UNIVERSITY AT STONY BROOK

CEAS Technical Report 668

RANDOM WALKS ON FINITELY STRUCTURED TRANSFINITE NETWORKS: PART II

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This work was supported by the National Science Foundation under Grants DMS-9200738 and MIP-9200748.

June 1, 1993

Preface

Let us comment on how the present report relates to previous studies of random walks on transfinite networks. This work is a generalization of a prior one [10] that was restricted to 1-networks. In this one, a general theory for random walks on μ-networks, where μ is any natural number, is achieved. Both [10] and this work generalize still earlier efforts [7], [8], in which the shorting of nodes of various ranks was not permitted; [10] and the present work allow such shortings, but this considerably complicates the theory. For instance, the "chainlike structure" used in [7] and [8] can no longer be employed; instead, the more general idea of a "finitely structured" transfinite network is introduced.

This report is written to make it independent of [10]. Moreover, it subsumes all the results of [10]. In addition, the five different rules that were used in [10] for the relative transition probabilities of transfinite random walks have now been simplified and consolidated into one rule given by Rule 10.1. This achievement requires a fairly long argument.

Table of Contents

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A.H.Zemanian

Abstract — A general theory for random walks on transfinite networks is established herein. In such networks, nodes of higher ranks connect together transfinite networks of lower ranks. The probabilities for transitions through such nodes are obtained as extensions of the Nash-Williams rule for random walks on ordinary infinite networks. The analysis is based on the theory of transfinite electrical networks, but it requires that the transfinite network have a structure that generalizes local-finiteness for ordinary infinite networks. The shorting together of nodes of different ranks are allowed; this complicates transitions through such nodes but provides a considerably more general theory. Other generalizations achieved herein are Kirchhoff's current law for nodes of any ranks, connections of pure voltage sources to such nodes, and the maximum principle for their node voltages. Finally, it is shown that, with respect to any finite set of nodes of any ranks, a transfinite random walk can be represented by an irreducible reversible Markov chain, whose state space is that set of nodes.

1 Introduction

This work presents a third version of a theory for random walks on a transfinite network. The first version [7], [8] did not allow the network to have embraced nodes, that is, two nodes of different ranks had to be totally disjoint. This is a severe restriction. The second version [10] removed it but on the other hand did not allow networks of ranks greater than 1. The present version removes both restrictions. This yields a far more general but considerably more complicated theory.

*This work was supported by the National Science Foundation under the Grants DMS-9200738 and MIP-9200748. The author is with the University at Stony Brook, Stony Brook, N.Y. 11794-2350.

However, our theory still does not encompass arbitrary transfinite networks. A structure has to be imposed that generalizes the idea of local-finiteness in ordinary networks. That structure is developed in Sections 3 to 6 and enables a generalized version of Kirchhoff's current law at nodes of higher ranks, as well as the excitation of the transfinite network by pure voltage sources appended to such nodes, a matter discussed in Section 7. These results are needed because our theory uses the nearest-neighbor and Nash-Williams rules [3] for random walks on ordinary infinite networks as paradigms for our theory of random walks on transfinite networks. Also needed is a maximum principle for node voltages at nodes of arbitrary ranks; this is established in Section 8. Deterministic and random transfinite walks are defined in Sections 9 and 10 respectively. The rest of this work (Sections 11 through 16) recursively develops the theory for transfinite random walks through an inductive argument that first shows how a random walker can reach a node of higher rank, then how it can leave that node, and finally how it can wander among such nodes. These processes are based upon definitions that continuously extend the nearest-neighbor and Nash-Williams rules. In prior works several definitions were employed for this purpose. Those definitions are herein consolidated into a single and, as it turns out, simpler definition (see Rule 10.1 in Section 10), but now more has to be proven to achieve this end. Our final result relates random walks on transfinite networks to random walks on ordinary networks in the following way. Given any arbitrarily chosen finite set of nodes in the transfinite network, the probabilities of transitions among those nodes are describable by an irreducible and reversible Markov chain, which in turn is representable by a finite network. Thus, the wanderings of a random walker among the chosen nodes of the transfinite network is mimicked by a random walk in the finite network.

2 Subsections and Cores

The idea of a transfinite graph was introduced in [5] and [6]. We freely use the definitions and results appearing therein. We also use some results appearing in subsequent works. For ready reference we summarize the latter with a word or two of explanation.

Lower Greek letters will denote natural numbers and μ will denote a positive natural number. $\mathcal{G}^\mu$ represents a μ-graph [5, Section 5.1]:

$$\mathcal{G}^\mu = \{\mathcal{B}, \mathcal{N}^0, \ldots, \mathcal{N}^\mu\}$$

where $\mathcal{B}$ is a countable set of branches and, for each $\theta = 0, \ldots, \mu$, $\mathcal{N}^\theta$ is a nonvoid set of θ-nodes. θ-nodes and $(\theta - 1)$-tips are defined in [5, pages 67-68 and 140-141]. The rank of an elementary tip is denoted either by -1 or by $\vec{0}$. A *reduction* of $\mathcal{G}^\mu$ is defined in [5, pages 71 and 143]; it is analogous to a subgraph. Borrowing the terminology of electrical circuits, we say that a θ-node n^θ *is shorted to* a ξ-node n^ξ or that n^θ and n^ξ are *shorted* if they are both embraced by the same node. (How nodes embrace is defined in [5, pages 69 and 141].) Here θ and ξ may be equal or unequal. The same terminology is used for two tips or for a node and a tip.

A node n is called *maximal* if there is no other node that embraces all the tips that n embraces. Thus, a *nonmaximal* node is a node n_1 whose embraced tips comprise a proper subset of the set of embraced tips of another node n_2. Hence, the rank of n_1 is no larger than the rank of n_2. Every node is embraced by a maximal node. A maximal 0-node is called an *ordinary* 0-node. A node may be maximal with respect to a reduction of $\mathcal{G}^\mu$ but not maximal with respect to $\mathcal{G}^\mu$. Henceforth, the maximality or nonmaximality of a node is understood to be with respect to $\mathcal{G}^\mu$ — unless some reduction of $\mathcal{G}^\mu$ is specified.

Let P be a path, t a tip, and n a node, whose ranks may be different. P *traverses* t if P embraces a representative of t when t is a nonelementary tip or if P embraces the branch for t when t is an elementary tip. P is said to *meet* n if P traverses a tip that is shorted to n. Similarly, P is said to *meet* a set of nodes or a reduced graph if P meets a node of that set or graph. At times, we use *reach* as a synonym for "meet."

A node n and a reduced graph $\mathcal{G}_r$ are said to be *incident* if $\mathcal{G}_r$ embraces a path that meets n. A θ-path is called *finite* if it has only finitely many θ-nodes. Two branches or two nodes or a branch and a node are called *θ-connected* if there is a finite θ-path that meets them. $\mathcal{G}^\mu$ is called *θ-connected* if every two of its branches are θ-connected. A *θ-section* is a reduction of $\mathcal{G}^\mu$ induced by a maximal set of branches that are pairwise θ-connected. Thus, a θ-section is also a λ-section for every $\lambda > \theta$.

It is important to remember that a sequence is a countable, totally ordered set that can be bijectively indexed by a set of integers whereby the ordering of the set and the natural ordering of the integers agree.

An infinite sequence $\{m_1, m_2, \ldots\}$ of nodes m_l is said to *approach* a tip t^θ if there is a representative

$$P^\theta = \{n_0^\theta, P_0^{\theta-1}, n_1^\theta, P_1^{\theta-1}, n_2^\theta, P_2^{\theta-1}, \ldots\}$$

for t^θ such that, for each natural number i, all but finitely many of the m_l are shorted in a one-to-one fashion to nodes embraced by P^θ lying to the right of n_i^θ. Let t_a and t_b be two tips (their ranks need not be the same). We say that t_a and t_b are *nondisconnectable* if there is an infinite sequence of nodes that approach both t_a and t_b.

Henceforth, we let β denote a natural number such that $0 < \beta \leq \mu$. Also, $\beta-$ and $\beta+$ will denote arbitrary and unspecified natural numbers such that $0 \leq (\beta-) < \beta \leq (\beta+) \leq \mu$. Two $(\beta-)$-nodes need not have the same rank, but their ranks will be less than β; similarly, two $(\beta+)$-nodes need not have the same ranks, but those ranks will be no less than β.

We also need the idea of a subsection. Subsections partition sections as well as $\mathcal{G}^\mu$ itself. To define it, choose and fix some β and then partition the branch set $\mathcal{B}$ by placing two branches in the same subset if they are $(\beta-)$-connected by a finite $(\beta-)$-path that does not meet any $(\beta+)$-node. As is shown in [11, Section 2], this is an equivalence relationship under Conditions 2.1 given below. A *$(\beta-)$-subsection* $\mathcal{S}_b^{\beta-}$ is defined as the reduction of $\mathcal{G}^\mu$ induced by the branches in one of the partitioning subsets. On the other hand, a $(0-)$-*subsection* is defined to be single branch. Let n be a node incident to $\mathcal{S}_b^{\beta-}$; n is called a *bordering* node of $\mathcal{S}_b^{\beta-}$ if n is embraced by a $(\beta+)$-node and is called an *internal* node of $\mathcal{S}_b^{\beta-}$ otherwise.

Immediate consequences of this definition of a $(\beta-)$-subsection are the following. The maximum rank α among the ranks of all the internal nodes of $\mathcal{S}_b^{\beta-}$ satisfies $0 \leq \alpha < \beta$, and the maximum rank of all the traversed tips of $\mathcal{S}_b^{\beta-}$ is either $\alpha - 1$ or α. (The rank -1 represents the rank $\vec{0}$ of an elementary tip.) A maximal bordering node $n^{\beta+}$ of $\mathcal{S}_b^{\beta-}$ is of some rank γ, where $\gamma \geq \beta$, and $n^{\beta+}$ must embrace at least one node of rank δ, where $\delta \leq \beta$, because this is the only way $n^{\beta+}$ can be incident to $\mathcal{S}_b^{\beta-}$. Thus, if n is a maximal node incident to a $(\beta-)$-subsection $\mathcal{S}_b^{\beta-}$, then n is internal node of $\mathcal{S}_b^{\beta-}$ if its rank is less than β, and n is a bordering node of $\mathcal{S}_b^{\beta-}$ if its rank is no less than β.

We will refer to the maximum rank α among the ranks of all internal nodes of $\mathcal{S}_b^{\beta-}$ as the *essential rank* of $\mathcal{S}_b^{\beta-}$ and to the rank of the reduced graph induced by all the branches of $\mathcal{S}_b^{\beta-}$ as the *induced rank* of $\mathcal{S}_b^{\beta-}$. The essential rank is less than β and no larger than the induced rank. It is possible for the induced rank to be any rank from α to μ. We shall also refer to a $(\beta-)$-subsection $\mathcal{S}_b^{\beta-}$ as an *α-subsection* $\mathcal{S}_b^{\alpha}$ when we wish to display the essential rank α of that subsection.

As a simple example, consider the 1-graph of Figure 1(a). It has only one $(1-)$-subsection, the 1-graph itself. Its essential rank is 0, and its induced rank is 1. Some more examples illustrating the foregoing ideas are given below and still more are given in [11, Section 2].

Throughout this work, we always impose the following conditions on the μ-graph $\mathcal{G}^{\mu}$. Conditions (d), (e), and (f) are understood to hold for every rank $\beta = 1, \ldots, \mu$.

Conditions 2.1.

(a) *$\mathcal{G}^{\mu}$ has no infinite 0-nodes, no self-loops, and no parallel branches. Moreover, every branch is incident to at least one ordinary 0-node.*

(b) *$\mathcal{G}^{\mu}$ is μ-connected.*

(c) *If two tips are nondisconnectable, then those tips are shorted.*

(d) *Every $(\beta-)$-tip of every maximal β-node has a representative lying in a single $(\beta-)$-subsection.*

(e) *Every $(\beta-)$-subsection has only finitely many maximal bordering $(\beta+)$-nodes.*

(f) *Every β-node is incident to only finitely many $(\beta-)$-subsections.*

Regarding Condition 2.1(d), the rank of the subsection may be equal to or higher than the rank of the tip.

Regarding Condition 2.1(a), the requirement of no parallel branches is hardly a restriction even when electrical parameters are assigned to branches because we can always combine parallel branches into a single branch by combining their electrical parameters appropriately. Similarly, any branch can be split into two series-connected branches with appropriately adjusted parameters in order to introduce an incident ordinary 0-node for each branch if need be. (This last condition will be needed when we deal with basic currents in $\mathcal{G}^{\mu}$ [5, condition (iv), page 154].)

We now define the "core" of a $(\beta-)$-subsection $\mathcal{S}_b^{\beta-}$. If $\mathcal{S}_b^{\beta-}$ has exactly one internal maximal node n, the *core* of $\mathcal{S}_b^{\beta-}$ is defined to be n. If $\mathcal{S}_b^{\beta-}$ has two or more internal maximal nodes, the *core* of $\mathcal{S}_b^{\beta-}$ is the reduction of $\mathcal{G}^{\mu}$ induced by all branches in $\mathcal{S}_b^{\beta-}$ that are not incident to $(\beta+)$-nodes.

Lemma 2.2. *Let $\mathcal{S}_b^{\beta-}$ be a $(\beta-)$-subsection. Then, the following hold.*

(i) *$\mathcal{S}_b^{\beta-}$ has at least one bordering node.*

(ii) *If $\mathcal{S}_b^{\beta-}$ has exactly one internal maximal node n, then n is an ordinary 0-node, and $\mathcal{S}_b^{\beta-}$ is a star graph with finitely many branches and with n as its central node.*

(iii) *If $\mathcal{S}_b^{\beta-}$ has two or more internal maximal nodes, then its core has at least one branch and is $(\beta-)$-connected through itself (i.e., through $(\beta-)$-paths that do not meet any bordering nodes).*

(iv) *The core of $\mathcal{S}_b^{\beta-}$ is not void, and $\mathcal{S}_b^{\beta-}$ has at least one internal node.*

(v) *If $\mathcal{S}_b^{\beta-}$ is incident to one of its bordering nodes through a nonelementary tip, then a representative of that tip lies in the core of $\mathcal{S}_b^{\beta-}$ and that core has an infinity of branches and an infinity of internal nodes.*

Proof. (i) If $\beta = \mu$ and if $\mathcal{S}_b^{\mu-}$ coincides with $\mathcal{G}^{\mu}$, then each μ-node is a bordering node of $\mathcal{S}_b^{\mu-}$. Otherwise, there is a branch b_1 in $\mathcal{S}_b^{\beta-}$ and a branch b_2 not in $\mathcal{S}_b^{\beta-}$. By Condition 2.1(b), there is a path that connects b_1 and b_2. That path must meet a bordering $(\beta+)$-node of $\mathcal{S}_b^{\beta-}$.

(ii) By Condition 2.1(a), every branch of $\mathcal{S}_b^{\beta-}$ must be incident to an ordinary 0-node. Since no bordering node can be an ordinary 0-node, n must be a 0-node, and moreover every branch of $\mathcal{S}_b^{\beta-}$ must be incident to n. Since there are no parallel branches and since all 0-nodes are of finite degree (Condition 2.1(a) again), our conclusion follows.

(iii) If a branch is incident to two internal nodes of $\mathcal{S}_b^{\beta-}$, it will lie in the core of $\mathcal{S}_b^{\beta-}$. Now, suppose that the core has no branch. Then, every branch of $\mathcal{S}_b^{\beta-}$ must be incident to a bordering node (as well as to an internal ordinary 0-node). This implies that all branches of $\mathcal{S}_b^{\beta-}$ must be incident to a single internal node of $\mathcal{S}_b^{\beta-}$, for otherwise at least two such branches would not be in the same $(\beta-)$-subsection. Thus, $\mathcal{S}_b^{\beta-}$ has only one internal node, in contradiction to the hypothesis. Consequently, the core of $\mathcal{S}_b^{\beta-}$ has at least one branch.

Furthermore, suppose two branches of the core are not $(\beta-)$-connected through the core. Then, they can be connected only by paths that meet $(\beta+)$-nodes and therefore must lie in different $(\beta-)$-subsections — another contradiction.

(iv) Suppose the core of $\mathcal{S}_b^{\beta-}$ is void. By (ii) and (iii), this is equivalent to supposing that $\mathcal{S}_b^{\beta-}$ has no internal node. Then, every branch of $\mathcal{S}_b^{\beta-}$ must be incident through both of its elementary tips to bordering $(\beta+)$-nodes. This violates the second sentence of Condition 2.1(a).

(v) The rank β is no less than 1. Since $\mathcal{S}_b^{\beta-}$ is incident to a bordering $(\beta+)$-node through a nonelementary tip, that tip will have a representative lying entirely in $\mathcal{S}_b^{\beta-}$ (Condition 2.1(d)). Since $\mathcal{S}_b^{\beta-}$ has only finitely many maximal bordering nodes (Condition 2.1(e)), that representative embraces a representative lying entirely within the core of $\mathcal{S}_b^{\beta-}$. This implies that the core has in infinity of branches and an infinity of maximal internal nodes. ♣

We need a certain property of paths, which is a consequence of Condition 2.1(d). To state it, we should recall some definitions. A path *embraces* a tip if it embraces a node that embraces that tip. A path *traverses* a tip if the path embraces a representative of that tip. A tip that is embraced by a path need not be traversed by that path, and conversely. The *essential rank* of a path is the smallest rank that is larger than the rank of every tip embraced and traversed by the path.

Moreover, if a path of essential rank α is finite, then it embraces every tip it traverses, it traverses at least one and at most finitely many $(\alpha-1)$-tips. and all the other tips it traverses are of ranks lower than $\alpha - 1$. In general, the α-nodes that embrace those $(\alpha-1)$-tips need not be maximal nodes of $\mathcal{G}^\mu$.

Lemma 2.3. *Let P^α be a finite path of essential rank α. Let θ be the largest rank* *among all the ranks of the maximal nodes that P^α meets. (Thus, $\alpha \le \theta$.) Then, P^α meets only finitely many maximal θ-nodes.*

Proof. The conclusion is obvious if $\alpha = 0$. So, let $\alpha > 0$. Suppose P^α meets an infinity of maximal θ-nodes. Then, exactly two cases arise. Either P^α traverses an $(\alpha - 1)$-tip none of whose representatives lie in a single $(\theta-)$-subsection, or every one of the finitely many $(\alpha - 1)$-tips traversed by P^α has a representative lying in a single $(\theta-)$-subsection. In the first case, Condition 2.1(d) is violated. In the second case, we can remove those finitely many representatives to find a finite path Q^λ of essential rank λ ($\lambda < \alpha$) such that Q^λ is embraced by P^α and Q^λ meets an infinity of maximal θ-nodes. We can repeat this argument for Q^λ — and continue repeating it if need be to find a sequence of paths whose essential ranks decrease and which possess the same properties as Q^λ. The process must stop before the decreasing sequence of ranks reaches 0, for no finite 0-path can meet an infinity of θ-nodes. Thus, we will eventually find a tip traversed by P^α, none of whose representatives lie in a single $(\theta-)$-subsection — in violation of Condition 2.1(d). ♣

3 Isolating Sets and Cuts

Let $\mathcal{G}_r$ be any reduced graph of $\mathcal{G}^\mu$, and let $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$ be three sets of nodes in $\mathcal{G}_r$. The nodes of these sets need not be maximal, and their ranks need not be the same — even within a single set. $\mathcal{N}_3$ is said to *separate* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *in* $\mathcal{G}_r$ if every path in $\mathcal{G}_r$ that meets $\mathcal{N}_1$ and $\mathcal{N}_2$ also meets a node of $\mathcal{N}_3$. This definition allows nodes of $\mathcal{N}_3$ to embrace nodes of $\mathcal{N}_1$ and/or $\mathcal{N}_2$, and conversely. For instance, if n_a^α is embraced by n_b^β, then n_a^α separates n_b^β from all other nodes, and n_b^β does the same for n_a^α. Similarly, two reductions of $\mathcal{G}_r$ are said to be *separated by* $\mathcal{N}_3$ *in* $\mathcal{G}_r$ if $\mathcal{N}_3$ separates their node sets. Finally, $\mathcal{N}_3$ will *separate* a reduction of $\mathcal{G}_r$ from a node n_0 of $\mathcal{G}_r$ if it separates the node set of the reduction from $\{n_0\}$.

In the following, β denotes a given natural number with $0 < \beta \le \mu$, as before. Also, $n^{\beta+}$ will be a maximal node whose rank is no less than β, and $\mathcal{S}_b^{\beta-}$ will be a $(\beta-)$-subsection incident to $n^{\beta+}$. Thus, $n^{\beta+}$ is a bordering node of $\mathcal{S}_b^{\beta-}$.

If $\mathcal{S}_b^{\beta-}$ is incident to $n^{\beta+}$ through one or more nonelementary tips (that is, if the core of $\mathcal{S}_b^{\beta-}$ is incident to $n^{\beta+}$), we let $\mathcal{V}$ denote a nonvoid finite set of ordinary 0-nodes in the core of $\mathcal{S}_b^{\beta-}$; none of these 0-nodes will be bordering nodes of $\mathcal{S}_b^{\beta-}$. If, in addition, $\mathcal{S}_b^{\beta-}$ is incident to $n^{\beta+}$ also through one or more elementary tips (that is, if a branch of $\mathcal{S}_b^{\beta-}$ is incident to $n^{\beta+}$), then $n^{\beta+}$ will embrace a 0-node n^0, and we set $\mathcal{W} = \mathcal{V} \cup \{n^0\}$; otherwise, we set $\mathcal{W} = \mathcal{V}$. On the other hand, if $\mathcal{S}_b^{\beta-}$ is incident to $n^{\beta+}$ only through elementary tips, we set $\mathcal{W} = \{n^0\}$ and make $\mathcal{V}$ void. In every case, $\mathcal{W}$ is not void.

Furthermore, when the core of $\mathcal{S}_b^{\beta-}$ is incident to $n^{\beta+}$, that is, when $\mathcal{V}$ is not void, we let $\mathcal{A}$ be the reduction of $\mathcal{S}_b^{\beta-}$ induced by all branches in the core of $\mathcal{S}_b^{\beta-}$ satisfying the following: Either the branch is incident to two nodes of $\mathcal{V}$ or the branch is connected to $n^{\beta+}$ by a path in the core of $\mathcal{S}_b^{\beta-}$ that does not meet $\mathcal{V}$. On the other hand, if $\mathcal{V}$ is void, we take $\mathcal{A}$ to be void as well.

If $\mathcal{V}$ is not void, neither is the branch set of $\mathcal{A}$. Indeed, $n^{\beta+}$ will be incident to $\mathcal{S}_b^{\beta-}$ through a tip having a representative lying in the core of $\mathcal{S}_b^{\beta-}$, and that representative can meet $\mathcal{V}$ only finitely often; hence, $\mathcal{A}$ will have an infinity of branches. The *complement* $\tilde{\mathcal{A}} = \mathcal{S}_b^{\beta-} \backslash \mathcal{A}$ *of* $\mathcal{A}$ *in* $\mathcal{S}_b^{\beta-}$ is the reduction of $\mathcal{S}_b^{\beta-}$ induced by all the branches of $\mathcal{S}_b^{\beta-}$ not in $\mathcal{A}$. Thus, within the core of $\mathcal{S}_b^{\beta-}$, $\mathcal{V}$ separates $\tilde{\mathcal{A}}$ from $\mathcal{A}$ and also from $n^{\beta+}$.

Two nodes (of any ranks) are said to be 0-*adjacent* if they are not shorted but are incident to the same branch. This is the usual idea of "adjacency." We shall generalize it later on.

Now, let $\mathcal{D}$ denote the set of all nodes of $\mathcal{S}_b^{\beta-}$ that are 0-adjacent to $n^{\beta+}$. $\mathcal{D}$ will be void if $\mathcal{W} = \mathcal{V}$ and will be nonvoid if $\mathcal{W} = \mathcal{V} \cup \{n^0\}$. In the latter case, $\mathcal{D}$ will be a finite set of ordinary 0-nodes according to Condition 2.1(a). Let $\mathcal{X} = \mathcal{V} \cup \mathcal{D}$. Thus, $\mathcal{X} = \mathcal{V}$ if $\mathcal{W} = \mathcal{V}$, and $\mathcal{X} = \mathcal{D}$ if $\mathcal{W} = \{n^0\}$.

Definition 3.1. *An isolating set in a subsection.* $\mathcal{W}$ is called an *isolating set for* $n^{\beta+}$ *in* $\mathcal{S}_b^{\beta-}$ if the following three conditions are satisfied whenever the finite set $\mathcal{V}$ of ordinary 0-nodes is nonvoid (so that $\mathcal{S}_b^{\beta-}$ is incident to $n^{\beta+}$ through at least one nonelementary tip — and possibly through elementary tips as well.)

(a) Within the core of $\mathcal{S}_b^{\beta-}$, $\mathcal{V}$ separates $\mathcal{A}$ from all the ($\beta+$)-nodes incident to that core other than $n^{\beta+}$ — as well as from all the nodes of the core that are 0-adjacent to ($\beta+$)-nodes (including $n^{\beta+}$). Moreover, no node of $\mathcal{V}$ is 0-adjacent to $n^{\beta+}$ (thus,

$\mathcal{V} \cap \mathcal{D} = \emptyset$).

(b) For every node n^0 of $\mathcal{V}$ there is a path in $\mathcal{A}$ that meets n^0 and $n^{\beta+}$ but does not meet $\mathcal{V} \backslash \{n^0\}$.

(c) Every node of $\mathcal{V}$ is incident to a branch in $\tilde{\mathcal{A}}$.

On the other hand, if $\mathcal{V}$ is void (so that $\mathcal{W} = \{n^0\}$, $n^{\beta+}$ is incident to $\mathcal{S}_b^{\beta-}$ only through one or more elementary tips embraced by n^0, and $\mathcal{A}$ is void), we call $\mathcal{W}$ the *trivial isolating set for $n^{\beta+}$ in $\mathcal{S}_b^{\beta-}$*.

Under these conditions (whether or not $\mathcal{V}$ is void), $\mathcal{X}$ will be called the *conjoining set for $n^{\beta+}$ in $\mathcal{S}_b^{\beta-}$ corresponding to $\mathcal{W}$*, the reduced graph $\mathcal{A}$ will be called the *arm in $\mathcal{S}_b^{\beta-}$ for $\mathcal{V}$* or *for $\mathcal{W}$*, and $\mathcal{V}$ will be called the *base of $\mathcal{A}$* or the *base of $\mathcal{W}$*. ♣

Note that $\mathcal{V}$ and $\mathcal{X}$ are both finite sets of ordinary 0-nodes; so too is $\mathcal{W}$ except for the 0-node embraced by $n^{\beta+}$ if such exists — that one will not be ordinary.

