STATE UNIVERSITY OF NEW YORK AT STONY BROOK

CEAS Technical Report 642

RANDOM WALKS ON FINITELY STRUCTURED TRANSFINITE NETWORKS: PART I

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This work was supported by the National Science Foundation under Grants MIP 8822774 AND DMS-9200738.

October 6, 1992

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Abstract — Defined and examined herein are transfinite random walks. These are random walks on a generalized version of a graph that consists of many infinite graphs connected together at their infinite extremities. Those connections are made by "1-nodes" and allow a random walker to "pass beyond infinity" through a 1-node. The probabilities for such transitions are obtained as extensions of the Nash-Williams law for random walks on ordinary infinite graphs under the nearest-neighbor rule. The analysis is based on the theory of transfinite electrical networks, but it requires that the transfinite graph have a structure that generalizes local-finiteness for ordinary infinite graphs. Branches that are incident to 1-nodes are allowed, which complicates the transitions through infinity. Another generalization achieved herein is an extension to transfinite networks of the maximum principle for node voltages. Finally, it is shown that a transfinite random walk can be represented by an irreducible reversible Markov chain, whose state space is the set of 1-nodes.

*Received by the Editors

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Key words. transfinite graphs, transfinite random walks, finitely structured graphs, maximum principle for node voltages, wandering through infinity

AMS (MOS) classifications. Primary: 05C99, 60J15, 94C05

Secondary: 05C40, 31C20, 94C15

1 Introduction

Analyses of random walks on countably infinite graphs are invariably restricted to graphs in which two nodes are either connected by a finite path or not connected at all; see the survey by Woess [8], which contains an extensive bibliography. This work develops the concept of a random walk on a transfinite graph, an idea initiated in [11] and [12]. Transfinite graphs in turn were defined and developed in [9] and [10]. The simplest kind of transfinite graph, called a "1-graph", can be obtained by connecting together ordinary countable infinite graphs (now called "0-graphs") at their infinite extremities. Those connections are made at "1-nodes" — a generalization of ordinary nodes, which are now called "0-nodes". As a result, two 0-nodes may not be connected in the usual sense — that is, through a finite path — but may instead be connected in a weaker and more general sense: One may get from one 0-node to the other by tracing along a finite number of infinite paths, where the transition from one infinite path to the next is via a 1-node.

Once graphs have been generalized to the transfinite case, so too can random walks be generalized. Indeed, an ordinary random walk can be defined on a locally finite 0-graph by assigning real positive numbers g_j, called "conductances," to the branches b_j and then using the nearest-neighbor rule: The probability that a random walker Ψ will proceed from a 0-node n_0 to an adjacent node n_k in one step is $g_k/\sum g_l$, where the summation is taken over all nodes n_l adjacent to n_0 and g_l is the conductance of the branch connected between n_0 and n_l. From this, one can derive relative probabilities of transition between nonadjacent nodes. For example, let n_0 be any node and let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two disjoint sets of 0-nodes such that $\mathcal{N}_1 \cup \mathcal{N}_2$ separates a finite subgraph containing n_0 from the rest of the graph. Nash-Williams [6] has shown that the probability of Ψ starting at n_0 and reaching some node of $\mathcal{N}_1$ before reaching any node of $\mathcal{N}_2$ can be obtained electrically; it is the voltage at n_0 when the nodes of $\mathcal{N}_1$ are held at 1 volt and the nodes of $\mathcal{N}_2$ are held at 0 volt. By taking certain limits and other extensions, one can extend this result to define a random walk on a transfinite graph [11], [12]. This requires new definitions for the probabilities for transitions to and from nodes of higher ranks. Appropriate ones can be devised using electrical criteria, much like the Nash-Williams law. In this case, the random walker Ψ may "wander through infinity" to reach nodes infinitely far away.

In this work, we develop a theory for random walks on a transfinite graph of rank 1. This is accomplished by relating those walks to an electrical network having a graph of that rank. In an earlier version of our theory [11], [12], a number of strong restriction were imposed. One of these was the imposition of Halin's finitely chainlike structure for ordinary infinite graphs [2], [3]. That assumption happens to be much stronger than need be, at least in the case of 1-graphs. The objective of this paper is to weaken it and to obtain thereby more general kinds of random walks on 1-graphs. For example, in [11] the only way Ψ could reach a 1-node was through an infinity of steps because no branch was allowed to be incident to a 1-node through an embraced 0-node. However, such branches can appear in transfinite graphs, and hence a random walk might also reach a 1-node in finitely many steps. Our objective now is to construct a theory for this more general kind of random walk. This requires substantially altered arguments because Ψ can now "wander through infinity" in different ways.

Actually, nodes of still higher ranks may also embrace nodes of lower ranks. Hence, the ideas developed herein may be extendible to transfinite random walks on graphs whose ranks exceed 1. This hopefully will be the subject of a subsequent work.

We use the definitions and terminology of transfinite graphs as given in [9] or [10], but all the ideas concerning transfinite random walks are defined herein. After conductances are assigned to branches, we will say "network" instead of "graph." Our arguments are based upon the theory of transfinite electrical networks.

The next section establishes a needed decomposition for 1-graphs arising from the removal of the connections at infinity. A structure that extends the idea of local finiteness to 1-nodes is presented in Section 3, and Section 4 adds the electrical assumptions that empower a theory of transfinite random walks. This permits the connection of pure voltage sources to infinite extremities of the network, as is proven in Section 5; in general, such connections are not permissible because some infinite networks effectively short those extremities [9, Sections 3.6 and 3.7]. Other requirements of our theory are the existence of node voltages and a maximum principle for them; these too are not in general available

[13], but our restrictions allow their establishment in Section 6. The definitions of transfinite walks are given in Section 7, and then a theory for transfinite random walks based on electrical networks is developed in Sections 8 and 9. Finally, it is shown in Section 10 that transitions between 1-nodes are governed by an irreducible and reversible Markov chain.

2 Subsections and Cores

Let $\mathcal{G}^1$ be a 1-connected 1-graph with no infinite 0-nodes, no self-loops, and no parallel branches. Embraced 0-nodes are allowed; that is, branches may be incident to 1-nodes. By definition of a 1-graph, $\mathcal{G}^1$ has a countable infinity of branches and at least one 1-node. The presence of branches incident to 1-nodes complicates matters considerably. In order to develop a theory for random walks on 1-graphs, we now have to identify a more detailed structure for the 1-graph.

The *opening* of a 1-node n^1 will mean the replacement of n^1 by singleton 1-nodes, one for each 0-tip in n^1, and by singleton 0-nodes, one for each elementary tip embraced by n^1 if there are any. Let us now partition the set of branches in $\mathcal{G}^1$ into subsets as follows: If two branches remain 0-connected after all the 1-nodes of $\mathcal{G}^1$ are opened, then those two branches are taken to be in the same subset. The reduced 0-graph induced by the branches in any one of those subsets will be called a *subsection* of $\mathcal{G}^1$. Some immediate consequences of this definition are the following: Every subsection lies entirely within some 0-section of $\mathcal{G}^1$; moreover, each 0-section is partitioned by some or all of the subsections, and so too is $\mathcal{G}^1$.

Every ordinary 0-node of a subsection $\mathcal{S}_b$ is identical with a 0-node in $\mathcal{G}^1$; that is, those two 0-nodes have the same incident branches. However, an embraced 0-node n^0 of $\mathcal{G}^1$ may have some incident branches in $\mathcal{S}_b$ and some not in $\mathcal{S}_b$. As a result, the corresponding reduced 0-node n_r^0 of $\mathcal{S}_b$ may be a proper subset of n^0. Nonetheless, we can uniquely identify n_r^0 to n^0 and will say that n^0 itself *belongs* to $\mathcal{S}_b$ — as well as to any other subsection having branches incident to n^0. We also say that the 1-node n^1 that embraces n^0 is *incident* to $\mathcal{S}_b$, and conversely.

Furthermore, if $\mathcal{G}_r$ is any reduction of $\mathcal{G}^1$ with respect to any subset of branches, we can identify each 0-tip t' of $\mathcal{G}_r$ with the unique 0-tip t of $\mathcal{G}^1$ that contains t' as a subset. In fact, $t' \mapsto t$ is an injection. We say that $\mathcal{G}_r$ *has* or *possesses* t as a 0-tip if there is at least one representative of t that lies entirely in $\mathcal{G}_r$. In this sense, every 0-tip of $\mathcal{G}_r$ is a 0-tip of $\mathcal{G}^1$. In the same way, we can identify each reduced 1-node with exactly one of the original 1-nodes of $\mathcal{G}^1$.

Lemma 2.1. *If a subsection $\mathcal{S}_b$ has no ordinary 0-node, it consists of a single branch.*

Proof. Since there are no self-loops, every branch of $\mathcal{S}_b$ is incident to two 1-nodes. Upon opening the 1-nodes, we disconnect such a branch from all other branches. Hence, that branch must be a subsection by itself. □

Lemma 2.2. *If a subsection $\mathcal{S}_b$ has exactly one ordinary 0-node n^0, it is a star graph with n^0 as its central node.*

Proof. By the preceding proof, no branch of $\mathcal{S}_b$ is incident to two 1-nodes, for otherwise that branch would be a subsection by itself with no ordinary 0-node. Hence, every branch of $\mathcal{S}_b$ is incident to n^0 and to a 1-node. Moreover, since there are no parallel branches, $\mathcal{S}_b$ must be a star graph, as stated. □

We will need still another idea. The *core* of a subsection $\mathcal{S}_b$ having two or more ordinary 0-nodes is the reduced 0-graph induced by all branches of $\mathcal{S}_b$ that are not incident to 1-nodes. (We will argue in a moment that there is at least one such branch.) In the special case where $\mathcal{S}_b$ has exactly one ordinary 0-node n^0, its *core* is taken to be n^0. When $\mathcal{S}_b$ has no ordinary 0-node, it core is void. Thus, all the nodes of a core of a subsection $\mathcal{S}_b$ are precisely the ordinary 0-nodes of $\mathcal{S}_b$ (where, as usual, we identify a reduced 0-node n_r^0 with the 0-node of $\mathcal{G}^1$ that contains n_r^0 as a subset).

Lemma 2.3. *If a subsection $\mathcal{S}_b$ has two or more ordinary 0-nodes, its core has at least one branch and is 0-connected through itself. Moreover, every embraced node of $\mathcal{S}_b$ is adjacent to a core node of $\mathcal{S}_b$.*

Proof. If the core has no branch or if the core has two nodes that are not 0-connected through the core, then the opening of the 1-nodes incident to $\mathcal{S}_b$ will change $\mathcal{S}_b$ into two or more components. This contradicts the hypothesis that $\mathcal{S}_b$ is a subsection.

The second sentence of the lemma follows from the fact that a branch that is incident to two 1-nodes is a subsection by itself. □

It follows now that every two nodes of a core are connected by a 0-path that remains within the core and therefore has no embraced nodes.

For an illustration of subsections and cores, refer to Figure 1. The heavy dots represent ordinary 0-nodes; the heavy lines represent 1-nodes, each of which embrace a 0-node (not shown); the other lines represent branches; and L_1 and L_2 label two doubly infinite ladders, whose 0-tips are embraced by the 1-nodes. All branches are 0-connected; for example, b_3 and b_4 are 0-connected through the 0-node embraced by n_2^1. Consequently, this entire 1-graph has only one 0-section. On the other hand, the branch b_0 is a subsection by itself; it has a void core. The star consisting of b_1, b_2, and b_3 is another subsection, and its core is the 0-node n_7^0. Another (degenerate) star taking the role of a subsection is induced by b_4 alone, and its core is n_8^0. The ladder L_1 along with b_5 and b_6 is still another subsection, and L_1 is its core. Similarly, L_2 is the core of the subsection consisting of L_2 along with b_7.

Finally, let us note that the idea of an "end" introduced by Halin [1] can also be defined for 1-graphs in terms of 0-tips. Let $\mathcal{B}_f$ be any finite set of branches in $\mathcal{G}^1$, and let $\mathcal{G}_f^1 = \mathcal{G}^1 \backslash \mathcal{B}_f$ denote the reduction of $\mathcal{G}^1$ induced by all branches of $\mathcal{G}^1$ that are not in $\mathcal{B}_f$. Since the removal of $\mathcal{B}_f$ disrupts at most a finite part of any one-ended path, we have that $\mathcal{G}^1$ and $\mathcal{G}_f^1$ possess exactly the same 0-tips. Two 0-tips of $\mathcal{G}^1$ will be called *end-equivalent* if, for every choice of $\mathcal{B}_f$, the two 0-tips have representatives lying in the same subsection of $\mathcal{G}_f^1$. This is an equivalence relationship, and the corresponding equivalence classes will be called the *ends* of $\mathcal{G}^1$. Clearly, the 0-tips in an end belong to a single subsection of $\mathcal{G}^1$; we say that the end *belongs to* that 0-section. Moreover, $\mathcal{G}^1$ and $\mathcal{G}_f^1$ have the same ends.

3 Finitely Structured 1-Graphs

Let the 1-graph $\mathcal{G}^1$ be as before and let $\mathcal{G}_r$ be any reduced graph of $\mathcal{G}^1$. A path P is said to *meet* a node n of $\mathcal{G}^1$ if P has or embraces a tip or node embraced by n.

Now, let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two (not necessarily disjoint) node sets in $\mathcal{G}^1$. A set $\mathcal{N}_s$ of nodes in $\mathcal{G}^1$ is said to *separate* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *within* $\mathcal{G}_r$ if every path P in $\mathcal{G}_r$ that meets a node of $\mathcal{N}_1$ and a node of $\mathcal{N}_2$ also meets a node of $\mathcal{N}_s$. (We also say that, within $\mathcal{G}_r$, $\mathcal{N}_s$ *separates* the nodes of $\mathcal{N}_s$ *from* the nodes of $\mathcal{N}_s$.) This definition allows nodes of $\mathcal{N}_s$ to embrace nodes of $\mathcal{N}_1$ and/or $\mathcal{N}_2$, and conversely. For instance, $(\mathcal{N}_1 \cap \mathcal{N}_2) \subset \mathcal{N}_s$ since the paths P can be trivial ones. Similarly, if a 1-node n^1 is incident to a subsection $\mathcal{S}_b$ only through an embraced 0-node n^0 — but not through any 0-tip of $\mathcal{S}_b$, then, within $\mathcal{S}_b$, n^0 separates n^1 from all the 0-nodes of $\mathcal{S}_b$. As another example, consider Figure 1; the 1-node n_0^1 is separated from the other two 1-nodes by the set of 0-nodes consisting of n_1^0, n_2^0, n_3^0, n_4^0, and the 0-node embraced by n_0^1. On the other hand, within the core L_1 the set of nodes n_1^0 and n_2^0 separates n_0^1 from n_1^1, and within the subsection having L_1 as its core the set of nodes n_1^0, n_2^0, and n_5^0 separates n_0^1 from n_1^1.

Similarly, two branches in $\mathcal{G}_r$ are said to be *separated by $\mathcal{N}_s$ in $\mathcal{G}_r$* if the two nodes of one branch are separated in $\mathcal{G}_r$ from the two nodes of the other branch by $\mathcal{N}_s$.

Now assume that the core $\mathcal{S}_c$ of a subsection $\mathcal{S}_b$ has infinitely many branches. Since all 0-nodes are of finite degree and since $\mathcal{S}_c$ is 0-connected (Lemma 2.3), it follows from Konig's lemma that $\mathcal{S}_c$ possesses at least one one-ended 0-path. Thus, there is at least one 1-node n^1 incident to $\mathcal{S}_c$. Moreover, by the definition of a core, no branch of $\mathcal{S}_c$ is incident to n^1: that is, n^1 is incident to $\mathcal{S}_c$ only through 0-tips.

We assume henceforth that there are only finitely many 1-nodes incident to any core $\mathcal{S}_c$. Now, assume in addition that there are at least two such 1-nodes. $\mathcal{V}$ will be called a *minimal separating set for n^1 in $\mathcal{S}_c$* if $\mathcal{V}$ is a finite nonvoid set of 0-nodes in $\mathcal{S}_c$ and separates n^1 from all the other 1-nodes incident to $\mathcal{S}_c$ and if for every node n^0 of $\mathcal{V}$ there is a path in $\mathcal{S}_c$ that meets n^1 and another 1-node incident to $\mathcal{S}_c$ but does not meet any node of $\mathcal{V}\backslash\{n^0\}$. If there is a finite nonvoid separating set, there will be a minimal separating set.

In the event $\mathcal{S}_c$ has only one incident 1-node n^1, we alter the last definition as follows. A set $\mathcal{V}$ of 0-nodes in $\mathcal{S}_c$ will be called a *minimal separating set for n^1 in $\mathcal{S}_c$* if $\mathcal{V}$ is finite and nonvoid and if there exists a nonvoid finite set $\mathcal{N}_a$ of 0-nodes in $\mathcal{S}_c$ such that $\mathcal{V}$ separates $\mathcal{N}_a$ from n^1 and for every node $n^0 \in \mathcal{V}$ there is a path in $\mathcal{S}_c$ that meets n^1 and a node of $\mathcal{N}_a$ but does not meet any node of $\mathcal{V}\backslash\{n^0\}$. A finite set $\mathcal{V}$ of this sort can always be found, for we can choose $\mathcal{V}$ to be some or all of the finitely many nodes of $\mathcal{N}_a$ that are adjacent to nodes of $\mathcal{S}_c$ not in $\mathcal{N}_a$.

