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NONSTANDARD ELECTRICAL NETWORKS
AND THE RESURRECTION OF KIRCHHOFF'S LAWS

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Abstract — Kirchhoff's laws fail to hold in general for infinite electrical networks. Standard calculus is simply incapable of resolving this paradox because it cannot provide the infinitesimals and more generally the hyperreal currents and voltages that such networks often require. However, nonstandard analysis can do precisely this. The idea of a nonstandard electrical network is introduced in this paper and is used to reestablish Kirchhoff's laws for a fairly broad class of infinite electrical networks. The second section herein presents a fairly brief tutorial on infinitesimals, hyperreal numbers, and the key ideas of nonstandard analysis needed for a comprehension of this work.

1 Introduction

It is fairly well-known now that Kirchhoff's laws need not always hold in infinite networks (see [12, Sections 1.6 and 3.4]). Perhaps the simplest example of this is provided by Figure 1, wherein an infinite parallel circuit of 1Ω resistors is fed by a 1 V voltage source in series with another 1Ω resistor. The infinite parallel circuit must be a short ($r = 0 \Omega$), and thus the voltage across it is $v = 0 \text{ V}$. Hence, the current i_k in each resistor ($k = 1, 2, 3, \dots$) is 0 A , and therefore $\sum_{k=1}^{\infty} i_k = 0 \text{ A}$ too. On the other hand, since $v = 0$, we have that $i_0 = 1 \text{ A}$. We conclude that 1 A flows toward node n_2 but 0 A flows away from it — in violation of Kirchhoff's current law.

How to explain this discrepancy would have been no problem two hundred years ago, for a conventional mathematical argument at that time (had electrical circuits been extant then) would assert that the current i_k in each parallel resistor is not exactly 0 A but is

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instead an infinitesimal and that the sum $\sum_{k=1}^{\infty} i_k$ of those infinitesimals is 1 A. Though intuitive and useful, infinitesimals were not legitimate — at least by later standards of mathematical rigor; in fact, they simply served as a “fudge factor,” allowing one to achieve apparently valid results. During the nineteenth century, infinitesimals were expunged from rigorous mathematics and replaced by limit processes based upon cumbersome epsilon-delta arguments. But, the facility of infinitesimals insured their survival at least for informal arguments. Not surprisingly, they were eventually reinvented, this time rigorously by Abraham Robinson [9], [10] about 35 years ago. Not only did this justify at last techniques that were so useful in the early development of calculus and differential equations during the 17th and 18th centuries, it also established an alternative method, called *nonstandard analysis*, for generating new mathematics, both pure and applied. As the literature survey of Lindstrom [8, pages 90-98] indicates, nonstandard analysis has been effectively used in a variety of mathematical subjects.

The objective of this paper is to introduce nonstandard circuit theory (perhaps for the first time?). One consequence will be the establishment of Kirchhoff's laws as inviolate rules for certain infinite networks. The kind of nonstandard network we shall consider is a finite interconnection of one-ports having nonstandard parameters. We shall show that certain one-ports, which are internally infinite, have nonstandard descriptions. The network of Figure 1 is of this sort, the infinite parallel circuit being a one-port with a nonstandard input resistance..

In Section 2 we provide some information about nonstandard analysis; this may serve as a brief tutorial on that subject. In Section 3 we lift Kirchhoff's laws and Ohm's law into the nonstandard realm and thereby obtain nonstandard versions of nodal analysis. This discussion also indicates that much more of conventional network analysis, such as fundamental-loop analysis and Thevenin's and Norton's theorems, can be extended in a nonstandard way. Certain one-ports having infinite internal circuits, such as infinite parallel circuits and infinite series circuits, are introduced in Section 4 and are shown to have nonstandard descriptions. These examples are generalized in Section 5 to obtain two broad classes of internally infinite one-ports that not only have nonstandard port-terminal de-

scriptions but also satisfy Kirchhoff's laws at their internal infinite nodes and infinite loops in a nonstandard way. Upon combining the results of Sections 3, 4, and 5, we finally obtain in Section 6 a nonstandard theory for a broad class of infinite networks in which Kirchhoff's laws are always satisfied and determine voltage-current regimes. This final conclusion is stated succinctly in the italicized paragraph of that section.

2 Some Elements of Nonstandard Analysis

Let us explain some of the elements of nonstandard analysis, enough to provide an understanding of the nonstandard circuit theory we introduce below. This section presents but a bare minimum of ideas. More detailed introductions to nonstandard analysis can be found in a number of sources. Davis and Hersh [2] provide a historical perspective along with a short description of some key ideas. Henle and Kleinberg [3] and Keisler [5] give elementary expositions. Introductions at a somewhat higher level are those of Keisler [6] and Lindstrom [8]. Far more is presented in the books by Davis [1] and Hurd and Loeb [4], but it may be advisable for the novice to read the aforementioned introductions before attempting these books. Robinson's original book [10] is at an advanced level and requires a knowledge of symbolic logic.

There are two approaches to nonstandard analysis, the axiomatic and the constructive. Actually, each subsumes the other, the only difference being the order in which ideas are introduced. The axiomatic approach is based upon symbolic logic and is the method used by Robinson in his seminal works [9], [10]. As a result, the earlier expositions were of this kind but did not provide much intuition as to the nature of infinitesimals. The constructive approach is probably more accessible to those accustomed to mathematical analysis rather than mathematical logic, in particular, to circuit theorists.

So how can one construct an infinitesimal? Well, how are the real numbers constructed? Answering the latter question may give us a clue about the former one. The constructions of the rational numbers as ratios of integers was accomplished by the ancients, but it was disconcerting when the Pythagoreans discovered that the diagonal of the unit square is not such a ratio [7, pages 104-105]. This situation remained unresolved from ancient times to

the late 19th century, when at last Cantor and Dedekind [7, page 179] proposed different but equivalent definitions for the real numbers. The definition most pertinent for us is that whereby a real number is defined as an equivalent class of Cauchy sequences of rational numbers, two Cauchy sequences being considered equivalent if their terms approach each other. In this way, the real numbers expand and fill out the set of rational numbers.

This idea can be reworked to define infinitesimals as certain equivalence classes of sequences of real numbers that approximate 0 in a certain way. More generally, all real sequences can be partitioned into equivalence classes, called *hyperreal numbers*, and they include not only infinitesimals but also numbers infinitesimally close to any real number, as well as infinitely large numbers.

