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A DISTRIBUTIONAL K TRANSFORMATION WITH APPLICATIONS TO TIME-VARYING ELECTRICAL NETWORKS

by

A. H. Zemanian

Contract No. AF 19(628) - 2981
Project No. 5628
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Scientific Report No. 8

SEPTEMBER 27, 1965

Air Force Cambridge Research Laboratories
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Abstract

This report describes another result in a continuing effort to extend the various classical integral transformations to distributions. Herein, we generalize the K transformation of order μ $(-\frac{1}{2} \leq Re\,\mu \leq \frac{1}{2})$. The procedure is to first construct a testing function space $\mathcal{K}_{\mu,a}$ that is closed under certain Bessel-type differentiation operators and contains the function $\sqrt{st}\, K_\mu(st)$ $(Re\, s > a > 0,\ 0 < t < \infty)$, where K_μ is the modified Bessel function of third kind and order μ. The dual $\mathcal{K}'_{\mu,a}$ of $\mathcal{K}_{\mu,a}$ consists of those distributions that can be transformed by our method. For $f \in \mathcal{K}'_{\mu,a}$, the transform $F(s)$ of f is defined by $F(s) = \langle f(t), \sqrt{st}\, K_\mu(st)\rangle$ $(Re\, s > a)$.

For this generalized transformation we establish an analyticity theorem, two inversion formulas, a uniqueness theorem, and a continuity theorem. Two applications to the analysis of certain time-varying electrical networks are given at the end of the report.

LIST OF PREVIOUS PUBLICATIONS PRODUCED UNDER CONTRACT AF19 (629)-2981

SCIENTIFIC REPORTS:

1. A. H. Zemanian, "A Time-Domain Characterization of Rational Positive-Real Matrices." First Scientific Report, AFCRL-63-390, College of Engineering Tech. Rep. 12, State University of New York at Stony Brook; August 5, 1963.

2. A. H. Zemanian, "A Time-Domain Characterization of Positive Real Matrices." Second Scientific Report, AFCRL-63-391, College of Engineering Tech. Rep. 13, State University of New York at Stony Brook; August 16, 1963.

3. A. H. Zemanian, "The Time-Domain Synthesis of Distributions." Third Scientific Report, AFCRL-64-191, College of Engineering Tech. Rep. 19, State University of New York at Stony Brook; February 1, 1964.

4. A. H. Zemanian, "The Distributional Laplace and Mellin Transformations." Fourth Scientific Report, AFCRL-64-685, College of Engineering Tech. Rep. 26, State University of New York at Stony Brook; August 15, 1964.

5. A. H. Zemanian, "Orthonormal Series Expansion of Certain Distributions and Distributional Transform Calculus." Fifth Scientific Report, AFCRL-64-995, College of Engineering Tech. Rep. 22, State University of New York at Stony Brook; November 15, 1964.

6. T. Laughlin, "A Table of Distributional Mellin Transforms." AFCRL-65-645, College of Engineering Tech. Rep. 40, State University of New York at Stony Brook; June 15, 1965.

7. A. H. Zemanian, "The Distributional Hankel Transformation" Scientific Report No. 7, AFCRL-65-646, College of Engineering Tech. Rep. 44, State University of New York at Stony Brook; September 27, 1965.

FINAL REPORT:

A. H. Zemanian, "Application of Generalized Function Theory to Network Realizability Theory and Time Domain Synthesis of Positive-Real Functions," AFCRL-65-316, College of Engineering Tech. Rep. 41, State University of New York at Stony Brook; May 31, 1965.

PAPERS:

1. A. H. Zemanian, "The Time-Domain Synthesis of Distributions." Proceedings of the First Allerton Conference on Circuit Theory, University of Illinois; 1963.

2. A. H. Zemanian, "The Approximation of Distributions by the Impulse Responses of RLC Two-ports." Proceedings of the International Conference of Microwaves, Circuit Theory, and Information Theory; Tokyo; September, 1964.

3. A. H. Zemanian, "The Time-Domain Synthesis of Distributions." IEEE Transactions on Circuit Theory, Vol. CT-11, pp. 487-493; December, 1964.

4. A. H. Zemanian, "A Characterization of the Inverse LaPlace Transforms of Rational Positive-Real Functions." J. Soc. Indust. Appl. Math., Vol. 13 (June, 1965), pp 463-468.

5. A. H. Zemanian, "Some Convergence Properties of Exponential Series Expansions of Distributions." J. Math. Anal. Appl., accepted for publication.

6. H. Konig and A. H. Zemanian, "Necessary and Sufficient Conditions for a Matrix Distribution to Have a Positive-Real Laplace Transform." J. Soc. Indust. Appl. Math., accepted for publication.

7. A. H. Zemanian, "The Distributional Laplace and Mellin Transformations." J. Soc. Indust. Appl. Math., accepted for publication.

8. A. H. Zemanian, "Inversion Formulas for the Distributional Laplace Transformation." J. Soc. Indust. Appl. Math., accepted for publication.

9. A. H. Zemanian, "Orthonormal Series Expansion of Certain Distributions and Distributional Transform Calculus," J. Math. Anal. Appl., accepted for publication.

10. A. H. Zemanian, "A Distributional Hankel Transformation," J. Soc. Indust. Appl. Math.

11. A. H. Zemanian, "The Hankel Transformation of Certain Distributions of Rapid Growth," J. Soc. Indust. Appl. Math.

1. Introduction.

This report describes another result in a continuing effort to extend the various classical integral transformations to distributions [1]-[5]. Here we generalize the K transformation of order μ $(-\frac{1}{2} \leq Re\,\mu \leq \frac{1}{2})$; this transformation for ordinary functions was first investigated by Meijer [6] and subsequently by Boas [7], [8], and Erdelyi [9].

Let z and μ be complex variables. As is customary, $K_\mu(z)$ denotes the modified Bessel function of third kind and order μ [10; p. 207]. If $f(t)$ is a suitably restricted function defined on $0 < t < \infty$, then its K transform of order μ is a function $F(s)$ of the complex variable $s = \sigma + iw$, defined on some half-plane $Re\ s > \sigma_f \geq 0$ by

$$F(s) = \int_0^\infty f(t) \sqrt{st}\ K_\mu(st)\, dt \qquad (1)$$

One of Meijer's results is the following inversion theorem for (1) [6; theorem 3]. We shall subsequently make use of it to get an inversion formula for our distributional transformation.

Meijer's theorem: Let $F(s)$ be an analytic function on the half-plane $Re\ s > a \geq 0$. For some real constant $\beta > a$, let the integral

$$\int_{-\infty}^{\infty} |F(\beta + iw)|\, dw$$

converge. Moreover, assume that $F(s)$ is bounded for Re $s \geq \beta$ and that $F(\sigma + iw) \to 0$ as $\sigma \to \infty$ uniformly for $-\infty < w < \infty$. Finally, assume that $-\frac{1}{2} \leq Re\,\mu \leq \frac{1}{2}$. Then, (1) holds for Re $s > \beta$ where $f(t)$ is given by

$$f(t) = \frac{1}{i\pi} \int_{\beta - i\infty}^{\beta + i\infty} F(s) \sqrt{st}\ I_\mu (st)\ ds \qquad (2)$$

and $I_\mu (z)$ is the modified Bessel function of first kind and order. [10; p. 207].

Another inversion formula for the ordinary K transformation has been developed by Boas [7]. It is a modification of the Post-Widder inversion formula for the Laplace transformation [11]. This too will be generalized to certain distributions.

Our procedure for generalizing (1) to distributions is a combination of the methods used in [1] and [4]. For each real positive number a and complex parameter μ $(0 \leq Re\,\mu \leq \frac{1}{2})$ we construct a testing function space $\mathcal{K}_{\mu,a}$ of infinitely differentiable functions $\varphi(t)$ on $0 < t < \infty$ which is closed with respect to certain Bessel-type differentiation operators and whose elements tend to zero at least as fast as $e^{-at} t^{\frac{1}{2} - \mu}$ as $t \to \infty$. It turns out that the kernel function $\sqrt{st}\ K_\mu (st)$ is in $\mathcal{K}_{\mu,a}$ for Re $s > a$. Moreover, since $K_\mu (z) = K_{-\mu} (z)$, our requirement that Re $\mu \geq 0$ imposes no restriction on the generality of our results.

The dual space $\mathcal{K}'_{\mu,a}$ of $\mathcal{K}_{\mu,a}$ consists of those distributions to which we may apply our generalized K transformation of order μ. The transform F(s) of $f \in \mathcal{K}'_{\mu,a}$ is simply defined as the application of f to $\sqrt{st}\ K_\mu(s,t)$

$$F(s) = \langle f(t), \sqrt{st}\ K_\mu(st) \rangle \qquad (\mathrm{Re}\ s>a) \quad (3)$$

Among the various properties of this transformation, which we shall develop, are the inversion formulas mentioned above, theorems on analyticity, continuity, and uniqueness, and an operational calculus that is useful when dealing with distributional differential equations containing certain Bessel-type differentiation operators. As an application of this latter result, two types of time-varying electrical networks are analyzed at the end of this work.

