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GENERALIZATIONS OF KONIG'S LEMMA FOR TRANSFINITE GRAPHS

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Abstract — Konig's lemma asserts that an ordinary infinite graph, which is locally finite and connected, contains a one-ended path. The present work extends that result to transfinite graphs. This requires a decomposition into certain reduced transfinite graphs in order to obtain a generalization of local-finiteness, in addition to transfinite connectedness. The conclusion is that, whatever be the choice of a node of rank ν in a graph of rank ν, there is a one-ended transfinite path of rank ν starting at that node.

1 Introduction

Konig's lemma [1, page 81] states that for each node n_0 in an ordinary, infinite, connected, locally finite graph there is a one-ended path starting at n_0. This is a basic result for infinite graphs and has a variety of ramifications [2], [3]. Our aim is to establish a generalization of Konig's lemma suitable for transfinite graphs. Such graphs were proposed and examined in [4] and [5], the key idea being that nodes of rank 1 or synonymously 1-nodes can be defined, which connect together infinite graphs at their extremities to obtain transfinite graphs of rank 1, i.e., 1-graphs. The latter can be connected together at their infinite extremities by 2-nodes to obtain 2-graphs. This process can be continued recursively to obtain ν-nodes and ν-graphs, where ν is any countable ordinal (finite or transfinite). The question arises as to whether a ν-graph contains a one-ended ν-path, that is, a transfinite path with infinitely many ν-nodes. The latter are needed if the ν-graph is to be connected to another ν-graph through a $(\nu + 1)$-node in order to obtain a $(\nu + 1)$-graph.

In order to obtain the desired extension of Konig's lemma, the idea of local-finiteness has to be extended to transfinite graphs, but one difficulty is that a transfinite node need not have any incident branches. Another complication is that a transfinite node may contain many transfinite nodes of lower ranks. Now, an ordinary graph can be viewed as being partitioned by its branches, which may meet in a locally-finite fashion. It turns out that there are certain subgraphs of a transfinite graph, which we will call "subsections" and which partition the transfinite graph and play a role analogous to that of branches in an ordinary graph. In particular, two ν-nodes will be called "$(\nu-)$-adjacent" if they are both incident to a subsection of rank less than ν. Then, a ν-graph can be viewed as being "locally finite" if every one of its ν-nodes is $(\nu-)$-adjacent to only finitely many ν-nodes. This leads to the conclusion that any ν-connected, "locally finite" ν-graph with infinitely many ν-nodes will have a one-ended ν-path.

We present a detailed proof of this result for the case where ν is any natural number and then indicate what few alterations are needed when ν is the first transfinite ordinal ω. The proofs are exactly the same for still larger countable ordinals; they just require a more complicated notation. An exposition of the theory of transfinite graphs used in this paper is given in [4, Chapters 3 and 5].

2 Sections and Subsections

Henceforth, Greek letters will denote natural numbers except for ω, which will denote the first transfinite ordinal; in addition, the natural number μ will be larger than 0. Furthermore, $\mathcal{G}^\mu$ will denote a μ-graph as defined in [4, Section 5.1]:

$$\mathcal{G}^\mu \;=\; \{\mathcal{B}, \mathcal{N}^0, \ldots, \mathcal{N}^\mu\} \tag{1}$$

Here, $\mathcal{B}$ is a countable set of branches, and, for each $\alpha = 0, \ldots, \mu$, $\mathcal{N}^\alpha$ is the set of α-nodes [4, page 141] in $\mathcal{G}^\mu$. All these sets are nonvoid.

Two nodes n_a and n_b are said to be *shorted* if there is a node that embraces both n_a and n_b; we also say n_a is *shorted to* n_b, and conversely. In this terminology n_a and/or n_b may be replaced by tips.

A *nonmaximal* node n_1 is a node such that its set of embraced tips is a proper subset of the set of embraced tips of another node n_2. Perforce, the rank of n_1 is no larger than the rank of n_2. A *maximal* node is a node that is not nonmaximal. Thus, an ordinary 0-node is a maximal 0-node. With regard to an α-node n_r^α in a reduction $\mathcal{G}_r$ of $\mathcal{G}^\mu$, n_r^α may be maximal with respect to $\mathcal{G}_r$ but not maximal with respect to $\mathcal{G}^\mu$ because $\mathcal{G}^\mu$ may have a node whose embraced tip set is a proper superset of the embraced tip set of n_r^α. Henceforth, when we call any node maximal we will mean that it is maximal with respect to $\mathcal{G}^\mu$ (or $\mathcal{G}^\omega$ for Section 5).

Let P be a path, t a tip, and n a node; their ranks need not be the same. P is said to *traverse* t if P embraces a representative of t. (If t is an elementary tip, its representative is the branch having that tip.) P is said to *meet* n if P traverses a tip such that both t and n are shorted, i.e., are embraced by some node. For example, if P is a one-ended 1-path, it is a representative of a 1-tip t^1. If there is a 2-node n^2 that embraces both t^1 and a 0-node n^0, then P meets both n^0 and n^2, even though it embraces neither n^0 nor n^2. Furthermore, we say that P *meets* a node set or a reduced graph if it meets a node of that node set or of that reduced graph.