In either case, let $\mathcal{V}$ be such a minimal set. A branch of $\mathcal{S}_c$ will be said to be *separated from* n^1 *by* $\mathcal{V}$ *within* $\mathcal{S}_c$ if both of its nodes are separated from n^1 by $\mathcal{V}$ within $\mathcal{S}_c$. The reduced 0-graph $\mathcal{A}$ induced by all branches of $\mathcal{S}_c$ that have both nodes in $\mathcal{V}$ or are not separated from n^1 by $\mathcal{V}$ will be called an *arm for* n^1 and $\mathcal{V}$ will be called the *base of* $\mathcal{A}$. The set of 0-tips for $\mathcal{A}$ will be called the *extremity of* $\mathcal{A}$. That extremity will be a subset of n^1, for otherwise $\mathcal{V}$ would not separate n^1 from all the other 1-nodes incident to $\mathcal{S}_c$.

For example, in Figure 1, $\mathcal{V} = \{n_9^0, n_{10}^0\}$ is a minimal separating set for n_1^1 within the core $\mathcal{S}_c = L_1$. The corresponding arm $\mathcal{A}$ is the 0-graph induced by the horizontal and vertical branches in L_1 lying to the left of $\mathcal{V}$ along with the branch connecting n_9^0 and n_{10}^0. $\mathcal{V}$ is the base of $\mathcal{A}$. The extremity of $\mathcal{A}$ consists of all the 0-tips of L_1 that are embraced by n_1^1.

When the said minimal separating sets exist in every core for every 1-node incident to that core through a 0-tip, we can set up an equivalence relationship between the 0-tips of $\mathcal{G}^1$ by calling two 0-tips "equivalent" if they belong to the same extremity of some arm of some core of $\mathcal{G}^1$; the equivalence classes are those extremities. Thus, the extremities of $\mathcal{G}^1$ partition the 0-tips of $\mathcal{G}^1$. Clearly, a core and the subsection in which it resides have the same extremities. As was noted above, an extremity is entirely contained in a single 1-node. Moreover, every 1-node n^1 will contain at least one extremity, one for every subsection to which n^1 is incident through a 0-tip. As usual, we say that a 1-node *embraces* its extremities.

With $\mathcal{S}_b$ still denoting a subsection, $\mathcal{S}_c$ its core, $\mathcal{A}$ an arm of $\mathcal{S}_c$ (assuming $\mathcal{S}_c$ is infinite), $\mathcal{V}$ the arm's base (by definition a minimal separating set within $\mathcal{S}_c$), and n^1 the 1-node incident to $\mathcal{A}$. let us set $\mathcal{W} = \mathcal{V} \cup \{n^0\}$ if n^1 is incident to $\mathcal{S}_b$ through an embraced 0-node n^0 (as well as through 0-tips), and let us set $\mathcal{W} = \mathcal{V}$ otherwise. If $\mathcal{S}_c$ is finite, n^1 can only be incident to $\mathcal{S}_b$ through an embraced 0-node n^0, in which case we set $\mathcal{W} = \{n^0\}$. We call $\mathcal{W}$ an *isolating set for* n^1 *within* $\mathcal{S}_b$. By definition, an isolating set is finite. It is also nonvoid because n^1 is incident to $\mathcal{S}_b$ either through 0-tips or through n^0 or both. Assume the following:

Conditions 3.1. *For each 1-node incident to* $\mathcal{S}_b$ *through a 0-tip, there is a sequence* $\{\mathcal{W}_p\}_{p=1}^{\infty}$ *of isolating sets* $\mathcal{W}_p$ *for* n^1 *within* $\mathcal{S}_b$ *such that the following two restrictions hold,* *wherein $\mathcal{A}_p$ denotes the arm corresponding to $\mathcal{W}_p$ and $\mathcal{V}_p$ denotes the base of $\mathcal{A}_p$.*

(a) *Given any branch b, there is a p such that b is not in $\mathcal{A}_q$ for all $q \geq p$.*

(b) *There exists a finite set $\{P_k^0\}_{k=1}^m$ of one-ended 0-paths, each of which meets exactly one node in $\mathcal{V}_p$ for every p, and every node in $\mathcal{V}_p$ is met by at least one of the P_k^0.*

Under Conditions 3.1, we will call $\{\mathcal{W}_p\}_{p=1}^\infty$ a *(nontrivial) contraction to n^1 within $\mathcal{S}_b$* and will say that $\{\mathcal{W}_p\}_{p=1}^\infty$ *isolates n^1 within $\mathcal{S}_p$*. Also, the P_k^0 will be called the *contraction paths to n^1 for $\{\mathcal{W}_p\}_{p=1}^\infty$*. An immediate consequence of Conditions 3.1 is that the cardinalities of the $\mathcal{W}_p$ are all bounded by the natural number $m+1$. Another is that every branch of $\mathcal{A}_p$ is 0-connected within $\mathcal{A}_p$ to one of the contraction paths, for otherwise $\mathcal{S}_b$ itself would not be 0-connected.

If a 1-node n^1 is incident to the subsection $\mathcal{S}_b$ only through a 0-node n^0 embraced by n^1, then we set $\mathcal{W}_p = \{n^0\}$ for all p. In this case, we call $\{\mathcal{W}_p\}_{p=1}^\infty$ a *trivial contraction to n^1 within $\mathcal{S}_b$*.

Now, let us assume that n^1 is incident to only finitely many subsections: $\mathcal{S}_{b1}, \ldots, \mathcal{S}_{bK}$ and that there is a (perhaps trivial) contraction $\{\mathcal{W}_{k,p}\}_{p=1}^\infty$ to n^1 within $\mathcal{S}_{bk}$ for each $k = 1, \ldots, K$. This time set $\mathcal{W}_p = \cup_{k=1}^K \mathcal{W}_{k,p}$. We now call $\mathcal{W}_p$ an *isolating set for n^1* and call $\{\mathcal{W}_p\}_{p=1}^\infty$ a *contraction to n^1*. Also, we say that $\{\mathcal{W}_p\}_{p=1}^\infty$ *isolates n^1*. Note that, since every 1-node n^1 embraces at least one 0-tip, it must be incident to at least one subsection through a 0-tip. Hence, for at least one k, $\{\mathcal{W}_{k,p}\}_{p=1}^\infty$ is a nontrivial contraction to n^1 within $\mathcal{S}_{bk}$.

Definition 3.2. A 1-graph $\mathcal{G}^1$ will be called *finitely structured* if it has the following properties:

(a) $\mathcal{G}^1$ is 1-connected and has no infinite 0-nodes, no self-loops, no parallel branches, and only finitely many 1-nodes.

(b) For each 1-node n^1 there is a contraction to n^1 (i.e., each 1-node is incident to only finitely many subsections, and Conditions 3.1 hold whenever a 1-node is incident to a subsection though a 0-tip).

In the following lemma, $\mathcal{A}_p$ and $\mathcal{A}_q$ will — as before — denote the pth and qth arms for a particular nontrivial contraction to a 1-node within a subsection $\mathcal{S}_b$, and $\mathcal{V}_p$ and $\mathcal{V}_q$ will denote the bases of $\mathcal{A}_p$ and $\mathcal{A}_q$ respectively.

Lemma 3.3. *Let the 1-graph $\mathcal{G}^1$ be finitely structured. Then the following statements hold:*

(i) *$\mathcal{G}^1$ has only finitely many subsections and only finitely many extremities.*

(ii) *For any $q > p$, the reduced graph $\mathcal{A}_p \backslash \mathcal{A}_q$ induced by all branches of $\mathcal{A}_p$ that are not in $\mathcal{A}_q$ is a finite 0-graph.*

(iii) *For each p, there exists a $q > p$ such that $\mathcal{A}_q \subset \mathcal{A}_p$ and $\mathcal{V}_q \cap \mathcal{V}_p = \emptyset$.*

(iv) *Every end is contained entirely within a single extremity, and each extremity contains only finitely many ends.*

(v) *Choose an arm for each extremity in $\mathcal{G}^1$. Then, every one-ended 0-path P^0 will eventually lie within one of those arms; that is, all but a finite part of P^0 will be in one of the chosen arms.*

Proof. (i) There can be only finitely many subsections in $\mathcal{G}^1$ because there are only finitely many 1-nodes, each 1-node is incident to only finitely many subsections, and every subsection meets at least one 1-node. Furthermore, the incidence between a subsection and a 1-node is either through a single extremity or through a 0-node or both; hence, there are only finitely many extremities.

(ii) Note that the boundary of $\mathcal{A}_p \backslash \mathcal{A}_q$ consists entirely of some 0-nodes on the finitely many contraction paths in $\mathcal{A}_p$. Hence, every branch of $\mathcal{A}_p \backslash \mathcal{A}_q$ must be 0-connected within $\mathcal{A}_p \backslash \mathcal{A}_q$ to one of the finitely many contraction paths in $\mathcal{A}_p$, for otherwise $\mathcal{S}_b$ would not be 0-connected — in violation of the definition of a subsection. Consequently, $\mathcal{A}_p \backslash \mathcal{A}_q$ can have only finitely many components.

Suppose now that $\mathcal{A}_p \backslash \mathcal{A}_q$ is an infinite 0-graph. Then, so too is one of its components. But, that component is locally finite and therefore by Konig's lemma must contain a one-ended 0-path. Thus, it must have a 0-tip that is not in the 1-node n^1 incident to $\mathcal{A}_p$. This means that $\mathcal{A}_p$ has two incident 1-nodes — in violation of the definition of an arm.

(iii) Consider the branches incident to $\mathcal{V}_p$. They are finite in number because $\mathcal{V}_p$ is a finite set and all 0-nodes are of finite degree. Hence, we can choose q so large that every such branch is not in $\mathcal{A}_q$. Since for each node n^0 of $\mathcal{V}_q$ there is at least one branch of $\mathcal{A}_q$ incident to n^0, $\mathcal{V}_p$ and $\mathcal{V}_q$ must be disjoint.

Now every branch b of $\mathcal{A}_q$ must be in $\mathcal{A}_p$, for otherwise we could choose a path that meets b and the 1-node n^1 incident to $\mathcal{A}_q$ and also another 1-node incident to $\mathcal{S}_b$ (or some 0-node not in $\mathcal{A}_p$ if n^1 is the only 1-node incident to $\mathcal{S}_b$) without meeting $\mathcal{V}_p$.

(iv) No end can be partly in one extremity of and partly in another, for, were this so, the removal of the finitely many branches incident to some separating set $\mathcal{V}_p$ would disconnect those two parts from each other — in violation of the definition of an end. Furthermore, since for each p every branch of $\mathcal{A}_p$ is 0-connected within $\mathcal{A}_p$ to one of the contraction paths in $\mathcal{A}_p$, the number of ends in the extremity of $\mathcal{A}_p$ can be no larger than the finite number of contraction paths in $\mathcal{A}_p$.

(v) The 0-tip of P^0 will be a member of one of those extremities. Let $\mathcal{A}$ be the corresponding chosen arm and let $\mathcal{V}$ be its base. P^0 will have at least one branch in $\mathcal{A}$. Furthermore, P^0 cannot pass into and out of $\mathcal{A}$ infinitely often, for each such passage must be through a different 0-node of $\mathcal{V}$ and $\mathcal{V}$ is a finite set. Hence, P^0 must eventually lie in $\mathcal{A}$. □

Definition 3.4. A nontrivial contraction $\{\mathcal{W}_p\}_{p=1}^{\infty}$ to a 1-node n^1 within a subsection $\mathcal{S}_b$ will be called *proper* if the following three conditions are satisfied by the arms $\mathcal{A}_p$ and the arm bases $\mathcal{V}_p$ corresponding to the $\mathcal{W}_p$:

(a) $\mathcal{A}_p \supset \mathcal{A}_{p+1}$ for all p.

(b) $\mathcal{V}_p \cap \mathcal{V}_{p+1} = \emptyset$ for all p.

(c) No node of $\mathcal{A}_1$ (and therefore of every $\mathcal{A}_p$) is adjacent of n^1.

Furthermore, a contraction $\{\mathcal{W}_p\}_{p=1}^{\infty}$ to n^1 is called *proper* if $\mathcal{W}_p = \cup_{k=1}^{K} \mathcal{W}_{k,p}$ for every p, where $\{\mathcal{W}_{k,p}\}_{p=1}^{\infty}$ is a proper contraction to n^1 in the kth subsection incident to n^1 whenever $\{\mathcal{W}_{k,p}\}_{p=1}^{\infty}$ is nontrivial; also, K denotes the number of subsections incident to n^1. As was noted above, $\{\mathcal{W}_{k,p}\}_{p=1}^{\infty}$ will be nontrivial for at least one k.

This definition does not impose any further restrictions upon $\mathcal{G}^1$ other than the assumption that $\mathcal{G}^1$ is finitely structured. It merely requires a judicious choice of $\{\mathcal{W}_p\}_{p=1}^{\infty}$. Indeed, Conditions (a) and (b) can be fulfilled by virtue of Lemma 3.3(iii). Also, if n^1 embraces a 0-node, there will be only finitely many 0-nodes adjacent to n^1; hence, we need only choose $\mathcal{A}_1$ small enough to avoid all those 0-nodes, thereby fulfilling Condition (c).

4 Finitely Structured Perceptible 1-Networks

A *0-network* or a *1-network* is respectively a 0-graph or a 1-graph whose branches have been assigned electrical parameters — as well as orientations, with respect to which branch voltages and branch currents are measured. We will use boldface notation for networks in place of the script notation used for graphs. Also, the terminology used for graphs is transferred directly to networks. Thus, for example, a 1-network is called *finitely structured* if its graph is finitely structured (Definition 3.2).

A branch b_j is called *sourceless* if it consists only of an electrical *conductance* g_j; by definition, g_j is the proportionality factor relating the current i through the conductance to the voltage v across the conductance: $i = g_j v$. Moreover, $r_j = g_j^{-1}$ is the branch *resistance*. In this work, all conductances and resistances will be real, positive numbers. A network or reduced network will be called *sourceless* if all its branches are sourceless.

Henceforth $\mathbf{N}^1$ will denote a 1-network that satisfies the following

Conditions 4.1.

(a) *The 1-graph of* $\mathbf{N}^1$ *is finitely structured.*

(b) *For every 1-node* n^1 *in* $\mathbf{N}^1$*, there exists a contraction to* n^1 *such that all its contraction paths are perceptible (i.e., the sum of all the resistances in each contraction path is finite).*

(c) $\mathbf{N}^1$ *is sourceless.*

A contraction to n^1 will be called *perceptible* if it satisfies Condition 4.1(b).

Lemma 4.2. *Between every two nodes (0-nodes or 1-nodes) of* $\mathbf{N}^1$ *there is a perceptible finite 1-path that terminates at those nodes.*

Proof. If n_a^0 and n_b^0 are two 0-nodes lying in the same 0-section of $\mathbf{N}^1$, they are connected by a finite 0-path P^0. Since P^0 has only finitely many branches, it is perceptible. But then, $\{n_a^0, P^0, n_b^0\}$ is the asserted 1-path.

So, assume that n_a and n_b are 0-nodes or 1-nodes that are infinitely distant from each other. The 1-connectedness of $\mathbf{N}^1$ implies that there is a finite 1-path

$$P^1 = \{n_a, P_1^0, n_2^1, P_2^0, \ldots, n_m^1, P_m^0, n_b\} \tag{1}$$

connecting n_a and n_b. Consider any 0-path P_k^0 in (1). If it is finite, it is perceptible. So, assume P_k^0 is one-ended and not perceptible. For every 1-node n^1 in $\mathbf{N}^1$, choose a perceptible contraction to n^1, Then, P_k^0 will eventually lie within an arm — according to Lemma 3.3(v). We can replace P_k^0 by a perceptible one-ended path, one that eventually follows a perceptible contraction path to reach the same 1-node that P^0 reaches. A similar replacement can be made for any endless 0-path in (1) by first partitioning it into one-ended 0-paths. Such replacements for all the nonperceptible 0-paths in (1) yield a perceptible finite 1-path that terminates at n_a and n_b. □

In manipulating networks, we will at times combine nodes. Their ranks need not be the same. Borrowing the terminology of electrical circuits, we will say that two or more 0-nodes have been *shorted* when the following is done: Replace those 0-nodes by a single 0-node n^0 and take a branch to be incident to n^0 if and only if that branch is incident to one or two of the original 0-nodes. Then remove any branch that becomes a self-loop, and combine parallel branches by adding their conductances. This may produce a 0-node n^0 of infinite degree if the original 0-nodes were infinitely many.