A few words about our terminology and notations: By a smooth function we mean a function that possesses ordinary derivatives of all orders at all points of its domain. We shall make considerable use of the following differentiation operators.

$$D = D_t = \frac{d}{dt}$$

$$Q_\mu = t^{-\mu+\frac{1}{2}}\, D\, t^{\mu-\frac{1}{2}}$$

$$R_\mu = t^{\mu-\frac{1}{2}}\, D\, t^{-\mu+\frac{1}{2}}$$

If $\mathcal{O}$ is a smooth function on $0 < t < \infty$, we shall also employ the notation:

$$\mathcal{U}^{[2k,\mu]}(t) = (R_\mu Q_\mu)^k \,\mathcal{U}(t)$$

$$\mathcal{U}^{[2k+1,\mu]}(t) = Q_\mu (R_\mu Q_\mu)^k \,\mathcal{U}(t)$$

$$(k = 0, 1, 2, \dots)$$

From the rule for the differentiation of products we see that $\mathcal{U}^{[n,\mu]}(t)$ has the form

$$\mathcal{U}^{[n,\mu]} = a_{n0} t^{-n} \mathcal{U} + a_{n1} t^{-n+1} D\mathcal{U} + \dots + a_{nn} D^n \mathcal{U}$$

where the a_{nk} are constants depending on the value of μ. This type of computation also shows that

$$\mathcal{U}^{[2,\mu]} = D^2 \mathcal{U} + \frac{1-4\mu^2}{4t^2} \mathcal{U}$$

from which it follows that $\mathcal{U}^{[2k,\mu]} = \mathcal{U}^{[2k,-\mu]}$.

When dealing with multivalued functions f(z) of the complex variable z , it will always be understood that we are restricting f(z) to its principal branch, and z is required to satisfy $-\pi < \arg z \le \pi$.

The notation f(t) for a singular distribution f merely indicates that the testing functions on which f is defined have t as their independent variable. $\langle f, \mathcal{U} \rangle$ denotes the number assigned to some element $\mathcal{U}$ in a certain testing function space by a distribution f in the dual space.

I is the open interval $(0, \infty)$. D_I denotes the space of smooth functions whose supports are compact subsets of I. We assign to D_I the topology that makes its dual D_I' the space of Schwartz distributions on I [12; Vol. I, p. 65]. $\mathcal{E}_I$ and $\mathcal{E}_I'$ are respectively the space of smooth functions on I and the space of distributions having compact supports with respect to I. These spaces have their customary topologies [12; Vol. I, pp. 88-90].

2. The Testing Function Space $\mathcal{K}_{\mu,a}$.

Throughout this work, a will denote a real positive number and μ a complex number satisfying $0 \le Re\,\mu \le \frac{1}{2}$. Let h(t) be a fixed smooth function on $0 < t < \infty$ such that $h(t) \le -1$ for $0 < t < \infty$, $h(t) = \log t$ for $0 < t < \frac{1}{3}$, and $h(t) = -1$ for $1 < t < \infty$. We define the testing function space $\mathcal{K}_{\mu,a}$ as the space of all complex-valued smooth functions $\varphi(t)$ on $0 < t < \infty$ for which all of the following quantities $\gamma_n^{\mu,a}(\varphi)$ (n= o, 1, 2, ...) exist.

$$\gamma_{2k}^{\mu,a}(\varphi) = \sup_{0<t<\infty} \left| e^{at} t^{\mu-\frac{1}{2}} \varphi^{[2k,\mu]}(t) \right| \quad (0 \le Re\,\mu \le \tfrac{1}{2}, \mu \ne 0)$$

$$\gamma_{2k}^{\mu,a}(\varphi) = \sup_{0<t<\infty} \left| \frac{e^{at} \varphi^{[2k,\mu]}(t)}{\sqrt{t}\; h(t)} \right| \quad (\mu = 0)$$

$$\gamma_{2k+1}^{\mu,a}(\varphi) = \sup_{0<t<\infty} \left| e^{at} t^{-\mu+\frac{1}{2}} \varphi^{[2k+1,\mu]}(t) \right| \quad (0 \le Re\,\mu \le \tfrac{1}{2})$$

(k= o, 1, 2, ...)

Note that the case where $\mu = 0$ is a special one; it will require an independent analysis at a number of points in the following development. Also, the precise behavior of $h(t)$ on $\frac{1}{3} < t < 1$ will not affect our results; throughout discussion it is understood that $h(t)$ is a fixed, though unspecified, function on $\frac{1}{3} < t < 1$.

$\mathcal{K}_{\mu,a}$ is a linear space over the field of complex numbers. We assign to it the topology generated by using all $\gamma_n^{\mu,a}$ $(n = 0, 1, 2, \ldots)$ as seminorms. Hence, $\mathcal{K}_{\mu,a}$ is a Hansdorff locally convex topological linear space that satisfies the first countability axiom.

An equivalent topology is generated in $\mathcal{K}_{\mu,a}$ by the following set of norms.

$$\rho_m^{\mu,a}(\varphi) = \max_{0 \le n \le m} \gamma_n^{\mu,a}(\varphi) \qquad (m = 0, 1, 2, \ldots)$$

These seminorms are in concordance [13; p. 5]. Indeed, let $\{\varphi_\nu\}_{\nu=1}^{\infty}$ be a Cauchy sequence with respect to both $\rho_p^{\mu,a}$ and $\rho_q^{\mu,a}$. For definiteness assume that $p < q$. If $\rho_q^{\mu,a}(\varphi_\nu) \to 0$ as $\nu \to \infty$, then obviously $\rho_p^{\mu,a}(\varphi_\nu) \to 0$. Conversely, assume that $\rho_p^{\mu,a}(\varphi_\nu) \to 0$ as $\nu \to \infty$; we wish to show that $\rho_q^{\mu,a}(\varphi_\nu) \to 0$. From the fact that $\rho_0^{\mu,a}(\varphi_\nu) \to 0$ it follows that $\lim_{\nu\to\infty} \varphi_\nu$ is the identically zero function and that the convergence of $\{\varphi_\nu\}$ is uniform on every compact subset Ω of $(0, \infty)$. Moreover, an inductive argument based upon (4) shows that $\{D^n \phi_\nu\}_{\nu=1}^{\infty}$ converges uniformly on each Ω for every n= 1, ..., q. Hence, we may interchange differentiation with the limiting process to conclude that $\lim_{\nu\to\infty} D^n \phi_\nu = D^n \lim_{\nu\to\infty} \phi_\nu = 0$ uniformly on each Ω for every n= 0, 1, ..., q. From (4) and the fact that $\{\phi_\nu\}$ is a Cauchy sequence with respect to $\rho_q^{\mu,a}$ we now infer that $\lim_{\nu\to\infty} \rho_q^{\mu,a}(\phi_\nu) = 0$.

The fact that the $\rho_m^{\mu,a}$ are all in concordance implies that $\mathcal{K}_{\mu,a}$ is a countably normed space [13; p. 6].

We now list some properties of the space $\mathcal{K}_{\mu,a}$.

(i) For Re s>a, $\sqrt{st}\, K_\mu(st) \in \mathcal{K}_{\mu,a}$. Indeed, $\sqrt{st}\, K_\mu(st)$ is analytic for Re s>0 and $0 < t < \infty$ and hence smooth on $0 < t < \infty$. Also, from the series expansion of $\sqrt{st}\, K_\mu(st)$ [10; p. 207] and the asymptotic behavior of $\sqrt{st}\, K_\mu(st)$ as $t \to \infty$ [14; p. 24] is is clear that the quantities $\gamma_n^{\mu,a}[\sqrt{st}\, K_\mu(st)]$ exist for all n= 0, 1, 2, ... (In fact, the seminorms $\gamma_n^{\mu,a}$ were devised to provide precisely this property.)

(ii) Let 0 < a < b. Then, $\mathcal{K}_{\mu,b} \subset \mathcal{K}_{\mu,a}$ and the topology of $\mathcal{K}_{\mu,b}$ is stronger than the topology induced on it by $\mathcal{K}_{\mu,a}$. This follows immediately from the inequality $\gamma_n^{\mu,a}(\phi) \le \gamma_n^{\mu,b}(\phi)$ for $\phi \in \mathcal{K}_{\mu,b}$.