Let n be a node of any rank; we do not require that n be maximal. Also, let $\mathcal{G}_r$ be a reduced graph of $\mathcal{G}^\mu$. We say that n and $\mathcal{G}_r$ are *incident* if n is shorted to a tip t having a representative lying entirely within $\mathcal{G}_r$ (i.e., all the branches of that representative belong to $\mathcal{G}_r$). This is equivalent to requiring that $\mathcal{G}_r$ contain a path that meets n. As a special case, when a node n of any rank embraces an elementary tip t_e, we take the branch b for t_e as the (one and only) representative for t_e and obtain the definition of incidence between a node n and a branch b.

Two branches, or two nodes, or a branch and a node are said to be *α-connected* if there is a finite α-path that meets them [4, page 146]. An *α-section* is a reduction [4, pages 142-143] of $\mathcal{G}^\mu$ induced by a maximal set of branches that are pairwise α-connected.

We need the idea of "nondisconnectable tips", which in [4] was only defined for 0-tips. Consider any representative of an α-tip t^α; this is a one-ended α-path P^α, more specifically, a one-way infinite alternating sequence of α-nodes n_i^α and $(\alpha-1)$-paths $P_i^{\alpha-1}$ (where $P_i^{\alpha-1}$ denotes a single branch if $\alpha = 0$):

$$P^\alpha \;=\; \{n_0^\eta, P_0^{\alpha-1}, n_1^\alpha, P_1^{\alpha-1}, n_2^\alpha, P_2^{\alpha-1}, \ldots\} \qquad (2)$$

Here, the first node n_0^η may have any rank η no larger than α. Also, certain additional requirements must be fulfilled if P^α is to be an α-path [4, page 144]. Next, consider an infinite sequence $\{m_1, m_2, m_3, \ldots\}$ of nodes m_l of possibly differing ranks, no two of which are shorted together. We say that the m_l *approach* the α-tip t^α if there exists a representative (2) for t^α such that, for each natural number i, all but finitely many of the m_l are shorted in a one-to-one fashion to nodes embraced by the members of (2) lying to the right of n_i^α.

Now let t_a and t_b be two tips, whose ranks need not be the same. We say that t_a and t_b are *nondisconnectable* if there is an infinite sequence of nodes that approach both t_a and t_b. In effect, t_a and t_b are nondisconnectable if each representative of t_a meets every representative of t_b infinitely often.

Henceforth, the following conditions are imposed upon $\mathcal{G}^\mu$.

Conditions 2.1.

(a) *$\mathcal{G}^\mu$ has no infinite 0-nodes, no self-loops, and no parallel branches.*

(b) *$\mathcal{G}^\mu$ is μ-connected.*

(c) *If two tips (of not necessarily the same rank) are nondisconnectable, then those two tips are shorted (i.e., are embraced by the same node).*

Restriction (c) insures that α-connectedness is a transitive — and thereby an equivalence relationship between branches and that α-sections partition $\mathcal{G}^\mu$ [6, Theorem 6.2 and Corollary 6.4].

Let β be a fixed natural number with $0 < \beta \leq \mu$; β will be so restricted henceforth. As a convenient notation, $\beta-$ and $\beta+$ will denote arbitrary and unspecified natural numbers such that $0 \leq (\beta-) < \beta \leq (\beta+) \leq \mu$. Thus, two $(\beta-)$-nodes need not have the same rank, but in any case their ranks will be less than β. Also, a $(\beta-)$-tip is not an elementary tip [4, page 7] because $(\beta-) \geq 0$.

We now turn to the definition of a "subsection". Subsections provide a finer partitioning of $\mathcal{G}^\mu$; indeed, each α-section is partitioned by certain subsections. We start by partitioning the branch set $\mathcal{B}$ by placing two branches in the same subset if there is a finite $(\beta-)$-path that meets them and does not meet any $(\beta+)$-node. This truly partitions $\mathcal{B}$. Indeed, the property of having such a $(\beta-)$-path between two branches establishes an equivalence relationship "=" within the set of all branches. Reflexivity and symmetry being obvious, consider transitivity. Let b_1, b_2, and b_3 be three branches with b_1"="b_2 and b_2"="b_3. This means that there are two finite $(\beta-)$-paths $P_1^{\beta-}$ and $P_2^{\beta-}$ (not necessarily of the same rank), which do not meet any $(\beta+)$-node and are such that $P_1^{\beta-}$ terminates at a node of b_1 and a node of b_2 and $P_2^{\beta-}$ terminates at a node of b_2 and a node of b_3. By virtue of Condition 2.1(c) and the proof of [6, Theorem 6.2], there is a finite $(\beta-)$-path $P_0^{\beta-}$ contained in $P_1^{\beta-} \cup P_2^{\beta-} \cup \{b_2\}$ that connects b_1 and b_3. Clearly, $P_0^{\beta-}$ also will not meet any $(\beta+)$-node. Whence, the transitivity of "=". Thus, "=" truly partitions $\mathcal{B}$ into subsets $\mathcal{B}_i$; that is, every $\mathcal{B}_i$ is nonvoid, $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ if $i \neq j$, and $\mathcal{B} = \cup \mathcal{B}_i$.

We define a *$(\beta-)$-subsection* $\mathcal{S}_{b\beta-}$ to be the reduction of $\mathcal{G}^\mu$ induced by the branches in any one of the subsets $\mathcal{B}_i$. Remember that $0 < \beta \leq \mu$.We also define a *$(0-)$-subsection* to be any individual branch.