To be more specific, let $N = \{1, 2, 3, \dots\}$ be the set of all positive natural numbers n , and let $\{a_n\} = \{a_n\}_{n \in N}$ denote a sequence of real numbers. Two (not necessarily convergent) sequences $\{a_n\}$ and $\{b_n\}$ will be called “equivalent” if they are the same on a “large enough” subset of N . In order to specify which subsets are “large enough,” we choose a measure m for all the subsets M of N as follows.

Conditions 2.1.

- (i) For each $M \subset N$, either $m(M) = 0$ or $m(M) = 1$.
- (ii) $m(N) = 1$.
- (iii) If M is a finite subset of N , then $m(M) = 0$.
- (iv) m is finitely additive; that is, if M_1 and M_2 are disjoint subsets, then $m(M_1 \cup M_2) = m(M_1) + m(M_2)$.

The “large enough” subsets are those having measure 1. These four conditions imply the following results: If $m(M) = 1$ (or $m(M) = 0$), then, for the complement $M^c = N \setminus M$ of M , $m(M^c) = 0$ (respectively, $m(M^c) = 1$). If $\{M_1, M_2, \dots, M_p\}$ is a finite partition of N , (that is, if these finitely many subsets are pairwise disjoint and if their union is N), then exactly one of the M_k has measure 1 and all the others have measure 0. If $m(M_1) = 1$ and $m(M_2) = 1$, then $m(M_1 \cap M_2) = 1$. If $M_1 \subset M_2$ and if $m(M_1) = 1$, then $m(M_2) = 1$. A

proof that such a measure can be assigned to the subsets of N is given in [8, pages 84-85]. In fact, there are many such measures.

Having chosen a particular measure m of this sort, we can partition the set of all sequences of real numbers into equivalence classes by taking two such sequences as being *equivalent* if the subset of N on which they agree has measure 1; in symbols, $\{a_n\}$ and $\{b_n\}$ are considered equivalent if $m\{n: a_n = b_n\} = 1$. An equivalence class will be denoted by $\langle a_n \rangle$ or by $\langle a_1, a_2, a_3, \dots \rangle$, where $\{a_n\}$ is any one of the sequences in that class. Any such sequence $\{a_n\}$ is called a *representative* of that class. These equivalence classes represent new entities, called *hyperreal numbers* or simply *hyperreals*. (Similarly, we often say just “real” in place of “real number.”) We let \mathcal{R} denote the set of reals and ${}^*\mathcal{R}$ the set of hyperreals. ${}^*\mathcal{R}$ can be viewed as an extension of \mathcal{R} as follows. If $a \in \mathcal{R}$, then the equivalence class $A = \langle a, a, a, \dots \rangle \in {}^*\mathcal{R}$ is the *image of a* in ${}^*\mathcal{R}$; A is also called the *hyperreal image of a* . On the other hand, the hyperreal number $\langle 1, 1/2, 1/3, 1/4, \dots \rangle$ is not the image of any real number but is in fact an “infinitesimal” (see below). Also, $\langle 1, 2, 3, \dots \rangle$ is an “infinite hyperreal” (in particular, an “infinite integer” because all its entries are integers). Furthermore, $\langle 1, 1/2, 1/3, 1/4, 1/5, 1/6, \dots \rangle$ and $\langle 1, 0, 1/3, 0, 1/5, 0, \dots \rangle$ denote the same hyperreal if the measure we have chosen assigns 1 to the set of odd positive natural numbers.

We shall use the following symbolism. n will always be an index varying through N . Thus, $\{a_n\} = \{a_1, a_2, a_3, \dots\}$ is a sequence, and $\langle a_n \rangle = \langle a_1, a_2, a_3, \dots \rangle$ is the hyperreal having $\{a_n\}$ as one of its representatives. A constant $c \in \mathcal{R}$ will have $\langle c \rangle = \langle c, c, c, \dots \rangle$ as its hyperreal image (the symbol *c is also used for $\langle c \rangle$). Moreover, a hyperreal symbol, such as $\langle a_k \rangle$, involving a subscript other than n will always denote the hyperreal image of a constant $a_k \in \mathcal{R}$; thus, $\langle a_k \rangle$ means $\langle a_k, a_k, a_k, \dots \rangle$, not $\langle a_1, a_2, a_3, \dots \rangle$. Also, real numbers will be denoted by lower-case Roman letters, and hyperreals by upper-case Roman letters. We will usually use corresponding lower-case and upper-case letters, as for example $A = \langle a \rangle = \langle a, a, a, \dots \rangle$ or $A = \langle a_n \rangle = \langle a_1, a_2, a_3, \dots \rangle$. If the hyperreal carries an index, that index will never be n ; thus, we may write “ A_k ” but never “ A_n .”

The hyperreals are ordered as follows. If $X = \langle x_n \rangle$ and $Y = \langle y_n \rangle$ and if $m\{n: x_n < y_n\} = 1$, then $X < Y$. Thus, the real numbers and their images in ${}^*\mathcal{R}$ have the same order, and that

order extends to all of ${}^*\mathcal{R}$. In this way, we have positive hyperreals $X > \langle 0 \rangle$ and negative hyperreals $X < \langle 0 \rangle$. Moreover, we define the arithmetic operations componentwise using any representatives for the hyperreals. In particular, $X + Y = \langle x_n + y_n \rangle$, $X - Y = \langle x_n - y_n \rangle$, $XY = \langle x_n y_n \rangle$, and $X/Y = \langle x_n / y_n \rangle$ if $Y \neq \langle 0 \rangle$ in the last case. The usual arithmetic laws hold in ${}^*\mathcal{R}$; in fact, ${}^*\mathcal{R}$ is an ordered field with the zero element $\langle 0 \rangle$ and the unit element $\langle 1 \rangle$. Therefore, we can manipulate the hyperreals as we do the real numbers. The absolute value $|X|$ of a hyperreal X is X if $X > \langle 0 \rangle$, is $-X$ if $X < \langle 0 \rangle$, and is $\langle 0 \rangle$ if $X = \langle 0 \rangle$.