(iii) $D_I \subset \mathcal{K}_{\mu,a}$, and the topology of D_I is stronger than that induced on it by $\mathcal{K}_{\mu,a}$.

(iv) $\mathcal{K}_{\mu,a}$ is an everywhere dense subspace of $\mathcal{E}_I$, and the topology of $\mathcal{K}_{\mu,a}$ is stronger than that induced on it by $\mathcal{E}_I$. Indeed, we obviously have that $\mathcal{D}_I \subset \mathcal{K}_{\mu,a} \subset \mathcal{E}_I$. Since $\mathcal{D}_I$ is everywhere dense in $\mathcal{E}_I$, the same may be said of $\mathcal{K}_{\mu,a}$. Moreover, an inductive argument based upon (4) shows that every neighborhood of 0 induced on $\mathcal{K}_{\mu,a}$ by the topology contains a neighborhood of 0 in the $\mathcal{K}_{\mu,a}$ topology.

(v) $\varphi \to \varphi^{[2,\mu]}$ is a continuous linear mapping of $\mathcal{K}_{\mu,a}$ into $\mathcal{K}_{\mu,a}$. For $\gamma_n^{\mu,a}(\varphi^{[2,\mu]}) = \gamma_{n+2}^{\mu,a}(\varphi)$ (n=0, 1, 2, ...) if $\varphi \in \mathcal{K}_{\mu,a}$.

(vi) $\mathcal{K}_{\mu,a}$ is a sequentially complete space. To see this, we again employ an inductive argument based upon (4) to first conclude that, if $\{\varphi_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{K}_{\mu,a}$, then, for each n, $\{D^n \varphi_\nu\}_{\nu=1}^{\infty}$ converges uniformly on every compact subset of I. Hence, there exists a smooth function φ on I such that $\lim_{\nu\to\infty} D^n\varphi_\nu(t) = D^n\varphi(t)$ at every point of I. Since $\gamma_n^{\mu,a}(\varphi_\nu)$ <B where B is independent of ν, it follows that $\gamma_n^{\mu,a}(\varphi)$ < B. That is $\varphi \in \mathcal{K}_{\mu,a}$.

3. The Distribution Space $\mathcal{K}'_{\mu,a}$.

The dual space $\mathcal{K}'_{\mu,a}$ of $\mathcal{K}_{\mu,a}$ consists of all continuous linear functionals on $\mathcal{K}_{\mu,a}$. We shall employ only the weak topology of $\mathcal{K}'_{\mu,a}$. Under the customary definitions of equality, addition, and multiplication by a complex number, $\mathcal{K}'_{\mu,a}$ is a linear space over the complex-number field. Since $\mathcal{K}_{\mu,a}$ is a sequentially complete countably normed space, $\mathcal{K}'_{\mu,a}$ is also sequentially complete [13; p. 13, theorem 16]. Other properties of $\mathcal{K}'_{\mu,a}$ are the following:

(i) Let $0 < a < b$. The restriction of $f \in \mathcal{K}'_{\mu,a}$ to $\mathcal{K}_{\mu,b}$ is in $\mathcal{K}'_{\mu,b}$. Also, convergence in $\mathcal{K}'_{\mu,a}$ implies convergence in $\mathcal{K}'_{\mu,b}$. These statements are consequences of note (ii) in the preceding section.

(ii) The restriction of $f \in \mathcal{K}'_{\mu,a}$ to $\mathcal{D}_I$ is in $\mathcal{D}'_I$, and convergence in $\mathcal{K}'_{\mu,a}$ implies convergence in $\mathcal{D}'_I$. This follows from note (iii) in the preceding section.

(iii) $\mathcal{E}'_I$ is a subspace of $\mathcal{K}'_{\mu,a}$. The weak topology of $\mathcal{E}'_I$ is stronger than the topology induced on $\mathcal{E}'_I$ by $\mathcal{K}'_{\mu,a}$. These statements follow readily from note (iv) in the preceding section. Note that, because $\mathcal{K}_{\mu,a}$ is everywhere dense in $\mathcal{E}_I$, the continuous functional $f \in \mathcal{E}'_I$ vanishes on $\mathcal{E}_I$ whenever it vanishes on $\mathcal{K}_{\mu,a}$. Hence, we can identify $\mathcal{E}'_I$ with a subspace of $\mathcal{K}'_{\mu,a}$.

(iv) For each $f \in \mathcal{K}'_{\mu,a}$, there exist a nonnegative integer r and a positive constant C such that, for all $\varphi \in \mathcal{K}_{\mu,a}$, $|\langle f,\varphi\rangle| \leq C\rho_r^{\mu,a}(\varphi)$. The proof of this is identical to that of theorem 3.3-1 of [15].

(v) Let a be a positive number and f a locally integrable function on I with the following property: If $\mu \neq 0$, then $e^{-at}t^{-\mu+\frac{1}{2}}f(t)$ is absolutely integrable on $0 < t < \infty$; if $\mu = 0$, then $e^{-at}\sqrt{t}\,h(t)f(t)$ is absolutely integrable on $0 < t < \infty$. Corresponding to the function f, we define on $\mathcal{K}_{\mu,a}$ a regular distribution, which we also denote by f, through the equation

$$\langle f, \varphi\rangle = \int_0^\infty f(t)\,\varphi(t)\,dt \qquad (\varphi \in \mathcal{K}_{\mu,a}).$$

That the distribution f is truly in $\mathcal{K}'_{\mu,a}$ follows from the inequality:

$$|\langle f, \varphi\rangle| \le \begin{cases} \gamma_0^{\mu,a}(\varphi)\int_0^\infty |e^{-at}\,t^{-\mu+\frac{1}{2}}f(t)|\,dt & (\mu \neq 0) \\ \gamma_0^{0,a}(\varphi)\int_0^\infty |e^{-at}\sqrt{t}\,h(t)f(t)|\,dt & (\mu = 0) \end{cases}$$

(iv) We define the self-adjoint operator $R_\mu Q_\mu$ on $\mathcal{K}'_{\mu,a}$ by

$$\langle f^{[2,\mu]}, \varphi\rangle = \langle f, \varphi^{[2,\mu]}\rangle \qquad (f \in \mathcal{K}'_{\mu,a},\ \varphi \in \mathcal{K}_{\mu,a}).$$

It readily follows from note (v) of the preceding section that $f \to f^{[2,\mu]}$ is a continuous linear mapping of $\mathcal{K}'_{\mu,a}$ into $\mathcal{K}'_{\mu,a}$. Also, note that $f^{[2,\mu]} = f^{[2,-\mu]}$.

Let us give an example of the <u>distributional</u> operation $R_\mu Q_\mu$ by computing $R_\mu Q_\mu \sqrt{t}\,K_\mu(t)$ where $0 < \operatorname{Re}\mu \le \frac{1}{2}$. (Note that $\sqrt{t}\,K_\mu(t) \in \mathcal{K}'_{\mu,a}$ for every $a > 0$.) For $\varphi \in \mathcal{K}_{\mu,a}$,

$$\langle R_\mu Q_\mu \sqrt{t}\,K_\mu(t), \varphi(t)\rangle = \langle \sqrt{t}\,K_\mu(t), R_\mu Q_\mu\,\varphi(t)\rangle$$

$$= \int_0^\infty t^\mu K_\mu(t)\,[D t^{-2\mu+1} D t^{\mu-\frac{1}{2}}\,\varphi(t)]\,dt$$

Integrating by parts we get

$$t^{-\mu+1} K_\mu(t)\, D t^{\mu-\frac{1}{2}} \varphi(t) \Big|_0^\infty + \int_0^\infty t^{-\mu+1} K_{\mu-1}(t)\, D t^{\mu-\frac{1}{2}} \varphi(t)\, dt$$

The upper limit term is clearly equal to zero. The lower limit term exists by virtue of the fact that both of the above integrals exist as their lower limits approach zero through positive values. We denote this lower limit term by

$$A = \lim_{t \to 0+} t^{-\mu+1} K_\mu(t)\, D t^{\mu-\frac{1}{2}} \varphi(t) = \lim_{t \to 0+} t^{-2\mu+1} D\, t^{\mu-\frac{1}{2}} \varphi(t)$$

It is readily seen that A defines a distribution f_A in $\mathcal{K}'_{\mu,a}$.