Some immediate implications of these definitions are the following: $\mathcal{G}^\mu$ is partitioned by the $(\beta-)$-subsections. Every $(\beta+)$-section is partitioned by some or all of the $(\beta-)$-subsections. For any natural number γ with $\beta < \gamma \leq \mu$, every $(\gamma-)$-subsection is partitioned by some or all of the $(\beta-)$-subsections. The ranks of the various $(\beta-)$-subsections as transfinite graphs can differ if $\beta > 1$, and those ranks can have any values up to μ. Moreover, given any $(\beta-)$-subsection $\mathcal{S}_{b\beta-}$, there will be some α with $\alpha < \beta$ such that every two nodes of $\mathcal{S}_{b\beta-}$ are α-connected. Furthermore, a path in $\mathcal{G}^\mu$ can pass from one $(\beta-)$-subsection to another $(\beta-)$-subsection only by meeting a $(\beta+)$-node as it makes that transition.

Examples illustrating these ideas are shown in Figures 1 to 3. Figure 1 indicates a 1-graph. Each maximal vertical 0-path — along with the 1-node it meets but with the embraced 0-node deleted — is a $(1-)$-subsection; its rank is 1. There are other $(1-)$-subsections; namely, each horizontal branch along with its reduced 0-nodes is a $(1-)$-subsection of rank 0. Except for n_0^1, every 1-node is incident to two or three $(1-)$-subsections. On the other hand, n_0^1 is a singleton 1-node whose 0-tip has as a representative the 0-path induced by all the horizontal branches; n_0^1 is not incident to any $(1-)$-subsection. Note that no representative of the 1-tip embraced by n_0^1 is contained in any single $(1-)$-

subsection.

Figure 2 shows a 2-graph that has only one $(2-)$-subsection and only one $(1-)$-subsection, both identical to the 2-graph itself. This situation arises because the 2-graph is branchwise 0-connected. All the 1-nodes and the 2-node as well are incident to both subsections.

The 2-graph of Figure 3(a) has an infinity of $(1-)$-subsections, each one consisting of an endless 0-path or a single horizontal branch. Each $(1-)$-subsection has two incident maximal 1-nodes. The two 2-nodes and their embraced nonmaximal 1-nodes are not incident to any $(1-)$-subsection. All the nodes are incident to the single $(2-)$-subsection, which is the 2-graph itself. This 2-graph is branchwise 1-connected. Note that here as well no representative of the single 0-tip embraced by either nonmaximal 1-node is contained in any single $(1-)$-subsection.

In the 2-graph of Figure 3(b), both 2-nodes and their embraced 1-nodes are incident to a $(1-)$-subsection consisting of the horizontal and vertical branches incident to the 0-nodes in the lower endless 0-path. The other $(1-)$-subsections are the upper endless 0-paths, infinitely many in number. Here too, the $(2-)$-subsection is the entire 2-graph.

Two of these examples show that there may be β-nodes that are not incident to any $(\beta-)$-subsections. This occurs when a β-node embraces a $(\beta-1)$-tip, none of whose representatives reside in a single $(\beta-)$-subsection. Such is the case for the 1-node n_0^1 in Figure 1 and for the two nonmaximal 1-nodes in Figure 3(a). We wish to avoid this situation for certain nodes and will do so by requiring that certain nodes be incident to certain subsections, where "incidence" bears the precise meaning defined above (see Condition 4.1 below).

3 β-Adjacency

As before, we require that $0 \leq (\beta-) < \beta \leq (\beta+) \leq \mu$. Two nodes n_1 and n_2 (of possibly differing ranks) are said to be *β-adjacent* if they are not shorted and if they are both incident to the same $(\beta-)$-subsection. Furthermore, two nodes are called *0-adjacent* if they are incident to the same branch. (Were we to view a single branch as a (-1)-subsection and were to allow $\beta = 0$, then the second definition would be a special case of the first one.)

We will need another lemma provided by some results of [6]. Let P^ρ and Q^ς be two finite oriented paths; their ranks need not be the same and may be either natural numbers or ω. Let P^ρ and Q^ζ both start at the same 0-node n_b^0 but end at two different 0-nodes, n_a^0 for P^ρ and n_c^0 for Q^ζ. Let $\{n_i\}_{i\in I}$ be the set of maximal nodes met by both P^ρ and Q^ζ, and order $\{n_i\}_{i\in I}$ in accordance with a tracing of P^ρ from n_b^0 to n_a^0. The 0-node n_b^0 is embraced by the first node in $\{n_i\}$.

Lemma 3.1. *There is a last node n_x in the ordered set $\{n_i\}$ that is met by both P^ρ and Q^ζ.*

The proof of this lemma is given in [6, Section 6] and in particular in the proofs of Theorem 6.2 and Corollary 6.3 of that reference.