A hyperreal X is called *infinitesimal* (resp. *infinite*) if $|X| < \langle a \rangle$ (resp. $|X| > \langle a \rangle$) for every positive real number a . A hyperreal that is not infinite is called *finite*. Two hyperreals X and Y are said to be *infinitesimally close* if $X - Y$ is infinitesimal, in which case we write $X \approx Y$. Altogether, the set of all hyperreals is called the *hyperreal line*, and that line consists of the infinitesimals (those hyperreals infinitesimally close to $\langle 0 \rangle$), the finite hyperreals (each of which is infinitesimally close to some image of a real number), and the infinite hyperreals. In fact, for each hyperreal X (finite or infinite), there is a set of infinitely close hyperreals called the *monad for X* , and there is a larger set of finitely close hyperreals Y (i.e., $X - Y$ is finite) called the *galaxy for X* . Furthermore, each finite hyperreal X has a *standard part* $\text{st}X \in \mathcal{R}$ such that $X \approx \langle \text{st}X \rangle$. There is only one such real $\text{st}X$ having an image in ${}^*\mathcal{R}$ to which the finite hyperreal X is infinitesimally close.

Let us at this point discuss the arbitrariness arising from the choice among many possibilities of the measure m . For example, consider the hyperreal

$$\langle 1, 0, 1, 0, 1, 0, 1, \dots \rangle. \quad (1)$$

One of the measures we may choose will assign 1 to the set of even positive integers, in which case (1) will be equal to the hyperreal $\langle 0 \rangle$. On the other hand, another permissible measure will assign 1 to the set of odd positive integers, in which case (1) will be equal to the hyperreal $\langle 1 \rangle$, and $\langle 0 \rangle$ will now be equal to $\langle 0, 1, 0, 1, \dots \rangle$. We can view this arbitrariness as simply different ways of arriving at the same hyperreal line. Moreover, the final result of this paper, namely, the resurrection of Kirchhoff's laws will be accomplished whatever be the choice of the measure m .

Another peculiarity arises from the fact (pointed out before) that, for any finite partition

of N , any chosen measure m assigns 1 to exactly one of the subsets in the partition and 0 to all the others. Consider for example the nested subsets of N of the form $N_p = \{n : n = 2^p k, k = 1, 2, 3, \dots\}$, where p is an even positive natural number; thus, if $p_j < p_l$, then $N_{p_l} \subset N_{p_j}$. We can choose a measure m such that $m(N_p) = 1$ for all p . Also, for any fixed p , let $a_{p,n} = 1$ for $n = 2^p k$, and let $a_{p,n} = 0$ for $n \neq 2^p k$. Because of our choice of m ,

$$\langle a_{1,n} \rangle = \langle 0, 1, 0, 1, 0, 1, \dots \rangle = \langle 1, 1, 1, \dots \rangle = \langle 0, 0, 0, 1, 0, 0, 0, 1, \dots \rangle = \langle a_{2,n} \rangle.$$

This indicates that a sequence of progressively sparser samplings of a given hyperreal $\langle a_n \rangle$ identify that hyperreal; in fact, the a_n can be changed outside any particular subset of measure 1 without altering that identification. Actually, for a fixed measure m , it is the asymptotic behavior of $\{a_n\}$ that determines $\langle a_n \rangle$, and that asymptotic behavior can be determined by a sampling within an arbitrarily sparse subset of N of measure 1.

It was noted above that, with the arithmetic operations transferred into ${}^*\mathcal{R}$ by means of componentwise definitions, ${}^*\mathcal{R}$ is a field in quite the same way as \mathcal{R} is a field. Consequently, rational functions of hyperreals and equations between such rational functions can be set up in the hyperreal realm. More specifically, consider two rational functions that involve some real coefficients a_k and some real variables x_j , which we may display together by means of a vector $\mathbf{s} = (a_1, \dots, a_K, x_1, \dots, x_J)$. Then, the equation between the rational functions will be true in the real realm when and only when \mathbf{s} lies in some subset \mathcal{S} of the Cartesian product \mathcal{R}^{K+J} of \mathcal{R} taken $K + J$ times. (Of course, that subset may be void.) It is an important property of nonstandard analysis that that equation will again be true in the subset ${}^*\mathcal{S}$ of ${}^*\mathcal{R}^{K+J}$ consisting of all $\langle \mathbf{s}_n \rangle$ such that $m\{n : \mathbf{s}_n \in \mathcal{S}\} = 1$. ${}^*\mathcal{S}$ is called an “internal set” in ${}^*\mathcal{R}^{K+J}$, but there are more general kinds of “internal sets” in ${}^*\mathcal{R}^{K+J}$ [6, page 43], [8, pages 10 and 24]. All this is a result of the so-called “transfer principle” of nonstandard analysis [6, pages 34-35], [8, page 77].

As a simple example, consider $a_1 x = a_2$ in the real realm. This equation is true whenever $a_1 \in \mathcal{R} \setminus \{0\}$, $a_2 \in \mathcal{R}$, and $x = a_2 \setminus a_1$. Thus, the set \mathcal{S} mentioned above is given by

$$\mathcal{S} = \{\mathbf{s} = (a_1, a_2, x) : a_1 \in \mathcal{R} \setminus \{0\}, a_2 \in \mathcal{R}, x = a_2 \setminus a_1\}.$$

Consequently, any $\langle \mathbf{s}_n \rangle$ of the form $\langle \mathbf{s}_n \rangle = (\langle a_{1n} \rangle, \langle a_{2n} \rangle, \langle x_n \rangle)$, where $a_{1n} \in \mathcal{R} \setminus \{0\}$, $a_{2n} \in \mathcal{R}$,

and $x_n = a_{2n} \setminus a_{1n}$ for all n in a subset of N of measure 1, will be a member of *S . We can succinctly restate this by setting $A_1 = \langle a_{1n} \rangle$, $A_2 = \langle a_{2n} \rangle$, and $X = \langle x_n \rangle$ and asserting that $A_1 X = A_2$ has the solution $X = A_2/A_1 \in {}^*\mathcal{R}$ when $A_1, A_2 \in {}^*\mathcal{R}$ and $A_1 \neq \langle 0 \rangle$. Here, A_1 and A_2 need not be the images of real numbers.

