline{1}_{2r}$, $(f^{n}_{k}x, x) \to (f^{\infty}_{k}x, x)$ as $n \to \infty$, we can conclude that

$$W(f^{\infty}_{k}) \subset \left[\frac{1}{\beta + \frac{1}{\alpha}}, \frac{1}{\alpha}\right], \quad 0 < \alpha < \beta < \infty \tag{5.6}$$

In fact, we have established most of

Theorem 5.1. Assume Rules I through IV for the grounded grid of Figure 2. Then, the driving-point impedances and admittances f^{∞}_{k} of the corresponding ladder network of operators shown in Figure 4 exist as the limits under the uniform operator topology of the infinite continued fractions

$$f^{\infty}_{k} = \frac{1}{f_k} + \frac{1}{f_{k+1}} + \frac{1}{f_{k+2}} + \cdots \tag{5.7}$$

The f^{∞}_{k} are all positive, invertible, Laurent operators whose numerical ranges are uniformly bounded according to (5.6).

Proof. There is only one thing left to prove; namely, each f^{∞}_{k} is a Laurent operator. An operator in $[\underline{1}_{2r}; \underline{1}_{2r}]$ is Laurent if and only if it commutes with the shifting operator s_q, for every $q \in N^n$ [1; Theorem 2]. s_q is defined as follows:

Let $x \in \underline{1}_{2r}$ and for $p \in N^n$ let x_p be the pth element of x. Then, by definition, $s_q x = y$, where $y_p = x_{p-q}$. Since each f_k^n is a Laurent operator, we have under the uniform operator topology that as $n \to \infty$

$$s_q f_k^\infty \leftarrow s_q f_k^n = f_k^n s_q \to f_k^\infty s_q.$$

This completes the proof.

VI. The Solution of the Ladder Network

Assume that the current source h of Figure 4 is a member of $\underline{1}_{2r}$. We now apply Kirchhoff's laws and Ohm's law to determine the $\underline{1}_{2r}$-valued currents and voltages in Figure 4.

For k = 0, 2, 4, ... , Kirchhoff's node law applied to the nodes of Figure 4 yields

$$i_{k+2} = i_k - g_{k+1} v_{k+1}.$$

By Ohm's law, $v_{k+1} = z_{k+1}^\infty i_k$. Therefore,

$$i_{k+2} = \theta_k i_k \tag{6.1}$$

where

$$\theta_k = 1 - g_{k+1} z_{k+1}^\infty \tag{6.2}$$

Here, 1 denotes the identity operator on $\underline{1}_{2r}$.

For k = 1, 3, 5, ... , Kirchhoff's loop law applied to the meshes of Figure 4 and Ohm's law yield

$$v_{k+2} = v_k - r_{k+1} i_{k+1}$$

$$i_{k+1} = y_{k+1}^\infty v_k$$

and thus

(6.3) $$v_{k+2} = \theta_k v_k$$

where

(6.4) $$\theta_k = 1 - r_{k+1} y_{k+1}^{\infty}$$

Given $h \in \underline{l}_{2r}$ in Figure 4, these equations allow us to determine every voltage and every current in that ladder network recursively. In particular, for $k = 2, 4, 6, \dots$

(6.5) $$i_k = \theta_{k-2}\theta_{k-4} \cdots \theta_0 h$$

and, for $k = 3, 5, 7, \dots$

(6.6) $$v_k = \theta_{k-2}\theta_{k-4} \cdots \theta_1 v_1 , \quad v_1 = z_1^{\infty} h.$$

Our next objective is to establish several properties of the operator θ_k. By Theorem 5.1, z_{k+1}^{∞} (k even) and y_{k+1}^{∞} (k odd) are Laurent operators. So too are g_{k+1} (k even) and r_{k+1} (k odd). Furthermore, the composition and sum of two Laurent operators are also Laurent operators. Hence, θ_k is a Laurent operator for every k.

Lemma 6.1. *If $A, B \in [H; H]$, where H is a Hilbert space, and if A is positive and commutes with B, then their numerical ranges satisfy $W(AB) \subset W(A)A(B)$.*

Proof. The square root $A^{\frac{1}{2}}$ commutes with every operator that commutes with A. Therefore,

$$(ABx, x) = (BA^{\frac{1}{2}}x, A^{\frac{1}{2}}x) \in W(B)\|A^{\frac{1}{2}}x\|^2 = W(B)(Ax, x).$$

Our lemma now follows immediately.