It follows from Definition 3.1 that, within $\mathcal{S}_b^{\beta-}$, the isolating set $\mathcal{W}$ separates $n^{\beta+}$ from every other ($\beta+$)-node that is incident to $\mathcal{S}_b^{\beta-}$, and so too does $\mathcal{X}$. Moreover, if $n^{\beta+}$ is incident to the core of $\mathcal{S}_b^{\beta-}$ (i.e., if $\mathcal{V}$ is not void), then within that core $\mathcal{V}$ separates $n^{\beta+}$ from $\mathcal{D}$ as well as from every other ($\beta+$)-node that is incident to that core.

Figure 1 illustrates these ideas. Part (a) shows a 1-graph consisting of a two-way-infinite ladder along with an extra branch b_0. All the 0-tips on the extreme left (and extreme right) are shorted through the 1-node n_1^1 (respectively, n_2^1). Furthermore, b_0 is incident to n_1^1 through the embraced 0-node n_1^0; it is also incident to the ordinary 0-node n_4^0. The entire graph is a ($1-$)-subsection. Its core is the ladder without b_0. $\mathcal{W} = \{n_1^0, n_2^0, n_3^0\}$ is an isolating set for n_1^1, and $\mathcal{X} = \{n_2^0, n_3^0, n_4^0\}$ is the corresponding conjoining set. The corresponding base is $\mathcal{V} = \{n_2^0, n_3^0\}$, and the arm $\mathcal{A}$ for $\mathcal{W}$ is induced by all ladder branches to the left and between n_2^0 and n_3^0. On the other hand, $\{n_2^0, n_3^0\}$ is not an isolating set for the 1-node n_2^1 because of the presence of b_0. Were b_0 incident to n_6^0 instead of n_4^0, $\{n_2^0, n_3^0\}$ would be an isolating set for n_2^1, but then $\{n_1^0, n_2^0, n_3^0\}$ would not be an isolating set for n_1^1 because Definition 3.1(a) would be violated.

Part (b) of Figure 1 is a 2-graph consisting of a ladder of ladders, an extra ladder L, and three additional branches b_0, b_1, and b_2. The entire graph is a $(2-)$-subsection with two maximal bordering 2-nodes n_1^2 and n_2^2. The essential rank of the $(2-)$-subsection is 1. In each of three ladders, we have selected a pair of ordinary 0-nodes as in part (a). This yields the isolating set $\mathcal{W} = \{n_1^0, n_2^0, \ldots, n_7^0\}$ for n_1^2. In this case, $\mathcal{V} = \{n_2^0, \ldots, n_7^0\}$; moreover, $\mathcal{A}$ is induced by all branches to the left and between the nodes of $\mathcal{V}$ (but not b_1 and b_2). The conjoining set $\mathcal{X}$ corresponding to $\mathcal{W}$ is $\{n_2^0, \ldots, n_7^0, n_{10}^0\}$. The core of the $(2-)$-subsection is induced by all the branches except b_1. This 2-graph has an infinity of $(1-)$-subsections. b_1 and b_2 together induce one of them. Each of the other $(1-)$-subsections consists of a single ladder, except for the two ladders connected by b_0; those two ladders along with b_0 comprise a single $(1-)$-subsection. The essential ranks of the $(1-)$-subsections are all 0. Note that every $(1-)$-subsection has only finitely many bordering nodes. Were branches like b_0 connected to all adjacent ladders, the ladder of ladders would become a single $(1-)$-subsection, but then that $(1-)$-subsection would have an infinity of bordering 1-nodes — in violation of Condition 2.1(e). It may also be worth noting that the conjoining set $\mathcal{X} = \{n_2^0, \ldots, n_7^0, n_{10}^0\}$ for n_1^2 happens to be an isolating set for the other 2-node n_2^2.

Lemma 3.2. *Let $\mathcal{V}$ be a nonvoid base of an isolating set in a $(\beta-)$-subsection, and let $\mathcal{A}$ be its arm. Then, every branch b in $\mathcal{A}$ is $(\beta-)$-connected within $\mathcal{A}$ to a node of $\mathcal{V}$.*

Proof. $\mathcal{A}$ is not void because $\mathcal{V}$ is not void. b and $\mathcal{V}$ both reside in the core of $\mathcal{S}_b^{\beta-}$. By Lemma 2.2(iii), there is a finite $(\beta-)$-path $P^{\beta-}$ in that core that connects a node of b to a node of $\mathcal{V}$. Moreover, within that core, $\mathcal{V}$ separates $\mathcal{A}$ from its complement $\tilde{\mathcal{A}}$. Hence, a tracing of $P^{\beta-}$ from a node of b to the first node of $\mathcal{V}$ that $P^{\beta-}$ meets yields a finite $(\beta-)$-path within $\mathcal{A}$. ♣

Definition 3.1 focuses on a $(\beta-)$-subsection and considers a bordering node $n^{\beta+}$ in an ancillary way. We now shift our attention to a maximal β-node n^{β} and treat its incident $(\beta-)$-subsections in an ancillary fashion. Such a node must be incident to at least one $(\beta-)$-subsection with a core having an infinity of branches and an infinity of internal nodes; this fact follows from Lemma 2.2(v) because that node will have at least one $(\beta-1)$-tip $t^{\beta-1}$ and because Condition 2.1(d) insures that a representative of $t^{\beta-1}$ will lie in a $(\beta-1)$-subsection. If each of the $(\beta-)$-subsections incident to n^{β} has an isolating set for n^{β}, we can consider n^β as being "isolated" by the union of those isolating sets. For subsequent purposes, we want that union to be finite; this is the reason for imposing Condition 2.1(f).

Definition 3.3. *An isolating set for a node.* Given a maximal β-node n^β, a set $\mathcal{W}$ of 0-nodes is called an *isolating set for* n^β if $\mathcal{W} = \bigcup_{k=1}^{K} \mathcal{W}_k$, where, for each k, $\mathcal{W}_k$ is an isolating set for n^β in a ($\beta-$)-subsection $\mathcal{S}_{b,k}^{\beta-}$ incident to n^β and where the $\mathcal{S}_{b,k}^{\beta-}$ ($k = 1, \ldots, K$) are all the ($\beta-$)-subsections incident to n^β. With $\mathcal{V}_k$ denoting the base for $\mathcal{W}_k$ and with $\mathcal{A}_k$ denoting the corresponding arm, we call $\mathcal{V} = \bigcup_{k=1}^{K} \mathcal{V}_k$ the *base of* $\mathcal{W}$ and call $\mathcal{A} = \bigcup_{k=1}^{K} \mathcal{A}_k$ the *arm for* $\mathcal{W}$ or *the arm for* $\mathcal{V}$. We also refer to $\mathcal{A}$ as an *arm for* n^β. Finally, with $\mathcal{X}_k$ denoting the conjoining set in $\mathcal{S}_{b,k}^{\beta-}$ corresponding to $\mathcal{W}_k$, we call $\mathcal{X} = \bigcup_{k=1}^{K} \mathcal{X}_k$ the *conjoining set for* n^β *corresponding to* $\mathcal{W}$. ♣

Thus, every isolating set for a maximal β-node n^β separates n^β from all the other maximal ($\beta+$)-nodes in $\mathcal{G}^\mu$.

Note that the $\mathcal{V}_k$ are disjoint from each other because each $\mathcal{V}_k$ is a set of internal ordinary 0-nodes of $\mathcal{S}_{b,k}^{\beta-}$ and because the $\mathcal{S}_{b,k}^{\beta-}$ meet only at bordering nodes. $\mathcal{V} = \bigcup_{k=1}^{K} \mathcal{V}_k$ is a finite set of ordinary 0-nodes within the cores of the $\mathcal{S}_{b,k}^{\beta-}$. However, $\mathcal{V}$ cannot be void — in contrast to any single $\mathcal{V}_k$ — because the core of at least one of the $\mathcal{S}_{b,k}^{\beta-}$ will be incident to n^β (Lemma 2.2(v)). Furthermore, $\mathcal{W} = \mathcal{V} \cup \{n^0\}$ if n^β embraces a 0-node n^0; otherwise $\mathcal{W} = \mathcal{V}$. Finally, with $\mathcal{D}$ denoting the set of all the 0-nodes that are 0-adjacent to n^β, we have $\mathcal{X} = \mathcal{V} \cup \mathcal{D}$; $\mathcal{D}$ may be void, but, if it is not void, all its 0-nodes are ordinary.

Our isolating sets have been so defined that maximal nodes are "isolated" not only by the nodes of those isolating sets but also by certain branches incident to those latter nodes. These branches may play a role analogous to that played by the branches incident to an ordinary 0-node. For instance, Kirchhoff's current law might be applicable to them, as we shall see.

Definition 3.4. *A cut in a subsection.* Under the notations and conditions of Definition 3.1, let $\mathcal{C}$ be the set of all branches in $\tilde{\mathcal{A}}$ that are incident to $\mathcal{W}$. Then. $\mathcal{C}$ is called a *cut for* $n^{\beta+}$ *at* $\mathcal{W}$ *in* $\mathcal{S}_b^{\beta-}$.

$\mathcal{C}$ is a finite set because $\mathcal{W}$ is finite and every 0-node is of finite degree. For example, in Figure 1(a), $\mathcal{C} = \{b_0, b_1, b_2\}$ is a cut for n_1^1 at $\mathcal{W} = \{n_1^0, n_2^0, n_3^0\}$. That same set $\mathcal{C}$ is also a cut for n_2^1 at $\mathcal{W}'$, where $\mathcal{W}' = \mathcal{V}' = \{n_4^0, n_5^0\}$. With regard to Figure 1(b), let $\mathcal{C}$ be b_1 along with the six branches incident to n_2^0 through n_7^0 and lying to the right of those nodes. Then, $\mathcal{C}$ is a cut for the 2-node n_1^2 at $\mathcal{W} = \{n_1^0, n_2^0, \ldots, n_7^0\}$. On the other hand, b_1 along with the branches incident to n_2^0 through n_7^0 and lying to the left of those nodes comprise a cut for the other 2-node n_2^2 at the isolating set $\{n_2^0, \ldots, n_7^0, n_{10}^0\}$ for n_2^2.

Definition 3.5. *A cut for a node.* Under the notations and conditions of Definition 3.3, let $\mathcal{C}_k$ be the cut for the β-node n^β at $\mathcal{W}_k$ in $\mathcal{S}_{b,k}^{\beta-}$ for each $k = 1, \ldots, K$. Then, $\mathcal{C} = \bigcup_{k=1}^K \mathcal{C}_k$ is called a *cut for n^β at $\mathcal{W}$*.

Note that $\mathcal{C}$ resides in $\tilde{\mathcal{A}} = \bigcup_{k=1}^K \tilde{\mathcal{A}}_k = \bigcup_{k=1}^K (\mathcal{S}_{b,k}^{\beta-} \backslash \mathcal{A}_k)$.

There is another result we shall need — the extension of Konig's lemma to transfinite graphs. This has already been established in a prior work [11] for a μ-graph. To apply it, we need the following definition. Two nodes (of not necessarily the same rank) are said to be *β-adjacent* if they are not shorted and if they are incident to the same $(\beta-)$-subsection.

Here now is the extension of Konig's lemma.

Lemma 3.6. *Let $\mathcal{G}^\mu$ satisfy Conditions 2.1 and also the following two conditions:*

(a) *Every μ-node is μ-adjacent to only finitely many μ-nodes.*

(b) *There are infinitely many μ-nodes.*

Then, for each μ-node n_0^μ, there is at least one one-ended μ-path starting at n_0^μ.

4 Contractions

For the moment, let us consider once again just a single $(\beta-)$-subsection $\mathcal{S}_b^{\beta-}$. Let $n^{\beta+}$ be a maximal bordering $(\beta+)$-node for $\mathcal{S}_b^{\beta-}$.

Definition 4.1. *A contraction in a subsection.*

Case 1: First, assume that $n^{\beta+}$ is incident to $\mathcal{S}_b^{\beta-}$ through one or more nonelementary tips (and possibly through elementary tips as well). A *contraction to $n^{\beta+}$ in $\mathcal{S}_b^{\beta-}$* is a sequence $\{\mathcal{W}_p\}_{p=1}^\infty$ of isolating sets $\mathcal{W}_p$ for $n^{\beta+}$ in $\mathcal{S}_b^{\beta-}$ (Definition 3.1) satisfying the following two conditions, wherein $\mathcal{A}_p$ is the arm for $\mathcal{W}_p$ and $\mathcal{V}_p$ is its base.

(a) Given any branch b, there is a p such that b is not in $\mathcal{A}_q$ for all $q \geq p$. Moreover, for $q > p$, $\mathcal{A}_q \subset \mathcal{A}_p$ and $\mathcal{V}_q \cap \mathcal{V}_p = \emptyset$.

(b) There is a finite set $\{P_k^{\beta-}\}_{k=1}^m$ of one-ended $(\beta-)$-paths in the core of $\mathcal{S}_b^{\beta-}$ such that $P_k^{\beta-}$ meets $n^{\beta+}$ and also meets exactly one node of $\mathcal{V}_p$ for every p. Moreover, every node of $\mathcal{V}_p$ is met by at least one of the $P_k^{\beta-}$.

The $P_k^{\beta-}$ are called the *contraction paths for* $\{\mathcal{W}_p\}_{p=1}^{\infty}$.

Case 2: On the other hand, if $n^{\beta+}$ is incident to $\mathcal{S}_b^{\beta-}$ only through elementary tips, set $\mathcal{W}_p = \{n^0\}$ for every p, where n^0 is the 0-node that contains those elementary tips. Then, $\{\mathcal{W}_p\}_{p=1}^{\infty}$ is called the *trivial contraction to* $n^{\beta+}$ *in* $\mathcal{S}_b^{\beta-}$. This ends Definition 4.1. ♣

In Case 1, the ranks of the $P_k^{\beta-}$ need not all be the same. Furthermore, with card$\mathcal{Y}$ denoting the cardinality of a set $\mathcal{Y}$, we have $0 < \text{card}\mathcal{V}_p \leq m$ and $0 < \text{card}\mathcal{W}_p \leq m+1$ for every p. Of course, $\text{card}\mathcal{V}_p \leq \text{card}\mathcal{W}_p$. Thus, in saying that there is a contraction to $n^{\beta+}$ in $\mathcal{S}_b^{\beta-}$, we are in fact imposing more structure upon $\mathcal{S}_b^{\beta-}$. Note also that the part of a contraction path between $\mathcal{V}_p$ and $n^{\beta+}$ will lie entirely within $\mathcal{A}_p$, for other wise $\mathcal{V}_p$ would not separate $n^{\beta+}$ from $\tilde{A}_p = \mathcal{S}_b^{\beta-} \backslash \mathcal{A}$.

In Case 2, $\mathcal{V}_p = \emptyset$ for all p, and there are no contraction paths and no arms corresponding to the trivial contraction $\{\mathcal{W}_p\}_{p=1}^{\infty}$.

Lemma 4.2. *In Case 1 of Definition 4.1, every node n_0 that is totally disjoint from $n^{\beta+}$ will not be incident to $\mathcal{A}_q$ for all sufficiently large q.*

Proof. The conclusion is obviously true if n_0 is not incident to $\mathcal{S}_b^{\beta-}$; so, assume it is incident to $\mathcal{S}_b^{\beta-}$. If n_0 is a 0-node, it is incident to a branch that will be excluded from $\mathcal{A}_q$ for all sufficiently large q; hence, n_0 will be too. So, let the rank of n_0 be larger than 0.

Suppose n_0 is incident to $\mathcal{A}_{q_i}$ for infinitely many q_i ($i = 0, 1, 2, \ldots$) with $q_0 < q_1 < q_2 < \ldots$. Since $\mathcal{A}_{q_i} \subset \mathcal{A}_q$ whenever $q_i > q$, n_0 is incident to $\mathcal{A}_q$ for all $q \geq 1$. Therefore, any path in $\mathcal{A}_1$ that meets n_0 must meet $\mathcal{V}_q$ for all sufficiently large q. In particular, within $\mathcal{A}_1$ there is a one-ended δ-path P^δ ($\delta > 0$), which meets every $\mathcal{V}_q$ ($q \geq 1$) and whose δ-tip t^δ is shorted to n_0. Thus, P^δ meets a contraction path in $\mathcal{A}_1$ infinitely often: that is, t^δ is nondisconnectable from the tip t_0 of that contraction path. By Condition 2.1(c), t^δ and t_0 are shorted. Since t_0 is embraced by $n^{\beta+}$, n_0 and $n^{\beta+}$ cannot be totally disjoint. Our supposition is false. ♣

Definition 4.3. Let n^β be a maximal β-node and let $\mathcal{S}_{b,k}^{\beta-}$ $(k = 1, \ldots, K)$ be its incident $(\beta-)$-subsections. Assume that there is a contraction $\{\mathcal{W}_{k,p}\}_{p=1}^\infty$ to n^β in each $\mathcal{S}_{b,k}^{\beta-}$. Set $\mathcal{W}_p = \bigcup_{k=1}^K \mathcal{W}_{k,p}$ for each p. Under these conditions, $\{\mathcal{W}_p\}_{p=1}^\infty$ is called a *contraction to* n^β. A *contraction path for* $\{\mathcal{W}_p\}_{p=1}^\infty$ is simply a contraction path for $\{\mathcal{W}_{k,p}\}_{p=1}^\infty$ in one of the $\mathcal{S}_{b,k}^{\beta-}$.

Here again, card$\mathcal{V}_p$ and card$\mathcal{W}_p$ are uniformly bounded with respect to p, and card$\mathcal{V}_p$ $\leq$ card$\mathcal{W}_p$. Also, for $q > p$, we have again $\mathcal{A}_q \subset \mathcal{A}_p$ and $\mathcal{V}_q \cap \mathcal{V}_p = \emptyset$.

As an example, we can construct a contraction to the 1-node n_1^1 in Figure 1(a) by choosing an infinite sequence of 0-node pairs, such as the base $\{n_2^0, n_3^0\}$, that shift progressively leftwards. Each such base along with the embraced 0-node n_1^0 comprise one of the isolating sets in the contraction to n_1^1. The corresponding contraction paths lie along the upper and lower horizontal parts of the ladder.

An example of a contraction to the 2-node n_1^2 in Figure 1(b) can be obtained by shifting the base $\{n_2^0, \ldots, n_7^0\}$ progressively leftwards and appending n_1^0 to each such base to get a sequence of isolating sets for n_1^2. Now, we have six contraction paths lying along the three upper parts and three lower parts of the horizontal ladders and the ladder L.

5 Finitely Structured μ-Graphs

Definition 5.1. *Locally finitely structured μ-graphs.* A μ-graph $\mathcal{G}^\mu$ is called *locally finitely structured* if $\mathcal{G}^\mu$ satisfies Conditions 2.1 and if there is a contraction to n^β for every maximal β-node n^β of every rank $\beta > 0$.

Lemma 5.2. *Let $\mathcal{G}^\mu$ be a locally finitely structured μ-graph. Then, for every rank $\beta > 0$, every maximal β-node n^β in $\mathcal{G}^\mu$ is β-adjacent to only finitely many maximal $(\beta+)$-nodes.*

Proof. This follows directly from Conditions 2.1(e) and (f). ♣

Definition 5.3. *Finitely structured μ-graphs.* A μ-graph $\mathcal{G}^\mu$ is called *finitely structured* if it is locally finitely structured and has only finitely many μ-nodes.

Lemma 5.4. *Assume $\mathcal{G}^\mu$ is finitely structured. Then, $\mathcal{G}^\mu$ has only finitely many $(\mu-)$-subsections.*

Proof. $\mathcal{G}^\mu$ has only finitely many μ-nodes, all of which are perforce maximal. Moreover, every μ-node is incident to only finitely many $(\mu-)$-subsections (Condition 2.1(f)). Furthermore, every $(\mu-)$-subsection is incident to at least one μ-node since $\mathcal{G}^\mu$ is μ-connected. Our conclusion follows. ♣

Given an arm $\mathcal{A}$ for a maximal β-node, we say that a one-ended α-path P^α $(\alpha < \beta)$ *eventually lies in* $\mathcal{A}$ if P^α contains a one-ended α-path that lies entirely in $\mathcal{A}$.

Lemma 5.5. *Let $\mathcal{G}^\mu$ be locally finitely structured. For every rank β $(0 < \beta \leq \mu)$ and for every maximal β-node n^β in $\mathcal{G}^\mu$, choose an arm $\mathcal{A}$ for n^β. Then, every one-ended α-path P^α $(0 \leq \alpha < \mu)$ will eventually lie in the arm for the maximal node n^β $(\alpha < \beta)$ that embraces the α-tip of P^α.*

Proof. Let $\mathcal{V}$ be the base of $\mathcal{A}$. P^α cannot pass infinitely often into and out of $\mathcal{A}$ because each such passage must be through a different node of $\mathcal{V}$ and $\mathcal{V}$ has only finitely many nodes. Since n^β embraces the α-tip of P^α, P^α must eventually lie in $\mathcal{A}$. ♣

Given two reduced graphs $\mathcal{H}$ and $\mathcal{M}$ of $\mathcal{G}^\mu$, $\mathcal{H}\backslash\mathcal{M}$ denotes the reduced graph induced by all the branches in $\mathcal{H}$ that are not in $\mathcal{M}$.

Lemma 5.6. *Let $\mathcal{G}^\mu$ be locally finitely structured. Let $\{\mathcal{W}_p\}_{p=1}^\infty$ be a contraction to the maximal β-node n^β $(\beta > 0)$ and let $\{\mathcal{A}_p\}_{p=1}^\infty$ be the corresponding sequence of arms. Then, for any $q > p$ the reduced graph $\mathcal{A}_p\backslash\mathcal{A}_q$ has at most finitely many nodes of rank $\beta - 1$ and no node of higher rank.*

Proof. Let $\mathcal{V}_p$ be the base of $\mathcal{A}_p$ and let $\mathcal{S}_c$ denote the union of the cores of all the $(\beta-)$-subsections to which n^β is incident. Now, all the nodes of $\mathcal{V}_p$, of $\mathcal{V}_q$, and of $\mathcal{A}_p\backslash\mathcal{A}_q$ are interior nodes of $\mathcal{S}_c$ and hence have ranks no larger than $\beta - 1$. Also, $\mathcal{V}_p \cup \mathcal{V}_q$ separates $\mathcal{A}_p\backslash\mathcal{A}_q$ from all $(\beta+)$-nodes, and all the nodes of $\mathcal{V}_p \cup \mathcal{V}_q$ are ordinary 0-nodes. It follows that no $(\beta+)$-node is incident to $\mathcal{A}_p\backslash\mathcal{A}_q$.

Now, suppose that $\mathcal{A}_p\backslash\mathcal{A}_q$ has an infinity of $(\beta-1)$-nodes. Each one has to be maximal. $\mathcal{A}_p\backslash\mathcal{A}_q$ can have only finitely many components because each branch of $\mathcal{A}_p\backslash\mathcal{A}_q$ is $(\beta-)$-connected to a node of $\mathcal{V}_p \cup \mathcal{V}_q$ according to Lemma 3.2 and because $\mathcal{V}_p \cup \mathcal{V}_q$ is a finite set. Thus, one of these components, say, $\mathcal{M}$ has an infinity of $(\beta - 1)$-nodes. Moreover, $\mathcal{M}$ is $(\beta - 1)$-connected since it is $(\beta-)$-connected. Since $\mathcal{G}^\mu$ is locally finitely structured, by Lemma 5.2, every $(\beta-1)$-node in $\mathcal{M}$ is $(\beta-1)$-adjacent to only finitely many $(\beta-1)$-nodes. Thus, the hypothesis of Lemma 3.6 is fulfilled when $\mathcal{G}^\mu$ is replaced by $\mathcal{A}_p \backslash \mathcal{A}_q$ and μ is replaced by $\beta - 1$. Consequently, $\mathcal{M}$ contains at least one one-ended $(\beta-1)$-path. Hence, $\mathcal{M}$ and thereby $\mathcal{A}_p \backslash \mathcal{A}_q$ has a $(\beta-1)$-tip. Since $\mathcal{A}_p \backslash \mathcal{A}_q$ is a reduction of a μ-graph where $\mu \geq \beta$, $\mathcal{A}_p \backslash \mathcal{A}_q$ must have a β-node. This contradicts our prior conclusion that no $(\beta+)$-node is incident to $\mathcal{A}_p \backslash \mathcal{A}_q$. Thus, our supposition is wrong, and $\mathcal{A}_p \backslash \mathcal{A}_q$ has at most finitely many $(\beta-1)$-nodes and no node of higher rank. ♣

6 Finitely Structured μ-Networks

An *electrical μ-network* $\mathbf{N}^\mu$ is a μ-graph $\mathcal{G}^\mu$ whose branches consist of electrical parameters [5, Section 1.4]. In this work $\mathbf{N}^\mu$ will not have any reactive elements. *Reductions* of $\mathbf{N}^\mu$ are reductions of $\mathcal{G}^\mu$ with the same assignment of electrical parameters to branches. Networks and reduced networks will be denoted by boldface capital letters — in contrast to the calligraphic capital letters used for graphs. Thus, if $\mathcal{A}$ is an arm in the graph $\mathcal{G}^\mu$, then $\mathbf{A}$ will denote $\mathcal{A}$ with the given assignment of electrical parameters to its branches. All the terminology and definitions used for graphs are carried over to networks. Thus, we may speak of subsections, cuts, contractions, and so on for the network $\mathbf{N}^\mu$.

In this work $\mathbf{N}^\mu$ will be *sourceless*; this means that every branch b_j of $\mathbf{N}^\mu$ consists only of a resistance r_j, whose value is a real positive number measured in ohms. The reciprocal $g_j = r_j^{-1}$ is the branch's conductance.

A path in $\mathbf{N}^\mu$ is called *perceptible* if the sum of all the resistances in the path is finite.

Definition 6.1. *Finitely Structured μ-Networks.* A μ-network $\mathbf{N}^\mu$ is called *finitely structured* if the graph of $\mathbf{N}^\mu$ is finitely structured (Definition 5.3) and if, for every β with $0 < \beta \leq \mu$ and for every maximal β-node n^β in $\mathbf{N}^\mu$, there is a contraction $\{\mathcal{W}_p\}_{p=1}^\infty$ to n^β all of whose contraction paths are perceptible.

It is assumed henceforth that $\mathbf{N}^\mu$ is sourceless and finitely structured. Also, when the contraction paths for the contraction $\{\mathcal{W}_p\}_{p=1}^\infty$ are all perceptible, the contraction itself will be called *perceptible*. We will always choose our contractions to be perceptible.

A *source branch* is a branch consisting of a voltage or current source and possibly a resistance as well. If that resistance is absent, the branch is called a *pure source* [5, Section 1.4]. $\mathbf{N}^\mu$ has no source branch. However, we will at times append source branches to $\mathbf{N}^\mu$, but the resulting network will be denoted by $\mathbf{N}_a^\mu$ — or by $\mathbf{N}_e^\mu$ if all appended branches are pure voltage sources.