This is a critically important result, for it allows us to find solutions in the hyperreal realm that do not have preimages in the real realm. For instance, Kirchhoff's current law does not hold for the network of Figure 1 because the total resistance r of the infinite parallel circuit cannot be obtained through finitely many arithmetic operations. Actually, the result $r = 0$ can be obtained in \mathcal{R} by passing to a limit as the number of parallel resistances increases toward infinity. But, that passage to a limit through an infinite sequence of parallel combinations is an operation falling outside of the theory of finite networks. On the other hand, we shall show in the next section that the total resistance of the infinite parallel circuit can be represented by an infinitesimal obtained by means of an infinite sequence of finite parallel combinations. Moreover, the other parameters of Figure 1 can be taken as the hyperreal images of constants. So, by finite circuit theory lifted into the hyperreal realm, we shall find a hyperreal solution for the current I_0 in the source. Furthermore, the current in each branch of the infinite parallel circuit will be found to be an infinitesimal, and the sum of all those infinitesimals will be equal to the hyperreal solution for I_0 . This latter result will be a consequence of how we define the hyperreal image of an infinite series. Thus, Kirchhoff's current law will hold in ${}^*\mathcal{R}$ at the nodes n_1 and n_2 .

But how, one may persist in asking, does nonstandard analysis of some other infinite network provide a hyperreal solution when standard analysis provides none at all? It does so as follows. The kind of infinite network we shall consider will be a finite network of one-ports, some or all of which consist internally of infinite circuits. Each infinite circuit may in turn be viewed as the end result of a sequence of finite circuits. So, standard analysis, applied to finite networks of internally finite one-ports, will provide solutions for the currents entering and within the one-ports at each stage of the sequence. Thus, upon considering the sequence all at once, those currents become sequences of real numbers. The current sequences need not converge, but they will always be representatives of hyperreal

currents. Moreover, sufficiently many components of those representatives will satisfy the standard Kirchhoff current law to warrant the satisfaction of the nonstandard Kirchhoff current law. The nonstandard Kirchhoff voltage law will be satisfied in the same way. The hyperreals that we obtain may depend upon the ways in which the internally finite one-ports are expanded into internally infinite ones. Nonetheless, whatever ways are chosen, the resulting hyperreals will satisfy the nonstandard Kirchhoff laws.

Actually, our analysis is more complicated than this brief synopsis indicates. Choosing a finite circuit to accommodate Kirchhoff's current law fails to work for Kirchhoff's voltage law, and conversely. Thus, any internally infinite one-port can have either infinite nodes or infinite loops — but not both.

One last matter: Subsequently we will meet infinite series of hyperreals, as for instance when we sum the conductances in the infinite parallel circuit of Figure 1 or the currents in those conductances. We can sum an infinite series of hyperreals just as we do a series of reals whenever the hyperreals are images of reals. In particular, let $X_k = \langle x_k \rangle = \langle x_k, x_k, x_k, \dots \rangle$ for each $k = 1, 2, 3, \dots$. Then, define $\sum_{k=1}^{\infty} X_k$ as the hyperreal having as a representative the sequence of partial sums of the reals taken in their natural order:

$$\sum_{k=1}^{\infty} X_k = \left\langle \sum_{k=1}^n x_k \right\rangle = \langle x_1, x_1 + x_2, x_1 + x_2 + x_3, \dots \rangle \quad (2)$$

This result may be either infinitesimal, finite, or infinite. For instance,

$$\left\langle -\frac{1}{2} \right\rangle + \sum_{k=2}^{\infty} \langle 3(-2)^{-k} \rangle = \left\langle \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \dots \right\rangle \approx \langle 0 \rangle,$$

$$\sum_{k=1}^{\infty} \langle 2^{-k} \rangle = \left\langle \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \dots \right\rangle \approx \langle 1 \rangle,$$

$$\sum_{k=1}^{\infty} \langle 1 \rangle = \langle 1, 2, 3, 4, \dots \rangle,$$

where the last series is an infinite hyperreal. Note however that an alteration in the order in which the terms of a hyperreal series is taken will in general change the hyperreal value of the series.

We will also encounter hyperreal infinite series whose terms are not images of reals. These too can be summed in ${}^*\mathcal{R}$ if each X_k has the form $X_k = \langle x_{k,n} \rangle$, where $x_{k,n} = 0$ when

$1 \leq n < k$. Any hyperreal X_k can be put into this form just by changing finitely many of its representative components. In this case, we have

$$\begin{aligned} \sum_{k=1}^{\infty} X_k &= \langle x_{1,1}, x_{1,2}, x_{1,3}, \dots \rangle \\ &+ \langle 0, x_{2,2}, x_{2,3}, \dots \rangle \\ &+ \langle 0, 0, x_{3,3}, \dots \rangle \\ &+ \dots \end{aligned} \tag{3}$$

(The series (2) has just this form with $x_{k,n}$ replaced by x_k whenever $n \geq k$.) Summing the X_k componentwise is the same as summing the rectangular array in (3) columnwise. This works because each column has only finitely many nonzero entries. The result is

$$\sum_{k=1}^{\infty} X_k = \langle \sum_{k=1}^n x_{k,n} \rangle = \langle x_{1,1}, x_{1,2} + x_{2,2}, x_{1,3} + x_{2,3} + x_{3,3}, \dots \rangle. \tag{4}$$

This then is our hyperreal definition of $\sum_{k=1}^{\infty} X_k$ when the X_k have the stated form. Moreover, this is just the form the X_k will have when we build our infinite circuits by inserting one branch at a time. Here too, the final result will depend in general on the order in which we insert the branches. Note that we are not free to alter arbitrarily the representatives of infinitely many of the X_k for this may change the hyperreal value of the series. This difficulty will be avoided by choosing and fixing a particular order for the insertion of branches.

Here ends our brief tutorial. Our explanation of the hyperreals has been cursory but hopefully informative. We have left out many essential concepts that one should know if one is to be knowledgeable about nonstandard analysis. We recommend [6] or [8] for an introduction to this subject.

3 Finite Nonstandard Electrical Networks

Our objective in this section is to show how nonstandard analysis can be applied to finite electrical networks. In other words, we wish to lift the theory of such networks from the real realm into the hyperreal realm.