Integrating by parts again, we get

$$A + \sqrt{t}\, K_{\mu-1}(t)\, \varphi(t) \Big|_0^\infty + \int_0^\infty \sqrt{t}\, K_\mu(t)\, \varphi(t)\, dt$$

Once again the upper limit term is zero. The lower limit term

$$B = \lim_{t \to 0+} \sqrt{t}\, K_{\mu-1}(t)\, \varphi(t) = \lim_{t \to 0+} t^{\mu-\frac{1}{2}} \varphi(t),$$

which exists for the same reasom as above, defines a distribution f_B in $\mathcal{K}'_{\mu,a}$. Thus, in the sense of distributional differentiation we have shown that

$$R_\mu Q_\mu \sqrt{t}\, K_\mu(t) = f_A + f_B + \sqrt{t}\, K_\mu(t).$$

4. The Distributional K Transformation.

Let f be a member of $\mathcal{K}'_{\mu,a}$ for some μ and a. In view of note (i) of Sec. 3, there exists some real number $\sigma_f \geq 0$ depending upon f such that $f \in \mathcal{K}'_{\mu,a}$ for all $a > \sigma_f$ and $f \notin \mathcal{K}'_{\mu,a}$

for $0 < a < \sigma_f$. Note (i) of Sec. 2 allows us to define the μ order K transform of f as

$$F(s) = R_\mu f = \langle f(t), \sqrt{st}\, K_\mu(st) \rangle \qquad (\mathrm{Re}\, s > \sigma_f) \qquad (5)$$

This distributional transformation will be denoted by R_μ. Also, we shall call (5) an R_μ transform and shall say that f is R_μ -transferable with a region (or half-plane) of definition $\mathrm{Re}\, s > \sigma_f$. The number σ_f will be called the abscissa of definition.

As a simple example, we compute the R_μ transform of the distribution f_B defined at the end of the preceding section.

Our distributional μ -order K transformation contains the ordinary transformation as a special case whenever f is a function that satisfies the requirements of note (v) in Sec. 3 for all $a > \sigma_f$. In this case (5) takes on the form of (1).

We now digress for a moment to establish some inequalities that we shall need.

<u>Lemma 1: Let μ be such that $\mu \neq 0$ and $0 \leq \mathrm{Re}\,\mu \leq \frac{1}{2}$. Then, for any</u> $a > 0$,

$$|e^{at}(st)^{\mu} K_\mu(st)| < A_\mu \qquad (\mathrm{Re}\, s \geq a,\ 0 < t < \infty) \qquad (6)$$

and
$$|e^{at}(\rho t)^{1-\mu} K_{\mu-1}(\rho t)| < B_\mu (1+|\rho|) \quad (Re\,\rho \geq b > a,\ 0 < t < \infty) \tag{7}$$

where A_μ and B_μ are constants with respect to s and t.

Proof: From the power series expansion of $K_\mu(z)$ [10; p. 207], the asymptotic behavior of $K_\mu(z)$ as $z \to \infty$ [14; p. 24], and the fact that $K_\mu(z)$ is analytic everywhere except at z=0 and $z = \infty$, it follows that

$$|z^\mu K_\mu(z)| < C_\mu \qquad (Re\, z > 0)$$

and

$$|z^{1-\mu} K_{\mu-1}(z)| < E_\mu (1+|z|) \quad (Re\, z > 0)$$

where C_μ and E_μ are constants with respect to z. Hence, given any T>0,

$$|e^{at}(st)^\mu K_\mu(st)| \leq C_\mu e^{aT} \quad (Re\,\rho \geq 0,\ 0 < t < T) \tag{8}$$

and

$$|e^{at}(st)^{1-\mu} K_{\mu-1}(st)| \leq e^{aT} E_\mu (1+|\rho|)(1+T) \ (Re\,\rho > 0, 0 < t < T). \tag{9}$$

Moreover, the asymptotic behavior of $K_\mu(z)$ as $z \to \infty$ shows that

$$e^{at}(\rho t)^\mu K_\mu(\rho t) = \sqrt{\frac{\pi}{2}}\, e^{(a-\rho)t} (\rho t)^{\mu-\frac{1}{2}} \left[1 + O\left(\frac{1}{|\rho t|}\right)\right] \tag{10}$$
$$(t \to \infty,\ Re\,\rho > 0)$$

Expressions (8) and (10) establish (6). The asymptotic behavior of $K_\mu(z)$ as $z \to \infty$ also shows that

$$e^{at}(\rho t)^{1-\mu} K_{\mu-1}(\rho t) = \rho \sqrt{\frac{\pi}{2}}\, t e^{(a-\rho)t} (\rho t)^{-\mu-\frac{1}{2}} \left[1 + O\left(\frac{1}{|\rho t|}\right)\right] \tag{11}$$
$$(t \to \infty,\ Re\,\rho > 0)$$

Expressions (9) and (11) establish (7). Q. E. D.

Lemma 2: For any $a > 0$,

$$\left|\frac{e^{at} K_0(\rho t)}{h(t)}\right| < A_0 \qquad (\mathrm{Re}\,\rho \geq a,\ 0 < t < \infty) \tag{12}$$

and

$$\left|e^{at} \rho t K_{-1}(\rho t)\right| < B_0 (1 + |\rho|) \quad (\mathrm{Re}\,\rho \geq b > a,\ 0 < t < \infty) \tag{13}$$

where A_0 and B_0 are constants with respect to s and t.

Proof: The power series expansions [10; p. 207], the asymptotic behavior [14; p. 24], and the analyticity of $K_0(z)$ and $K_1(z)$ show that

$$\left|\frac{K_0(z)}{h(|z|)}\right| < C_0 \qquad (\mathrm{Re}\, z > 0)$$

and

$$\left|z K_{-1}(z)\right| < E_0 (1 + |z|) \qquad (\mathrm{Re}\, z > 0)$$

where C_0 and E_0 are constants with respect to z. Moreover, it follows from the definition of h(t) that

$$\left|\frac{h(|\rho t|)}{h(t)}\right| < B \qquad (\mathrm{Re}\,\rho \geq a,\ 0 < t < \infty).$$

Consequently, for any given $T > 0$,

$$\left|\frac{e^{at} K_0(\rho t)}{h(t)}\right| = \left|\frac{e^{at} K_0(\rho t)}{h(|\rho t|)} \cdot \frac{h(|\rho t|)}{h(t)}\right| < C_0 B e^{at} \tag{14}$$

$$(\mathrm{Re}\,\rho \geq a,\ 0 < t < T)$$

and

$$\left| e^{at} \rho t\, K_{-1}(\rho t) \right| < e^{at} E_0 (1+|\rho|)(1+T) \qquad (15)$$
$$(Re\, \rho > 0,\ 0 < t < T)$$

Furthermore, by the asymptotic behavior of K_0 (z) as $z \to \infty$,

$$\frac{e^{at} K_0(\rho t)}{h(t)} = \sqrt{\frac{\pi}{2\rho t}}\, e^{(a-\rho)t} \left[1 + O\left(\frac{1}{|\rho t|}\right)\right] \quad (t \to \infty,\ Re\,\rho > 0) \qquad (16)$$

Expressions (14) and (16) imply (12). Also, the asymptotic behavior of K_{-1} (z) as $z \to \infty$ yields

$$e^{at} \rho t\, K_{-1}(\rho t) = \rho \sqrt{\frac{\pi}{2}}\, t\, e^{(a-\rho)t} (\rho t)^{-\frac{1}{2}} \left[1 + O\left(\frac{1}{|\rho t|}\right)\right] \qquad (17)$$
$$(t \to \infty,\ Re\,\rho > 0)$$

Expressions (15) and (17) imply (13). Q. E. D.

Theorem 1: Let $F(s) = R_\mu f$ for $Re\ s > \sigma_f$. Then, $F(s)$ is an analytic function for $Re\ s > \sigma_f$ and

$$\frac{dF}{d\rho} = \left\langle f(t), \frac{d}{d\rho} \sqrt{\rho t}\, K_\mu(\rho t) \right\rangle \quad (Re\,\rho > \sigma_f). \qquad (18)$$

Proof: Let s be an arbitrary but fixed point in the region of definition. Choose the real positive numbers a, b, r, and r_1 such that $\sigma_f < a < b = Re\ s - r_1 < Re\ s - r < Re\ s$. Finally, let Δs be a nonzero complex increment such that $|\Delta \rho| < r$ and consider the expression

$$\frac{F(\rho + \Delta\rho) - F(\rho)}{\Delta\rho} - \left\langle f(t), \frac{d}{d\rho} \sqrt{\rho t}\, K_\mu(\rho t) \right\rangle = \qquad (19)$$

$$\langle f(t), \psi_{\Delta\rho}(t) \rangle$$

where

$$\psi_{\Delta\mu}(t) = \sqrt{\mu t + \Delta\mu t}\; K_\mu(\mu t + \Delta\mu t) - \sqrt{\mu t}\; K_\mu(\mu t) - \frac{d}{d\mu}\sqrt{\mu t}\; K_\mu(\mu t).$$

The series expansion and asymptotic behavior of $K_\mu(st)$ shows that $\frac{d}{d\mu}\sqrt{\mu t}\; K_\mu(\mu t)$

is in $K_{\mu, a}$ so that (18) and (19) have a sense.