We now examine some numerical ranges. Let $k = 0, 2, 4, \dots$. According to (5.6) and (5.1),

$$W(y^{\infty}_{k+2}) \subset \left[\frac{\alpha}{1 + \alpha\beta}, \frac{1}{\alpha}\right],$$

$$W(g_{k+1}) \subset [\alpha, \beta].$$

Therefore,

$$W(g^{-1}_{k+1}) \subset \left[\frac{1}{\beta}, \frac{1}{\alpha}\right].$$

Since we are dealing with positive Laurent operators and Laurent operators commute, we can invoke Lemma 6.1.

$$W(y^{\infty}_{k+2}g^{-1}_{k+1}) \subset \left[\frac{\alpha}{\beta + \alpha\beta^2}, \frac{1}{\alpha^2}\right]$$

It can be seen from Figure 4 that

$$(6.7) \qquad z^{\infty}_{k+1} = (g_{k+1} + y^{\infty}_{k+2})^{-1} = g^{-1}_{k+1}(1 + y^{\infty}_{k+2}g^{-1}_{k+1})^{-1}.$$

Therefore,

$$g_{k+1}z^{\infty}_{k+1} = (1 + y^{\infty}_{k+2}g^{-1}_{k+1})^{-1}.$$

So,

$$W(g_{k+1}z^{\infty}_{k+1}) \subset \left[\left(1 + \frac{1}{\alpha^2}\right)^{-1}, \left(1 + \frac{\alpha}{\beta + \alpha\beta^2}\right)^{-1}\right]$$

The last closed interval is contained in the open interval (0, 1). So, in view of (6.2),

$$(6.8) \qquad W(\theta_k) \subset \left[1 - \left(1 + \frac{\alpha}{\beta + \alpha\beta^2}\right)^{-1}, 1 - \left(1 + \frac{1}{\alpha^2}\right)^{-1}\right] = \left[\frac{1}{1 + \alpha^{-1}\beta + \beta^2}, \frac{1}{1 + \alpha^2}\right].$$

This shows that θ_k is a positive, invertible, strictly contractive operator with

(6.9) $$\|\theta_k\| \leq \frac{1}{1 + \alpha^2}$$

For k = 1, 3, 5, ... , (6.7) is replaced by

$$y^{\infty}_{k+1} = (r_{k+1} + z^{\infty}_{k+2})^{-1}$$

We can now apply the same argument to (6.4) to obtain (6.8) and (6.9) once again. This establishes

<u>Theorem 6.1. Under Rules I through IV and for every k = 0, 1, 2, ... , θ_k is a positive, invertible, Laurent operator in $[\underline{1}_{2r}; \underline{1}_{2r}]$ satisfying (6.9).</u>

Note that, by Rule IV(ii), α is independent of k.

We can now show that the solution given by (6.5) and (6.6) is precisely the one dictated by Theorem 3.1. Indeed, let the H_r of that theorem be $\underline{1}_{2r}$. Since there is only one current source, the vector of current sources is a member of $\underline{1}_2(\underline{1}_{2r})$. We have already noted in Section V that $W(g_k) \subset [\alpha, \beta]$ and $W(r_k^{-1}) \subset [\beta^{-1}, \alpha^{-1}]$. By the analysis in the second and third paragraphs of this section, the solution given by (6.5) and (6.6) satisfies Kirchhoff's laws and Ohm's law. The rest of the hypothesis of Theorem 3.1 is clearly satisfied except perhaps for the requirements that the vector of all branch voltages and the vector of all branch currents be members of $\underline{1}_2(\underline{1}_{2r})$; this we now verify.

Summing over all odd k and using (6.6), we get for the vertical branches of Figure 4

$$\sum \|v_k\|^2 = \|v_1\|^2 + \|\theta_1 v_1\|^2 + \|\theta_3\theta_1 v_1\|^2 + \|\theta_5\theta_3\theta_1 v_1\|^2 + \cdots$$

By (6.9), $\|\theta_k\| \le K = (1 + \alpha^2)^{-1} < 1$ for every odd k. Therefore,

$$\sum \|v_k\|^2 \le (1 + K^2 + K^4 + K^6 + \cdots)\|v_1\|^2$$
$$= (1 - K^2)^{-1}\|v_1\|^2 < \infty.$$

For the horizontal branches of Figure 4, we have $v_k = r_k i_k$, where now k is even. Summing over all such k and using (5.1) and (6.5), we get .

$$\sum \|v_k\|^2 = \sum \|r_k i_k\|^2 \le \beta^2 \sum \|i_k\|^2$$
$$= (\|h\|^2 + \|\theta_0 h\|^2 + \|\theta_2\theta_0 h\|^2 + \|\theta_4\theta_2\theta_0 h\|^2 + \cdots)\beta^2$$
$$\le (1 + K^2 + K^4 + K^6 + \cdots)\beta^2\|h\|^2$$
$$= (1 - K^2)^{-1}\beta^2\|h\|^2 < \infty.$$

So truly, the vector of all branch voltages is a member of $\underline{l}_2(\underline{l}_{2r})$.

Quite the same argument shows that the vector of all branch currents is also a member of $\underline{l}_2(\underline{l}_{2r})$. Thus, we have

Theorem 6.2. Under Rules I through IV, the solution for the network of Figure 4.1 given by (6.5) and (6.6) is the unique (finite power) solution dictated by Theorem 3.1.

