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TRANSFINITE RANDOM WALKS  
BASED ON ELECTRICAL NETWORKS: II

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## PREFACE

This report encompasses a prior research report\*, which examined transfinite random walks on 1-networks, the first rank of transfiniteness for networks. In this work, transfinite random walks on  $\nu$ -networks, where  $\nu$  is any natural number, are also examined.

\*A.H.Zemanian, *Transfinite Random Walks Based on Electrical Networks*, SUNY - Stony Brook, CEAS Report 601, April 7, 1991.

# TRANSFINITE RANDOM WALKS BASED ON ELECTRICAL NETWORKS

A. H. Zemanian

*Abstract* — The idea of a transfinite graph was recently established [A.H.Zemanian, Transfinite Graphs and Electrical Networks, *Trans. Amer. Math. Soc.*, in press]. This is a graph that extends “beyond infinity” in a manner roughly analogous to the extension of the natural numbers to the transfinite ordinals. As a result, there is the possibility of random walks on a transfinite network, walks that may pass beyond infinity through “ $\nu$ -nodes” that represent connections at infinite extremities of the transfinite network. This concept is first explored for a certain kind of transfinite network of the first rank (i.e., for a certain kind of 1-network) having at most a finite number of 1-nodes. Those random walks on 1-networks that succeed in passing through 1-nodes without consecutively reencountering any 1-node are “roving 1-walks”. It is shown that random roving 1-walks comprise an irreducible reversible Markov chain whose state space is the finite set of 1-nodes. A finite electrical network is synthesized whose random walks in the ordinary sense correspond to random roving 1-walks. These ideas are then extended recursively to random roving walks on certain transfinite networks of higher ranks  $\nu$ , where  $\nu$  is any natural number. The definitions and proofs of this paper are based upon the electrical analogue for random walks.

## 1 Introduction

So far, the theory of random walks on connected, countably infinite graphs has been restricted to walks on graphs of the “usual kind”, usual in the sense that between any two nodes there is a finite path. An undoubtedly incomplete list of references on this subject is

[1] - [7], [12] - [23].

How about random walks on the recently devised idea of a transfinite graph [25]? Conceptually, such a graph is constructed by connecting together infinite graphs of the usual kind at their infinite extremities. The result is called a “1-graph” to distinguish it from the usual infinite graph, which is now called a “0-graph”. A 1-graph may have pairs of nodes that are connected only through “paths” that pass through infinite extremities. As a special case, a 1-graph may also be defined simply by specifying some “connections at infinity” for a single 0-graph. When specifying these connections, we are in fact distinguishing between different infinite extremities of a 0-graph and may therefore ask such questions as: “What is the probability that a random walk starting at some node may reach ‘one part of infinity’ before it reaches ‘another part of infinity’?” “What is the probability of it then ‘passing through that part of infinity’ and reaching a node transfinitely far away before it returns to the starting node?”

Answering these and other such question is the objective of this paper. Moreover, we do so for a hierarchy of transfinite graphs obtained by connecting together an infinity of 1-graphs at their extremities to obtain a “2-graph”, then doing the same with 2-graphs to get a “3-graph, and so forth. In this way, we recursively generate “ $\nu$ -graphs”, where  $\nu$  may be any natural number.

Let us sketch out how we extend random walks to transfinite graphs. Infinite extremities of a 0-graph, where connections to other 0-graphs may be made, are called “1-nodes”, and ordinary nodes are now called “0-nodes”. A positive number, called a “conductance”, is assigned to each branch, rendering the graph into a transfinite electrical network. (For this reason, we shall always say “network” instead of “graph”.) For infinite networks of the usual kind, transition probabilities between ordinary nodes are certain 0-node voltages, and they can be determined by connecting pure voltage sources to the boundaries of finite subnetworks [15]. To obtain transition probabilities from 0-nodes to 1-nodes, we expand the finite subnetworks indefinitely and take limits. To obtain the transition probabilities from 1-nodes to 0-nodes, we interchange the connections of voltage sources and determine certain 0-node voltages. The consistency of these definitions is verified. Finally, the transition

probabilities between 1-nodes are obtained by combining these procedures. To empower all this, we have to extend the maximum principle to the node (both 0-node and 1-node) voltages of a transfinite network. A more severe difficulty arises from the fact that a pure voltage source cannot in general be connected between the 1-nodes of a transfinite network, in contrast to a pure current source [24]. The reason is that the infinity of paths between 1-nodes may prevent some 1-nodes from having different voltages (i.e., may in effect “short” those 1-nodes). To avoid this possibility and other similar problems, we restrict our transfinite networks appropriately, one condition being that there be only a finite number of 1-nodes. In summary, our approach is to define transition probabilities by extending some established formulas for the usual kind of infinite network. The resulting random walks that succeed in passing through 1-nodes are “transient”, and walks that do not return to any 1-node without first passing through a different 1-node are “roving 1-walks”. Such walks comprise an irreducible reversible finite Markov chain. The latter in turn leads to a “surrogate network”, a finite 0-network whose random walks (in the usual sense) correspond to the roving 1-walks on the original transfinite network.

As for random walks on transfinite networks of higher ranks, the existence and properties of such walks can be established recursively. For example, our prior results concerning random walks on 1-networks having only finitely many 1-nodes can be taken as local behavior for a 2-network with only a finite number of 2-nodes. More generally, the local behavior of a  $\nu$ -network can be based upon random walks in a  $(\nu - 1)$ -network so long as  $\nu$  is a successor ordinal. We insure the latter requirement by restricting  $\nu$  to the natural numbers. When  $\nu$  is a limit ordinal, a more complicated construction will be needed.

The definitions and proofs of this paper are based upon the electrical analogue for irreducible reversible Markov chains [4], [12, pages 303-310]. Moreover, the paper is written as a sequel to [25]. We freely use the definitions of that work, which are rather extensive. A repetition of them here does not seem warranted. Please refer to that work for an explication of our terminology.

A particular kind of 0-graph we shall employ is Halin’s finitely chainlike structure [9], [10]. It can be defined as follows: A graph (i.e., a 0-graph)  $M$  is called *finitely chainlike*

or synonymously *m-times chainlike*, where  $m$  is any positive natural number, if it is locally finite and can be partitioned into a sequence of finite subgraphs  $M_p$ :

$$M = \bigcup_{p=0}^{\infty} M_p, \quad (1)$$

where each branch of  $M$  belongs to one and only one  $M_p$  and in addition

$$M_{p-1} \cap M_p = \mathcal{V}_p, \quad p = 1, 2, 3, \dots, \quad (2)$$

where  $\mathcal{V}_p$  is a finite set of nodes satisfying the following three conditions:

- (a) The cardinality  $m$  of each  $\mathcal{V}_p$  does not depend upon  $p$ .
- (b) For every  $p \geq 2$ ,  $\mathcal{V}_p$  shares no nodes in common with  $\bigcup_{q=0}^{p-2} M_q$ .
- (c) In each  $M_p$  ( $p \geq 1$ ) there are  $m$  pairwise disjoint paths from the nodes in  $\mathcal{V}_p$  to the nodes in  $\mathcal{V}_{p+1}$ .

Upon taking the union for all  $p$  of all the disjoint paths of condition (c), we obtain  $m$  disjoint one-ended paths. We call each of them a *spine*, and the set of all of them a *full set of spines*. Neither a spine nor a full set of spines need be unique because the paths of condition (b) may not be unique.

## 2 A Special Kind of 1-Network

Consider a countably infinite, locally finite, connected network of the usual kind without parallel branches or self-loops, each branch of which has a positive number—called a *conductance*—assigned to it. Let  $\Psi$  be a random walker that wanders from node to node in accordance with the following transition probabilities. For any node  $n_0$  with the adjacent nodes  $n_1, \dots, n_K$ , the probability that  $\Psi$  will proceed from  $n_0$  to an adjacent node  $n_k$  in one step is by definition  $g_k / \sum_{l=1}^K g_l$ , where  $g_l$  is the conductance of the branch incident to  $n_0$  and  $n_l$ . As Nash-Williams [15] has pointed out, the resulting random walk can be analyzed by treating the network as an electrical network with a 1-volt voltage source connected between various nodes. For example, the aforementioned transition probability can

be obtained electrically by maintaining  $n_k$  at 1 volt and all the other nodes adjacent to  $n_0$  at 0 volt. The resulting node voltage at  $n_0$  is equal to  $g_k / \sum_{l=1}^K g_l$ .

On the other hand, to obtain a random walk on a 1-network, we need to ascertain how  $\Psi$  can wander through a 1-node. The above formula for determining transition probabilities is unavailable for a 1-node because 1-nodes need not have incident branches. Nonetheless, if the 1-network is sufficiently restricted, probabilities for transitions between any two nodes—whether they be 0-nodes or 1-nodes—can be obtained electrically.

Let  $\mathbf{N}^1$  be a 1-connected 1-network with no infinite 0-nodes, no 1-nodes that embrace 0-nodes, no self-loops, and no parallel branches. By definition of a 1-network,  $\mathbf{N}^1$  has a countable infinity of branches and at least one 1-node. Moreover, since no 0-node is embraced by a 1-node, the 1-connectedness of  $\mathbf{N}^1$  implies that every 0-section has at least one 0-tip.

If  $\mathbf{K}$  is any reduction of  $\mathbf{N}^1$  with respect to any subset of branches, we can identify each 0-tip  $t'$  in  $\mathbf{K}$  with the unique 0-tip  $t$  in  $\mathbf{N}^1$  that contains  $t'$  as a subset, and  $t' \mapsto t$  is an injection. We say that  $\mathbf{K}$  *has* or *possesses*  $t$  as a 0-tip if there is at least one representative of  $t$  that lies entirely in  $\mathbf{K}$ . In this sense, every 0-tip of  $\mathbf{K}$  is a 0-tip of  $\mathbf{N}^1$ .

Let  $\mathcal{B}_f$  be any finite set of branches in  $\mathbf{N}^1$ , and let  $\mathbf{N}_f^1 = \mathbf{N}^1 \setminus \mathcal{B}_f$  denote the reduction of  $\mathbf{N}^1$  induced by all branches of  $\mathbf{N}^1$  that are not in  $\mathcal{B}_f$ . Since the removal of  $\mathcal{B}_f$  disrupts at most a finite part of any one-ended path, we have that  $\mathbf{N}^1$  and  $\mathbf{N}_f^1$  possess exactly the same 0-tips.

The idea of an “end” introduced by Halin [8] can also be defined for 1-networks in terms of 0-tips. Two 0-tips of  $\mathbf{N}^1$  will be called *end-equivalent* if, for every choice of  $\mathcal{B}_f$ , the two 0-tips have representatives lying in the same 0-section of  $\mathbf{N}_f^1$ . This is an equivalence relationship, and the corresponding equivalence classes will be called the *ends* (later on, the *0-ends*) of  $\mathbf{N}^1$ . Clearly, the 0-tips in an end belong to a single 0-section of  $\mathbf{N}^1$ ; we say that the end *belongs to* that 0-section. On the other hand, since there are no embraced 0-nodes, every 0-section has at least one end and may have more than one end. As an immediate consequence of all this, we can state

**Lemma 2.1.**  $\mathbf{N}^1$  and  $\mathbf{N}_f^1$  have the same 0-tips and the same ends.

Given an end  $d$  of  $\mathbf{N}^1$ , assume that a particular finite set  $\mathcal{B}_f$  of branches can be so chosen that within  $\mathbf{N}_f^1 = \mathbf{N}^1 \setminus \mathcal{B}_f$  the 0-section  $\mathbf{S}_d$  that possesses  $d$  as an end has no other end. Then,  $\mathbf{S}_d$  will be called a *spur* (later on, a *0-spur*) for the end  $d$ , and  $d$  will be said to *have*  $\mathbf{S}_d$  as a spur. If an end has a spur, it will have an infinity of spurs; indeed, another spur can be obtained by appending to  $\mathcal{B}_f$  any branch of  $\mathbf{S}_d$ .

**Lemma 2.2.** *Assume that  $\mathbf{N}^1$  has only finitely many ends. Then, a finite set  $\mathcal{B}_f$  of branches can be so chosen that every end of  $\mathbf{N}^1$  belongs to a 0-section of  $\mathbf{N}^1 \setminus \mathcal{B}_f$  having no other end (i.e., each such 0-section is a spur for that end) and the spurs are mutually disjoint.*

**Proof.** We shall construct a spur for an arbitrary end  $d$  by reducing the 0-section  $\mathbf{S}$  to which  $d$  belongs. If  $\mathbf{S}$  has only one end, then it is already a spur for  $d$ . So, assume  $\mathbf{S}$  has at least two ends  $d_1$  and  $d_2$ . By the definition of an end, there is a finite set  $\mathcal{B}_f$  of branches such that  $d_1$  and  $d_2$  belong to different 0-sections of  $\mathbf{S} \setminus \mathcal{B}_f$ . Thus for each pair of ends of  $\mathbf{S}$ —say, the  $k$ th pair—such a finite set  $\mathcal{B}_{fk}$  can be chosen. Since  $\mathbf{N}^1$  and therefore  $\mathbf{S}$  possess only finitely many ends, the union  $\cup \mathcal{B}_{fk}$  is again a finite set. Moreover, each end  $d$  of  $\mathbf{S}$  belongs to a 0-section  $\mathbf{S}_d$  of  $\mathbf{S} \setminus \cup \mathcal{B}_{fk}$ , and  $\mathbf{S}_d$  has no other end. Thus,  $\mathbf{S}_d$  is a spur for  $d$ . These spurs are mutually disjoint.  $\square$

We now gather together the assumptions that we impose on the 1-network  $\mathbf{N}^1$ . These assumptions enable a transfinite random walk that visits all the 1-nodes of  $\mathbf{N}^1$ .

**Conditions 2.3.**

- (a)  $\mathbf{N}^1$  is a 1-connected 1-network with no infinite 0-nodes, no self-loops, no parallel branches, and no 1-nodes that embrace 0-nodes.  $\mathbf{N}^1$  has at least two 1-nodes.
- (b)  $\mathbf{N}^1$  has only finitely many ends.
- (c) Each end is embraced by some 1-node (i.e., all the 0-tips in that end are members of a single 1-node).
- (d) Every branch  $b_j$  of  $\mathbf{N}^1$  has assigned to it a positive number  $g_j$  called the branch conductance;  $r_j = g_j^{-1}$  is called the branch resistance. Every branch also has an orientation.



- (e) *Every end has a spur that is finitely chainlike and possesses a full set of spines, each of which is perceptible (i.e., the sum of the resistances of all branches in each spine is finite).*

The kind of network we are considering is partially illustrated in Figure 1, which shows a 0-section, and in Figure 2, which shows two adjacent 0-sections. The crosshatched areas represent regions where branches occur, and the heavy lines represent 1-nodes  $n_k^1$ . Each protrusion of the crosshatched areas can be a spur, and where it touches a heavy line is where an end exists. The heavy dots (other than  $n_0$  in Figure 1) represent the nodes of certain sets " $\mathcal{V}_{k,p_k}$ " for the finitely chainlike structures of some spurs.

**Lemma 2.4.** *Under Conditions 2.3, the following statements hold.*

- (i) *Every spur has the properties indicated in Conditions 2.3(e).*
- (ii) *Every one-ended 0-path lies within a spur.*
- (iii) *Between every two nodes (0-nodes or 1-nodes) of  $\mathbb{N}^1$  there is a perceptible 1-path that terminates at those nodes.*
- (iv)  *$\mathbb{N}^1$  has only finitely many 0-sections and 1-nodes.*

**Proof.** (i) Given any spur  $S$ , let  $d$  be the end for that spur. Let  $S'$  be the spur for  $d$  indicated in Conditions 2.3(e).  $S$  and  $S'$  differ by no more than a finite set of branches. Hence,  $S$  can be assigned a finitely chainlike structure with perceptible spines just by choosing its initial finite subnetwork  $M_0$  appropriately.

(ii) Choose a  $\mathcal{B}_f$  in accordance with Lemma 2.2 to obtain one spur for each end. Since each spur is finitely chainlike, any one-ended 0-path  $P^0$  can enter and leave a spur only a finite number of times. Since the chosen  $\mathcal{B}_f$  produces only finitely many spurs (Condition 2.3(b)),  $P^0$  must eventually remain within a single spur. But then we can expand that spur, if necessary, by appending a finite number of branches to obtain a spur that contains all of  $P^0$ .

(iii) If  $n_1$  and  $n_2$  are two 0-nodes lying in the same 0-section of  $\mathbb{N}^1$ , they are connected by a finite 0-path  $P^0$ . But then,  $\{n_1, P^0, n_2\}$  is the asserted 1-path. So, assume that

$n_1$  and  $n_2$  are either 0-nodes or 1-nodes that are infinitely distant from each other. The 1-connectedness of  $\mathbf{N}^1$  implies that there is a finite 1-path

$$P^1 = \{n_1, P_1^0, x_2^1, P_2^0, \dots, x_m^1, P_m^0, n_2\} \quad (3)$$

connecting  $n_1$  and  $n_2$ . Since there are no embraced nodes, each 0-path  $P_k^0$  ( $k = 2, \dots, m-1$ ) is endless; on the other hand,  $P_1^0$  and  $P_m^0$  can be either endless or one-ended, the latter occurring when  $n_1$  or  $n_2$  is a 0-node. Therefore, every 0-path in (3) is either a one-ended 0-path or the union of two one-ended 0-paths. But, every 0-path lies in a spur according to (ii). By virtue of (i), we can replace each of these one-ended 0-paths  $P^0$  by one that eventually follows a perceptible spine in order to reach the same end that  $P^0$  reaches. These replacements yield a new 1-path that is perceptible and terminates at  $n_1$  and  $n_2$ .

(iv) Since every 0-section has at least one end and there are only finitely many ends (Condition 2.3(b)), there are only finitely many 0-sections. Finally, all the 1-nodes comprise a partition of all 0-tips, and so too do all the ends. That there are only finitely many 1-nodes now follows from Condition 2.3(c).  $\square$

Let us now define what we mean by “shorting nodes together”. To *short* a finite number of 0-nodes of  $\mathbf{N}^1$  means the following: Replace those 0-nodes by a single 0-node and take a branch to be incident to the new 0-node if and only if that branch is incident to one of the original 0-nodes; then remove any branch that becomes a self-loop, and combine parallel branches by adding their conductances. As for 1-nodes, first note that  $\mathbf{N}^1$  has only a finite number of them according to Lemma 2.4(iv). To *short* any number of 1-nodes of  $\mathbf{N}^1$ , just take their union, that is, those 1-nodes are replaced by a single 1-node consisting of all the 0-tips in the original 1-nodes. As with the original 1-nodes, the new 1-node will not embrace a 0-node. Finally, to *short* a finite collection of both 1-nodes and 0-nodes, create a new 1-node consisting of all the 0-tips in those 1-nodes and also consisting of the 0-node obtained by shorting the original 0-nodes; incident branches are defined as before, self loops are removed, and parallel branches are combined. In this last case, the resulting 1-node will embrace a 0-node.

If  $\mathbf{N}$  satisfies Conditions 2.3, it continues to do so after finitely many of its 1-nodes are shorted. The same is true after finitely many of its 0-nodes are shorted.

### 3 Excitations at 1-Nodes

A pure current source can always be connected between two nodes, whether they be 0-nodes or 1-nodes, so long as a perceptible path exists between those nodes [24]. This is so for our 1-network  $\mathbf{N}^1$  according to Lemma 2.4(iii). On the other hand, a sufficient condition for the connection of a pure voltage source to two nodes is that one of them be a finite 0-node [24]. However, a pure voltage source cannot in general be connected to two 1-nodes. The reason is that the 1-network, having possibly an infinity of conductive paths between the 1-nodes, may in effect act as a short between the 1-nodes and may thereby prevent their voltages from being different. However, for 1-networks that satisfy Conditions 2.3, this problem does not arise. To establish this fact, we shall show that the resistance matrix relating any pure current sources applied at the 1-nodes to the resulting voltages between the 1-nodes is nonsingular. Hence, any set of voltage differences between the 1-nodes is possible and can be obtained by imposing them as pure voltage sources.

If a voltage source  $e$  (current source  $h$ ) is connected to nodes  $n_a$  and  $n_b$ , we shall say that the source is connected *from*  $n_a$  *to*  $n_b$  if  $e$  is measured as a potential rise from  $n_a$  to  $n_b$  (respectively, if  $h$  is measured as a current directed from  $n_a$  to  $n_b$ ). Append to  $\mathbf{N}^1$  a pure voltage source  $e_0$  by connecting it from node  $n_a$  to node  $n_b$  of  $\mathbf{N}^1$ . This entails the shorting of nodes. Denote the resulting network by  $\mathbf{N}_e^1$ . At this point, we shall also require that at least one of the nodes  $n_a$  and  $n_b$  be a 0-node. Later on, this condition will be relaxed: Both  $n_a$  and  $n_b$  will be allowed to be 1-nodes.

Let us recall the fundamental theorem [25, Theorem 10.2] for voltage-current regimes in a form suitable for  $\mathbf{N}_e^1$ . (That this theorem continues to hold with the appended pure voltage source  $e_0$  is shown by transferring the source through its incident 0-node [24, Sections VII and XII].) Let  $b_0$  denote the branch for  $e_0$ ; orient  $b_0$  from  $n_a$  to  $n_b$ . Also, let  $b_1, b_2, \dots$  be all the other branches—those of  $\mathbf{N}^1$ . As before,  $r_j$  denotes the branch resistance of  $b_j$ ; thus,  $r_0 = 0$  and  $r_j = g_j^{-1} > 0$ . If  $\mathbf{i} = (i_0, i_1, i_2, \dots)$  is a branch-current vector for  $\mathbf{N}_e^1$ , we let  $\mathbf{i}' = (i_1, i_2, \dots)$  be the corresponding branch-current vector for  $\mathbf{N}^1$ . Under Kirchhoff's current law applied to one end of  $e_0$ , a knowledge of  $\mathbf{i}'$  uniquely determines  $i_0$  and thereby  $\mathbf{i}$ .

$\mathcal{I}$  is the Hilbert space of all branch-current vectors  $\mathbf{i} = (i_0, i_1, i_2, \dots)$  for  $\mathbf{N}_e^1$  such that  $\sum_{j=1}^{\infty} i_j^2 r_j < \infty$  and with  $i_0$  determined from  $i_1, i_2, \dots$  as stated. The inner product for  $\mathcal{I}$  is  $(\mathbf{i}, \mathbf{s}) = \sum_{j=1}^{\infty} r_j i_j s_j$ . A 0-loop current (or 1-loop current) in  $\mathbf{N}_e^1$  is a current flow of constant value around a 0-loop (respectively, 1-loop) in  $\mathbf{N}_e^1$  with zero branch currents outside the loop. A 1-loop current is called *proper* if it is not a 0-loop current. Under Conditions 2.3, a 1-basic current is a countable superposition of proper 1-loop currents such that no more than a finite number of the 1-loop currents flow through any 0-node.  $\mathcal{K}^0$  is the span of all 0-loop currents and 1-basic currents in  $\mathcal{I}$ , and  $\mathcal{K}$  is the closure of  $\mathcal{K}^0$  in  $\mathcal{I}$ .  $\mathcal{K}$  too is a Hilbert space.