For the sake of specificity, let us do so for a nodal analysis of a particular kind of network. We start in the real realm with a finite connected network of $p + 1$ nodes and q branches, with each branch assigned an orientation and consisting of either a positive conductance g_j alone, or an independent current source h_j alone, or a parallel combination of g_j and h_j as shown in Figure 2(a). This is the Norton form of a branch. The branch index varies from 1 to q . We will call this the *standard network*. With the superscript T denoting matrix transpose, we let the column vectors

$$\mathbf{v} = [v_1, \dots, v_q]^T \quad (5)$$

and

$$\mathbf{i} = [i_1, \dots, i_q]^T \quad (6)$$

represent respectively the vector of branch voltages and the vector of branch currents. Next, choose a ground node n_g and index the remaining nodes by $k = 1, \dots, p$. Let $[a_{kj}]$ be the $p \times q$ incidence matrix, wherein $a_{kj} = 1$ (or -1) if branch b_j is incident toward (respectively, away from) node n_k and $a_{kj} = 0$ if b_j is not incident to n_k . It follows that Kirchhoff's current law holds at every node if and only if

$$[a_{kj}]\mathbf{i} = \mathbf{0} \quad (7)$$

Next, assign $u_g = 0$ V as the voltage at the ground node n_g , and choose the other node voltages u_k ($k = 1, \dots, p$) arbitrarily — at least for the moment. If branch b_j is incident away from node n_k and toward node n_l , then its branch voltage is related to the corresponding node voltages by $v_j = u_k - u_l$; one of these nodes may be ground. Let

$$\mathbf{u} = [u_1, \dots, u_p]^T \quad (8)$$

be the vector of voltages at all the nodes other than ground. Then, the relationship between node voltages and branch voltages can be written succinctly as

$$\mathbf{v} = [a_{kj}]^T \mathbf{u} \quad (9)$$

It is a fact that Kirchhoff's voltage law holds around every loop of the network if and only if \mathbf{v} is given by (9) for some vector \mathbf{u} of node voltages.

Furthermore, let $[g_{jj}]$ be the diagonal $q \times q$ matrix whose main-diagonal entries are $g_{jj} = g_j$ and whose other entries are 0; possibly, some of the main diagonal entries may be 0 too (i.e., $g_j = 0$ for some j). By Ohm's law, the vector of currents in the conductances is $[g_{jj}]\mathbf{v}$. With $\mathbf{h} = [h_1, \dots, h_q]^T$ being the vector of branch current sources, the branch current vector \mathbf{i} is then

$$\mathbf{i} = [g_{jj}]\mathbf{v} - \mathbf{h}. \quad (10)$$

So, given the finite network, that is, its graph and all the values g_j and h_j , a combination of (7), (9), and (10) comprises a nodal analysis yielding the following expression for \mathbf{v}

$$\mathbf{v} = [a_{kj}]^T ([a_{kj}][g_{jj}][a_{kj}]^T)^{-1} [a_{kj}]\mathbf{h} \quad (11)$$

so long as $[a_{kj}][g_{jj}][a_{kj}]^T$ is nonsingular. A sufficient condition for that nonsingularity is that every branch have a positive conductance: $g_j > 0$ for all j . These are all conventional results in the real realm. In summary, *if Ohm's law (10) is satisfied and if $[a_{kj}][g_{jj}][a_{kj}]^T$ is nonsingular — or, more particularly, if every branch has a positive conductance, then Kirchhoff's laws are satisfied throughout the network if and only if (11) holds.*

Note that the right-hand side of (11) involves no more than a finite number of arithmetic operations. Since these have been lifted into the hyperreal realm through componentwise definitions, we can transfer (11) into that realm under the caveat of nonsingularity. Indeed, we can now analyze in the very same way a finite connected network whose every branch is either a hyperreal positive conductance G_j , or a hyperreal independent current source H_j , or a parallel combination of these two elements. Such a network will be called a *nonstandard network* or synonymously a *network in the hyperreal realm*. All voltages and currents will now be hyperreals. So, upon replacing all lower-case symbols for the source currents, conductances, and voltages by upper-case symbols, we have the following results. Kirchhoff's current law becomes

$$[a_{kj}]\mathbf{I} = \langle \mathbf{0} \rangle = [\langle 0 \rangle, \dots, \langle 0 \rangle]^T, \quad (12)$$

where

$$\mathbf{I} = [G_{jj}]\mathbf{V} - \mathbf{H}. \quad (13)$$

Kirchhoff's voltage law is assured whenever

$$\mathbf{V} = [a_{kj}]^T \mathbf{U} \quad (14)$$

for some node voltage vector \mathbf{U} . Finally, (11) becomes

$$\mathbf{V} = [a_{kj}]^T ([a_{kj}][G_{jj}][a_{kj}]^T)^{-1} [a_{kj}] \mathbf{H}. \quad (15)$$

(We could also replace the 0's, 1's, and -1 's in $[a_{kj}]$ by $\langle 0 \rangle$'s, $\langle 1 \rangle$'s, and $\langle -1 \rangle$'s to get $[A_{kj}]$ in place of $[a_{kj}]$, but the operations with the $[A_{kj}]$ merely represent additions and subtractions, which we have already defined on the hyperreals componentwise; thus, there is no need for this latter change in symbols.)

In order for (15) to be meaningful in the hyperreal realm, we need to confine the determinant of $[a_{kj}][G_{jj}][a_{kj}]^T$ to the internal set ${}^*\mathcal{R} \setminus \langle 0 \rangle$. This will certainly be so if G_j is a positive hyperreal for every j . Other than this, there is no restriction on the hyperreal elements of $[G_{jj}]$ and \mathbf{H} . We can conclude as follows: *If Ohm's law (13) is satisfied and if the determinant of $[a_{kj}][G_{jj}][a_{kj}]^T$ is not equal to $\langle 0 \rangle$ or, more particularly, if every branch of the nonstandard network has a positive hyperreal conductance G_j , then, for every choice of the hyperreal independent current-source vector \mathbf{H} , (15) holds, and the hyperreal branch-voltage vector \mathbf{V} and the hyperreal branch-current vector \mathbf{I} will satisfy Kirchhoff's voltage and current laws throughout the network.* Note that, the G_j and H_j need not be hyperreal images of real numbers, and therefore the nonstandard network need not be the hyperreal image of a standard network. In particular, any conductance, voltage, or current in the nonstandard network may be either an infinitesimal, finite, or infinite hyperreal.

We have been considering a nonstandard nodal analysis. Nevertheless, much more of standard network theory can be transferred into the hyperreal realm to get, for example, a nonstandard mesh analysis or more generally a fundamental-loop analysis [11]. Also, many of the usual network theorems, such as Thevenin's and Norton's theorems, can be transferred into the hyperreal realm as well. We will not explicitly do so here because the needed arguments are much the same as those used for a nodal analysis. All the nonstandard finite-network theory we employ in this paper is truly available.