As was promised at the beginning of Section 5, we now have a two-step procedure for determining the solution for the grid of Figure 2. We first determine the solution for the operator network of Figure 4 and then determine the interior branch currents of each ∞-port to get the currents in the branches of Figure 2. However, there is one more thing we should verify; namely, the solution for the grid of Figure 2 given by this two-step procedure is the same as the solution specified by Theorem 3 when that theorem is applied directly to the grid of Figure 2 with H_r now being the real line. This can be established in a completely straightforward way, The details of the argument are spelled out in [18; Section 5].

VII. A Computational Procedure

So far, we have established the existence and uniqueness of the solution in $\underline{l}_{2r}$ (i.e., the finite-power solution) for the network of Figure 2. However, the question remains how one might compute the numerical values of the voltages and currents in the network, given the current sources h_j of Figure 2. For this purpose, we use Equation (6.6) to compute all the node voltages, from which all the currents in the grid can be determined. The first step is to determine $v_1 \in \underline{l}_{2r}$, and this is facilitated by the isomorphism between $\underline{l}_{2r}$ and its corresponding space of Fourier series. Let's quickly survey that isomorphism and its effect on Laurent operators [1].

Let S denote the unit circle and S^n the Cartesian product of n replicates of S. $L_2(S^n)$ is as usual the Hilbert space of (equivalence classes of) quadratically integrable functions f on S^n with the norm

$$\|f\| = \left[\frac{1}{(2\pi)^n} \int_{S^n} |f(\omega)|^2 \, d\omega \right]^{\frac{1}{2}}$$

$$\omega = (\omega_1, \ldots, \omega_n), \; 0 \leq \omega_j < 2\pi$$

Let $\mathfrak{F}$ denote the transformation that assigns to each $x = \{x_p : p \in N^n\} \in \underline{l}_{2r}$ the function

$$\tilde{x}(\omega) = \sum_{p \in N^n} x_p e^{i(p,\, \omega)}$$

where $(p, \omega) = p_1\omega_1 + \cdots + p_n\omega_n$. A standard result is that $\mathcal{F}$ is a (topological linear) isomorphism from $\underline{l}_{2r}$ onto $L_2(S^n)$ such that $\|x\| = \|\tilde{x}\|$.

Let $z \in [\underline{l}_{2r}; \underline{l}_{2r}]$ and let $\hat{z}$ be the mapping of $L_2(S^n)$ into $L_2(S^n)$ induced by $\mathcal{F}$; that is, $\hat{z} = \mathcal{F}z\mathcal{F}^{-1}$. Then, $\|z\| = \|\hat{z}\|$. It is a fact that z is a Laurent operator (that is, $z \in [\underline{l}_{2r} ; \underline{l}_{2r}]$ and satisfies (4.1)) if and only if $\hat{z}$ is a multiplication. More specifically,

$$(7.1) \qquad (\hat{z}\tilde{x})(\omega) = \tilde{z}(\omega)\tilde{x}(\omega)$$

where

$$(7.2) \qquad \tilde{z}(\omega) = \sum_{q \in N^n} z_{0,q}\, e^{-i(q,\omega)}.$$

Here, the subscript 0 denotes the origin in N^n and $z_{0,q}$ is the 0,q entry of the matrix representation $[z_{p,q}]$ of z.

The mapping $z \mapsto \hat{z}$ of $[\underline{l}_{2r}; \underline{l}_{2r}]$ into $[L_2(S^n); L_2(S^n)]$ is linear, continuous, and norm preserving; that is, $\|z\| = \|\hat{z}\|$. Also,

$$(7.3) \qquad \|z\| = \text{ess sup } |\tilde{z}(\omega)|.$$

Moreover, if z is positive and invertible, then ess inf $\tilde{z}(\omega) > 0$ and z^{-1} corresponds to multiplication by $[\tilde{z}(\omega)]^{-1}$. Finally, the numerical range of $\hat{z}$ is the closed interval between the essential supremum and the essential infimum of $\tilde{z}(\omega)$.

These results imply that z_1^{∞}, which exists as a Laurent operator according to Theorem 5.1, corresponds to multiplication by the function

$$(7.4) \qquad \tilde{z}_1^{\infty}(\omega) = \frac{1}{\tilde{g}_1(\omega)} + \frac{1}{b_2} + \frac{1}{\tilde{g}_3(\omega)} + \frac{1}{b_4} + \cdots$$

where for k odd $\tilde{g}_k(\omega)\cdot$ is the multiplication corresponding to g_k

and for k even $b_k\cdot$ is the multiplication corresponding to $r_k = b_k 1$. By virtue of Rule II, each $\tilde{g}_k(\omega)$ is a finite Fourier series and hence a continuous function. Also, the range of $\tilde{g}_k(\omega)$ is contained in $[\alpha, \beta]$ where $\alpha > 0$.