Here is the desired version of the fundamental theorem.

**Theorem 3.1.** *Let a single pure voltage source  $e_0$  be connected to two nodes of  $\mathbf{N}^1$ , at least one of which is a 0-node. Then, there is a unique current vector  $\mathbf{i} \in \mathcal{K}$  for  $\mathbf{N}_e^1$  such that, for every  $\mathbf{s} \in \mathcal{K}$ ,*

$$e_0 s_0 = \sum_{j=1}^{\infty} r_j i_j s_j \quad (4)$$

Now, consider any 1-node  $n^1$  in  $\mathbf{N}_e^1$ —possibly a 1-node that embraces a 0-node of the source branch  $b_0$ . Choose a spur for every end embraced by  $n^1$ . By Lemma 2.4(i), every such spur has a finitely chainlike structure. Since there are only finitely many ends (Condition 2.3(b)), we can take the union of those spurs to obtain another chainlike structure  $\mathbf{M} = \bigcup_{p=0}^{\infty} \mathbf{M}_p$  with the 0-node sets  $\mathcal{V}_p = \mathbf{M}_{p-1} \cap \mathbf{M}_p$ . We assign any branch having both of its nodes in  $\mathcal{V}_p$  to  $\mathbf{M}_p$ . Every branch of  $\mathbf{M}_{p-1}$  ( $p > 2$ ) having one node in  $\mathcal{V}_p$  and one node not in  $\mathcal{V}_p$  will be called a *cut-branch at  $\mathcal{V}_p$* . The set  $\mathcal{C}$  of them will be called a *cut for  $n^1$  at  $\mathcal{V}_p$* . We say that  $\mathcal{C}$  *isolates  $n^1$*  within  $\mathbf{N}^1$  from all other 1-nodes.  $\mathcal{C}$  is a finite set of branches. Every 1-path that embraces  $n^1$  and a node of  $\mathbf{M}_0$  must embrace at least one cut-branch at  $\mathcal{V}_p$ . Thus, within  $\mathbf{M}$  the removal of a cut for  $n^1$  disconnects  $n^1$  from  $\mathbf{M}_0$ . A branch of  $\mathcal{C}$  is said to be *oriented away from (toward)  $n^1$*  if it is oriented away from (toward)  $\mathcal{V}_p$ .

An extension of Kirchhoff's current law to the cut  $\mathcal{C}$ —along with  $b_0$  if  $b_0$  is incident to  $n^1$ —asserts that

$$\sum \pm i_j = 0 \quad (5)$$

where the summation is for the branches in  $\mathcal{C}$  and for  $b_0$  as well if  $b_0$  is incident to  $n^1$ ,  $i_j$  is

the current in those branches, and the plus (minus) sign is used if the branch's orientation is away from (toward)  $n^1$ .

**Lemma 3.2.** *Kirchhoff's current law (5) holds whenever  $\mathbf{i} \in \mathcal{K}$ .*

**Proof.** Any 0-loop or 1-loop can embrace branches of  $\mathcal{C}$  at most finitely often. Hence, each 0-loop current or 1-loop current enters the summation of (5) an even number of times, half with plus signs and the other half with minus signs. Hence, its contribution to that summation is zero. The same is true for any 1-basic current because such a current is a countable superposition of 1-loop currents, only finitely many of which flow through the branches of  $\mathcal{C}$  and  $b_0$ . Now,  $\mathcal{K}^0$  is the span of all 0-loop and 1-basic currents. Thus, for any  $\mathbf{i} \in \mathcal{K}^0$ , (5) holds. Finally, we can choose a sequence from  $\mathcal{K}^0$  convergent toward  $\mathbf{i} \in \mathcal{K}$  and can argue as in the proof of [25, Theorem 11.1] to conclude the proof.  $\square$

Now, let  $n_k^1$  ( $k = 1, \dots, K$ ) denote the 1-nodes of  $\mathbf{N}^1$ . Any pure current source can be connected between any two 1-nodes because those 1-nodes are connected by a perceptible 1-path (Lemma 2.4(iii)). Furthermore, any set of pure current sources connected between the 1-nodes can be represented by a set of  $K - 1$  current sources connected from  $n_1^1$  to  $n_2^1, \dots, n_K^1$ . Let  $h_k$  be the value of the current source from  $n_1^1$  to  $n_k^1$ , and set  $\mathbf{h} = (h_2, \dots, h_K)$ . This creates a unique voltage-current regime in  $\mathbf{N}^1$  in accordance with [24]. It follows from Lemma 2.4(iii) that every  $n_k^1$  possesses a unique node voltage  $u_k^1$  with respect to  $n_1^1$  [25, Section 14]. ( $n_1^1$  is assigned the zero node voltage). Set  $\mathbf{u} = (u_2^1, \dots, u_K^1)$ . Thus,  $\mathbf{N}^1$  as seen from its 1-nodes is a  $(K - 1)$ -port with a common ground  $n_1^1$  and has a  $(K - 1) \times (K - 1)$  resistance matrix  $Z$ . In short,  $\mathbf{u} = Z\mathbf{h}$ . If  $Z$  is invertible, then  $\mathbf{h} = Z^{-1}\mathbf{u}$ , and this signifies that any set of  $K - 1$  pure voltage sources can be connected from  $n_1^1$  to  $n_k^1$  ( $k = 2, \dots, K$ ) to get an  $\mathbf{h}$ . It is the connection of pure voltage sources to the 1-nodes that will determine the transition probabilities for  $\Psi$  wandering between 1-nodes.

**Lemma 3.3.**  *$Z$  is positive-definite and therefore nonsingular.*

**Proof.** Let  $\mathbf{h} = (h_2, \dots, h_K)$  be an arbitrary vector of  $K - 1$  current sources connected as above. For the  $k$ th 1-node  $n_k^1$ , let  $\mathcal{C}$  be a cut for  $n_k^1$  as above. According to Lemma 3.2, the net current flowing through the branches of  $\mathcal{C}$  away from  $n_k^1$  is  $h_k$ . Therefore, there exists at least one branch of  $\mathcal{C}$  carrying a current no less than  $h_k/c$ , where  $c$  is the

cardinality of the finite set  $\mathcal{C}$ . The power dissipated in that branch is no less than  $\omega_k h_k^2$ , where  $\omega_k = r_{min}/c^2 > 0$  and  $r_{min}$  is the smallest of the resistances for the branches of  $\mathcal{C}$ .

Now, let  $(\cdot, \cdot)$  denote the inner product for  $(K-1)$ -dimensional Euclidean space. Then,  $(\mathbf{u}, \mathbf{h}) = (Z\mathbf{h}, \mathbf{h})$  is both the power generated by the applied current sources and the power dissipated within  $\mathbf{N}^1$ . The latter can be seen by transferring the pure current sources into  $\mathbf{N}^1$  along perceptible 1-paths and then invoking [25, Corollary 10.3 and Theorem 13.2]. So,

$$(Z\mathbf{h}, \mathbf{h}) \geq \sum_{k=2}^K \omega_k h_k^2 \geq \omega \|\mathbf{h}\|^2$$

where  $\omega = \min(\omega_2, \dots, \omega_K) > 0$ . Thus,  $Z$  is positive definite.  $\square$

Finally, we wish to extend Theorem 3.1 to the 1-network  $\mathbf{N}_e^1$  obtained by appending a pure voltage source  $e_0$  to two 1-nodes of  $\mathbf{N}^1$ . We can do this by inserting a positive resistance  $\rho$  in series with  $e_0$ , thereby rendering  $b_0$  into a resistive branch, and then letting  $\rho \rightarrow 0+$ . Let  $\mathbf{N}_\rho^1$  denote the 1-network with the resistive source branch connected to two 1-nodes. For  $\mathbf{N}_\rho^1$  the fundamental theorem reads exactly as before except that the summation in (4) and for the inner product for  $\mathcal{I}$  is over  $j = 0, 1, 2, \dots$ . Thus, (4) is replaced by

$$e_0 s_0 = \rho i_0^\rho s_0 + \sum_{j=1}^{\infty} r_j i_j^\rho s_j \quad (6)$$

where  $\mathbf{i}^\rho$  denotes the unique branch-current vector for  $\mathbf{N}_\rho^1$ .

By Lemma 3.3,  $e_0 - \rho i_0^\rho = z i_0^\rho$ , where  $z$  is the positive resistance of  $\mathbf{N}^1$  as seen from the two 1-nodes to which  $b_0$  is connected. If  $\lambda$  is another value for the resistance in  $b_0$ ,

$$i_0^\rho - i_0^\lambda = \frac{e_0}{\rho + z} - \frac{e_0}{\lambda + z} \rightarrow 0$$

as  $\rho, \lambda \rightarrow 0+$  independently. From (6) we have

$$\sum_{j=1}^{\infty} r_j (i_j^\rho - i_j^\lambda) s_j = (\lambda i_0^\lambda - \rho i_0^\rho) s_0 = z (i_0^\rho - i_0^\lambda) s_0.$$

Upon setting  $s_j = i_j^\rho - i_j^\lambda$  for all  $j$ , we get

$$\|\mathbf{i}^\rho - \mathbf{i}^\lambda\|^2 = \sum_{j=1}^{\infty} r_j (i_j^\rho - i_j^\lambda)^2 = z (i_0^\rho - i_0^\lambda)^2 \rightarrow 0$$

as  $\rho, \lambda \rightarrow 0+$ . Hence,  $\{\mathbf{i}^\rho : \rho > 0\}$  is a Cauchy directed function in  $\mathcal{K}$ . Therefore, there is an  $\mathbf{i} \in \mathcal{K}$  to which the directed function converges in  $\mathcal{K}$ . Upon passing to the limit in (6) and invoking the bicontinuity of the inner product, we get (4) again.

There is no other  $\hat{\mathbf{i}} \in \mathcal{K}$  that satisfies (4) for all  $\mathbf{s} \in \mathcal{K}$ . Indeed, from (4) we have  $\sum_{j=1}^{\infty} r_j(i_j - \hat{i}_j)s_j = e_0s_0 - e_0s_0 = 0$ . By setting  $s_j = i_j - \hat{i}_j$ , we get  $i_j = \hat{i}_j$  for all  $j$ . All this establishes

**Theorem 3.4.** *Theorem 3.1 continues to hold for  $\mathbf{N}_e^1$  even when the pure voltage source is connected to two 1-nodes of  $\mathbf{N}^1$ .*

Henceforth, we take it that voltage-current regimes produced by pure voltage sources are those dictated by Theorems 3.1 and 3.4. Because  $\mathbf{N}^1$  is a linear network, these theorems determine through superposition a unique voltage-current regime when finitely many pure voltage sources are connected to 0-nodes and 1-nodes of  $\mathbf{N}^1$ .

## 4 Node Voltages

We shall eventually argue that node voltages in a 1-network excited by a 1-volt voltage source correspond to certain transition probabilities. For this purpose, we show in this section that node voltages are bounded by 0 and 1 when the negative terminal of the 1-volt source is assigned a node voltage of 0 volt. This is in fact an extension of the maximum principal for node voltages to 1-networks.

In the following, a “0-section” will always mean a 0-section with respect to  $\mathbf{N}^1$ , not with respect to  $\mathbf{N}_e^1$ . Henceforth, we take it that the single source exciting  $\mathbf{N}_e^1$  is a 1-volt source. Also, if  $n$  is a node,  $u$  will denote its node voltage—and  $u$  will have the same subscripts and superscripts as  $n$ . Let  $n_g$  (and  $n_e$ ) denote the node of  $\mathbf{N}_e^1$  to which the negative (respectively, positive) terminal of the 1-volt source is incident. We call  $n_g$  *ground* and set  $u_g = 0$ . Let  $n_0$  be any other node of  $\mathbf{N}_e^1$ , either a 0-node or a 1-node. By Lemma 2.4(iii) there exists at least one perceptible 1-path  $P$  starting at  $n_0$ , ending at  $n_g$ , and remaining within  $\mathbf{N}^1$  (thus avoiding the source branch  $b_0$ ).  $P$  may in fact represent a 0-path if  $n_0$  and  $n_g$  are in the

same 0-section of  $\mathbf{N}^1$ . We assign to  $n_0$  the node voltage:

$$u_0 = \sum_P \pm v_j \quad (7)$$

where  $\sum_P$  denotes a summation over the branch indices for the perceptible 1-path  $P$ , the  $v_j$  are the branch voltages for  $P$  as dictated by Theorem 3.1 or Theorem 3.4, and the plus (minus) sign is chosen if a branch orientation agrees (disagrees) with a tracing of  $P$  from  $n_0$  to  $n_g$ . According to [25, Section 14], the series (7) converges absolutely, and  $u_0$  is independent of the choice of the perceptible 1-path  $P$ .

**Lemma 4.1.** *The node voltages in  $\mathbf{N}_e^1$  along any one-ended 0-path  $P^0$  (whether perceptible or not) converge to the voltage  $u^1$  of the 1-node  $n^1$  that  $P^0$  meets.*

**Proof.** By Lemma 2.4(i) and (ii),  $P^0$  lies within a spur  $\mathbf{M}$ , which is finitely chainlike with perceptible spines. Thus,  $n^1$  is the 1-node that embraces the end of  $\mathbf{M}$ . Since every spine is perceptible, the 0-node voltages along any spine of  $\mathbf{M}$  converge to the 1-node voltage  $u^1$  for  $n^1$ .

Now, let  $\mathbf{M}_p$  and  $\mathcal{V}_p$  be the finite subnetworks and the 0-node sets of the chainlike structure of  $\mathbf{M}$  (see (1) and (2)). There are only finitely many spines—all disjoint, and the spines contain all the nodes of all the  $\mathcal{V}_p$ . Let  $u_p^{max}$  (or  $u_p^{min}$ ) be the maximum (respectively, minimum) node voltage for the nodes of  $\mathcal{V}_p$ . It follows that  $\lim_{p \rightarrow \infty} u_p^{max} = u^1$  and  $\lim_{p \rightarrow \infty} u_p^{min} = u^1$ .

Given any natural number  $q$ ,  $P^0$  eventually remains within  $\bigcup_{p=q}^{\infty} \mathbf{M}_p$ . Moreover, we can choose  $q$  so large that  $n_g$  and  $n_e$  are excluded from  $\bigcup_{p=q}^{\infty} \mathbf{M}_p$ . Since for  $p \geq q$  each  $\mathbf{M}_p$  is a finite, resistive, sourceless subnetwork, its node voltages all lie between the maximum and minimum of the node voltages for the  $\mathcal{V}_p \cup \mathcal{V}_{p+1}$ . Consequently, the node voltages along  $P^0$  also converge to  $u^1$ .  $\square$

The supposition that there is a node voltage larger than one in  $\mathbf{N}_e^1$  leads to a contradiction. The next lemma, which invokes this hypothesis, is but a step toward obtaining that contradiction. We say that a 1-node and a 0-section are *incident* if the 1-node contains a 0-tip of the 0-section.

**Lemma 4.2.** *Suppose there is a 0-node or 1-node  $n_0$  in  $\mathbf{N}_e^1$  with a voltage larger than one. Let  $\mathbf{S}$  be a 0-section that either contains  $n_0$  or is incident to  $n_0$ . Two and only two*



*possibilities arise:*

- (i) *The 0-node voltages in  $\mathbf{S}$  are all the same and larger than one (and therefore by Lemma 4.1 all the voltages for the 1-nodes incident to  $\mathbf{S}$  have the same value as well).*
- (ii) *All the 0-node voltages in  $\mathbf{S}$  are strictly less than the largest of the voltages for the 1-nodes incident to  $\mathbf{S}$ ; that largest 1-node voltage is also larger than one.*

**Proof.** If there is a 1-node with voltage larger than one, then by Lemma 4.1 there is a 0-node in  $\mathbf{S}$  with voltage larger than one. If (i) does not hold, then at least some of the 0-node voltages for  $\mathbf{S}$  differ. It follows that there are two 0-nodes  $n_0$  and  $n'_0$  in  $\mathbf{S}$  with voltages satisfying  $u_0 > u'_0$  and  $u_0 > 1$ . We may trace some 0-path in  $\mathbf{S}$  from  $n_0$  to  $n'_0$  to find the first 0-node  $n_1$  whose voltage  $u_1$  is no less than  $u_0$  and is strictly larger than that of the next 0-node in the path. By Kirchhoff's current law there is another 0-node  $n_2$  adjacent to  $n_1$  with  $u_2 > u_1$ . By the same law applied to  $n_2$ , there is a third 0-node  $n_3$  adjacent to  $n_2$  with  $u_3 > u_2$ . Continuing in this way, we find a one-ended 0-path  $P^0$  whose successive node voltages are strictly increasing. By Lemma 4.1, the 1-node that contains the 0-tip having  $P^0$  as a representative has a node voltage strictly larger than  $u_0$ . Since all this is true for every  $n_0$  with  $u_0 > 1$ , (ii) follows.  $\square$

Since there are only finitely many 0-sections, we have shown that, under the supposition of Lemma 4.2, there is a 1-node  $n_{max}$  with a voltage  $u_{max}$  that is larger than one and no less than any other 0-node voltage or 1-node voltage in  $\mathbf{N}_e^1$ . Now, we can trace a path from  $n_{max}$  to the positive terminal of the 1-volt source to find a 1-node  $n_a$  with voltage  $u_{max}$  and incident to a 0-section all of whose 0-node voltages are strictly less than  $u_{max}$ .

Now consider all the 0-sections to which  $n_a$  is incident. By Lemma 4.2, there may be some such 0-sections having all their 0-node voltages equal to  $u_{max}$ . As far as the flow of current is concerned, they can be ignored. All the other 0-sections will have 0-node voltages strictly less than  $u_{max}$ . Choose a spur for every end of those latter 0-sections that is embraced by  $n_a$ . By Lemma 2.4(i), every such spur has a finitely chainlike structure. Since there are only finitely many ends (Conditions 2.3(b)), we can take the union of those spurs to obtain another finitely chainlike structure  $\mathbf{M} = \bigcup_{p=0}^{\infty} \mathbf{M}_p$  with the 0-node sets  $\mathcal{V}_p$ .

As before, we assign any branch having both of its nodes in  $\mathcal{V}_p$  to  $\mathbf{M}_p$ . A *cut-branch* at  $\mathcal{V}_p$  is again any branch of  $\mathbf{M}_{p-1}$  ( $p > 1$ ) having one node in  $\mathcal{V}_p$  and one node not in  $\mathcal{V}_p$ .

Since the 0-node voltages along each spine of  $\mathbf{M}$  are strictly less than  $u_{max}$  and converge to  $u_{max}$  and since there are only finitely many spines in  $\mathbf{M}$ , we can choose two sets  $\mathcal{V}_p$  and  $\mathcal{V}_q$  with  $p < q$  such that the least node voltage for  $\mathcal{V}_p$  is larger than one and in addition the largest node voltage for  $\mathcal{V}_p$  is strictly less than the least node voltage for  $\mathcal{V}_q$ . So, consider next the finite subnetwork  $\mathbf{M}_{p,q} = \bigcup_{k=p}^{q-1} \mathbf{M}_k$ .  $\mathbf{M}_{p,q}$  is sourceless. We can generate the same voltage-current regime in  $\mathbf{M}_{p,q}$  as it has as a reduced network of  $\mathbf{N}_e^1$  by connecting pure voltage sources as follows. Let  $n_{p,1}$  be a 0-node of  $\mathcal{V}_p$  with the largest node voltage  $u_{p,1}$  for  $\mathcal{V}_p$ . Let  $n_{p,k}$  be any other 0-node of  $\mathcal{V}_p$  and let  $u_{p,k}$  be its voltage. Connect a pure voltage source of value  $u_{p,1} - u_{p,k}$  from  $n_{p,k}$  to  $n_{p,1}$  (positive terminal at  $n_{p,1}$ ). Do this for all  $n_{p,k}$ . Similarly, connect a pure voltage source from a node  $n_{q,1}$  of  $\mathcal{V}_q$  with the least node voltage  $u_{q,1}$  for  $\mathcal{V}_q$  to each of the other nodes of  $\mathcal{V}_q$  to establish their node voltages at the values they have in  $\mathbf{N}_e^1$ . Finally, connect a pure voltage source  $e_{p,q}$  of value  $u_{q,1} - u_{p,1} > 0$  from  $n_{p,1}$  to  $n_{q,1}$ .  $\mathbf{M}_{p,q}$  with these appended sources is a connected finite network.

Let us now examine the cut-branches for  $\mathcal{V}_q$ ; we orient them away from  $\mathcal{V}_q$ . Set  $i = \sum_q i_j$ , where  $\sum_q$  denotes a summation over the branch indices for those cut-branches. Apply Kirchhoff's current law and superposition. The sum  $i$  will be zero when each appended voltage source is acting alone (all other appended sources set equal to zero) and has both of its nodes in  $\mathcal{V}_p$  or both of its nodes in  $\mathcal{V}_q$ . However, for  $e_{p,q}$  acting alone,  $i$  will be positive. Thus, by superposition, for the voltage-current regime in  $\mathbf{N}_e^1$ , the net current  $i = \sum_q i_j$  in the cut-branches for  $\mathcal{V}_q$  will be positive (i.e., will represent a net flow in those branches away from  $\mathcal{V}_q$ ). This is a result of the supposition of Lemma 4.2.

But, this contradicts Kirchhoff's current law (Lemma 3.2). Indeed, the 1-volt source that excites  $\mathbf{N}_e^1$  is not incident to  $n_a$ . Hence, Lemma 3.2 dictates that  $i = 0$ . Consequently, no node voltage in  $\mathbf{N}_e^1$  can be larger than one.

With just minor modifications, our arguments can be reapplied to show that no node voltage can be less than zero. We have established

**Theorem 4.3.** *Under Conditions 2.3, every 0-node voltage and 1-node voltage in  $\mathbf{N}_e^1$*

is no less than zero and no greater than one.

Let  $\alpha$  be the rank of  $n_e$  and  $\beta$  be the rank of  $n_g$ . In the following,  $n_0$  is another node of rank  $\gamma$ . Each of these ranks are either 0 or 1.

**Corollary 4.4.**

- (i) Let there be a  $\mu$ -path  $P^\mu$ , where  $\mu = \max(\beta, \gamma)$ , that terminates at  $n_0$  and  $n_g$  and does not embrace  $n_e$ . Then,  $u_0 < 1$ .
- (ii) Let there be a  $\mu$ -path  $P^\mu$ , where  $\mu = \max(\alpha, \gamma)$ , that terminates at  $n_0$  and  $n_e$  and does not embrace  $n_g$ . Then,  $u_0 > 0$ .

**Proof.** Under the hypothesis of (i), suppose  $u_0 = 1$ . Exactly two cases arise:

*Case 1.*  $P^\mu$  embraces two adjacent 0-nodes  $n_a$  and  $n_b$  with  $u_a = 1$  and  $u_b < 1$ . But then, by Kirchhoff's current law for 0-nodes, there must be another 0-node  $n_1$  adjacent to  $n_a$  with  $u_1 > 1$ , in violation of Theorem 4.3.