Now, assume that $\mu \neq 0$ and let C denote the circle with center at s and radius equal to r_1. Since

$$[\sqrt{st}\; K_\mu(st)]^{[2k,\mu]} = s^{2k}\sqrt{st}\; K_\mu(st) \tag{20}$$

and since we may interchange our differentiations with respect to s and t, $\psi_{\Delta\mu}^{[2k,\mu]}(t)$ can be written as a closed integral on C as follows [16; pp. 120-121].

$$\psi_{\Delta\mu}^{[2k,\mu]}(t) = \frac{\Delta\mu}{2\pi i}\int_C \frac{\zeta^{2k}\sqrt{\zeta t}\; K_\mu(\zeta t)}{(\zeta-\mu)^2(\zeta-\mu-\Delta\mu)}\, d\zeta$$

Let A_μ be a bound on $e^{at}(\zeta t)^\mu K_\mu(\zeta t)$ for $0 < t < \infty$ and all $\zeta \in C$ (see lemma 1). Then, setting $\mu = \mu_R + i\mu_x$, we may write

$$\left|e^{at}\, t^{\mu-\frac{1}{2}}\psi_{\Delta\mu}^{[2k,\mu]}(t)\right| \leq \frac{|\Delta\mu|\, A_\mu}{2\pi}\int_C \left|\frac{\zeta^{2k+\frac{1}{2}-\mu}}{(\zeta-\mu)^2(\zeta-\mu-\Delta\mu)}\right| d\zeta$$

$$\leq \frac{|\Delta\mu|\, A_\mu\, r_1\,(|\mu|+r_1)^{2k+\frac{1}{2}-\mu_R}\, e^{|\mu_x|\pi/2}}{r_1^2\,(r_1-r)}$$

This proves that $\gamma_{2k}^{\mu,a}(\psi_{\Delta\mu}) \to 0$ as $|\Delta\mu| \to 0$.

Using the fact that

$$[\sqrt{\mu t}\; K_\mu(\mu t)]^{[2k+1,\mu]} = -\mu^{2k+1}\sqrt{\mu t}\; K_{\mu-1}(\mu t), \tag{21}$$

we can show in a similar fashion that

$$\left| e^{at} t^{-\mu+\frac{1}{2}} \psi_{\Delta\rho}{}^{[2k+1,\mu]}(t) \right| \le \frac{|\Delta\rho|\, B_\mu\, r_1\, (|\rho|+r_1)^{2k+\frac{1}{2}+\mu_R}\, e^{|\mu_x|\pi/2}}{r_1^2\,(r_1-r)}$$

where B_μ is a bound on $e^{at}(\zeta t)^{1-\mu} K_{\mu-1}(\zeta t)$ for $0 < t < \infty$ and all $\zeta \in C$ (see lemma 1). Thus,

$$\gamma_{2k+1}^{\mu,a}(\psi_{\Delta\rho}) \to 0 \quad \text{as } |\Delta\rho| \to 0.$$

Altogether, $\psi_{\Delta\rho}$ converges to zero in $K_{\mu,a}$ as $|\Delta\rho| \to 0$. Consequently, (19) implies (18) when $\mu \neq 0$. The proof for the case where $\mu = 0$ is the same as the above except for some minor modifications and the use of lemma 2 in place of lemma 1.

In the next proof we shall have need of the following useful operation-transform formula.

<u>Lemma 3</u>: <u>If</u> $F(s) = R_\mu f$ <u>for</u> $\operatorname{Re} s > \sigma_f$, <u>then</u> $R_\mu f^{[2k,\mu]} = R_\mu f^{[2k,-\mu]} = s^{2k} F(s)$ <u>for</u> $\operatorname{Re} s > \sigma_f$ <u>and</u> $k = 1, 2, 3, \ldots.$

<u>Proof</u>: This is an immediate consequence of note (iv) of Sec. 3 and equation (20).

We now derive a characterization of the R_μ transforms in terms of their growth as $s \to \infty$.

<u>Theorem 2</u>: <u>A necessary and sufficient condition for a function</u> $F(s)$ <u>to be a</u> R_μ <u>transform is that there be a half-plane</u> $\operatorname{Re} s \geq b > 0$ <u>on which</u> $F(s)$ <u>is analytic and bounded</u>

according to

$$|F(s)| \leqslant P_b(|s|) \tag{22}$$

where $P_b(|s|)$ is a polynomial in $|s|$.

As we shall see in the following proof, b can be any real point in the half-plane of definition. However, $P_b(|s|)$ will depend in general on the choice of b.

Proof: Necessity: Assume that $F(s) = R_\mu f$ for $\operatorname{Re} s > \sigma_f$. Theorem 1 states that $F(s)$ is analytic for $\operatorname{Re} s > \sigma_f$. Choose two real numbers a and b such that $\sigma_f < a < b$. By note (i) of Sec. 2, $\sqrt{st}\, K_\mu(st) \in \mathcal{K}_{\mu,a}$ for $\operatorname{Re} s \geq b$. Also, by note (iv) of Sec. 3, there exists a constant C and an integer r such that

$$|F(s)| \leq C \max_{0\leq n\leq r} \gamma_n^{\mu,a}\left[\sqrt{st}\, K_\mu(st)\right].$$

Moreover, if $\operatorname{Re} s > 0$, we can use (20) and (21) to write

$$\gamma_{2k}^{\mu,a}\left[\sqrt{st}\, K_\mu(st)\right] \leq \begin{cases} |s|^{2k+\frac{1}{2}-\mu_R}\, e^{|\mu_x|\pi/2} \sup\limits_{0<t<\infty} \left| e^{at}(st)^\mu K_\mu(st)\right| & (\mu \neq 0) \\ |s|^{2k+\frac{1}{2}} \sup\limits_{0<t<\infty} \left| \dfrac{e^{at}\, K_0(st)}{h(t)}\right| & (\mu = 0) \end{cases}$$

and

$$\gamma_{2k+1}^{\mu,a}\left[\sqrt{st}\, K_\mu(st)\right] \leq |s|^{2k+\frac{1}{2}+\mu_R}\, e^{|\mu_x|\pi/2} \sup_{0<t<\infty} \left| e^{at}(st)^{1-\mu} K_{\mu-1}(st)\right|.$$

The inequality (22) now follows from lemmas 1 and 2.

<u>Sufficiency</u>: Let m be an even integer such that $m-2$ is no less than the degree of $P_b(|s|)$. Then, $s^{-m}F(s)$ satisfies the hypothesis of Meijer's theorem, which was stated in the introduction. Consequently, for $\operatorname{Re} s > c > b$,

$$s^{-m} F(s) = \int_0^\infty g(t)\sqrt{st}\, K_\mu(st)\, dt \qquad (23)$$

where

$$g(t) = R_{\mu,c}^{-1}\, s^{-m} F(s) = \frac{1}{i\pi}\int_{c-i\infty}^{c+i\infty} s^{-m}F(s)\sqrt{st}\, I_\mu(st)\, ds \qquad (24)$$

From [10; p. 207] and [14; p. 86, eqn. (5)], we see that

$$e^{-ct}\sqrt{st}\, I_\mu(st$$

is a continuous bounded function of (s,t) for all s on the line $s= c+iw$ $(-\infty< w < \infty)$ and for $0 \leq t <\infty$, Hence, the integral when multiplied by e^{-ct}, in (24) converges uniformly for $0 < t <\infty$, and $e^{-ct}g(t)$ is continuous and bounded for $0 < t <$. By note (v) of Sec. 3, $g(t)$ is a continuous function in $\mathcal{K}'_{\mu,d}$ for $d>c$. Hence, (23) is a particular distributional R_μ transform whose region of definition contains the half-plane $\operatorname{Re} s \geq d$. From lemma 3 we get $F(s) = R_\mu\, g^{[m,\mu]}$ for at least $\operatorname{Re} s \geq d$. This completes the proof.

The preceding proof has established the following inversion formula:

Corollary 2a: Let $F(s) = R_\mu f$ for Re $s > \sigma_f$ and let m be an even integer such that $m-2$ is no less than the degree of $P_b(|s|)$, where P_b is a polynomial as specified in theorem 2. Then, $f = (R_\mu Q_\mu)^{m/2}[R_{\mu,c}^{-1} s^{-m} F(s)]$ where the transformation $R_{\mu,c}^{-1}$ is defined in (24) and c is any real number greater than b.

It may be worth emphasizing that the differentiation in $R_\mu Q_\mu$ are the distributional operations defined in note (vi) of Sec. 3.

We are now prepared to state a uniqueness theorem.