The function in $L_2(S^n)$ corresponding to $v_1 = z_1^\infty h$ for a given $h = \{h_p : p \in N^n\} \in \underline{\underline{1}}_{2r}$ is

$$\tilde{v}_1(\omega) = \tilde{z}_1^\infty(\omega)\tilde{h}(\omega) \tag{7.5}$$

where

$$\tilde{h}(\omega) = \sum_{p \in N^n} h_p \, e^{i(p,\omega)}. \tag{7.6}$$

Thus, the node voltages $(v_1)_p$ for the nodes on the first box of Figure 2 are

$$(v_1)_p = \frac{1}{(2\pi)^n} \int_{S^n} \tilde{v}_1(\omega)\, e^{-i(p,\omega)}\, d\omega \,. \tag{7.7}$$

The next step is the computation of the functions $\tilde{\theta}_k(\omega)$ for k odd by means of the following analogue to (6.4):

$$\tilde{\theta}_k(\omega) = 1 - \tilde{r}_{k+1}(\omega)\tilde{y}_{k+1}^\infty(\omega) \tag{7.8}$$

Here, $\tilde{r}_{k+1}(\omega) = b_{k+1}$, and

$$\tilde{y}_{k+1}^\infty(\omega) = \frac{1}{b_{k+1}} + \frac{1}{\tilde{g}_{k+2}(\omega)} + \frac{1}{b_{k+3}} + \frac{1}{\tilde{g}_{k+4}(\omega)} + \cdots \tag{7.9}$$

The analogue to Equation (6.6) then yields

$$\tilde{v}_k(\omega) = \tilde{\theta}_{k-2}(\omega)\tilde{\theta}_{k-4}(\omega) \cdots \tilde{\theta}_1(\omega)\tilde{v}_1(\omega) \tag{7.10}$$

Finally, the node voltages in the kth box of Figure 2 are given by

$$(7.11) \qquad (v_k)_p = \frac{1}{(2\pi)^n} \int_{S^n} \tilde{v}_k(\omega)\, e^{-i(p,\omega)}\, d\omega.$$

In practical computations we must either determine the continued fraction (7.4) in closed form, usually an unlikely prospect, or truncate it by an open-circuit or short-circuit operator admittance and estimate the resulting error, or, in the case where the grid approaches a uniform grid as $k \to \infty$, perhaps truncate it with the characteristic operator admittance of the uniform grid [17].

Let's consider how that truncation error might be estimated when the grid is terminated by an open-circuit or short-circuit operator after the nth section. In fact, let us approximate $\tilde{z}_1^\infty(\omega)$ by

$$(7.12) \qquad \tilde{z}_1^n(\omega) = \frac{1}{\tilde{g}_1(\omega)} + \frac{1}{b_2} + \cdots + \frac{1}{f_n(\omega)}$$

where $f_n(\omega) = b_n$ for n even and $f_n(\omega) = \tilde{g}_n(\omega)$ for n odd. Now, a property of convergent infinite continued fractions with positive terms is that its limit lies between any two consecutive truncations: Thus, $\tilde{z}_1^\infty(\omega)$ lies between $\tilde{z}_1^{n-1}(\omega)$ and $\tilde{z}_1^n(\omega)$. Hence,

$$(7.13) \qquad \left|\tilde{z}_1^\infty(\omega) - \tilde{z}_1^n(\omega)\right| \leq \left|\tilde{z}_1^{n-1}(\omega) - \tilde{z}_1^n(\omega)\right|.$$

Also, by virtue of Rule II, every $\tilde{g}_k(\omega)$ and therefore every $\tilde{z}_1^n(\omega)$ is a continuous function.

This allows us to bound the error generated by truncating (7.4) as follows. Let $v_1^a \in \underline{l}_{2r}$ be the approximation of $v_1 \in \underline{l}_{2r}$ resulting from the replacement of $\tilde{z}_1^\infty(\omega)$ by $\tilde{z}_1^n(\omega)$. Then, letting $(x)_p$ denote the pth component of the vecotr $x \in \underline{l}_{2r}$ and using (7.12), we may write

$$
(7.14)\qquad
\begin{aligned}
&\left|(v_1)_p - (v_1^a)_p\right| \\
&\quad \le \frac{1}{(2\pi)^n} \int_{S^n} \left|\tilde{z}_1^{\infty}(\omega) - \tilde{z}_1^{n}(\omega)\right| \left|\tilde{h}(\omega)\right| \, d\omega \\
&\quad \le \sup_{\omega} \left|\tilde{z}_1^{n-1}(\omega) - \tilde{z}_1^{n}(\omega)\right| \sup_{\omega} \left|\tilde{h}(\omega)\right| \frac{1}{(2\pi)^n} \int_{S^n} d\omega \\
&\quad \le \sup_{\omega} \left|\tilde{z}_1^{n-1}(\omega) - \tilde{z}_1^{n}(\omega)\right| \sum \left|h_p\right| .
\end{aligned}
$$

Here, $\sum |h_p|$ denotes the sum over all the nonzero $|h_p|$, these usually being finite in number **in practical cases.** Because of (7.3) and the fact that (5.7) converges in the uniform operator topology, given the h_p with $\sum |h_p|$ convergent, we can make the right-hand side of (7.14) as small as we wish by choosing n large enough. That is, (7.14) can be used to control the error generated by truncating (7.4). However, this is a conservative approach; the bound (7.14) will be in general much larger than the actual error.