*Case 2.*  $P^\mu$  embraces a 1-node  $n_a$  with  $u_a = 1$ , and  $n_a$  is incident to a 0-section  $S$  that contains at least one 0-node with a voltage less than one. Whether or not  $S$  contains  $n_e$ , the maximum principle for node voltages in a 0-network ensures that every end of  $S$  embraced by  $n_a$  has a spur all of whose 0-node voltages are strictly less than one. By virtue of Lemma 4.1, we can again argue that Kirchhoff's current law (Lemma 3.2) will be violated at a cut that isolates  $n_a$  from all other 1-nodes.

Thus,  $u_0 < 1$ , as asserted by (i). (ii) is established similarly.  $\square$

## 5 Transfinite Walks

We turn now to the idea of a walk on a 1-network that satisfies Conditions 2.3. We wish to define matters in such a fashion that those walks may "pass through infinity" via 1-nodes.

First we define walks on a 0-section. A 0-walk on  $\mathbf{N}^1$  is a conventional kind of walk contained within a 0-section  $\mathbf{S}$  of  $\mathbf{N}^1$ . It is an alternating sequence of 0-nodes  $n_m^0$  and branches  $b_m$ :

$$W^0 = \{\dots, n_m^0, b_m, n_{m+1}^0, b_{m+1}, \dots\} \quad (8)$$

such that each branch is incident to the two nodes adjacent to it in the sequence. 0-nodes and branches may repeat in (6.1). However,  $n_m^0$  and  $n_{m+1}^0$  are different 0-nodes for every  $m$  because  $\mathbf{N}^1$  contains no self-loops; to express this fact, we shall say that every 0-walk *roves*. Since (8) is a sequence, the indices  $m$  are restricted to the integers.  $W^0$  may be finite, one-ended, or endless; in the first two cases each terminal element is a 0-node, and we say that  $W^0$  *starts at* (*stops at*) its terminal 0-node on the left (respectively, right). Moreover,  $W^0$  *reaches* each of its 0-nodes and *passes through* each of its 0-nodes other than its terminal nodes.  $W^0$  is *nontrivial* if it has at least one branch. We say that  $W^0$  *embraces* itself and all its elements.

If  $W^0$  is one-ended or endless, we denote one-ended portions of  $W^0$  by

$$W_{-\infty, m}^0 = \{\dots, b_{m-2}, n_{m-1}^0, b_{m-1}, n_m^0\}$$

and

$$W_{m, \infty}^0 = \{n_m^0, b_m, n_{m+1}^0, b_{m+1}, \dots\}.$$

Let  $d$  be an end of  $\mathbf{S}$ , and let  $\mathbf{M} = \bigcup_{p=0}^{\infty} \mathbf{M}_p$  be a finitely chainlike representation for a spur  $\mathbf{M}$  for  $d$ . We say that  $W^0$  *starts at*  $d$  if, given any integer  $q \geq 0$ , there is an  $m$  such that  $W_{-\infty, m}^0$  remains within  $\bigcup_{p=q}^{\infty} \mathbf{M}_p$ ; we also say that  $W^0$  *starts at* the 1-node that embraces  $d$ . Similarly, we say that  $W^0$  *stops at*  $d$  and also *stops at* the 1-node that embraces  $d$ , if, given any integer  $q \geq 0$ , there is an  $m$  such that  $W_{m, \infty}^0$  remains within  $\bigcup_{p=q}^{\infty} \mathbf{M}_p$ . In both cases, we also say that  $W^0$  *reaches*  $d$  and the said 1-node. Any 0-walk that reaches a 1-node will be called *transient*. This use of the adjective “transient” differs from customary usage; indeed, we are now applying it to a deterministic walk rather than to a random walk.

According to these definitions, a 0-walk may keep expanding within a 0-section  $\mathbf{S}$  without reaching any 1-node (i.e., without being transient). Consider for example a 0-walk  $W^0$  that satisfies the following condition: Given a finitely chainlike structure  $\mathbf{M} = \bigcup_{p=0}^{\infty} \mathbf{M}_p$  for all of  $\mathbf{S}$ , there is a positive integer  $k$  such that, for every choice of the positive integers  $m$  and  $q$  with  $q > k$ , the one-ended portion  $W_{m, \infty}^0$  of  $W^0$  meets both  $\mathbf{M}_k$  and  $\mathbf{M}_q$ —and therefore all the intervening  $\mathbf{M}_{k+1}, \dots, \mathbf{M}_{q-1}$  as well. In other words,  $W^0$  keeps getting into ever-larger portions of  $\mathbf{S}$  but also keeps returning to  $\mathbf{M}_k$ . Thus, no matter how large we choose  $m$ ,

$W_{m,\infty}^0$  will definitely return to  $M_k$  before it reaches any 1-node. We might call such a deterministic walk “recurrent”; this kind of 0-walk will not arise as an embraced 0-walk when we examine transfinite random walks that reach 1-nodes.

Consider now a (finite, one-ended, or endless) alternating sequence of the form

$$W^1 = \{\dots, n_m^1, W_m^0, n_{m+1}^1, W_{m+1}^0, \dots\} \quad (9)$$

where each  $W_m^0$  is a nontrivial 0-walk and each  $n_m^1$  is a 1-node—except possibly when (9) terminates on the left and/or on the right, in which case the terminal element is either a 0-node or a 1-node. (We shall drop the superscript 1 whenever a node is allowed to be of either rank.) Again terms may repeat in (9); in fact, consecutive 1-nodes  $n_m^1$  and  $n_{m+1}^1$  may be the same—in contrast to the situation for (8). As before, since this is a sequence,  $m$  is restricted to the integers.

**Definition 5.1.**  $W^1$ , as given by (9), is called a 1-walk if, for every  $m$ ,  $W_m^0$  starts at  $n_m$  and stops at  $n_{m+1}$ . Also,  $W^1$  is said to perform a *one-step transition* from the 1-node  $n_m^1$  to the 1-node  $n_{m+1}^1$ . Finally,  $W^1$  is said to *rove* if, for every  $m$ ,  $n_m^1$  and  $n_{m+1}^1$  are different 1-nodes.

Since we have not allowed any 1-node to embrace a 0-node, a 0-walk  $W_m^0$  can start from or stop at a 1-node adjacent to  $W_m^0$  in (9) only if it starts or stops at an end embraced by that 1-node. In the event that  $W^1$  is finite or one-ended, we also say that  $W^1$  *starts at* (*stops at*) its terminal element on the left (respectively, right). As with other entities, we say that a 1-walk *embraces* itself, all its elements, and all elements embraced by its elements. Also,  $W^1$  is said to *pass through* each of its embraced branches, 0-nodes, and 1-nodes other than its terminal nodes. Furthermore,  $W^1$  is *nontrivial* if (9) contains at least three elements. By these definitions, every 0-walk in a 1-walk is transient unless it is the last 0-walk and stops at a 0-node; in fact, every 0-walk is also *transient-in-reverse* because it starts at a 1-node—except possibly when it is the first 0-walk.

## 6 Random 0-Walks

We now discuss a random 0-walk, which may reach a 1-node incident to the 0-section  $S$  to which the 0-walk is confined. Our definitions will generalize the customary random walk

on the usual kind of infinite network because we will establish comparative probabilities of reaching different 1-nodes through a limiting process. The random 0-walk may be discussed in terms of an entity  $\Psi$  that wanders through 0-nodes and possibly reaches a 1-node. Terminology that we shall define for random walks is also used for  $\Psi$  as the agent performing that walk.

Let  $n_0$  be a 0-node and let  $n_k$  ( $k = 1, \dots, L$ ) be its adjacent 0-nodes. Let  $g_{0,k} = g_{k,0}$  be the conductance of the branch connecting  $n_0$  and  $n_k$ . The standard rule for the probability  $P_{0,k}$  that  $\Psi$  starting at  $n_0$  will reach  $n_k$  in one step is  $P_{0,k} = g_{0,k} / \sum_{l=1}^L g_{0,l}$ . This probability can also be obtained from node voltages as follows. Let  $n_k$  be held at 1 volt and let all the other  $n_l$  ( $l = 1, \dots, L; l \neq k$ ) be held at 0 volt. Then, the node voltage at  $n_0$  is  $P_{0,k}$  according to Kirchhoff's laws and Ohm's law. This governs the wandering of  $\Psi$  within any 0-section  $\mathbf{S}$  of  $\mathbf{N}^1$ . That wandering is described by a Markov chain whose infinite state space consists of the 0-nodes of  $\mathbf{S}$  [12, Chapter 9, Section 10].

Under Conditions 2.3, it is possible for  $\Psi$  to reach any 1-node  $n_k^1$  incident to the 0-section  $\mathbf{S}$  by following a one-ended 0-walk. We wish to determine the probability of it reaching  $n_k^1$  before it reaches any other 1-node incident to  $\mathbf{S}$ , given that it starts at the 0-node  $n_0$  and does reach some 1-node and given that  $\mathbf{S}$  has two or more incident 1-nodes.

Choose a spur for each end of  $\mathbf{S}$  that is embraced by  $n_k^1$  in such a way that the spurs are mutually disjoint (Lemma 2.2). Then, choose a finitely chainlike representation  $\bigcup_{p=0}^{\infty} \mathbf{M}_{k,p}$  for the union of those spurs. As before,  $\mathcal{V}_{k,p} = \mathbf{M}_{k,p-1} \cap \mathbf{M}_{k,p}$ , but now all the nodes of  $\mathcal{V}_{k,p}$  lie in  $\mathbf{S}$ ;  $\mathcal{V}_{k,p}$  separates  $n_k^1$  from  $\mathbf{M}_{k,0}$  within  $\mathbf{S}$  but not from any part of any other 0-section incident to  $n_k^1$ . Perform this construction of a finitely chainlike representation for the set of ends of  $\mathbf{S}$  embraced by each 1-node  $n_l^1$  ( $l = 1, \dots, K$ ) that is incident to  $\mathbf{S}$ , maintaining disjoint spurs throughout. Then, within  $\mathbf{S}$  and for any choices of the positive integers  $p_1, \dots, p_K$ ,  $\bigcup_{l=1}^K \mathcal{V}_{l,p_l}$  separates all the ends of  $\mathbf{S}$  from a finite subnetwork  $\mathbf{F}(p_1, \dots, p_K)$  of  $\mathbf{S}$  (see Figure 1) that is maximal in the sense that the nodes of  $\bigcup_{l=1}^K \mathcal{V}_{l,p_l}$  are in  $\mathbf{F}(p_1, \dots, p_K)$ . Let us assume that the 0-node  $n_0$  is in  $\mathbf{F}(p_1, \dots, p_K)$  too. As has been shown by Nash-Williams [15, Corollary 4A], the probability of  $\Psi$  reaching any node of  $\mathcal{V}_{k,p_k}$  before it reaches any node of  $\bigcup\{\mathcal{V}_{l,p_l} : l = 1, \dots, K; l \neq k\}$ , given that  $\Psi$  starts at  $n_0$ , is equal to the node

voltage  $v_{0,k}(p_1, \dots, p_K)$  at  $n_0$  (as determined by Kirchhoff's and Ohm's laws) when the nodes of  $\mathcal{V}_{k,p_k}$  are held at 1 volt and the other said nodes are held at 0 volt.

By virtue of Lemma 3.4, another node voltage  $u_{0,k}$  can be obtained at  $n_0$  by holding  $n_k^1$  at 1 volt and all other 1-nodes incident to  $\mathbf{S}$  at 0 volt.

**Lemma 6.1.**  $v_{0,k}(p_1, \dots, p_K)$  converges to  $u_{0,k}$  as the  $p_1, \dots, p_K$  tend to infinity independently.

**Proof.** For each  $l = 1, \dots, K$ , let  $n_{l,p_l,i}$  denote the  $i$ th node of  $\mathcal{V}_{l,p_l}$ , and let  $u_{l,p_l,i}$  denote the corresponding node voltage resulting from 1 volt at  $n_k^1$  and 0 volt at all the other 1-nodes incident to  $\mathbf{S}$ . Then, by superposition,  $v_{0,k}(p_1, \dots, p_K) - u_{0,k}$  is the voltage at  $n_0$  resulting from  $1 - u_{k,p_k,i}$  imposed at the  $i$ th node of  $\mathcal{V}_{k,p_k}$  for every  $i$  and  $-u_{l,p_l,i}$  imposed at the  $i$ th node of  $\mathcal{V}_{l,p_l}$  for every  $i$  and every  $l \neq k$ . Now,  $1 - u_{k,p_k,i}$  and  $u_{l,p_l,i}$  are nonnegative by virtue of Theorem 4.3. By the maximum principal for node voltages in a finite network,

$$-\max_{i;l \neq k} u_{l,p_l,i} \leq v_{0,k}(p_1, \dots, p_K) - u_{0,k} \leq \max_i (1 - u_{k,p_k,i})$$

where the maximum on the right-hand side is taken over all node indices  $i$  for  $\mathcal{V}_{k,p_k}$  and the maximum on the left-hand side is taken over all node indices  $i$  for all  $\mathcal{V}_{l,p_l}$  other than  $\mathcal{V}_{k,p_k}$ . Recall that the  $\mathcal{V}$ 's are finite sets whose cardinalities are constant with respect to the  $p$ 's. By Lemma 4.1,  $\max_i u_{k,p_k,i} \rightarrow 1$  as  $p_k \rightarrow \infty$  and  $\max_{i;l \neq k} u_{l,p_l,i} \rightarrow 0$  as  $p_l \rightarrow \infty$  when  $l \neq k$ .  $\square$

In the following we shall use a notation like

$$\text{Prob}(sn_1, rn_2, bn_3 \mid A)$$

to indicate the probability that  $\Psi$ , after starting from node  $n_1$ , reaches node  $n_2$  before reaching node  $n_3$ , given the restriction  $A$ . The ranks of these nodes may differ. Moreover,  $n_2$  and  $n_3$  may be replaced by sets of nodes.

We take Lemma 6.1 as the basis for the following definition of a comparative transition probability for  $\Psi$  starting at a 0-node  $n_0$  of a 0-section  $\mathbf{S}$  and reaching a 1-node  $n_k^1$  incident to  $\mathbf{S}$  before reaching any of the other 1-nodes  $n_l^1$  ( $l = 1, \dots, L; l \neq k$ ) incident to  $\mathbf{S}$ . We let  $\mathcal{N}_g^1$  denote the set of those other 1-nodes. (Actually, we should also establish that  $\Psi$  truly

can reach any such 1-node; this will be shown shortly by showing that, under Conditions 2.3, every 0-section  $\mathbf{S}$  is “transient”.)

**Definition 6.2.** Let the 0-section  $S$  have at least two incident 1-nodes. Given that  $\Psi$  starts at a 0-node  $n_0$  in  $S$  and reaches some 1-node, the probability that  $\Psi$  will reach  $n_k^1$  before reaching any node of  $\mathcal{N}_g^1$ :

$$Prob(sn_0, rn_k^1, b\mathcal{N}_g^1 \mid \Psi \text{ reaches some 1-node}) \quad (10)$$

is defined to be the node voltage  $u_0$  at  $n_0$  when  $n_k^1$  is held at 1 volt and all the  $n_i^1 \in \mathcal{N}_g^1$  are held at 0 volt.

In short, this definition arises as a limiting case of the aforementioned Nash-Williams result.

Variations of Lemma 6.1 can be established in the same way and lead to definitions of other comparative transitions probabilities. For example, we can compare transitions to 1-nodes with transitions to 0-nodes as in the next definition. In this case, the 0-section  $\mathbf{S}$  may have just one incident 1-node.

**Definition 6.3.** Let  $\mathcal{N}_g^0$  be any finite set of 0-nodes in  $\mathbf{S}$ , let  $n_0$  be another 0-node in  $\mathbf{S}$  with  $n_0 \notin \mathcal{N}_g^0$ , and let  $\mathcal{N}_e^1$  be the set of all 1-nodes incident to  $S$ . Given that  $\Psi$  starts at  $n_0$ , the probability that  $\Psi$  will reach any 1-node incident to  $\mathbf{S}$  before reaching any node of  $\mathcal{N}_g^0$ :

$$Prob(sn_0, r\mathcal{N}_e^1, b\mathcal{N}_g^0) \quad (11)$$

is defined to be the voltage  $u_0$  at  $n_0$  obtained when all the 1-nodes of  $\mathcal{N}_e^1$  are held at 1 volt and all the nodes of  $\mathcal{N}_g^0$  are held at 0 volt.

With this latter definition in hand we can examine the transiency of any 0-section. In particular,  $\mathbf{S}$  is called *transient* if, given that  $\Psi$  starts at any arbitrarily chosen 0-node  $n_g$  in  $\mathbf{S}$ , there is a positive probability that  $\Psi$  will reach a 1-node incident to  $\mathbf{S}$  before returning to  $n_g$ .

**Theorem 6.4.** *Under Conditions 2.3, every 0-section  $\mathbf{S}$  of  $\mathbf{N}^1$  is transient.*

**Proof.** Let  $n_g$  be chosen arbitrarily as a 0-node of  $\mathbf{S}$ .  $\Psi$  starting at  $n_g$  perforce reaches a 0-node adjacent to  $n_g$  in one step. Now, hold all the 1-nodes incident to  $\mathbf{S}$  at 1 volt and



hold  $n_g$  at 0 volt by connecting a 1-volt source from  $n_g$  to a short among all those 1-nodes. Then, by Definition 6.3,  $\mathbf{S}$  will be transient if at least one 0-node adjacent to  $n_g$  has a positive voltage.

Suppose all such adjacent nodes have zero voltages. (By Theorem 4.3, they cannot have negative voltages.) Set  $\mathbf{s} = \mathbf{i}$ , the current regime produced by the 1-volt source in accordance with Theorem 3.1. By Kirchhoff's current law applied at  $n_g$  and Ohm's law,  $s_0 = i_0 = 0$  in (4), and therefore  $\sum_{j=1}^{\infty} r_j i_j^2 = 0$ . Hence,  $i_j = 0$  for all  $j$ . This means that there can be no voltage difference between  $n_g$  and the said 1-nodes—in contradiction to the facts that  $n_g$  is at 0 volt and those 1-nodes are all at 1-volt.  $\square$

Finally, we note that the “recurrent” 0-walk cited in Section 5 has zero probability of reaching any 1-node before returning to any 0-node of the finite subnetwork  $\mathbf{M}_k$  used in that section. This is why such 0-walks do not occur in the random 1-walks that will be discussed in the next section.

## 7 Random 1-Walks

So far, we have examined random 0-walks that may stop at a 1-node. To obtain random 1-walks that pass through 1-nodes and wander in general from one 0-section to another, we have to first define how a random 0-walk may start at a 1-node.

Let  $n^1$  be any 1-node of  $\mathbf{N}^1$ , choose a spur for every end embraced by  $n^1$ —making those spurs mutually disjoint, and then choose a finitely chainlike representation  $\mathbf{M} = \bigcup_{p=0}^{\infty} \mathbf{M}_p$  for the union of those spurs. Within  $\mathbf{M}$ ,  $\mathcal{V}_p$  separates  $n^1$  from  $\mathbf{M}_0$ , and, for  $q > p$ ,  $\mathcal{V}_q$  separates  $n^1$  from  $\mathcal{V}_p$ . Now, however,  $\mathcal{V}_p$  and  $\mathcal{V}_q$  will lie in many 0-sections whenever  $n^1$  is incident to many 0-sections. For the next definition we assume that  $\mathbf{M}$  has two or more spines, and thus  $\mathcal{V}_p$  has two or more nodes. Also,  $\mathcal{V}_p \setminus n_{p,k}$  denotes the set of all nodes in  $\mathcal{V}_p$  other than the node  $n_{p,k}$ .

**Definition 7.1.** Given that  $\Psi$  starts at  $n_0^1$  and reaches a node of  $\mathcal{V}_p$ , the probability:

$$P(n_0^1; n_{p,k}) = \text{Prob}(sn_0^1, rn_{p,k}, b\mathcal{V}_p \setminus n_{p,k} \mid \Psi \text{ reaches } \mathcal{V}_p) \quad (12)$$

that  $\Psi$  reaches the node  $n_{p,k}$  of  $\mathcal{V}_p$  before it reaches any of the other nodes  $n_{p,l}$  ( $l = 1, \dots, m; l \neq k$ ) of  $\mathcal{V}_p$  is defined to be the node voltage  $u^1(p, k)$  at  $n^1$  when  $n_{p,k}$  is held at 1

volt and every  $n_{p,l}$  ( $l \neq k$ ) is held at 0 volt. Similarly, given that  $\Psi$  starts at a node  $n_{q,i}$  of  $\mathcal{V}_q$  ( $q > p$ ) and reaches a node of  $\mathcal{V}_p$ , the probability:

$$P(n_{q,i}; n_{p,k}) = \text{Prob}(sn_{q,i}, rn_{p,k}, b\mathcal{V}_p \setminus n_{p,k} \mid \Psi \text{ reaches } \mathcal{V}_p) \quad (13)$$

that  $\Psi$  reaches  $n_{p,k}$  before reaching any other node  $n_{p,l}$  of  $\mathcal{V}_p$  is defined to be the node voltage  $u_{q,i}(p, k)$  at  $n_{q,i}$  when  $n_{p,k}$  is held at 1 volt and every  $n_{p,l}$  ( $l \neq k$ ) is held at 0 volt.

The second sentence of this definition is needed because  $n_{q,i}$  resides in an infinite network exterior to  $\mathcal{V}_p$ —in contrast to the Nash-Williams result which holds for finite interior networks.

Definition 7.1 assigns comparative probabilities for transitions from  $n^1$  to the nodes of any  $\mathcal{V}_p$ . Since  $\Psi$ , when proceeding from  $n^1$  to a node  $n_{p,k}$  of  $\mathcal{V}_p$ , must first meet at least one node of  $\mathcal{V}_q$ , where  $q > p$ , we should now prove the consistency of our definition in the following sense: The comparative probability for the transitions from  $n^1$  to  $n_{p,k}$  is the same as that obtained by combining the comparative probabilities for transitions from  $n^1$  to the various nodes of  $\mathcal{V}_q$  with the comparative probabilities for transitions from the nodes of  $\mathcal{V}_q$  to  $n_{p,k}$ . More specifically, by conditional probabilities, we should have for  $q > p$

$$P(n^1; n_{p,k}) = P(n_{q,1}; n_{p,k})P(n^1; n_{q,1}) + \cdots + P(n_{q,m}; n_{p,k})P(n^1; n_{q,m}) \quad (14)$$

if Definition 7.1 is to be consistent. This equation can be established electrically by using the conditions in Definition 7.1 as follows.