Theorem 3: If $F(s) = R_\mu f$ for Re $s > \sigma_f$, if $G(s) = R_\mu g$ for Re $s > \sigma_g$, and if $F(s) = G(s)$ on some half-plane Re $s \geq b > \max(\sigma_f, \sigma_g)$, then $f=g$ in the sense of equality in $\mathcal{K}'_{\mu,b}$.

Proof: Choose $c > b$. Then, by the inversion formula (25),

$$f - g = (R_\mu Q_\mu)^{m/2}\{R_{\mu,c}^{-1} s^{-m}[F(s) - G(s)]\} = 0$$

Q. E. D.

As an example of the use of formula (25), consider the function $F(s) = \sqrt{s}$ (Re $s > 0$). By theorem 2, this is a R_o transform. To compute the corresponding member $f \in \mathcal{K}'_{0,a}$ (for any $a > 0$), choose $m = 4$. Then, by [17; p. 127, eqn, (1)],

$$g(t) = R_{0,c}^{-1} s^{-7/2} = \frac{1}{4} t^{5/2} \qquad (0 < t < \infty).$$

To compute the distributional differentiations in $(R_oQ_o)^2 g$, choose an arbitrary $\varphi \in \mathcal{K}_{0,a}$. Through some integrations by parts and the use of the order conditions on $\varphi(t)$ as $t \to 0+$ and $t \to \infty$ that are implicit in the seminorms $\gamma_n^{\mu,a}$, we obtain

$$\langle (R_oQ_o)^2 g(t), \varphi(t) \rangle = \int_0^\infty g(t) (R_oQ_o)^2 \varphi(t)\, dt$$

$$= \lim_{t \to 0+} tD \frac{\varphi(t)}{\sqrt{t}} = \lim_{t\to 0+} tD \left[\frac{\varphi(t)}{\sqrt{t}\log t} \log t\right]$$

$$= \lim_{t \to 0+} \frac{\varphi(t)}{\sqrt{t}\log t} = \langle f, \varphi \rangle$$

The last equality can be used as the defining expression for $f \in \mathcal{K}'_{0,a}$. A direct computation readily shows that $R_o f = \sqrt{s}$ (Re $s > 0$).

The sequential completeness of $\mathcal{K}'_{\mu,a}$ and note (i) of Sec. 2 immediately yield the following continuity theorem.

<u>Theorem 4</u>: <u>If</u> $\{f_\nu\}_{\nu=1}^\infty$ <u>converges in</u> $\mathcal{K}'_{\mu,a}$ <u>for some</u>

$a \geq 0$ and if $R_\mu f_\nu = F_\nu(s)$, then $R_\mu \lim_{\nu\to\infty} f_\nu = F(s)$ exists for at least $\operatorname{Re} s > a$, and $F_\nu(s) \to F(s)$ pointwise on the half-plane $\operatorname{Re} s > a$.

We conclude this section with a generalization of Boas' extension to the ordinary K transformation [7] of the Post-Widder inversion formula for the Laplace transformation [11; p. 288]. The similarity of the following result with theorem 3 of [3] may be noted.

Theorem 5: Let f be a R_μ-transformable distribution whose support is contained in the interval $T < t < \infty$ $(T>0)$. Let $F(s) = R_\mu f$ $(\operatorname{Re} s > \sigma_f)$. Then, in the sense of convergence in $\mathcal{D}'_I$,

$$f(t) = \lim_{k\to\infty} \sqrt{\frac{2}{\pi}}\, \frac{1}{(2k)!} \left(\frac{2k}{t}\right)^{2k+1} F^{[2k,\mu]}\left(\frac{2k}{t}\right). \tag{26}$$

It is understood here that the distributional differentiations in $F^{[2k,\mu]}$ are with respect to the argument of F.

Proof: Let φ be a member of $\mathcal{D}_I$ with its support contained in the closed interval $[A,B]$ where $0 < A < B < \infty$. By making the change of variable $a = 2k/t$, we may write [15; p. 30]

$$\left\langle \sqrt{\tfrac{2}{\pi}}\,\frac{1}{(2k)!}\left(\frac{2k}{t}\right)^{2k+1} F^{[2k,\mu]}\left(\frac{2k}{t}\right),\ \alpha(t)\right\rangle$$
$$= \left\langle F^{[2k,\mu]}(u),\ \sqrt{\tfrac{2}{\pi}}\,\frac{u^{2k-1}}{(2k-1)!}\,\alpha\left(\frac{2k}{u}\right)\right\rangle \tag{27}$$
$$= \left\langle F(u),\ \theta_k(u)\right\rangle$$

where

$$\theta_k(u) = \sqrt{\tfrac{2}{\pi}}\,\frac{1}{(2k-1)!}\left[u^{2k-1}\,\alpha\left(\frac{2k}{u}\right)\right]^{[2k,\mu]} \tag{28}$$

The differentiations in the right-hand side of (28) are with respect to u. $\theta_k(u)$ is a member of $\mathcal{D}_I$ and its support is contained in the closed interval [2k/B, 2k/A]. Also, F(u) is a smooth function on $\sigma_f < u < \infty$. So, for all sufficiently large k, the right-hand side of (27) has a sense and is equal to

$$\left\langle \theta_k(u)\,,\ \left\langle f(x)\,,\ \sqrt{ux}\ K_\mu\ (ux)\right\rangle\right\rangle \tag{29}$$

Now, we shall show that we may interchange the order of the "inner products" in (29). Let $\lambda(x)$ be a smooth function on $0 < x < \infty$ that is identically equal to 1 on a neighborhood of the support of f and is identically equal to 0 on $0 < x < x.$ for some x_1. Then, letting 1 (u) denote the function that equals 1 everywhere, we may equate (29) to

$$\left\langle 1(u), \left\langle e^{-\sigma x} f(x), \theta_k(u)\lambda(x)\, e^{\sigma x}\sqrt{ux}\, K_\mu(ux)\right\rangle\right\rangle \tag{30}$$

Since as $x \to \infty$

$$e^{\sigma x} \sqrt{ux}\, K_\mu(ux) = O(e^{(\sigma-\mu)x}),$$

it follows that, for all k such that $2k > \sigma B$,

$$\theta_k(u)\, \lambda(x)\, e^{\sigma x} \sqrt{ux}\, K_\mu(ux) \tag{31}$$

is a testing function of rapid descent on the (u,x) plane [15; Sec. 4.2]. It is understood here that (31) is extended to nonpositive values of u and x as the zero function.

Moreover, if a and σ are such that $f \in \mathcal{K}'_{\mu,a}$ and $\sigma \geq a > 0$, then $e^{-\sigma x} f(x)$ is a tempered distribution on $-\infty < x < \infty$ [15; Sec. 4.3]. (Here also, we extend f(x) onto the nonpositive x axis as the zero distribution.) To see this, let $\psi(x) \in \mathcal{S}_x$, where $\mathcal{S}_x$ is the space of testing functions of rapid descent on the x-axis. Consider

$$\langle e^{-\sigma x} f(x), \psi(x) \rangle = \langle f(x), e^{-\sigma x} \lambda(x)\, \psi(x) \rangle$$

Clearly, $\psi \to e^{-\sigma x} \lambda \psi$ is a continuous linear mapping from $\mathcal{S}_x$ into $\mathcal{K}_{\mu,a}$ whenever $\sigma \geq a$. Hence, $f \to e^{-\sigma x} f$ is a mapping from $\mathcal{K}'_{\mu,a}$ into $\mathcal{S}'_x$, the space of tempered distributions on the x axis.

These results allow us to invoke corollary 5.3-2a of [15] and interchange the order of inner products in (30) and therefore in (29). Hence, for all sufficiently large k, (27) is equal to

$$\langle f(x), \lambda(x)\, \rho_k(x) \rangle \tag{32}$$

where

$$\rho_k(x) = \int_{2k/B}^{2k/A} \theta_k(u)\sqrt{ux}\, K_\mu(ux)\, dx \qquad (33)$$

To complete the proof, we need merely show that as $k \to \infty$, $\lambda(x)\rho_k(x) \to \lambda(x)\phi(x)$ in $K_{\mu,a}$ for every $a > 0$. Note that through some integrations by parts and the change of variable $y = ux/2k$, we get

$$\rho_k(x) = \sqrt{\frac{2}{\pi}}\, \frac{(2k)^{2k+1}}{(2k)!} \int_{x/B}^{x/A} y^{2k-1}\sqrt{2ky}\, K_\mu(2ky)\, \phi\left(\frac{x}{y}\right) dy.$$

In a moment we shall show that, for every nonnegative integer n and positive constant a,

$$e^{ax}\left[\rho_k(x) - \phi(x)\right]^{[n,\mu]} \to 0$$

as $k \to \infty$ uniformly on $X < x < \infty$ where $0 < X < T$. In view of (4) this implies that, for every n, $D^n(\rho_k - \phi) \to 0$ as $k \to \infty$ uniformly on every compact subset of $X \leq x < \infty$. It then follows that, as $k \to \infty$, $\lambda\rho_k - \lambda\phi \to 0$ in $K_{\mu,a}$, which is our desired conclusion.