Bounds on the error generated in the computation of the $(v_k)_p$ by the continued-fraction expressions for the $\tilde{y}_{k+1}^{\infty}(\omega)$ can be estimated in exactly the same way, but now an error appears for each factor $\tilde{\theta}_k(\omega)$ as well as for $\tilde{v}_1(\omega) = \tilde{z}_1^{\infty}(\omega)\tilde{h}(\omega)$ in (7.10). Finally, when our nonuniform grid approaches a uniform one as $k \to \infty$, we will generate less error by terminating in the characteristic operator immittance of the uniform grid [17], and so our aforementioned bounds on the error will still be valid. Of course, other errors are generated by the numerical integrations of (7.7) and (7.11); these can be estimated by standard methods.

VIII. An Example

We illustrate our computational procedure with an example. We assign values to the parameters in the grid of Figure 1 as follows:

$$h_j = \begin{cases} 1 & \text{for } j = 0 \\ 0 & \text{for } j \neq 0 \end{cases}$$

$$a = 1$$

$$c_{2m+1} = 1 + e^{-m} \quad \text{for} \quad m = 0, 1, 2, \ldots$$

Consequently, the various functions of ω generated by the isomorphism $\mathcal{J}$ are

$$\tilde{h}(\omega) = 1$$

$$\tilde{r}_{2m}(\omega) = 1 \quad \text{for} \quad m = 1, 2, 3, \ldots$$

$$\tilde{g}_{2m+1}(\omega) = 3 + e^{-m} - 2\cos\omega \quad \text{for} \quad m = 0, 1, 2, \ldots$$

To compute approximately the driving-point impedance $\tilde{z}_1^{\infty}(\omega)$, we use the fact that our grid approaches a uniform grid as $m \to \infty$. So, we may replace the ladder network beyond node 2M+1, where M is chosen sufficiently large, by its characteristic impedance $\tilde{z}_0(\omega)$. The latter can be determined by the method given in [17]; it is

$$\tilde{z}_0(\omega) = \frac{1}{2}\left\{-3 + 2\cos\omega + \left[(3 - 2\cos\omega)^2 + 4(3 - 2\cos\omega)\right]^{\frac{1}{2}}\right\}.$$

Then, for sufficiently large M, we have to a high order of accuracy

$$\tilde{z}_1^{\infty}(\omega) \approx \frac{1}{\tilde{g}_1(\omega) +} \; \frac{1}{1 +} \; \frac{1}{\tilde{g}_3(\omega) +} \; \frac{1}{1 +} \cdots \; \frac{1}{+\, \tilde{g}_{2M-1}(\omega) +} \; \frac{1}{1 + \tilde{z}_0(\omega)} \,.$$

Similarly, for k odd and $k \ll 2M + 1$,

$$\tilde{y}_{k+1}^{\infty} \approx \frac{1}{1} + \frac{1}{\tilde{g}_{k+2}(\omega)} + \frac{1}{1} + \frac{1}{\tilde{g}_{k+4}(\omega)} + \cdots + \frac{1}{\tilde{g}_{2M-1}(\omega)} + \frac{1}{1 + \tilde{z}_0(\omega)} .$$

We now use (7.8), (7.10), and (7.11) to compute the node voltages for the nonuniform grid in the vicinity of the single current source $h_0 = 1$.

For the sake of illustration, we have chosen $M = 12$ and have computed the node voltages for the first five rows of nodes (that is, for the first five boxes) and for the first eight columns of nodes on either side of the 0th column where $h_0 = 1$ appears. The results are displayed in Table 1. Since the node values have even symmetry around the 0th column, We have indicated their values only to the right of the 0th column. Computer execution time for these results was 38.5 seconds.

TABLE 1

Column

Row	0	1	2	3	4	5	6	7	8
1:	.28076	.07717	.02217	.00661	.00203	.00064	.00020	.00007	.00002
2:	.07199	.03413	.01308	.00465	.00161	.00055	.00019	.00006	.00002
3:	.02069	.01303	.00621	.00261	.00102	.00038	.00014	.00005	.00002
4:	.00643	.00474	.00265	.00127	.00056	.00023	.00009	.00004	.00001
5:	.00211	.00170	.00107	.00057	.00028	.00012	.00005	.00002	.00001

IX. Nonuniform Grids of Positive-real Impedances

We now turn from purely resistive networks to ones that may contain inductors, capacitors, transformers, and so on. By using the results of [13], we can extend our analysis to grids of the form of Figure 2 where now each branch is a positive-real driving-point impedance with no coupling between branches. We shall in fact examine the transient behavior of such networks. The arguments needed to do this are quite similar to those given in Section 8 of [17]. Therefore, we shall at this point merely define the needed concepts and then indicate how the arguments in Section 8 of [17] have to be modified to make them suitable for our present considerations.