More generally, given any set of 0-nodes and 1-nodes, we *short* them as follows. First short all the 0-nodes — including those embraced by the 1-nodes — to get a new 0-node n^0. Then, create a new 1-node n^1 by taking the union of all the 0-tips embraced by the original 1-nodes and letting n^0 be the single 0-node embraced by n^1. Of course, n^0 will be absent when there are no 0-nodes — ordinary or embraced — among the original of nodes.

5 Voltage-Current Regimes with Pure Sources

We need some existence and uniqueness theorems for the voltage-current regimes when $\mathbf{N}^1$ is excited by various sources. They will be obtained by modifying [9, Theorem 3.3-5], which assumed that all sources had resistances. This will also yield a form of Kirchhoff's current law suitable for a finite set of branches that separate a 1-node from all other 1-nodes.

With $\mathbf{N}^1$ satisfying Conditions 4.1, let us denote its branches by b_j, where $j = 1, 2, 3, \ldots$. Let b_0 be the branch for a pure voltage source e_0, which we shall append to $\mathbf{N}^1$ by shorting the nodes of b_0 to two nodes of $\mathbf{N}^1$. For the moment we require that one of those nodes of $\mathbf{N}^1$ be an ordinary 0-node. The other may be a 1-node or an embraced 0-node. Later on, we will relax this restriction (see Theorem 5.5). $\mathbf{N}^1_e$ will denote $\mathbf{N}^1$ with b_0 appended as stated. Thus, $r_0 = 0$ and $r_j > 0$ for $j > 0$.

We now transfer e_0 through the ordinary 0-node n^0 to which it is incident. This does not change the branch currents, but it does render b_0 into a short circuit. We may then invoke [9, Theorem 3.3-5]. Upon restoring e_0 to b_0, we obtain the following fundamental theorem.

First some notation: $\mathbf{i} = (i_0, i_1, i_2, \ldots)$ is a branch current vector for $\mathbf{N}^1_e$. At this point, we only require that Kirchhoff's current law be satisfied at n^0, not necessarily at other nodes. This uniquely determines i_0 from $i_1, i_2, i_3, \ldots$. $\mathcal{I}$ is the Hilbert space of all such branch current vectors for $\mathbf{N}^1_e$ with the inner product $(\mathbf{i}, \mathbf{s}) = \sum_{j=1}^{\infty} r_j i_j s_j$. Convergence in $\mathcal{I}$ implies branchwise convergence. $\mathcal{K}^0$ is the span of all 0-loop currents and 1-basic currents [9, page 75] in $\mathcal{I}$, and $\mathcal{K}$ is the closure of $\mathcal{K}^0$ in $\mathcal{I}$. Thus, $\mathcal{K}^0 \subset \mathcal{K} \subset \mathcal{I}$, and $\mathcal{K}$ is a Hilbert space by itself under the same inner product.

Theorem 5.1. *With the branch b_0 incident to an ordinary 0-node of $\mathbf{N}^1$, there is a unique $\mathbf{i} \in \mathcal{K}$ for $\mathbf{N}^1_e$ such that, for every $\mathbf{s} \in \mathcal{K}$,*

$$e_0 s_0 \;=\; \sum_{j=1}^{\infty} r_j i_j s_j. \tag{2}$$

Henceforth, we assume that the voltage-current regime in $\mathbf{N}^1_e$ is dictated either by this theorem of by an extension of it wherein b_0 may be incident to two 1-nodes. That extension is derived below (Theorem 5.5).

Under Conditions 4.1, any node n_0 in $\mathbf{N}_e^1$ can be assigned a unique node voltage u_0 with respect to some arbitrarily chosen ground node u_g, whatever be the ranks of n_0 and n_g. To do this, first assign the node voltage $u_g = 0$ to n_g. Then, choose a perceptible path P within $\mathbf{N}^1$ that terminates at n_g and n^0. Such a path exists by virtue of Lemma 4.2. The node voltage u_0 is defined to be

$$u_0 = \sum_P \pm v_j \tag{3}$$

where the summation is over the indices j for the branches embraced by P, v_j is the jth branch voltage, and the plus (minus) sign is chosen when a branch orientation agrees (disagrees) with a tracing of P from n_0 to n_g. When P has infinitely many branches, (3) will converge absolutely because P is perceptible [9, page 83]. Moreover, if two 0-tips of $\mathbf{N}^1$ are nondisconnectable [9, page 104], they must be in the same extremity because their 1-nodes cannot be separated by any finite 0-node set. This allows us to invoke [13, Corollary 8.3] to conclude that u_0 does not depend upon the choice of the perceptible path P. Thus, once the ground node has been chosen, every node in $\mathbf{N}_e^1$ has a unique voltage, as determined by (3).

We now consider how Kirchhoff's current law may be applied indirectly to any 1-node n^1 — actually, to a certain set of branches that separates n^1 from all the other 1-nodes. Let $\{\mathcal{W}_p\}_{p=1}^{\infty}$ be a contraction to n^1. Choose some p. There is a finite set of arms, one for each extremity embraced by n^1, whose bases are contained in $\mathcal{W}_p$. Let $\mathbf{A}_p$ be the union of those arms and let $\mathcal{V}_p$ be the union of their bases. We define a *cut-branch at* $\mathcal{W}_p$ to be a branch that is separated from n^1 by $\mathcal{W}_p$ and has one node in $\mathcal{W}_p$ and one node not in $\mathcal{W}_p$. Thus, such a branch is not in $\mathbf{A}_p$ but is incident either to $\mathcal{V}_p$ or to the possible 0-node embraced by n^1 with one of its nodes not in $\mathcal{A}_p$. Let $\mathbf{C}$ be the set of all cut-branches at $\mathcal{W}_p$. $\mathbf{C}$ is a finite set. We call $\mathbf{C}$ the *cut for* n^1 *at* $\mathcal{W}_p$, or simply a *cut for* n^1.

For example, in Figure 1 let n_1^1 be the 1-node under consideration. We may choose $\mathcal{W}_p$ to be $\{n_9^0, n_{10}^0, n_x^0\}$ where n_x^0 is the 0-node embraced by n_1^1. Then, $\mathbf{C} = \{b_0, b_1, b_9, b_{10}\}$, but $b_5 \notin \mathbf{C}$. (In this case, $\mathcal{W}_p$ cannot be a member of a proper contraction because of the presence of branch b_5; were b_5 absent and $\mathcal{W}_p = \mathcal{W}_1$, $\mathcal{W}_1$ would be the first isolating set of a proper contraction.)

Kirchhoff's current law for $\mathbf{C}$ is

$$\sum \pm i_j = 0 \tag{4}$$

where the summation is over the indices of the branches in $\mathbf{C}$ and the plus (minus) sign is chosen if branch b_j is oriented away from (toward) a node of $\mathcal{W}_p$.

Lemma 5.2. *Kirchhoff's current law (4) holds whenever* $\mathbf{i} \in \mathcal{K}$.

Proof. Since $\mathbf{C}$ is a finite branch set, any 0-loop or 1-loop can embrace branches of $\mathbf{C}$ at most finitely often. Moreover, each 0-loop current or 1-loop current appears as additive terms to the $\pm i_j$ in (3) an even number of times, positively for half of those times and negatively for the other half. Hence, its total contribution to the left-hand side of (3) is zero.

The same is true for any 1-basic current $\mathbf{i} = \sum \mathbf{i}_m$. Indeed, by the definition of such a current vector [9, page 75], each $\mathbf{i}_m$ is a proper 1-loop current (i.e., its 1-loop is not a 0-loop), and only finitely many of the 1-loops corresponding to the $\mathbf{i}_m$ meet any given ordinary 0-node. This implies that only finitely many of the $\mathbf{i}_m$ pass through $\mathbf{C}$, as we shall now show.

Let $\mathcal{J}$ denote the set of 1-loops corresponding to the $\mathbf{i}_m$. $\mathcal{J}$ is in general an infinite set. For the chosen $\mathcal{W}_p$, let $\mathbf{C}'$ be the set of branches in $\mathbf{C}$ incident to the union $\mathcal{V}_p$ of bases in $\mathcal{W}_p$ and let $\mathbf{C}''$ be the other branches in $\mathbf{C}$. Since $\mathcal{V}_p$ is a finite set and since the nodes of $\mathcal{V}_p$ are all ordinary 0-nodes, only finitely many of the 1-loops in $\mathcal{J}$ pass through branches of $\mathbf{C}'$.

Next, note that no proper 1-loop can be confined only to the branches that are incident only to 1-nodes because there are only finitely many such branches; this follows from the facts that there are only finitely many 1-nodes, every 1-node embraces at most one 0-node, and every 0-node is of finite degree. Thus, every proper 1-loop in $\mathcal{J}$ that passes through $\mathbf{C}''$ must also pass through a branch that is incident to both a 1-node and to an ordinary 0-node. But, for the same reasons as those just given, there are only finitely many ordinary 0-nodes adjacent to 1-nodes. We can conclude that only finitely many of the 1-loops in $\mathcal{J}$ pass through $\mathbf{C}''$. Hence, the same is true for $\mathbf{C} = \mathbf{C}' \cup \mathbf{C}''$. It now follows that every 1-basic current makes a zero contribution to the left-hand side of (4).

Consequently, the same is true for every member of $\mathcal{K}^0$ and therefore of $\mathcal{K}$ as well since convergence in $\mathcal{K}$ implies branchwise convergence. □

We turn to the case where the appended source branch b_0 is a pure current source h_0 connected to any two nodes of $\mathbf{N}^1$. Such a connection is permissible whenever there is a perceptible path P between the two nodes of b_0 [9, Sections 3.6 and 3.7], as is the case for $\mathbf{N}^1$ (Lemma 4.2). By transferring h_0 into $\mathbf{N}^1$ to get current sources across all the resistances in P and an open in place of b_0, we induce thereby a voltage-current regime in $\mathbf{N}^1$ whose current vector $\mathbf{k} = (k_0, k_1, k_2, \ldots)$ is a member of $\mathcal{K}$ with $k_0 = 0$. But, the current vector $\mathbf{i}$ induced in $\mathbf{N}^1_e$ (i.e., in $\mathbf{N}^1$ augmented with b_0) is equal to $\mathbf{k}$ plus the 1-loop current whose value is h_0 and which flows around P and b_0. Hence, $\mathbf{i} \in \mathcal{K}$ too. We may now invoke Lemma 5.2 to conclude with

Lemma 5.3. *Kirchhoff's current law (4) continues to hold when $\mathbf{N}^1$ is augmented with a pure current source appended to any two nodes of $\mathbf{N}^1$.*

We shall now show that, for any network $\mathbf{N}^1$ satisfying Conditions 4.1, a pure voltage source may also be connected to any two 1-nodes of $\mathbf{N}^1$. (Actually, we will prove something more general.)

Let $n^1_1, \ldots, n^1_K$ denote all the 1-nodes of $\mathbf{N}^1$ and let there be pure current sources connected between these 1-nodes. Without loss of generality, we can take them to be $K-1$ current sources feeding the currents $h_2, \ldots, h_K$ from n^1_1 to $n^1_2, \ldots, n^1_K$ respectively. This creates a unique voltage-current regime in $\mathbf{N}^1$. Moreover, as was noted above for $\mathbf{N}^1_e$ and by virtue of the superposition principle, every node in $\mathbf{N}^1$ will have a unique node voltage with respect to n^1_1. Denote the node voltage at n^1_k by u_k and set $\mathbf{h} = (h_2, \ldots, h_K)$ and $\mathbf{u} = (u_2, \ldots, u_K)$. In this way, $\mathbf{N}^1$ acts as an internally transfinite, resistive $(K-1)$-port with n^1_1 acting as the common ground for all the ports. Moreover, the mapping $Z \colon \mathbf{h} \mapsto \mathbf{u}$ is the $(K-1) \times (K-1)$ resistance matrix for this $(K-1)$-port. We will now show that Z is invertible. This will imply that any choice of the node-voltage vector $\mathbf{u}$ can be obtained by setting $\mathbf{h} = Z^{-1}\mathbf{u}$. In other words, it will follow that any set of pure voltage sources u_k, where $k = 2, \ldots, K$, can be connected from n^1_1 to the n^1_k to produce the currents h_k passing from n^1_1 through the sources to the n^1_k, yielding thereby a unique voltage-current regime within $\mathbf{N}^1$.

Lemma 5.4. *Z is symmetric and positive-definite and therefore nonsingular.*

Proof. The symmetry of Z follows from the reciprocity principle [9, page 80]. We will prove that Z is positive-definite. Choose any vector $\mathbf{h} = (h_1, \ldots, h_K)$ for the current sources connected as above. For any n_k^1 $(k > 0)$, choose as above a cut $\mathbf{C}$ that separates n_k^1 from all other 1-nodes. Thus, the source branch b_k for h_k is a member of $\mathbf{C}$, but the other source branches are not. Hence, $\mathbf{C} = \mathbf{D} \cup \{b_k\}$, where $\mathbf{D}$ is the set of branches in $\mathbf{C}$ other than b_k. Let d_k be the number of branches in $\mathbf{D}$. By Lemma 5.3 and superposition, the net current flowing through $\mathbf{D}$ oriented away from n_k^1 is h_k. Therefore, there is at least one branch of $\mathbf{D}$ carrying a current no less than h_k/d_k. With r_{min} denoting the least value for all the resistances in $\mathbf{D}$, we can conclude that the power dissipated in all the resistances of $\mathbf{D}$ is no less than $\delta_k h_k^2$, where $\delta_k = r_{min} d_k^{-2} > 0$. Hence, with a cut chosen for each of the 1-nodes $n_2^1, \ldots, n_K^1$, we see that the power dissipated in the resistances in all those cuts is no less than $\sum_{k=2}^{\infty} \delta_k h_k^2$.

Now let $(\cdot, \cdot)$ be the inner product for $(K-1)$-dimensional Euclidean space. Tellegen's equation holds for transfinite networks [9, page 79], a consequence of which is that the power $(\mathbf{u}, \mathbf{h}) = (Z\mathbf{h}, \mathbf{h})$ supplied by the sources appended to $\mathbf{N}^1$ is equal to the power dissipated in all the resistances of $\mathbf{N}^1$. Thus, $(Z\mathbf{h}, \mathbf{h}) \geq \sum_{k=2}^{\infty} \delta_k h_k^2$, which proves that Z is positive-definite. □

The next theorem asserts that the conclusion of Theorem 5.1 continues to hold even when the pure voltage source e_0 is connected to two 1-nodes of $\mathbf{N}^1$.

Theorem 5.5. *Let $\mathbf{N}_e^1$ now denote $\mathbf{N}^1$ with a pure voltage source e_0 connected to any two nodes of $\mathbf{N}^1$. Then, there is a unique $\mathbf{i} \in \mathcal{K}$ for $\mathbf{N}_e^1$ such that, for every $\mathbf{s} \in \mathcal{K}$, (2) holds.*

Proof. To prove this theorem, we will insert a resistance $\rho > 0$ in series with the voltage source e_0 in the branch b_0 to obtain the unique current vector $\mathbf{i}^\rho = (i_0^\rho, i_1^\rho, i_2^\rho, \ldots)$ dictated by [9, Theorem 3.3-5], and then will take $\rho \to 0$ to obtain (2) in the limit.

With ρ inserted as stated, [9, Theorem 3.3-5] asserts that

$$e_0 s_0 \;=\; \rho i_0^\rho s_0 + \sum_{j=1}^{\infty} r_j i_j^\rho s_j. \tag{5}$$

By virtue of Lemma 5.4, $\mathbf{N}^1$ appears as a positive driving-point resistance z between the two nodes to which the source branch b_0 is connected. Hence,

$$e_0 - \rho i_0^\rho \;=\; z i_0^\rho. \tag{6}$$

With λ being another positive value for the resistance inserted into b_0,

$$i_0^\rho - i_0^\lambda \;=\; \frac{e_0}{\rho + z} - \frac{e_0}{\lambda + z} \;\to\; 0 \tag{7}$$

as $\rho, \lambda \to 0+$ independently. From (5) and (6), we obtain

$$\sum_{j=1}^{\infty} r_j (i_j^\rho - i_j^\lambda) s_j \;=\; (\lambda i_0^\lambda - \rho i_0^\rho) s_0 \;=\; z(i_0^\rho - i_0^\lambda) s_0. \tag{8}$$

Note now that both $\mathbf{i}^\rho$ and $\mathbf{i}^\lambda$ are members of $\mathcal{K}$. Indeed, the definition of $\mathcal{K}$ only imposes an inner product upon the currents within $\mathbf{N}^1$ — with the current in the source branch b_0 being uniquely determined by Kirchhoff's current law. That law has now been extended to the case where b_0 is incident to two 1-nodes. Also, recall that the norm $\|\mathbf{i}\|$ for any $\mathbf{i} \in \mathcal{K}$ is given by $\|\mathbf{i}\|^2 = \sum_{j=1}^{\infty} r_j i_j^2$. Consequently, we may set $s_j = i_j^\rho - i_j^\lambda$ for all j in (8) and then invoke (7) to get

$$\|\mathbf{i}^\rho - \mathbf{i}^\lambda\|^2 \;=\; \sum_{j=1}^{\infty} r_j (i_j^\rho - i_j^\lambda)^2 \;=\; z(i_0^\rho - i_0^\lambda)^2 \;\to\; 0$$

as $\rho, \lambda \to 0+$ independently. Hence, $\{\mathbf{i}^\rho : \rho > 0\}$ is a Cauchy directed function in $\mathcal{K}$ and therefore converges in $\mathcal{K}$ to an $\mathbf{i} \in \mathcal{K}$. Since the inner product of $\mathcal{K}$ is bicontinuous, we may pass to the limit in (5) to obtain (2).