To proceed. Because of the smoothness of ϕ, we may differentiate under the integral sign to get for each n...

$$\rho_k^{[n,\mu]}(x) = \sqrt{\frac{2}{\pi}}\, \frac{(2k)^{2k+1}}{(2k)!} \int_{x/B}^{x/A} y^{2k-n-1}\sqrt{2ky}\, K_\mu(2ky)\, \phi^{[n,\mu]}\left(\frac{x}{y}\right) dy$$

We have proved in a previous paper [3] that as $k \to \infty$

$$e^{ax}\, \frac{(2k)^{2k+1}}{(2k)!} \int_{x/B}^{x/A} y^{2k-n-1} e^{-2ky}\, \phi^{[n,\mu]}\left(\frac{x}{y}\right) dy \to e^{ax}\phi^{[n,\mu]}(x)$$

uniformly on $X < x < \infty$. Thus, our proof will be complete when we show that

$$\mathfrak{Z}_k(x) = e^{ax}\sqrt{\tfrac{2}{\pi}}\,\frac{(2k)^{2k+1}}{(2k)!}\int_{x/B}^{x/A} y^{2k-n-1}e^{-2ky}\left[\sqrt{\tfrac{\pi}{2}} - e^{2ky}\sqrt{2ky}\,K_\mu(2ky)\right]\mathcal{C}^{[n,\mu]}\left(\tfrac{x}{y}\right)dy \to 0$$

uniformly on $X < x < \infty$.

By the asymptotic behavior for K_μ [14; p. 24], for a given $\varepsilon > 0$, there exists a $k_1 > 0$ such that for all $k > k_1$ and $y \geq X/B$

$$\left|\sqrt{\tfrac{\pi}{2}} - e^{2ky}\sqrt{2ky}\,K_\mu(2ky)\right| < \varepsilon \tag{34}$$

Let C_n be a bound on $z^{n+1}\mathcal{C}^{[n,\mu]}(z)$ for $0 < z < \infty$. Then, for all $k > k_1$ and $X \leq x < \infty$

$$|\mathfrak{Z}_k(x)| < \frac{\varepsilon C_n e^{ax}}{x^{n+1}}\sqrt{\tfrac{2}{\pi}}\,\frac{(2k)^{2k+1}}{(2k)!}\int_{x/B}^{\infty} y^{2k}e^{-2ky}\,dy \tag{35}$$

Since

$$0 < \frac{(2k)^{2k+1}}{(2k)!}\int_{x/B}^{\infty} y^{2k}e^{-2ky}\,dy < 1,$$

it follows that $\mathfrak{Z}_k(x) \to 0$ uniformly on every compact subset of $X \leq x < \infty$ as $k \to \infty$. Moreover, we have proved in [3] that the right-hand side of (35) converges uniformly to zero for $x > 3B/2$ as $k \to \infty$. Q. E. D.

5. An Operational Calculus.

Let $P(x)$ be any polynomial and consider the differential equation

$$P(R_\mu Q_\mu) \quad u = g$$

when g is a R_μ-transformable distribution and the differentiations are in the distributional sense in accordance with note (vi) of Sec. 3. We wish to find a solution u to (36). Applying our R_μ transformation and lemma 3, we obtain $R_\mu u = G(s)/P(s^2)$ where $G(s) = R_\mu g$ for $\operatorname{Re} s > \sigma_g$. Theorem 2 indicates that $G(s)/P(s^2)$ is truly a R_μ transform. Let σ_I be the largest of the real parts of all the roots of $P(s^2)$. Then, u can be computed by choosing the constants b and c ($c > b > \max(\sigma_I, \sigma_g)$) and applying the inversion formula (25). By theorem 3 there is no other distribution in $\mathcal{K}'_{\mu,b}$ that satisfies (36).

The extension of this technique to simultaneous differential equations of the form (36) is straightforward.

Let us compare our R_μ operational calculus with the h_μ operational calculus discussed in [4]. The R_μ operational calculus solves the same type of differential equation as does the h_μ operational calculus. However, the allowable solutions in the former case may be of exponential growth as $t \to \infty$ whereas they must be at most of slow growth in the latter cases. Hence, so far as the behavior as $t \to \infty$

is concerend, the $\mathfrak{R}_\mu$ transformation extends the $\mathfrak{h}_\mu$ transformation is much the same way as does the Laplace transformation extend the Fourier transformation.

On the other hand, the situation concerning the behavior of the allowable solutions as $t \to 0+$ is somewhat different as is seen from the following facts. The $\mathfrak{h}_\mu$ transformation is defined for all real values of the order parameter μ in the range $-\frac{1}{2} \leq \mu < \infty$. Also, the testing functions for this case behave as does the function $\sqrt{t}\, J_\mu(t)$ when $t \to 0+$; here, J_μ is the Bessel function of first kind and order μ. In contrast to this, the R_μ transformation is defined for those real or complex values of μ that satisfy $-\frac{1}{2} \leq Re\,\mu \leq \frac{1}{2}$. (We have used the fact that $R_\mu = R_{-\mu}$ to restrict our above analysis to the cases where $0 \leq Re\,\mu \leq \frac{1}{2}$.) Moreover, the testing functions in the present work behave as does $\sqrt{t}\, K_\mu(t)$ when $t \to 0+$.

Finally, let us note that for the $\mathfrak{h}_\mu$ operational calculus the solutions of (36) is no longer unique in the distribution space $\mathcal{H}'_\mu$ when $P(x)$ has roots on the axis $-\infty \leq x \leq 0$. In the present work the solution of (36) is unique in $\mathcal{K}'_{\mu,b}$ for every $b > \max(\sigma_I, \sigma_g)$ no matter where the roots of $P(x)$ are.

6. Applications to Certain Time-varying Electrical Networks Under Distributional Excitations.

Consider the time-varying electrical network shown in Fig. 1 for the time interval $0 \leq t < \infty$. The symbols a, b, and c denote real nonzero constants, and μ is a real parameter restricted to the range $-\frac{1}{2} \leq \mu \leq \frac{1}{2}$. Gerardi [18] has analyzed networks of this type using the ordinary Hankel transformation in the case where the exciting sources are described by ordinary functions of time t. The distributional K transformation that has been developed in this work allows one to analyze such networks when the sources are described by certain distributions. We use the circuit of Fig. 1 to illustrate the procedure, even though any network consisting of inductances and capacitances, whose time-variations are proportional to $t^{-2\mu+1}$ and $t^{2\mu-1}$ respectively, can be analyzed in the same way.

Assume the network is initially at rest. Let q_1 and q_2 be the mesh charges; that is, q_1 and q_2 are distributions such that $Dq_1 = i_1$ and $Dq_2 = i_2$ where i_1 and i_2 are the indicated mesh currents. Applying a mesh analysis we obtain the simultaneous differential equations

$$v = D\, L_1(t)\, Dq_1 + \frac{q_1 - q_2}{C(t)}$$

$$0 = \frac{q_2 - q_1}{C(t)} + DL_2(t)\, Dq_2$$

Making the change of variable,

$$u_1 = t^{-\mu-\frac{1}{2}} q_1 \, , \quad u_2 = t^{-\mu+\frac{1}{2}} q_2 \, ,$$

and inserting the time variations for $L_1(t)$, $L_2(t)$, and $C(t)$, we convert these equations into

$$t^{\mu-\frac{1}{2}} v(t) = a t^{\mu-\frac{1}{2}} D t^{-2\mu+1} D t^{\mu-\frac{1}{2}} u_1 + c(u_1 - u_2)$$

$$0 = c(u_2 - u_1) + b t^{\mu-\frac{1}{2}} D t^{-2\mu+1} D t^{\mu-\frac{1}{2}} u_2$$

The application of our distributional R_μ transformation and lemma 3 yields the following algebraic equations whenever the source voltage is such that $t^{\mu-\frac{1}{2}} v(t)$ is an R_μ -transformable distribution.

$$\begin{aligned} G(s) &= (as^2 + c)\, U_1(s) - cU_2(s) \\ 0 &= -cU_1(s) + (bs^2 + c)\, U_2(s) \end{aligned} \qquad (37)$$

Here,

$$G(s) = R_\mu t^{\mu-\frac{1}{2}} v(t) \qquad (\mathrm{Re}\, s > \sigma_g)$$

$$U_1(s) = R_\mu u_1(t)$$

$$U_2(s) = R_\mu u_2(t)$$

(Note that the cases where $-\frac{1}{2} \le \mu < 0$ can also be transformed in this way since $R_\mu = R_{-\mu}$.) Solving (37), we get

$$U_1(s) = \frac{bs^2 + c}{abs^4 + c(a+b)\, s^2}\, G(s) \qquad (\mathrm{Re}\ s > \sigma_1)$$

$$U_2(s) = \frac{c}{abs^4 + c(a+b)s^2}\, G(s) \qquad (\mathrm{Re}\ s > \sigma_2)$$

Here, σ_1 and σ_2 are no longer then max (σ_g, σ_d) where σ_d denotes the largest of the real parts of the roots of $abs^4 + c(a+b)\, s^2$. The application of the inversion formula (25) yields u_1 and u_2, which in turn determine q_1, q_2, i_1 and i_2.