Here are some concepts from [13]. C_+ denotes the open right half of the complex plane C:

$$C_+ = \{s \in C: \text{Re } s > 0\}$$

For $s \in C_+$, Ω_s is the closed cone:

$$\Omega_s = \{z \in C: |\arg z| \leq |\arg s|\},$$

where it is understood that the origin is a member of Ω_s. $\underline{l}_2$ denotes the complexification of $\underline{l}_{2r}$. By an "operator", we henceforth mean a continuous linear mapping of $\underline{l}_2$ into $\underline{l}_2$.

P is the set of all analytic operator-valued functions F on C_+ such that, for every $s \in C_+$, the numerical range $W[F(s)]$ of $F(s)$ is contained in C_+. Thus, if $F \in P$, $F(\sigma)$ is a positive operator for each $\sigma > 0$. P_i is the set of all $F \in P$ such that, for every fixed $s \in C_+$, $W[F(s)]$ is bounded away from the origin, that is, there exists a $\delta > 0$ depending in general on s such that

Re $(F(s)x, x) \geq \delta \|x\|^2$ for all $x \in \underline{1}_2$. Thus, for each $s \in C_+$ and $F \in P_i$, $F(s)$ is an invertible operator. It was shown in [13] that, if $F \in P_i$ and $G \in P$, then $F + G \in P_i$; also, if $F \in P_i$ and if F^{-1} denotes the function $s \mapsto [F(s)]^{-1}$, then $F^{-1} \in P_i$.

Next, let $F_1, F_2, F_3, \ldots \in P_i$ and let $Z_n(s)$ be the operator-valued finite continued fraction

$$(9.1) \qquad Z_n(s) = \frac{1}{F_1(s)} + \frac{1}{F_2(s)} + \cdots + \frac{1}{F_n(s)}.$$

Then, by what has just been stated, $Z_n \in P_i$. The following theorem is a somewhat simplified version of Theorem 1 and Corollary 1a in [13].

<u>Theorem 9.1. Assume the following three conditions:</u>

<u>(i)</u> $F_k \in P_i$ <u>for every</u> $i = 1, 2, 3, \ldots$.

<u>(ii) Given any compact set</u> $\Xi \subset C_+$, <u>there exists a constant</u> $\delta > 0$, <u>depending upon</u> Ξ, <u>such that</u> $\inf \operatorname{Re} W[F_k(s)] > \delta$ <u>for</u> $k = 1, 2$ <u>and</u> $s \in \Xi$.

<u>(iii) For each</u> $\sigma > 0$ <u>and all</u> $k = 1, 2, 3, \ldots$, <u>the operators</u> $F_k(\sigma)$ <u>commute with each other and</u> $W[F_k(\sigma)] > \delta_k(\sigma)$, <u>where the</u> $\delta_k(\sigma)$ <u>are positive numbers satisfying</u> $\sum_{k=1}^{\infty} \delta_k(\sigma) = \infty$.

<u>Then, for every</u> $s \in C_+$, <u>the sequence</u> $\{Z_n(s)\}_{n=1}^{\infty}$ <u>converges in the uniform operator topology, and the convergence is uniform with respect to</u> s <u>in any compact subset of</u> C_+. <u>Moreover, the limit function</u> $Z = \lim_{n\to\infty} Z_n$ <u>is a member of</u> P_i.

Next, we turn to the time domain and define the space $L_2(R, \underline{1}_{2r})$. Consider the mappings of the real line R into $\underline{1}_{2r}$ which are quadratically integrable on R under the $\underline{1}_{2r}$ norm, and let two such functions be in the same equivalence class if they differ on no more than a set of Lebesgue measure zero. $L_2(R, \underline{1}_{2r})$ is the real Hilbert space of all such equivalence classes supplied with the inner product

$$(a, b) = \int_{-\infty}^{\infty} \sum_{p \in N^n} a_p(t) b_p(t)\, dt; \quad t \in R; \quad a, b \in L_2(R, \underline{1}_{2r}).$$

(See [11; Appendix G].)

In [17], we proved

Lemma 8.1. Assume that the vector h is a member of $L_2(R, \underline{1}_{2r})$ and that the support of h is bounded on the left. Let H be the Laplace transform of h, that is, the vector of Laplace transforms H_p of the components h_p, $p \in N^n$. Then, for each $s \in C_+$, $H(s)$ exists and is a member of $\underline{1}_2$. Also, for each $\sigma > 0$, $H(\sigma)$ is a member of $\underline{1}_{2r}$.

These are all the results we need to generalize our discussion to grids of impedances. To proceed, replace Rules I through IV with the following, where $s \in C_+$.

Rule I'. Same as Rule I except that the conductances c_k are replaced by scalar positive-real admittances $C_k(s)$.

Rule II'. Same as Rule II except that the conductance $a_{k\mu}$ (k odd) for every branch class $\Gamma_{k\mu}$ is replaced by a scalar positive-real admittance $A_{k\mu}(s)$.