$\mathbf{i}$ is uniquely determined by (2) because its right-hand side is the inner product $(\mathbf{i}, \mathbf{s})$ determined for all $\mathbf{s} \in \mathcal{K}$ by the left-hand side. $\square$

6 A Maximum Principle for Node Voltages

Our objective now is to extend the maximum principle to the node voltages in a transfinite 1-network $\mathbf{N}_\epsilon^1$ specified as follows:

Conditions 6.1. *Let* $\mathbf{N}^1$ *satisfy Conditions 4.1.* $\mathbf{N}^1_e$ *is a transfinite network obtained by appending finitely many (voltage and/or current) sources to* $\mathbf{N}^1$ *by shorting the nodes of those sources to some of the nodes of* $\mathbf{N}^1$*.*

Since $\mathbf{N}^1$ presents a positive driving-point resistance z between any two of its nodes, each current source h can be replaced by an equivalent voltage source $e = zh$, which does not alter the voltage-current regime. Thus, that regime in $\mathbf{N}^1_e$ is the superposition of the regimes induced by each of the sources — as dictated by Theorem 5.5.

A subsection $\mathbf{S}_b$ is sourceless if none of its ordinary 0-nodes is incident to a source. (However, its embraced 0-nodes and incident 1-nodes may be incident to sources). A sourceless subsection is perforce a subsection of both $\mathbf{N}^1$ and $\mathbf{N}^1_e$, in contrast to 0-sections, which may differ in those two networks. When speaking of a 0-section, we will mean a 0-section of $\mathbf{N}^1$ — not of $\mathbf{N}^1_e$.

Let us now assume that a ground node n_g has been chosen in $\mathbf{N}^1_e$ and assigned the voltage $u_g = 0$. Then, every other node n^0 has a unique node voltage u_0 given by (5.2), and u_0 is independent of the choice of the perceptible path P between n_g and n^0.

Lemma 6.2. *Under Conditions 6.1, the node voltages in* $\mathbf{N}^1_e$ *along any one-ended 0-path* P^0 *(whether perceptible or not) converge to the voltage* u^1 *of the 1-node* n^1 *that* P^0 *meets terminally with a 0-tip.*

Proof. Choose a proper contraction for every 1-node in $\mathbf{N}^1$. By Lemma 3.3(v), P^0 will eventually remain within every arm for one of those proper contractions. Denote that contraction by $\{\mathcal{W}_p\}_{p=1}^{\infty}$ and let $\{\mathbf{A}_p\}_{p=1}^{\infty}$ be the corresponding sequence of arms. The arm base $\mathcal{V}_p$ of $\mathbf{A}_p$ is a finite set of 0-nodes, each of which lies on a perceptible contraction path for $\mathcal{W}_p$.

Since there are only finitely many sources, an integer q can be chosen so large that $\mathbf{A}_q$ contains no nodes incident to sources. So, let $p > q$. Each $\mathbf{A}_p \backslash \mathbf{A}_{p+1}$ is a finite resistive sourceless 0-network, and therefore its node voltages lie between the maximum $u_{max,p}$ and minimum $u_{min,p}$ of all the node voltages for the finite node set $\mathcal{V}_p \cup \mathcal{V}_{p+1}$. Since all contraction paths for $\{\mathcal{W}_p\}_{p=1}^{\infty}$ are perceptible, the voltages along any one of them converge to the node voltage u^1 for n^1. Since the number of those contraction paths is finite, $u_{max,p} \to u^1$ and

$u_{min,p} \to u^1$ as $p \to \infty$.

So, consider again P^0. The node voltages for $P^0 \cap (\mathbf{A}_p \backslash \mathbf{A}_{p+1})$ lie between $u_{max,p}$ and $u_{min,p}$. It follows that the node voltages of P^0 also converge to u^1 — even when P^0 is not perceptible. □

Let $\mathbf{S}_b$ denote a subsection. Since the number of 1-nodes is finite, we can let u^1_{max} (and u^1_{min}) be the largest (respectively, least) node voltage for all the 1-nodes incident to $\mathbf{S}_b$.

Theorem 6.3. *Let* $\mathbf{N}^1_e$ *satisfy Conditions 6.1 and let* $\mathbf{S}_b$ *denote a sourceless subsection of* $\mathbf{N}^1_e$. *Assume* $\mathbf{S}_b$ *has a (nonvoid) core* $\mathbf{C}$. *Then, exactly one of the following statements is true:*

(i) *All the (ordinary and embraced) 0-nodes of* S_b *have the same voltage, namely,* $u^1_{min} = u^1_{max}$.

(ii) *There are at least two 1-nodes incident to* $\mathbf{S}_b$ *with different voltages, and every node of the core* $\mathbf{C}$ *has a voltage strictly larger than* u^1_{min} *and strictly less than* u^1_{max}.

Proof. Either (i) holds or it does not. Assume it does not. We will show that (ii) must hold. We consider two cases, exactly one of which must hold.

Case 1: The node voltages in the core $\mathbf{C}$ *are all the same.* Hence, the current in every branch connected between two core nodes will be zero. Let v_c be that common value for the core node voltages. By Lemma 6.2, any 1-node that embraces a 0-tip of $\mathbf{S}_b$ must also have the same voltage v_c. So, the only way an incident 1-node can have a different voltage is when it is incident to $\mathbf{S}_b$ only through an embraced node (and not through a 0-tip). Let us refer to such a 1-node as being *nodally incident* to $\mathbf{S}_b$. Suppose $u^1_{max} = u^1_{min} \neq v_c$. Then, all the 1-nodes incident to $\mathbf{S}_b$ are nodally incident. Thus, all the branches of $\mathbf{S}_b$ that are incident to 1-nodes will all carry positive currents in the same direction with respect to the core: that is, those currents will all flow toward the core or all flow away from it. This implies that Kirchhoff's current law must be violated at a core node. So, our supposition is false. Since we have assumed that (i) does not hold, we must have that $u^1_{min} < u^1_{max}$. Hence, there are at least two 1-nodes incident to $\mathbf{S}_b$.

Next, suppose that $v_c \leq u^1_{min}$. Then, every branch of $\mathbf{S}_b$ incident to a 1-node will carry a nonnegative current toward the core, and at least one of those branches will carry a positive current, as for instance any branch that is incident to a 1-node with a voltage equal to u^1_{max}. Again Kirchhoff's law will be violated at a core node.

Similarly, we cannot have $v_c \geq u^1_{max}$. It follows that $u^1_{min} < v_c < u^1_{max}$, as asserted in (ii).

Case 2: The node voltages in the core **C** *are not all the same.* Recall that all the nodes of a core are ordinary finite 0-nodes and therefore satisfy Kirchhoff's current law. Choose a 0-node n^0_a in the core arbitrarily. There will be another 0-node n^0_b in the core with a different node voltage. By Lemma 2.3, there is a 0-path P^0 lying entirely in the core and terminating at those two nodes. We can trace along P^0 starting from n^0_a to find a 0-node n^0_1 (possibly n^0_a itself) with the same voltage $u^0_1 = u^0_a$ as that of n^0_a and lying adjacent to a 0-node with a different voltage. By Kirchhoff's current law applied to n^0_1, there is a 0-node n^0_2 adjacent to n^0_1 with a voltage larger than u^0_1. If n^0_2 is embraced, we have $u^0_a = u^0_1 < u^0_2 \leq u^1_{max}$. If n^0_2 is ordinary, then Kirchhoff's current law applied to n^0_2 implies that there is still another 0-node n^0_3 with $u^0_3 > u^0_2$. If n^0_3 is embraced, $u^0_a < u^0_3 \leq u^1_{max}$. If n^0_3 is ordinary, we continue this process. Either an embraced 0-node is reached in a finite number of steps, in which case $u^0_a \leq u^1_{max}$, or a one-ended 0-path Q^0 of ordinary 0-nodes in the core with successively strictly increasing node voltages is generated. In the latter case, Q^0 will — through a 0-tip — meet a 1-node n^1_0 incident to $\mathbf{S}_b$, and, by Lemma 6.2, the node voltages along Q^0 will converge to the node voltages u^1_0 at n^1_0. Thus, $u^0_a < u^1_0 \leq u^1_{max}$. Since n^0_a was chosen arbitrarily, u^1_{max} is strictly larger than the voltage at every core node of $\mathbf{S}_b$.

The strict lower bound $u^1_{min} < u^0_a$ for every core node n^0_a can be established similarly. Since $u^1_{min} < u^1_{max}$, we again must have at least two 1-nodes incident to $\mathbf{S}_b$. □

Our next objective is to show that, when $\mathbf{N}^1_e$ is driven by a single 1 V voltage source, all node voltages remain within 1 V of each other. We shall prove this by supposing otherwise and then constructing a contradiction. The next lemma is a step toward that goal.

Conditions 6.4. *Let* $\mathbf{N}^1_e$ *satisfy Conditions 6.1, but let there be only one source — a pure voltage source of value 1 V. Let that source's negative terminal be the ground node* n_g *with voltage* $u_g = 0$ *and let* n_e *denote its positive terminal.*

Lemma 6.5. *Let* $\mathbf{N}^1_e$ *satisfy Conditions 6.4 and let* $\mathbf{S}_b$ *be the subsection of* $\mathbf{N}^1_e$ *containing* *the source. Let n_a be either a 0-node in $\mathbf{S}_b$ or a 1-node incident to $\mathbf{S}_b$ with a node voltage $u_a > 1$. Then, there is a 1-node incident to $\mathbf{S}_b$ whose voltage is larger than 1, is no less than the voltages at all the other 1-nodes incident to $\mathbf{S}_b$, and is strictly larger than the core node voltages for $\mathbf{S}_b$.*

Proof. $\mathbf{S}_b$ must have a core, for otherwise it is a single branch, the source branch, and cannot have a node with a voltage larger than 1. If the core consists of a single 0-node, it is a star network (Lemma 2.2), one of whose branches is the source branch. So, the voltage at the single core node is either 1 V or 0 V. Our conclusion then follows.

So, let $\mathbf{S}_b$ have a core with two or more nodes. At least one of the nodes of the source branch must be in the core, for otherwise $\mathbf{S}_b$ would be sourceless. If n_a is a 0-node in the core, we can — according to Lemma 2.3 — choose a finite 0-path P^0 that connects n_a to a node of the source. If n_a is an embraced 0-node, it is adjacent to a node of the core (Lemma 2.3), and P^0 can be chosen such that all its nodes are in the core except for n_a. If n_a is a 1-node incident to $\mathbf{S}_b$ through a 0-tip, then by Lemma 6.2 there is a 0-node n_b in the core with a voltage greater than 1; P^0 can again be chosen as a finite 0-path connecting n_b to a source node through the core. In every case, we can trace along P^0 to find a node with the highest voltage on P^0 and then can argue as in Case 2 of the preceding proof to assert that there is a 1-node incident to $\mathbf{S}_b$ with a voltage strictly larger than the core node voltages on P^0. Since n_a can be chosen arbitrarily, this implies the conclusion of Lemma 6.5. □

Theorem 6.6. *Let $\mathbf{N}_e^1$ satisfy Conditions 6.4. Then, every node in $\mathbf{N}_e^1$ has a voltage that is no larger than 1 and no less than 0.*

Proof. Suppose there is a node somewhere in $\mathbf{N}_e^1$ with a voltage larger than 1. An immediate consequence of Theorem 6.3 and Lemma 6.5 in conjunction with the fact that $\mathbf{N}_e^1$ has only finitely many 1-nodes is that there is a 1-node n_c^1 in $\mathbf{N}_e^1$ with a voltage U_{max} larger than 1 and no less than the voltages at all the other 1-nodes and 0-nodes of $\mathbf{N}_e^1$. Once again, we can trace a path from n_c^1 to a node of the source branch to find a 1-node n_d^1 with the voltage U_{max} and incident to a subsection S_d having at least one node voltage less than U_{max}. By Theorem 6.3 and Lemma 6.5 again, if S_d has a core, its core node voltages will all be less than U_{max}. If S_d has no core, it consists of a single branch with one of its nodes at a voltage less than U_{max}.

We wish to apply Kirchhoff's current law to n_d^1. Since its voltage is larger than 1, the source branch cannot be incident to n_d^1. Let us examine all the subsections that are incident to n_d^1. According to Theorem 6.3(i), some of them may have all their node voltages equal to U_{max}. This can only happen if they are sourceless. Moreover, all their branches will carry zero current. Hence, we can ignore those subsections so far as Kirchhoff's current law is concerned.

As for the other subsections incident to n_d^1, their node voltages will vary as do the node voltages in S_d: if any one of them has a core, its core node voltages will be less than U_{max}. Now, if one or more of those subsections are incident to n_d^1 only through branches (not through any 0-tips), those branches incident to n_d^1 will carry positive currents away from n_d^1.

All the remaining subsections will have extremities embraced by n_d^1. Choose a proper contraction to n^1 (Definition 3.4). This yields sequences of arms for the said extremities embraced by n_d^1, one sequence for each extremity. Set $\mathbf{M} = \cup_{p=1}^{\infty} \mathbf{M}_p$, where $\mathbf{M}_p = \mathbf{A}_p \backslash \mathbf{A}_{p+1}$ and $\mathbf{A}_p$ is the union of the pth arms in the said sequences. All the nodes of $\mathbf{M}$ are core nodes, and none of them are adjacent to n_d^1. Moreover, every $\mathbf{M}_p$ is a finite 0-network (Lemma 3.3(ii)), and, for $p > 1$, $\mathbf{M}_{p-1} \cap \mathbf{M}_p = \mathcal{V}_p$ where $\mathcal{V}_p$ is the union of the arm bases for those pth arms. The node voltages in $\mathbf{M}$ will be strictly less than U_{max}, and, by Lemma 6.2, the node voltages along each contraction path in $\mathbf{M}$ will converge to U_{max}. Consequently, we can choose two integers p and q with $p < q$ and so large that the following two conditions are satisfied: The largest node voltage for $\mathcal{V}_p$ is less than the least node voltage for $\mathcal{V}_q$. The finite network $\mathbf{M}_{p,q} = \cup_{k=p}^{q-1} \mathbf{M}_k$ is not incident to the source.

We can generate the same voltage-current regime in $\mathbf{M}_{p,q}$ as it has as a finite part of $\mathbf{N}_\epsilon^1$ by connecting pure voltage sources as follows: Let $n_{p,1}^0$ be a 0-node of $\mathcal{V}_p$ with the largest node voltage $u_{p,1}$ for $\mathcal{V}_p$. Let $n_{p,k}^0$ be any other 0-node of $\mathcal{V}_p$ and let $u_{p,k}$ be its voltage. Connect a pure voltage source of value $u_{p,1} - u_{p,k}$ from $n_{p,k}^0$ to $n_{p,1}^0$ with its positive terminal at $n_{p,1}^0$. (That source will be a short if $u_{p,1} = u_{p,k}$.) Do this for all $n_{p,k}^0$. Similarly, connect a pure voltage source from a node $n_{q,1}^0$ of $\mathcal{V}_q$ with the least node voltage $u_{q,1}$ for $\mathcal{V}_q$ to each of the other nodes of $\mathcal{V}_q$ to establish their relative node voltages at the values they have in $\mathbf{N}_e^1$. Finally, connect a pure voltage source $e_{p,k}$ of value $u_{q,1} - u_{p,1} > 0$ from $n_{p,1}$ to $n_{q,1}$. $\mathbf{M}_{p,q}$ with these appended sources is a connected finite network.

Now, choose a cut $\mathbf{C}_q$ for n_d^1 at $\mathcal{W}_q$, the isolating set that contains $\mathcal{V}_q$. That cut may contain branches incident to n_d^1 and to nodes not in $\cup_{k=p}^{\infty} \mathbf{M}_k$. Since the voltage at n_d^1 is no less than all the node voltages in $\mathbf{N}_e^1$, those branches will carry nonnegative currents away from n_d^1. All the other branches of the cut $\mathbf{C}_q$ will be cut-branches at $\mathcal{V}_q$, the set of which we denote by $\mathbf{C}_q'$. Orient $\mathbf{C}_q'$ away from $\mathcal{V}_q$. The total current in $\mathbf{C}_q'$ will be zero when any appended voltage source is acting alone (all other appended sources set equal to zero) and has both of its nodes in $\mathcal{V}_p$ or both of its nodes in $\mathcal{V}_q$. However, for $e_{p,q}$ acting alone, the total current in $\mathbf{C}_q'$ will be positive. So, by superposition, with all the appended sources acting simultaneously, the total current in $\mathbf{C}_q'$ will be positive. Hence, the total current in $\mathbf{C}_q$ will be positive too. This is the same current that $\mathbf{C}_q$ will carry for the voltage-current regime in $\mathbf{N}_e^1$.