Let  $\mathbf{1}_k$  denote the vector of  $m$  real numbers with 1 as the  $k$ th entry and 0 for all other entries. Let  $\mathbf{u}_p = (u_{p,1}, \dots, u_{p,m})$  where  $u_{p,k}$  is the node voltage at  $n_{p,k}$ , and similarly for  $\mathbf{u}_q = (u_{q,1}, \dots, u_{q,m})$ . As before, let  $u^1(p, k)$  be the voltage at  $n^1$  when  $\mathbf{u}_p = \mathbf{1}_k$ , and let  $u^1(q, k)$  be the voltage at  $n^1$  when  $\mathbf{u}_q = \mathbf{1}_k$ . Now, let  $\mathbf{u}_q(p, k)$  be in particular the vector of node voltages for  $\mathcal{V}_q$  when  $\mathbf{u}_p = \mathbf{1}_k$ , and let  $u_{q,i}(p, k)$  be the  $i$ th component of  $\mathbf{u}_q(p, k)$ . Since  $\mathcal{V}_q$  separates  $\mathcal{V}_p$  from  $n^1$ , the voltage  $u^1(p, k)$  is the same as the voltage induced at  $n^1$  by imposing  $\mathbf{u}_q(p, k)$  as the vector of node voltages for  $\mathcal{V}_q$ . But,  $\mathbf{u}_q(p, k) = u_{q,1}(p, k)\mathbf{1}_1 + \cdots + u_{q,m}(p, k)\mathbf{1}_m$ , and so by superposition

$$u^1(p, k) = u_{q,1}(p, k)u^1(q, 1) + \cdots + u_{q,m}(p, k)u^1(q, m). \quad (15)$$

Now, by Definition 7.1,  $u^1(p, k) = P(n^1; n_{p,k})$ ,  $u^1(q, k) = P(n^1; n_{q,k})$ , and  $u_{q,i}(p, k) = P(n_{q,i}; n_{p,k})$ . Thus, (14) is justified by (15), and Definition 7.1 has the stated consistency.

A similar use of conditional probabilities and electrical-network manipulations shows that Definition 7.1 is consistent for transitions from  $\mathcal{V}_p$  to  $\mathcal{V}_q$  through  $\mathcal{V}_n$  where  $p < n < q$ .

With regard to walks that start at  $n^1$ , we have so far restricted ourselves to those that do reach  $\mathcal{V}_p$ . The next natural question is: What is the probability that  $\Psi$ , after starting at  $n^1$ , will reach  $\mathcal{V}_p$  before returning to  $n^1$ ? The answer is zero. Indeed, that  $\Psi$  starts from  $n^1$  means that  $\Psi$  reaches a node  $n_{q,k}$  of  $\mathcal{V}_q$  for some  $q > p$ . The probability that  $\Psi$  then returns to  $n^1$  before reaching  $\mathcal{V}_p$  is, according to Definition 6.3, the voltage  $u_{q,k}$  at  $n_{q,k}$  when  $n^1$  is held at 1 volt and all nodes of  $\mathcal{V}_p$  are held at 0 volt. But,  $u_{q,k} \rightarrow 1$  as  $q \rightarrow \infty$  according to Lemma 4.1. In other words, only a vanishingly small proportion of the 1-walks that start at  $n^1$  reach  $\mathcal{V}_p$  without first returning to  $n^1$ , whatever be  $p$ . In this sense,  $n^1$  is not a transient node. It follows of course that the probability that  $\Psi$ , given that it starts at  $n^1$ , will reach another 1-node before returning to  $n^1$  is zero.

This does not mean however that there are no random 1-walks that, starting from some 1-node, penetrate a 0-section and continue on to reach another 1-node. It simply means that we are dealing with the exceptional case among all the random 1-walks that start at  $n^1$  when we examine those that reach  $\mathcal{V}_p$  for any given  $p$ . Thus, we are free to compare transition probabilities for those 1-walks that do penetrate a 0-section from an incident 1-node.

We can for example compare transition probabilities for 1-walks that rove (Definition 5.1), that is, for 1-walks that, after starting from a 1-node  $n_0^1$ , reach another 1-node before returning to  $n_0^1$ . As our last task in this section, we shall show that a random roving 1-walk is a Markov chain with a finite state space consisting of the 1-nodes of  $\mathbf{N}^1$ .

For this purpose, consider now a 1-node  $n_0^1$  and all its incident 0-sections  $\mathbf{S}_a$  ( $a = 1, \dots, A$ ). This is illustrated in Figure 2 wherein we have taken  $A = 2$ . Let  $n_1^1, \dots, n_K^1$  be the 1-nodes incident to those 0-sections  $\mathbf{S}_a$  other than  $n_0^1$ ; we say that those 1-nodes are *adjacent to*  $n_0^1$ . Choose a spur for every end of the  $\mathbf{S}_a$  that is embraced by an  $n_k^1$  ( $k = 1, \dots, K$ ). Those spurs can be chosen so small that they are disjoint from all the

other spurs; do so. Let  $\bigcup_{p=0}^{\infty} M_{k,p}$  now be a finitely chainlike representation for the union of the spurs for the ends of the  $S_a$  that are embraced by the single 1-node  $n_k^1$ . Once again,  $V_{k,p} = M_{k,p-1} \cup M_{k,p}$  ( $p > 0$ ).

Let us choose a positive integer  $p_k$  for each  $k = 1, \dots, K$ . The nodes of  $\bigcup_{k=1}^K V_{k,p_k}$  lie in all the 0-sections incident to  $n_0^1$  and separate  $n_0^1$  from all the  $n_k^1$ . (See Figure 2.) As a direct extension of Definition 7.1, we can assign comparative probabilities for transitions from  $n_0^1$  to the various  $V_{k,p_k}$ . In particular, given that  $\Psi$  starts at  $n_0^1$  and reaches a node of  $\bigcup_{k=1}^K V_{k,p_k}$ , the probability that  $\Psi$  reaches any node of  $V_{k,p_k}$  before it reaches any node of  $\cup\{V_{l,p_l} : l = 1, \dots, K; l \neq k\}$  is equal to the node voltage  $v_{0,k}(p_1, \dots, p_K)$  at  $n_0^1$  when the nodes of  $V_{k,p_k}$  are held at 1 volt and the nodes of all the  $V_{l,p_l}$  ( $l \neq k$ ) are held at 0 volt. As before, by virtue of Lemma 3.4, another node voltage  $u_{0,k}$  is obtained at  $n_0^1$  by holding  $n_k^1$  at 1 volt and the other 1-nodes  $n_l^1$  ( $l \neq k$ ) adjacent to  $n_0^1$  at 0 volt. We can repeat the proof of Lemma 6.1, substituting  $n_0^1$  for  $n_0$ , all the 0-sections  $S_a$  incident to  $n_0^1$  for the single 0-section  $S$ , and the 1-nodes adjacent to  $n_0^1$  for the 1-nodes incident to  $S$ . The proof proceeds exactly as before, the only difference being that we need a maximum principal for the node voltages in a 1-network. This is provided by Theorem 4.3. All this leads to the conclusion that  $v_{0,k}(p_1, \dots, p_K)$  converges to  $u_{0,k}$  as the  $p_1, \dots, p_K$  tend to infinity independently. Hence, we are led to the following definition, wherein  $\mathcal{N}_g^1$  denotes all the 1-nodes adjacent to  $n_0^1$  other than  $n_k^1$ .

**Definition 7.2.** Assume there are two or more 1-nodes adjacent to the 1-node  $n_0^1$ . For any random roving 1-walk, the probability:

$$P(n_0^1; n_k^1) = \text{Prob}(sn_0^1, rn_k^1, b\mathcal{N}_g^1 \mid \Psi \text{ roves}) \quad (16)$$

that  $\Psi$ , starting from  $n_0^1$ , reaches an adjacent 1-node  $n_k^1$  before it reaches any of the other adjacent 1-node  $n_l^1$  ( $l \neq 0, k$ ) is defined to be the node voltage at  $n_0^1$  when  $n_k^1$  is held at 1 volt and all the 1-nodes of  $\mathcal{N}_g^1$  are held at 0 volt.

**Lemma 7.3.** Under the conditions of Definition 7.2,  $0 < P(n_0^1; n_k^1) < 1$ .

*Proof.* This follows directly from Corollary 4.4. For instance, to conclude that  $P(n_0^1; n_k^1) < 1$ , choose the 1-path  $P^1$  of that corollary to be  $P^1 = \{n_0^1, P^0, n_g^1\}$ , where  $n_g^1$  is the 1-node obtained by shorting the nodes of  $\mathcal{N}_g^1$  and  $P^0$  is an endless 0-path that reaches  $n_0^1$  and  $n_g^1$ .

□

If  $n_0^1$  has only one adjacent 1-node  $n_1^1$ ,  $P(n_0^1; n_1^1)$  is set equal to one.

This then defines the one-step transition probabilities for a random roving 1-walk from any 1-node  $n_0^1$  to its adjacent 1-nodes. Obviously the probability that a 1-walk will go in one step from  $n_0^1$  to a 1-node that is not adjacent to  $n_0^1$  is zero. Moreover, by definition of a roving 1-walk, the probability of a transition from  $n_0^1$  to  $n_0^1$  in one step is zero too. Finally, to show that we have a Markov chain, we have to show that these probabilities for one-step transitions from any given 1-node sum to one. By superposition, this sum is equal to the voltage  $u_0^1$  at  $n_0^1$  when all the 1-nodes adjacent to  $n_0^1$  are held at 1 volt and all other 1-nodes and 0-nodes are left floating (i.e., no source connections to them). But then, all branch currents in the 0-sections incident to  $n_0^1$  are zero, and therefore  $u_0^1 = 1$  too, as required.

**Theorem 7.4.** *A random roving 1-walk  $W^1$  for a network  $N^1$  that satisfies Conditions 2.3 is a Markov chain with a state space consisting of the 1-nodes  $n_k^1$  of  $N^1$  and with the following one-step transition probabilities:  $P_{k,k} = 0$ ;  $P_{k,l} = 0$  if  $n_k^1$  and  $n_l^1$  are not adjacent;  $P_{k,l}$  is given by Definition 7.2 if  $n_k^1$  and  $n_l^1$  are adjacent.*

By definition the following two conditions have also been imposed: Any 0-walk  $W^0$  embraced by  $W^1$  follows the customary rules of transition for a random walk on the 0-section to which  $W^0$  is confined. The transition in (9) from one 0-walk  $W_m^0$  to the next 0-walk  $W_{m+1}^0$  through the 1-node  $n_{m+1}^1$  is governed by Definitions 6.2, 6.3, and 7.1.

## 8 Reversibility and the Surrogate Network

**Theorem 8.1.** *The Markov chain of Theorem 7.4 is irreducible and reversible.*

**Proof.** The case where  $N^1$  has just two 1-nodes is trivial. So, let  $N^1$  have more than two 1-nodes.

For any two adjacent 1-nodes  $n_1^1$  and  $n_2^1$ , the probability that a roving 1-walk will pass from  $n_1^1$  to  $n_2^1$  in one step is positive (Lemma 7.3). The irreducibility [11] of the Markov chain now follows from the 1-connectedness of  $N^1$ .

As for reversibility, we start by recalling the definition of a cycle—adapted for 1-nodes. This is a finite sequence  $C = (n_1^1, n_2^1, \dots, n_c^1, n_{c+1}^1 = n_1^1)$  of 1-nodes  $n_k^1$ , where all 1-nodes

are distinct except for the first and last, there are at least three 1-nodes (i.e.,  $c > 2$ ), and consecutive 1-nodes in  $C$  are adjacent in  $\mathbf{N}^1$ . A Markov chain is reversible if, for every cycle  $C$ , the product  $\prod_{k=1}^c P_{k,k+1}$  of transition probabilities  $P_{k,k+1}$  from  $n_k^1$  to  $n_{k+1}^1$  remains the same when every  $P_{k,k+1}$  is replaced by  $P_{k+1,k}$  [11, Section 1.5]. Thus, we need only show that

$$P_{1,2}P_{2,3}\cdots P_{c,1} = P_{1,c}\cdots P_{3,2}P_{2,1}. \quad (17)$$

According to Definition 7.2,  $P_{k,k+1}$  is obtained by holding  $n_{k+1}^1$  at 1 volt, by holding all the 1-nodes adjacent to  $n_k^1$  other than  $n_{k+1}^1$  at 0 volt, and setting  $P_{k,k+1} = u_k^1$ , where  $u_k^1$  is the resulting voltage at  $n_k^1$ . For this situation,  $u_k$  will remain unchanged when still other 1-node voltages are arbitrarily specified.

To simplify notation, let us denote  $n_k^1$  by  $m_0$  and  $n_{k+1}^1$  by  $m_1$ . Also, let  $m_2, \dots, m_K$  denote all the 1-nodes different from  $n_k^1$  and  $n_{k+1}^1$  but adjacent to either  $n_k^1$  or  $n_{k+1}^1$  or both. Since the cycle has at least three 1-nodes, we have  $K \geq 2$ . Now, consider the  $K$ -port obtained from  $\mathbf{N}^1$  by choosing  $m_k, m_0$  as the pair of terminals for the  $k$ th port ( $k = 1, \dots, K$ ) with  $m_0$  being the common ground for all ports. To obtain the required node voltages for measuring  $P_{k,k+1}$ , we externally connect a 1-volt source to  $m_1$  from all of the  $m_2, \dots, m_K$ , with  $m_0$  left floating (i.e.,  $m_0$  has no external connections). The resulting voltage  $u_0$  at  $m_0$  is  $P_{k,k+1}$ .

With respect to  $m_0$ , the voltage at  $m_1$  is  $1 - u_0$  and the voltage at  $m_k$  ( $k = 2, \dots, K$ ) is  $-u_0$ . Moreover, with  $i_k$  denoting the current entering  $m_k$  ( $k = 1, \dots, K$ ), the sum  $i_1 + \dots + i_K$  is zero. (Apply Kirchhoff's current law at  $m_1$ .) Furthermore, the port currents and voltages are related by  $\mathbf{i} = Y\mathbf{u}$ , where  $\mathbf{i} = (i_1, \dots, i_K)$ ,  $\mathbf{u} = (1 - u_0, -u_0, \dots, -u_0)$ , and  $Y = [Y_{a,b}]$  is a  $K \times K$  matrix of real numbers that is positive-definite (Lemma 3.3). Hence,  $Y$  is symmetric. Upon expanding  $\mathbf{i} = Y\mathbf{u}$  and adding the  $i_k$ , we get

$$0 = i_1 + \dots + i_K = \sum_{a=1}^K Y_{a,1} - u_0 \sum_{a=1}^K \sum_{b=1}^K Y_{a,b}.$$

Therefore,

$$P_{k,k+1} = u_0 = \frac{\sum_{a=1}^K Y_{a,1}}{\sum_{a=1}^K \sum_{b=1}^K Y_{a,b}}. \quad (18)$$

Upon setting  $G_k = \sum_{a=1}^K \sum_{b=1}^K Y_{a,b}$ , we can rewrite (18) as

$$G_k P_{k,k+1} = \sum_{a=1}^K Y_{a,1}. \quad (19)$$

Now,  $\sum_{a=1}^K Y_{a,1}$  is the sum  $i_1 + \dots + i_K$  when  $\mathbf{u} = (1, 0, \dots, 0)$ ; that is,  $\sum_{a=1}^K Y_{a,1}$  is the sum of the currents entering  $m_1, m_2, \dots, m_K$  from external connections when 1-volt sources are connected to  $m_1$  from all of the  $m_0, m_2, \dots, m_K$ .

By reversing the roles of  $m_0$  and  $m_1$ , we have by the same analysis that  $G_{k+1} P_{k+1,k}$  is the sum  $i_0 + i_2 + \dots + i_K$  of the currents entering  $m_0, m_2, \dots, m_K$  from external connections when 1-volt sources are connected to  $m_0$  from all of the  $m_1, m_2, \dots, m_K$ . With respect to the ground node  $m_0$ , we now have  $u_1 = \dots = u_K = -1$ , and therefore  $i_1 = -\sum_{a=1}^K Y_{1,a}$ . Moreover, under this latter connection, the sum  $-i_1 - i_2 - \dots - i_K$  of the currents leaving  $m_1, m_2, \dots, m_K$  is equal to the current  $i_0$  entering  $m_0$ . Hence,  $-i_1 = i_0 + i_2 + \dots + i_K$ . Thus,

$$G_{k+1} P_{k+1,k} = -i_1 = \sum_{a=1}^K Y_{1,a}. \quad (20)$$

Since the matrix  $Y$  is symmetric, we have  $Y_{1,a} = Y_{a,1}$ . So, by (19) and (20),

$$G_{k+1} P_{k+1,k} = G_k P_{k,k+1}. \quad (21)$$

Finally, we may now write

$$P_{1,2} P_{2,3} \dots P_{c,1} = \frac{G_2}{G_1} P_{2,1} \frac{G_3}{G_2} P_{3,2} \dots \frac{G_1}{G_c} P_{1,c} = P_{2,1} P_{3,2} \dots P_{1,c}$$

This verifies (17) and completes the proof.  $\square$

Because the Markov chain is irreducible and reversible, we can synthesize a finite 0-network  $\mathbf{N}^{1 \mapsto 0}$  whose 0-nodes correspond bijectively to the 1-nodes of  $\mathbf{N}^1$  and whose random 0-walks are governed by the same transition matrix as that for the random roving 1-walks of  $\mathbf{N}^1$ .  $\mathbf{N}^{1 \mapsto 0}$  acts as a *surrogate* for  $\mathbf{N}^1$ . A realization for it can be obtained by connecting a conductance  $g_{k,l} = g_{l,k}$  between the 0-nodes  $n_k^0$  and  $n_l^0$  in  $\mathbf{N}^{1 \mapsto 0}$ , where  $g_{k,l}$  is given as follows: Let  $n_k^1 \mapsto n_k^0$  denote the bijection from the 1-nodes of  $\mathbf{N}^1$  to the 0-nodes of  $\mathbf{N}^{1 \mapsto 0}$ . If  $n_k^1$  and  $n_l^1$  are not adjacent in  $\mathbf{N}^1$ , set  $g_{k,l} = 0$ . If  $n_k^1$  and  $n_l^1$  are adjacent in  $\mathbf{N}^1$ , relabel  $n_k^1$  as  $m_0$ ,  $n_l^1$  as  $m_1$ , and let  $m_2, \dots, m_K$  be the other 1-nodes that are adjacent to either

$m_0$  or  $m_1$  or both. Then, with our prior notation, set  $G_k = \sum_{a=1}^K \sum_{b=1}^K Y_{a,b}$ . Also, set  $G = \sum_k G_k$ , where this latter sum is over all indices for all the 1-nodes of  $\mathbf{N}^1$ . Finally, set  $g_{k,l} = P_{k,l}G_k/G$ . By (21),  $g_{k,l} = g_{l,k}$ . This yields the surrogate network [4, page 43].

## 9 A Special Kind of $\nu$ -Network

Let us turn now to random walks on  $\nu$ -networks, where  $\nu$  is any positive natural number. Our theory for transfinite random walks of the higher ranks is constructed recursively and is much like that for random 1-walks. However, modifications and extensions are needed throughout the development. Because of this, we will be specific about our definitions, lemmas, and theorems but will explicate our proofs only when the arguments involve significant deviations or expansions from the foregoing.

Let  $\mu$  be any natural number, and let  $\mathbf{K}^\mu$  be a given  $\mu$ -network with the branch set  $\mathcal{B}$ . Let  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be two subsets of  $\mathcal{B}$ , and let  $\mathbf{K}_1$  and  $\mathbf{K}_2$  be the reduction of  $\mathbf{K}^\mu$  induced by  $\mathcal{B}_1$  and  $\mathcal{B}_2$  respectively. Then, the *union*  $\mathbf{K}_1 \cup \mathbf{K}_2$  is the reduction of  $\mathbf{K}^\mu$  induced by  $\mathcal{B}_1 \cup \mathcal{B}_2$ . Also, the *intersection*  $\mathbf{K}_1 \cap \mathbf{K}_2$  is the reduction of  $\mathbf{K}^\mu$  induced by  $\mathcal{B}_1 \cap \mathcal{B}_2$ . If  $\mathcal{B}_0, \mathcal{B}_1, \dots$  is a partition of  $\mathcal{B}$ , we say that  $\mathbf{K}^\mu$  is *partitioned* into  $\mathbf{K}_0, \mathbf{K}_1, \dots$ , where each  $\mathbf{K}_p$  is the reduction of  $\mathbf{K}^\mu$  induced by  $\mathcal{B}_p$ . Thus, we may write  $\mathbf{K}^\mu = \cup_p \mathbf{K}_p$ .

Furthermore, let a reduction  $\mathbf{K}_r$  of  $\mathbf{K}^\mu$  have a tip or node (of whatever rank) that is embraced by a node  $n_0$  (also of some unspecified rank) of  $\mathbf{K}^\mu$ ; then,  $\mathbf{K}_r$  and  $n_0$  are said to be *incident*. If  $\mathbf{K}_r$  and  $\mathbf{L}_r$  are two reductions of  $\mathbf{K}^\mu$ , then  $\mathbf{K}_r \odot \mathbf{L}_r$  will denote the set of all nodes of  $\mathbf{K}^\mu$  that are incident to both  $\mathbf{K}_r$  and  $\mathbf{L}_r$ .

As with 1-networks, a critical construct in our development is an extension to  $\mu$ -networks of Halin's finitely chainlike structure for 0-networks. Now, let  $\mu \geq 1$ . A  $\mu$ -network  $\mathbf{M}^\mu$  will be called *finitely  $\mu$ -chainlike* if it can be partitioned into a sequence  $\{\mathbf{M}_p^\mu\}_{p=0}^\infty$  of reductions  $\mathbf{M}_p^\mu$  of  $\mathbf{M}^\mu$ :

$$\mathbf{M}^\mu = \bigcup_{p=0}^{\infty} \mathbf{M}_p^\mu \quad (22)$$

where each  $\mathbf{M}_p^\mu$  is of rank  $\mu$  and has finitely many  $(\mu - 1)$ -sections and where every  $(\mu - 1)$ -section of  $\mathbf{M}^\mu$  appears in its entirety in one and only one of the  $\mathbf{M}_p^\mu$ ; it is also required



that

$$\mathbf{M}_p^\mu \odot \mathbf{M}_{p-1}^\mu = \mathcal{V}_p^\mu, \quad p = 1, 2, \dots, \quad (23)$$

where each  $\mathcal{V}_p^\mu$  is a finite set of  $\mu$ -nodes in  $\mathbf{M}^\mu$  and that the following three conditions be fulfilled.

- (a) The cardinality  $m^\mu$  of each  $\mathcal{V}_p^\mu$  does not depend upon  $p$ .
- (b) For every  $p \geq 2$ ,  $\mathbf{M}_p^\mu \odot \bigcup_{q=0}^{p-2} \mathbf{M}_q^\mu = \emptyset$ .
- (c) In each  $\mathbf{M}_p^\mu$  ( $p \geq 1$ ) there are  $m^\mu$  pairwise totally disjoint  $\mu$ -paths from the  $\mu$ -nodes in  $\mathcal{V}_p^\mu$  to the  $\mu$ -nodes in  $\mathcal{V}_{p+1}^\mu$ .