Lemma 6.2. *Between every two nodes (of any ranks) in* $\mathbf{N}^\mu$ *there is a perceptible path that terminates at those two nodes.*

Proof. Let n_a and n_b be the two nodes. By the μ-connectedness of $\mathbf{N}^\mu$, there is a finite path P^α of essential rank α $(0 \le \alpha \le \mu)$ that terminates at n_a and n_b. Thus, $\alpha - 1$ is the largest rank for all the tips traversed by P^α. If $\alpha = 0$, that path has only finitely many branches, and our conclusion follows. So, consider the case where $\alpha \ge 1$.

Let θ be the maximum rank among all the ranks of the maximal nodes that P^α meets. By Lemma 2.3, P^α will meet only finitely many maximal θ-nodes. We can choose an isolating set for each of those θ-nodes such that their corresponding arms are pairwise totally disjoint and such that neither n_a and nor n_b is incident to an arm except when n_a or n_b is embraced by a θ-node. Let n^θ denote one of the maximal θ-nodes that P^α meets. Also, let $\mathbf{A}$ be the corresponding chosen arm, and $\mathcal{V}$ its base. Orient P^α from n_a to n_b. Then, there will either be a first node $n_1 \in \mathcal{V}$ that P^α will meet before meeting n^θ, or a last node $n_2 \in \mathcal{V}$ that P^α will meet after leaving n^θ, or both. In each case, replace that part of P^α between n_1 and n^θ or between n^θ and n_2 by a perceptible path in $\mathbf{A}$ that terminates at n_1 and n^θ or terminates at n^θ and n_2. This can be done because of Definition 6.1.

Do this for all of the finitely many θ-nodes that P^α meets. Let P_r^α be the finite α-path resulting from the said replacements. Upon deleting the substituted α-paths from P_r^α, we are left with finitely many finite paths. For each of them, the maximum rank θ' among the ranks of all the maximal nodes that path meets is less than θ (i.e., $\theta' \le \theta - 1$). We can treat each one of those paths in the same way to obtain finitely many substituted perceptible paths, whose deletions leave finitely many finite paths. For each of those, the maximum rank θ'' as described above is still lower (i.e., $\theta'' \le \theta - 2$) Continuing this process, we are finally left with finitely many finite 0-paths, which are perforce perceptible. Upon connecting together all of those 0-paths and the finitely many substituted paths in accordance with the tracing from n_a to n_b, we obtain a perceptible path that terminates at those two nodes. ♣

We have already defined in Section 2 what is meant by saying that two nodes are "shorted." We now wish to extend this idea, for, while manipulating networks, we will at times combine two or more maximal nodes by *shorting* them. This is accomplished by creating a new node that embraces all the tips of all ranks that are embraced by the said nodes. With regard to the elementary tips, something more is done to complete the shorting: If any branch becomes a self-loop, it is eliminated. Also, if parallel resistors arise, they are combined by adding conductances. Finally, if a branch arises that is not incident to an ordinary 0-node, that branch is replaced by two branches in series by introducing another ordinary 0-node; this may be needed to maintain the second sentence of Condition 2.1(a).

An important and easily checked consequence of Definition 6.1 is

Lemma 6.3. *Let us short finitely many nodes (of any ranks) of the finitely structured* μ*-network* $\mathbf{N}^\mu$*. Then, the resulting network is also a finitely structured* μ*-network.*

7 Excitations by Pure Sources

The fundamental theory for voltage-current regimes in transfinite networks [5, Chapters 3 and 5], [6] takes pure voltage sources into account by transferring them into resistive branches. That transference can be accomplished only when one node of the voltage source is an ordinary 0-node. For our purposes, we need a fundamental theory that encompasses pure voltage sources without transferring them. In particular, we wish to append a pure voltage source to $\mathbf{N}^\mu$ possibly at two nodes of ranks higher than 0. Consequently, our next objective is to generalize the fundamental theory accordingly. In doing so, we shall also generalize Kirchhoff's current law to make it applicable to a cut that isolates an α-node.

Index the branches of the sourceless μ-network $\mathbf{N}^\mu$ by $j = 1, 2, 3, \ldots$. Let b_0 be a source branch and append it to any two nodes of $\mathbf{N}^\mu$, whatever be their ranks. Denote the resulting augmented network by $\mathbf{N}^\mu_a$. At this time, no restriction is placed on the kind of source branch b_0 may be; it may be either a pure voltage source, or a pure current source, or a source with a resistance. The current vector for $\mathbf{N}^\mu_a$ is denoted by $\mathbf{i} = (i_0, i_1, i_2, \ldots)$, where i_j is the current in branch b_j, $j = 0, 1, 2, \ldots$. We wish to construct a Hilbert space $\mathcal{K}_a$ of permissible branch-current vectors for $\mathbf{N}_a^\mu$, which is like the space $\mathcal{K}$ [5, page 154] but without the branch b_0 contributing a term to the inner product for $\mathcal{K}_a$.

As the first step toward that objective, we consider Kirchhoff's current law applied to a cut $\mathbf{C}$ in $\mathbf{N}_a^\mu$ that isolates one of the maximal nodes n^α $(0 \leq \alpha \leq \mu)$ in $\mathbf{N}_a^\mu$. If n^α is an ordinary 0-node (i.e., $\alpha = 0$), we let $\mathbf{C}$ denote the set of all branches incident to n^α and we orient $\mathbf{C}$ toward n^α. If however n^α is of higher rank (i.e., $0 < \alpha \leq \mu$), we let $\mathbf{C}$ be a cut that isolates n^α from all other $(\alpha+)$-nodes (Definition 3.5) and we orient $\mathbf{C}$ toward the isolating set $\mathcal{W}$ at which $\mathbf{C}$ exists — effectively toward n^α again. In both cases, $\mathbf{C}$ is a finite set. If the source branch b_0 is incident to n^α, then $b_0 \in \mathbf{C}$. Kirchhoff's current law for $\mathbf{C}$ asserts that

$$\sum_{\mathbf{C}} \pm i_j \;=\; 0 \tag{1}$$

where i_j is the current in branch b_j, the summation is over the indices of the branches in $\mathbf{C}$, and the plus (minus) sign is used if the orientations of $\mathbf{C}$ and branch b_j agree (respectively, disagree). We have yet to establish whether (1) holds.

The next step is to construct a Hilbert space $\mathcal{K}_a$ of current vectors $\mathbf{i} = (i_0, i_1, i_2, \ldots)$ in $\mathbf{N}_a^\mu$ for which the total power dissipation $\sum_{j=1}^\infty i_j^2 r_j$ within $\mathbf{N}^\mu$ is finite and also for which (1) holds whatever be the choice of $\mathbf{C}$. Let $\mathcal{I}_a$ be the set of all current vectors $\mathbf{i}$ for $\mathbf{N}_a^\mu$ such that $\sum_{j=1}^\infty i_j^2 r_j < \infty$. (In contrast to the constructions in [5, pages 74 and 154], $\mathcal{I}_a$ cannot now be identified as an inner product space; for instance, the nonzero vector $\mathbf{i}$ for which $i_0 = 1$ and $i_j = 0$, $j = 1, 2, 3, \ldots$, has zero power dissipation within $\mathbf{N}^\mu$.) Next, let $\mathcal{K}_a^0$ denote the span of all basic currents [5, page 154] in $\mathcal{I}_a$.

Lemma 7.1. *Kirchhoff's current law (1) holds at every cut* $\mathbf{C}$ *whenever* $\mathbf{i} \in \mathcal{K}_a^0$.

Proof. Since there are only finitely many branches in $\mathbf{C}$, any α-loop $(0 \leq \alpha \leq \mu)$ can embrace branches of $\mathbf{C}$ at most finitely often. Moreover, each α-loop current contributes additive terms to the left-hand side of (1) an even number of times, positively for half of those times and negatively for the other half. Hence, its total contribution to the left-hand side of (1) is 0.

The same is true for any basic current i. Indeed, a basic current is an α-basic current for some α with $0 \leq \alpha \leq \mu$. Moreover, a 0-basic current is simply a 0-loop current. On the other hand, for $\alpha > 0$, an α-basic current $\mathbf{i}$ is a countable sum $\mathbf{i} = \sum \mathbf{i}_m$ of α-loop currents such that only finitely many of the $\mathbf{i}_m$ meet any given 0-node (with two other conditions imposed as well) [5, page 154]. This implies that only finitely many of the $\mathbf{i}_m$ pass through $\mathbf{C}$ because every branch of $\mathbf{C}$ is incident to an ordinary 0-node and therefore can carry only finitely many of the $\mathbf{i}_m$ and because $\mathbf{C}$ is a finite set. Since each $\mathbf{i}_m$ contributes 0 to the left-hand side of (1), $\mathbf{i}$ does too.

We can now conclude that the same is true for every member of $\mathcal{K}_a^0$. ♣

We now define an inner product for $\mathcal{K}_a^0$ by

$$(\mathbf{i}, \mathbf{s}) \;=\; \sum_{j=1}^{\infty} i_j s_j r_j \tag{2}$$

where $\mathbf{i}, \mathbf{s} \in \mathcal{K}_a^0$. Even though (2) does not contain i_0 and s_0, it is nonetheless positive-definite. Indeed, $(i, i) \geq 0$ obviously. Moreover, if $(i, i) = \sum_{j=1}^{\infty} i_j^2 r_j = 0$, then $i_j = 0$ for all $j > 0$. Now, choose a cut $\mathbf{C}$ that isolates one of the two nodes to which b_0 is incident. Thus, b_0 is a member of $\mathbf{C}$, and by (1) we have

$$i_0 \;=\; -\sum_{\mathbf{C}\backslash\{b_0\}} i_j \tag{3}$$

where the summation is over the indices for the branches in $\mathbf{C}$ other than b_0. By (3), if $i_j = 0$ for all $j > 0$, then $i_0 = 0$ too. Therefore, $\mathbf{i} = 0$. Whence, the positive-definiteness of $(\mathbf{i}, \mathbf{s})$. The other inner-product axioms are also fulfilled.

Let $\mathcal{K}_a$ denote the completion of $\mathcal{K}_a^0$ under the norm $\|\mathbf{i}\| = (\mathbf{i}, \mathbf{i})^{\frac{1}{2}}$. Convergence under that norm obviously implies branchwise convergence for every $j > 0$. Since $\mathbf{C}\backslash\{b_0\}$ is a finite set, (3) implies branchwise convergence for $j = 0$ too. Thus, the members of $\mathcal{K}_a$ can be identified through branchwise convergence as current vectors in $\mathbf{N}_a^{\mu}$. Moreover, it follows in a standard way [4, pages 263 and 350] that $\mathcal{K}_a$ is a Hilbert space with an inner product given by (2).

The branchwise convergence coupled with the finiteness of the set $\mathbf{C}$ allows us to extend Lemma 7.1 immediately:

Lemma 7.2. *Kirchhoff's current law (1) holds at every cut* $\mathbf{C}$ *whenever* $\mathbf{i} \in \mathcal{K}_a$.

Let $\mathcal{I}$ be the subset of $\mathcal{I}_a$ consisting of all $\mathbf{i} \in \mathcal{I}_a$ for which $i_0 = 0$. With (2) as the inner product, $\mathcal{I}$ is a Hilbert space. Let $\mathcal{K}$ be the corresponding subset of $\mathcal{K}_a$. In fact, $\mathcal{K}$ is a linear subspace of $\mathcal{K}_a$. Now, augment $\mathbf{N}^\mu$ with an appended branch b_0 consisting of a pure current source h_0 to obtain the augmented network $\mathbf{N}^\mu_a$. This is permissible according to [5, pages 98 and 156] because there is a perceptible path between any two nodes of $\mathbf{N}^\mu$ (Lemma 6.2). Moreover, we can transfer h_0 to within $\mathbf{N}^\mu$ along a perceptible path P to obtain a unique current vector $(i'_1, i'_2, \ldots)$ in the resulting network in accordance with [5, Theorem 3.5-2]. (That is, in the resulting network the branch b_0 has been eliminated by removing it after the said transference is made.) We are now free to append the component $i'_0 = 0$ to that current vector and will obtain thereby $\mathbf{i}' = (0, i'_1, i'_2, \ldots) \in \mathcal{K}$. Upon restoring b_0 and transferring the current source back to b_0, we obtain the corresponding current vector $\mathbf{i}$ for $\mathbf{N}^\mu_a$ as the superposition of $\mathbf{i}'$ and a loop current of value h_0 flowing around the loop $P \cup \{b_0\}$. Hence, $\mathbf{i}$ is a member of $\mathcal{K}_a$ and therefore satisfies Kirchhoff's current law according to Lemma 7.2. It is a fact that $\mathbf{i}$ is independent of the choice of the perceptible path P in $\mathbf{N}^\mu$ by which $\mathbf{i}$ was constructed [5, pages 98 and 156]. In this way, a pure current source in b_0 generates a unique current vector $\mathbf{i} \in \mathcal{K}_a$ in accordance with [5, Theorem 3.5-2].

We have yet to establish that a pure voltage source can be connected to any two nodes of $\mathbf{N}^\mu$. This would not be possible if $\mathbf{N}^\mu$ acted as a short between those nodes. However, as was just noted, we are permitted to connect a pure current source between those two nodes. We shall do so and then will show that $\mathbf{N}^\mu$ acts as a positive resistance between those nodes. This will justify the application of a pure voltage source to them. Actually, it is just as easy to show something more general; namely, at K arbitrarily selected nodes, $\mathbf{N}^\mu$ behaves as a $(K-1)$-port with a nonsingular driving-point resistance matrix.

The voltage-current regime induced by appending finitely many pure current sources to $\mathbf{N}^\mu$ can be described by a set of node voltages as follows. Choose any node n_g of any rank in $\mathbf{N}^\mu$ and fix it. We call n_g *ground* and assign to it the node voltage $u_g = 0$. Next, choose any other node n_0 of any rank and select a perceptible path P in $\mathbf{N}^\mu$ terminating at n_g and n_0 (Lemma 6.2). Orient P from n_0 to n_g. Then, the node voltage u_0 at n_0 is defined to be

$$u_0 = \sum_P \pm v_j \tag{4}$$

where the summation is over the indices of the branches in P, v_j is the voltage of the branch b_j in P, and the plus (minus) sign is used if the orientations of P and b_j agree (respectively, disagree). The perceptibility of P ensures that (4) converges absolutely whenever it has an infinity of terms [5, page 83]. Moreover, (4) is independent of the choice of P so long as P is perceptible; indeed, Condition 2.1(c) allows us to invoke [9, Corollary 8.3], which asserts that independence.

Arbitrarily select finitely many nodes $n_1, \ldots, n_K$ of any ranks in $\mathbf{N}^\mu$ and connect pure current sources between them. Without loss of generality, we can take them to be $K-1$ current sources feeding the currents $h_2, \ldots, h_K$ from n_1 to $n_2, \ldots, n_K$ respectively. We designate n_1 as the ground node and obtain thereby the respective node voltages $u_2, \ldots, u_K$. In this way, $\mathbf{N}^\mu$ acts as an internally transfinite and sourceless, resistive $(K-1)$-port with n_1 as the common ground for the various ports . Moreover, $\mathbf{h} = (h_2, \ldots, h_K)$ is the imposed port-current vector, $\mathbf{u} = (u_2, \ldots, u_K)$ is the resulting port-voltage vector, and the mapping $Z : \mathbf{h} \mapsto \mathbf{u}$ is the $(K-1) \times (K-1)$ resistance matrix for this $(K-1)$-port. We will now show that Z is nonsingular. This will imply that any choice of the port-voltage vector $\mathbf{u}$ can be obtained by setting $\mathbf{h} = Z^{-1}\mathbf{u}$. This will also imply that any finite set of pure voltage sources can be appended to any nodes of $\mathbf{N}^\mu$ to obtain a unique voltage-current regime throughout $\mathbf{N}^\mu$.

Lemma 7.3. *Z is symmetric and positive-definite and therefore nonsingular.*

Proof. The symmetry of Z follows from the reciprocity principle [5, page 80], which extends to our present situation of pure current sources. (The latter can be seen by transferring current sources along perceptible paths in $\mathbf{N}^\mu$, invoking reciprocity for resistive branches, and also invoking the absolute convergence of the infinite series that arise in order to rearrange summations.)

We now prove that Z is positive-definite. Choose any vector $\mathbf{h} = (h_2, \ldots, h_K)$ of current sources h_k applied at the ports. For each n_k $(k = 2, \ldots, K)$, choose a cut $\mathbf{C}_k$ that isolates n_k. Thus, the source branch b_k for h_k is a member of $\mathbf{C}_k$, but the other source branches are not in $\mathbf{C}_k$. Hence, $\mathbf{C}_k = \mathbf{D}_k \cup \{b_k\}$, where $\mathbf{D}_k$ is the set of branches in $\mathbf{C}_k$ that lie within $\mathbf{N}^\mu$. As was noted in the paragraph after Lemma 7.2, the current regime induced when h_k is acting alone satisfies Kirchhoff's current law at $\mathbf{C}_k$. Moreover, the same is true at $\mathbf{C}_k$ when any other appended current sources is acting alone. Consequently, by superposition, Kirchhoff's current law is satisfied at $\mathbf{C}_k$ when all the current sources are acting simultaneously. We can conclude that in the latter case the net current flowing through $\mathbf{D}_k$ away from n_k is h_k. Therefore, there is at least one branch of $\mathbf{D}_k$ that carries a current of absolute value no less than $|h_k|/d_k$, where $d_k = \text{card}\mathbf{D}_k$. Let $r_{min,k}$ be the least resistance among the branches of $\mathbf{D}_k$. Hence, the power dissipated in all the resistances of $\mathbf{D}_k$ is no less than $\delta_k h_k^2$, where $\delta_k = r_{min,k} d_k^{-2} > 0$. Hence, with a cut chosen for each of the nodes $n_2, \ldots, n_K$, we see that the power dissipated in all those cuts is no less than $\sum_{k=2}^{K} \delta_k h_k^2$. The last expression is positive if $\mathbf{h} \neq \mathbf{0}$.

Now. let $(\cdot, \cdot)$ be the inner product for $(K-1)$-dimensional Euclidean space. Tellegen's equation holds for transfinite networks even when pure current sources are present. A consequence is that the power $(\mathbf{u}, \mathbf{h}) = (Z\mathbf{h}, \mathbf{h})$ supplied by the sources to $\mathbf{N}^\mu$ is equal to the power dissipated in all the resistances in $\mathbf{N}^\mu$. Thus, $(Z\mathbf{h}, \mathbf{h}) \geq \sum_{k=2}^{K} \delta_k h_k^2$, which proves that Z is positive-definite. ♣

Finally, we come to the result we seek. Remember that the source branch b_0 can be connected to any two nodes of $\mathbf{N}^\mu$ to obtain the augmented network $\mathbf{N}_a^\mu$.

Theorem 7.4. *Let b_0 be a pure voltage source e_0. Then, there is a unique $\mathbf{i} \in \mathcal{K}_a$ such that*

$$e_0 s_0 \;=\; \sum_{j=1}^{\infty} r_j i_j s_j \tag{5}$$

for every $\mathbf{s} \in \mathcal{K}_a$.

Proof. To prove this theorem, we will insert a resistance $\rho > 0$ in series with the voltage source e_0 within the branch b_0 to obtain the unique current vector $\mathbf{i}^\rho = (i_0^\rho, i_1^\rho, i_2^\rho, \ldots)$ dictated by [5, Theorem 3.3-5], and then we will take $\rho \to 0$ to obtain (5) in the limit.

With ρ inserted as stated, [5, Theorem 3.3-5] asserts that

$$e_0 s_0 \;=\; \rho i_0^\rho s_0 + \sum_{j=1}^{\infty} r_j i_j^\rho s_j. \tag{6}$$

By virtue of Lemma 7.3, $\mathbf{N}^\mu$ appears as a positive driving-point resistance z between the two nodes to which the source branch b_0 is connected. Hence,

$$e_0 - \rho i_0^\rho = z i_0^\rho. \tag{7}$$

With λ being another positive value for the resistance inserted into b_0, we have that

$$i_0^\rho - i_0^\lambda = \frac{e_0}{\rho + z} - \frac{e_0}{\lambda + z} \to 0 \tag{8}$$

as $\rho, \lambda \to 0+$ independently. From (6) and (7), we obtain

$$\sum_{j=1}^{\infty} r_j (i_j^\rho - i_j^\lambda) s_j = (\lambda i_0^\lambda - \rho i_0^\rho) s_0 = z(i_0^\rho - i_0^\lambda) s_0. \tag{9}$$

Note now that both $\mathbf{i}^\rho$ and $\mathbf{i}^\lambda$ are members of $\mathcal{K}_a$. (For instance, $\mathbf{i}^\rho$ can be obtained by using i_0^ρ as a pure current source in b_0 in accordance with the paragraph following Lemma 7.2 again.) Also, recall that the norm $\|\mathbf{i}\|$ for any $\mathbf{i} \in \mathcal{K}_a$ is given by $\|\mathbf{i}\|^2 = \sum_{j=1}^{\infty} r_j i_j^2$. Consequently, we may set $s_j = i_j^\rho - i_j^\lambda$ in (9) and then invoke (8) to get

$$\|\mathbf{i}^\rho - \mathbf{i}^\lambda\|^2 = \sum_{j=1}^{\infty} r_j (i_j^\rho - i_j^\lambda)^2 = z(i_0^\rho - i_0^\lambda)^2 \to 0$$

as $\rho, \lambda \to 0+$ independently. Hence, $\{\mathbf{i}^\rho : \rho > 0\}$ is a Cauchy directed function in $\mathcal{K}_a$ and therefore converges in $\mathcal{K}_a$ to an $\mathbf{i} \in \mathcal{K}_a$. Since the inner product of $\mathcal{K}_a$ is bicontinuous, we may pass to the limit in (6) to obtain (5).

$\mathbf{i}$ is uniquely determined by (5) because the right-hand side of (5) is the inner product $(\mathbf{i}, \mathbf{s})$ determined for all $\mathbf{s} \in \mathcal{K}_a$ by the left-hand side. ♣

Henceforth, we take it that the voltage-current regime in $\mathbf{N}^\mu$ excited by a single pure voltage source e_0 appended to any two nodes of $\mathbf{N}^\mu$ is dictated by Theorem 7.4. We may alter $\mathbf{N}^\mu$ by shorting a finite set of maximal nodes to obtain one terminal for e_0 and by shorting another disjoint, finite set of maximal nodes to obtain the other terminal for e_0; this is permissible according to Lemma 6.3. That possibly altered network with e_0 appended will be denoted by $\mathbf{N}_e^\mu$. Furthermore, the voltage-current regime corresponding to finitely many pure voltage sources appended to the linear network $\mathbf{N}^\mu$ is determined from Theorem 7.4 by superposition. Finally, when we speak of node voltages in $\mathbf{N}_e^\mu$, it is understood that some ground node has been selected.

We have already used the following fact [5, page 83] regarding (4), but, since we wish to use it again, let us state it explicitly.

Lemma 7.5. *Let P be a perceptible path with an infinity of branches. Then, the sum of its branch voltages converges absolutely.*

8 A Maximum Principle for Node Voltages in a μ-Network

Let $\mathbf{N}^0$ be an ordinary, locally finite, infinite network whose branches are purely resistive (i.e., all sources are at infinity), and let $\mathbf{N}^0$ carry a current that satisfies Kirchhoff's current law at every node, Kirchhoff's voltage law around every 0-loop (that is, around every finite loop), and Ohm's law. Upon choosing any node of $\mathbf{N}^0$ as ground and setting its voltage at 0, we obtain unique node voltages throughout $\mathbf{N}^0$. The maximum principle for those node voltages asserts that either they are all equal to 0 or there is no maximum value and no minimum value among them. If $\mathbf{N}^0$ were to be embedded as a 0-subsection in a larger, finitely structured μ-network $\mathbf{N}^\mu$ ($\mu \geq 1$), this principle would assert that the maximum and minimum node voltages for that subsection would have to occur at bordering nodes of $\mathbf{N}^0$. The objective of this section is to extend this principle to subsections of higher ranks in $\mathbf{N}^\mu$.

Again let $\mathbf{S}_b^{\beta-}$ be a $(\beta-)$-subsection of the finitely structured μ-network $\mathbf{N}^\mu$ with an assigned ground node at 0 V. $\mathbf{S}_b^{\beta-}$ will be called *sourceless* if none of its internal nodes is incident to a source branch. (However, we allow bordering nodes of $\mathbf{S}_b^{\beta-}$ to be so incident.) Since $\mathbf{S}_b^{\beta-}$ has only finitely many bordering $(\beta+)$-nodes, we can let u_{max} (and u_{min}) be the largest (respectively, least) node voltage at those bordering $(\beta+)$-nodes. Consider the following. Part (a) is a transfinite generalization of the maximum principle.

Properties 8.1. *For all $\beta = 1, \ldots, \mu$, the following hold.*

(a) *There are exactly two possibilities for all the node voltages of any sourceless $(\beta-)$-subsection $\mathbf{S}_b^{\beta-}$ of $\mathbf{N}^\mu$.*

(a1) *All those node voltages (for both internal and bordering nodes of $\mathbf{S}_b^{\beta-}$) have the same value.*

(a2) $\mathbf{S}_b^{\beta-}$ *has at least two incident* $(\beta+)$*-nodes with differing node voltages, and the internal node voltages of* $\mathbf{S}_b^{\beta-}$ *are strictly less than* u_{max} *and strictly larger than* u_{min}.

(b) *Let* P^α $(\alpha < \beta)$ *be any one-ended* α*-path in* $\mathbf{S}_b^{\beta-}$ *and let* t^α *be its* α*-tip. (*P^α *need not be perceptible.) Let* $\{m_1, m_2, \ldots\}$ *be any sequence of nodes embraced by* P^α *that approaches* t^α*. Then, the node voltages at the* m_l *converge to the node voltage of the* $(\alpha+1)$*-node that embraces* t^α.

We will show that Properties 8.1 always hold. We start with a lemma.

Lemma 8.2. *Assume that Properties 8.1(a) hold. Let* n^β *be a maximal* β*-node* $(\beta > 0)$*, whose incident* $(\beta-)$*-subsections are all sourceless. Assume also that at least one of those incident* $(\beta-)$*-subsections satisfies (a2). If the voltage* u^β *at* n^β *is no less than all the node voltages at the* $(\beta+)$*-nodes that are* β*-adjacent to* n^β*, then Kirchhoff's current law (1) is violated at some cut* $\mathbf{C}$ *for* n^β.

Proof. So far as Kirchhoff's current law is concerned, the $(\beta-)$-subsections incident to n^β that satisfy (a1) can be ignored because all their branch currents are 0. So, let us consider only those $(\beta-)$-subsections incident to n^β that satisfy (a2). In their union $\mathbf{U}$ choose a perceptible contraction $\{\mathcal{W}_p\}_{p=1}^\infty$ to n^β. If n^β is 0-adjacent to one or more internal nodes of $\mathbf{U}$, then the voltages at those internal nodes are strictly less than the voltage u^β at n^β. Thus, the current leaving n^β through the branches incident to n^β will be positive. Let $\mathcal{V}_p$ be the base of $\mathcal{W}_p$. Since we may be dealing with only some of the $(\beta-)$-subsections incident to n^β, it is possible for $\mathcal{V}_p$ to be void. In that case, n^β must be 0-adjacent to internal nodes of $\mathbf{U}$, and the positive currents leaving n^β through the corresponding branches shows that Kirchhoff's current law is violated at n^β.