But, this violates Kirchhoff's current law (4), which must hold according to Lemma 5.2 and Theorem 5.5. Hence, our supposition that there is a node in $\mathbf{N}_e^1$ with a voltage larger than 1 is false.

In a similar way, we can show that no node voltage in $\mathbf{N}_e^1$ can be negative. □

We will need a refinement of Theorem 6.6. As before, n_e and n_g are the nodes to which the source branch is incident (positive terminal at n_e); n_0 is another node, and u_0 is its voltage.

Corollary 6.7. *Assume $\mathbf{N}_e^1$ satisfies Conditions 6.4.*

(i) *Let there be a path P that terminates at n_0 and n_g and does not embrace n_e. Then,* $u_0 < 1$.

(ii) *Let there be a path that terminates at n_0 and n_e and does not embrace n_g. Then,* $u_0 > 0$.

Proof. Under the hypothesis of (i), suppose $u_0 = 1$. Trace P from n_0 to n_g. Three cases arise:

Case 1. We find an ordinary 0-node n_a^0 with a voltage equal to 1 and adjacent to a 0-node with a voltage less than 1. This is impossible, for by Kirchhoff's current law applied to n_a^0 there must be another 0-node adjacent to n_a^0 with a voltage larger than 1 — in violation of Theorem 6.6.

Case 2. We find a 1-node n^1 with a voltage equal to 1 and incident to a subsection having a core with voltages less than 1. We can choose an arm for each extremity embraced by n^1 (if there are any such extremities) such that the arm is not incident to the source branch. In each arm the node voltages will all be equal to 1 or will all be less than 1. Similarly, the branches incident to n^1 (if there are any such branches) will carry nonnegative currents away from n^1. Moreover, there will be such an arm with voltages less than 1, or such a branch incident to a core node with a voltage less than 1, or both. Whatever be the case, there will be a cut that isolates n^1 from all the other 1-nodes and carries a positive current away from n^1. This violates Kirchhoff's current law at that cut for n^1.

Case 3. We find a 1-node n^1 with a voltage equal to 1 and adjacent to a 1-node with a voltage less than 1. The branch connecting those two 1-nodes will carry a positive current away from n^1. Moreover, as in Case 2, a cut that isolates n^1 from all the other 1-nodes can be so chosen that all its branches carry nonnegative currents away from n^1. Again Kirchhoff's current law will be violated at that cut.

These three cases exhaust all possibilities. Hence, $u_0 < 1$. A similar argument establishes (ii). □

7 Transfinite Walks

In this section we define walks of ranks 0 and 1 on a sourceless 1-network satisfying Conditions 4.1. A walk of either rank may "arrive at infinity" by reaching a 1-node — and can do so in two different ways, namely by reaching an embraced node through an incident branch or by passing along an arm. Moreover, a 1-walk may then "pass through infinity" by leaving the 1-node again in one of two possible ways. Furthermore, if it passes from one 0-section to another, at most one of the two transitions to and from the 1-node can be along a branch incident to the 1-node. As compared to the simpler transfinite walks discussed in

[11, Section 5], we have here some complications.

A 0-*walk* is a walk of the conventional sort; it is an an alternating sequence of 0-nodes n_m^0 and branches b_m:

$$W^0 = \{\ldots, n_m^0, b_m, n_{m+1}^0, b_{m+1}, \ldots\} \tag{9}$$

such that, for each m, b_m is incident to both n_m^0 and n_{m+1}^0. Moreover, if W^0 terminates on either side, we require that it terminate at a 0-node.

Since there are no self-loops in $\mathbf{N}^1$, n_m^0 and n_{m+1}^0 are different 0-nodes whatever be m. Except for this restriction, the elements of W^0 may repeat. That (9) is a sequence means that the indices $\ldots, m, m+1, \ldots$ traverse a strictly increasing, consecutive set of integers. This also implies that W^0 is restricted to a single 0-section because a transition from one 0-section to another would require the conjunction of two sequences, at least one of which meets the other with an infinitely extending subsequence.

W^0 is called *nontrivial* if it has at least one branch. We say that W^0 *embraces* itself and all its elements. W^0 may be either *finite* with two terminal nodes, or *one-ended* with exactly one terminal node, or *endless* without any terminal node. When W^0 has a terminal node, we say that W^0 *starts at* (*stops at*) its terminal node on the left (respectively, on the right). We also say that W^0 *reaches* each of its elements and *passes through* each of its elements other than any terminal node. If a 0-node of W^0 is embraced by a 1-node n^1, we use the same terminology with respect to n^1. Thus, a 0-walk may pass through a 1-node via incident branches, but nonetheless it will remain within a single 0-section because all the branches incident to a 1-node are 0-connected.

On the other hand, a one-ended or endless 0-walk W^0 may "reach" a 1-node by proceeding infinitely through an arm. To be more precise, let us denote one-ended parts of W^0 by

$$W_{-\infty,m}^0 = \{\ldots, b_{m-2}, n_{m-1}^0, b_{m-1}, n_m^0\}$$

and

$$W_{m,\infty}^0 = \{n_m^0, b_m, n_{m+1}^0, b_{m+1}, \ldots\}.$$

Let $\mathbf{S}$ be the 0-section to which W^0 is confined, let x be an extremity of $\mathbf{S}$, let $\mathbf{S}_b$ be the subsection to which x belongs, and let n^1 be the 1-node that embraces x. By the definition of an extremity, no other extremity of $\mathbf{S}_b$ will be embraced by n^1. Choose any proper contraction $\{\mathcal{W}_p\}_{p=1}^{\infty}$ that isolates n^1 within $\mathbf{S}_b$, and let $\{\mathbf{A}_p\}_{p=1}^{\infty}$ be the corresponding sequence of arms. Thus, for every p, $\mathbf{A}_p = \cup_{q=p}^{\infty}\mathbf{A}_q$ and $\mathbf{A}_p \backslash \mathbf{A}_{p-1}$ is a nonvoid finite 0-network. We say that W^0 *starts at* (*stops at*) x — as well as at the 1-node n^1 that embraces x — if, given any natural number q, there is an m such that $W^0_{-\infty,m}$ (respectively, $W^0_{m,\infty}$) remains within $\mathbf{A}_q$. In either case, we also say that W^0 *reaches* x and its embracing 1-node. This definition does not depend upon the choice of $\{\mathcal{W}_p\}_{p=1}^{\infty}$; if the defining condition is fulfilled for one choice, it will be fulfilled for every choice. When W^0 reaches a 1-node through an arm in this way, we at times call W^0 *transient*. This use of the adjective "transient" differs from customary usage: indeed, we are now applying it to a deterministic walk rather than to a random walk.

It is worth pointing out that W^0 may intermittently reach further and further away from some starting 0-node without ever reaching a 1-node. For instance, choose a proper contraction within $\mathbf{S}_b$ for every 1-node n_k^1 ($k = 1, \ldots, K$) incident to $\mathbf{S}_b$, and for every p let $\mathbf{M}_p = \cup_{k=1}^{K}\mathbf{A}_{k,p}$ be the union of the corresponding pth arms $\mathbf{A}_{k,p}$. It is possible for a 0-walk W^0 to have the following property: For every choice of the natural numbers m and q, the one-ended part $W^0_{m,\infty}$ of W^0 meets both $\mathbf{M}_1 \backslash \mathbf{M}_2$ and $\mathbf{M}_q$. Thus, W^0 keeps getting into ever-larger portions of $\mathbf{S}_b$ but also keeps returning to $\mathbf{M}_1 \backslash \mathbf{M}_2$ before it reaches any 1-node. We might call such a 0-walk "recurrent" even though W^0 is a deterministic walk. As we shall see later on when we discuss random walks, the probability of a 0-walk on $\mathbf{N}^1$ being recurrent is zero.

We turn now to walks that "pass through" 1-nodes and possibly through different 0-sections. Consider the following alternating sequence of 1-nodes n_m^1 and nontrivial 0-walks W_m^0:

$$W^1 = \{\ldots, n_m^1, W_m^0, n_{m+1}^1, W_{m+1}^0, \ldots\} \tag{10}$$

where, for each m, W_m^0 reaches n_m^1 and n_{m+1}^1 under the following restrictions: No more than one of the 0-walks W_m^0 and W_{m+1}^0 reaches n_{m+1}^1 through an embraced 0-node; the other one must do so through a one-ended part of itself that passes through an arm. (This insures that each W_m^0 is maximal as a 0-walk in W^1 and is in fact transient.) Furthermore, we allow the sequence (10) to be *finite* or *one-ended* or *endless*; in the first two cases, each terminal element is required to be either a 0-node or a 1-node. Under these conditions, W^1 will be called a 1-*walk*.

Note that, according to this definition, various entries in (10) may be the same 1-node or the same 0-walk. We say that W^1 *embraces* itself, and all its elements, and also all the elements embraced by its elements. Thus, any branch or node may occur many times as an embraced element of W^1. W^1 is called *nontrivial* if (10) has at least three elements, and W^1 is said to perform a *one-step transition* from n_m^1 to n_{m+1}^1. Also, W^1 is said to *rove* if it has at least two 1-nodes and every two consecutive 1-nodes in (10) are always different. When W^1 is finite or one-ended, we say that W^1 *starts at* (*stops at*) its terminal node on its left (respectively, on its right). We also say that W^1 *passes through* all its elements other than its terminal nodes and *reaches* all its elements.

Finally, note that a finite 0-walk W^0 with the terminal nodes n_a^0 and n_b^0 is a special case of a 1-walk, namely, $\{n_a^0, W^0, n_b^0\}$.

8 Random 0-Walks

Consider a random walker Ψ that wanders around $\mathbf{N}^1$ in such a fashion that the comparative probabilities of the one-step transitions from any ordinary 0-node n_0^0 are governed by the *nearest-neighbor rule*: Let n_l^0 $(k = 1, \ldots, L)$ be the (ordinary or embraced) 0-nodes adjacent to n_0^0 and let g_l denote the branch conductance between n_0^0 and n_l^0; then the probability $P_{0,k}$ of Ψ making the one-step transition from n_0^0 to n_k^0 is defined to be $P_{0,k} = g_k / \sum_{l=1}^{L} g_l$. This probability can be measured electrically. Let n_k^0 be held at 1 V and let all the other n_l^0 $(l \neq k)$ be held at 0 V. By Kirchhoff's laws and Ohm's law, the resulting node voltage at n_0^0 is $P_{0,k}$.

As Ψ wanders through a subsection $\mathbf{S}_b$ of $\mathbf{N}^1$, it generates a conventional random walk, which can be described by a Markov chain whose state space consists of the ordinary and embraced 0-nodes of $\mathbf{S}_b$ [5, Chapter 9, Section 10]. Thus, that state space may be either finite or infinite. Nash-Williams [6] has shown that the nearest-neighbor rule generalizes in the following way. Let $\mathcal{N}_a$ be any finite set of 0-nodes in $\mathbf{S}_b$. We define the *boundary of* $\mathcal{N}_a$

to be the set of all 0-nodes of $\mathcal{N}_a$ that either are embraced by a 1-node, or are adjacent to 0-nodes that are not in $\mathcal{N}_a$, or are both. Also, let $\mathcal{N}_e$, $\mathcal{N}_g$, and $\mathcal{N}_s$ be three disjoint sets of 0-nodes in $\mathbf{S}_b$ such that $\mathcal{N}_e \cup \mathcal{N}_g$ contains the boundary of $\mathcal{N}_a$, and $\mathcal{N}_s$ is entirely contained in $\mathcal{N}_a$. Then, the *Nash-Williams rule* [6, Corollary 4A] states that the probability of Ψ reaching some node of $\mathcal{N}_e$ before reaching any node of $\mathcal{N}_g$, given that Ψ starts at some node of $\mathcal{N}_s$, is equal to the voltage at $\mathcal{N}_s$ when the nodes of $\mathcal{N}_s$ have been shorted together while the nodes of $\mathcal{N}_e$ are held at 1 V and the nodes of $\mathcal{N}_g$ are held at 0 V.

As Ψ wanders through the subsection $\mathbf{S}_b$, it may eventually reach a 1-node incident to $\mathbf{S}_b$. In fact, it may do so either in a finite number of steps by meeting an embraced 0-node or in an infinity of steps by passing along an arm to reach an extremity. We will generalize the Nash-Williams rule through a limiting process in order to establish comparative probabilities for transitions to the 1-nodes incident to $\mathbf{S}_b$.

Assume for now that $\mathbf{S}_b$ has at least two incident 1-nodes. $\mathbf{S}_b$ may not have any extremities (that is, $\mathbf{S}_b$ may be finite), in which case the Nash-Williams rule can be applied directly to find comparative probabilities for transitions to the various embraced 0-nodes of $\mathbf{S}_b$. On the other hand, if $\mathbf{S}_b$ does have extremities, we can choose a proper contraction $\{\mathcal{W}_{k,p}\}_{p=1}^{\infty}$ within $\mathbf{S}_b$ for every 1-node n_k^1 ($k = 1, \ldots, K$) incident to $\mathbf{S}_b$. Let $\{\mathbf{A}_{k,p}\}_{p=1}^{\infty}$ be the sequence of arms for the kth contraction, and let $\mathcal{V}_{k,p}$ be the base of $\mathbf{A}_{k,p}$. We can and do select those contractions such that $\mathbf{A}_{k,1} \cap \mathbf{A}_{l,1} = \emptyset$ whenever $k \neq l$. Now, choose a positive integer p_k for each k and set $\mathbf{M}(p_1, \ldots, p_K) = \cup_{k=1}^{K} \mathbf{A}_{k,p_k}$.

Next, set $\mathbf{F}(p_1, \ldots, p_K) = \mathbf{S}_b \backslash \mathbf{M}(p_1, \ldots, p_K)$. $\mathbf{F}(p_1, \ldots, p_k)$ is illustrated in Figure 2: it is the reduced finite network induced by all the branches of $\mathbf{S}_b$ that are separated from all the 1-nodes by the nodes of all the $\mathcal{W}_{l,p_l}$. For example, the branches b_1 and b_2 of Figure 2 will be in $\mathbf{F}(p_1, \ldots, p_k)$, and so too will the embraced 0-nodes of n_k^1 and n_l^1. For a particular k, let $\mathcal{N}_e$ be the nodes of $\mathcal{W}_{k,p_k}$, let $\mathcal{N}_g$ be the nodes of all the other $\mathcal{W}_{l,p_l}$ ($l \neq k$), and let $\mathcal{N}_s$ be a singleton whose element is an ordinary 0-node n_0^0 in $\mathbf{F}(p_1, \ldots, p_k)$ but not in $\mathcal{N}_e$ or $\mathcal{N}_g$. Now, hold all the nodes of $\mathcal{N}_e$ at 1 V and all the nodes of $\mathcal{N}_g$ at 0 V. By the Nash-Williams rule, the resulting voltage $v_{0,k}(p_1, \ldots, p_K)$ at n_0^0 is the probability of Ψ starting from n_0^0 and reaching some node of $\mathcal{N}_e$ before reaching any node of $\mathcal{N}_g$.

As was established in Section 5, we are free to apply pure voltage sources to the 1-nodes of $\mathbf{N}^1$. Consequently, another voltage $u_{0,k}$ will be induced at n_0^0 when n_k^1 is held at 1 V, and all the other n_l^1 $(l \neq k)$ are held at 0 V.

Lemma 8.1. *$v_{0,k}(p_1, \ldots, p_K)$ converges to $u_{0,k}$ as the $p_1, \ldots, p_K$ tend to infinity independently.*

Proof. For each $l = 1, \ldots K$, let $n_{l,p_l,i}^0$ be the ith node of $\mathcal{W}_{l,p_l}$ and let $u_{l,p_l,i}$ denote the voltage at $n_{l,p_l,i}^0$ when the 1-node n_k^1 is held at 1 V and and all the other 1-nodes n_l^1 $(l \neq k)$ are held at 0 V. (Thus, if $n_{l,p_l,i}^0$ is an embraced node, $u_{l,p_l,i}$ will be 1 for $l = k$ and 0 for $l \neq k$.) By the superposition principle, $v_{0,k}(p_1, \ldots, p_K) - u_{0,k}$ is the voltage at n_0^0 when every $n_{k,p_k,i}$ is held at $1 - u_{k,p_k,i}$ and when every $n_{l,p_l,i}$ $(l \neq k)$ is held at $-u_{l,p_l,i}$. Theorem 6.6 asserts that $1 - u_{k,p_k,i}$ and $u_{l,p_l,i}$ are nonnegative. By the maximum principle for the node voltages in a finite sourceless network,

$$\min_{i;l\neq k}(-u_{l,p_l,i}) \leq v_{0,k}(p_1, \ldots, p_K) - u_{0,k} \leq \max_i(1 - u_{k,p_k,i}) \tag{11}$$

where the maximum is taken over all indices i for the nodes in $\mathcal{W}_{k,p_k}$ and the minimum is taken over all the indices for all the nodes in all the other $\mathcal{W}_{l,p_l}$. Now, the nodes of all the $\mathcal{W}$'s lie on finitely many contraction paths. Therefore, by Lemma 6.2, both sides of (11) tend to zero as the p's tend to infinity independently. Consequently, so too does the middle. □

It will be helpful to use the notation

$$Prob(\mathrm{s}\mathcal{N}_1, \mathrm{r}\mathcal{N}_2, \mathrm{b}\mathcal{N}_3 \mid A) \tag{12}$$

to denote a comparative transition probability under a given restriction A, where $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$ are three disjoint node sets. More specifically, let all the nodes of $\mathcal{N}_1$ be shorted together and let the random walker Ψ start from that short; then, (12) will denote the probability that Ψ will reach some node of $\mathcal{N}_2$ before reaching any node of $\mathcal{N}_3$. If $\mathcal{N}_1$ is a singleton $\{n_1\}$, we will replace $\mathcal{N}_1$ by n_1, and similarly for $\mathcal{N}_2$ and $\mathcal{N}_3$. Also, we will delete the notation "$\mid A$"when no restriction needs to be specified.