Once again the union for all  $p$  of all the totally disjoint  $\mu$ -paths of condition (c) yields  $m^\mu$  totally disjoint, one-ended  $\mu$ -paths, which we will call *spines* or  $\mu$ -*spines*, and the set of all  $m^\mu$  of them will be called a *full set of  $\mu$ -spines* for the finitely  $\mu$ -chainlike structure.

A significant difference between this definition of a finitely  $\mu$ -chainlike structure for a  $\mu$ -network and Halin's definition corresponding to a 0-network is that  $\mu$ -sections now take on the role played previously by branches. This distinction will remain in force throughout our development of random  $\nu$ -walks on a  $\nu$ -network  $\mathbf{N}^\nu$ .

As before, we require that there be no infinite 0-nodes, no self-loops, and no parallel branches in  $\mathbf{N}^\nu$ . Now, let  $\nu$  be a natural number no less than 2. For every  $\mu = 1, \dots, \nu$ , we also require that no  $\mu$ -node of  $\mathbf{N}^\nu$  embrace a node of lower rank. Moreover, we shall require that  $\mathbf{N}^\nu$  be  $\nu$ -connected. As a result, every  $(\mu - 1)$ -section will have at least one incident  $\mu$ -node and will be  $\mu$ -connected to other parts of  $\mathbf{N}^\nu$  through its incident  $\mu$ -nodes.

**Lemma 9.1.** *Under the just-stated conditions on  $\mathbf{N}^\nu$ , let  $\nu \geq 2$ . Assume in addition that, for every  $\mu = 0, \dots, \nu - 2$ , each  $\mu$ -section has finitely many incident  $(\mu + 1)$ -nodes. Then, there are infinitely many  $\mu$ -sections in every  $(\mu + 1)$ -section for each  $\mu = 0, \dots, \nu - 2$ .*

**Proof.** By definition of a  $\nu$ -network [25],  $\mathbf{N}^\nu$  has at least one  $\nu$ -node and therefore at least one  $(\nu - 1)$ -tip. This in turn implies that there is at least one one-ended  $(\nu - 1)$ -path and therefore an infinity of  $(\nu - 1)$ -nodes. Our conclusion for  $\mu = \nu - 2$  now follows from the assumption that each  $(\nu - 2)$ -section is incident to finitely many  $(\nu - 1)$ -nodes and from the fact that every  $(\nu - 1)$ -node must be incident to at least one  $(\nu - 2)$ -section. Our conclusion

for all lower values of  $\mu$  is established by repeating this argument for  $\mu = \nu - 3, \dots, 0$  in turn with  $N^\nu$  replaced by a  $(\mu + 2)$ -section.  $\square$

Now, let  $1 \leq \mu < \nu$ . We say that  $N^\nu$  *possesses* a  $\mu$ -tip  $t^\mu$  and that  $t^\mu$  is *in*  $N^\nu$  if  $N^\nu$  has a  $(\mu + 1)$ -node that contains  $t^\mu$ ; a similar wording is used for any reduction of  $N^\nu$ . Let  $K$  be any reduction of  $N^\nu$  and let the rank of  $K$  be  $\mu + 1$ . Hence,  $K$  possesses at least one  $\mu + 1$ -node and therefore at least one  $\mu$ -tip  $\tau^\mu$ . There is a unique  $\mu$ -tip  $t^\mu$  in  $N^\nu$  that contains  $\tau^\mu$  as a subset. In fact,  $\tau^\mu \mapsto t^\mu$  is an injection from all the  $\mu$ -tips of  $K$  to some or all of the  $\mu$ -tips of  $N^\nu$ . Because of this, we say that  $K$  *possesses*  $t^\mu$  as a  $\mu$ -tip if there is a representative of  $t^\mu$  that lies entirely within  $K$ .

For  $1 \leq \mu < \nu$  again, let  $S_f^{\mu-1}$  denote any finite set of  $(\mu - 1)$ -sections in  $N^\nu$  and let  $N_f^\nu = N^\nu \setminus S_f^{\mu-1}$  denote the reduction of  $N^\nu$  induced by all branches of  $N^\nu$  that are not in the members of  $S_f^{\mu-1}$ . Since the deletion of the branches in the members of  $S_f^{\mu-1}$  removes only a finite part of any one-ended  $\mu$ -path,  $N^\nu$  and  $N_f^\nu$  possess exactly the same  $\mu$ -tips.

As the next step, we extend the definition of an end. With  $1 \leq \mu < \nu$  still, two  $\mu$ -tips of  $N^\nu$  will be called  *$\mu$ -end-equivalent* if, for every choice of  $S_f^{\mu-1}$ , the two  $\mu$ -tips have representatives lying in the same  $\mu$ -section of  $N_f^\nu$ . The corresponding equivalence classes of  $\mu$ -tips will be called  *$\mu$ -ends* of  $N^\nu$ . Since all  $\mu$ -tips in a given  $\mu$ -end must belong to a single  $\mu$ -section of  $N^\nu$ , we say that the  $\mu$ -end *belongs* to that  $\mu$ -section. Because every  $\mu$ -section of  $N^\nu$  has at least one incident  $(\mu + 1)$ -node and therefore at least one  $\mu$ -tip, every  $\mu$ -section has at least one  $\mu$ -end. As with Lemma 2.1, we have

**Lemma 9.2.** *Under the stated assumptions on  $N^\nu$ ,  $N^\nu$  and  $N_f^\nu = N^\nu \setminus S_f^{\mu-1}$  have the same  $\mu$ -tips and the same  $\mu$ -ends.*

Let a  $\mu$ -end  $d^\mu$  of  $N^\nu$  be such that an  $S_f^{\mu-1}$  can be so chosen that, within  $N_f^\nu = N^\nu \setminus S_f^{\mu-1}$ , the  $\mu$ -section  $S_d^\mu$  that possess  $d^\mu$  as a  $\mu$ -end has no other  $\mu$ -end. In this case,  $S_d^\mu$  is called a  *$\mu$ -spur* for  $d^\mu$ , and  $d^\mu$  will be said to *possess*  $S_d^\mu$  as a  $\mu$ -spur. As before, we can alter  $S_d^\mu$  and still have a  $\mu$ -spur for the given  $d^\mu$  by removing or perhaps appending  $(\mu - 1)$ -sections. Some obvious modifications of the proof of Lemma 2.2 establishes the following under our stated assumptions on  $N^\nu$ .

**Lemma 9.3.** *Assume that a  $\mu$ -section  $S^\mu$  of  $N^\nu$  has only finitely many  $\mu$ -ends. Then,*

a finite set  $\mathcal{S}_f^{\mu-1}$  of  $(\mu-1)$ -sections can be so chosen that every  $\mu$ -end of  $\mathbf{S}^\mu$  is the one and only  $\mu$ -end of some  $\mu$ -section  $\mathbf{S}_d^\mu$  of  $\mathbf{N}^\nu \setminus \mathcal{S}_f^{\mu-1}$  (i.e., each such  $\mu$ -section is a  $\mu$ -spur for its  $\mu$ -end) and the  $\mu$ -spurs are mutually disjoint (i.e., mutually not  $\mu$ -connected).

Let us now gather together all the assumptions we impose upon the  $\nu$ -network  $\mathbf{N}^\nu$  throughout the remainder of this work.

**Conditions 9.4.** Let  $\mu$  and  $\nu$  be natural numbers with  $\nu \geq 1$  and  $\mu = 0, \dots, \nu - 1$ .

- (a)  $\mathbf{N}^\nu$  is a  $\nu$ -connected  $\nu$ -network having no infinite 0-nodes, no self-loops, and no parallel branches. For every  $\mu$ , no  $(\mu+1)$ -node embraces a node of lower rank.  $\mathbf{N}^\nu$  has at least two  $\nu$ -nodes.
- (b)  $\mathbf{N}^\nu$  has only finitely many  $(\nu-1)$ -ends, and for each  $\mu$  every  $\mu$ -section has only finitely many  $\mu$ -ends.
- (c) Each  $\mu$ -end is embraced by some  $(\mu+1)$ -node (i.e., all the  $\mu$ -tips in that  $\mu$ -end are members of a single  $(\mu+1)$ -node), and every  $(\mu+1)$ -node embraces only finitely many  $\mu$ -ends.
- (d) Every branch  $b_j$  of  $\mathbf{N}^\nu$  has assigned to it a positive number  $g$ , called the branch conductance;  $r_j = g_j^{-1}$  is called the branch resistance. Every branch has an orientation.
- (e) Every  $\mu$ -end has a  $\mu$ -spur that is finitely  $\mu$ -chainlike and possesses a full set of  $\mu$ -spines, each of which is perceptible.

The proof of Lemma 2.4 extends directly to yield the following. Here, a 0-spur is a spur as defined in Section 2, and similarly for a 0-end.

**Lemma 9.5.** Under Conditions 9.4, the following is true.

- (i) Every  $\mu$ -spur has the properties indicated in Condition 9.4(e).
- (ii) Every one-ended  $\mu$ -path lies within a  $\mu$ -spur.
- (iii) Between every two nodes of  $\mathbf{N}^\nu$  there is a perceptible  $\nu$ -path that terminates at those nodes, whatever be the ranks of those nodes.

(iv)  $N^\nu$  has only finitely many  $(\nu - 1)$ -sections and  $\nu$ -nodes. Moreover, for each  $\mu$ , every  $\mu$ -section is incident to only finitely many  $(\mu + 1)$ -nodes, and every  $(\mu + 1)$ -node is incident to only finitely many  $\mu$ -sections.

We have already defined in Section 2 what we mean by shorting together finitely many 0-nodes. As for the *shorting* of a finite number of  $\mu$ -nodes, where  $\mu$  is fixed, positive, and possibly equal to  $\nu$ , just take their union to create a new  $\mu$ -node and thereby a new  $\nu$ -network. Note that the new  $\nu$ -network also satisfies Conditions 9.4. As for the *shorting* of finitely many nodes of various ranks, first short all the nodes of the same rank, doing this for every rank; then the resulting node of highest rank is taken to embrace the resulting node of next highest rank, which in turn is taken to embrace the resulting node of third highest rank, and so forth.

## 10 Excitations at Nodes of Arbitrary Ranks

The objective of this section is to establish that a pure voltage source can be connected to any two nodes of  $N^\nu$ , whatever be the ranks of those nodes. We start with a resistive voltage source. Let  $b_0$  denote a source branch consisting of a pure voltage source  $e_0$  in series with a positive resistance  $\rho$ . Let us append  $b_0$  to  $N^\nu$  by connecting  $b_0$  to any two nodes (of any ranks) of  $N^\nu$ . (We now index the branches of  $N^\nu$  by  $j = 1, 2, \dots$ ) The resulting network will be denoted by  $N_e^\nu$ . As with 0-sections, a “ $\mu$ -section” of whatever rank  $\mu$  will always be understood to be a  $\mu$ -section with respect to  $N^\nu$ , not with respect to  $N_e^\nu$ . The spaces  $\mathcal{K}^0$  and  $\mathcal{K}$  are defined for  $N_e^\nu$  as in [25, section 10], and so too for  $q$ -loop currents and  $q$ -basic currents. That paper has established

**Theorem 10.1.** *Let the resistive-voltage-source branch  $b_0$  be connected to any two nodes of  $N_e^\nu$ . Then, there is a unique current vector  $\mathbf{i}^\rho = (i_0^\rho, i_1^\rho, i_2^\rho, \dots) \in \mathcal{K}$  such that*

$$e_0 s_0 = \rho i_0^\rho s_0 + \sum_{j=1}^{\infty} r_j i_j^\rho s_j \quad (24)$$

for every  $\mathbf{s} \in \mathcal{K}$ .

The next step is to extend Kirchoff’s current law to a “cut” in  $N_e^\nu$  that “isolates” a node  $n^{\mu+1}$  of rank  $\mu + 1 \leq \nu$ . We allow  $n^{\mu+1}$  to be one of the nodes to which the source branch

$b_0$  is connected. By Condition 9.4(c),  $n^{\mu+1}$  embraces only finitely many  $\mu$ -ends. Choose a  $\mu$ -spur for every  $\mu$ -end embraced by  $n^{\mu+1}$  (Lemma 9.3). From the finitely  $\mu$ -chainlike structure of the union of those spurs, we can select a finite set  $\mathcal{V}_{p_\mu}^\mu = \mathbf{M}_{p_\mu}^\mu \odot \mathbf{M}_{p_{\mu-1}}^\mu$  of  $\mu$ -nodes. Now each  $\mu$ -node in  $\mathcal{V}_{p_\mu}^\mu$  embraces only finitely many  $(\mu-1)$ -ends, and therefore is incident to only finitely many  $(\mu-1)$ -sections. By the definition of the finitely  $\mu$ -chainlike structure (see Section 9), each of the latter  $(\mu-1)$ -sections lies entirely within either  $\mathbf{M}_{p_\mu}^\mu$  or  $\mathbf{M}_{p_{\mu-1}}^\mu$  (not in both). If there is any  $(\mu-1)$ -section that is incident only to  $\mu$ -nodes in  $\mathcal{V}_{p_\mu}^\mu$ , we assign that  $(\mu-1)$ -section to  $\mathbf{M}_{p_\mu}^\mu$ . Thus, any  $(\mu-1)$ -section  $\mathbf{S}_a^{\mu-1}$  in  $\mathbf{M}_{p_{\mu-1}}^\mu$  that is incident to one or more  $\mu$ -nodes in  $\mathcal{V}_{p_\mu}^\mu$  is also incident to some  $\mu$ -nodes of  $\mathbf{M}_{p_{\mu-1}}^\mu$  that are not in  $\mathcal{V}_{p_\mu}^\mu$ . Consider all of the  $(\mu-1)$ -sections like  $\mathbf{S}_a^{\mu-1}$ . Choose a  $(\mu-1)$ -spur for every one of their  $(\mu-1)$ -ends that is embraced by a  $\mu$ -node of  $\mathcal{V}_{p_\mu}^\mu$ . The union of all those  $(\mu-1)$ -spurs is finitely chainlike. Therefore, we can choose a finite set  $\mathcal{V}_{p_{\mu-1}}^{\mu-1}$  of  $(\mu-1)$ -nodes from that chainlike structure, and then consider all  $(\mu-2)$ -sections  $\mathbf{S}_b^{\mu-2}$  in  $\mathbf{M}_{p_{\mu-1}-1}^{\mu-1}$  each of which is incident to at least one node of  $\mathcal{V}_{p_{\mu-1}}^{\mu-1}$  and to at least one node not in  $\mathcal{V}_{p_{\mu-1}}^{\mu-1}$ . Continuing in this way toward lower ranks, we finally reach a finite set  $\mathcal{V}_{p_0}^0 = \mathbf{M}_{p_0}^0 \odot \mathbf{M}_{p_0-1}^0$  of 0-nodes which *separates*  $n^{\mu+1}$  from all other  $(\mu+1)$ -nodes in the following way: If  $P^{\mu+1}$  is a  $(\mu+1)$ -path in  $\mathbf{N}^\nu$  that meets  $n^{\mu+1}$  and any other  $(\mu+1)$ -node, then  $P^{\mu+1}$  embraces a 0-node of  $\mathcal{V}_{p_0}^0$ . (In a similar sense, we will say later on that  $\mathcal{V}_{p_0}^0$  *separates*  $n^{\mu+1}$  from nodes of other ranks.)

All the branches of  $\mathbf{M}_{p_0-1}^0$  that have one node in  $\mathcal{V}_{p_0}^0$  and one node not in  $\mathcal{V}_{p_0}^0$  comprise a finite set  $\mathcal{C}$ . We call  $\mathcal{C}$  a *cut for*  $n^{\mu+1}$  at  $\mathcal{V}_{p_0}^0$ , and refer to the branches of  $\mathcal{C}$  as *cut-branches at*  $\mathcal{V}_{p_0}^0$ . We also say that, within  $\mathbf{N}^\nu$ ,  $\mathcal{C}$  *isolates*  $n^{\mu+1}$  from all other  $(\mu+1)$ -nodes. Again a cut-branch is said to be *oriented away from (toward)*  $n^{\mu+1}$  if it is oriented away from (toward) its node in  $\mathcal{V}_{p_0}^0$ . Kirchhoff's current law for  $\mathcal{C}$  is again expressed by (5), except that  $n^1$  is now replaced by  $n^{\mu+1}$ . To be specific, for  $\mathbf{N}_e^\nu$ , that law states that

$$\sum \pm i_j = 0, \quad (25)$$

where the summation is for the branches in  $\mathcal{C}$  and for  $b_0$  as well if  $b_0$  is incident to  $n^{\mu+1}$ ,  $i_j$  is the current in those branches, and the plus (minus) sign is used if the branch's orientation is away from (toward)  $n^{\mu+1}$ . With obvious modifications to its proof, Lemma 3.2 extends

to the present case of a  $\nu$ -network  $N_e^\nu$ :

**Lemma 10.2.** *Kirchhoff's current law (25) holds whenever  $i \in \mathcal{K}$ .*

Lemma 3.3 also extends directly. In particular, we can produce a unique voltage-current regime within  $N^\nu$  by connecting finitely many pure current sources  $h_2, \dots, h_K$  from any node  $n_1$  in  $N^\nu$  to  $K - 1$  other nodes  $n_2, \dots, n_K$ . The ranks of these nodes may differ. The resulting vector  $\mathbf{u} = (u_2, \dots, u_K)$  of node voltages at the latter nodes measured with respect to  $n_1$  (i.e.,  $u_1 = 0$ ) is related to the vector  $\mathbf{h} = (h_2, \dots, h_K)$  by  $\mathbf{u} = Z\mathbf{h}$ , where  $Z$  is a  $(K - 1) \times (K - 1)$  real matrix. In the same way as before, we can prove

**Lemma 10.3.**  *$Z$  is positive-definite and therefore nonsingular.*

This implies that pure voltage sources can be connected from  $n_1$  to  $n_2, \dots, n_k$  to produce a unique voltage-current regime within  $N^\nu$ .

Furthermore, the voltage-current regime produced by a single pure voltage source  $e_0$  connected between any two nodes can be approached by connecting a resistive voltage source, as in Theorem 10.1, and then letting  $\rho \rightarrow 0$ . Exactly as in Section 3, the following generalization of Theorem 3.4 can be obtained. (Here, the same branch numbering system as that for Theorem 10.1 is used. Also, in the limit the inner product for  $\mathcal{I}$  becomes  $(\mathbf{i}, \mathbf{s}) = \sum_{j=1}^{\infty} i_j s_j$ , and  $i_0$  is determined from the other branch currents by Kirchhoff's current law.)

**Theorem 10.4.** *Let a pure voltage source  $e_0$  be connected to any two nodes of  $N^\nu$  of arbitrary and possibly differing ranks. Then, for the resulting network  $N_e^\nu$ , there is a unique  $i \in \mathcal{K}$  such that, for every  $\mathbf{s} \in \mathcal{K}$ ,*

$$e_0 s_0 = \sum_{j=1}^{\infty} r_j i_j s_j. \quad (26)$$

## 11 Node Voltages in $N_e^\nu$

From now on,  $N_e^\nu$  will denote  $N^\nu$  with a single pure 1-volt voltage source connected to any two nodes of  $N^\nu$ . The negative terminal of the 1-volt source will always be assigned the zero node voltage. As was the case for 1-networks, we will need the fact that every node voltage in  $N_e^\nu$  is bounded between zero and one. We shall establish this through an

inductive argument that extends the results of Section 4 to higher ranks. In doing so, we will obtain a maximum principle for node voltages in a  $\nu$ -network.

Let  $0 \leq \mu < \nu$ , and let  $S^\mu$  be a  $\mu$ -section in  $N^\nu$ .  $S^\mu$  will be called *sourceless* if no node (of whatever rank  $\leq \mu$ ) in  $S^\mu$  is incident to the source branch  $b_0$  of  $N^\nu$ . (A sourceless  $S^\mu$  may have  $b_0$  incident to one of its incident  $(\mu + 1)$ -nodes.) Consider the following. Part (a) is a transfinite generalization of the maximum principle for node voltages.

**Properties 11.1.**

(a) *There exist exactly two possibilities for all node voltages in any sourceless  $\mu$ -section  $S^\mu$ .*

(a1) *All the  $\mu$ -nodes in  $S^\mu$  have the same voltage.*

(a2) *The  $\mu$ -node voltages in  $S^\mu$  are strictly less (and strictly larger) than the largest (respectively, least) voltage for the  $(\mu + 1)$ -nodes that are incident to  $S^\mu$ .*

(b) *The  $\mu$ -node voltages along any one-ended  $\mu$ -path  $P^\mu$  in  $S^\mu$  (whether or not  $P^\mu$  is perceptible) converge to the voltage of the  $(\mu + 1)$ -node that  $P^\mu$  meets.*

Note that these properties are fulfilled for  $\mu = 0$  according to Lemmas 4.1 and the classical maximum principle for node voltages in the usual kind of network. We now use strong induction. Assume that Properties 11.1 hold for all ranks up to and including any chosen  $\mu$ , where  $\mu \leq \nu - 2$ . We argue that they also hold for  $\mu$  replaced by  $\mu + 1$ . If possibility (a1) is not fulfilled by some sourceless  $(\mu + 1)$ -section  $S^{\mu+1}$ , then any arbitrarily chosen  $(\mu + 1)$ -node  $n_a^{\mu+1}$  in  $S^{\mu+1}$  will have a voltage  $u_a^{\mu+1}$  different from that of some other  $(\mu + 1)$ -node  $n_b^{\mu+1}$  in  $S^{\mu+1}$ . We can choose a  $(\mu + 1)$ -path in  $S^{\mu+1}$  that terminates at  $n_a^{\mu+1}$  and  $n_b^{\mu+1}$  and trace along it to find a  $(\mu + 1)$ -node (possibly  $n_a^{\mu+1}$  itself) with the same voltage as  $n_a^{\mu+1}$  but adjacent to a  $(\mu + 1)$ -node with a different voltage. Let  $S_i^\mu$  be the union of all  $\mu$ -sections, each of which is incident to both of the latter two  $(\mu + 1)$ -nodes. According to Lemma 9.5(iv), there are only finitely many  $\mu$ -sections in the union  $S_i^\mu$ . Then, by the assumed Properties 11.1, all the  $\mu$ -node voltages in  $S_i^\mu$  are strictly less than the largest voltage  $u_1^{\mu+1}$  for the  $(\mu + 1)$ -nodes incident to  $S_i^\mu$  (i.e., incident to a  $\mu$ -section of  $S_i^\mu$ ). Let  $n_1^{\mu+1}$  be a  $(\mu + 1)$ -node incident to  $S_i^\mu$  with the largest voltage  $u_1^{\mu+1}$ . (Possibly,  $n_1^{\mu+1} = n_a^{\mu+1}$ .) We have  $u_1^{\mu+1} \geq u_a^{\mu+1}$ .