Let us now restate Kirchhoff's laws in terms of hyperreals. His current law (12) can be rewritten as

$$\sum_{(n_0)} \pm \langle i_{j,n} \rangle = \langle 0 \rangle, \quad (16)$$

where n_0 is any node in the nonstandard finite network, the summation is over the indices j for the branches incident to n_0 , $\langle i_{j,n} \rangle = \langle i_{j,1}, i_{j,2}, \dots \rangle$ is the hyperreal current in the j th branch, and the plus (minus) sign is used if branch j is incident toward (respectively, away from) node n_0 . The equality in (16) holds componentwise for almost all n ; that is, there is a subset N_1 of N of measure 1 such that $\sum_{(n_0)} \pm i_{j,n} = 0$ for all $n \in N_1$.

As for Kirchhoff's hyperreal voltage law, we have

$$\sum_{(L)} \pm \langle v_{j,n} \rangle = \langle 0 \rangle, \quad (17)$$

where L is any oriented loop in the nonstandard finite network, the summation is over the indices j for the branches in L , $\langle v_{j,n} \rangle = \langle v_{j,1}, v_{j,2}, \dots \rangle$ is the hyperreal voltage in the j th branch, and the plus (minus) sign is used if the orientation of branch j agrees (respectively, disagrees) with the orientation of loop L . Finally, the equality in (17) holds componentwise for almost all n , as indicated above.

Finally, let us note that not all finite networks of hyperreal one-ports are amenable to a nonstandard analysis – as, for example, when there is a loop of pure voltage sources that violate (16), or a node with only pure current sources as its incident branches which violate (17), or simply a three-branch network consisting of a parallel circuit of a $\langle 1 \rangle \Omega$ resistor, a $\langle -1 \rangle \Omega$ resistor, and a $\langle 1 \rangle$ V voltage source. So, we have to restrict the kinds of nonstandard networks we consider. With regard to nodal analysis, our assumption about the nonsingularity of $[a_{kj}][G_{jj}][a_{kj}]^T$ suffices for this purpose.

4 Nonstandard One-Ports and Their Hyperreal Representations

We have seen in the last section that a nonstandard finite network of one-ports with hyperreal parameters has a hyperreal voltage-current regime, but we have not looked inside the one-ports to see if their hyperreal parameters represent any standard electrical circuits,

finite or infinite. If the hyperreal parameters of a given nonstandard one-port are images of real parameters, then the one-port is internally the hyperreal image of a real circuit. For example, if the nonstandard one-port is described by $I = GV - H$, where $G = \langle g \rangle$ and $H = \langle h \rangle$ are hyperreal images of the reals g and h , then the nonstandard one-port is just the hyperreal image of the real Norton branch such as that of Figure 2(a). If in addition $0 < g < \infty$, then the nonstandard one-port is also the hyperreal image of the equivalent Thevenin branch of Figure 2(b).

We will now examine several cases wherein the parameters of a nonstandard one-port are hyperreals that need not be images of reals.

4a. An infinite parallel circuit of conductances: Let us assume that the nonstandard one-port is described by $I = GV$, where I , V , and G are the one-port's hyperreal current, voltage, and input conductance. In particular, let $G = \langle g_1, g_2, g_3, \dots \rangle$ be any hyperreal. By virtue of the definition (2) of an infinite series of hyperreals, G is synthesized as the infinite parallel circuit of hyperreal images $\langle a_k \rangle$ of real conductances a_k , where $a_1 = g_1$ and $a_k = g_k - g_{k-1}$ for $k = 2, 3, 4, \dots$. Thus,

$$G = \langle a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots \rangle. \quad (18)$$

This is illustrated in Figure 3(a). There are no restrictions on the g_k and thus none on the a_k either; they can be positive or negative and have any order of growth or decay. Consequently, G can be either an infinitesimal, finite, or infinite hyperreal. Since each $\langle a_k \rangle$ is the hyperreal image of the real conductance a_k , we may take the preimage of the hyperreal circuit branch by branch to obtain the real parallel circuit shown in Figure 3(b), but now that real circuit may not have a real input conductance. If the sequence

$$\{g_k\}_{k=1}^{\infty} = \{a_1 + \dots + a_k\}_{k=1}^{\infty} \quad (19)$$

converges, then its limit will be the standard part $\text{st}G$ of G [6, Theorem 8(i), page 56], and the difference between G and the hyperreal image of $\text{st}G$ will be the infinitesimal

$$\langle \text{st}G \rangle - G = \left\langle \sum_{k=2}^{\infty} a_k, \sum_{k=3}^{\infty} a_k, \sum_{k=4}^{\infty} a_k, \dots \right\rangle.$$

This will be so whatever be the choice of the measure m . However, if the sequence (19) strictly increases in absolute value unboundedly, then G will be infinite whatever be m and therefore will not have a standard part [6, Theorem 9(i), page 57]. Finally, if the sequence (19) oscillates boundedly, then G will be finite as long as the measure m is chosen appropriately [6, Theorem 9(ii), page 57], in which case it will have a standard part. A particular case of interest is when all the a_k are positive; now (19) either increases to a limit or increases unboundedly, and G is either a positive finite hyperreal or respectively a positive infinite hyperreal.

Now, let $G \neq \langle 0 \rangle$. The hyperreal input resistance R of the circuit of Figure 3(a) is

$$R = G^{-1} = \langle r_1, r_2, r_3, \dots \rangle,$$

where $r_k = g_k^{-1} = (a_1 + \dots + a_k)^{-1}$. R has a nonzero standard part when and only when G has a nonzero standard part; otherwise, $\text{st}R = 0$ when G is infinite, and R is infinite when $\text{st}G = 0$.

There is an important feature of these infinite parallel circuits that must be taken into account: The hyperreal G depends in general upon the order in which the conductances are connected into the parallel circuit. We may alter that order for finitely many of the a_k without changing G , but infinitely many such changes may affect G . For example, let $a_k = k$ for all $k \in \mathbb{N}$ and insert these a_k in accordance with the natural order of their indices. Then, $G = \langle a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots \rangle$ becomes

$$G = \langle 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \dots \rangle. \quad (20)$$

On the other hand, let us instead reorder those conductances by first connecting a_1 , then a_3 and a_5 , then a_2 , then a_7 and a_9 , then a_4 , and so forth. That is, we continue choosing sequentially two odd-indexed conductances and then one even-indexed conductance, repeating this cycle indefinitely. This corresponds to reindexing the conductances according to $1 \mapsto 1, 3 \mapsto 2, 5 \mapsto 3, 2 \mapsto 4, 7 \mapsto 5, 9 \mapsto 6, 4 \mapsto 7, \dots$. With this reindexing, G becomes

$$G' = \langle 1, 4, 9, 11, 18, 27, 31, 42, 55, 61, \dots \rangle. \quad (21)$$

No two corresponding subsequences of (20) and (21) will be identical, and therefore G and G' must represent different hyperreals. This shows that one cannot rearrange the conductances

of an nonstandard infinite parallel circuit without in general changing its input conductance. Because of this, it is not enough to specify the set of conductances in a nonstandard infinite parallel circuit if the hyperreal input conductance is to be uniquely determined; the order in which the conductances are introduced into the parallel circuit must also be specified — except in certain special cases, such as the case where all the conductances have the same value.