Lemma 3.2. *Let n_1 and n_2 be two β-adjacent $(\beta+)$-nodes and let $\mathcal{S}_{b\beta-}$ be a $(\beta-)$-subsection to which they are both incident. Assume that $\mathcal{S}_{b\beta-}$ has only finitely many incident $(\beta+)$-nodes. Then, there is a $(\beta-)$-path $P^{\beta-}$ that meets n_1 and n_2, lies within $\mathcal{S}_{b\beta-}$, and does not meet any $(\beta+)$-nodes except for n_1 and n_2.*

Proof. Since n_1 is incident to $\mathcal{S}_{b\beta-}$, it embraces an α-tip t_1^α with a representative lying entirely within $\mathcal{S}_{b\beta-}$. Similarly, n_2 embraces a γ-tip t_2^γ with such a representative. Since $\mathcal{S}_{b\beta-}$ has only finitely many incident $(\beta+)$-nodes, those representatives cannot meet infinitely many $(\beta+)$-nodes and therefore cannot have ranks of β or larger. Hence, $\alpha < \beta$ and $\gamma < \beta$. We can choose a representative P_1^α for t_1^α and a representative P_2^γ for t_2^γ such that they are totally disjoint, do not meet any $(\beta+)$-nodes other than n_1 for P_1^α and n_2 for P_2^γ, and terminate at 0-nodes n_b^0 for P_1^α and n_c^0 for P_2^γ within $\mathcal{S}_{b\beta-}$ (see Figure 4). Hence, there is a finite $(\beta-)$-path $P_{bc}^{\beta-}$ in $\mathcal{S}_{b\beta-}$ that terminates at n_b^0 and n_c^0 and does not meet any $(\beta+)$-node. Furthermore, there is a 0-node n_a^0 in P_1^α such that the path in P_1^α from n_1 to n_a^0 is totally disjoint from $P_{bc}^{\beta-}$, for otherwise $P_{bc}^{\beta-}$ would traverse a tip that is nondisconnectable from n_1 [6, Lemma 4.2] and would therefore meet n_1 according to Condition 2.1(c). We can now invoke Lemma 3.1: Upon tracing P_1^α from n_b^0 to n_a^0, we will find a last maximal node n_x that is also embraced by $P_{bc}^{\beta-}$.

The same argument yields the following: Upon tracing P_2^γ from n_c^0 to n_2, we will find a last maximal node n_y that is also embraced by $P_{bc}^{\beta-}$. Now the part P_{1x}^α from n_1 to n_x does not meet any maximal node of $P_{bc}^{\beta-}$ other than n_x, and similarly the part P_{y2}^γ of P_2^γ from

n_y to n_2 also does not meet any maximal node of $P_{bc}^{\beta-}$ other than n_y. So, the part $P_{xy}^{\beta-}$ of $P_{bc}^{\beta-}$ between n_x and n_y is totally disjoint from P_{1x}^{α} and P_{y2}^{γ} except terminally. In fact, $P_{1x}^{\alpha} \cup P_{xy}^{\beta-} \cup P_{y2}^{\gamma}$ is a $(\beta-)$-path, and is the path $P^{\beta-}$ we seek. ♣

4 Extension of Konig's Lemma to μ-Graphs

We will first obtain a generalization of Konig's lemma for the μ-graph $\mathcal{G}^{\mu}$ and subsequently will indicate how it can be extended to any reduced graph of $\mathcal{G}^{\mu}$. We assume throughout that $\mathcal{G}^{\mu}$ satisfies Conditions 2.1, that every $(\mu-)$-subsection has only finitely many incident μ-nodes, and that the following is fulfilled as well:

Condition 4.1. *Every $(\mu-)$-tip of every μ-node has a representative lying in a single $(\mu-)$-subsection.*

This means that each μ-node is incident to a subsection through every one of its $(\mu-)$-tips (but those subsections may be different for different $(\mu-)$-tips of the μ-node). Here again, the rank of the subsection need not be the same as that of the corresponding tip but it cannot be any less than the rank of the tip.

We define a metric $d_{\mu}(\cdot,\cdot)$ for the set of μ-nodes of $\mathcal{G}^{\mu}$ as follows: $d_{\mu}(n^{\mu}, n^{\mu}) = 0$ for every μ-node n^{μ}; for any two distinct μ-nodes n_a^{μ} and n_b^{μ}, $d_{\mu}(n_a^{\mu}, n_b^{\mu}) = m$ if there exists a μ-path P^{μ} terminating at n_a^{μ} and n_b^{μ} with exactly $m+1$ μ-nodes (counting n_a^{μ} and n_b^{μ}) and if there does not exist any such μ-path with fewer μ-nodes. Since $\mathcal{G}^{\mu}$ is μ-connected, there is such an m for each choice of n_a^{μ} and n_b^{μ}. We call $d_{\mu}(n_a^{\mu}, n_b^{\mu})$ the *μ-distance between n_a^{μ} and n_b^{μ}*. If n_a^{μ} and n_b^{μ} are μ-adjacent, then $d_{\mu}(n_a^{\mu}, n_b^{\mu}) = 1$; indeed, by virtue of Lemma 3.2 and [4, Lemma 5.1-6], there is a μ-path terminating at n_a^{μ} and n_b^{μ} and having no other μ-nodes.