Suppose all the  $(\mu + 1)$ -nodes adjacent to  $n_1^{\mu+1}$  have voltages no larger than  $u_1^{\mu+1}$ . Ignore the  $\mu$ -sections incident to  $n_1^{\mu+1}$  with constant  $\mu$ -node voltages; all their currents are zero. With regard to the remaining  $\mu$ -sections incident to  $n_1^{\mu+1}$ , from the finitely  $\mu$ -chainlike structure for a union of  $\mu$ -spurs for all the  $\mu$ -ends embraced by  $n_1^{\mu+1}$ , choose a  $\mathcal{V}_p^\mu$  and a  $\mathcal{V}_q^\mu$  ( $q > p$ ), where  $\mathcal{V}_q^\mu$  separates  $\mathcal{V}_p^\mu$  from  $n_1^{\mu+1}$ . Next, consider the network between  $\mathcal{V}_p^\mu$  and  $\mathcal{V}_q^\mu$ . It has  $(\mu - 1)$ -spurs for its  $(\mu - 1)$ -ends embraced by the nodes of  $\mathcal{V}_q^{\mu-1}$ , and the union of those spurs has a  $(\mu - 1)$ -chainlike structure. Choose a  $\mathcal{V}_q^{\mu-1}$  from it. Thus,  $\mathcal{V}_q^{\mu-1}$  separates  $\mathcal{V}_p^\mu$  from  $\mathcal{V}_q^\mu$  and therefore from  $n_1^{\mu+1}$ . Repeating this procedure for still lower ranks, we finally obtain a  $\mathcal{V}_q^0$ , where  $\mathcal{V}_q^0$  separates  $\mathcal{V}_p^\mu$  from  $n_1^{\mu+1}$ . Moreover, these choices can be so made that the largest  $\mu$ -node voltage for  $\mathcal{V}_p^\mu$  is strictly less than the least 0-node voltage in  $\mathcal{V}_q^0$ . The latter requirement can be fulfilled by virtue of (a2) and (b) of Properties 11.1 and our strong-induction hypothesis. Therefore, we can connect pure voltage sources to the nodes of  $\mathcal{V}_p^\mu \cup \mathcal{V}_q^0$  as in our argument for Theorem 4.3 to derive a contradiction to Kirchhoff's current law as applied to a cut at  $\mathcal{V}_q^0$  for  $n_1^{\mu+1}$ . This shows that adjacent to  $n_1^{\mu+1}$  there is a  $(\mu + 1)$ -node  $n_2^{\mu+1}$  with  $u_2^{\mu+1} > u_1^{\mu+1}$ . We choose  $n_2^{\mu+1}$  such that  $u_2^{\mu+1}$  is the largest voltage for all  $(\mu + 1)$ -nodes adjacent to  $n_1^{\mu+1}$ .

Repeating this argument, we can find another  $(\mu + 1)$ -node  $n_3^{\mu+1}$  with the following properties:  $n_3^{\mu+1}$  is adjacent to  $n_2^{\mu+1}$  but not to  $n_1^{\mu+1}$ ;  $u_3^{\mu+1} > u_2^{\mu+1}$ ;  $u_3^{\mu+1}$  is the largest voltage for all  $(\mu + 1)$ -nodes adjacent to  $n_2^{\mu+1}$ .

Further repetitions lead to an infinite sequence  $\{n_k^{\mu+1}\}_{k=1}^\infty$  of  $(\mu + 1)$ -nodes wherein two  $(\mu + 1)$ -nodes are adjacent in the sequence if and only if they are adjacent in  $N_\nu$ . Moreover,  $u_k^{\mu+1} < u_{k+1}^{\mu+1}$  for every  $k$ . It follows that we can find a  $(\mu + 1)$ -path  $P^{\mu+1}$  having  $n_1^{\mu+1}, n_2^{\mu+1}, \dots$  as its consecutive  $(\mu + 1)$ -nodes.  $P^{\mu+1}$  is a representative of a  $(\mu + 1)$ -tip embraced by some  $(\mu + 2)$ -node  $n_0^{\mu+2}$  incident to  $\mathbf{S}^{\mu+1}$ , the sourceless  $(\mu + 1)$ -section with which we started. Now, the  $(\mu + 1)$ -node voltages along  $P^{\mu+1}$  are strictly increasing. Moreover,  $P^{\mu+1}$  meets infinitely often various nodes of  $\bigcup_{p=0}^\infty \mathcal{V}_p^{\mu+1}$  for a finitely  $(\mu + 1)$ -chainlike structure for a union of  $(\mu + 1)$ -spurs (chosen according to Condition 9.4(e)) for all  $(\mu + 1)$ -ends embraced by  $n_0^{\mu+2}$ . The voltages for the nodes of  $\bigcup_{p=0}^\infty \mathcal{V}_p^{\mu+1}$  comprise a directed function and converge to  $u_0^{\mu+2}$  because every spine is perceptible and there are only



finitely many spines. Consequently, the  $(\mu + 1)$ -node voltages along  $P^{\mu+1}$  converge to  $u_0^{\mu+2}$ . Thus,  $u_0^{\mu+2} > u_1^{\mu+1} \geq u_a^{\mu+1}$ . Since  $n_a^{\mu+1}$  was chosen arbitrarily in  $S^{\mu+1}$ , we can conclude that every  $(\mu + 1)$ -node voltage for  $S^{\mu+1}$  is strictly less than the largest of the voltages for the finitely many  $(\mu + 2)$ -nodes incident to  $S^{\mu+1}$ .

A similar argument shows that, if part (a1) of Properties 11.1 is not fulfilled by  $S^{\mu+1}$ , then every  $(\mu + 1)$ -node voltage for  $S^{\mu+1}$  is strictly larger than the least of the voltages for the finitely many  $(\mu + 2)$ -nodes incident to  $S^{\mu+1}$ . This establishes part (a) of Properties 11.1 for  $\mu$  replaced by  $\mu + 1$ .

As for part (b), let  $\mu \leq \nu - 2$  again and let  $P^{\mu+1}$  be a one-ended  $(\mu + 1)$ -path in  $S^{\mu+1}$ . By Lemma 9.5(i) and (ii),  $P^{\mu+1}$  lies in a  $(\mu + 1)$ -spur, which is finitely  $(\mu + 1)$ -chainlike in accordance with Condition 9.4(e). Let  $n^{\mu+2}$  be the  $(\mu + 2)$ -node that  $P^{\mu+1}$  meets. Now consider any  $M_p^{\mu+1}$  in that chainlike structure. By our preceding argument, all the voltages for the  $(\mu + 1)$ -nodes in  $M_p^{\mu+1}$  are no larger (no less) than the largest (respectively, least)  $(\mu + 1)$ -node voltage for  $\mathcal{V}_p^{\mu+1} \cup \mathcal{V}_{p+1}^{\mu+1}$ . Since there are only finitely many spines and they are all perceptible, these largest and least voltages converge to the voltage at  $n^{\mu+2}$  as  $p \rightarrow \infty$ . Thus, part (b) holds for  $\mu$  replaced by  $\mu + 1$ . We have established inductively

**Theorem 11.2.** *Under Conditions 9.4, Properties 11.1 are fulfilled by any sourceless  $\mu$ -section  $S^\mu$  with any rank  $\mu = 0, \dots, \nu - 1$ .*

**Corollary 11.3.** *If  $S^\mu$  is a sourceless  $\mu$ -section with  $0 \leq \mu < \nu$  and if  $S^\mu$  has only one incident  $(\mu + 1)$ -node, then all the node voltages for  $S^\mu$  are the same, namely, the voltage of the incident  $(\mu + 1)$ -node.*

We can draw a stronger conclusion regarding convergence of voltages along one-ended paths even when there are finitely many sources appended to  $S^\mu$ , both of the voltage and current types and both pure and resistive.

**Lemma 11.4.** *If no more than finitely many sources are appended to nodes of  $S^\mu$ , part (b) of Properties 11.1 remains true for every  $\mu < \nu$ .*

**Proof.** In the prior inductive proof of part (b),  $P^{\mu+1}$  can be restricted to a  $(\mu + 1)$ -spur which is sourceless. Then, every  $M_p^{\mu+1}$  is sourceless and the prior argument can be applied again.  $\square$

Let us now return to  $N_e^\nu$ , that is, to  $N^\nu$  with a single source—a pure 1-volt source—appended to two nodes of arbitrary ranks; the negative terminal of the source is assigned the zero node voltage. The next objective is another extension to  $N_e^\nu$  of the maximum principle for node voltages. We start with a lemma, which will be used later on to construct a contradiction.

**Lemma 11.5.** *Let  $S^\mu$  be a  $\mu$ -section with  $0 \leq \mu < \nu$  and let it contain one or both of the source nodes. Assume there is a  $\mu$ -node  $n_0^\mu$  in  $S^\mu$  with  $u_0^\mu > 1$ . Then, there is a  $(\mu + 1)$ -node  $n_1^{\mu+1}$  incident to  $S^\mu$  such that  $u_1^{\mu+1} > 1$ ,  $u_1^{\mu+1}$  is no less than the voltages at all the  $(\mu + 1)$ -nodes incident to  $S^\mu$ , and  $u_1^{\mu+1}$  is strictly larger than the voltages at all the  $\mu$ -nodes within  $S^\mu$ .*

**Proof.** We argue inductively. For  $\mu = 0$  this lemma follows from Lemma 4.2 for the case where the 0-section  $S^0$  contains both  $n_0^0$  and a source node and therefore has differing 0-node voltages. Now, assuming the lemma is true for all ranks up to and including some chosen rank  $\mu \leq \nu - 2$ , we shall prove that it is true for the  $(\mu + 1)$ -section  $S^{\mu+1}$  that contains  $S^\mu$ .

Let there be a  $(\mu + 1)$ -node  $n_0^{\mu+1}$  in  $S^{\mu+1}$  with  $u_0^{\mu+1} > 1$ . It follows that there will be two adjacent  $(\mu + 1)$ -nodes  $n_a^{\mu+1}$  and  $n_b^{\mu+1}$  in  $S^{\mu+1}$  with  $u_a^{\mu+1} \geq u_0^{\mu+1} > 1$  and  $u_a^{\mu+1} > u_b^{\mu+1}$ . Let  $S_i^{\mu+1}$  denote the finite union of all the  $\mu$ -sections that are incident to both  $n_a^{\mu+1}$  and  $n_b^{\mu+1}$ . Then, by Lemma 11.4 in conjunction either with our inductive hypothesis (if a  $\mu$ -section of  $S_i^{\mu+1}$  contains a source node) or with Theorem 11.2 (if a  $\mu$ -section of  $S_i^{\mu+1}$  is sourceless), there exists a  $(\mu + 1)$ -node incident to  $S_i^{\mu+1}$  such that  $u_1^{\mu+1}$  is no less than the voltages at the finitely many  $(\mu + 1)$ -nodes incident to  $S_i^{\mu+1}$  and  $u_1^{\mu+1}$  is strictly larger than the voltages at all the  $\mu$ -nodes in  $S_i^{\mu+1}$ . Thus,  $u_1^{\mu+1} \geq u_a^{\mu+1} \geq u_0^{\mu+1} > 1$ .

Using Lemma 11.4 again, we now proceed exactly as in the argument that led to Theorem 11.2 to conclude that, if Kirchhoff's current law is not to be violated at a cut that isolates  $n_1^{\mu+1}$  from all other  $(\mu + 1)$ -nodes and from the source nodes, there must be a  $(\mu + 1)$ -node  $n_2^{\mu+1}$  adjacent to  $n_1^{\mu+1}$  such that the following hold:  $u_2^{\mu+1} > u_1^{\mu+1} \geq u_0^{\mu+1} > 1$ ;  $u_2^{\mu+1}$  is no less than all the voltages at the finitely many  $(\mu + 1)$ -nodes adjacent to  $n_1^{\mu+1}$ ;  $u_2^{\mu+1}$  is strictly greater than the voltages at all  $\mu$ -nodes within the  $\mu$ -sections incident to  $n_1^{\mu+1}$ .

A repetition of the last paragraph shows that, if Kirchhoff's current law is not to be violated at a cut that isolates  $n_2^{\mu+1}$  from all other  $(\mu + 1)$ -nodes and from the source nodes, there must be another  $(\mu + 1)$ -node  $n_3^{\mu+1}$  adjacent to  $n_2^{\mu+1}$  such that the stated conditions are again satisfied with  $n_1^{\mu+1}$  replaced by  $n_2^{\mu+1}$  and  $n_2^{\mu+1}$  replaced by  $n_3^{\mu+1}$ .

Continuing in this way, we find a sequence  $n_1^{\mu+1}, n_2^{\mu+1}, n_3^{\mu+1}, \dots$  of consecutively adjacent  $(\mu + 1)$ -nodes in  $\mathbf{S}^{\mu+1}$  with  $1 < u_0^{\mu+1} \leq u_a^{\mu+1} \leq u_1^{\mu+1} < u_2^{\mu+1} < u_3^{\mu+1} < \dots$ . It follows that these are the  $(\mu + 1)$ -nodes along a  $(\mu + 1)$ -path  $P^{\mu+1}$  that meets a  $(\mu + 2)$ -node  $n_0^{\mu+2}$  incident to  $\mathbf{S}^{\mu+1}$  and whose voltages converge to  $u_0^{\mu+2}$  (Lemma 11.4 again). Since this is so whatever be the node  $n_0^{\mu+1}$  with which we started, we can furthermore conclude that there is a  $(\mu + 2)$ -node  $n_1^{\mu+2}$  incident to  $\mathbf{S}^{\mu+1}$  such that the conclusion of Lemma 11.5 is fulfilled when  $\mu$  is replaced by  $\mu + 1$ . This completes the inductive proof.  $\square$

**Theorem 11.6.** *The voltage at every node (of any rank) in  $\mathbf{N}_e^\nu$  is no less than zero and no greater than one.*

**Proof.** Suppose there is a node with a voltage larger than one. We can apply Theorem 11.2 and Lemma 11.5 recursively to conclude that there is a  $\nu$ -node  $n_0^\nu$  in  $\mathbf{N}_e^\nu$  with the following two properties:  $u_0^\nu > 1$ ;  $u_0^\nu$  is no less than all the voltages at all the nodes of all ranks in  $\mathbf{N}_e^\nu$ . By tracing a  $\nu$ -path from  $n_0^\nu$  to a source node, we can find a  $\nu$ -node  $n_1^\nu$  such that  $u_1^\nu = u_0^\nu$  and  $n_1^\nu$  is incident to a  $(\nu - 1)$ -section whose nodes of all ranks less than  $\nu$  have voltages strictly less than  $u_1^\nu$  (invoke Lemma 11.5 again). However, this implies a contradiction to Kirchhoff's current law at a cut that isolates  $n_0^\nu$  from all the other  $\nu$ -nodes and also from the source nodes. Hence, our supposition is false.

A similar argument involving a modification of Lemma 11.5 establishes that there is no node with a negative voltage. (Alternatively, we can obtain this second conclusion from the first one by reversing the source's polarity and adding 1 to all node voltages. This merely reverses all branch currents.)  $\square$

As before,  $n_e^\alpha$  denotes the source node at 1-volt and  $n_g^\beta$  denotes the source node at 0 volt,  $\alpha$  and  $\beta$  being their ranks.

**Corollary 11.7.**

(i) *Let  $n_0^\gamma$  be a  $\gamma$ -node for which there is  $\mu$ -path  $P^\mu$ , where  $\mu \geq \max(\beta, \gamma)$ , that terminates*

at  $n_0^\gamma$  and  $n_g^\beta$  and does not embrace  $n_e^\alpha$ . Then,  $u_0^\gamma < 1$ .

(ii) Let  $n_0^\gamma$  be a  $\gamma$ -node for which there is a  $\mu$ -path  $P^\mu$ , where  $\mu \geq \max(\alpha, \gamma)$ , that terminates at  $n_0^\gamma$  and  $n_e^\alpha$  and does not embrace  $n_g^\beta$ . Then,  $u_0^\gamma > 0$ .

**Proof.** Under the hypothesis of (i), suppose  $u_0^\gamma = 1$ . Let us trace  $P^\nu$  from  $n_0^\gamma$  to  $n_g^\beta$  and examine the node voltages. They must eventually fall below one. By virtue of Lemma 11.4 and Theorems 11.2 and 11.6, exactly two cases arise.

*Case 1:*  $P^\mu$  embraces two adjacent  $\lambda$ -nodes  $n_a^\lambda$  and  $n_b^\lambda$  with  $u_a^\lambda = 1$  and  $u_b^\lambda < 1$ , and the  $(\lambda - 1)$ -sections to which  $n_a^\lambda$  and  $n_b^\lambda$  are both incident are sourceless. As in the proof of Lemma 11.5 (with  $\mu + 1$  replaced by  $\lambda$ ), we can find a  $\lambda$ -node  $n_2^\lambda$  with  $u_2^\lambda > u_a^\lambda = 1$ . But,  $u_2^\lambda > 1$  contradicts Theorem 11.6.

*Case 2:*  $P^\mu$  embraces a  $\lambda$ -node  $n_a^\lambda$  that is incident to at least one sourceless  $(\lambda - 1)$ -spur whose  $(\lambda - 1)$ -nodes have voltages strictly less than one. Since all node voltages are no larger than one (Theorem 11.6), this violates Kirchhoff's current law at a cut that isolates  $n_a^\lambda$  from all other  $\lambda$ -nodes and from the source nodes.

Hence,  $u_0^\gamma < 1$ . A similar argument works for (ii).  $\square$

## 12 $\mu$ -Walks

With the results of Sections 9 through 11 in hand, we can extend our discussion of random roving 1-walks to walks on the transfinite network  $N^\nu$  quite directly. In this section,  $\mu \leq \nu$ . Having defined 0-walks and 1-walks, we may now define  $\mu$ -walks recursively. We take it that, for  $\mu \geq 2$ ,  $(\mu - 1)$ -walks have already been defined as the alternating sequence

$$W^{\mu-1} = \{\dots, n_m^{\mu-1}, W_m^{\mu-2}, n_{m+1}^{\mu-1}, W_{m+1}^{\mu-2}, \dots\}$$

of  $(\mu - 1)$ -nodes  $n_m^{\mu-1}$ , nontrivial  $(\mu - 2)$ -walks  $W_m^{\mu-2}$ , and possibly a terminal element on the left and/or on the right. A terminal element is required to be an  $\eta$ -node where  $0 \leq \eta \leq \mu - 1$ . In this way,  $W^{\mu-1}$  may be finite, one-ended, or endless.  $W^{\mu-1}$  is called *nontrivial* if it has at least one  $(\mu - 2)$ -walk. When denoting a terminal node, we will drop the superscript if that node's rank is arbitrary and unspecified. (Shortly, we shall complete our recursive definition of a  $\mu$ -walk by stating all the conditions that a  $\mu$ -walk must fulfill.)

One-ended portions of a one-ended or endless  $W^{\mu-1}$  are denoted by

$$W_{-\infty, m}^{\mu-1} = \{\dots, W_{m-2}^{\mu-2}, n_{m-1}^{\mu-1}, W_{m-1}^{\mu-2}, n_m^{\mu-1}\}$$

and

$$W_{m, \infty}^{\mu-1} = \{n_m^{\mu-1}, W_m^{\mu-2}, n_{m+1}^{\mu-1}, W_{m+1}^{\mu-2}, \dots\}.$$

Let  $d^{\mu-1}$  denote a  $(\mu-1)$ -end. Choose a finitely  $(\mu-1)$ -chainlike representation  $M^{\mu-1} = \bigcup_{p=0}^{\infty} M_p^{\mu-1}$  for a  $(\mu-1)$ -spur for  $d^{\mu-1}$ . We say that  $W^{\mu-1}$  *starts at*  $d^{\mu-1}$  and also *starts at* the  $\mu$ -node that embraces  $d^{\mu-1}$  if, given any integer  $q \geq 0$ , there is an  $m$  such that  $W_{-\infty, m}^{\mu-1}$  remains within  $\bigcup_{p=q}^{\infty} M_p^{\mu-1}$ . Under the same definition but with  $W_{-\infty, m}^{\mu-1}$  replaced by  $W_{m, \infty}^{\mu-1}$ , we say that  $W^{\mu-1}$  *stops at*  $d^{\mu-1}$  and also *stops at* the  $\mu$ -node that embraces  $d^{\mu-1}$ . Also,  $W^{\mu-1}$  is called *transient* if it stops at a  $\mu$ -node and *transient-in-reverse* if it starts at a  $\mu$ -node.

Now let  $W^\mu$  be a (finite, one-ended, or endless) alternating sequence of the form

$$W^\mu = \{\dots, n_m^\mu, W_m^{\mu-1}, n_{m+1}^\mu, W_{m+1}^{\mu-1}, \dots\} \quad (27)$$

where  $m$  is restricted to the integers, every  $W_m^{\mu-1}$  is a nontrivial  $(\mu-1)$ -walk, and every  $n_m^\mu$  is a  $\mu$ -node—with a possible exception arising if  $W^\mu$  terminates on the left and/or on the right. In the latter case, the terminal element is a node, and its rank may be any integer from 0 to  $\mu$ . Terms in (27) may repeat. For example, consecutive  $\mu$ -nodes may be the same. (The latter was also allowed when  $\mu = 1$ , but could not occur when  $\mu = 0$  because of the absence of self-loops.)

We need to define terminal behavior when (27) is one-ended or finite and terminates at a node of rank less than  $\mu$ . In particular, consider the one-way infinite sequence

$$W^\mu = \{n_0^\eta, W_0^{\mu-1}, n_1^\mu, W_1^{\mu-1}, \dots\} \quad (28)$$

where  $0 \leq \eta < \mu$ . This will signify that there is a finite sequence of  $\lambda$ -walks, where  $\lambda = \eta, \dots, \mu$ , of the form:

$$\begin{aligned} W_0^\eta &= \{n_0^\eta, W_0^{\eta-1}, n_1^\eta, W_1^{\eta-1}, n_2^\eta, \dots\} \\ W_0^{\eta+1} &= \{n_0^\eta, W_0^\eta, n_1^{\eta+1}, W_1^\eta, n_2^{\eta+1}, \dots\} \end{aligned}$$

$$\begin{aligned}
W_0^{\mu-1} &= \{n_0^\eta, W_0^{\mu-2}, n_1^{\mu-1}, W_1^{\mu-2}, n_2^{\mu-1}, \dots\} \\
W^\mu &= \{n_0^\eta, W_0^{\mu-1}, n_1^\mu, W_1^{\mu-1}, n_2^\mu, \dots\}
\end{aligned}$$

where  $W_0^{\eta-1}$  starts at  $n_0^\eta$ ,  $W_0^{\lambda-1}$  stops at  $n_1^\lambda$ , and finally  $W_m^{\lambda-1}$  for  $m > 0$  starts at  $n_m^\lambda$  and stops at  $n_{m+1}^\lambda$ . In this case, we shall say that (28) *starts at*  $n_0^\eta$ . A similar definition is used when (27) terminates on the right at an  $\eta$ -node  $n_z^\eta$ , in which case we say that (27) *stops at*  $n_z^\eta$ .

**Definition 12.1.**  $W^\mu$ , as given by (27), is called a  $\mu$ -walk if, for every  $m$ ,  $W_m^{\mu-1}$  starts at the node on its left and stops at the node on its right. Also,  $W^\mu$  is said to perform a *one-step  $\mu$ -transition* from  $n_m^\mu$  to  $n_{m+1}^\mu$ .