Rule III'. Same as Rule III except that every resistance b_k (k even) connecting box $k-1$ to box $k+1$ is replaced be a scalar positive-real impedance $B_k(s)$.

Rule IV'. (i) In the time-domain, the vector $h = \{h_p : p \in N^n\}$ of current sources at the input of Figure 2 is a member of $L_2(R, \underline{1}_{2r})$, and the support of h is bounded on the left.

(ii) There exist two continuous functions $\alpha(\sigma)$ and $\gamma(\sigma)$ on the half line $R_+ = \{\sigma \in R: \sigma > 0\}$ with $0 < \alpha(\sigma) < \gamma(\sigma)$ for every $\sigma \in R_+$ such that, for $k = 1, 3, 5, \ldots$, we have $\alpha(\sigma) \leq C_k(\sigma) \leq \gamma(\sigma)$ and

$$\sum_{\mu=1}^{j_k} A_{k\mu}(\sigma) \leq \gamma(\sigma),$$

and, for every $k = 2, 4, 6, \ldots$, we have $\alpha(\sigma) \leq B_k(\sigma) \leq \gamma(\sigma)$.

We now decompose the grid of Figure 2 into ∞-ports as before to get the ladder network of Figure 4. In the frequency domain, the g_k for k odd and the r_k for k even are replaced by $[\underline{1}_2; \underline{1}_2]$-valued functions $Y_k(s)$ and $Z_k(s) = B_k(s)1$ respectively, where $s \in C_+$. Upon modifying the arguments that established (4.3) with manipulations suitable for a complex Hilbert space, we obtain for $x = \{x_p : p \in N^n\} \in \underline{1}_2$

$$(9.2) \qquad (Y_k(s)x, x) = C_k(s) \sum_p |x_p|^2 + \sum_{\mu=1}^{j_k} A_{k\mu}(s) \sum_{k \in \Gamma_{k\mu}} |x_{p_b} - x_{q_b}|$$

Using this equation and Rules I' through IV', we can argue in virtually the same way as in [17; Section 8] to establish that every Y_k and Z_k is a member of P_i, that $Y_k(\sigma)$ and $Z_k(\sigma)$ are Laurent operators for each $\sigma \in R_+$, that the hypothesis of Theorem 9.1 is satisfied when $F_k(s) = Y_k(s)$ for k even and $F_k(s) = Z_k(s)$ for k odd, and that the driving-point impedance $Z_1^\infty = \lim_{n\to\infty} Z_1^n$ is also a member of P_i.

Proceeding still further as in [17; Section 8], we conclude that the Y_k, Z_k, Z_1^∞, and θ^k, where

$$\theta^k(s) = 1 - Y_{k+1}(s)Z_{k+1}^\infty(s), \quad k = 0, 2, 4, \ldots$$

$$\theta^k(s) = 1 - Z_{k+1}(s)Y_{k+1}^\infty(s), \quad k = 1, 3, 5, \ldots,$$

are all Laplace transforms of $[\underline{1}_{2r}; \underline{1}_{2r}]$-valued right-sided distributions whose supports are bounded on the left at the origin. Next, we generalize Ohm's law by means of distributional convolution [11; Section 5.2]:

$$v = r * i \ , \quad i = g * v$$

Finally, we conclude with

Theorem 9.2. For the network of Figure 2, assume that Rules I' through IV' hold. Then, there exists one and only one set of right-sided Laplace-transformable distributions for the branch voltages v_m in the network of Figure 2 such that Kirchhoff's node and loop laws and (generalized) Ohm's law are satisfied in the time domain and such that, for at least one $\sigma > 0$ and for V_m denoting the Laplace transform of v_m, we have

$$\sum_m [V_m(\sigma)]^2 < \infty. \tag{9.3}$$

In this case, (9.3) holds for all $\sigma > 0$.

The branch voltages and branch currents for Figure 2 can be determined from the components of the $\underline{1}_{2r}$-valued distributions i_k and v_k, which are given in turn by (6.5) and (6.6) appropriately rewritten as distributional convolutions. Alternatively, we can work in the frequency domain, in which case (6.5) and (6.6) should be rewritten as multiplications of Laplace transforms.

X. The Computation of Transient Responses.

By using the method described in Section VII, we can compute the voltage $V(\sigma)$ at any node or the current $I(\sigma)$ in any branch at a finite set of points $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_q$ on the real axis in C_+. Then, using these values of $V(\sigma)$ or $I(\sigma)$, we can apply Papoulis' method [7], [8] to compute the corresponding transient response $v(t)$ or $i(t)$. This requires however that $V(s)$ and $I(s)$ tend to zero fast enough as $s \to \infty$ in C_+ to ensure that $v(t)$ or $i(t)$ be a sufficiently well-behaved function to allow the convergence of Papoulis' method.