Another discrepancy can arise when one tries to simplify an infinite parallel circuit by combining conductances. For instance, consider again the infinite parallel circuit of $1\text{-}\mathcal{U}$ conductances. Its input conductance is $G = \langle 1, 2, 3, \dots \rangle$. However, if we combine pairs of conductances into $2\text{-}\mathcal{U}$ conductances, the resulting parallel circuit of $2\text{-}\mathcal{U}$ conductances will have the input conductance $G' = \langle 2, 4, 6, \dots \rangle$. Clearly, G and G' are different hyperreals.

It is noteworthy however that, if

$$G = \langle a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots \rangle,$$

if $\sum_{k=1}^{\infty} a_k$ converges absolutely, and if G' is obtained by combining some of the conductances a_k before inserting them, then $G - G'$ will be an infinitesimal. (To see this, compare the terms of $G - G'$ with $\sum_{k=n}^{\infty} a_k$.)

All this indicates that there is more subtlety to infinite parallel circuits in the hyperreal realm than there is in the real realm. Actually, it is this subtlety that allows us to resurrect Kirchhoff's current law. In the real realm that law simply collapses at times.

So, let us consider the currents in the nonstandard infinite parallel circuit of Figure 3(a), where the order of the conductances $\langle a_k \rangle$ is taken to be that of its indices $k = 1, 2, 3, \dots$. Its input conductance G is given by (18), and its input current I_0 is related to its input voltage V by Ohm's law: $I_0 = VG$. By virtue of (3), this can be expanded into

$$\begin{aligned} I_0 &= V \langle a_1, a_1, a_1, \dots \rangle \\ &+ V \langle 0, a_2, a_2, \dots \rangle \\ &+ V \langle 0, 0, a_3, \dots \rangle \\ &+ \dots \end{aligned} \tag{22}$$

Each row in this array is the hyperreal current in the k th hyperreal conductance $\langle a_k \rangle$. Thus, we now have Kirchhoff's current law in the form of a nonstandard infinite series $I_0 = I_1 + I_2 + I_3 + \dots$.

We can use this technique to resolve the paradox concerning Figure 1. Let us consider the more general case where the conductances and voltage source are arbitrary reals. Thus, in the hyperreal realm we have the infinite parallel circuit of Figure 3(a) being fed from a source branch with the voltage source $E = \langle e \rangle$ in series with the conductance $G_0 = \langle g_0 \rangle$. Upon making a Thevenin-to-Norton conversion, that source branch becomes a current source $EG_0 = \langle eg_0 \rangle$ in parallel with $G_0 = \langle g_0 \rangle$. As before, we take it that the conductances $\langle a_k \rangle$ of the parallel circuit are built up in the order of their indices. By the current division law for a finite nonstandard network,

$$\begin{aligned} I_0 &= EG_0 \frac{G}{G_0 + G} \\ &= \langle eg_0 \rangle \frac{\langle a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots \rangle}{\langle g_0 \rangle + \langle a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots \rangle} \\ &= \left\langle \frac{eg_0 a_1}{g_0 + a_1}, \frac{eg_0(a_1 + a_2)}{g_0 + a_1 + a_2}, \frac{eg_0(a_1 + a_2 + a_3)}{g_0 + a_1 + a_2 + a_3}, \dots \right\rangle \\ &= \sum_{k=1}^{\infty} I_k \end{aligned}$$

where

$$I_k = \left\langle 0, \dots, 0, \frac{eg_0 a_k}{g_0 + a_1 + \dots + a_k}, \frac{eg_0 a_k}{g_0 + a_1 + \dots + a_{k+1}}, \frac{eg_0 a_k}{g_0 + a_1 + \dots + a_{k+2}}, \dots \right\rangle.$$

(In the last expression, there are $k - 1$ initial 0's.) So again, the nonstandard Kirchhoff's current law is satisfied.

In the special case of Figure 1 where $e = g_0 = a_1 = a_2 = \dots = 1$, we have

$$I_0 = \left\langle \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \right\rangle = \left\langle \frac{n}{n+1} \right\rangle$$

and

$$I_k = \left\langle 0, \dots, 0, \frac{1}{k+1}, \frac{1}{k+2}, \frac{1}{k+3}, \dots \right\rangle = \left\langle \frac{1}{n+1} \right\rangle.$$

We can interpret all of this by saying that, at every stage of the construction of the parallel circuit, Kirchhoff's current law is satisfied in the real realm, and therefore, for the infinite network, Kirchhoff's current law is satisfied in the hyperreal realm.

4b. An infinite parallel circuit of Norton branches: Let us reverse our approach now and start with an arbitrary nonstandard Norton branch consisting of a current source $H = \langle h_n \rangle = \langle h_1, h_2, h_3, \dots \rangle$ in parallel with a conductance $G = \langle g_n \rangle = \langle g_1, g_2, g_3, \dots \rangle$. Set $j_1 = h_1$ and $j_k = h_k - h_{k-1}$ for $k = 2, 3, 4, \dots$. Then, $H = \sum_{k=1}^{\infty} \langle j_k \rangle$. Also, set $a_1 = g_1$ and $a_k = g_k - g_{k-1}$ for $k = 2, 3, 4, \dots$. Then, $G = \sum_{k=1}^{\infty} \langle a_k \rangle$. It follows that any nonstandard Norton branch with the representation

$$I = VG - H = \left\langle v_n \sum_{k=1}^n a_k - \sum_{k=1}^n j_k \right\rangle$$

can be synthesized by the infinite parallel circuit of Figure 4, whose elements are images of reals; it is understood that the conductances and the current sources in Figure 4 are summed separately. As before, the order in which the elements are summed cannot be altered in general if those alterations extend over infinitely many elements. In particular, it is also understood that the infinite parallel circuit is built up through a sequence of finite parallel circuits by introducing each pair of elements $\langle a_k \rangle$ and $\langle j_k \rangle$ a pair at a time in the order of their subscripts.