$d_{\mu}(n_a^{\mu}, n_b^{\mu})$ satisfies the metric axioms. The only axiom that is not obviously satisfied is the triangle inequality. To verify that one, let P_{ab}^{μ} (or P_{bc}^{μ}) be a μ-path that terminates at n_a^{μ} and n_b^{μ} (respectively, at n_b^{μ} and n_c^{μ}). Then, $P_{ab}^{\mu} \cup P_{bc}^{\mu}$ is a tracing through $\mathcal{G}^{\mu}$ that terminates at n_a^{μ} and n_c^{μ}. If P_{ab}^{μ} and P_{bc}^{μ} embrace the same μ-node n_d^{μ}, then a part of that tracing that starts and ends at n_d^{μ} can be removed to get a shorter tracing from n_a^{μ} to n_c^{μ}. A finite number of such removals will yield a tracing from n_a^{μ} to n_c^{μ} in which no μ-node repeats. Furthermore, if in the last tracing two μ-nodes are μ-adjacent, then by Lemma 3.2 the tracing between those two μ-nodes can be replaced by a μ-path that terminates at them and passes through exactly one $(\mu-)$-subsection without meeting any other μ-node and therefore without meeting any other node incident to another $(\mu-)$-subsection. A finite number of replacements of the latter kind finally yields a tracing in which no $(\mu-)$-subsection is traversed more than once. Hence, the last tracing is a μ-path Q^μ terminating at n_a^μ and n_b^μ. The number of μ-nodes in Q^μ will be no larger than the sum of those numbers for P_{ab}^μ and P_{bc}^μ. We can conclude that $d_\mu(\cdot,\cdot)$ satisfies the triangle inequality.

Here is an extension of Konig's lemma (promoted to the rank of "proposition") for μ-graphs:

Proposition 4.2. *Assume Conditions 2.1 and 4.1. Let the μ-graph $\mathcal{G}^\mu$ $(\mu \geq 1)$ be such that the following hold.*

(i) *Every μ-node is μ-adjacent to only finitely many μ-nodes.*

(ii) *There are infinitely many μ-nodes.*

Then, for each μ-node n_0^μ there is at least one one-ended μ-path starting at n_0^μ.

Proof. Corresponding to $\mathcal{G}^\mu$, we set up a *surrogate* 0-graph $\mathcal{G}_s^0$ by setting up one and only one 0-node s_a^0 in $\mathcal{G}_s^0$ for each μ-node n_a^μ in $\mathcal{G}^\mu$ and inserting branches as follows: Arbitrarily choose but then fix a μ-node n_0^μ and let s_0^0 be its corresponding 0-node. In the following we shall say that another μ-node n_a^μ of $\mathcal{G}^\mu$ is *at the distance* m ("from n_0^μ" being understood) when $d_\mu(n_0^\mu, n_a^\mu) = m$. Also, the correspondence $n_a^\mu \mapsto s_a^0$ is designated by identical subscripts. Insert a branch from s_0^0 to a 0-node s_a^0 whenever n_a^μ is adjacent to n_0^μ (i.e., is at the distance 1). Continue recursively: If n_b^μ is at the distance m, insert a branch between s_b^0 and every s_c^0 for which n_c^μ is μ-adjacent to n_b^μ and is at the distance $m+1$ — if any such s_c^μ exists. Because of hypotheses (i) and (ii) and the μ-connectedness of $\mathcal{G}^\mu$ this process will never cease and every 0-node of $\mathcal{G}_s^0$ will be incident to a branch. In fact, $\mathcal{G}_s^0$ will be 0-connected, locally finite, and infinite. Consequently, we may invoke Konig's lemma to conclude that $\mathcal{G}_s^0$ contains a one-ended 0-path P^0 starting at s_0^0.

Note now that, with $d_0(\cdot,\cdot)$ denoting the distance function for a 0-graph, we have $d_\mu(n_0^\mu, n_a^\mu) = d_0(s_0^0, s_a^0)$. Trace along P^0 starting from s_0^0. There will be a last node s_1^0

in P^0 that is adjacent to s_0^0. Let b_0 be the branch in $\mathcal{G}_s^0$ between s_0^0 and s_1^0. In the subpath of P^0 beyond s_1^0 there will be a last node s_2^0 adjacent to s_1^0 and at the distance 2. Let b_1 be the branch in $\mathcal{G}_s^0$ between s_1^0 and s_2^0. Continue recursively: Let the branches $b_0, b_1, \ldots, b_{m-1}$ be chosen with s_m^0 being the node incident to b_{m-1} and at the distance m. In the subpath of P^0 beyond s_m^0 there will be a last node s_{m+1}^0 adjacent to s_m^0 and at the distance of $m+1$. Let b_m be the branch in $\mathcal{G}_s^0$ between s_m^0 and s_{m+1}^0. We obtain in this way infinitely many branches $b_0, b_1, \ldots$, which induce a one-ended 0-path Q^0 in $\mathcal{G}_s^0$. Moreover, with respect to a tracing of Q^0 starting at s_0^0, the consecutive 0-nodes of Q^0 will be at strictly increasing distances.