For example, assume that, as $s \to \infty$ in C_+, every $C_k(s)$ acts capacitively, that is, it is asymptotic to a constant times s, and every $A_{k\mu}(s)$ and $B_k(s)$ acts resistively, that is, it is asymptotic to a constant. Then, using the transformation $\mathcal{J}$ to manipulate the Laurent operators, each $Y_k(s)$ or each $Z_k(s)$ transforms into a function $\tilde{Y}_k(s, \omega)$ or $\tilde{Z}_k(s, \omega)$ respectively, where $\omega \in S^n$. Moreover,

$$(10.1) \quad \tilde{Z}_1^\infty(s, \omega) = \frac{1}{\tilde{Y}_1(s, \omega) +} \; \frac{1}{\tilde{Z}_2(s, \omega) +} \; \frac{1}{\tilde{Y}_3(s, \omega) +} \; \frac{1}{\tilde{Z}_4(s, \omega) +} \cdots$$

In this case, as $s \to \infty$ in C_+,

$$(10.2) \qquad \tilde{Y}_k(s, \omega) \sim c_k s \, , \quad k = 1, 3, 5, \ldots$$

and

$$(10.3) \qquad \tilde{Z}_k(s, \omega) \sim b_k \, , \quad k = 2, 4, 6, \ldots$$

where c_k and b_k are positive constants, and these asymptoticities are uniform with respect to all ω. (In fact, $\tilde{Z}(s, \omega)$ is independent of ω since $Z_k(s) = B_k(s)1$.) It now follows from (10.1) that $\tilde{Z}_1^\infty(s. \omega)$ is asymptotic to $(c_1 s)^{-1}$ uniformly for all ω.

Similarly, under $\mathcal{F}$, $\theta_k(s)$ transforms into $\tilde{\theta}_k(s, \omega)$. For $k = 0, 2, 4, \ldots$,

$$\tilde{\theta}_k(s, \omega) = 1 - \tilde{Y}_{k+1}(s, \omega)\tilde{Z}_{k+1}^\infty(s, \omega)$$

$$= \frac{\tilde{Y}_{k+2}^\infty(s, \omega)}{\tilde{Y}_{k+1}(s, \omega) + \tilde{Y}_{k+2}^\infty(s, \omega)}$$

where $\tilde{Y}_{k+2}^\infty(s, \omega)$ is asymptotic to a constant uniformly for all ω. So, by (10.2), $\tilde{\theta}_k(s, \omega)$ is asymptotic to a constant divided by s uniformly for all ω. Similarly, for $k = 1, 3, 5, \ldots$,

$$\tilde{\theta}_k(s, \omega) = 1 - \tilde{Z}_{k+1}(s, \omega)\tilde{Y}_{k+1}^\infty(s, \omega)$$

$$= \frac{\tilde{Z}_{k+2}^\infty(s, \omega)}{\tilde{Z}_{k+1}(s, \omega) + \tilde{Z}_{k+2}^\infty(s, \omega)}$$

where, as with $\tilde{Z}_1^\infty(s, \omega)$, $\tilde{Z}_{k+2}^\infty(s, \omega)$ is asymptotic to $(c_k s)^{-1}$ uniformly for all ω. So, by (10.3), $\tilde{\theta}_k(s, \omega)$ is again asymptotic to a constant divided by s uniformly for all ω.

Now, assume in addition that the Laplace transform $H_p(s)$ of every current generator is of order $O(|s|^{-j})$ where j is an integer greater than one. Also, assume that only a finite number of the $H_p(s)$ are not identically equal to zero.

Set

$$\tilde{H}(s, \omega) = \sum_p H_p(s)\, e^{i(p,\omega)}.$$

Then, for k = 2, 4, 6, ... ,

$$\tilde{I}_{k-2}(s, \omega) = \tilde{\theta}_{k-2}(s, \omega)\, \tilde{\theta}_{k-4}(s, \omega) \cdots \tilde{\theta}_0(s, \omega)\, \tilde{H}(s, \omega)$$

is of order $O(|s|^{-j-k/2})$ uniformly for all ω so that every current flowing between box k-1 and box k has continuous derivatives up to the $(j + k/2 - 2)$th derivative. (This follows from [10; Lemma 3.6-1 and differentiation under the integral sign.) Similarly,

$$\tilde{V}_1(s, \omega) = \tilde{Z}_1^{\infty}(s, \omega)\tilde{H}(s, \omega)\ , \quad k = 1$$

or

$$\tilde{V}_k(s, \omega) = \tilde{\theta}_{k-2}(s, \omega)\, \tilde{\theta}_{k-4}(s, \omega) \cdots \tilde{\theta}_1(s, \omega)\tilde{Z}_1^{\infty}(s, \omega)\check{H}(s, \omega),$$

$$k = 3, 5, 7, \ldots$$

is of order $O(|s|^{-j-k/2-1/2})$ so that every node voltage in box k has continuous derivatives up to the $(j + k/2 - 3/2)$th derivative.

Thus, these transients are quite smooth, and we may apply Papoulis' method as indicated in the first paragraph of this section.

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