So assume that $\mathcal{V}_p$ is nonvoid. Therefore, it is nonvoid for every p and we have $\mathrm{card}\mathcal{V}_p \leq m$, where m is finite and independent of p. Let $\mathbf{A}_p$ be the arm for $\mathcal{V}_p$ and set $\mathbf{M}_p = \mathbf{A}_p \backslash \mathbf{A}_{p+1}$. All the nodes of $\mathbf{M}_p$ are core nodes and none of them are 0-adjacent to n^β. Moreover, for $p > 1$, $\mathcal{V}_p$ separates $\mathbf{M}_{p-1}$ and $\mathbf{M}_p$. Now, each contraction path for $\{\mathcal{W}_p\}_{p=1}^\infty$ is perceptible, and the 0-node voltages at the $\mathcal{V}_p$ along any contraction path are strictly less than u^β (see Property 8.1(a2)) and converge to u^β (see (4) and Lemma 7.5). Since there are only finitely many contraction paths and since every node of $\mathcal{V}_p$ lies on a contraction path, it follows that we can choose two natural numbers p and q with $p < q$ such that the largest node voltage for $\mathcal{V}_p$ is less than the least node voltage for $\mathcal{V}_q$.

Set $\mathbf{M}_{p,q} = \bigcup_{k=p}^{q-1} \mathbf{M}_k$. We can generate the same voltage-current regime in $\mathbf{M}_{p,q}$ as it has in $\mathbf{N}_e^\mu$ by appending pure voltage sources to $\mathcal{V}_p \cup \mathcal{V}_q$ as follows: Let $n_{p,1}^0$ be a node of $\mathcal{V}_p$ with the largest node voltage $u_{p,1}$ for $\mathcal{V}_p$. Let $n_{p,k}^0$ be any other node of $\mathcal{V}_p$ and let $u_{p,k}$ be its voltage. Connect a pure voltage source of value $u_{p,1} - u_{p,k} \geq 0$ from $n_{p,k}^0$ to $n_{p,1}^0$ with its positive terminal at $n_{p,1}^0$. (That source will be a short if $u_{p,1} = u_{p,k}$.) Do this for all $n_{p,k}^0$ in $\mathcal{V}_p$. Similarly, connect a pure voltage source from a node $n_{q,1}^0$ of $\mathcal{V}_q$ with the least node voltage $u_{q,1}$ for $\mathcal{V}_q$ to each of the other nodes of $\mathcal{V}_q$ to establish their relative node voltages at the values they have in $\mathbf{N}_e^\mu$. Finally, for the same purpose, connect a pure voltage source $e_{p,q}$ of value $u_{q,1} - u_{p,1} > 0$ from $n_{p,1}^0$ to $n_{q,1}^0$, positive terminal at $n_{q,1}^0$. All these appended voltages are nonnegative, and moreover $e_{p,q}$ is positive.

We shall argue by superposition. Assume that $e_{p,q}$ is acting alone (i.e., all other voltage sources set equal to zero). Then, $\mathcal{V}_p$ (and $\mathcal{V}_q$) is shorted into a single ordinary 0-node n_p (respectively, n_q). The resulting network $\mathbf{M}'_{p,q}$ has the pure voltage source $e_{p,q}$ connected between n_p and n_q and is an α-network ($\alpha < \beta$) with finitely many α-subsections.

We are free to assume that n_p and n_q are of rank β. Indeed, one-ended $(\beta - 1)$-paths could be appended to n_p and n_q through their $(\beta - 1)$-tips with those paths being otherwise totally disjoint from each other and from $\mathbf{M}'_{p,q}$. Those paths will carry no currents and will not alter the voltage-current regimes in $\mathbf{M}'_{p,q}$. In this way, $\mathbf{M}'_{p,q}$ can be viewed as a union of finitely many $(\beta-)$-subsections.

So, by Property 8.1(a2), all nodes of $\mathbf{M}'_{p,q}$ other than n_p and n_q have node voltages strictly larger than that at n_p and strictly less than that at n_q. Let $\mathbf{C}'_q$ be the set of branches of $\mathbf{M}_{p,q}$ that are incident to $\mathcal{V}_q$. It follows that in $\mathbf{M}'_{p,q}$ the current in each such branch is nonzero and directed away from n_q.

Next, let the voltage source e connected between $n_{p,k}^0$ and $n_{p,1}^0$ act alone. Virtually the same argument leads to the same conclusion concerning the currents in the branches of $\mathbf{C}'_q$ so long as $e \neq 0$. If $e = 0$, those currents are 0.

On the other hand, let e' be the voltage source connected between the nodes $n^0_{q,1}$ and $n^0_{q,k}$ of $\mathcal{V}_q$. When e' is acting alone and is not 0, we have positive currents flowing away from $n^0_{q,k}$ through the branches of $\mathbf{C}'_q$ incident to $n^0_{q,k}$ and positive currents flowing toward the other nodes of $\mathcal{V}_q$ through the remaining branches of $\mathbf{C}'_q$ as well as toward the nodes of $\mathcal{V}_p$ through the branches of $\mathbf{M}_{p,q}$ incident to $\mathcal{V}_p$. By Kirchhoff's current law applied at the shorts imposed to make e' act alone (that is, at the nodes of e' with the said shorts in place), the algebraic sum of the currents in the branches of $\mathbf{C'}_q$ measured away from $\mathcal{V}_q$ is positive if $e' \neq 0$ and is 0 if $e' = 0$.

By superposition, the algebraic sum of the currents in $\mathbf{C}'_q$ is positive. Altogether then, we can now conclude that, whether or not the bases $\mathcal{V}_p$ are void, Kirchhoff's current law will be violated at some cut $\mathbf{C}$ for n^β. ♣

We will use an inductive argument to establish Properties 8.1. We could start with the maximum principle for ordinary infinite networks as described in the first paragraph of this section. But, we won't. Instead, we will start at an even earlier stage and will thereby establish that ordinary maximum principle as well. This will be accomplished by taking each branch to be a more primitive kind of subsection, namely, a $(0-)$-subsection having no internal nodes, exactly two $(0-)$-tips, and two 0-nodes as its only bordering nodes. In fact, we can view each branch as being $(0-)$-connected to itself but not $(0-)$-connected to any other branch. In this way, the branch satisfies the definition of a subsection in a trivial way. Moreover, all of Properties 8.1 hold for that branch. In particular, part (b) holds vacuously.

To proceed. Assume that Properties 8.1 hold for β replaced by $\beta - 1$ and for all ranks lower than $\beta - 1$ as well. (Thus, if $\beta = 1$, we have that Properties 8.1 hold for $\beta - 1 = 0$, which means we are dealing with a $(0-)$-subsection, that is, a branch.) We will prove that Properties 8.1 also hold for β.

Clearly, Properties (a1) and (a2) are mutually exclusive for any arbitrarily chosen $(\beta-1)$-subsection $\mathbf{S}^{\beta-}_b$. Assuming (a1) does not hold, we prove that (a2) must hold. Let the largest rank among all the internal nodes of $\mathbf{S}^{\beta-}_b$ be α. Thus, $\alpha < \beta$. If $\alpha < \beta - 1$, then $\mathbf{S}^{\beta-}_b$ is also a $((\beta-1)-)$-subsection , and therefore (a2) holds by the inductive hypothesis (or trivially if $\beta = 1$, for then we have a $(0-)$-subsection, namely, a single branch). So, consider the case where $\alpha = \beta - 1$. Some or all of the $(\alpha-)$-subsections in $\mathbf{N}^\mu$ partition $\mathbf{S}_b^{\beta-}$, and Properties 8.1 (with β replaced by α) hold for each of them — by the inductive hypothesis again. In fact, (a2) holds for at least one of them since (a1) does not hold for $\mathbf{S}_b^{\beta-}$.

Let n_a^α be any arbitrarily chosen internal α-node of $\mathbf{S}_b^{\beta-}$. There will be another node n_b, which is either an internal α-node of $\mathbf{S}_b^{\beta-}$ or is a bordering $(\beta+)$-node of $\mathbf{S}_b^{\beta-}$, such that the voltage at n_b differs from the voltage u_a^α of n_a^α. We can choose an α-path that terminates at n_a^α and meets n_b and lies in the core of $\mathbf{S}_b^{\beta-}$ except possibly for a last branch incident to n_b (Lemma 2.2(iii)). Upon tracing along that path starting from n_a^α, we will find an internal α-node n_0^α (possibly n_a^α itself) with the same voltage as n_a^α but α-adjacent to either n_b or to an internal α-node with a voltage different from u_a^α. Let $\mathbf{S}_u$ be the union of all the $(\alpha-)$-subsections that are incident to those two α-adjacent nodes with differing voltages. According to Condition 2.1(f) — or Condition 2.1(a) if $\alpha- = 0-$, $\mathbf{S}_u$ is a finite union and has only finitely many incident $(\alpha+)$-nodes. By the inductive hypothesis again, (a2) holds for each member of $\mathbf{S}_u$. Let $n_x^{\alpha+}$ be an $(\alpha+)$-node incident to $\mathbf{S}_u$ with the largest voltage $u_x^{\alpha+}$ as compared to the other $(\alpha+)$-nodes incident to $\mathbf{S}_u$. We have $u_x^{\alpha+} \geq u_a^\alpha$. If $n_x^{\alpha+}$ is a bordering node of $\mathbf{S}_b^{\beta-}$, then either $u_x^{\alpha+} > u_a^\alpha = u_0^\alpha$ or $u_x^{\alpha+} = u_a^\alpha = u_0^\alpha$. In the latter case, the internal node n_0^α will be α-adjacent to another $(\alpha+)$-node with a voltage less than u_0^α. All this implies that either there is a bordering node of $\mathbf{S}_b^{\beta-}$ with a voltage larger than u_a^α or there is an internal α-node n_1^α with $u_1^\alpha \geq u_a^\alpha$ and with n_1^α incident to an $(\alpha-)$-subsection whose internal node voltages are strictly less than u_1^α.

Now, consider the set $\mathcal{N}_1^{\alpha+}$ of all the $(\alpha+)$-nodes that are α-adjacent to n_1^α. At least one node of $\mathcal{N}_1^{\alpha+}$ must have a voltage larger than u_1^α, for otherwise Kirchhoff's current law would be violated at a cut for n_1^α according to Lemma 8.2 (with β replaced by $\alpha = \beta - 1$). Thus, either we have a bordering node for $\mathcal{S}_b^{\beta-}$ in $\mathcal{N}_1^{\alpha+}$ with a voltage larger than u_1^α or, failing that, we can select an internal α-node $n_2^\alpha \in \mathcal{N}_1^{\alpha+}$ with a voltage u_2^α no less than the voltages at all the other α-nodes α-adjacent to n_1^α and with $u_2^\alpha > u_1^\alpha \geq u_a^\alpha$. Note also that we can connect n_1^α and n_2^α by an $(\alpha-)$-path since they are α-adjacent.

In the case where we have selected n_2^α, we can invoke the inductive hypothesis (a2) and Lemma 8.2 again to deduce that either there is a bordering node with a voltage larger than

u_2^α or there is an internal α-node n_3^α with the following properties: n_3^α is α-adjacent to n_2^α but not to n_1^α; $u_3^\alpha > u_2^\alpha > u_1^\alpha \geq u_a^\alpha$; u_3^α is the largest voltage for all the α-nodes that are α-adjacent to n_2^α. Here too, we can connect n_2^α to n_3^α by an $(\alpha-)$-path. This yields an α-path from n_1^α to n_3^α.

Further repetitions of this argument generate an α-path P^α whose α-node voltages are no less than u_a^α and strictly increasing when ordered according to a tracing that starts at n_1^α. Either P^α is finite and terminates at a bordering node of $\mathbf{S}_b^{\beta-}$ with a voltage larger than u_a^α or it is one-ended. In the latter case, we know from Lemma 5.5 that P^α will eventually lie in every arbitrarily chosen arm for the bordering $(\beta+)$-node $n^{\beta+}$ $(\beta > \alpha)$ that embraces the α-tip of P^α. However, we cannot yet assert that the α-node voltages of P^α converge to the voltage $u^{\beta+}$ of $n^{\beta+}$ because P^α may not be perceptible. Nor can we use Property 8.1(b) yet because our inductive hypothesis has it that Property 8.1(b) holds for β replaced by $\beta - 1$, that is, for an α less than $\beta - 1$. We need that property for $\alpha = \beta - 1$.

So, as our next step, we show that Property 8.1(b) holds for $\alpha = \beta - 1$. Choose a perceptible contraction $\{\mathcal{W}_p\}_{p=1}^\infty$ to $n^{\beta+}$ and let $\{\mathbf{A}_p\}_{p=1}^\infty$ be the corresponding sequence of arms. By Lemma 5.5, P^α eventually lies in $\mathbf{A}_1$. Let $\mathbf{M}_p = \mathbf{A}_p \backslash \mathbf{A}_{p+1}$. By Definition 4.1, $\mathbf{M}_p$ is not void. We also have that, for $p > 1$, $\mathcal{V}_p$ separates $\mathbf{M}_{p-1}$ and $\mathbf{M}_p$. Moreover, $\mathbf{M}_p$ has only finitely many α-nodes (perhaps, none at all). Indeed, if it had infinitely many, we could invoke Lemma 3.6 with μ replaced by α to conclude that $\mathbf{M}_p$ has an α-tip and thereby a β-node. This would violate the fact that $\mathcal{W}_1$ separates $n^{\beta+}$ from all other $(\beta+)$-nodes.

As in the proof of Lemma 8.2, $\mathbf{M}_p$ can be viewed as being a finite union of $(\beta-)$-subsections in some expanded μ-network with the bordering nodes of $\mathbf{M}_p$ being $(\beta+)$-nodes that embrace the nodes of $\mathcal{V}_p \cup \mathcal{V}_{p+1}$. Indeed, we can append finitely many pure voltage sources to the nodes of $\mathcal{V}_p \cup \mathcal{V}_{p+1}$ to produce the same voltage-current regime in $\mathbf{M}_p$ as it has as a part of $\mathbf{N}_e^\mu$. Moreover, we can append finitely many, one-ended $(\beta - 1)$-paths to the nodes of $\mathcal{V}_p \cup \mathcal{V}_{p+1}$ through their $(\beta - 1)-$tips and with those paths being otherwise totally disjoint from each other and from $\mathcal{M}_p$. The appending of those paths will not alter the voltage-current regime in $\mathbf{M}_p$ and will yield a larger, finitely structured μ-network.

Since $\mathbf{M}_p$ has only finitely many α-nodes, the argument we have already constructed for

$\mathbf{S}_b^{\beta-}$ can be applied to $\mathbf{M}_p$. This will lead to a finite α-path that must terminate at a node of $\mathcal{V}_p \cup \mathcal{V}_{p+1}$, and the conclusion is that every α-node voltage for $\mathbf{M}_p$ is no larger than the largest voltage for the nodes of $\mathcal{V}_p \cup \mathcal{V}_{p+1}$. By a similar argument "no larger" and "largest" can be replaced by "no less" and "least".

Now, let $u_{max,p}$ and $u_{min,p}$ be respectively the largest and least voltage for the nodes of $\mathcal{V}_p$. There are only finitely many contraction paths for the chosen contraction, each being perceptible, and every node of $\mathcal{V}_p$ lies on a perceptible contraction path. It follows from Lemma 7.5 and Equation (4) that the voltages at the various $\mathcal{V}_p$ along any contraction path converge to the voltage $u^{\beta+}$ at $n^{\beta+}$ and that therefore $u_{max,p} \to u^{\beta+}$ and $u_{min,p} \to u^{\beta+}$ as $p \to \infty$.

As a consequence of the last two paragraphs, the α-node voltages along the path P^α converge to $u^{\beta+}$. This conclusion extends immediately to the voltages along any sequence of nodes embraced by P_α and approaching $n^{\beta+}$. Indeed, all but finitely many of those nodes will lie in $\cup_{p=1}^{\infty}\mathbf{M}_p$. If one of those nodes n^δ is of rank δ ($0 \leq \delta < \alpha$) and lies in $\mathbf{M}_p$, its voltage u^δ will be no larger (no less) than the largest (respectively, least) bordering voltage for the ($\alpha-$)-subsection in which n^δ resides — according to our inductive hypothesis again. (In the same way as before, we can view the nodes of $\mathcal{V}_p \cup \mathcal{V}_{p+1}$ as being α-nodes.) By what we have shown above, this in turn implies that

$$\min(u_{min,p-1}, u_{min,p}) \;\leq\; u^\delta \;\leq\; \max(u_{max,p-1}, u_{max,p}).$$

Our asserted extension follows.

Altogether then, we have established inductively that Property 8.1(b) holds for $\alpha = \beta - 1$.

We can now complete our proof of Property 8.1(a2) under the assumption that (a1) does not hold. We now have it that, in the event that P^α is a one-ended α-path, the α-node voltages along P^α converge to the node voltage $u^{\beta+}$ at $n^{\beta+}$. Consequently, $u^{\beta+} > u_a^\alpha$. Since n_a^α was arbitrarily chosen as an internal α-node of $\mathbf{S}_b^{\beta-}$, it now follows that the voltages at every internal α-node is strictly less than the maximum voltage u_{max} for the bordering ($\beta+$)-nodes of $\mathbf{S}_b^{\beta-}$. By the inductive hypothesis, this is also true for every internal node, whatever be its rank. A similar argument allows us to replace "strictly less" by "strictly greater" and "maximum voltage u_{max}" by "minimum voltage u_{min}". Whence Property

8.1(a2).

The next theorem summarizes what has so far been established. Remember that $\mathbf{N}^\mu$ is sourceless and finitely structured (Definition 6.1), that $\mathbf{N}_e^\mu$ is $\mathbf{N}^\mu$ with finitely many pure voltage sources appended, and that the voltage-current regime is determined by Theorem 7.4 along with superposition if there are more than one pure voltage source.

Theorem 8.3. *For any $\beta = 1, \ldots, \mu$, let $\mathbf{S}_b^{\beta-}$ be a $(\beta-)$-subsection of $\mathbf{N}^\mu$ and let it be sourceless (i.e., none of its internal nodes is incident to a source branch of $\mathbf{N}_e^\mu$). Then, Properties 8.1 hold for it.*

Corollary 8.4. *Under the hypothesis of Theorem 8.3, assume also that $\mathbf{S}_b^{\beta-}$ has only one incident $(\beta+)$-node $n^{\beta+}$. Then, the node voltages for $\mathbf{S}_b^{\beta-}$ are all the same, namely, $u^{\beta+}$.*

Corollary 8.5. *Let $\mathbf{N}_e^\mu$ now denote $\mathbf{N}^\mu$ with exactly one pure voltage source appended to any two nodes. Let that source's value be 1 V and let the negative terminal of the source be ground with a node voltage of 0. Then, the voltage at every node of $\mathbf{N}_e^\mu$ is no less than 0 and no larger than 1.*

Proof. Let n_e and n_g be the two maximal nodes to which the 1 V source is appended, with n_g being the ground node. As we have done before, we can append one-ended μ-paths to n_e and n_g through their μ-tips in such a fashion that the μ-paths are otherwise totally disjoint from each other and from $\mathbf{N}_e^\mu$. This yields a larger finitely structured $(\mu + 1)$-network $\mathbf{N}_t^{\mu+1}$. The result of this trick is that $\mathbf{N}_e^\mu$ becomes a finite union of sourceless $((\mu + 1)-)$-subsections inside $\mathbf{N}_t^{\mu+1}$. Moreover, each such subsection has no more than two bordering nodes, namely, one or both of the two $(\mu + 1)$ nodes that embrace n_e and n_g.

We may now apply Theorem 8.3 with μ replaced by $\mu+1$ and $\mathbf{N}^\mu$ replaced by $\mathbf{N}_t^{\mu+1}$. Our conclusion then follows from Property 8.1(a) as applied to each of the $((\mu+1)-)$-subsections that comprise $\mathbf{N}_e^\mu$. ♣

The next corollary sharpens the last one. Let n_e and n_g be as stated in the first sentence of the last proof.

Corollary 8.6. *Assume the hypothesis of Corollary 8.5 and let n_0 be another maximal node of $\mathbf{N}_e^\mu$ different from n_e and n_g.*

(i) *If there is a path P in $\mathbf{N}_e^\mu$ that meets n_0 and n_g but does not meet n_e, then $u_0 < 1$.*

(ii) *If there is a path P in $\mathbf{N}_e^\mu$ that meets n_0 and n_e but does not meet n_g, then $u_0 > 0$.*

Proof. By means of the trick used in the last proof, we can take n_e to be embraced by a $(\mu+1)$-node, and similarly for n_g. This makes $\mathbf{N}_e^\mu$ a finite union of sourceless $((\mu+1)-)$-subsections in a larger $(\mu+1)$-network satisfying Corollary 8.5 (with appropriate rewording — remember that we have now established Corollary 8.5 for any natural number μ). Each such subsection will have one or both of n_e and n_g as its only bordering nodes. Moreover, the path P will lie within one of those subsections.

Under the hypothesis of (i), suppose $u_0 = 1$. Upon tracing P starting at n_0, we will meet a node with a voltage less than 1. Two cases arise:

Case 1. We meet an ordinary 0-node n_1^0 with a voltage equal to 1 and adjacent to a 0-node with a voltage less than 1. By Kirchhoff's current law, there must be another 0-node in $\mathbf{N}_e^\mu$ that is 0-adjacent to n_1^0 and has a voltage greater than 1. This violates Corollary 8.5.

Case 2. We meet a maximal δ-node n_1^δ with $0 < \delta \leq \mu+1$, $u_1^\delta = 1$, and n_1^δ incident to a $(\delta-)$-subsection having at least some of its internal or bordering node voltages less than 1. By Property 8.1(a2), at least one of the bordering node voltages of that $(\delta-)$-subsection will be less than 1. By Lemma 8.2, if Kirchhoff's law is to be satisfied at a cut that isolates n_1^δ, there must be another $(\delta+)$-node that is δ-adjacent to n_1^δ and has a voltage larger than 1. Again Corollary 8.5 is violated.

These two are the only possible cases. Hence, our supposition is false, and (i) is true. (ii) is established similarly. ♣

9 Transfinite Deterministic Walks

A *0-walk* W^0 is a walk of the conventional sort; it is an alternating sequence of 0-nodes n_m^0 and branches b_m:

$$W^0 = \{\ldots, n_m^0, b_m, n_{m+1}^0, b_{m+1}, \ldots\} \qquad (10)$$

such that the following are satisfied.

(i) For each m, b_m is incident to both n_m^0 and n_{m+1}^0.

(ii) If the sequence terminates on either side, it terminates at a 0-node.

(iii) Other than the terminal nodes, the nodes of (10) are ordinary. (However, the terminal nodes may be embraced by nodes of higher ranks.)

That the nonterminal nodes of (10) are ordinary implies that W^0 is restricted to the interior of a 0-subsection except possibly terminally. Furthermore, since there are no self-loops, the adjacent 0-nodes n_m^0 and n_{m+1}^0 are different for each m. Except for this, the elements of W^0 may repeat.

Some terminology regarding a 0-walk W^0: We may refer to (10) as a *deterministic* 0-walk in order to distinguish it from a random 0-walk, which will be discussed in the next section. W^0 is called *nontrivial* if it has at least one branch. We say that W^0 *embraces* itself, all its elements, all elementary tips embraced by its nodes, and all its subsequences that are 0-walks by themselves. W^0 may either be *finite* with two terminal nodes, or *one-ended* with exactly one terminal node, or *endless* without any terminal node. When W^0 has a terminal node, we say that W^0 *starts at* (*stops at*) its terminal node on the left (respectively, on the right). We say that W^0 *leaves* a terminal node if it starts at that node. We also say that W^0 *meets* or *reaches* each of its elements and *passes through* each of its elements other than any terminal node. However, a 0-walk cannot pass through a β-node n^β; it can only reach n^β either by terminating at a 0-node embraced by n^β or by proceeding infinitely through an arm for n^β.

To be more precise about this last idea of "reaching through an arm," let us choose a contraction $\{\mathcal{W}_p\}_{p=1}^{\infty}$ to a maximal β-node n^β and let $\{\mathbf{A}_p\}_{p=1}^{\infty}$ be the corresponding sequence of arms. Also, let us denote one-ended parts of a one-ended or endless 0-walk W^0 by

$$W_{-\infty,m}^0 \;=\; \{\ldots, b_{m-2}, n_{m-1}^0, b_{m-1}, n_m^0\}$$

and

$$W_{m,\infty}^0 \;=\; \{n_m^0, b_m, n_{m+1}^0, b_{m+1}, \ldots\}.$$

We say that W^0 *starts at* or *leaves* (*stops at*) n^β if either W^0 terminates on the left (respectively, on the right) at a 0-node embraced by n^β or, for every natural number q, there is an m depending on q such that $W^0_{-\infty,m}$ (respectively, $W^0_{m,\infty}$) remains within $\mathbf{A}_q$. In both cases, we also say that W^0 *meets* n^β or synonymously *reaches* n^β. In the latter case, we say that W^0 *eventually lies in* $\mathbf{A}_q$ and reaches n^β *through* $\mathbf{A}_q$.

This definition of starting and stopping does not depend upon the choice of the contraction. Indeed, let $\{\mathbf{A}_p\}_{p=1}^\infty$ and $\{\mathbf{A}'_p\}_{p=1}^\infty$ be two sequences of arms corresponding to two choices of the contraction. Then, given any q, we can find an r such that $\mathbf{A}'_r \subset \mathbf{A}_q$. This is so because of Definition 4.1(a) and the fact that there are only finitely many branches incident to the base $\mathcal{V}_q$ of $\mathbf{A}_q$. Thus, we can choose r so large that none of those branches are in $\mathbf{A}'_r$. But, this implies that all of $\mathbf{A}'_r$ is in $\mathbf{A}_q$, for otherwise $\mathcal{W}_q$ would not separate n^β from the complement $\tilde{\mathbf{A}}_q$ of $\mathbf{A}_q$. Hence, to insure that W^0 eventually lies in $\mathbf{A}_q$, we need merely ascertain that it eventually lies in $\mathbf{A}'_r$ for some sufficiently large r — whence the asserted independence from the choice of the contraction.

With 0-walks in hand, we can define β-walks recursively. A β-walk W^β is a (finite, one-ended, or endless) alternating sequence:

$$W^\beta = \{\ldots, n_m^\beta, W_m^{\beta-}, n_{m+1}^\beta, W_{m+1}^{\beta-}, \ldots\} \tag{11}$$

where, for every m, the following conditions hold:

(i) n_m^β is a maximal β-node — except that, if (11) terminates on the left or on the right, the terminal element is a node of rank β or less and need not be maximal (that is, the terminal node may be embraced by a node of any higher rank, perhaps higher than β).

(ii) $W_m^{\beta-}$ denotes a "nontrivial" α-walk, where $0 \le \alpha < \beta$.