By hypothesis (i), every $(\mu-)$-subsection has only finitely many incident μ-nodes. Hence, we may now invoke Lemma 3.2. For each branch b_m with incident nodes s_m^0 and s_{m+1}^0 in $\mathcal{G}_s^0$, there is a $(\mu-)$-path $P_{m,m+1}^{\mu-}$ that resides in a single $(\mu-)$-subsection of $\mathcal{G}^\mu$ and meets n_m^μ and n_{m+1}^μ. Moreover, for $m \neq i$, $P_{m,m+1}^{\mu-}$ and $P_{i,i+1}^{\mu-}$ will be totally disjoint — except terminally when $i = m+1$ — because they will reside in different $(\mu-)$-subsections. Furthermore, $P_{m,m+1}^{\mu-}$ and $P_{m+1,m+2}^{\mu-}$ will both meet n_{m+1}^μ. Upon replacing s_m^0 by n_m^μ and b_m by $P_{m,m+1}^{\mu-}$ for every m, we convert Q^0 into a one-ended μ-path in $\mathcal{G}^\mu$ starting at n_0^μ. ♣

Proposition 4.2 extends directly to reduced graphs of $\mathcal{G}^\mu$ such as $(\beta-)$-subsections because any reduced graph is an α-graph $\mathcal{G}_r^\alpha$ $(\alpha \leq \mu)$ by itself. We need merely replace μ by α and $\mathcal{G}^\mu$ by $\mathcal{G}_r^\alpha$ in Conditions 2.1 and 4.1 and in Proposition 4.2 in order to get the appropriate statements.

5 Extension of Konig's Lemma to ω-Graphs

Our arguments extend directly to ω-graphs $\mathcal{G}^\omega$. Such graphs are defined in [4, Section 5.2]. So too are $\vec{\omega}$-tips, ω-nodes, ω-connectedness, and ω-sections. The nondisconnectability of two $\vec{\omega}$-tips or of an $\vec{\omega}$-tip and a μ-tip is defined exactly as above after the idea of a sequence $\{m_1, m_2, m_3, \ldots\}$ of nodes approaching an $\vec{\omega}$-tip is defined. The latter is done by replacing (2) by an $\vec{\omega}$-path [4, Equation (5.5)]. (See also [6, Section 2].) Conditions 2.1 are now replaced by

Conditions 5.1.

(a) *$\mathcal{G}^\omega$ has no infinite 0-nodes, no self-loops, and no parallel branches.*

(b) *$\mathcal{G}^\omega$ is ω-connected.*

(c) *Let t_1 and t_2 be two tips whose ranks may differ but are both no larger than $\vec{\omega}$. If t_1 and t_2 are nondisconnectable, then t_1 and t_2 are shorted.*

Our definition of incidence requires no changes so far as its wording is concerned, but now the tip t may have the rank $\vec{\omega}$. As for the definitions of an $(\omega-)$-subsection and of ω-adjacency , replace β by ω in our prior definitions of a $(\beta-)$-subsection and β-adjacency. Thus, $\beta+$ is now ω, and $\beta-$ is now $\omega-$; $\omega-$ denotes an arbitrary and unspecified natural number or possibly $\vec{\omega}$. As before, two $(\beta-)$-entities may have different ranks. Lemma 3.1 has already been stated in a fashion suitable for ω-graphs. On the other hand, Lemma 3.2 is replaced by

Lemma 5.2. *Let n_1 and n_2 be two ω-adjacent ω-nodes and let $\mathcal{S}_{b\omega-}$ be an $(\omega-)$-subsection to which those two nodes are both incident. Assume that $\mathcal{S}_{b\omega-}$ has only finitely many incident ω-nodes. Then, there is an $(\omega-)$-path $P^{\omega-}$ that meets n_1 and n_2, lies within $\mathcal{S}_{b\omega-}$, and does not meet any ω-node except for n_1 and n_2.*

The definition of the metric $d_\omega(\cdot,\cdot)$ for the set of ω-nodes in $\mathcal{G}^\omega$ reads exactly as does that for $d_\mu(\cdot,\cdot)$ but with μ replaced by ω. Condition 4.1 is replaced by

Condition 5.3. *Every $(\omega-)$-tip of every ω-node has a representative lying in a single $(\omega-)$-subsection.*

Finally, our extension of Konig's lemma to ω-graphs is the following:

Proposition 5.4. *Assume Conditions 5.1 and 5.3. Let the ω-graph $\mathcal{G}^\omega$ be such that the following hold.*

(i) *Every ω-node is ω-adjacent to only finitely many ω-nodes.*

(ii) *There are infinitely many ω-nodes.*

Then, for each ω-node n_0^ω there is at least one one-ended ω-path starting at n_0^ω.

The proof of this proposition is the same as that of Proposition 4.2 but with the aforementioned changes in notation and replacements of lemmas and conditions. Also, replace [6, Lemma 4.2] by [6, Lemma 4.3].

6 A Final Note

Propositions 4.2 and 5.4 can be extended recursively to any ν-graph whose rank ν is any countable ordinal. The arguments for a successor ordinal ν or a limit ordinal ν hold exactly as they do for a natural number μ or respectively for ω.

References

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Figure Legends

Figure 1. A 1-graph in which the 1-node n_0^1 is not incident to any $(1-)$-subsection. The heavy dots are 0-nodes, the small circles are 1-nodes, and the lines between the 0-nodes are branches. Note also that this graph does not contain a one-ended 1-path even though it has an infinity of 1-nodes.

Figure 2. A 2-graph. The symbolism is the same as that of Figure 1 except for the indicated 2-node. The upper line of branches induce a one-ended 1-path with the 1-nodes $n_1^1, n_2^1, n_3^1, \ldots$; its 1-tip is embraced by the 2-node n_0^2. There are other 1-tips, and they too are taken to be embraced by n_0^2. All the 0-tips on the extreme right are embraced by the nonmaximal 1-node n_0^1, which in turn is embraced by n_0^2. The branches connecting the upper 0-nodes to the lower 0-nodes induce a bijection between those two sets of 0-nodes.