Lemma 8.1 motivates a rule for the comparative probabilities for the transitions from a 0-node of a subsection to its incident 1-nodes: it arises as a limiting case of the Nash-

Williams rule. In the next definition, $\mathbf{S}_b$ is a subsection of $\mathbf{N}^1$ having two or more incident 1-nodes, n_k^1 is one of them, $\mathcal{N}_g^1$ is the set of all the other 1-nodes n_l^1 $(l \neq k)$ incident to $\mathbf{S}_b$, and A represents the condition that Ψ does reach some 1-node. We will show through Theorem 8.4 below that A can occur with a positive probability.

Definition 8.2. Given that Ψ starts at an ordinary 0-node n_0^0 of $\mathbf{S}_b$ and reaches some 1-node, the probability

$$Prob(\mathrm{s}n_0^0, \mathrm{r}n_k^1, \mathrm{b}\mathcal{N}_g^1 \mid A)$$

that Ψ will reach n_k^1 before reaching any node of $\mathcal{N}_g^1$ is defined to be the voltage at n_0^0 when n_k^1 is held at 1 V and all the nodes of $\mathcal{N}_g^1$ are held at 0 V.

A variation of Lemma 8.1 leads to a rule for comparing the probability of transition to the set $\mathcal{N}_e^1$ of all 1-nodes incident to $\mathbf{S}_b$ with the probability of transition to some finite set $\mathcal{N}_g^0$ of ordinary 0-nodes in $\mathbf{S}_b$. Choose the isolating sets $\mathcal{W}_{l,p_l}$ $(l = 1, \ldots, K)$ as before and let n_0^0 be any 0-node in $F(p_1, \ldots, p_K)$ with $n_0^0 \notin \mathcal{N}_g^0$. Hold all the nodes of $\mathcal{N}_g^0$ at 0 V, and let $v_0(p_1, \ldots, p_K)$ be the voltage at n_0^0 when all the nodes in all the $\mathcal{W}_{l,p_l}$ are held at 1 V. On the other hand, with the nodes of $\mathcal{N}_g^0$ still held at 0 V and all the nodes of $\mathcal{N}_e^1$ held at 1 V, let u_0 be the resulting voltage at n_0^0 and let $u_{l,p_l,i}$ be the resulting voltage at the ith node of $\mathcal{W}_{l,p_l}$. By the superposition principle, Theorem 6.6, and the maximum principle for the node voltages in the finite network $\mathbf{F}(p_1, \ldots, p_K)$, we get

$$\min_{i,l}(1 - u_{l,p_l,i}) \;\leq\; v_0(p_1, \ldots, p_K) - u_0 \;\leq\; \max_{i,l}(1 - u_{l,p_l,i}) \tag{13}$$

as the replacement for (11). By virtue of Lemma 6.2 and the fact that all the nodes of all the $\mathcal{W}_{l,p_l}$ lie on finitely many contraction paths, both sides of (13) tend to zero as the $p_l \to \infty$ independently. Hence, $v_0(p_1, \ldots, p_K) \to u_0$.

This motivates the next definition as a limiting form of the Nash-Williams rule. In this case, the subsection $\mathbf{S}_b$ of $\mathbf{N}^1$ may have only one incident 1-node. Also, $\mathcal{N}_e^1$ is the set of all 1-nodes incident to $\mathbf{S}_b$, $\mathcal{N}_g^0$ is any finite set of ordinary 0-nodes in $\mathbf{S}_b$, and $n_0^0 \notin \mathcal{N}_g^0$ is another ordinary 0-node in $\mathbf{S}_b$.

Definition 8.3. The probability

$$Prob(\mathrm{s}n_0^0, \mathrm{r}\mathcal{N}_e^1, \mathrm{b}\mathcal{N}_g^0)$$

that Ψ, after starting from n_0^0, reaches some 1-node incident to $\mathbf{S}_b$ before reaching any 0-node in $\mathcal{N}_g^0$ is defined to be the voltage u_0 at n_0^0 when all the 1-nodes incident to $\mathbf{S}_b$ are held at 1 V and all the 0-nodes of $\mathcal{N}_g^0$ are held at 0 V.

As promised, we will now check condition A in Definition 8.2. To this end, we define a subsection $\mathbf{S}_b$ as being *transient* if Ψ, after starting from any arbitrarily chosen ordinary 0-node n_g^0 in $\mathbf{S}_b$, always has a positive probability of reaching some 1-node incident to $\mathbf{S}_b$ before returning to n_g^0.

Theorem 8.4. *Under Conditions 4.1, every subsection of* $\mathbf{N}^1$ *is transient.*

Proof. Choose arbitrarily an ordinary 0-node n_g^0 in $\mathbf{S}_b$. The nearest-neighbor rule insures that, for every 0-node n_a^0 adjacent to n_g^0, there is a positive probability that Ψ, after starting from n_g^0, will reach n_a^0 in one step. So, if any such n_a^0 is embraced, the theorem follows immediately. On the other hand, to show that $\mathbf{S}_b$ is transient when no such n_a^0 is embraced, we need only show that there is a positive probability that Ψ, after starting from n_a^0, will reach some 1-node incident to $\mathbf{S}_b$ before reaching n_g^0. By Definition 8.3, this can be accomplished by showing that, for some choice of the adjacent 0-node n_a^0, the voltage at n_a^0 is positive when n_g^0 is held at 0 V and all the 1-nodes incident to $\mathbf{S}_b$ are held at 1 V.

Suppose this is not so. In view of Theorem 6.6, we suppose that all the voltages at the nodes adjacent to n_g^0 are zero. Hence, the currents in the resistive branches incident to n_g^0 are zero too. Let $\mathbf{i}$ be the current vector produced in $\mathbf{N}_e^1$ by the stated assignment of node voltages. $\mathbf{i}$ is determined by Theorem 5.5 when all the 1-nodes incident to $\mathbf{S}_b$ are shorted together and a 1-V source in the branch b_0 is connected from n_g^0 to that short. Since the resistive branches incident to n_g^0 carry zero current, we have from Kirchhoff's current law that $i_0 = 0$. We are free to set $\mathbf{s} = \mathbf{i}$ in (2). Therefore, $s_0 = i_0 = 0$, and $\sum_{j=1}^{\infty} r_j i_j^2 = 0$, which implies that $i_j = 0$ for all j. Consequently, there can be no voltage difference between any two nodes — in contradiction to the presence of the 1 V source. □

Let us note that the probability of a 0-walk on $\mathbf{N}^1$ being "recurrent" — as described in Section 7 — is zero. Indeed, such a walk starts at $\mathbf{M}_1 \backslash \mathbf{M}_2$, reaches some $\mathbf{M}_q$ $(q > 1)$, and returns to $\mathbf{M}_0$ before reaching a 1-node; moreover, it does so infinitely often. But, the probability of one such round trip occurring is less than 1, and therefore the probability of it occurring infinitely often is zero. (However, this does not mean that such a recurrent 0-walk is impossible; it merely means that such walks are "rare".)

Finally, it is worth pointing out that, according to [7], the ends of any 0-section $\mathbf{S}^0$ in $\mathbf{N}^1$ can be related in certain cases to the Martin boundary for $\mathbf{S}^0$ arising from our adopted nearest-neighbor rule. Assume now that there are no embraced nodes in $\mathbf{S}^0$. Also assume that, for every end d of $\mathbf{S}^0$, a finitely chainlike representation of a spur for d [11] can be chosen such that the following two conditions are satisfied: (i) For each $p > 1$ and every two nodes n_a^0 and n_b^0 in $\mathcal{V}_p$, there exists a path connecting n_a^0 and n_b^0 that does not meet $\mathcal{V}_{p-1} \cup \mathcal{V}_{p+1}$. (Here, the $\mathcal{V}_p$ are the node sets for the finitely chainlike structure [9, page 33] and are analogous to our arm bases.) (ii) Comparative transition probabilities satisfy

$$\sum_{p=1}^{\infty} \min\{Prob(\mathrm{s}n_a^0, \mathrm{r}n_b^0, \mathrm{b}\mathcal{V}_{p+1}) : n_a^0, n_b^0 \in \mathcal{V}_p\} \;=\; \infty.$$

Then, by virtue of the theorem in [7], the finitely many ends of $\mathbf{S}^0$ correspond bijectively to the points of the Martin boundary for $\mathbf{S}^0$ with regard to a random 0-walk on $\mathbf{S}^0$. The conditions (i) and (ii) can be satisfied by imposing appropriate restrictions on the spurs and the resistances of $\mathbf{S}^0$, some examples of which can also be found in [7].

Under the stated conditions, each 0-section of $\mathbf{N}^1$ will have its corresponding Martin boundary. On the other hand, we are free to choose the 1-nodes quite arbitrarily in order to short various ends together, thereby imposing a structure beyond that of the Martin boundaries. To put this another way, just as the 0-nodes (i.e., the shorts between elementary tips) determine the random 0-walks on the various 0-sections and thereby their Martin boundaries under appropriate conditions, so too do the shorts between 0-tips determine a random 1-walk through $\mathbf{N}^1$, as we shall see. It has been shown in [11] that, when there are no embraced nodes, the 1-walks on certain infinite 1-networks can be described as 0-walks on a "surrogate" infinite 0-network. Consequently, under appropriate conditions again, the ends of that surrogate 0-network determine another Martin boundary, one for a random 1-walk on the infinite 1-network. This would be a Martin Boundary of, say, rank 1. More generally, for a certain infinite ν-network, there can be a hierarchy of Martin boundaries of differing ranks.

9 Random 1-Walks

Now that we have examined how a random 0-walk may stop at a 1-node, we need to examine how it may start at a 1-node. This will allow us to piece together random 0-walks to obtain a random 1-walk that wanders through $\mathbf{N}^1$.

Given any 1-node n^1 of $\mathbf{N}^1$, choose a proper contraction $\{\mathcal{W}_p\}_{p=1}^{\infty}$ to n^1. Let $\{\mathbf{S}_{bk}\}_{k=1}^{\infty}$be the set of subsections incident to n^1. Thus, $\mathcal{W}_p = \cup_{k=1}^{K} \mathcal{W}_{k,p}$, where $\{\mathcal{W}_{k,p}\}_{p=1}^{\infty}$ is a contraction to n^1 within $\mathbf{S}_{bk}$. Let $\mathbf{A}_{k,p}$ be the arm corresponding to $\mathcal{W}_{k,p}$, and set $\mathbf{A}_p = \cup_{k=1}^{K} \mathbf{A}_{k,p}$. We now let $\mathcal{V}_p$ be the union of the arm bases $\mathcal{V}_{k,p}$ for all the arms $\mathbf{A}_{k,p}$, $k = 1, \ldots, K$. If n^1 embraces a 0-node, append to $\mathcal{V}_p$ every 0-node that is adjacent to n^1, and let $\mathcal{X}_p$ be the resulting set of 0-nodes. If n^1 does not embrace a 0-node, set $\mathcal{X}_p = \mathcal{V}_p$.

For example, consider the central 1-node n_0^1 in Figure 1. Choose $\mathcal{W}_1$ as the isolating set $\{n_1^0, n_2^0, n_3^0, n_4^0, n_a^0\}$, where n_a^0 is the 0-node embraced by n_0^1. Thus, $\mathcal{V}_1 = \{n_1^0, n_2^0, n_3^0, n_4^0\}$. Similarly, we can choose $\{\mathcal{W}_p\}_{p=1}^{\infty}$ as a contraction to n_0^1 such that $\mathcal{V}_p$ is a set of four 0-nodes that form a rectangular pattern and are closer to n_0^1 than are the nodes of $\mathcal{V}_{p-1}$ ($p > 1$). Finally, for each $p = 1, 2, \ldots$, $\mathcal{X}_p$ consists of the four nodes of $\mathcal{V}_p$, the 0-node embraced by n_1^1, and the ordinary 0-nodes n_5^0, n_6^0, and n_7^0.

We return to the general case. $\mathcal{X}_p$ separates n^1 from the nodes of any branch that is neither in $\mathbf{A}_p$ nor incident to n^1. Similarly, for $q > p$. $\mathcal{X}_q$ separates n^1 from $\mathcal{X}_p$.

In the next two definitions, it is assumed that $\mathcal{X}_p$ has two or more nodes. Also, $\mathcal{Y}_P$ denotes a proper subset of $\mathcal{X}_p$.

Definition 9.1. Given that Ψ starts at n^1 and reaches a node of $\mathcal{X}_p$, the probability:

$$P(n^1; \mathcal{Y}_p) \;=\; Prob(\mathrm{s}n^1, \mathrm{r}\mathcal{Y}_P, \mathrm{b}\mathcal{X}_p \backslash \mathcal{Y}_p \mid \Psi \text{ reaches } \mathcal{X}_p) \qquad (14)$$

that Ψ reaches some node in $\mathcal{Y}_p$ before it reaches any node of $\mathcal{X}_p \backslash \mathcal{Y}_p$ is defined to be the voltage at n^1 when the nodes of $\mathcal{Y}_p$ are held at 1 V and the nodes of $\mathcal{X}_p \backslash \mathcal{Y}_p$ are held at 0 V.

With $q > p$. number the nodes of $\mathcal{X}_p$ and of $\mathcal{X}_q$ and let $n_{p,k}^0$ and $n_{q,i}^0$ be the kth and ith nodes of $\mathcal{X}_p$ and $\mathcal{X}_q$ respectively.

Definition 9.2. Given that Ψ starts at $n_{q,i}^0$ and reaches some node of $\mathcal{X}_p$, the probability:

$$P(n^0_{q,i}; n^0_{p,k}) \ = \ Prob(\mathrm{s}n^0_{q,i}, \ \mathrm{r}n^0_{p,k}, \ \mathrm{b}\mathcal{X}_p\backslash\{n^0_{p,k}\} \mid \Psi \text{ reaches } \mathcal{X}_p) \tag{15}$$

that Ψ reaches $n^0_{p,k}$ before reaching any other node of $\mathcal{X}_p$ is defined to be the voltage $u_{q,i}(p,k)$ at $n^0_{q,i}$ when $n^0_{p,k}$ is held at 1 V and every other node of $\mathcal{X}_p$ is held at 0 V.

Although Definition 9.2 is much like the Nash-Williams rule, it is needed because $n^0_{q,i}$ resides in an exterior infinite network rather than in an interior finite network. Moreover, $\mathcal{X}_p$ and $\mathcal{X}_q$ will intersect when they both contain 0-nodes adjacent to the 1-node n^1; thus, it may happen that $n^0_{q,i}$ and $n^0_{p,k}$ are the same node, in which case $P(n^0_{q,i}; n^0_{p,k}) = 1$.

Definition 9.1 assigns comparative probabilities for transitions from n^1 to the nodes of any $\mathcal{X}_p$. Since Ψ, when proceeding from n^1 to a node $n^0_{p,k}$ of $\mathcal{X}_p$, must meet at least one node of $\mathcal{X}_q$, where $q > p$, we should now prove the consistency of our definitions in the following sense: The comparative probability for the transition from n^1 to $n^0_{p,k}$ is the same as that obtained by combining the comparative probabilities for transitions from n^1 to the various nodes of $\mathcal{X}_q$ with the comparative probabilities for transitions from the nodes of $\mathcal{X}_q$ to $n^0_{p,k}$. More specifically, let us replace $\mathcal{Y}_p$ by $n^0_{p,k}$ in Definition 9.1. Then, by conditional probabilities, we should have

$$P(n^1; n^0_{p,k}) \ = \ \sum_i P(n^1; n^0_{q,i}) P(n^0_{q,i}; n^0_{p,k}) \tag{16}$$

if Definitions 9.1 and 9.2 are to be consistent. This equation can be established electrically.