(iii) Finally, $W_m^{\beta-}$ "starts at"' the node on its left and "stops at" the node on its right.

To complete this recursive definition of a β-walk, we have to define "starts at," "stops at," and "nontrivial" for a β-walk. This has already been done for a 0-walk.

First of all, if W^β terminates on the left (right), we say that W^β *starts at* or *leaves* (respectively, *stops at*) its terminal node. Next, we define one-ended portions of a one-ended or endless β-walk W^β as follows.

$$W^\beta_{-\infty,m} = \{\ldots, W^{\beta-}_{m-2}, n^\beta_{m-1}, W^{\beta-}_{m-1}, n^\beta_m\}$$

$$W^\beta_{m,\infty} = \{n^\beta_m, W^{\beta-}_m, n^\beta_{m+1}, W^{\beta-}_{m+1}, \ldots\}$$

Choose a contraction $\{\mathcal{W}_p\}_{p=1}^\infty$ to some maximal γ-node n^γ, where $\gamma > \beta$, and let $\{\mathbf{A}_p\}_{p=1}^\infty$ be the corresponding sequence of arms. We say that W^β *starts at* (*stops at*) n^γ if either $\mathcal{W}^\beta$ terminates on the left (respectively, right) at some node embraced by n^γ or, for every natural number q, there is an m depending on q such that $W^\beta_{-\infty,m}$ (respectively, $W^\beta_{m,\infty}$) remains within $\mathbf{A}_q$. In the latter case, we say that, for both starting and stopping at n^γ, W^β *eventually lies in* $\mathbf{A}_q$ and reaches n^γ *through* $\mathbf{A}_q$.

The same argument as that used for 0-walks shows that this definition of starting and stopping does not depend upon the choice of the contraction $\{\mathcal{W}_p\}_{p=1}^\infty$.

W^β is said to *embrace* itself, all its elements, all the elements embraced by its elements, and so forth down to all the elementary tips embraced by its embraced 0-nodes. It will also *embrace* walks that are subsequences of (11) or subsequences of embraced walks of lower ranks. Thus, "embrace" has the same meaning for walks as it does for paths and for transfinite graphs in general. Finally, W^β is called *nontrivial* if it embraces at least one branch. This completes our recursive definition of a β-walk.

Note that a β-walk W^β is perforce restricted to the interior of a $((\beta+1)-)$-subsection except possibly terminally. This is because every nonterminal β-node n^β_m in (11) is maximal and moreover, by our recursive definition, every other node embraced by W^β is also not embraced by a $((\beta+1)+)$-node except possibly for the terminal nodes of W^β.

Some more terminology: We say that W^β *meets* or *reaches* its embraced elements and the nodes at which it starts or stops. "*Passes through*" is also used in place of "meets" if "starting at" and "stopping at" do not apply. Thus, a β-walk cannot pass through a γ-node n^γ if $\gamma > \beta$; it can only start or stop at n^γ. Furthermore, the β-walk (11) is said to perform a *one-step β-transition* from n^β_m to n^β_{m+1}. Just as with β-paths [9, Section 4], a finite β-walk W^β with the terminal nodes n_a and n_b can be viewed as a three-element γ-walk $\{n_a, W^\beta, n_b\}$ for any $\gamma > \beta$.

We now define "roving." Note that a β-walk W^β will embrace walks of various ranks no larger than β. An α-walk embraced by W^β is called *maximal* if it cannot be extended within (11) into a longer α-walk.

Definition 9.2. We shall say that W^β *β-roves* if, for every maximal α-walk ($0 \leq \alpha \leq \beta$) embraced by W^β, the following conditions hold. (For the following statements, replace β by α in (11).)

(i) If n_m^α and n_{m+1}^α are nonterminal nodes in (11), then they are different maximal α-nodes.

(ii) If n_a is a terminal node on the left in (11), then the next node n_b in (11) is embraced by an ($\alpha+$)-node (n_b itself will do if n_b is of rank α), and n_a and that ($\alpha+$)-node are totally disjoint.

(iii) If n_c is a terminal node on the right in (11), it is embraced by an (($\alpha + 1$)+)-node, which is totally disjoint from the node that precedes n_c in (11). ♣

With regard to (ii), n_b can be of rank less than α and the embracing ($\alpha+$)-node can be of rank greater than α only if n_b is a terminal node on the right.

An informal way of stating this condition of roving is as follows. For each maximal α-walk embraced by (11), if the α-walk enters an ($\alpha-$)-subsection $\mathbf{S}_b^{\alpha-}$ from a bordering ($\alpha+$)-node, it continues on through $\mathbf{S}_b^{\alpha-}$ to meet a different ($\alpha+$)-node bordering $\mathbf{S}_b^{\alpha-}$. Also, if that maximal α-walk starts at an internal ($\alpha-$)-node of $\mathbf{S}_b^{\alpha-}$, it continues through $\mathbf{S}_b^{\alpha-}$ to meet an ($\alpha+$)-node bordering $\mathbf{S}_b^{\alpha-}$. Finally, the α-walk either continues indefinitely or it meets an (($\alpha+1$)+)-node, at which point that α-walk terminates (but then another α-walk may start at that (($\alpha + 1$)+)-node).

Note also that a 0-walk that either stops at a (1+)-node or continues without stopping automatically 0-*roves*; indeed, after leaving a 0-node n^0, it perforce meets another 0-node before returning to n^0.

Note still further that, if a β-walk β-roves, its embraced α-walks α-rove for every α less than β; we shall simply say that the *β-walk α-roves* for each α less than β.

10 Transfinite Random Walks — Outline of the Inductive Argument

Our objective throughout the rest of this work is to establish a theory for transfinite random walks that wander among nodes of various ranks in a finitely structured μ-network $\mathbf{N}^{\mu}$. It will be convenient at times to speak in terms of a *random walker* Ψ that performs the random walk. We assume throughout that Ψ adheres to the *nearest-neighbor rule* when leaving an ordinary 0-node n_0^0. That rule specifies the probabilities $P_{0,k}$ $(k = 1, \ldots, K)$ of one-step transitions from n_0^0 to the 0-nodes n_k^0 that are 0-adjacent to n_0^0. Specifically, with g_k denoting the conductance of the branch between n_0^0 and n_k^0, we have $P_{0,k} = g_k / \sum_{l=1}^{K} g_l$ as a definition. This probability can be measured electrically. Let n_k^0 be held at 1V and let all the other n_l^0 $(l = 1, \ldots, K; l \neq k)$ be held at 0 V. By Kirchhoff's laws and Ohm's law, the resulting voltage at n_0^0 is equal to $P_{0,k}$.

Our theory for transfinite random walks will be established inductively. With regard to random 0-walks, we will use a result obtained by Nash-Williams [3, Corollary 4A] in his study of a random walk on a locally finite 0-network under the nearest-neighbor rule. We can restate his result in a form more suitable for our purposes by noting the following two facts: (i) The shorting of finitely many 0-nodes does not disturb the local-finiteness property. (ii) By holding his "co-finite" set v at a single voltage, he in effect dealt with a finite 0-network in his Corollary 4A. Thus, we have the following *Nash-Williams rule*: Let $\mathcal{N}_e$ and $\mathcal{N}_g$ be two disjoint sets of nodes in a finite connected 0-network $\mathbf{N}^0$ and let n_s^0 be another node in $\mathbf{N}^0$ but not in $\mathcal{N}_e \cup \mathcal{N}_g$; then, the probability of Ψ reaching some node of $\mathcal{N}_e$ before reaching any node of $\mathcal{N}_g$, given that Ψ starts at n_s^0, is equal to the voltage at n_s^0 when the nodes of $\mathcal{N}_e$ are held at 1 V and the nodes of $\mathcal{N}_g$ are held at 0 V. (It happens to be a certainty that Ψ will eventually reach at least one node of $\mathcal{N}_e \cup \mathcal{N}_g$. We establish a more general result with Theorem 16.2 below.)

Note that the Nash-Williams rule becomes the nearest-neighbor rule when we set $n_s^0 = n_0^0$, $\mathcal{N}_e = \{n_k^0\}$, and $\mathcal{N}_g = \{n_l^0 : l = 1, \ldots, K; l \neq k\}$.

A first step of generalization, namely, from random 0-walks to random 1-walks was made in [10] in the following way. We started with a truncation of a 0-subsection $\mathbf{S}_b^0$

in a 1-network N^1. That truncation was along isolating sets in $\mathbf{S}_b^0$, one such set for each bordering 1-node of $\mathbf{S}_b^0$. The Nash-Williams rule gave relative probabilities of reaching those isolating sets for a random walker Ψ starting at some node within the truncation. Then, by replacing the isolating sets by contractions, we expanded the truncation to fill out $\mathbf{S}_b^0$ — and through a limiting process obtained relative probabilities of transitions to the bordering 1-nodes of $\mathbf{S}_b^0$. Next, through a similar limiting process, we obtained relative probabilities of transitions from a 1-node n_0^1 to the 1-nodes that are 1-adjacent to n_0^1. The latter comprise a generalization to 1-nodes for the nearest-neighbor rule. However, it required that the admissible 1-walks be restricted to those that 1-rove. Finally, by examining the Markov chain corresponding to a 1-roving random 1-walk, we were able to extend the Nash-Williams rule to the wanderings of Ψ through a finitely structured 1-network.

We shall inductively extend all this to higher ranks of random roving walks by defining the probabilities of transitions among nodes of higher ranks in a fashion consistent with the definitions for the lower ranks. We will show that the defined probabilities arise as limiting cases of prior generalizations. This however does not eliminate the need for those definitions because transfinite random walks whose transition probabilities are not such continuous extensions may be conceivable. In short, we are using the nearest-neighbor rule coupled with the Nash-Williams rule as the paradigm for our transfinite random walks and are thereby basing those walks on the theory of transfinite electrical networks.

We will restrict the kinds of walks that the random walker Ψ is permitted to make by saying that Ψ *roves*. This will mean that, whatever walk Ψ follows, that walk α-roves for every α no larger than the rank of the walk. The reason for — and the feasibility of — this restriction will be explained in Section 13.

It will be convenient to use a certain concise notation for relative probabilities of transitions. Consider a random walker Ψ roving through $\mathbf{N}^\mu$. We say that Ψ *reaches a nodal set* $\mathcal{N}$ if Ψ reaches any node of $\mathcal{N}$. Let $\mathcal{N}_e$ and $\mathcal{N}_g$ be two disjoint finite sets of maximal nodes in $\mathbf{N}^\mu$ and let n_s be another maximal node of $\mathbf{N}^\mu$ that is not in $\mathcal{N}_e \cup \mathcal{N}_g$. Then,

$$Prob(\mathrm{s}n_s, \mathrm{r}\mathcal{N}_e, \mathrm{b}\mathcal{N}_g) \tag{12}$$

will denote the probability that Ψ, having started at n_s, will reach $\mathcal{N}_e$ before reaching $\mathcal{N}_g$.

There is a tacit condition regarding (10.1), namely, that Ψ truly reaches $\mathcal{N}_e \cup \mathcal{N}_g$. This will always be so whenever we use (10.1) because of the roving of Ψ and the way n_s, $\mathcal{N}_e$, and $\mathcal{N}_g$ are chosen.

Let us now explain the inductive assumption (Rule 10.1 below) upon which our arguments will be based. As always, we assume that the μ-network $\mathbf{N}^\mu$ is sourceless and finitely structured (Definition 6.1).

With the natural number β fixed with $0 < \beta \leq \mu$, consider any $(\beta-)$-subsection $\mathbf{S}_b^\alpha$ of essential rank α; thus, $0 \leq \alpha < \beta$. Let $n_k^{\beta+}$ $(k = 1, \ldots, K)$ be the bordering nodes of $\mathbf{S}_b^\alpha$. Choose a perceptible contraction $\{\mathcal{W}_{k,p_k}\}_{p_k=1}^\infty$ within $\mathbf{S}_b^\alpha$ for each $n_k^{\beta+}$. We define $\mathbf{F}(p_1, \ldots, p_K)$ as the reduction of $\mathbf{S}_b^\alpha$ induced by all branches of $\mathbf{S}_b^\alpha$ that do not reside in the arms $\mathbf{A}_{1,p_1}, \ldots, \mathbf{A}_{K,p_K}$ corresponding to the isolating sets $\mathcal{W}_{1,p_1}, \ldots, \mathcal{W}_{K,p_K}$. This is illustrated in Figure 3. For sufficiently large $p_1, \ldots, p_K$, the rank of $\mathbf{F}(p_1, \ldots, p_K)$ will be α (Lemma 4.2). $\mathbf{F}(p_1, \ldots, p_K)$ may be incident to a bordering node of $\mathbf{S}_b^\alpha$ through (at most finitely many) branches, but it will possess only the 0-node embraced by that bordering node — not the bordering node itself.

Furthermore, $\mathbf{F}(p_1, \ldots, p_K)$ will be α-connected so long as the arms $\mathbf{A}_{k,p_k}$ are chosen small enough (i.e., the p_k are chosen large enough). Indeed, if this were not so, some branches of some cuts for the bordering nodes of $\mathbf{S}_b^\alpha$ would not be connected through paths that do not meet those bordering nodes.

When the p_k are chosen so large that $\mathbf{F}(p_1, \ldots, p_K)$ is of rank α and is α-connected, we call $\mathbf{F}(p_1, \ldots, p_K)$ a *truncation of* $\mathbf{S}_b^\alpha$.

Note also that the truncation $\mathbf{F}(p_1, \ldots, p_K)$ will have only finitely many α-nodes, for otherwise it would be incident to an $(\alpha + 1)$-node according to Lemma 3.6; the later possibility was eliminated by the deletion of the arms $\mathbf{A}_{k,p_k}$. Altogether then, we have it that $\mathbf{F}(p_1, \ldots, p_K)$ is finitely structured as an α-network.

We wish to apply a generalized form of the Nash-Williams rule to a truncation $\mathbf{F}(p_1, \ldots, p_K)$ of the $(\beta-)$-subsection $\mathbf{S}_b^\alpha$. In doing so, we shall say that a nodal set $\mathcal{N}$ is held at a voltage u if every node of $\mathcal{N}$ is held at u.

Rule 10.1. *Let $\mathcal{N}_e$ and $\mathcal{N}_g$ be two disjoint finite sets of maximal nodes in the truncation*

$\mathbf{F}(p_1, \ldots, p_K)$ *such that* $\mathcal{N}_e \cup \mathcal{N}_g$ *contains* $\cup_{k=1}^{K} \mathcal{W}_{k,p_k}$ *for given* $p_1, \ldots, p_K$. *Also, let* n_s *be a maximal node in* $\mathbf{F}(p_1, \ldots, p_K)$ *that is not in* $\mathcal{N}_e \cup \mathcal{N}_g$. *The probability (12) of* Ψ *reaching* $\mathcal{N}_e$ *before reaching* $\mathcal{N}_g$, *given that* Ψ *starts from* n_s *and* β*-roves, is the voltage* v_s *at* n_s *when* $\mathcal{N}_e$ *is held at 1 V and* $\mathcal{N}_g$ *is held at 0 V.*

Note that the condition that Ψ β-roves insures that Ψ reaches $\mathcal{N}_e \cup \mathcal{N}_g$ because $\cup_{k=1}^{K} \mathcal{W}_{k,p_k}$ separates n_s from the bordering nodes of $\mathcal{S}_b^{\alpha}$ and because the β-roving of Ψ insures that Ψ reaches one of those bordering nodes. Actually, for fixed $p_1, \ldots, p_K$, the assumption of β-roving for Ψ can be weakened to α-roving; this will be established in Section 16. In the next section we will send the p_k to infinity ($p_k \to \infty$), in which case β-roving will be needed.

Rule 10.1, when restricted to a truncation of a 0-section, is exactly the Nash-Williams rule, which in turn encompasses the nearest-neighbor rule as a special case. The rest of this paper is aimed at showing that, Rule 10.1 can be extended through a series of limiting processes to any truncation of any β-subsection whenever it holds for truncations of $(\beta-)$-subsections. This will inductively extend Rule 10.1 to all truncations of all subsections.

Before leaving this section, let us check the consistency of Rule 10.1 for the following case. Let n_s, $\mathcal{N}_e$, and $\mathcal{N}_g$ be as stated, and let $\mathcal{M}$ be another finite set of maximal nodes in $\mathbf{F}(p_1, \ldots, p_K)$ that separates n_s from $\mathcal{N}_e \cup \mathcal{N}_g$ with $n_s \notin \mathcal{M}$ and $\mathcal{M} \cap (\mathcal{N}_e \cup \mathcal{N}_g) = \emptyset$. Assume that Ψ, after starting from n_s, reaches $\mathcal{N}_e \cup \mathcal{N}_g$. Then, Ψ must meet at least one node of $\mathcal{M}$ before reaching $\mathcal{N}_e \cup \mathcal{N}_g$. Let m_i ($i = 1, \ldots, I$) be the nodes of $\mathcal{M}$. By conditional probabilities, we should have

$$Prob(\mathrm{s}n_s, \mathrm{r}\mathcal{N}_e, \mathrm{b}\mathcal{N}_g) \;=\sum_{i=1}^{I} Prob(\mathrm{s}n_s, \mathrm{r}m_i, \mathrm{b}(\mathcal{M}\backslash\{m_i\}))Prob(\mathrm{s}m_i, \mathrm{r}\mathcal{N}_e, \mathrm{b}\mathcal{N}_g). \qquad (13)$$

This equation can be established electrically.

Let u_s (and u_{m_i}) be the voltage at n_s (respectively, at m_i) when $\mathcal{N}_e$ is held at 1 V and $\mathcal{N}_g$ is held at 0 V. Also, let $v_s(i)$ be the voltage at n_s when m_i is held at 1 V and $\mathcal{M}\backslash\{m_i\}$ is held at 0 V. By the superposition principle for electrical networks.

$$u_s \;=\; \sum_{i=1}^{I} v_s(i) u_{m_i}. \qquad (14)$$

By Rule 10.1, the voltages in (14) correspond to the probabilities in (13). This verifies (13).

11 Reaching a Bordering Node

As before, let $\mathbf{S}_b^\alpha$ be any $(\beta-)$-subsection; α is its essential rank. We wish to obtain relative probabilities of transitions from an internal maximal node n_s of $\mathbf{S}_b^\alpha$ to the various $(\beta+)$-nodes $n_k^{\beta+}$ $(k = 1, \ldots, K)$ bordering $\mathbf{S}_b^\alpha$ — and possibly to other internal nodes of $\mathbf{S}_b^\alpha$ as well. (For an illustration, see Figure 3 again.) Let $\{\mathcal{W}_{k,p_k}\}_{p_k=1}^{\infty}$ and $\mathbf{F}(p_1, \ldots, p_K)$ be as in the preceding section. However, with regard to the nodal sets $\mathcal{N}_e$ and $\mathcal{N}_g$ of Rule 10.1, we now impose the additional condition that, for each k, $\mathcal{W}_{k,p_k}$ lies entirely within $\mathcal{N}_e$ or alternatively entirely within $\mathcal{N}_g$ — and stays therein as $p_k \to \infty$.

Let $\mathcal{M}_e$ be the set obtained by deleting from $\mathcal{N}_e$ all nodes of every set $\mathcal{W}_{k,p_k}$ contained in $\mathcal{N}_e$ and adding in their place $n_k^{\beta+}$. Also, construct $\mathcal{M}_g$ from $\mathcal{N}_g$ in the same way. (All the $\mathcal{W}_{k,p_k}$ may be in $\mathcal{N}_e$, in which case $\mathcal{M}_g = \mathcal{N}_g$; similarly, $\mathcal{M}_e = \mathcal{N}_e$ if all the $\mathcal{W}_{k,p_k}$ are in $\mathcal{N}_g$.) Also, replace $\mathbf{F}(p_1, \ldots, p_K)$ of Rule 10.1 by $\mathbf{S}_b^\alpha$. Let u_s be the voltage at n_s when $\mathcal{M}_e$ is held at 1 V and $\mathcal{M}_g$ is held at 0 V. Through a formal application of Rule 10.1, we have $u_s = Prob(\mathrm{s}n_s, \mathrm{r}\mathcal{M}_e, \mathrm{b}\mathcal{M}_g)$, a result we wish to obtain through a limiting process.

In the next lemma, we send $p_k \to \infty$ for every k. It is understood that $\mathcal{N}_e \cup \mathcal{N}_g$ is adjusted accordingly; that is, the nodes of $\mathcal{M}_e \cup \mathcal{M}_g$ that are not bordering nodes of $\mathbf{S}_b^\alpha$ are fixed nodes of $\mathcal{N}_e \cup \mathcal{N}_g$, and all other nodes of $\mathcal{N}_e \cup \mathcal{N}_g$ are those of the $\mathcal{W}_{k,p_k}$; as the p_k increase, the nodes in the bases $\mathcal{V}_{k,p_k}$ are changed as well.

Furthermore, let us denote the voltage v_s at n_s specified in Rule 10.1 by $v_s(p_1, \ldots, p_K)$ in order to display its dependence upon the p_k.

Lemma 11.1. *$v_s(p_1, \ldots, p_K)$ converges to u_s as the $p_1, \ldots, p_K$ tend to infinity independently.*

Proof. For each $k = 1, \ldots, K$, let $n_{k,p_k,i}^0$ denote the ith node in $\mathcal{W}_{k,p_k}$ and let $u_{k,p_k,i}$ denote the corresponding node voltage resulting from 1 V at $\mathcal{M}_e$ and 0 V at $\mathcal{M}_g$. Theorem 8.5 as applied to $\mathbf{S}_b^\alpha$ implies that $0 \leq u_{k,p_k,i} \leq 1$ for all k and i. By superposition. $v_s(p_1, \ldots, p_K) - u_s$ is the voltage at n_s resulting from the following application of node voltages for all k and i: $n_{k,p_k,i}^0$ is held at $1 - u_{k,p_k,i}$ if $n_k^{\beta+} \in \mathcal{M}_e$ and $n_{k,p_k,i}^0$ is held at $-u_{k,p_k,i}$ if $n_k^{\beta+} \in \mathcal{M}_g$. All other nodes of $\mathcal{M}_e$ and $\mathcal{M}_g$ are held at 0 V.

Now, let u_{max} be the maximum of all the voltages $1 - u_{k,p_k,i}$ at the nodes $n_{k,p_k,i}^0$ in the

$\mathcal{W}_{k,p_k}$ corresponding to all the $n_k^{\beta+} \in \mathcal{M}_e$, and let u_{min} be the minimum of all the voltages $-u_{k,p_k,i}$ at the nodes $n_{k,p_k,i}^0$ in the $\mathcal{W}_{k,p_k}$ corresponding to all the $n_k^{\beta+} \in \mathcal{M}_g$. We have $u_{max} \geq 0$ and $u_{min} \leq 0$. Moreover, u_{max} and u_{min} depend in general on $p_1, \ldots, p_K$. Since $\mathbf{F}(p_1, \ldots, p_K)$ of Rule 10.1 is a linear network, Theorem 8.5 as applied to $\mathbf{F}(p_1, \ldots, p_K)$ also implies that

$$u_{min} \;\leq\; v_s(p_1, \ldots, p_K) - u_s \;\leq\; u_{max}.$$

Recall that, for each fixed k but varying p_k, the cardinalities of the $\mathcal{W}_{k,p_k}$ are uniformly bounded (in fact, are no larger than $m+1$, where m is the number of contraction paths for the chosen contraction to $n_k^{\beta+}$). Hence, by Property 8.1(b), which holds in this case according to Theorem 8.3, we have that $u_{min} \to 0$ and $u_{max} \to 0$ as the $p_1, \ldots p_K$ tend to infinity independently. ♣

The last lemma immediately yields the following extension of Rule 10.1.

Theorem 11.2. *As the $p_k \to \infty$, Rule 10.1 extends continuously from truncations $\mathbf{F}(p_1, \ldots, p_K)$ of an α-subsection $\mathbf{S}_b^\alpha$ to all of $\mathbf{S}_b^\alpha$ and thereby yields the following result: Let n_s be an internal maximal node of $\mathbf{S}_b^\alpha$ and let $\mathcal{M}_e$ and $\mathcal{M}_g$ be finite disjoint nodal sets with $\mathcal{M}_e \cup \mathcal{M}_g$ containing all the maximal bordering nodes of $\mathbf{S}_b^\alpha$ and with $n_s \notin (\mathcal{M}_e \cup \mathcal{M}_g)$. Then, for a β-roving Ψ ($\beta > \alpha$), $Prob(\mathrm{s}n_s, \mathrm{r}\mathcal{M}_e, \mathrm{b}\mathcal{M}_g)$ is the voltage at n_s when $\mathcal{M}_e$ is held at 1 V and $\mathcal{M}_g$ is held at 0 V.*

We take this continuous extension of Rule 10.1 as the definition of the probabilities of transitions from an internal node of an α-subsection $\mathbf{S}_b^\alpha$ to the bordering ($\beta+$)-nodes of $\mathbf{S}_b^\alpha$.

The assumption that Ψ β-roves insures that Ψ, having started from any internal node of $\mathbf{S}_b^\alpha$, will reach a bordering node of $\mathbf{S}_b^\alpha$. Let us relax this assumption somewhat and examine the probability of Ψ reaching a bordering node when it only α-roves but does not necessarily β-rove. In this case, we shall call an α-subsection $\mathbf{S}_b^\alpha$ *transient* if Ψ, after starting from any arbitrarily chosen internal node n_s of $\mathbf{S}_b^\alpha$, always has a positive probability of reaching some bordering node of $\mathbf{S}_b^\alpha$ before returning to n_s.

Theorem 11.3. *Assume that Ψ α-roves but does not necessarily β-rove for any $\beta > \alpha$. Then, every α-subsection $\mathbf{S}_b^\alpha$ of $\mathbf{N}^\mu$ is transient.*

Proof. Let n_s be any internal maximal node of $\mathbf{S}_b^\alpha$ and assume that Ψ starts at n_s.

If n_s is an ordinary 0-node, then, by the nearest-neighbor rule and for each node n_a that is 0-adjacent to n_s, the probability is positive that Ψ will reach n_a in one 0-step. If n_a is embraced by an $((\alpha+1)+)$-node, then the theorem follows immediately. So, assume the latter is not the case when n_s is a 0-node.

On the other hand, if n_s is a maximal δ-node $(1 \leq \delta \leq \alpha)$, let $\mathcal{X}$ be a conjoining set for n_s. All the nodes of $\mathcal{X}$ are ordinary 0-nodes. It is a certainty that Ψ will reach $\mathcal{X}$ before returning to n_s because Ψ α-roves and because the rank of n_s is no larger than α. Choose $n_a \in \mathcal{X}$ arbitrarily and let all the other nodes of $\mathcal{X}$ comprise $\mathcal{N}_g$. There is a path (possibly just a single branch) that meets n_s and n_a and does not meet $\mathcal{N}_g$. Hence, by Rule 10.1 and by Corollary 8.6, $Prob(\mathrm{s}n_s, \mathrm{r}n_a, \mathrm{b}\mathcal{N}_g)$ is positive.