This 2-graph has exactly one $(1-)$-subsection, namely, itself because the entire 2-graph is 0-connected. Similarly, its $(2-)$-subsection is also the 2-graph itself. Thus, the ranks of both subsections are equal to 2.

Figure 3. Two 2-graphs. The symbolism is the same as before. The 0-paths within the endless 1-paths are endless.

(a) Neither the nonmaximal 1-nodes nor the maximal 2-nodes are incident to any $(1-)$-subsection, that is, to any 0-subsection. All the 1-nodes and both of the 2-nodes are incident to the one and only $(2-)$-subsection, namely, the entire 2-graph.

(b) All of the 1-nodes and both of the 2-nodes are incident to the $(1-)$-subsection induced by the horizontal and vertical branches incident to the lower 0-nodes. The other $(1-)$-subsections are the upper endless 0-paths along with their incident reduced 1-nodes. There is only one $(2-)$-subsection, the entire 2-graph itself.

Figure 4. Illustration for the proof of Lemma 3.2.

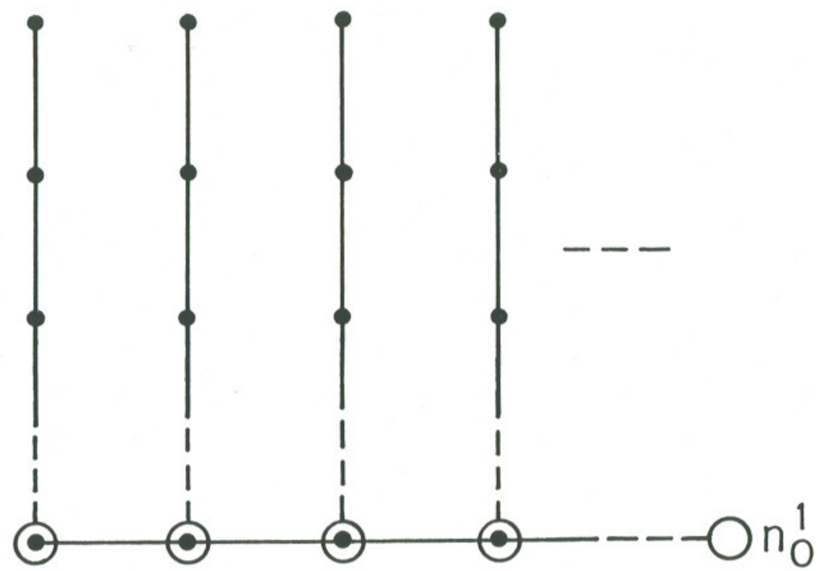


FIG. 1

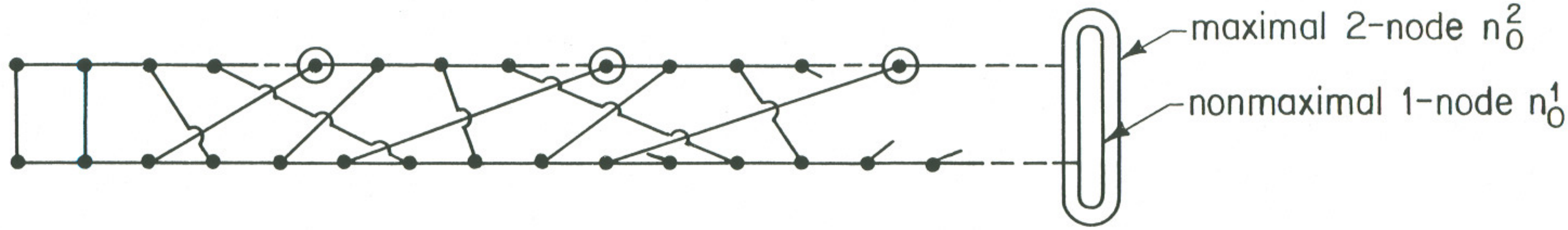
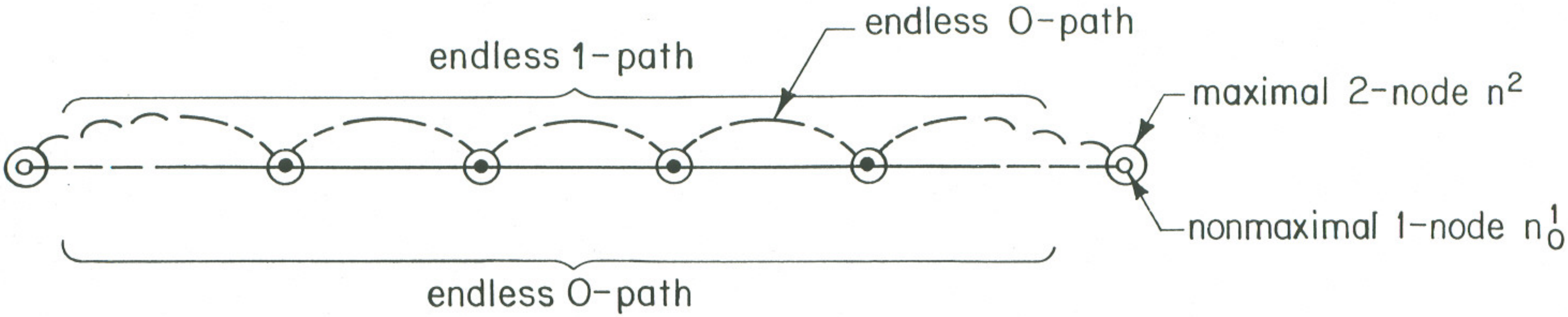
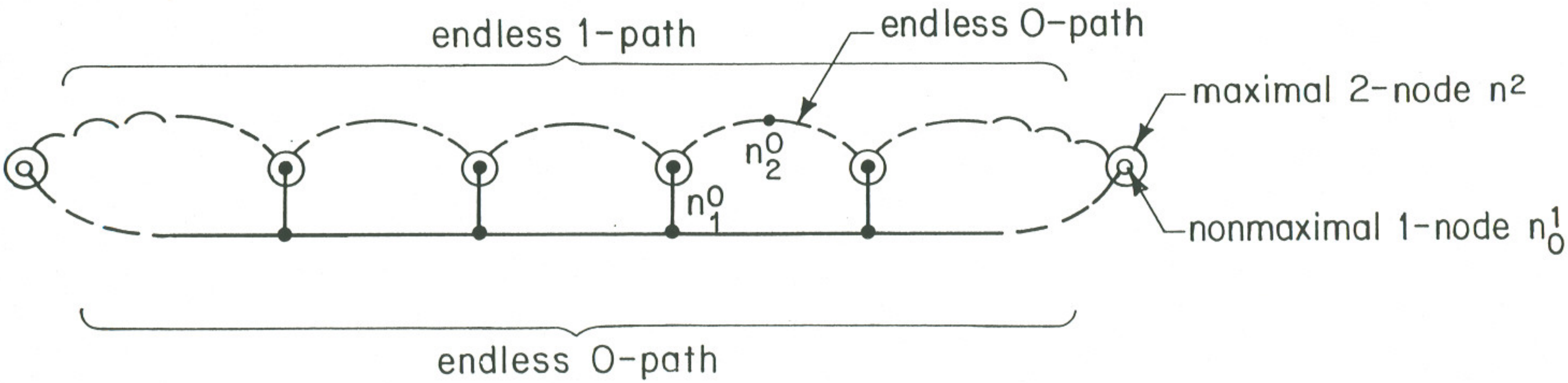