Let $u^1(p,k)$ be the voltage at n^1 when $n^0_{p,k}$ is held at 1 V and all the other nodes of $\mathcal{X}_p$ are held at 0 V. Let $u^1(q,i)$ be defined similarly in terms of the node voltages for $\mathcal{X}_q$ (replace p by q and k by i). Furthermore, let $u_{q,i}(p,k)$ be as in Definition 9.2. By the superposition principle for electrical networks,

$$u^1(p,k) \ = \ \sum_i u^1(q,i) u_{q,i}(p,k). \tag{17}$$

According to Definitions 9.1 and 9.2, $u^1(p,k) = P(n^1; n^0_{p,k})$, $u^1(q,i) = P(n^1, n^0_{q,i})$, and $u_{q,i}(p,k) = P(n^0_{q,i}, n^0_{p,k})$. Thus, (16) is justified by (17), and therefore Definitions 9.1 and 9.2 are consistent.

There is still another matter we should examine. What is the probability that Ψ, after starting from n^1, reaches $\mathcal{X}_p$ for some given p before returning to n^1? It is zero. To see this, first note that "Ψ starting from n^1" means that Ψ reaches $\mathcal{X}_q$ for some sufficiently large q. It may do so by leaving n^1 either along a branch incident to n^1 or along an arm between n^1 and $\mathcal{V}_q$. (See Figure 3.)

We take it that $q > p$ and that a proper contraction to n^1 has been chosen. For each p, this yields a union of arms $\mathbf{A}_p = \cup_{k=1}^{K} \mathbf{A}_{k,p}$ as above. Let $\mathcal{D}$ be the set of all 0-nodes adjacent to n^1. Hence, no node of $\mathcal{D}$ is in $\mathbf{A}_p$ or in $\mathbf{A}_q$. Moreover, $\mathcal{X}_q = \mathcal{D} \cup \mathcal{V}_q$. (In Figure 3, $\mathcal{D} = \{n_1^0, n_a^0\}$, where n_a^0 is the 0-node embraced by n_a^1.)

Case 1. Ψ leaves n^1 along an incident branch: By Definition 9.1, the probability $Prob(\mathrm{s}n^1, \mathrm{r}\mathcal{D}, \mathrm{b}\mathcal{V}_q \mid \Psi \text{ reaches } \mathcal{X}_q)$ is the voltage u^1 at n^1 when the nodes of $\mathcal{D}$ are held at 1 V and the nodes of $\mathcal{V}_q$ are held at 0 V. By the voltage-divider rule, $u_1 = R_q/(R_d + R_q)$, where R_q is the resistance of the union of arms between n^1 and a short at $\mathcal{V}_q$ and R_d is the parallel resistance of the branches incident to n^1. By Condition 4.1(b), every node of $\mathcal{V}_q$ lies on a perceptible contraction path for the chosen contraction to n^1. Let m be the number of such paths. Now, replace every branch in the said union of arms that is not in a contraction path by an open circuit, and let R_j be the sum of the resistances in the jth contraction path between n^1 and $\mathcal{V}_q$. By Rayleigh's monotonicity law [9, page 103], $R_q \leq (\sum_{j=1}^{m} R_j^{-1})^{-1}$. As $q \to \infty$, each $R_j \to 0$. Hence, $R_q \to 0$, and therefore $u^1 \to 0$. This means that the probability of Ψ leaving n^1 through a branch incident to n^1 instead of along an arm is zero.

Case 2. Ψ leaves n^1 along an arm: Thus, Ψ reaches $\mathcal{V}_q$ for some sufficiently large q greater than p. Let $n_{q,i}^0$ be any node of $\mathcal{V}_q$, as before. We will now show that, as $q \to \infty$, $Prob(\mathrm{s}n_{q,i}^0, \mathrm{r}\mathcal{X}_p, \mathrm{b}n^1)$ tends to zero. Indeed, by Definition 8.3, that probability is the voltage $u_{q,i}$ at $n_{q,i}^0$ when the nodes of $\mathcal{X}_p$ are held at 1 V and n^1 is held at 0 V. But, by Lemma 6.2, $u_{q,i} \to 0$ as $q \to \infty$. This means that Ψ, after starting from n^1 along an arm, will almost surely return to n^1 before it reaches $\mathcal{X}_p$ for any given p.

Both cases taken together imply that only a vanishingly small proportion of the random 1-walks that start at n^1 will reach $\mathcal{X}_p$ without first returning to n^1, whatever be p. In this sense, no 1-node is a transient node. Thus, there is a zero probability that Ψ will rove. However, this does not mean that there are no random roving 1-walks. It simply means that we are dealing with the exceptional case when we compare transition probabilities for roving 1-walks.

As our last task in this section, we shall now show that a random roving 1-walk is a Markov chain with a finite state space consisting of the 1-nodes of $\mathbf{N}^1$. For this purpose, consider now a 1-node n_0^1 and all its incident subsections, which we denote by $\mathbf{S}_{bi}$ where $i = 1, \ldots, I$. (This is illustrated in Figure 4 wherein n_0^1 has four incident subsections: $\mathbf{S}_{b1}$, $\mathbf{S}_{b2}$, and the two subsections having only one branch each, namely, b_1 and b_2.) Let $n_1^1, \ldots, n_K^1$ be the 1-nodes incident to those subsections $\mathbf{S}_{bi}$ other than n_0^1; we say that those 1-nodes are *adjacent to* n_0^1. For each n_k^1, choose a proper contraction to n_k^1 within every subsection $\mathbf{S}_{bi}$ that is incident to n_k^1 through an arm and is also incident to n_0^1. Let I_k be the index set for all such $\mathbf{S}_{bi}$. (I_k will be void when there are no such subsections $\mathbf{S}_{bi}$.) For each p this yields the pth arm base $\mathcal{V}_{k,i,p}$ in $\mathbf{S}_{bi}$, $i \in I_k$. Let $\mathcal{V}_{k,p} = \cup_{i \in I_k} \mathcal{V}_{k,i,p}$. Then, let $\mathcal{Z}_{k,p} = \mathcal{V}_{k,p} \cup \{n_k^0\}$ if n_k^1 embraces a 0-node n_k^0 incident to at least one of the $\mathbf{S}_{bi}$; otherwise, let $\mathcal{Z}_{k,p} = \mathcal{V}_{k,p}$. On the other hand, let $\mathcal{Z}_{k,p} = \{n_k^0\}$ if I_k is void and if n_k^0 is incident to at least one of the $\mathbf{S}_{bi}$ and is embraced by n_k^1. It follows in every case that $\mathcal{Z}_{k,p}$ separates n_k^1 from n_0^1.

Let us choose a positive integer p_k for each $k = 1, \ldots, K$. The nodes of $\cup_{k=1}^K \mathcal{Z}_{k,p_k}$ lie in all the 0-sections incident to n_0^1 and separate n_0^1 from all the 1-nodes n_k^1 adjacent to n_0^1. As a direct extension of Definition 9.1, we can assign comparative probabilities for transitions from n_0^1 to the various $\mathcal{Z}_{k,p_k}$. In particular, given that Ψ starts at n_0^1 and reaches a node of $\cup_{l=1}^K \mathcal{Z}_{l,p_l}$, the probability the Ψ reaches any node of $\mathcal{Z}_{k,p_k}$ before it reaches any node of $\cup\{\mathcal{Z}_{l,p_l} : l = 1, \ldots, K; l \neq k\}$ is equal to the node voltage $v_{0,k}(p_1, \ldots, p_K)$ at n_0^1 when the nodes of $\mathcal{Z}_{k,p_k}$ are held at 1 V and the nodes of all the $\mathcal{Z}_{l,p_l}$ $(l \neq k)$ are held at 0 V. As before, by virtue of Theorem 5.5, another voltage $u_{0,k}$ is obtained at n_0^1 by holding n_k^1 at 1 V and the other 1-nodes n_l^1 $(l \neq k)$ adjacent to n_0^1 at 0 V. We can repeat the proof of Lemma 8.1, substituting n_0^1 for n_0^0, all the subsections $\mathbf{S}_{bi}$ incident to n_0^1 for the single subsection $\mathbf{S}_b$, and the 1-nodes adjacent to n_0^1 for the 1-nodes incident to $\mathbf{S}_b$. The proof proceeds exactly as before, the only difference being that we need a maximum principle for the node voltages in a 1-network. This is provided by Theorem 6.6. All this leads to the conclusion that

$v_{0,k}(p_1, \ldots, p_K)$ converges to $u_{0,k}$ as the $p_1, \ldots, p_K$ tend to infinity independently. Hence, as a limiting case of Definition 9.1, we are led to the following definition, wherein $\mathcal{N}_g^1$ denotes the set of all the 1-nodes adjacent to n_0^1 other than n_k^1.

Definition 9.3. Assume there are two or more 1-nodes adjacent to the 1-node n_0^1. For any random roving 1-walk, the probability:

$$P(n_0^1; n_k^1) \;=\; Prob(\mathrm{s}n_0^1, \mathrm{r}n_k^1, \mathrm{b}\mathcal{N}_g^1 \mid \Psi \text{ roves}) \tag{18}$$

that Ψ, starting from n_0^1, reaches an adjacent 1-node n_k^1 before reaching any of the 1-nodes in $\mathcal{N}_g^1$ is defined to be the node voltage at n_0^1 when n_k^1 is held at 1 V and all the 1-nodes of $\mathcal{N}_g^1$ are held at 0 V.

Lemma 9.4. *Under the conditions of Definition 9.3,* $0 < P(n_0^1; n_k^1) < 1$.

Proof. This follows directly from Corollary 6.7. For instance, to conclude that $P(n_0^1; n_k^1) < 1$, choose the path of part (i) of that corollary to be the 1-path $P^1 = \{n_0^1, P^0, n_g^1\}$, where n_g^1 is the 1-node obtained by shorting the nodes of $\mathcal{N}_g^1$ together, and P^0 is a 0-path that reaches n_0^1 and n_g^1. Since n_k^1 is adjacent to n_0^1, we can choose P^0 such that it does not meet $n_e = n_k^1$. □

By our definition of "roving", we have the following one-step transition probabilities: $P(n_0^1; n_0^1) = 0$. If there is only one 1-node n_1^1 adjacent to n_0^1, $P(n_0^1; n_1^1) = 1$. Furthermore, if n_1^1 is not adjacent to n_0^1, $P(n_0^1; n_1^1) = 0$ obviously. These results along with Definition 9.3 give all the one-step transition probabilities for the roving Ψ.

Finally, to establish that we have a Markov chain, we have to show that these probabilities for one-step transitions from any given 1-node n_0^1 sum to 1. By superposition, this sum is equal to the voltage u_0^1 at n_0^1 when all the 1-nodes adjacent to n_0^1 are held at 1 V and all other 1-nodes and 0-nodes are left floating (i.e., no source connections are made to them). But then, all branch currents in the 0-sections incident to n_0^1 are zero, and therefore $u_0^1 = 1$ too, as required. We have established.

Theorem 9.5. *Let the 1-network* $\mathbf{N}^1$ *satisfy Conditions 4.1. Let* Ψ *be a random roving walker on* $\mathbf{N}^1$ *that follows the nearest-neighbor rule at every ordinary 0-node and follows the five definitions in Sections 8 and 9 for comparative probabilities of transitions between the 0-nodes and 1-nodes. These five definitions arise as limiting cases or consistent variants* *of the Nash-Williams rule. Moreover, Ψ's transitions among the 1-nodes is governed by a Markov chain with a state space consisting of the 1-nodes n_k^1 of $\mathbf{N}^1$ and with the following one-step transition probabilities: $P_{k,k} = 0$, $P_{k,l} = 0$ if n_k^1 and n_l^1 are not adjacent; $P_{k,l} = 1$ if n_l^1 is the only 1-node adjacent to n_k^1; $P_{k,l}$ is given by Definition 9.3 if n_k^1 and n_l^1 are adjacent and there are two or more 1-nodes adjacent to n_k^1.*

10 Reversibility and the Surrogate Network

Theorem 10.1. *The Markov chain of Theorem 9.5 is irreducible and reversible.*

Proof. The case where $\mathbf{N}^1$ has just two 1-nodes is trivial. So, let $\mathbf{N}^1$ have more than two 1-nodes.

For any two adjacent 1-nodes n_1^1 and n_2^1, the probability that a roving 1-walk will pass from n_1^1 to n_2^1 in one step is positive (Lemma 9.4). The irreducibility [4] of the Markov chain now follows from the 1-connectedness of $\mathbf{N}^1$.

As for reversibility, we start by recalling the definition of a cycle—adapted for 1-nodes. This is a finite sequence $C = (n_1^1, n_2^1, \cdots, n_c^1, n_{c+1}^1 = n_1^1)$ of 1-nodes n_k^1, where all 1-nodes are distinct except for the first and last, there are at least three 1-nodes (i.e., $c > 2$), and consecutive 1-nodes in C are adjacent in $\mathbf{N}^1$. A Markov chain is reversible if, for every cycle C, the product $\prod_{k=1}^{c} P_{k,k+1}$ of transition probabilities $P_{k,k+1}$ from n_k^1 to n_{k+1}^1 remains the same when every $P_{k,k+1}$ is replaced by $P_{k+1,k}$ [4, Section 1.5]. Thus, we need only show that

$$P_{1,2}P_{2,3}\cdots P_{c,1} \;=\; P_{1,c}\cdots P_{3,2}P_{2,1}. \tag{19}$$

According to Definition 9.3, $P_{k,k+1}$ is obtained by holding n_{k+1}^1 at 1 volt, by holding all the 1-nodes adjacent to n_k^1 other than n_{k+1}^1 at 0 volt, and setting $P_{k,k+1} = u_k^1$, where u_k^1 is the resulting voltage at n_k^1. For this situation, u_k will remain unchanged when still other 1-node voltages are arbitrarily specified.

To simplify notation, let us denote n_k^1 by m_0 and n_{k+1}^1 by m_1. Also, let $m_2, \cdots, m_K$ denote all the 1-nodes different from n_k^1 and n_{k+1}^1 but adjacent to either n_k^1 or n_{k+1}^1 or both. Since the cycle has at least three 1-nodes, we have $K \geq 2$. Now, consider the K-port obtained from $\mathbf{N}^1$ by choosing m_k, m_0 as the pair of terminals for the kth port

$(k = 1, \cdots, K)$ with m_0 being the common ground for all ports. To obtain the required node voltages for measuring $P_{k,k+1}$, we externally connect a 1-volt source to m_1 from all of the $m_2, \cdots m_K$, with m_0 left floating (i.e., m_0 has no external connections). The resulting voltage u_0 at m_0 is $P_{k,k+1}$.

With respect to m_0, the voltage at m_1 is $1 - u_0$ and the voltage at m_k $(k = 2, \cdots, K)$ is $-u_0$. Moreover, with i_k denoting the current entering m_k $(k = 1, \cdots, K)$, the sum $i_1 + \cdots + i_K$ is zero. (Apply Kirchhoff's current law at m_1.) Furthermore, the port currents and voltages are related by $\mathbf{i} = Y\mathbf{u}$, where $\mathbf{i} = (i_1, \cdots, i_K)$, $\mathbf{u} = (1 - u_0, -u_0, \cdots, -u_0)$, and $Y = [Y_{a,b}]$ is a $K \times K$ matrix of real numbers that is symmetric (Lemma 5.4). Upon expanding $\mathbf{i} = Y\mathbf{u}$ and adding the i_k, we get

$$0 = i_1 + \cdots + i_K = \sum_{a=1}^{K} Y_{a,1} - u_0 \sum_{a=1}^{K} \sum_{b=1}^{K} Y_{a,b}.$$

Therefore,

$$P_{k,k+1} = u_0 = \frac{\sum_{a=1}^{K} Y_{a,1}}{\sum_{a=1}^{K} \sum_{b=1}^{K} Y_{a,b}}. \tag{20}$$

Upon setting $G_k = \sum_{a=1}^{K} \sum_{b=1}^{K} Y_{a,b}$, we can rewrite (20) as

$$G_k P_{k,k+1} = \sum_{a=1}^{K} Y_{a,1}. \tag{21}$$

Now, $\sum_{a=1}^{K} Y_{a,1}$ is the sum $i_1 + \cdots + i_K$ when $\mathbf{u} = (1, 0, \cdots, 0)$; that is, $\sum_{a=1}^{K} Y_{a,1}$ is the sum of the currents entering $m_1, m_2, \cdots, m_K$ from external connections when 1-volt sources are connected to m_1 from all of the $m_0, m_2, \cdots, m_K$.