One consequence of this definition is that  $W^\mu$  is confined to some  $\mu$ -section. Also, since no node embraces a node of lower rank according to Condition 9.4(a), a  $(\mu - 1)$ -walk  $W_m^{\mu-1}$  in (27) can start or stop at a  $\mu$ -node adjacent to  $W_m^{\mu-1}$  in (27) only if it starts or stops at a  $(\mu - 1)$ -end embraced by that  $\mu$ -node. We use the words “to reach” and “to pass through” as before.  $W^\mu$  will be called *nontrivial* if (27) has at least one  $(\mu - 1)$ -walk. Also,  $W^\mu$  is said to *embrace* itself, all of its elements, all elements embraced by its elements, all elements embraced by the elements embraced by its elements, and so forth. By these definitions, for any  $\eta < \mu$  every  $\eta$ -walk embraced by  $W^\mu$  must be both transient and transient-in-reverse unless it is respectively the last or first  $\eta$ -walk embraced by  $W^\mu$ .

**Definition 12.2.** The  $\mu$ -walk  $W^\mu$  (and  $\Psi$  too) is said to *rove* if, for every  $\eta$ -walk  $W^\eta$  embraced by  $W^\mu$  ( $\eta \leq \mu$ ), every two consecutive  $\eta$ -nodes in  $W^\eta$  are different.

Henceforth, we always assume that  $\Psi$  roves whatever be the rank  $\mu \leq \nu$ .

### 13 Random $\mu$ -Walks

We will now discuss how  $\Psi$  may wander in  $\mathbf{N}^\nu$  by passing through nodes of various ranks. Transitions between and through nodes of ranks 0 and 1 have already been discussed in

Sections 6 and 7.

In accordance with our recursive approach, we now take it that random roving  $\eta$ -walks have been defined for  $\eta = 1, \dots, \mu$ , where  $1 \leq \mu < \nu$ . This means in particular that appropriate generalizations of Definitions 6.2, 6.3, 7.1, and 7.2 have been established for each  $\eta$  less than  $\mu$ . These are stated in Definition 13.1 below (for  $\mu$  instead of  $\eta$ ). In effect, the behavior of a random roving  $\eta$ -walk that is transient and also transient-in-reverse is taken as the local behavior of a random roving  $(\eta + 1)$ -walk. Those definitions are then used to establish random roving  $(\mu + 1)$ -walks.

As always,  $\mathbf{S}^\mu$  will denote a  $\mu$ -section in  $\mathbf{N}^\nu$ , where  $\mu < \nu$ .

**Definition 13.1.**

- (a) Let  $\mathbf{S}^\mu$  have two or more incident  $(\mu + 1)$ -nodes. Let  $n_k^{\mu+1}$  be a  $(\mu + 1)$ -node incident to  $\mathbf{S}^\mu$  and let  $\mathcal{N}_g^{\mu+1}$  be the set of all the other  $(\mu + 1)$ -nodes incident to  $\mathbf{S}^\mu$ . Given that  $\Psi$  starts at a  $\mu$ -node  $n_0^\mu$  of  $\mathbf{S}^\mu$  and reaches some  $(\mu + 1)$ -node (i.e.,  $W^\mu$  is transient), the probability that  $\Psi$  will reach  $n_k^{\mu+1}$  before reaching some node of  $\mathcal{N}_g^{\mu+1}$ :

$$Prob(sn_0^\mu, rn_k^{\mu+1}, b\mathcal{N}_g^{\mu+1} \mid W^\mu \text{ is transient}) \quad (29)$$

is defined as the voltage at  $n_0^\mu$  when  $n_k^{\mu+1}$  is held at 1 volt and all the nodes of  $\mathcal{N}_g^{\mu+1}$  are held at 0 volt.

- (b) Now,  $\mathbf{S}^\mu$  is allowed to have just one incident  $(\mu + 1)$ -node. Let  $\mathcal{N}_g^\mu$  be any finite set of  $\mu$ -nodes in  $\mathbf{S}^\mu$ . Let  $n_0^\mu$  be another  $\mu$ -node in  $\mathbf{S}^\mu$  with  $n_0^\mu \notin \mathcal{N}_g^\mu$ . Let  $\mathcal{N}_e^{\mu+1}$  be the (finite) set of all  $(\mu + 1)$ -nodes incident to  $\mathbf{S}^\mu$ . Given that  $\Psi$  starts at  $n_0^\mu$ , the probability that  $\Psi$  will reach some node of  $\mathcal{N}_e^{\mu+1}$  before reaching some node of  $\mathcal{N}_g^\mu$ :

$$Prob(sn_0^\mu, r\mathcal{N}_e^{\mu+1}, b\mathcal{N}_g^\mu) \quad (30)$$

is defined as the voltage at  $n_0^\mu$  when all the nodes of  $\mathcal{N}_e^{\mu+1}$  are held at 1 volt and all the nodes of  $\mathcal{N}_g^\mu$  are held at 0 volt.

- (c) Let  $n_0^{\mu+1}$  be a  $(\mu + 1)$ -node. Choose a  $\mu$ -spur for every  $\mu$ -end embraced by  $n_0^{\mu+1}$ , making those  $\mu$ -spurs mutually disjoint (apply Lemma 9.3 to every  $\mu$ -section incident

to  $n_0^{\mu+1}$ ). Then, choose a finite  $\mu$ -chainlike representation  $\bigcup_{p=0}^{\infty} M_p^\mu$  for the union of those  $\mu$ -spurs. For  $p \geq 1$ , set  $V_p^\mu = M_p^\mu \odot M_{p-1}^\mu$ . Assume that  $V_p^\mu$  has two or more nodes, and let  $V_p^\mu \setminus n_{p,k}^\mu$  denote the set of all nodes in  $V_p^\mu$  other than  $n_{p,k}^\mu \in V_p^\mu$ . Given that  $\Psi$  starts at  $n_0^{\mu+1}$  and reaches a node of  $V_p^\mu$ , the probability that  $\Psi$  will reach the node  $n_{p,k}^\mu \in V_p^\mu$  before it reaches any of the other nodes in  $V_p^\mu$ :

$$Prob(sn_0^{\mu+1}, rn_{p,k}^\mu, bV_p^\mu \setminus n_{p,k}^\mu \mid \Psi \text{ reaches } V_p^\mu) \quad (31)$$

is defined as the voltage at  $n_0^{\mu+1}$  when  $n_{p,k}^\mu$  is held at 1 volt and all the other nodes of  $V_p^\mu$  are held at 0 volt.

- (d) Assume that there are two or more  $(\mu + 1)$ -nodes adjacent to the  $(\mu + 1)$ -node  $n_0^{\mu+1}$ . Let  $n_k^{\mu+1}$  be a  $(\mu + 1)$ -node adjacent to  $n_0^{\mu+1}$  and let  $\mathcal{N}_g^{\mu+1}$  be the set of all the other  $(\mu + 1)$ -nodes adjacent to  $n_0^{\mu+1}$ . Given that  $\Psi$  starts at  $n_0^{\mu+1}$  (remember that  $\Psi$  roves), the probability that  $\Psi$  will reach  $n_k^{\mu+1}$  before it reaches some node of  $\mathcal{N}_g^{\mu+1}$ :

$$Prob(sn_0^{\mu+1}, rn_k^{\mu+1}, b\mathcal{N}_g^{\mu+1}) \quad (32)$$

is defined as the voltage at  $n_0^{\mu+1}$  when  $n_k^{\mu+1}$  is held at 1 volt and all the nodes of  $\mathcal{N}_g^{\mu+1}$  are held at 0 volt.

- (e) Let  $n_g^\mu$  be a  $\mu$ -node in  $S^\mu$  and let  $n_0^{\mu+1}$  be incident to  $S^\mu$ . Let  $\mathcal{N}_e^{\mu+1}$  denote the set of all  $(\mu + 1)$ -nodes adjacent to  $n_0^{\mu+1}$ . Given that  $\Psi$  starts at  $n_0^{\mu+1}$  (and roves), the probability that  $\Psi$  reaches some node of  $\mathcal{N}_e^{\mu+1}$  before reaching  $n_g^\mu$ :

$$Prob(sn_0^{\mu+1}, r\mathcal{N}_e^{\mu+1}, bn_g^\mu) \quad (33)$$

is defined as the voltage at  $n_0^{\mu+1}$  when all the nodes of  $\mathcal{N}_e^{\mu+1}$  are held at 1 volt and  $n_g^\mu$  is held at 0 volt.

Parts (a), (b), (c), and (d) of this description generalize respectively Definitions 6.2, 6.3, 7.1, and 7.2. Part (e) represents an new situation that we shall encounter. Also, part (d) generalizes the standard rule for one-step transitions for random 0-walks.

To justify Definition 13.1(a) as a limiting case of the Nash-Williams rule, choose a  $\mu$ -spur for every  $\mu$ -end of  $S^\mu$  with each  $\mu$ -spur disjoint from the other  $\mu$ -spurs (Lemma 9.3). Only



finitely many such  $\mu$ -spurs are needed. Now, consider any single  $n_k^{\mu+1}$  of the  $(\mu + 1)$ -nodes that are incident to  $\mathbf{S}^\mu$ . The union of the  $\mu$ -spurs for the  $\mu$ -ends of  $\mathbf{S}^\mu$  that are embraced by  $n_k^{\mu+1}$  is finitely  $\mu$ -chainlike (Lemma 9.5(e)), and so we can choose a chainlike representation  $\bigcup_{p=0}^{\infty} \mathbf{M}_{k,p}^\mu$  for that union, wherein the spines are all perceptible. Then, for any given  $\mu$ -node  $n_0^\mu$  in  $\mathbf{S}^\mu$ , a  $p_k$  can be chosen such that  $n_0^\mu$  is not in  $\bigcup_{p=p_k}^{\infty} \mathbf{M}_{k,p}^\mu$ . Let  $\mathcal{V}_{k,p_k}^\mu = \mathbf{M}_{p_k}^\mu \odot \mathbf{M}_{p_k-1}^\mu$ ; this is finite set of  $\mu$ -nodes that within  $\mathbf{S}^\mu$  separates  $n_k^{\mu+1}$  from  $n_0^\mu$ . With  $n_l^{\mu+1}$  ( $l = 1, \dots, K$ ) denoting all of the  $(\mu + 1)$ -nodes incident to  $\mathbf{S}^\mu$ , perform this construction for each union of  $\mu$ -spurs for the  $\mu$ -ends of  $\mathbf{S}^\mu$  embraced by an  $n_l^{\mu+1}$ . The  $\mathcal{V}_{l,p_l}^\mu$  are mutually disjoint, and together they separate  $n_0^\mu$  from all the  $(\mu + 1)$ -nodes; in fact, they separate the  $(\mu + 1)$ -nodes from the reduced  $\mu$ -network  $F^\mu(p_1, \dots, p_K)$  induced by all branches in  $\mathbf{S}^\mu$  that are not in  $\bigcup_{l=1}^K \bigcup_{p=p_l}^{\infty} \mathbf{M}_{l,p_l}^\mu$ .  $F^\mu(p_1, \dots, p_K)$  has only finitely many  $\mu$ -nodes.

Now, in accordance with our recursive construction, we also take it that there is a finite surrogate 0-network  $F^{\mu \rightarrow 0}(p_1, \dots, p_K)$  whose 0-nodes correspond bijectively to the  $\mu$ -nodes of  $F^\mu(p_1, \dots, p_K)$  and whose 0-walks are governed by the same probability transition matrix as that which governs the random roving  $\mu$ -walks on  $F^\mu(p_1, \dots, p_K)$ . Thus, by applying the Nash-Williams rule, we can determine the probability  $v_{0,k}^\mu(p_1, \dots, p_K)$  of  $\Psi$  reaching any node of  $\mathcal{V}_{k,p_k}^\mu$  before it reaches any node of  $\bigcup_{l \in L} \mathcal{V}_{l,p_l}^\mu$ , where  $L = \{l : l = 1, \dots, K; l \neq k\}$ . (Hold the nodes of  $\mathcal{V}_{k,p_k}^\mu$  at 1 volt, hold the nodes of  $\bigcup_{l \in L} \mathcal{V}_{l,p_l}^\mu$  at 0 volt, and measure the voltage  $v_{0,k}^\mu(p_1, \dots, p_K)$  at  $n_0^\mu$ .) Finally, exactly as in the proof of Lemma 6.1 but using Theorem 11.6 in place of Theorem 4.3 and Theorem 11.2 (for Property 11.1(b)) in place of Lemma 4.1, we can show that, as the  $p_1, \dots, p_K$  tend to infinity independently,  $v_{0,k}^\mu(p_1, \dots, p_K)$  converges to  $u_{0,k}^\mu$ , the voltage indicated in Definition 13.1(a). This then is the justification of that definition as a limiting case of the Nash-Williams rule extended to  $\mu$ -networks.

In much the same way, Definition 13.1(b) can be justified.

Definition 13.1(c) indicates how a random  $\mu$ -walk may start at a  $(\mu + 1)$ -node  $n^{\mu+1}$ . Just as in Section 7, it can be shown that this definition yields a consistent set of probabilities when the comparative probabilities for transitions from  $n^{\mu+1}$  to the nodes of  $\mathcal{V}_p^\mu$  are first obtained directly from the definition and are then obtained by using conditional

probabilities to combine probabilities for transitions from  $n^{\mu+1}$  to the nodes of  $\mathcal{V}_q^\mu$  ( $q > p$ ) with probabilities for transitions from the nodes of  $\mathcal{V}_q^\mu$  to the nodes of  $\mathcal{V}_p^\mu$ . The latter transition probabilities are defined as in Definition 7.1 (except that 0-nodes are now replaced by  $\mu$ -nodes.).

Furthermore, for  $p$  fixed and given that  $\Psi$  starts from  $n^{\mu+1}$  and reaches a node of  $\mathcal{V}_q^\mu$ , the probability that  $\Psi$  then reaches a node of  $\mathcal{V}_p^\mu$  before returning to  $n^{\mu+1}$  vanishes as  $q \rightarrow \infty$ . In other words, we are again dealing with the exceptional case when discussing random roving  $(\mu + 1)$ -walks: Such walks comprise but a vanishingly small proportion of all random  $(\mu + 1)$ -walks whose consecutive nodes in (27) are allowed to be the same.

Definition 13.1(d) states comparative probabilities for  $\Psi$  passing from a  $(\mu + 1)$ -node  $n_0^{\mu+1}$  to any of its adjacent  $(\mu + 1)$ -nodes  $n_k^{\mu+1}$  ( $k = 1, \dots, K$ ). To obtain it, we again proceed as in Section 7. Consider the  $\mu$ -ends of all the  $\mu$ -sections incident to  $n_0^{\mu+1}$  other than the  $\mu$ -ends embraced by  $n_0^{\mu+1}$ . Choose mutually disjoint  $\mu$ -spurs for those  $\mu$ -ends. Next, for each  $k$ , choose a  $(\mu + 1)$ -chainlike structure  $\bigcup_{p=0}^{\infty} M_{k,p}^\mu$  for the union of the  $\mu$ -spurs corresponding to  $n_k^{\mu+1}$ . Set  $\mathcal{V}_{k,p}^\mu = M_{k,p}^\mu \odot M_{k,p-1}^\mu$ . Finally, choose a natural number  $p_k$  for each  $k$ . Then,  $\bigcup_{k=1}^K \mathcal{V}_{k,p_k}^\mu$  is a set of  $\mu$ -nodes that separates  $n_0^{\mu+1}$  from all of the  $n_k^{\mu+1}$ . As a direct extension of Definition 13.1(c), we take the voltage at  $n_0^{\mu+1}$ , when the nodes of  $\mathcal{V}_{k,p_k}^\mu$  are held at 1 volt and the nodes of  $\bigcup_{l \in L} \mathcal{V}_{l,p_l}^\mu$ , where  $L = \{l : l = 1, \dots, K; l \neq k\}$ , are held at 0 volt, as the probability that  $\Psi$  will reach a node of  $\mathcal{V}_{k,p_k}^\mu$  before it reaches a node of any  $\mathcal{V}_{l,p_l}^\mu$ . Arguing as in the proof of Lemma 6.1 but using now the maximum principle of Theorem 11.6, we can show that these comparative transition probabilities converge as the  $p_1, \dots, p_K$  tend to infinity independently. Thus, Definition 13.1(d) also arises as a limiting case of a prior definition.

Finally, Definition 13.1(e) is obtained in the same way as Definition 13.1(d). Using the same  $\mathcal{V}_{k,p_k}^\mu$  as in the preceding paragraph, we now examine the voltage at  $n_0^{\mu+1}$  when  $n_g^\mu$  is held at 0 volt and either the nodes of  $\mathcal{V}_{k,p_k}^\mu$  or  $n_k^{\mu+1}$  is held at 1 volt for every  $k$ . Upon sending  $p_1, \dots, p_K$  to infinity, we obtain this definition as a limiting case again.

## 14 Transience

A  $\mu$ -section is called *transient* if, given that  $\Psi$  starts at any arbitrarily chosen 0-node  $n_g^0$  in  $\mathbf{S}^\mu$  and then roves, there is a positive probability that  $\Psi$  will reach some  $(\mu + 1)$ -node incident to  $\mathbf{S}^\mu$  before returning to  $n_g^0$ .

Let  $\mathbf{S}^{-1}$  denote the reduced network induced by the branches incident to  $n_g^0$ . We shall say that a 0-node is *incident to  $\mathbf{S}^{-1}$*  if it is adjacent to  $n_g^0$ . Also, for each  $\eta = 1, \dots, \mu + 1$ , let  $\mathbf{S}^{\eta-1}$  be the  $\eta$ -section in which  $n_g^0$  resides. Let  $P_t^{\mu+1}$  denote a three-term  $(\mu + 1)$ -path  $\{n_g^0, P_0^\mu, n_f^{\mu+1}\}$ ; thus,  $n_f^{\mu+1}$  is incident to  $\mathbf{S}^\mu$ , and  $P_t^{\mu+1}$  does not contain any other  $(\mu + 1)$ -node. Finally, for  $\eta = 1, \dots, \mu$ , let  $n_f^\eta$  be the first  $\eta$ -node embraced by  $P_t^{\mu+1}$  and let  $n_f^0$  be the first node after  $n_g^0$  embraced by  $P_t^{\mu+1}$ . (Figure 3 illustrates some of this.)

**Lemma 14.1.**  *$P_t^{\mu+1}$  can be so chosen that, for each  $\eta = 0, \dots, \mu + 1$ , every  $\eta$ -node embraced by  $P_t^{\mu+1}$  other than  $n_f^\eta$  is not incident to  $\mathbf{S}^{\eta-1}$ .*

**Proof.** Because there are only finitely many  $\mu$ -nodes incident to  $\mathbf{S}^{\mu-1}$ , there is a three-term  $(\mu + 1)$ -path  $P_f^{\mu+1} = \{n_f^\mu, P_0^\mu, n_f^{\mu+1}\}$  that starts at some  $\mu$ -node  $n_f^\mu$  incident to  $\mathbf{S}^{\mu-1}$ , stops at some  $(\mu + 1)$ -node  $n_f^{\mu+1}$  incident to  $\mathbf{S}^\mu$ , and embraces no other  $\mu$ -node incident to  $\mathbf{S}^{\mu-1}$ . For a similar reason, there is a three-term  $\mu$ -path  $P_f^\mu = \{n_f^{\mu-1}, P_0^{\mu-1}, n_f^\mu\}$  that starts at some node  $n_f^{\mu-1}$  incident to  $\mathbf{S}^{\mu-2}$ , stops at the chosen  $n_f^\mu$ , and embraces no other  $(\mu - 1)$ -node incident to  $\mathbf{S}^{\mu-1}$ . In this fashion, we can continue selecting three-term  $\eta$ -paths  $P_f^\eta = \{n_f^{\eta-1}, P_0^{\eta-1}, n_f^\eta\}$  for  $\eta = \mu, \dots, 1$ , where  $P_f^\eta$  embraces no other  $(\eta - 1)$ -node incident to  $\mathbf{S}^{\eta-1}$ ; also,  $n_f^\eta$  is the  $\eta$ -node previously chosen when selecting  $P_f^{\eta+1}$ . As the last selection, we choose the 0-path  $P_f^0 = \{n_g^0, b, n_f^0\}$ , where  $n_f^0$  is the previously chosen 0-node adjacent to  $n_g^0$ .

We now construct the  $(\mu + 1)$ -path  $P_t^{\mu+1}$  as follows: Append  $P_f^0$  as the initial 0-path to  $P_f^1$  to get a 1-path  $P_t^1$ . Append  $P_t^1$  as the initial 1-path to  $P_f^2$  to get a 2-path  $P_t^2$ . Continuing in this way, we finally append  $P_t^\mu$  to  $P_f^{\mu+1}$  to get the  $(\mu + 1)$ -path  $P_t^{\mu+1}$  that we seek.  $\square$

**Theorem 14.2.** *Under Conditions 9.4, if  $\Psi$  roves, then every  $\mu$ -section of every rank  $\mu$  is transient.*

**Proof.** With  $n_g^0$  chosen arbitrarily, choose a  $(\mu + 1)$ -path  $P_t^{\mu+1}$  in accordance with

Lemma 14.1. There is a positive probability that  $\Psi$ , after starting from  $n_g^0$  will reach  $n_f^0$  before reaching any other 0-node adjacent to  $n_g^0$ . The path  $P_f^1$  (specified in the proof of Lemma 14.1) embraces  $n_f^0$  and  $n_f^1$  and does not embrace  $n_g^0$ . So, by Definition 13.1(b) and Lemma 11.7(ii), there is a positive probability that  $\Psi$  will reach some 1-node incident to  $S^0$  before returning to  $n_g^0$ :

$$\text{Prob}(sn_f^0, r\mathcal{N}_g^1, bn_g^0) > 0,$$

where  $\mathcal{N}_g^1$  now denotes the set of all 1-nodes incident to  $S_0$ . If there is more than one 1-node incident to  $S^0$ , we invoke Definition 13.1(a): Since  $P_f^1$  does not embrace any 1-node incident to  $S^0$  other than  $n_f^1$ , Lemma 11.7(ii) implies that there is a positive probability that  $\Psi$  will reach  $n_f^1$  before reaching any other 1-node incident to  $S^0$ :

$$\text{Prob}(sn_f^0, rn_f^1, b\mathcal{N}_g^1 \setminus n_f^1 \mid \Psi \text{ reaches } \mathcal{N}_g^1) > 0$$

Thus, with positive probability  $\Psi$  will reach  $n_f^1$  before returning to  $n_g^0$ .