Now, consider both cases where either n_s is an ordinary 0-node with $\mathcal{X}$ being the set of nodes adjacent to n_s or n_s is a maximal δ-node $(1 \leq \delta \leq \alpha)$ with $\mathcal{X}$ being a conjoining set for n_s. In order to show that $\mathbf{S}_b^\alpha$ is transient, we need only show that $Prob(\mathrm{s}n_a, \mathrm{r}\mathcal{N}_e, \mathrm{b}n_s)$ is positive, where now $\mathcal{N}_e$ is the set of bordering nodes of $\mathbf{S}_b^\alpha$. By Rule 10.1 again — as extended by Theorem 11.2, the latter can be accomplished by showing that, for some choice of $n_a \in \mathcal{X}$, the voltage at n_a is positive when n_s is held at 0 V and all the bordering nodes of $\mathbf{S}_b^\alpha$ are held at 1 V.

Suppose there is no $n_a \in \mathcal{X}$ fulfilling the last condition. In view of Theorem 8.5, we suppose that all the voltages at the nodes of $\mathcal{X}$ are 0. Hence, the currents in all branches between n_s and $\mathcal{X}$ (i.e., in the arm corresponding to $\mathcal{X}$ — if such exists — and also in the branches incident to n_s) are 0. Let $\mathbf{i}$ be the current vector produced in $\mathbf{S}_b^\alpha$ when all the bordering nodes of $\mathbf{S}_b^\alpha$ are shorted and a pure voltage source e_0 of value 1 V is connected from n_s to that short to hold n_s at 0 V and the short at 1 V. $\mathbf{i}$ is determined by Theorem 7.4 as applied to $\mathbf{S}_b^\alpha$ with the appended source. Apply Kirchhoff's current law to n_s if n_s is an ordinary 0-node or to a cut (i.e., use (7.1)) at the isolating set corresponding to $\mathcal{X}$ if the rank of n_s is larger than 0. We obtain $i_0 = 0$, where i_0 is the current through e_0. We are free to set $\mathbf{s} = \mathbf{i}$ in Theorem 7.4. Therefore, $s_0 = i_0 = 0$, and $\sum r_j i_j^2 = 0$, where the summation is over the branches in $\mathbf{S}_b^\alpha$. This implies that $i_j = 0$ for all branches b_j in $\mathbf{S}_b^\alpha$. Consequently, there can be no voltage difference between any two nodes of $\mathbf{S}_b^\alpha$ — in contradiction to the presence of the 1 V source. ♣

The last theorem shows that no difficulty arises in making the further assumption that Ψ β-roves — at least so far as a transition from an internal node to a bordering node of a $(\beta-)$-subsection is concerned.

12 Leaving a β-Node

Having examined probabilities as Ψ reaches a bordering $(\beta+)$-node, let us now consider probabilities as Ψ leaves a β-node n_s^β. Let $\mathcal{N}_e$ and $\mathcal{N}_g$ be two finite sets of maximal $(\beta-)$-nodes such that $\mathcal{N}_e \cap \mathcal{N}_g = \emptyset$, $\mathcal{N}_e \cup \mathcal{N}_g$ separates n_s^β from all other $(\beta+)$-nodes, and n_s^β is totally disjoint from every node of $\mathcal{N}_e \cup \mathcal{N}_g$. (See Figure 4.) We can choose a perceptible contraction $\{\mathcal{W}_p\}_{p=1}^\infty$ to n_s^β such that no node of $\mathcal{N}_e \cup \mathcal{N}_g$ lies in the first arm $\mathbf{A}_1$ and thereby in any arm $\mathbf{A}_p$ of that contraction.

Next, let $\mathbf{M}$ be the reduced network induced by all the branches that are not separated from n_s^β by $\mathcal{N}_e \cup \mathcal{N}_g$. (In Figure 4, $\mathbf{M}$ is the reduced network lying within the ring of nodes comprising $\mathcal{N}_e \cup \mathcal{N}_g$.) The only node of $\mathbf{M}$ whose rank is no less than β is n_s^β. We wish to know $Prob(\mathrm{s}n_s^\beta, \mathrm{r}\mathcal{N}_e, \mathrm{b}\mathcal{N}_g)$. However, we cannot as yet use Rule 10.1 because our inductive hypothesis with respect to Rule 10.1 is presently assumed only for a starting node n_s of rank less than β. What we can do, however, is short the arm $\mathbf{A}_p$ to obtain a single 0-node n_p^0 and then examine $Prob(\mathrm{s}n_p^0, \mathrm{r}\mathcal{N}_e, \mathrm{b}\mathcal{N}_g)$. As $p \to \infty$, the $\mathcal{V}_p$ contract to n_s^p, and hopefully the corresponding $Prob(\mathrm{s}n_p^0, \mathrm{r}\mathcal{N}_e, \mathrm{b}\mathcal{N}_g)$ converge. (They will.) The limit can then be taken as the definition of $Prob(\mathrm{s}n_s^\beta, \mathrm{r}\mathcal{N}_e, \mathrm{b}\mathcal{N}_g)$.

To proceed. Let $\mathbf{M}_p$ be the network obtained from $\mathbf{M}$ by replacing every branch of $\mathbf{A}_p$ by a short. That shorting creates an ordinary 0-node n_p^0 and eliminates n_s^β. Thus, the essential rank of $\mathbf{M}_p$ is α, where $\alpha < \beta$. So, we can apply Rule 10.1 to $\mathbf{M}_p$ with $n_s = n_p^0$ to get $Prob(\mathrm{s}n_p^0, \mathrm{r}\mathcal{N}_e, \mathrm{b}\mathcal{N}_g)$ as the voltage v_p at n_p^0 when $\mathcal{N}_e$ is held at 1 V and $\mathcal{N}_g$ is held at 0 V. Let us refer to these imposed voltages at $\mathcal{N}_e$ and $\mathcal{N}_g$ as the *excitation* E. On the other hand, we can formally apply Rule 10.1 to $\mathbf{M}$ to get $Prob(\mathrm{s}n_s^\beta, \mathrm{r}\mathcal{N}_e, \mathrm{b}\mathcal{N}_g)$ as the voltage u_s^β at n_s^β under excitation E. In this section, we will show that $v_p - u_s^\beta$ as $p \to \infty$. This will prove that Rule 10.1 extends continuously to a formula for determining $Prob(\mathrm{s}n_s^\beta, \mathrm{r}\mathcal{N}_e, \mathrm{b}\mathcal{N}_g)$

with respect to $\mathbf{M}$.

The excitation E can be produced by a pure 1-V voltage source $e_0 = 1$ in a branch b_0 appended to $\mathbf{M}$ or $\mathbf{M}_p$; b_0 is connected from a short at $\mathcal{N}_g$ to a short at $\mathcal{N}_e$. Let $\mathbf{M}_a = \mathbf{M} \cup \{b_0\}$ and let $\mathbf{M}_{pa} = \mathbf{M}_p \cup \{b_0\}$. In accordance with Theorem 7.4, we obtain a unique voltage-current regime $\mathbf{v}, \mathbf{i}$ in $\mathbf{M}_a$ and another one $\mathbf{v}_p, \mathbf{i}_p$ in $\mathbf{M}_{pa}$. We have $\mathbf{i} \in \mathcal{K}_a$ (and $\mathbf{i}_p \in \mathcal{K}_{pa}$), where $\mathcal{K}_a$ (respectively, $\mathcal{K}_{pa}$) is the Hilbert space indicated in Theorem 7.4 with respect to the network $\mathbf{M_a}$ (respectively, $\mathbf{M}_{pa}$) in place of $\mathbf{N}_a^\mu$.

We can extend the current vector $\mathbf{i}_p \in \mathcal{K}_{pa}$ for $\mathbf{M}_{pa}$ into a current vector $\mathbf{i}_p^e \in \mathcal{K}_a$ for $\mathbf{M}_a$ as follows. On $\mathbf{M}_{pa}$, $\mathbf{i}_p^e$ and $\mathbf{i}_p$ agree. Next, let $\mathbf{C}_p$ be the cut at $\mathcal{W}_p$. $\mathbf{C}_p$ resides in $\mathbf{M}_{pa}$. Let n_k be any node of $\mathcal{V}_p$ and let P_k be that part of a contraction path that connects n_k to n_s^β. Let c_{pk} be the algebraic sum of the currents in the branches of $\mathbf{C}_p$ that are incident to n_k. (Measure those currents as directed toward n_k.) Assign to P_k the current flow of value c_{pk} directed toward n_s^β. Do the same thing for every node of P_k to obtain a current flow from each node of $\mathcal{V}_p$ along a contraction path to n_s^β. In $\mathbf{M}_{pa}$, $\mathbf{C}_p$ is simply the finite set of branches incident to the ordinary 0-node n_p^0. Therefore, Kirchhoff's current law is satisfied at $\mathbf{C}_p$, and it follows that the algebraic sum of all the flows c_{pk} is 0. As for those branches of $\mathbf{A}_p$ that are not in any P_k, let their currents be 0. In this way we extend $\mathbf{i}_p \in \mathcal{K}_{pa}$ into a current vector $\mathbf{i}_p^e$ for $\mathbf{M}_a$.

On the other hand, $\mathbf{v}_p$ is the voltage vector for $\mathbf{M}_{pa}$ dictated by Theorem 7.4; that is, $v_{p0} = -e_0$ and, when $j \neq 0$ and the branch b_j lies in $\mathbf{M}_{pa}$, $v_{pj} = r_j i_{pj}$. We extend $\mathbf{v}_p$ into a voltage vector $\mathbf{v}_p^e$ for $\mathbf{M}_a$ simply by assigning 0 V as the branch voltage for each branch in $\mathbf{A}_p$. Note that, for the regime $\mathbf{v}_p^e, \mathbf{i}_p^e$, Ohm's law is not satisfied along the paths P_k, but for our purposes this is of no concern.

Given any voltage vector $\mathbf{w}$ and any current vector $\mathbf{s}$ for $\mathbf{M}_a$, we define a coupling of $\mathbf{w}$ and $\mathbf{s}$ by $\langle \mathbf{w}, \mathbf{s} \rangle = \sum w_j s_j$, where the summation is over the indices of the branches in $\mathbf{M}_a$. Our next objective is to show that

$$\langle \mathbf{v} - \mathbf{v}_p^e, \mathbf{i} - \mathbf{i}_p^e \rangle \; = \; 0. \tag{15}$$

That $\langle \mathbf{v}, \mathbf{i} \rangle = 0$ follows directly from the application of Theorem 7.4 to $\mathbf{M}_a$; see (5). Similarly, upon applying that theorem to $\mathbf{M}_{pa}$, we get $\langle \mathbf{v}_p, \mathbf{i}_p \rangle = 0$. Since $\mathbf{v}_p^e$ vanishes throughout

$\mathbf{A}_p$, we also have $\langle \mathbf{v}_p^e, \mathbf{i}_p^e \rangle = 0$.

We now wish to show that

$$\langle \mathbf{v}, \mathbf{i}_p^e \rangle \ = \ 0. \tag{16}$$

For this purpose, we use the space $\mathcal{K}_a$ indicated in Theorem 7.4 but now with $\mathbf{N}_a^\mu$ replaced by $\mathbf{M}_a$. If we can show that $\mathbf{i}_p^e \in \mathcal{K}_a$, then (16) will follow from (5) with $\mathbf{s} = \mathbf{i}_p^e$.

Lemma 12.1. $\mathbf{i}_p^e \in \mathcal{K}_a$.

Proof. Upon applying Theorem 7.4 for $\mathbf{M}_{pa}$, we obtain the current vector $\mathbf{i}_p \in \mathcal{K}_{pa}$; $\mathbf{i}_p^e$ extends $\mathbf{i}_p$ as described above. Assume at first that $\mathbf{i}_p \in \mathcal{K}_{pa}^0$, where $\mathcal{K}_{pa}^0$ is the span of basic currents in $\mathbf{M}_{pa}$. Thus, the flow of $\mathbf{i}_p$ through the single ordinary 0-node n_p^0, obtained by shorting $\mathbf{A}_p$ and thereby $\mathcal{W}_p$, can be represented by a finite number of loop currents in $\mathbf{M}_{pa}$; thus, the currents in the branches of the cut $\mathbf{C}_p$ arise from the superposition of finitely many loop currents. If such a loop current passes along a short between two different nodes of $\mathcal{W}_p$, that loop current can be taken to flow — not along that short — but instead along one or two contraction paths between $\mathcal{W}_p$ and n_s^β, and thereby from one node of $\mathcal{W}_p$ through n_s^β to another node of $\mathcal{W}_p$. (Possibly n_s^β embraces a 0-node of $\mathcal{W}_p$, in which case just one contraction path might be traversed.) With such an alteration for every one of the said loop currents, we will obtain $\mathbf{i}_p^e$. Moreover, this procedure will convert any basic current for $\mathbf{M}_{pa}$ into a basic current for $\mathbf{M}_a$. We can conclude that $\mathbf{i}_p^e \in \mathcal{K}_a^0$.

Next, assume that $\mathbf{i}_p \in \mathcal{K}_{pa}$. The flow through the aforementioned short (i.e., through the ordinary 0-node n_p^0) can again be represented by finitely many loop currents in $\mathbf{M}_{pa}$. Once again, each such loop current can be extended into a loop current passing through n_s^β via one or two contraction paths. The result is an extension $\mathbf{i}_p^e$ of $\mathbf{i}_p \in \mathcal{K}_{pa}$.

Moreover, $\mathbf{i}_p$ is the limit in $\mathcal{K}_{pa}$ of a sequence of current vectors in $\mathcal{K}_{pa}^0$, each of which can be extended in the same way into a member of $\mathcal{K}_a^0$. Furthermore. convergence in $\mathcal{K}_{pa}$ implies branchwise convergence. Since the current c_{pk} in each path P_k is the algebraic sum of the currents in the finitely many cut-branches that are incident to the node $n_k \in \mathcal{V}_p$ at which P_k terminates, we get branchwise convergence on P_k as well.

We now argue that, since each P_k is perceptible, we get convergence on P_k in accordance with the norm of $\mathcal{K}_a$. Indeed, corresponding to a sequence $\{\mathbf{i}_{p\nu}\}_{\nu=1}^\infty$ of vectors in $\mathcal{K}_{pa}^0$

that converges to $\mathbf{i}_p \in \mathcal{K}_{pa}$, we have a sequence $\{\mathbf{i}^e_{p\nu}\}_{\nu=1}^{\infty}$ of extended current vectors in $\mathcal{K}^0_a$ converging branchwise to $\mathbf{i}^e_p$ and also a sequence $\{c_{k\nu}\}_{\nu=1}^{\infty}$ of current flows in P_k that converge to c_k. With $\sum_{P_k}$ denoting a summation over the branch indices for P_k, we have

$$\sum_{P_k} r_j (i^e_{pj} - i^e_{p\nu j})^2 \;=\; (c_k - c_{k\nu})^2 \sum_{P_k} r_j. \tag{17}$$

By the perceptibility of P_k, $\sum_{P_k} r_j < \infty$. Consequently, the right-hand side and thereby the left-hand side of (17) tend to 0 as $\nu \to \infty$. But that left-hand side arises from the restriction of the squared norm of $\mathcal{K}_a$ to the branches of P_k. Whence our assertion.

Moreover, the norm for $\mathcal{K}_{pa}$ is the restriction of the norm for $\mathcal{K}_a$ to the branches of $\mathbf{M}_{pa}$. We can conclude that $\mathbf{i}^e_p$ is the limit in $\mathcal{K}_a$ of a sequence of vectors in $\mathcal{K}^0_a$. Hence, $\mathbf{i}^e_p \in \mathcal{K}_a$. ♣

As was indicated above, Lemma 12.1 establishes (16).

To finally establish (15), we have to show that

$$\langle \mathbf{v}^e_p, \mathbf{i} \rangle \;=\; 0 \tag{18}$$

whenever $\mathbf{i} \in \mathcal{K}_a$. By definition, $\mathbf{v}^e_p$ vanishes throughout $\mathbf{A}_p$. Hence, the left-hand side of (18) is equal to $\langle \mathbf{v}_p, \mathbf{i}_s \rangle$, where $\mathbf{v}_p$ is the voltage regime dictated by Theorem 7.4 as applied to $\mathbf{M}_{pa}$ and $\mathbf{i}_s$ is the restriction of $\mathbf{i} \in \mathcal{K}_a$ to the branches of $\mathbf{M}_{pa}$.

Lemma 12.2. $\mathbf{i}_s \in \mathcal{K}_{pa}$.

Proof. Again let $\mathbf{C}_p$ be the cut at $\mathcal{W}_p$. Now, any loop current in $\mathbf{M}_a$ that passes through some branches of $\mathbf{C}_p$ can be truncated into a loop current in $\mathbf{M}_{pa}$ that flows along the short at $\mathcal{W}_p$. Thus, when $\mathbf{i} \in \mathcal{K}^0_a$, we have $\mathbf{i}_s \in \mathcal{K}^0_{pa}$. We can now use a limiting process as in the preceding proof to conclude that $\mathbf{i}_s \in \mathcal{K}_{pa}$ whenever $\mathbf{i} \in \mathcal{K}_a$ ♣

By Theorem 7.4 as applied to $\mathbf{M}_{pa}$ (in particular, by (5)), $\langle \mathbf{v}_p, \mathbf{i}_s \rangle = 0$. This establishes (18). Altogether then, we have established (15).

The components of $\mathbf{v}$ and $\mathbf{v}^e_p$ for the source branch b_0 are both $-\epsilon_0 = -1$. Thus, the left-hand side of (15) can be written as a summation just for the branches in $\mathbf{M}$. In fact, upon rearranging that summation into two sums, one over the branch indices for $\mathbf{M}\backslash\mathbf{A}_p$

and the other for $\mathbf{A}_p$, and using the branch conductances g_j, we can rewrite (15) as

$$\sum_{\mathbf{M}\backslash\mathbf{A}_p} g_j(v_j - v_{pj}^e)^2 + \sum_{\mathbf{A}_p}(v_j - v_{pj}^e)(i_j - i_{pj}^e) \; = \; 0.$$

Since $\mathbf{v}_p^e$ vanishes on $\mathbf{A}_p$, the second summation is equal to

$$\begin{aligned}\sum_{\mathbf{A}_p} v_j(i_j - i_{pj}^e) &= \sum_{\mathbf{A}_p} g_j v_j^2 - \sum_{\mathbf{A}_p} v_j i_{pj}^e \\ &= \sum_{\mathbf{A}_p} g_j(v_j - v_{pj}^e)^2 - \sum_{\mathbf{A}_p} v_j i_{pj}^e.\end{aligned}$$

Thus, (15) becomes

$$\sum_{\mathbf{M}} g_j(v_j - v_{pj}^e)^2 \; = \; \sum_{\mathbf{A}_p} v_j i_{pj}^e.$$

By Schwarz's inequality,

$$\sum_{\mathbf{M}} g_j(v_j - v_{pj}^e)^2 \; = \; \sum_{\mathbf{A}_p} \sqrt{g_j} v_j \sqrt{r_j} i_{pj}^e \; \leq \; \left[\sum_{\mathbf{A}_p} g_j v_j^2 \sum_{\mathbf{A}_p} r_j (i_{pj}^e)^2 \right]^{1/2}. \qquad (19)$$

Since the voltage-current regime for $\mathbf{M}$ is of finite power, $\sum_{\mathbf{A}_p} g_j v_j^2 \to 0$ as $p \to \infty$. Our next objective is to show that $\sum_{\mathbf{A}_p} r_j (i_{pj}^e)^2$ remains bounded as $p \to \infty$.

For this purpose, we need a particular form of the reciprocity theorem for the two-port illustrated in Figure 5. The needed equation is derived from the general reciprocity relation [5, Corollary 3.3-7 and page 155] exactly as it is derived in [1, pages 776-778], even though we are now dealing with an internally transfinite two-port rather than a finite one. With regard to the port variables indicated in Figure 5, the needed result is

$$V_1 \hat{I}_1 + V_2 \hat{I}_2 \; = \; \hat{V}_1 I_1 + \hat{V}_2 I_2 \qquad (20)$$

We can choose variables as follows. The symbols with "hats" are the port variables when a 1-V voltage source is applied to port 2 ($\hat{V}_2 = 1$) and an open circuit is maintained at port 1 ($\hat{I}_1 = 0$). By Corollary 8.5, $|\hat{V}_1| \leq 1$. The symbols without "hats" are the port variables when a current source of value I_1 is applied at port 1 and a short is imposed on port 2 ($V_2 = 0$). Consequently, (20) yields

$$|I_2| \; = \; |\hat{V}_1 I_1| \; \leq \; |I_1| \qquad (21)$$

We may now apply (21) to a rearrangement of $\mathbf{M}_p$ into a two-port. The two terminals of port 1 will be the short at $\mathcal{N}_e$ and the short at $\mathcal{N}_g$; $\mathcal{N}_e$ and $\mathcal{N}_g$ are not shorted together. To obtain port 2, do the following to $\mathbf{M}$. Remove the branches in the arm $\mathbf{A}_p$. Then, choose any node n_k^0 of $\mathcal{V}_p$ and let it be one terminal of port 2. Also, short the nodes of $\mathcal{W}_p\backslash\{n_k^0\}$ and let that short be the other terminal of port 2. Now, when an external short is imposed upon port 2, we will have $\mathbf{M}_p$ again, and moreover the current in that external short will be the current c_{pk} in the path P_k used when we were constructing $\mathbf{i}_p^e$.

To apply (21) to this rearrangement of $\mathbf{M}_p$, short port 2 and impose a pure current source at port 1 whose value is the current h_p flowing through the 1-V voltage source under excitation E. By (21), $|c_{pk}| \leq |h_p|$. Since the choice of n_k^0 was arbitrary, we can conclude that the component of $\mathbf{i}_p^e$ for each branch b_j in $\mathbf{A}_p$ satisfies $|i_{pj}^e| \leq |h_p|$.

Let us now return to the case where $\mathbf{M}_p$ is obtained simply by replacing every branch of $\mathbf{A}_p$ by a short. Let p_0 be a fixed natural number and let $p > p_0$. We can change $\mathbf{M}_{p_0}$ into $\mathbf{M}_p$ by adding resistances to the branches in $\mathbf{A}_{p_0}\backslash\mathbf{A}_p$. By Rayleigh's monotonicity principle [5, pages 103 and 156], $|h_p| \leq |h_{p_0}|$. Thus, $|i_{pj}^e| \leq |h_{p_0}|$. Since every path P_k is perceptible, we can now conclude that the last summation $\sum_{\mathbf{A}_p} r_j(i_{pj}^e)^2$ in (19) remains bounded as $p \to \infty$. Consequently, the left-hand side of (19) tends to 0 as $p \to \infty$.

Finally, let u_s^β be the voltage at n_s^β under a formal application of Rule 10.1 to $\mathbf{M}$ — as before. Also, for $\mathbf{M}_p$, let v_p be the voltage at the node n_p^0 produced by shorting all of $\mathbf{A}_p$ — again as before. Let P be a perceptible path in $\mathbf{M}$ from the short at $\mathcal{N}_g$ to n_s^β. Since $\mathbf{v}_p^e$ vanishes on $\mathbf{A}_p$, we can write the following, where $\sum_P$ is a summation over the branch indices for P.

$$
\begin{aligned}
|u_s^\beta - v_p| &= \left|\sum_P \pm v_j - \sum_P \pm v_{pj}^e\right| = \left|\sum_P \pm\sqrt{g_j}(v_j - v_{pj}^e)\sqrt{r_j}\right| \\
&\leq \left[\sum_P g_j(v_j - v_{pj}^e)^2 \sum_P r_j\right]^{1/2} \leq \left[\sum_M g_j(v_j - v_{pj}^e)^2 \sum_P r_j\right]^{1/2}
\end{aligned}
$$

Since P is perceptible, $\sum_P r_j < \infty$. Thus, $v_p \to u_s^\beta$ as $p \to \infty$. This is what we needed to show in order to justify an application of Rule 10.1 for a random walker Ψ starting at a β-node and reaching $\mathcal{N}_e \cup \mathcal{N}_g$, when $\mathcal{N}_e$ and $\mathcal{N}_g$ are specified as in the first paragraph of this section. We summarize all this through

Theorem 12.3. *For* M *and* $\mathbf{M}_p$ *defined as above and with* n_p^0 *being the 0-node obtained by shorting all of* $\mathbf{A}_p$*, Rule 10.1 extends continuously as* $p \to \infty$ *from the case where* Ψ *starts at* n_p^0 *and reaches* $\mathcal{N}_e \cup \mathcal{N}_g$ *to the case where* Ψ *starts at* n_s^β *and reaches* $\mathcal{N}_e \cup \mathcal{N}_g$*. That is, for a* β*-roving* Ψ *and as* $p \to \infty$,

$$Prob(\mathrm{s}n_p^0, \mathrm{r}\mathcal{N}_e, \mathrm{b}\mathcal{N}_g) \to Prob(\mathrm{s}n_s^\beta, \mathrm{r}\mathcal{N}_e, \mathrm{b}\mathcal{N}_g)$$

where the probability on the left is for $\mathbf{M}_p$ *and the probability on the right is for* M.

13 Roving

Let us now indicate the reason why the condition of roving is being imposed upon the walks that Ψ is permitted to take. Assume that Ψ α-roves for every α less than β but does not necessarily β-rove. We have seen (Theorem 11.3) that Ψ has a positive probability of reaching a β-node n_s^β from within a $(\beta-)$-subsection incident to n_s^β. Can Ψ leave n_s^β? The answer is "no" — in the following sense. Let us assume the Rule 10.1 — as expressed by Theorem 12.3 — still governs the wanderings of Ψ as it starts from n_s^β even though Ψ only α-roves. Then, given any other node, the probability that Ψ will reach that other node before returning to n_s^β is 0.

To show this, choose a perceptible contraction to n_s^β. Let $\mathcal{X}_p$ and $\mathcal{X}_q$ be two conjoining sets for that contraction, let $\mathcal{V}_p$ and $\mathcal{V}_q$ be the corresponding bases, and let $p < q$. Also, let $\mathcal{D}$ be the set of nodes that are 0-adjacent to n_s^β. Two cases arise.

Case 1. Ψ *leaves* n_s^β *along an incident branch:* We are seeking the probability that Ψ will reach $\mathcal{D}$ before reaching $\mathcal{V}_q$ for any q. By a formal application of Rule 10.1, $Prob(\mathrm{s}n_s^\beta, \mathrm{r}\mathcal{D}, \mathrm{b}\mathcal{V}_q)$ is the voltage u_s^β at n_s^β when $\mathcal{D}$ is held at 1 V and $\mathcal{V}_q$ is held at 0 V. By the voltage-divider rule, $u_s^\beta = R_q/(R_d + R_q)$, where R_q is the resistance of the arm $\mathbf{A}_q$ between n_s^β and a short at $\mathcal{V}_q$ and R_d is the parallel resistance of the branches incident to n_s^β. Since the contraction paths in $\mathbf{A}_q$ are perceptible, R_q is finite, and moreover $R_q \to 0$ as $q \to \infty$. Therefore, $u_s^\beta \to 0$ as $q \to \infty$. This implies that the probability of Ψ leaving n_s^β through a branch incident to n_s^β instead of along the arm $\mathbf{A}_q$ is 0.