By reversing the roles of m_0 and m_1, we have by the same analysis that $G_{k+1} P_{k+1,k}$ is the sum $i_0 + i_2 + \cdots + i_K$ of the currents entering $m_0, m_2, \cdots, m_K$ from external connections when 1-volt sources are connected to m_0 from all of the $m_1, m_2, \cdots, m_K$. With respect to the ground node m_0, we now have $u_1 = \cdots = u_K = -1$, and therefore $i_1 = -\sum_{a=1}^{K} Y_{1,a}$. Moreover, under this latter connection, the sum $-i_1 - i_2 - \cdots - i_K$ of the currents leaving $m_1, m_2, \cdots m_K$ is equal to the current i_0 entering m_0. Hence, $-i_1 = i_0 + i_2 + \cdots + i_K$. Thus,

$$G_{k+1} P_{k+1,k} = -i_1 = \sum_{a=1}^{K} Y_{1,a}. \tag{22}$$

Since the matrix Y is symmetric, we have $Y_{1,a} = Y_{a,1}$. So, by (21) and (22),

$$G_{k+1} P_{k+1,k} = G_k P_{k,k+1}. \tag{23}$$

Finally, we may now write

$$P_{1,2} P_{2,3} \cdots P_{c,1} = \frac{G_2}{G_1} P_{2,1} \frac{G_3}{G_2} P_{3,2} \cdots \frac{G_1}{G_c} P_{1,c} = P_{2,1} P_{3,2} \cdots P_{1,c}$$

This verifies (19) and completes the proof. □

Because the Markov chain is irreducible and reversible, we can synthesize a finite 0-network $\mathbf{N}^{1\mapsto 0}$ whose 0-nodes correspond bijectively to the 1-nodes of $\mathbf{N}^1$ and whose random 0-walks are governed by the same transition matrix as that for the 1-node to 1-node transitions of the random roving 1-walks of $\mathbf{N}^1$. $\mathbf{N}^{1\mapsto 0}$ acts as a *surrogate* for $\mathbf{N}^1$. A realization for it can be obtained by connecting a conductance $g_{k,l} = g_{l,k}$ between the 0-nodes n_k^0 and n_l^0 $(k \neq l)$ in $\mathbf{N}^{1\mapsto 0}$, where $g_{k,l}$ is given as follows: Let $n_k^1 \mapsto n_k^0$ denote the bijection from the 1-nodes of $\mathbf{N}^1$ to the 0-nodes of $\mathbf{N}^{1\mapsto 0}$. If n_k^1 and n_l^1 are not adjacent in $\mathbf{N}^1$, set $g_{k,l} = 0$. If n_k^1 and n_l^1 are adjacent in $\mathbf{N}^1$, relabel n_k^1 as m_0. n_l^1 as m_1, and let $m_2, \cdots, m_K$ be the other 1-nodes that are adjacent to either m_0 or m_1 or both. Then, with our prior notation. set $G_k = \sum_{a=1}^{K} \sum_{b=1}^{K} Y_{a,b}$. Also, set $G = \sum_k G_k$. where this latter sum is over all indices for all the 1-nodes of $\mathbf{N}^1$. Finally, set $g_{k,l} = P_{k,l} G_k / G$. By (23), $g_{k,l} = g_{l,k}$. This yields the surrogate network $\mathbf{N}^{1\mapsto 0}$. The one-step transition probabilities for a random 0-walk on $\mathbf{N}^{1\mapsto 0}$ following the nearest-neighbor rule are the same as the probabilities indicated in Theorem 9.5 for a random roving 1-walk on $\mathbf{N}^1$.

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Figure Captions

Figure 1. A 1-graph. The heavy dots denote ordinary 0-nodes. The heavy lines denote 1-nodes, each of which embrace 0-nodes; the latter are not shown. The other lines denote branches, except for the long braces which point out two ladder networks L_1 and L_2 that comprise the cores of two subsections. It is understood here that all the 0-tips on the left-hand side of L_1 are embraced by n_1^1, and similarly for the other 0-tips of both ladders.

Figure 2. A subsection $\mathbf{S}_b$ in $\mathbf{N}^1$. The heavy lines denote 1-nodes incident to $\mathbf{S}_b$. The dash-dot lines denote branches of $\mathbf{S}_b$ incident to 1-nodes. $\mathcal{V}_{k,p_k}$ denotes an arm base for an arm with an extremity embraced by the 1-node n_k^1. ($\mathcal{W}_{k,p_k}$ consists of the 0-nodes in $\mathcal{V}_{k,p_k}$ along with the embraced 0-node of n_k^1 if the latter 0-node exists).

Figure 3. Illustrations for the sets $\mathcal{X}_p$ and $\mathcal{X}_q$. $\mathcal{V}_p$ is an arm base for a proper contraction and similarly for $\mathcal{V}_q$. The heavy lines denote 1-nodes, the dash-dot lines denote branches, the heavy dots denote 0-nodes, and the cross-hatched areas denote arms. $\mathcal{X}_p = \mathcal{V}_p \cup \{n_1^0, n_a^0\}$ and $\mathcal{X}_q = \mathcal{V}_q \cup \{n_1^0, n_a^0\}$, where n_a^0 is the 0-node embraced by n_a^1.

Figure 4. A 1-node n_0^1, its incident subsections, and its adjacent 1-nodes: $n_1^1, n_l^1, n_k^1, n_j^1, n_K^1$. The subsections incident to n_0^1 are $\mathbf{S}_{b1}$, $\mathbf{S}_{b2}$, and the two subsections consisting of single branches b_1 and b_2 respectively. Here again, the dash-dot lines denote branches incident to 1-nodes.

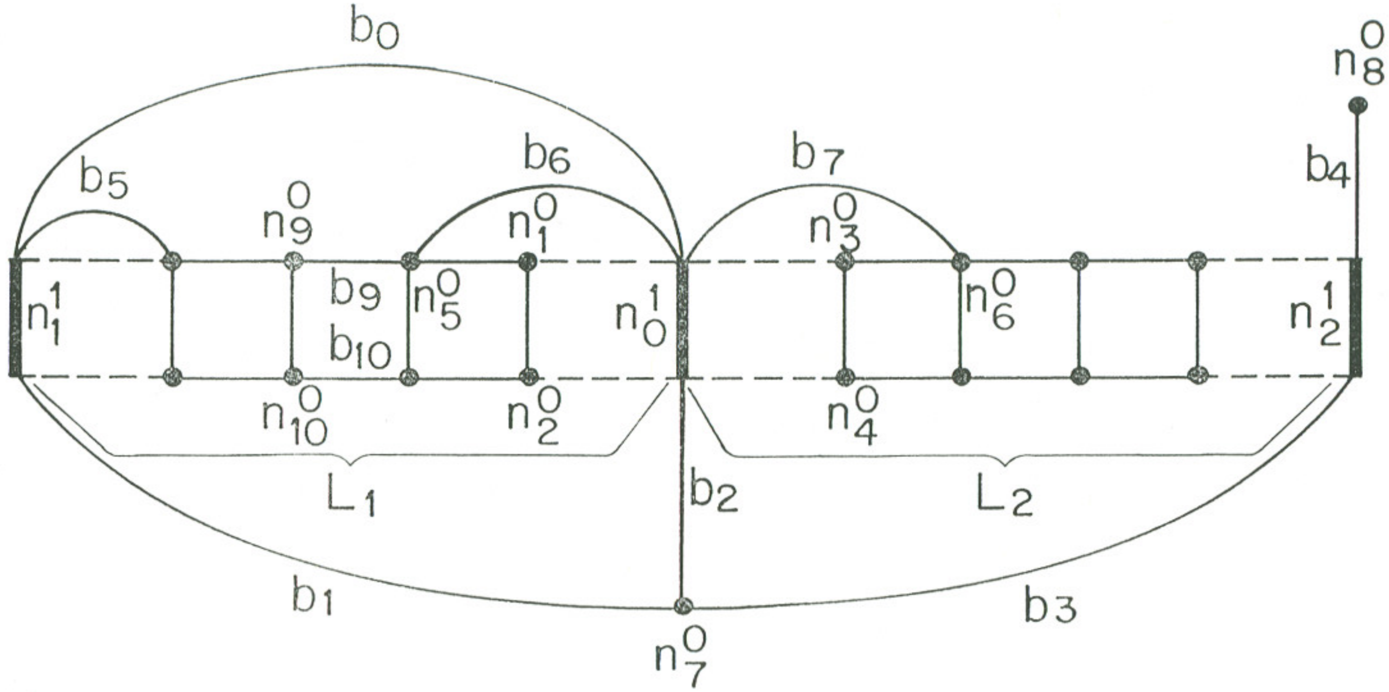
b0
b5
b6
b7
n9 0
n1 0
n3 0
n8 0
b4
b9
n5 0
b10
n1 1
n0 1
n6 0
n2 1
n10 0
n2 0
n4 0
L1
b2
L2
b1
b3
n7 0

FIG. 1

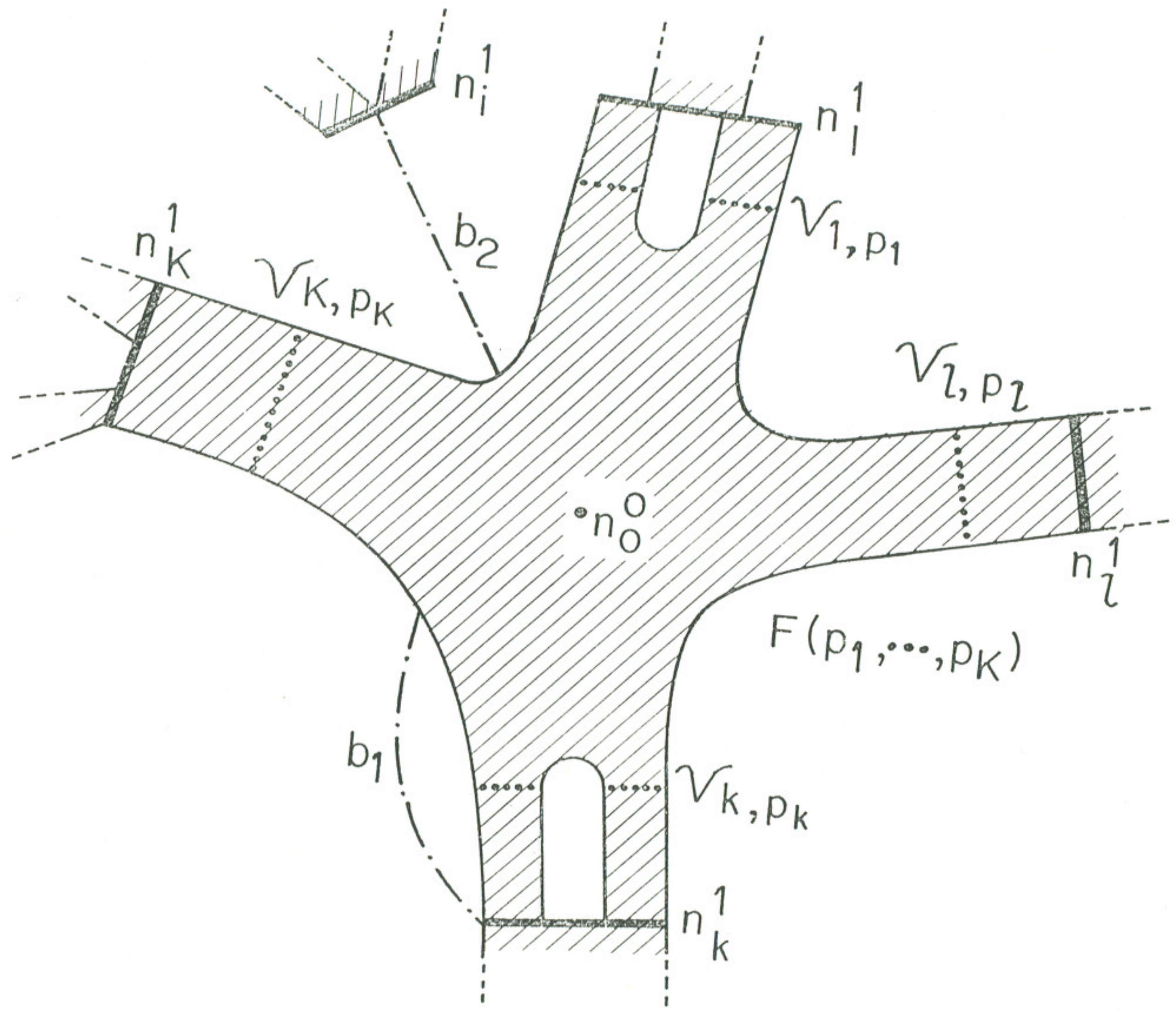

FIG. 2

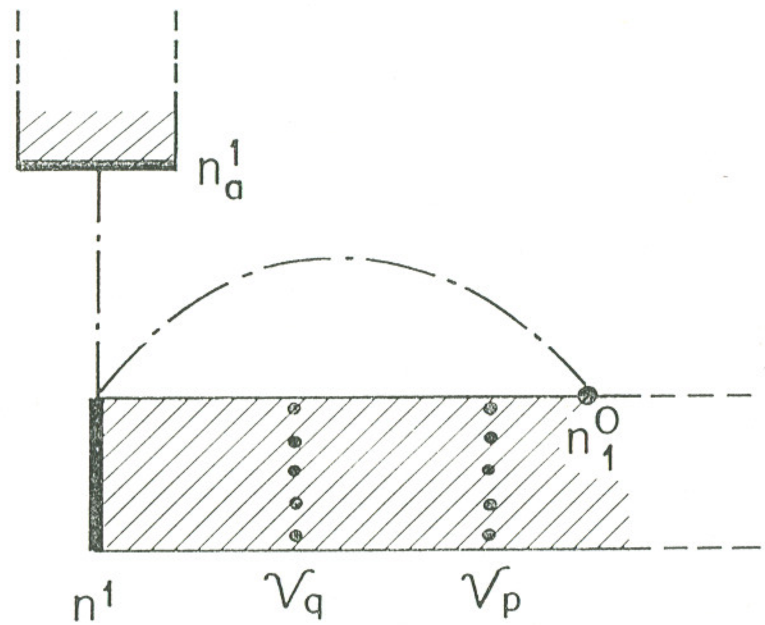

FIG. 3

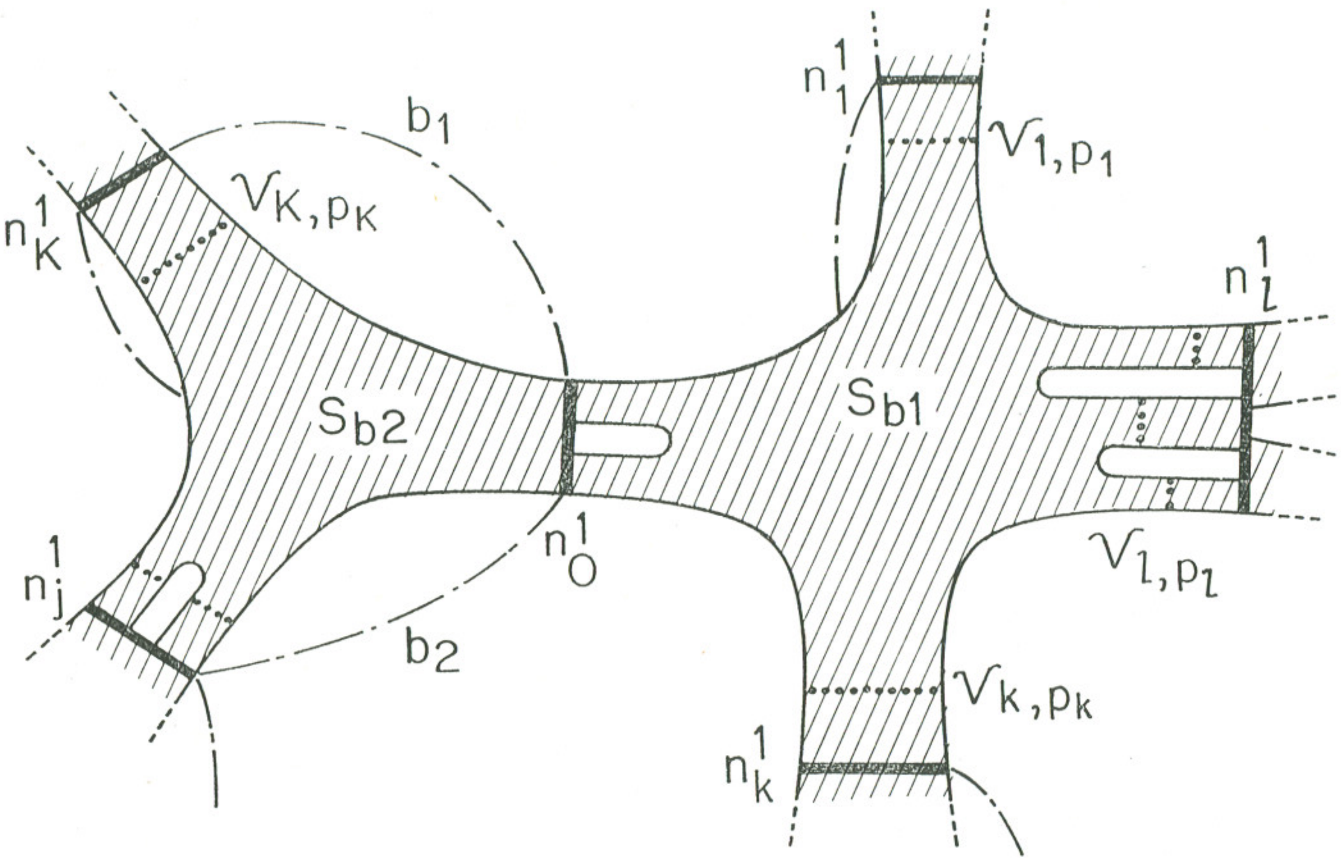

FIG. 4