As a final application consider the circuit of Fig. 2 for the time interval $0 \leq t < \infty$. This circuit has been analyzed for ordinary-function voltages v(t) by Aseltine [19]. The inductance L and capacitance C have fixed positive values, but the resistance R(t) varies according to R/t where R is a fixed value restricted to the range $0 \leq R \leq 2L$. Assume the circuit is initially at rest. In terms of the change q on the capacitor, the differential equation for this circuit is

$$D^2 q + \frac{R}{L\,t} Dq + \frac{q}{LC} = \frac{v(t)}{L}$$

or

$$t^{-1} Dt^{-2\mu+1} Dt^{2\mu} q + \frac{q}{LC} = \frac{v(t)}{L}$$

where $2\mu + 1 = R/L$. Hence, we have $-\frac{1}{2} \leq \mu \leq \frac{1}{2}$. Using the change of variable $u = t^{\mu+\frac{1}{2}} q$, this becomes

$$t^{\mu-\frac{1}{2}} Dt^{-2\mu+1} Dt^{\mu-\frac{1}{2}} u + \frac{u}{LC} = \frac{1}{L} t^{\mu+\frac{1}{2}} v(t).$$

If $t^{\mu+\frac{1}{2}}$ v(t) is an $\mathcal{R}_\mu$ -transformable distribution, we may apply $\mathcal{R}_\mu$ and lemma 3 to get

$$U(p) = \frac{G(p)}{p^2 + \frac{1}{LC}} \qquad (\operatorname{Re} p > \sigma_1)$$

where

$$G(s) = \mathcal{R}_\mu \frac{1}{L} t^{\mu+\frac{1}{2}} v(t) \qquad (\mathrm{Re}\, p > \sigma_g)$$

and

$$U(s) = \mathcal{R}_\mu u(t).$$

In this case σ_1 is no greater than max (σ_g, 0). The application of the inversion formula (25) to U(s) yields u(t) and thereby q(t) and i(t) = Dq(t).

References

1. A. H. Zemanian, "The Distributional Laplace and Mellin Transformations," J. Soc. Indust. Appl. Math.,

2. A. H. Zemanian, "Orthonormal Series Expansions of Certain Distributions and Distributional Transform Calculus," J. Math. Anal. Appl.

3. A. H. Zemanian, "Inversion Formulas for the Distributional Laplace Transformation," J. Soc. Indust. Appl. Math.

4. A. H. Zemanian, "A Distributional Hankel Transformation," J. Soc. Indust. Appl. Math.

5. A. H. Zemanian, "The Hankel Transformation of Certain Distributions of Rapid Growth," J. Soc. Indust. Appl. Math.

6. C. S. Meijer, "Ueber eine Erweiterung der Laplace- Transformation," Proc. Amsterdam Akad. Wet., 43 (1940) 599-608, 702-711.

7. R. P. Boas, Jr., "Inversion of a Generalized Laplace Integral," Proc. Nat. Acad. of Sciences, U. S. A., 28 (1942) 21-24.

8. R. P. Boas, Jr., "Generalized Laplace Integrals," Bull. Amer. Math. Soc., 48 (1942) 286-294.

9. A. Erdelyi, "On Some Functional Transformation," Rend. Sem. Mat. Univ. Torino, 10 (1950-1951) 217-234.

10. E. Jahnke, F. Emde, and F. Losch, "Tables of Higher Functions," Sixth Edition, McGraw-Hill Book Co., New York, 1960.

11. D. V. Widder, "The Laplace Transform," Princeton University Press, Princeton, N. J., 1946.

12. L. Schwartz, "Theorie des Distributions," Vols. I and II, Hermann, Paris, 1957 and 1959.

13. A. Friedman, "Generalized Functions and Partial Differential Equations," Prentice-Hall, Englewood Cliffs, N. J., 1963.

14. A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, "Higher Transcendental Functions," Vol. II, McGraw-Hill Book Co., New York, 1953.

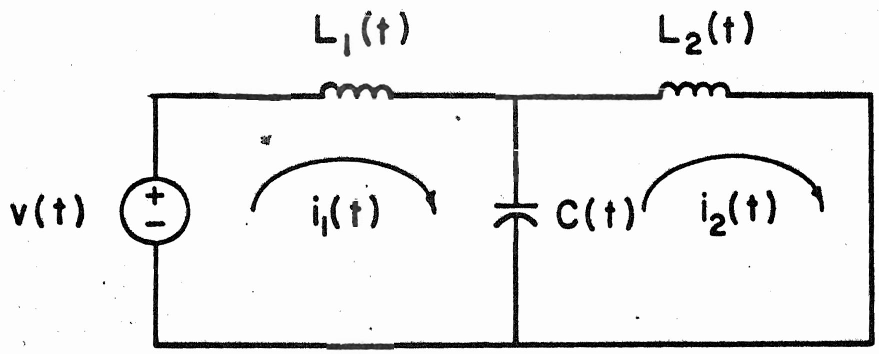
15. A. H. Zemanian, "Distribution Theory and Transform Analysis," McGraw-Hill Book Co., New York, 1965.

16. R. V. Churchill, "Complex Variables and Applications," McGraw-Hill Book Co., New York, 1965.

17. A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, "Tables of Integral Transforms," Vol.II, McGraw-Hill Book Co., New York, 1954.

18. F. R. Gerardi, "Application of Mellin and Hankel Transforms to Networks with Time-varying Parameters," IRE Trans. on Circuit Theory, CT-6 (1959), 197-208.

19. J. A. Aseltine, "A Transform Method for Linear Time-varying Systems" J. Appl. Phys. 25 (1954) 761-764.



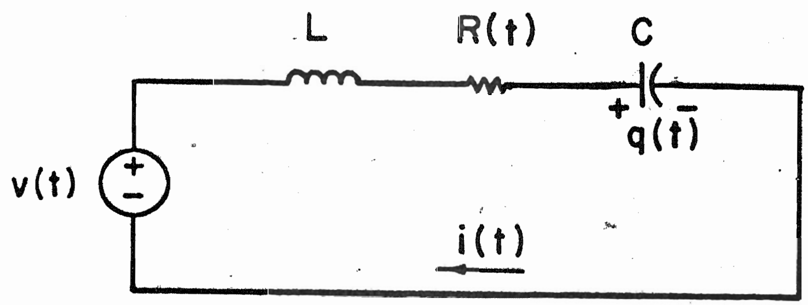
$L_1(t) = a\ t^{-2\mu+1}$ henrys

$L_2(t) = b\ t^{-2\mu+1}$ henrys

$C(t) = c^{-1}\ t^{2\mu-1}$ farads

$-\frac{1}{2} \leq \mu \leq \frac{1}{2}$

Fig. 1



$$R(t) = \frac{R}{t} \text{ ohms}$$

Fig. 2

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13. ABSTRACT

This report describes another result in a continuing effort to extend the various classical integral transformations to distributions. Herein, we generalize the K transformation of order μ $(-\frac{1}{2} \leq \mathrm{Re}\,\mu \leq \frac{1}{2})$. The procedure is to first construct a testing function space $\mathcal{K}_{\mu,a}$ that is closed under certain Bessel-type differentiation operators and contains the function $\sqrt{st}\, K_\mu(st)$ $(\mathrm{Re}\, s > a > 0,\ 0 < t < \infty)$, where K_μ is the modified Bessel function of third kind and order μ. The dual $\mathcal{K}'_{\mu,a}$ of $\mathcal{K}_{\mu,a}$ consists of those distributions that can be transformed by our method. For $f \in \mathcal{K}'_{\mu,a}$, the transform $F(s)$ of f is defined by $F(s) = \langle f(t), \sqrt{st}\, K_\mu(st) \rangle$ $(\mathrm{Re}\, s > a)$.

For this generalized transformation we establish an analyticity theorem, two inversion formulas, a uniqueness theorem, and a continuity theorem. Two applications to the analysis of certain time-varying electrical networks are given at the end of the report.

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