This argument can be continued inductively. (See Figure 3. The dots represent nodes and the closed curves represent sections—of various ranks.) Let us assume the following for any  $\eta$  such that  $1 \leq \eta \leq \mu$ : Given that  $\Psi$  starts at  $n_g^0$  and roves, there is a positive probability that  $\Psi$  reaches  $n_f^\eta$  before returning to  $n_g^0$ . Let  $\mathcal{N}_g^{\eta-1}$  be the set of all  $(\eta - 1)$ -nodes incident to the  $(\eta - 2)$ -section  $S^{\eta-2}$  that contains  $n_g^0$ . (If  $\eta = 1$ ,  $\mathcal{N}_g^{\eta-1}$  is replaced by  $n_g^0$ .) Also, let  $n_a^\eta$  be the next  $\eta$ -node after  $n_f^\eta$  that is embraced by  $P_t^{\mu+1}$ . Thus, there is a three-term  $\eta$ -path  $P_{fa}^\eta$  embraced by  $P_t^{\mu+1}$  that terminates at  $n_f^\eta$  and  $n_a^\eta$  and does not embrace any node of  $\mathcal{N}_g^{\eta-1}$ . Let  $\mathcal{N}_e^\eta$  denote the set of all  $\eta$ -nodes adjacent to  $n_f^\eta$ . By Definition 13.1(e) (we can view the nodes of  $\mathcal{N}_g^{\eta-1}$  as being shorted at this point) and by Corollary 11.7(ii), there is a positive probability:

$$\text{Prob}(sn_f^\eta, r\mathcal{N}_e^\eta, b\mathcal{N}_g^{\eta-1}) > 0$$

that  $\Psi$  will reach some  $\eta$ -node adjacent to  $n_f^\eta$  before reaching any node of  $\mathcal{N}_g^{\eta-1}$  and therefore before returning to  $n_g^0$ .

Note that  $P_{fa}^\eta$  also does not embrace any other  $\eta$ -node adjacent to  $n_f^\eta$ . Hence, we can combine Definition 13.1(d) and Corollary 11.7(ii) to conclude that

$$\text{Prob}(sn_f^\eta, rn_a^\eta, b\mathcal{N}_e^\eta \setminus n_a^\eta) > 0.$$

Therefore, with positive probability  $\Psi$  reaches  $n_g^\eta$  before returning to  $n_g^0$ .

Next, let  $\mathcal{N}_g^\eta$  ( $\mathcal{N}_g^{\eta+1}$ ) be the set of all  $\eta$ -nodes (respectively,  $(\eta + 1)$ -nodes) incident to  $\mathbf{S}^{\eta-1}$  (respectively,  $\mathbf{S}^\eta$ ).  $P_t^{\mu+1}$  embraces an  $\eta$ -path  $P_{af}^\eta$  that terminates at  $n_a^\eta$ , reaches  $n_f^{\eta+1} \in \mathcal{N}_g^{\eta+1}$ , and does not embrace any node of  $\mathcal{N}_g^\eta$ . Hence, by Definition 13.1(b) and Corollary 11.7(ii) again, there is a positive probability:

$$\text{Prob}(sn_a^\eta, r\mathcal{N}_g^{\eta+1}, b\mathcal{N}_g^\eta) > 0$$

that  $\Psi$  will reach some node of  $\mathcal{N}_g^{\eta+1}$  before reaching any node of  $\mathcal{N}_g^\eta$  and therefore before reaching  $n_g^0$ .

Now, note that  $P_{af}^\eta$  also does not reach any node of  $\mathcal{N}_g^{\eta+1}$  other than  $n_f^{\eta+1}$ . So, by Definition 13.1(a) and Corollary 11.7(ii) once again,

$$\text{Prob}(sn_a^\eta, rn_f^{\eta+1}, b\mathcal{N}_g^{\eta+1} \setminus n_f^{\eta+1} \mid \Psi \text{ reaches } \mathcal{N}_g^{\eta+1}) > 0.$$

Therefore, there is a positive probability that  $\Psi$  will reach  $n_f^{\eta+1}$  before returning to  $n_g^0$ . Hence, by induction the last statement is true for  $n_f^{\eta+1}$  replaced by  $n_f^{\mu+1}$ . Thus,  $S^\mu$  is transient.  $\square$

**Theorem 14.3.** *Under Condition 9.4, if  $\Psi$  roves, then, for any two nodes  $n_a^\alpha$  and  $n_b^\beta$  of whatever ranks  $\alpha$  and  $\beta$ , there is a positive probability that  $\Psi$ , after leaving  $n_a^\alpha$ , will reach  $n_b^\beta$  before returning to  $n_a^\alpha$ .*

**Proof.** There is a finite  $\mu$ -path  $P_0^\mu$  with  $\mu \geq \max(\alpha, \beta)$  that terminates at  $n_a^\alpha$  and  $n_b^\beta$ . With regard to a tracing of  $P_0^\mu$  from  $n_a^\alpha$  to  $n_b^\beta$ , if  $\alpha < \mu$ , let  $n_1^\mu$  be the last  $\mu$ -node in  $P_0^\mu$  that is incident to the  $(\mu - 1)$ -section that contains  $n_a^\alpha$ ; also, if  $\beta < \mu$ , let  $n_z^\mu$  be the first  $\mu$ -node in  $P_0^\mu$  that is incident to the  $(\mu - 1)$ -section that contains  $n_b^\beta$ . Proceeding as in the proof of Lemma 14.1, we replace the  $\mu$ -path embraced by  $P_0^\mu$  that terminates at  $n_a^\alpha$  and  $n_1^\mu$  by a  $\mu$ -path  $P_{a1}^\mu$  with the same terminations and fulfilling the conclusion of that lemma for  $\eta = \alpha + 1, \dots, \mu$ , where  $n_a^\alpha$  takes the role of  $n_g^0$ . That is,  $P_{a1}^\mu$  does not embrace any  $\eta$ -node that is incident to the  $(\eta - 1)$ -section containing  $n_a^\alpha$  other than a single  $\eta$ -node. In the same way, we replace the  $\mu$ -path embraced by  $P_0^\mu$  that terminates at  $n_z^\mu$  and  $n_b^\beta$  by a  $\mu$ -path  $P_{zb}^\mu$  with the same terminations and fulfilling similar conditions. Finally, if  $n_1^\mu$  and

$n_z^\mu$  are distinct, let  $P_{1z}^\mu$  be a  $\mu$ -path embraced by  $P_0^\mu$  that terminates at those nodes. Any of the paths  $P_{a1}^\mu$ ,  $P_{1z}^\mu$ , and  $P_{zb}^\mu$  may be trivial if  $\alpha = \mu$ ,  $n_1^\mu = n_z^\mu$ , and  $\beta = \mu$  respectively.

By the proof of Theorem 14.2, there is a positive probability that  $\Psi$ , after starting from  $n_a^\alpha$  will reach  $n_1^\mu$  before returning to  $n_a^\alpha$ .

Now, let  $n_1^\mu, n_2^\mu, \dots, n_z^\mu$  be the consecutive  $\mu$ -nodes in  $P_{1z}^\mu$ . By the definition of  $n_1^\mu$ , the  $\mu$ -path  $P_{12}^\mu$  embraced by  $P_{1z}^\mu$  that terminates at  $n_1^\mu$  and  $n_2^\mu$  does not embrace any nodes of the  $(\mu - 1)$ -section that contains  $n_a^\alpha$ . So, by Definition 13.1(e) and Corollary 11.7(ii), there is a positive probability that  $\Psi$  will reach a  $\mu$ -node adjacent to  $n_1^\mu$  before reaching any  $(\mu - 1)$ -node incident to the  $(\mu - 2)$ -section containing  $n_a^\alpha$  and therefore before reaching  $n_a^\alpha$ . Also, by Definition 13.1(d) and Corollary 11.7(ii),  $\Psi$  will reach  $n_2^\mu$  before reaching any other  $\mu$ -node adjacent to  $n_1^\mu$ , will reach  $n_3^\mu$  before reaching any other  $\mu$ -node adjacent to  $n_2^\mu$ , and so forth. Thus, with positive probability,  $\Psi$  will reach  $n_z^\mu$  before returning to  $n_a^\alpha$ .

Finally, we argue that, with positive probability,  $\Psi$ , after starting from  $n_z^\mu$ , will reach  $n_b^\beta$  before returning to  $n_a^\alpha$ . Let  $n_z^\mu, n_z^{\mu-1}, \dots, n_z^\beta = n_b^\beta$  be the last nodes of ranks  $\mu, \mu - 1, \dots, \beta$  in  $P_{zb}^\mu$ . As in the construction of Definition 13.1(c), we can choose a set  $\mathcal{V}_p^{\mu-1}$  of  $(\mu - 1)$ -nodes that isolates  $n_z^\mu$  from all other  $\mu$ -nodes and from  $n_z^{\mu-1}$  as well. Moreover, we can modify  $P_{zb}^\mu$ , if need be, so that it embraces only one node of  $\mathcal{V}_p^{\mu-1}$ ; let that node be  $n_1^{\mu-1}$ . Thus, there is a  $(\mu - 1)$ -path that terminates at  $n_1^{\mu-1}$ , reaches  $n_z^\mu$ , and does not embrace any other node of  $\mathcal{V}_p^{\mu-1}$ . So, by Definition 13.1(c), Corollary 11.7(ii), and the fact that  $\Psi$  roves, there is a positive probability that  $\Psi$  will reach  $n_1^{\mu-1}$  before returning to  $n_a^\alpha$ .

Next, note that the embraced  $(\mu - 1)$ -nodes in  $P_{zb}^\mu$  lying between  $n_1^{\mu-1}$  and  $n_z^{\mu-1}$  are only finite in number. Upon repeatedly applying Definition 13.1(d) and Corollary 11.7(ii) to those  $(\mu - 1)$ -nodes, we conclude that with positive probability  $\Psi$  will reach  $n_z^{\mu-1}$  before returning to  $n_a^\alpha$ .

The same argument works for transitions from  $n_z^{\mu-1}$  to  $n_z^{\mu-2}$ , from  $n_z^{\mu-2}$  to  $n_z^{\mu-3}$ , and so forth down to  $n_b^\beta$ . This completes the proof.  $\square$

## 15 Markov Chains and the Surrogate Network

Our final conclusions about the random roving  $\mu$ -walks on any  $\mu$ -section  $S^\mu$  of  $N^\nu$  follow from arguments virtually the same as those given in Section 8. Two obvious statements are: (i) The probability of a one-step  $\mu$ -transition between nonadjacent  $\mu$ -nodes is zero. (ii) Since  $\Psi$  roves, the probability of a one-step  $\mu$ -transition from a  $\mu$ -node back to itself is zero. Finally, by superposition of the sources specified in Definition 13.1(d), we also have: (iii) The probabilities for one-step  $\mu$ -transitions from a  $\mu$ -node to its adjacent  $\mu$ -nodes sum to one. These results yield

**Theorem 15.1.** *Under Definition 13.1, the random roving  $\mu$ -walks on any  $\mu$ -section  $S^\mu$  of  $N^\nu$  comprise a Markov chain with a countable state space consisting of the  $\mu$ -nodes of  $S^\mu$  and having the following transition probabilities:  $P_{k,k} = 0$ ;  $P_{k,l} = 0$  if  $n_k^\mu$  and  $n_l^\mu$  are not adjacent;  $P_{k,l}$  is given by Definition 13.1(d) when  $n_k^\mu$  and  $n_l^\mu$  are adjacent. When  $\mu = \nu$ , the state space consists of the finitely many  $\nu$ -nodes of  $N^\nu$ .*

**Theorem 15.2.** *The Markov chain of Theorem 15.1 is irreducible and reversible.*

**Proof.** Consider the Markov chain for the random roving  $\mu$ -walks on a  $\mu$ -section  $S^\mu$  of  $N^\nu$ . Its irreducibility follows from Theorem 14.3. Its reversibility can be proven exactly as in the proof of Theorem 8.1 by substituting Definition 13.1(d) for Definition 7.2 and Lemma 10.3 for Lemma 3.3.  $\square$

As a result of this last theorem, a finite “surrogate” 0-network  $N^{\nu \rightarrow 0}$  can be derived exactly as in Section 8. The random 0-walks on  $N^{\nu \rightarrow 0}$  are governed by the same transition matrix as that for the random roving  $\nu$ -walks on  $N^\nu$ . Similarly, for  $\mu < \nu$  an infinite “surrogate” 0-network  $N^{\mu \rightarrow 0}$  exists for the random roving  $\mu$ -walks on any  $\mu$ -section  $S^\mu$  of  $N^\nu$ .

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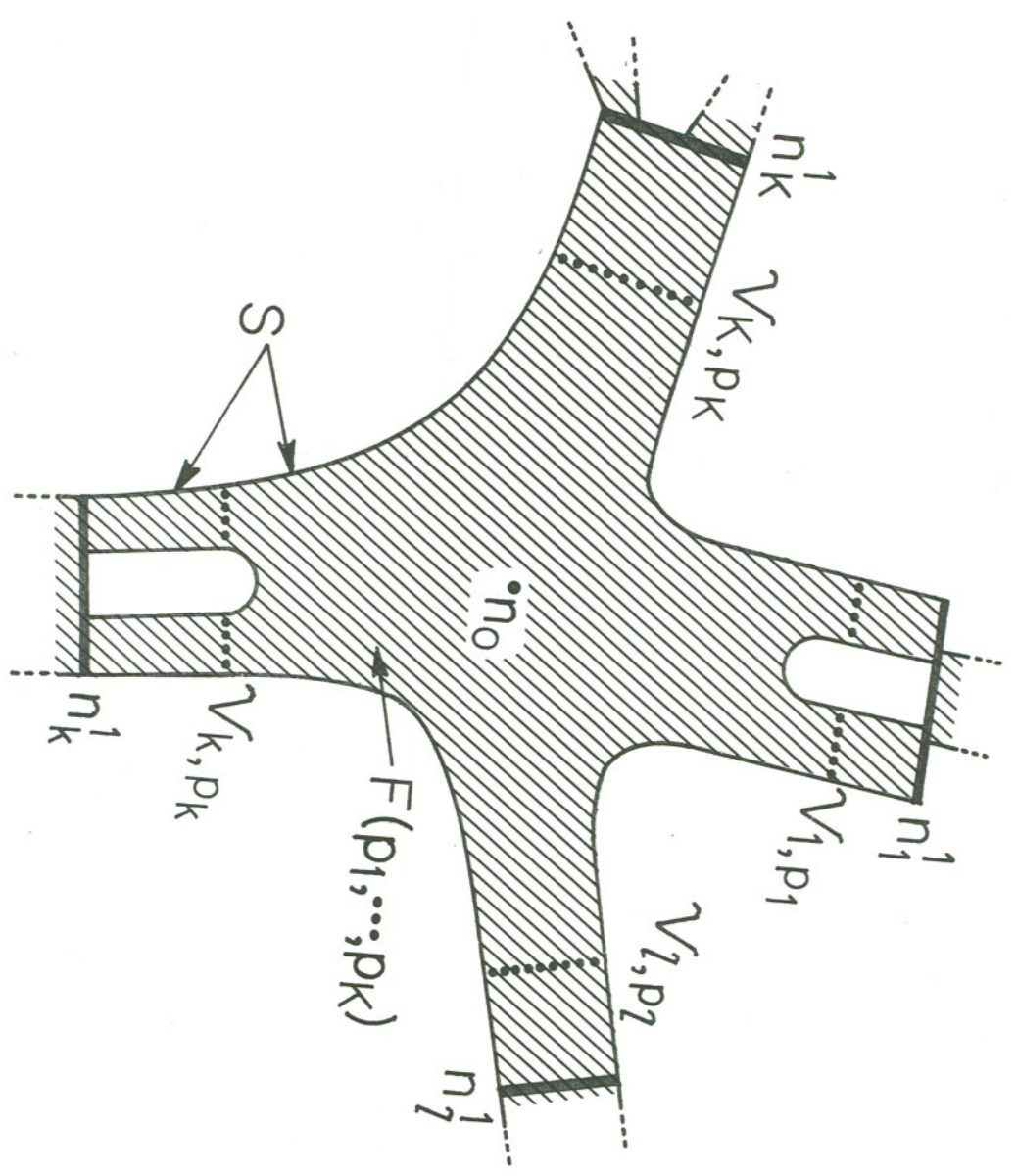


FIG. 1

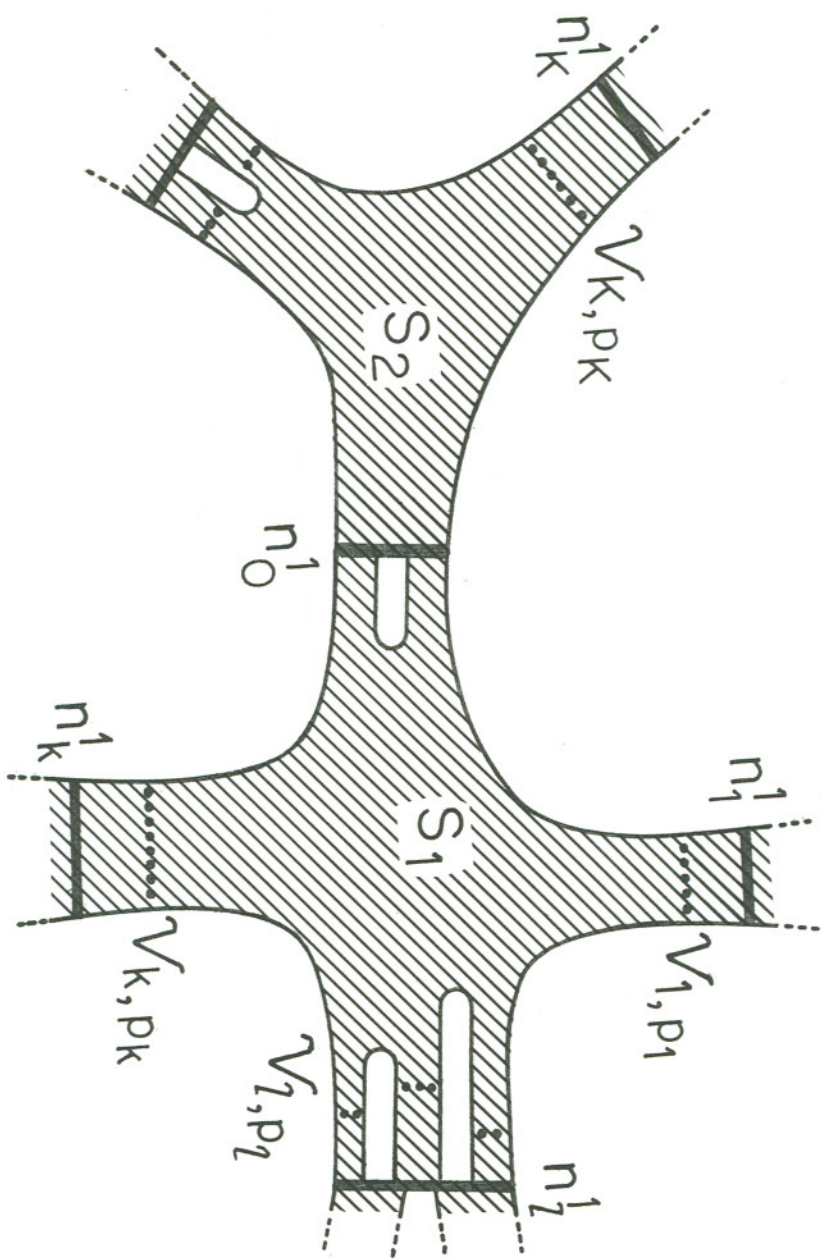


FIG 2

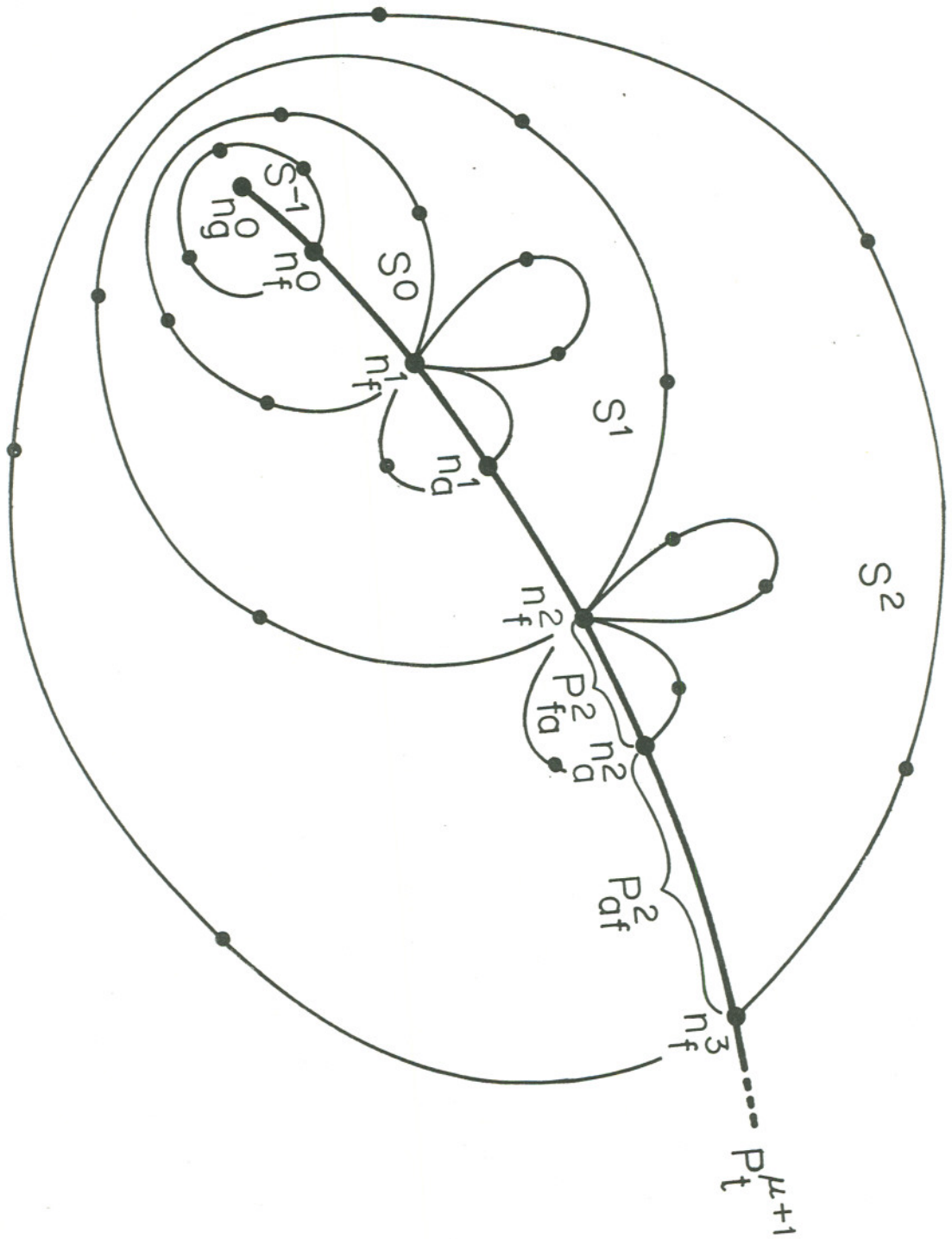


FIG. 3