Case 2. Ψ *leaves* n_s^β *along the arm* $\mathbf{A}_q$*:* That is, Ψ reaches $\mathcal{V}_q$ for some sufficiently large q greater than p before reaching $\mathcal{D}$. Let $n_{q,i}^0$ be any node of $\mathcal{V}_q$. With Ψ starting from

$n^0_{q,i}$, how probable is it that Ψ reaches $\mathcal{V}_p$ before returning to n^β_s? To answer this, we can invoke Theorem 11.2 with $\mathcal{M}_e$ containing $\mathcal{V}_p$ and $\mathcal{M}_q = \{n^\beta_s\}$ — even though we are not assuming that Ψ necessarily β-roves. Indeed, by Theorem 11.3 again, there is a positive probability that Ψ will reach $\mathcal{V}_p \cup \{n^\beta_s\}$. (The only other possibility is that Ψ wanders indefinitely without ever reaching $\mathcal{V}_p$ or n^β_s. We will show later on —see Lemma 16.1 — that the probability of this happening is 0.)

We argue that, as $q \to \infty$, $Prob(\mathrm{s}n^0_{q,i}, \mathrm{r}\mathcal{V}_p, \mathrm{b}n^\beta_s)$ tends to 0. That probability is the voltage $u_{q,i}$ at $n^0_{q,i}$ when $\mathcal{V}_p$ is held at 1 V and n^β_s is held at 0 V. But, by Theorem 8.3 and Property 8.1(b), $u_{q,i} \to 0$ as $q \to \infty$. This implies that it is a certainty that Ψ, after starting from n^β_s along the arm $\mathbf{A}_q$, will return to n^β_s before reaching $\mathcal{V}_p$ for any given p.

Both cases taken together mean that only a vanishingly small proportion of the random walks that start at n^β_s will reach $\mathcal{X}_p$ without first returning to n^β_s, whatever be p. This in turn implies the following for the random walker Ψ that α-roves for every α less than β but does not necessarily β-rove: Once Ψ reaches a β-node n^β, the probability that Ψ will reach any other node before returning to n^β is 0.

However, this does not mean the there are no roving β-walks. It simply means that we are dealing with the exceptional case when we consider the random roving β-walks among all the transfinite random walks. In short, Definition 9.2 for roving β-walks remains valid, and we are free to restrict Ψ to such walks and to assign probabilities for transitions from a β-node in accordance with Rule 10.1.

14 From a β-node to a β-Adjacent $(\beta+)$-Node

So far in our inductive argument, we have examined transitions from within a $(\beta-)$-subsection to a bordering $(\beta+)$-node and from a β-node to within a $(\beta-)$-subsection. As the next step, we discuss transitions from a β-node to its β-adjacent $(\beta+)$-nodes. The argument now needed is the same as that leading up to Theorem 11.2.

Let the random walker Ψ start at the β-node n^β_s and let $n^{\beta+}_k$ $(k = 1, \ldots, K)$ be the $(\beta+)$-nodes that are β-adjacent to n^β_s. (See Figure 6.) Choose a perceptible contraction $\{\mathcal{W}_{k,p_k}\}_{p_k=1}^\infty$ for each $n^{\beta+}_k$. Also, let $\mathbf{H}(p_1, \ldots, p_K)$ be the reduced network induced by all branches that are not separated from n_s^β by $\bigcup_{k=1}^{K} \mathcal{W}_{k,p_k}$. $\mathbf{H}(p_1, \ldots, p_K)$ is of rank β because it contains n_s^β — the one and only $(\beta+)$-node in it. Moreover, $\mathbf{H}(p_1, \ldots, p_K)$ will be β-connected when the $p_1, \ldots, p_K$ are chosen sufficiently large. $\mathbf{H}(p_1, \ldots, p_K)$ plays the same role as the α-network $\mathbf{F}(p_1, \ldots, p_K)$ did before. We define $\mathcal{N}_e$ and $\mathcal{N}_g$ as two finite disjoint nodal sets such that $n_s^\beta \notin (\mathcal{N}_e \cup \mathcal{N}_g)$ and, for each k, $\mathcal{W}_{k,p_k}$ lies entirely within $\mathcal{N}_e$ or alternative within $\mathcal{N}_g$. (It will be understood that, as $p_k \to \infty$, $\mathcal{W}_{k,p_k}$ remains in $\mathcal{N}_e$ or in $\mathcal{N}_g$; the latter sets adjust accordingly.) Furthermore, $\mathcal{M}_e$ and $\mathcal{M}_g$ are similarly defined except that $n_k^{\beta+}$ replaces $\mathcal{W}_{k,p_k}$.

Under the present meanings of these symbols, we can use the argument of Section 11 — virtually word for word — to get Lemma 11.1 again and thereby the following extension of Rule 10.1.

Theorem 14.1. *As the $p_k \to \infty$, Rule 10.1 extends continuously from the result achieved in Theorem 12.3 to the following result: For a β-roving Ψ, $Prob(\mathrm{s}n_s^\beta, \mathrm{r}\mathcal{M}_e, \mathrm{b}\mathcal{M}_g)$ is the voltage at n_s^β when $\mathcal{M}_e$ is held at 1 V and $\mathcal{M}_g$ is held at 0 V.*

Once again, we take this as the definition of probabilities of transitions from a β-node to its β-adjacent $(\beta+)$-nodes.

15 Transitions within a Truncation of a β-Subsection; the Surrogate Network

We continue our inductive argument by showing that Rule 10.1 holds for a truncation $\mathbf{T}$ of any β-subsection $\mathbf{S}_b^\beta$ when $\beta < \mu$. (The case where $\beta = \mu$ will be considered in the next section.) We shall do this by setting up a "surrogate" 0-network $\mathbf{T}^{\beta\mapsto 0}$, whose behavior mimics that of $\mathbf{T}$ in a certain way, and then by applying the Nash-Williams rule to $\mathbf{T}^{\beta\mapsto 0}$. $\mathbf{T}^{\beta\mapsto 0}$ in turn is obtained from a Markov chain encompassing the probabilities for transitions among finitely many maximal nodes of $\mathbf{T}$.

Let $n_l^{(\beta+1)+}$ $(l = 1, \ldots, L)$ be the maximal bordering nodes of $\mathbf{S}_b^\beta$. Choose an isolating set $\mathcal{W}_l$ within $\mathbf{S}_b^\beta$ for each $n_l^{(\beta+1)+}$ and let $\mathbf{T}$ be the reduced network induced by all branches of $\mathbf{S}_b^\beta$ that are not in the arms corresponding to the $\mathcal{W}_l$. As before, it is understood that the $\mathcal{W}_l$ are chosen sufficiently close to the bordering nodes to ensure that $\mathbf{T}$ is β-connected.

$\mathbf{T}$ will have only finitely many β-nodes; this fact is a consequence of Lemma 3.6.

As usual, let n_s be any maximal node of $\mathbf{T}$ and let $\mathcal{N}_e$ and $\mathcal{N}_g$ be two disjoint finite sets of maximal nodes of $\mathbf{T}$ such that $n_s \notin (\mathcal{N}_e \cup \mathcal{N}_g)$ and $\mathcal{N}_e \cup \mathcal{N}_g$ contains $\bigcup_{l=1}^{L} \mathcal{W}_l$. Furthermore, let $\mathcal{M}$ be a finite set of maximal nodes in $\mathbf{T}$ containing n_s, all the nodes of $\mathcal{N}_e \cup \mathcal{N}_g$, and all the β-nodes of $\mathbf{T}$. $\mathcal{M}$ will be the state space of a certain Markov chain. Two nodes n_a and n_b of $\mathcal{M}$ will be called *$\mathcal{M}$-adjacent* if there is a path in $\mathbf{T}$ that terminates at n_a and n_b and does not meet any other node of $\mathcal{M}$. Thus, each node n_a of $\mathcal{M}$ will have a unique set $\mathcal{M}_a$ of nodes $\mathcal{M}$-adjacent to n_a.

Lemma 15.1. *If two nodes of $\mathcal{M}$ are $\mathcal{M}$-adjacent, then they are also β-adjacent.*

Proof. Let n_a and n_b be $\mathcal{M}$-adjacent nodes of $\mathcal{M}$. If n_a and n_b are not β-adjacent, they are not incident to the same $(\beta-)$-subsection. Consequently, every path between n_a and n_b must meet a β-node distinct from n_a and n_b. Since $\mathcal{M}$ contains all the β-nodes of $\mathbf{T}$, n_a and n_b are not $\mathcal{M}$-adjacent. ♣

We now assume that the random walker Ψ β-roves. This ensures that Ψ, having started at $n_a \in \mathcal{M}$, will surely meet the set $\mathcal{M}_a$ of nodes that are $\mathcal{M}$-adjacent to n_a. We say that Ψ makes a *one-step transition from n_a to n_b* if $n_b \in \mathcal{M}_a$ and if Ψ starts at n_a and reaches n_b before reaching any other node of $\mathcal{M}_a$. To construct our desired Markov chain, we need the probabilities of the one-step transitions. By virtue of Lemma 15.1, we can apply to $\mathbf{T}$ Rule 10.1 as extended by Theorem 14.1 to get those probabilities.

Note first of all that the β-roving of Ψ implies still more; namely, if Ψ starts at a β-node n_a, it will surely meet a node of $\mathcal{M}_a$ before returning to n_a. However, if $n_a \in \mathcal{M}$ is of lower rank than β, Ψ may return to n_a before reaching $\mathcal{M}_a$. We will simply ignore such returns when setting up our Markov chain. Thus, we restrict ourselves to the one-step transitions from n_a to its $\mathcal{M}$-adjacent nodes when assigning positive probabilities to the one-steps of the Markov chain..

Let n_a and n_b be any two nodes of $\mathcal{M}$, where now n_a and n_b need not be $\mathcal{M}$-adjacent. We let $P_{a,b}$ be the probability of a one-step transition from n_a to n_b. As was just explained, we set $P_{a,a} = 0$. Also, $P_{a,b} = 0$ if n_a and n_b are not $\mathcal{M}$-adjacent because it is then impossible for Ψ to make a one-step transition from n_a to n_b. On the other hand, if there is only one node n_b in $\mathcal{M}_a$, we have $P_{a,b} = 1$. However, if there are many nodes in $\mathcal{M}_a$, we use the extension of Rule 10.1 given by Theorem 14.1 to obtain the probability $P_{a,b}$ of a one-step transition from n_a to n_b in $\mathcal{M}_a$. In this last case, we have $0 < P_{a,b} < 1$ by virtue of Corollary 8.6 and the fact that there is a path from n_a to each node of $\mathcal{M}_a$ that does not meet any other node of $\mathcal{M}_a$. Finally, to conclude that we have the one-step probabilities of a Markov chain, we have to show that these one-step probabilities sum to 1. This follows immediately from the fact that it is a certainty that our β-roving Ψ will reach $\mathcal{M}_a$ after starting from n_a. It can also be shown electrically as follows: Measure the voltage u_a at n_a when one node n_b of $\mathcal{M}_a$ is held at 1 V and $\mathcal{M}_a \backslash \{n_b\}$ is held at 0 V. Then sum the various values of u_a obtained as n_b varies through $\mathcal{M}_a$. By the superposition principle, that sum is the voltage at n_a when all of $\mathcal{M}_a$ is held at 1 V. Consequently, that sum equals 1. This confirms our assertion.

Thus, for any choice as specified above of the finite set $\mathcal{M}$, we have a Markov chain with $\mathcal{M}$ as its state space. We denote that chain by $\mathbf{M}(\mathcal{M})$. We can examine the wanderings of Ψ among the nodes of $\mathcal{M}$ by analyzing $\mathbf{M}(\mathcal{M})$ — but only up to the point where Ψ arrives at a node of $\bigcup_{l=1}^{L} \mathcal{W}_l$. After that Ψ may leave $\mathbf{T}$ when wandering in $\mathbf{N}^{\mu}$, in which case $\mathbf{M}(\mathcal{M})$ is no longer relevant. For our purposes, this is of no concern because we are presently only interested in the wanderings of Ψ from the point where it starts at a node n_s of $\mathbf{T}$ up to the point where it reaches $\mathcal{N}_e \cup \mathcal{N}_g$. The last set contains $\bigcup_{l=1}^{L} \mathcal{W}_l$.

Theorem 15.2. *The Markov chain* $\mathbf{M}(\mathcal{M})$ *is irreducible and reversible.*

Proof. The case where $\mathcal{M}$ has just two nodes is trivial. So, let $\mathcal{M}$ have more than two nodes.

For any two $\mathcal{M}$-adjacent nodes n_a and n_b of $\mathcal{M}$, $P_{a,b} > 0$ — as we have noted above. The irreducibility of $\mathbf{M}(\mathcal{M})$ now follows from the β-connectedness of $\mathbf{T}$ [2].

As for reversibility, we start by recalling the definition of a *cycle in* $\mathcal{M}$. This is a finite sequence

$$C = (n_1, n_2, \ldots, n_c, n_{c+1} = n_1)$$

of nodes n_k in $\mathcal{M}$ with the following properties: All the nodes of C are distinct except for the first and last; there are at least three nodes in C (i.e., $c > 2$); consecutive nodes in

C are $\mathcal{M}$-adjacent. A Markov chain is called *reversible* if, for every cycle C, the product $\prod_{k=1}^{c} P_{k,k+1}$ of transition probabilities $P_{k,k+1}$ from n_k to n_{k+1} remains the same when every $P_{k,k+1}$ is replaced by $P_{k+1,k}$ [2, Section 1.5]. Thus, we need only show that

$$P_{1,2}P_{2,3}\cdots P_{c,1} = P_{1,c}\cdots P_{3,2}P_{2,1} \tag{22}$$

According to Rule 10.1 as extended by Theorem 14.1, $P_{k,k+1}$ is the voltage u_k at n_k obtained by holding n_{k+1} at 1 V and by holding all the other nodes of $\mathcal{M}$ that are $\mathcal{M}$-adjacent to n_k at 0 V. For this situation, u_k will remain unchanged when the voltages at still other nodes of $\mathcal{M}$ are specified.

To simplify notation, let us denote n_k by m_0 and n_{k+1} by m_1. Thus, m_0 and m_1 are $\mathcal{M}$-adjacent. Also, let $m_2, \cdots, m_K$ denote all the nodes of $\mathcal{M}$ that are different from n_k and n_{k+1} but are $\mathcal{M}$-adjacent to either n_k or n_{k+1} or both. Since the cycle has at least three nodes, we have $K \geq 2$. Now, consider the K-port obtained from $\mathbf{T}$ by choosing m_k, m_0 as the pair of terminals for the kth port ($k = 1, \cdots, K$) with m_0 being the common ground for all ports. To obtain the required node voltages for measuring $P_{k,k+1}$, we externally connect a 1-V voltage source to m_1 from all of the $m_2, \cdots m_K$, with m_0 left floating (i.e., m_0 has no external connections). The resulting voltage u_0 at m_0 is $P_{k,k+1}$.

With respect to m_0 (taken as the ground node), the voltage at m_1 is $1 - u_0$ and the voltage at m_k ($k = 2, \cdots, K$) is $-u_0$. Moreover, with i_k denoting the current entering m_k ($k = 1, \cdots, K$), the sum $i_1 + \cdots + i_K$ is 0. (Apply Kirchhoff's current law at m_1.) Furthermore, the port currents and voltages are related by $\mathbf{i} = Y\mathbf{u}$, where $\mathbf{i} = (i_1, \cdots, i_K)$, $\mathbf{u} = (1 - u_0, -u_0, \cdots, -u_0)$, and $Y = [Y_{a,b}]$ is a $K \times K$ matrix of real numbers that is symmetric (Lemma 7.3). Upon expanding $\mathbf{i} = Y\mathbf{u}$ and adding the i_k, we get

$$0 = i_1 + \cdots + i_K = \sum_{a=1}^{K} Y_{a,1} - u_0 \sum_{a=1}^{K}\sum_{b=1}^{K} Y_{a,b}.$$

Therefore,

$$P_{k,k+1} = u_0 = \frac{\sum_{a=1}^{K} Y_{a,1}}{\sum_{a=1}^{K}\sum_{b=1}^{K} Y_{a,b}}. \tag{23}$$

Upon setting $G_k = \sum_{a=1}^{K}\sum_{b=1}^{K} Y_{a,b}$, we can rewrite (23) as

$$G_k P_{k,k+1} = \sum_{a=1}^{K} Y_{a,1}. \tag{24}$$

Now, $\sum_{a=1}^{K} Y_{a,1}$ is the sum $i_1 + \cdots + i_K$ when $\mathbf{u} = (1, 0, \cdots, 0)$; that is, $\sum_{a=1}^{K} Y_{a,1}$ is the sum of the currents entering $m_1, m_2, \cdots, m_K$ from external connections when 1-V voltage sources are connected to m_1 from all of the $m_0, m_2, \cdots, m_K$.

By reversing the roles of m_0 and m_1, we have by the same analysis that $G_{k+1}P_{k+1,k}$ is the sum $i_0 + i_2 + \cdots + i_K$ of the currents entering $m_0, m_2, \cdots, m_K$ from external connections when 1-V voltage sources are connected to m_0 from all of the $m_1, m_2, \cdots, m_K$. With respect to m_0 acting as the ground node again, we now have $u_1 = \cdots = u_K = -1$, and therefore $i_1 = -\sum_{a=1}^{K} Y_{1,a}$. Moreover, under this latter connection, the sum $-i_1 - i_2 - \cdots - i_K$ of the currents leaving $m_1, m_2, \cdots m_K$ is equal to the current i_0 entering m_0. Hence, $-i_1 = i_0 + i_2 + \cdots + i_K$. Thus,

$$G_{k+1}P_{k+1,k} \;=\; -i_1 \;=\; \sum_{a=1}^{K} Y_{1,a}. \tag{25}$$

Since the matrix Y is symmetric, we have $Y_{1,a} = Y_{a,1}$. So, by (24) and (25),

$$G_{k+1}P_{k+1,k} \;=\; G_k P_{k,k+1}. \tag{26}$$

Finally, we may now write

$$P_{1,2}P_{2,3}\cdots P_{c,1} \;=\; \frac{G_2}{G_1}P_{2,1}\frac{G_3}{G_2}P_{3,2}\cdots\frac{G_1}{G_c}P_{1,c} \;=\; P_{2,1}P_{3,2}\cdots P_{1,c}.$$

This verifies (22) and completes the proof. ♣

Because the Markov chain $\mathbf{M}(\mathcal{M})$ is irreducible and reversible, we can synthesize a finite connected 0-network $\mathbf{T}^{\beta\mapsto 0}$ whose 0-nodes correspond bijectively to the nodes of $\mathcal{M}$ and whose random 0-walks are governed by the same one-step probability matrix as that for $\mathbf{M}(\mathcal{M})$ [2, page 126]. We call $\mathbf{T}^{\beta\mapsto 0}$ the *surrogate network for* $\mathbf{T}$ *with respect to* $\mathcal{M}$. A realization for $\mathbf{T}^{\beta\mapsto 0}$ can be obtained by connecting a conductance $g_{k,l} = g_{l,k}$ between the 0-nodes x_k^0 and x_l^0 ($k \neq l$) in $\mathbf{T}^{\beta\mapsto 0}$, where $g_{k,l}$ is determined as follows: Let $n_k \mapsto x_k^0$ denote the bijection from the nodes of $\mathcal{M}$ to the 0-nodes of $\mathbf{T}^{\beta\mapsto 0}$. If n_k and n_l are not $\mathcal{M}$-adjacent in $\mathbf{T}$, set $g_{k,l} = 0$. If n_k and n_l are $\mathcal{M}$-adjacent in $\mathbf{T}$, relabel n_k as m_0, relabel n_l as m_1, and let $m_2, \ldots, m_K$ be the other nodes of $\mathcal{M}$ that are $\mathcal{M}$-adjacent to either m_0 or m_1, or both. Then, with our prior notation, set $G_k = \sum_{a=1}^{K}\sum_{b=1}^{K} Y_{a,b}$. Also set $G = \sum_k G_k$, where the latter sum is over all indices for all the nodes of $\mathcal{M}$. Finally, set $g_{k,l} = P_{k,l}G_k/G$, where $P_{k,l}$

is the probability of a one-step transition from n_k to n_l (before the relabeling). By (26), $g_{k,l} = g_{l,k}$. This yields the surrogate network $\mathbf{T}^{\beta \mapsto 0}$.

The one-step transition probabilities for a random 0-walk on $\mathbf{T}^{\beta \mapsto 0}$ that follows the nearest-neighbor rule are the same as the one-step transition probabilities $P_{k,l}$ for $\mathbf{M}(\mathcal{M})$. Indeed, the nearest-neighbor rule asserts that the probability of a one-step transition in $\mathbf{T}^{\beta \mapsto 0}$ from a node x_k^0 to an adjacent node x_l^0 is the ratio $g_{k,l} / \sum_{\lambda=1}^{L} g_{k,\lambda}$, where $\sum_{\lambda=1}^{L} g_{k,\lambda}$ is the sum of all conductances incident to x_k^0. Since $g_{k,l} = P_{k,l} G_k / G$, that ratio is equal to $P_{k,l} / \sum_{\lambda=1}^{L} P_{k,\lambda}$. But, as we have noted earlier, $\sum_{\lambda=1}^{L} P_{k,\lambda} = 1$. This verifies our assertion.

The point of all this is that we can now apply the Nash-Williams rule to $\mathbf{T}^{\beta \mapsto 0}$ to get an extension of Rule 10.1 for the wandering of Ψ on any truncation $\mathbf{T}$ of a β-subsection, where $\beta < \mu$. That is, we have extended that rule from rank α to rank β. Indeed, we have inductively established the following theorem. (It is now understood that α is replaced by β in the definition of symbols given before Rule 10.1.) However, the assumption of β-roving for Ψ in that rule will now be replaced by $(\beta+1)$-roving to insure that Ψ reaches $\mathcal{N}_e \cup \mathcal{N}_g$. But, this is not really necessary; as will be shown in the next section (Lemma 16.1 with μ replaced by β), β-roving alone suffices to insure that Ψ will reach $\mathcal{N}_e \cup \mathcal{N}_g$.

Theorem 15.3. *Rule 10.1 holds for any truncation of any β-subsection, when $\beta < \mu$ and Ψ $(\beta+1)$-roves.*

16 Wandering on a μ-Network

In the arguments up to and including Section 14 we allowed $\beta = \mu$, but in Section 15 we required that $\beta < \mu$. So, we now need a replacement for Section 15 for the case where $\beta = \mu$. We cannot now require that Ψ $(\mu+1)$-rove because there are no $(\mu+1)$-nodes. Instead we only assume that Ψ μ-roves and exploit the fact that there are only finitely many μ-nodes in $\mathbf{N}^\mu$.

Let $\mathcal{N}_t$ be a finite set of maximal nodes in $\mathbf{N}^\mu$; $\mathcal{N}_t$ will play the role that $\mathcal{N}_e \cup \mathcal{N}_g$ did before, but now we allow $\mathcal{N}_t$ to be a singleton. Let n_s be a maximal node not in $\mathcal{N}_t$. As before, we let $\mathcal{M}$ be a finite set of maximal nodes that contains n_s, all the nodes of $\mathcal{N}_t$, and all the μ-nodes of $\mathbf{N}^\mu$. $\mathcal{M}$-adjacency for the nodes of $\mathcal{M}$ is defined as before, and, in accordance with Lemma 15.1, if two nodes of $\mathcal{M}$ are $\mathcal{M}$-adjacent, they are also μ-adjacent. Moreover, given any $n_a \in \mathcal{M}$, there is a unique set $\mathcal{M}_a$ of nodes in $\mathcal{M}$ that are $\mathcal{M}$-adjacent to n_a. The fact that Ψ μ-roves insures that Ψ, having started from n_a, will surely meet $\mathcal{M}_a$.

Exactly as in Section 15, we can assign probabilities of one-step transitions between the nodes of $\mathcal{M}$ and conclude that we have a Markov chain $\mathbf{M}(\mathcal{M})$ governing the wandering of the μ-roving Ψ among the nodes of $\mathcal{M}$. Moreover, the proof of Theorem 15.2 holds again and shows that $\mathbf{M}(\mathcal{M})$ is irreducible and reversible. Thus, $\mathbf{M}(\mathcal{M})$ can be represented by the wandering of a random walker Ψ' of a finite connected 0-network $\mathbf{N}^{\mu\mapsto 0}$ — a surrogate for $\mathbf{N}^{\mu}$, which of course depends upon the choice of $\mathcal{M}$. Ψ' obeys the nearest-neighbor rule. Moreover, we have a bijection between the 0-nodes x_k of $\mathbf{N}^{\mu\mapsto 0}$ and the nodes n_k of $\mathbf{M}$. The Markov chain governing the wandering of Ψ' on $\mathbf{N}^{\mu\mapsto 0}$ has the same probability transition matrix as that for $\mathbf{M}(\mathcal{M})$.

As was mentioned above, we cannot invoke $(\mu+1)$-roving for Ψ in order to assert the the certainty that Ψ, having started from n_s, will meet $\mathcal{N}_t$. Nonetheless, this conclusion is true. To verify this, we employ a result of Nash-Williams [3, Lemma 3], whose proof we repeat here. In the next lemma $\mathcal{N}_t'$ is any finite nonvoid set of nodes in $\mathbf{N}^{\mu\mapsto 0}$ and x_s is a node of $\mathbf{N}^{\mu\mapsto 0}$ not in $\mathcal{N}_t'$.

Lemma 16.1. *It is a certainty that the random walker Ψ' on $\mathbf{N}^{\mu\mapsto 0}$, after starting from x_s, will reach $\mathcal{N}_t'$ in finitely many steps.*

Proof. Let $\mathcal{M}'$ be the set of nodes for $\mathbf{N}^{\mu\mapsto 0}$, let $x \in (\mathcal{M}'\backslash\mathcal{N}_t')$, and let $\delta(x, \mathcal{N}_t')$ denote the distance in $\mathbf{N}^{\mu\mapsto 0}$ from x to $\mathcal{N}_t'$, that is, the length of the shortest path from x to $\mathcal{N}_t'$, where the length of a path is measured by the number of branches in the path. Also, let $Q_k(x, \mathcal{N}_t')$ be the probability that Ψ', after starting from x, will reach $\mathcal{N}_t'$ in k or fewer steps (i.e., in k or fewer single-branch transitions). Set

$$d = \max\{\delta(x, \mathcal{N}_t') : x \in (\mathcal{M}'\backslash\mathcal{N}_t')\}$$

and

$$Q_{min} = \min\{Q_d(x, \mathcal{N}_t') : x \in (\mathcal{M}'\backslash\mathcal{N}_t')\}.$$

Thus, Q_{min} is a lower bound on the probability that Ψ', after starting from some node of $\mathcal{M}'\backslash\mathcal{N}_t'$, will reach $\mathcal{N}_t'$ in d or fewer steps. Therefore $1 - Q_{min}$ is an upper bound on the probability that Ψ', after starting as stated, will not reach $\mathcal{N}_t'$ in d or fewer steps.