FIG. 2


endless 1-path
endless 0-path
maximal 2-node n2
nonmaximal 1-node n10
endless 0-path

(a)


endless 1-path
endless 0-path
maximal 2-node n2
n02
n01
nonmaximal 1-node n10
endless 0-path

(b)

FIG. 3

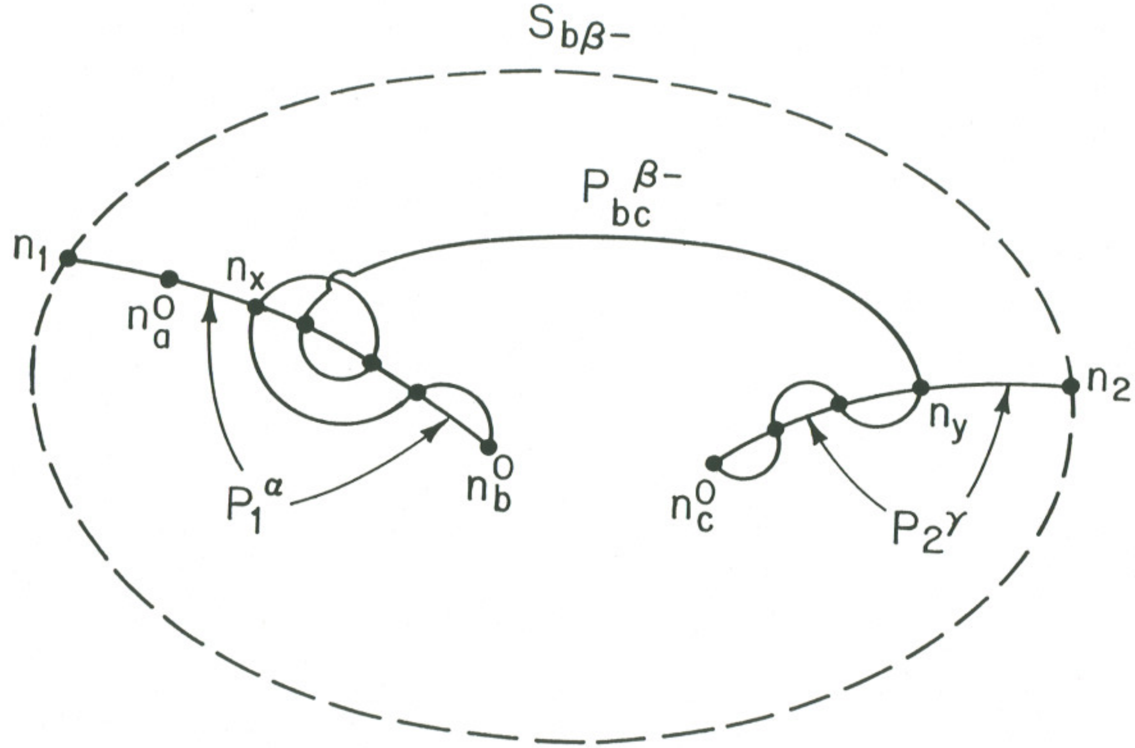


FIG. 4