For any given $x \in (\mathcal{M}'\backslash\mathcal{N}_t')$, we can trace a path P from x to $\mathcal{N}_t'$ whose length is no larger than d. By the nearest-neighbor rule, for each node x_a in P, there is a positive probability that Ψ' will proceed from x_a to the next node in P in one step. Consequently, there is a positive probability that Ψ' will proceed along P to reach $\mathcal{N}_t'$ in d or fewer steps. Since there are only finitely many nodes in $\mathcal{M}'\backslash\mathcal{N}_t'$, we can infer that $Q_{min} > 0$.

Now, let $R_k(x_s, \mathcal{N}_t') = 1 - Q_k(x_s, \mathcal{N}_t')$; this is the probability that Ψ', after starting from x_s, will not reach $\mathcal{N}_t'$ in k or fewer steps. Thus, we may write

$$0 \leq R_{kd}(x_s, \mathcal{N}_t') \leq (1 - Q_{min})^k \rightarrow 0$$

as $k \rightarrow \infty$. This shows that the probability of Ψ' never reaching $\mathcal{N}_t'$ after starting from x_s is 0. Thus, it is a certainty that Ψ' will reach $\mathcal{N}_t'$ in at most finitely many steps. ♣

We can now invoke the bijection between the nodal set $\mathcal{M}'$ of $\mathbf{N}^{\mu\mapsto 0}$ and the nodal set $\mathcal{M}$ to obtain immediately the following consequence of Lemma 16.1. As before, $\mathcal{N}_t \subset \mathcal{M}$ and $n_s \in (\mathcal{M}\backslash\mathcal{N}_t)$.

Theorem 16.2. *For a μ-roving random walker Ψ on $\mathbf{N}^\mu$, it is a certainty that Ψ, having started from n_s, will reach $\mathcal{N}_t$ after passing through nodes of $\mathcal{M}$ at most finitely many times — and therefore after passing through μ-nodes at most finitely many times.*

Furthermore, we are now free to apply the Nash-Williams rule to $\mathbf{N}^{\mu\mapsto 0}$ and invoke the said bijection in order to obtain the following result.

Theorem 16.3. *With $\mathcal{N}_e$ and $\mathcal{N}_g$ being any disjoint nonvoid finite sets of maximal nodes in $\mathbf{N}^\mu$ and with n_s being a maximal node in $\mathbf{N}^\mu$ with $n_s \notin (\mathcal{N}_e \cup \mathcal{N}_g)$, the following is true for the wandering of a μ-roving random walker on $\mathbf{N}^\mu$: $Prob(\mathrm{s}n_s, \mathrm{r}\mathcal{N}_e, \mathrm{b}\mathcal{N}_g)$ is the voltage at n_s when $\mathcal{N}_e$ is held at 1 V and $\mathcal{N}_g$ is held at 0 V.*

With regard to the promised strengthening of Rule 10.1, note that, with μ replaced by α, the truncation $F(p_1, \ldots, p_K)$ of the α-subsection satisfies Definition 6.1 with μ replaced by α and therefore can play the role of $\mathbf{N}^\mu$. (For instance, $F(p_1, \ldots, p_K)$ has only finitely many α-nodes.) Thus, Theorem 16.3 can be restated as follows.

Corollary 16.4 *Rule 10.1 continues to hold for any truncation $F(p_1, \ldots, p_K)$ of an α-subsection when the assumption of β-roving for Ψ is replaced by the weaker assumption of α-roving for Ψ.*

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Figure Legends

Figure 1.

(a) A 1-graph consisting of a two-way infinite ladder shorted at its extremities by two 1-nodes and with an appended branch b_0. The heavy dots denote 0-nodes, the two small circles denote 1-nodes, and the line segments are branches.

(b) A 2-graph consisting of a ladder of ladders, to which are appended three extra branches b_0, b_1, and b_2 and also another ladder L. Each bar of two closely spaced lines denotes a ladder of branches like that of part (a). The two larger circles denote 2-nodes at the two extremities of this 2-graph.

Figure 2.

A 1-graph in which there is no path between the 1-nodes n_a^1 and n_b^1.

Figure 3.

An α-subsection $\mathbf{S}_b^\alpha$. The heavy lines denote the $(\beta+)$-nodes bordering $\mathbf{S}_b^\alpha$. The dot-dash lines denote branches b_1 and b_2 of $\mathbf{S}_b^\alpha$ incident to bordering $(\beta+)$-nodes. $\mathcal{V}_{k,p_k}$ denotes an arm base in $\mathbf{S}_b^\alpha$ corresponding to the $(\beta+)$-node $n_k^{\beta+}$; it is indicated by a line of dots. $\mathbf{F}(p_1,\ldots,p_K)$ is the reduced network within the boundary consisting of the $\mathcal{V}_{k,p_k}$ $(k = 1,\ldots,K)$; $\mathbf{F}(p_1,\ldots,p_K)$ contains the branches b_1 and b_2. ($\mathcal{W}_{k,p_k}$ is $\mathcal{V}_{k,p_k} \cup \{n_k^0\}$, where n_k^0 is the 0-node embraced by $n_k^{\beta+}$ and incident to $\mathbf{S}_b^\alpha$. In general, either $\mathcal{V}_{k,p_k}$ may be void or n_k^0 may be absent, but not both.)

Figure 4.

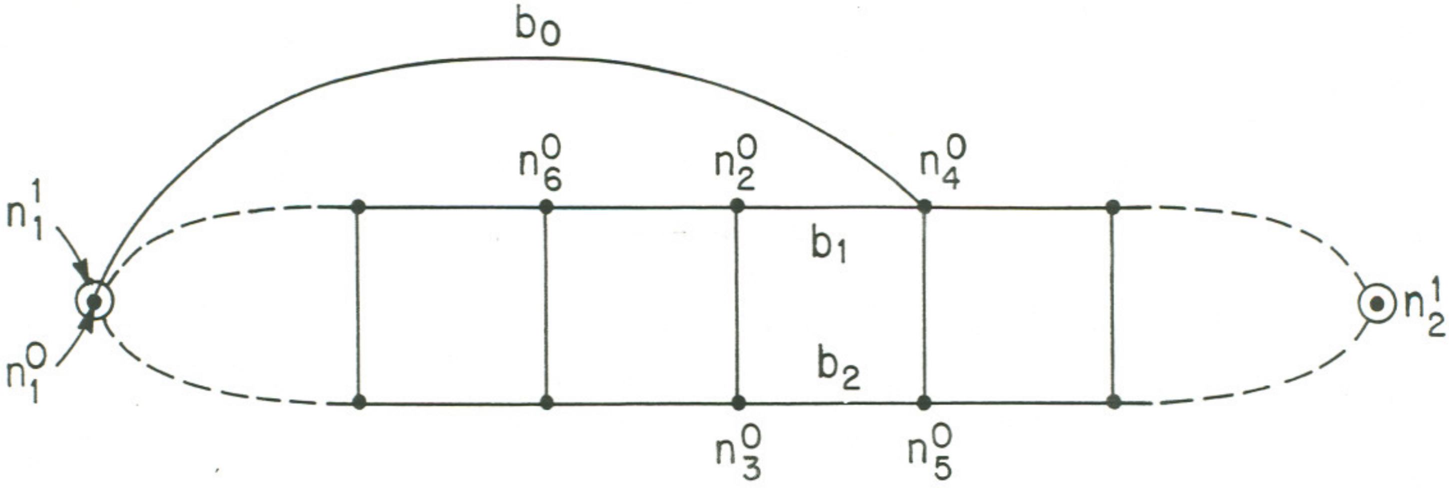
A β-node n_s^β with its incident $(\beta-)$-subsections. Branches incident to n_s^β are denoted by dot-dash lines. The base $\mathcal{V}_p$ is denoted by a ring of dots. The isolating set $\mathcal{W}_p$ is $\mathcal{V}_p$ along with the 0-node embraced by n_s^β. The arm $\mathbf{A}_p$ is shown cross-hatched. $\mathcal{N}_e \cup \mathcal{N}_g$ is also shown by a ring of dots. $\mathcal{N}_e \cup \mathcal{N}_g$ does not meet $\mathbf{A}_p$ and separates $\mathcal{W}_p$ from all the $(\beta+)$-nodes other than n_s^β; moreover, $n_s^\beta \notin \mathcal{N}_e \cup \mathcal{N}_g$.

Figure 5.

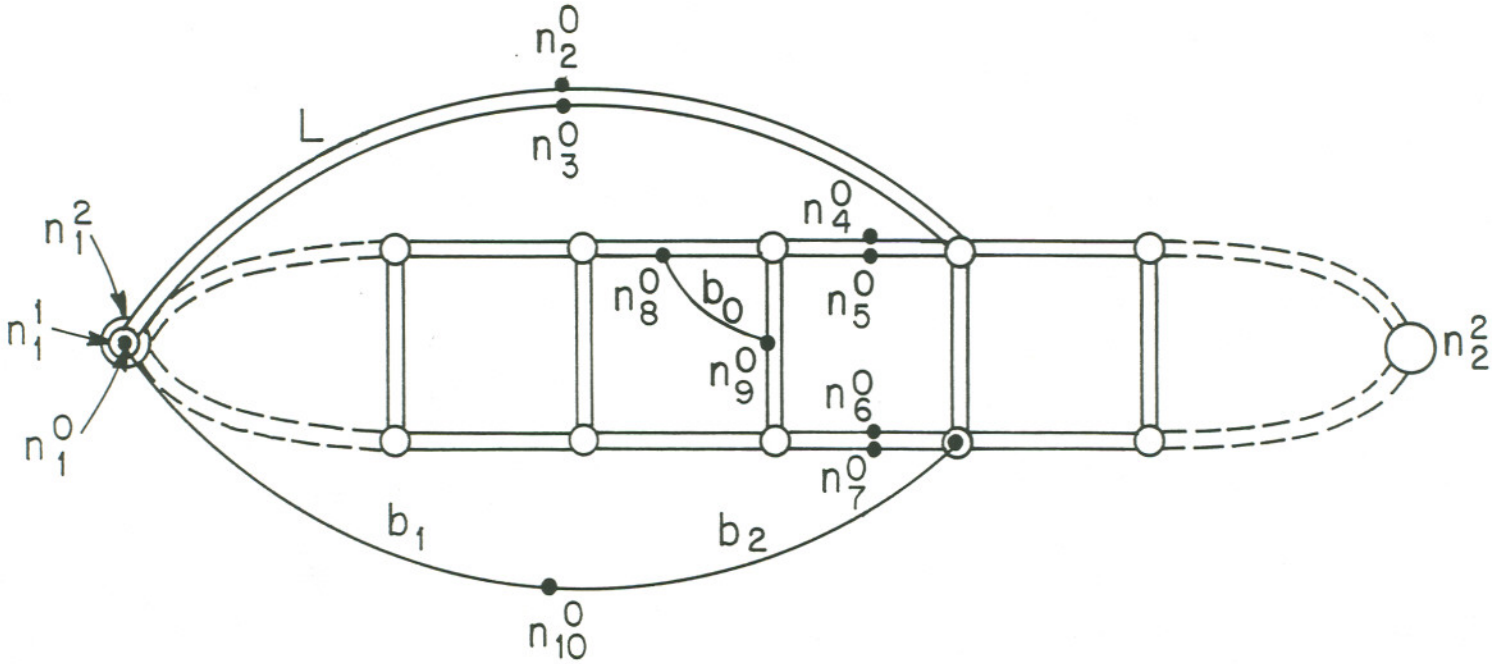
A two-port which is internally a transfinite sourceless network.

Figure 6.

A β-node n_s^β, its incident ($\beta-$)-subsections $\mathbf{S}_{b1}$ and $\mathbf{S}_{b2}$, and its β-adjacent ($\beta+$)-nodes $n_1^{\beta+}, \ldots, n_K^{\beta+}$. The notation is the same as that used in Figure 3, but now $\mathbf{H}(p_1, \ldots, p_K)$ replaces $\mathbf{F}(p_1, \ldots, p_K)$.


b_0
n_6^0
n_2^0
n_4^0
n_1^1
b_1
n_2^1
n_1^0
b_2
n_3^0
n_5^0

(a)


n_2^0
L
n_3^0
n_4^0
n_1^2
n_8^0
b_0
n_5^0
n_1^1
n_2^2
n_9^0
n_6^0
n_1^0
n_7^0
b_1
b_2
n_{10}^0

(b)

FIG. 1

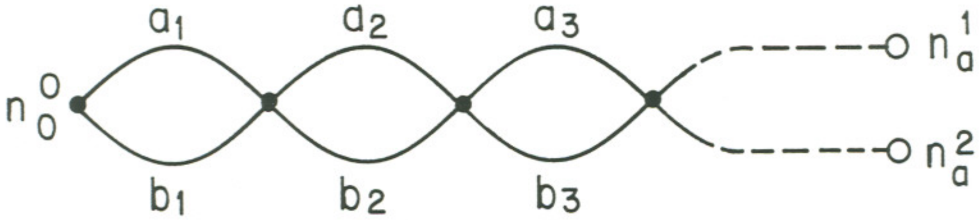


FIG. 2

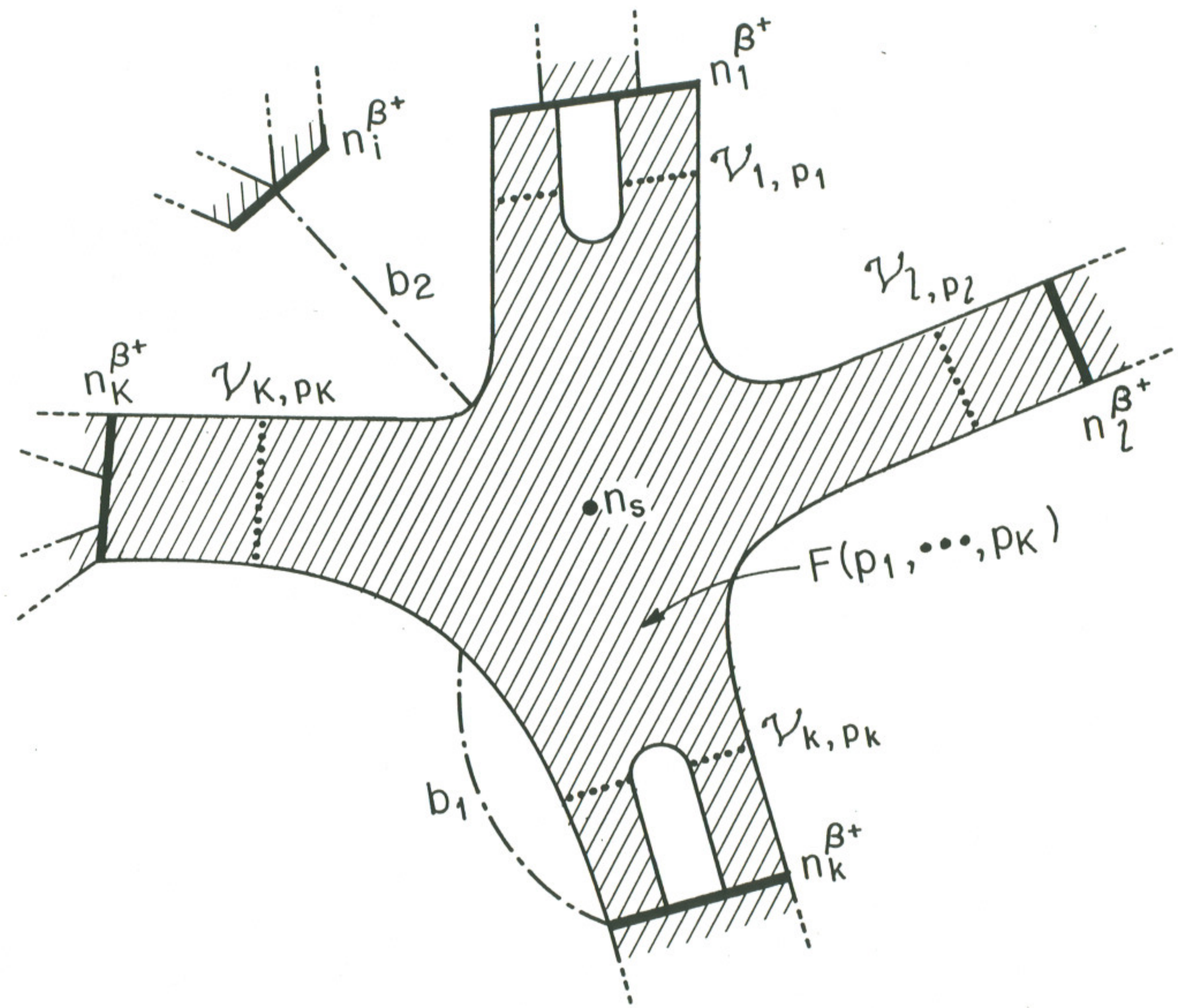
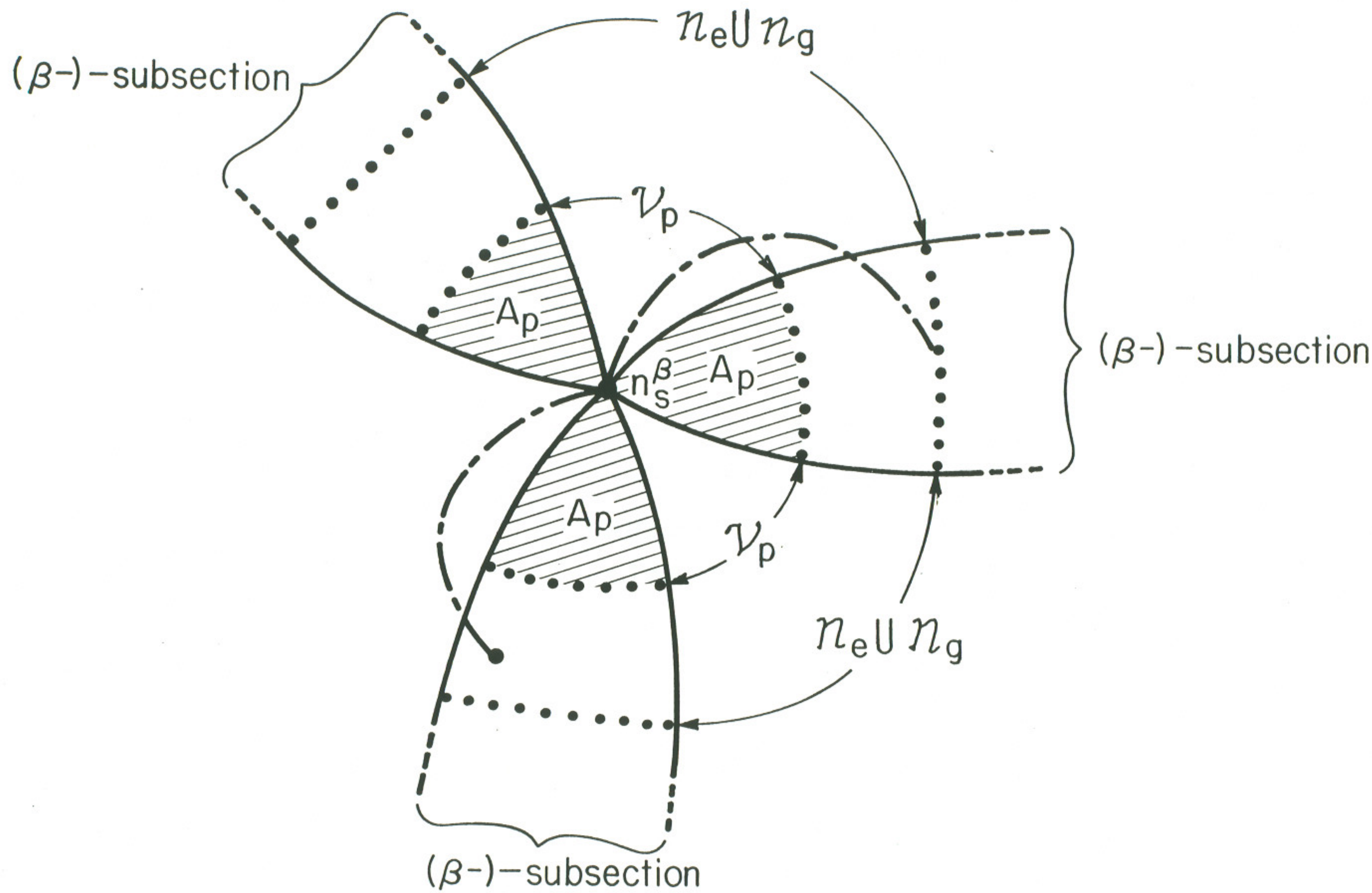

$n_1^{\beta^+}$
$n_i^{\beta^+}$
$\mathcal{V}_{1,p_1}$
b_2
$\mathcal{V}_{l,p_l}$
$n_K^{\beta^+}$
$\mathcal{V}_{K,p_K}$
$n_l^{\beta^+}$
n_s
$F(p_1, \cdots, p_K)$
$\mathcal{V}_{k,p_k}$
b_1
$n_k^{\beta^+}$

FIG. 3



$\mathcal{n}_e \cup \mathcal{n}_g$
(β-)-subsection
ν_p
A_p
(β-)-subsection
n_s^β
A_p
A_p
ν_p
$\mathcal{n}_e \cup \mathcal{n}_g$
(β-)-subsection

FIG. 4

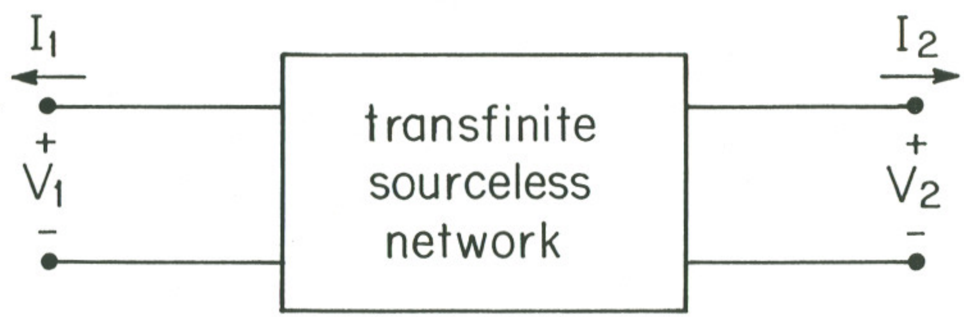
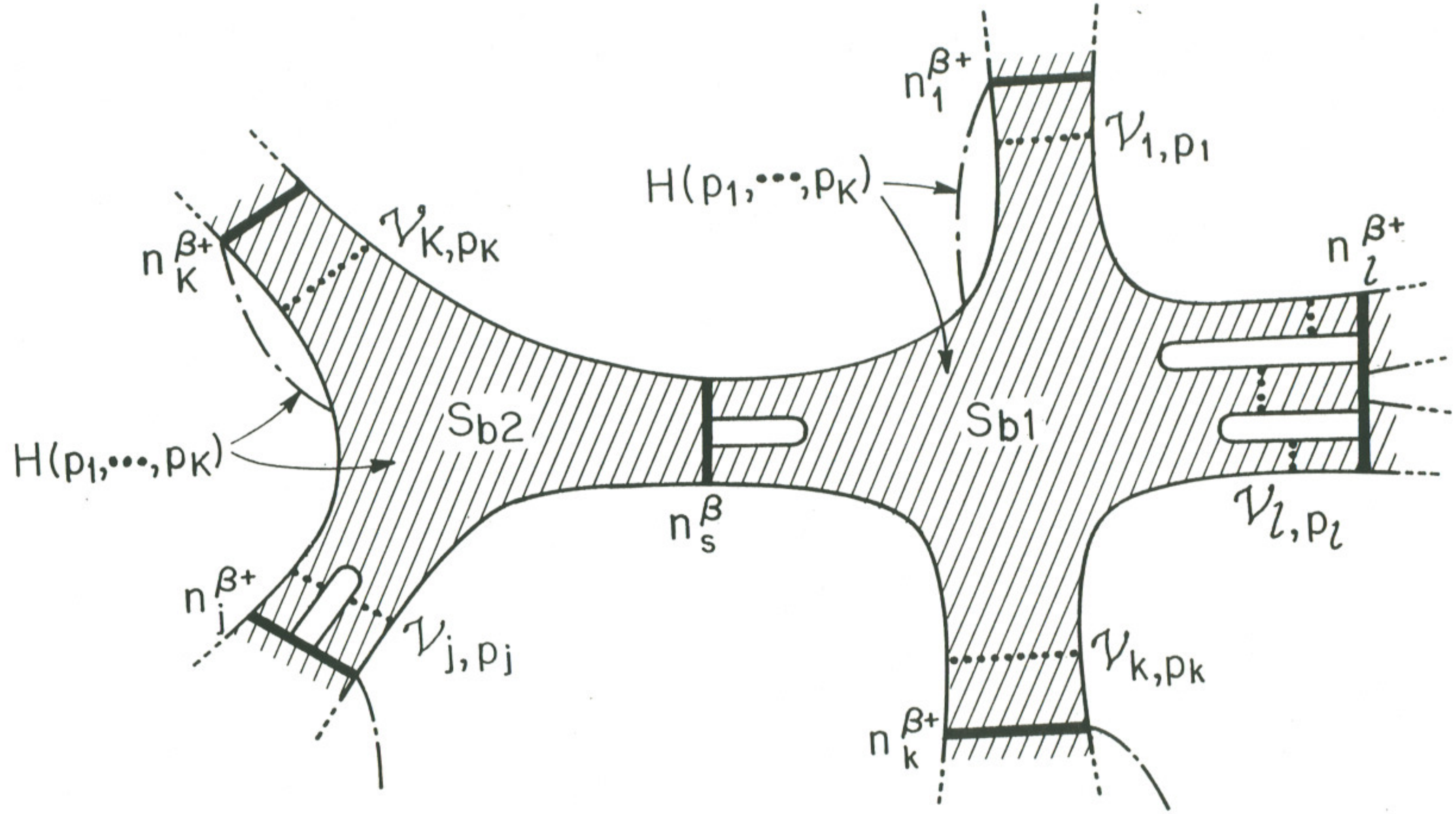


FIG. 5



$n_1^{\beta+}$
$\mathcal{V}_{1,p_1}$
$H(p_1,\cdots,p_K)$
$n_l^{\beta+}$
$n_K^{\beta+}$
$\mathcal{V}_{K,p_K}$
S_{b2}
S_{b1}
$H(p_1,\cdots,p_K)$
n_s^{β}
$\mathcal{V}_{l,p_l}$
$n_j^{\beta+}$
$\mathcal{V}_{j,p_j}$
$\mathcal{V}_{k,p_k}$
$n_k^{\beta+}$

FIG. 6