4c. An infinite series circuit of resistances: The kind of circuit we now wish to consider is a one-port consisting of a one-way infinite series circuit of resistances whose infinite extremity is one of the port's terminals. This is illustrated in Figure 5. The theory of transfinite graphs justifies such a circuit [12, Sections 3.2 and 3.3], but under standard analysis Kirchhoff's voltage law may fail if the resistances are not appropriately chosen [12, Section 3.4]. On the other hand, any hyperreal resistance $R \in {}^*\mathcal{R}$ can be synthesized by an infinite series circuit of images of real resistances x_k as follows. Let $R = \langle r_1, r_2, r_3, \dots \rangle$. Set $x_1 = r_1$ and $x_k = r_k - r_{k-1}$ for $k = 2, 3, 4, \dots$. Then, by (2), $R = \sum_{k=1}^{\infty} \langle x_k \rangle$.

Actually, all of the ideas we have discussed for infinite parallel circuits of conductances carry over to the present case in a dual fashion. For example, in order to specify the input resistance R uniquely, the infinite series circuit should be viewed as the result of inserting the resistances x_k into a finite series circuit one at a time, and the order in which this is done may affect R ; therefore, that order should be specified. Similarly, combining some of the resistances before insertion may affect R . However, once these specifications are made, the nonstandard Kirchhoff voltage law $V = \sum_{k=1}^{\infty} V_k$ will hold around the infinite series

circuit, where the summation is defined by (3) and an equation dual to that of (22) occurs.

4d. An infinite series circuit of Thevenin branches: More generally, given any nonstandard Thevenin branch with the representation $V = IR - E$, where $V = \langle v_n \rangle$, $I = \langle i_n \rangle$, $R = \langle r_n \rangle$, and $E = \langle e_n \rangle$, we can set $R = \sum_{k=1}^{\infty} \langle x_k \rangle$ as before and in addition $E = \sum_{k=1}^{\infty} \langle f_k \rangle$, where $f_1 = e_1$ and $f_k = e_k - e_{k-1}$ for $k = 2, 3, 4, \dots$. Thus,

$$V = IR - E = \left\langle i_n \sum_{k=1}^n x_k - \sum_{k=1}^n f_k \right\rangle.$$

This expression is realized by an infinite series circuit whose elements are images of reals; this is illustrated in Figure 6. Here too, it is understood that the infinite series circuit is built up through a sequence of finite series circuits by introducing each pair of elements $\langle x_k \rangle$ and $\langle f_k \rangle$ a pair at a time in the order of their subscripts.

4e. An infinite binary tree with terminating resistances: Consider the infinite resistive binary tree T connected at infinity through 1-nodes to 2^{\aleph_0} many resistors of value r . Figure 7 illustrates this network. The resistors at the k th level of the tree all have the same value r_k . What is the input resistance R between nodes n_1 and n_2 ?

Since there are no restrictions on r and the r_k , this question can only be answered through a nonstandard analysis. To do so, we must first specify the order in which the network was built up. Let us assume that this was done through a sequence $\{T_n\}_{n=1}^{\infty}$ of finite binary trees T_n of n levels ending in 2^n many resistances. It follows that the currents in each level of every T_n are all the same, and therefore the same is true for the infinite binary tree T . At each step of this construction, let us drive the network with a constant current source h ; thus, the infinite network will be driven by the hyperreal current source $\langle h \rangle$. By the symmetry of the infinite network, the current in each resistor $\langle r_k \rangle$ is $\langle h \rangle / \langle 2^k \rangle$ and the current in each resistor $\langle r \rangle$ is $\langle h \rangle / \langle 2^n \rangle$. (Remember that $\langle 2^k \rangle = \langle 2^k, 2^k, 2^k, \dots \rangle$, whereas $\langle 2^n \rangle = \langle 2, 4, 8, \dots \rangle$.) Now, the sum V of the voltages along any path from node n_1 to node n_2 is $V = \langle \sum_{k=1}^n r_k h / 2^k \rangle + \langle r h / 2^n \rangle$. Thus,

$$\begin{aligned} R &= \frac{V}{\langle h \rangle} = \left\langle \frac{r}{2^n} + \sum_{k=1}^n \frac{r_k}{2^k} \right\rangle \\ &= \left\langle \frac{r}{2} + \frac{r_1}{2}, \frac{r}{4} + \frac{r_1}{2} + \frac{r_2}{4}, \frac{r}{8} + \frac{r_1}{2} + \frac{r_2}{4} + \frac{r_3}{8}, \dots \right\rangle. \end{aligned}$$

The reason we have presented this example is that it does not fit into the class of “finitely constructible” one-ports, which we will be introducing in the next section. By definition, the finitely constructible one-ports have internally no more than countably many branches. Even though the network of Figure 7 is constructed out of a sequence of finite circuits, the final result has uncountably many branches. The difficulty here is that no single terminating resistor in the final result can be identified with a particular terminating resistor of an intermediate finite circuit. Such an identification is essential for the finitely constructible one-ports. Note also that the symmetry in resistance values was essential to our analysis; without such symmetry, the infinitesimal currents in the terminating resistors are difficult (impossible?) to determine.

4f. Other one-ports: Nonstandard one-ports can be constructed out of other infinite circuits by using a sequences of finite subcircuits that “fill out” the infinite circuits. This is done in the next section in two different ways.

5 Finitely Constructible One-Ports

Henceforth, we will always assume that *every standard branch has a positive resistance*. Thus, every branch can be converted either into the Norton form of Figure 2(a) with $g_j > 0$ or into the Thevenin form of Figure 2(b) with $r_j > 0$. This prevents the occurrence of a node incident only to current sources that do not satisfy Kirchhoff’s current law; it also prevents the occurrence of loop containing only voltage sources that do not satisfy Kirchhoff’s voltage law.

We shall now show how some more general kinds of internally infinite one-ports can be taken as the final result of a sequence of internally finite one-ports. We have two versions of this.

5a. Internally infinite one-ports constructible from opens: Let M be a one-port, which is internally a countably infinite, 0-connected 0-network (in the terminology of [12]); n_1 and n_2 will be the nodes of M that serve as the port terminals. 0-connectedness is the same as the ordinary connectedness of customary graph theory. That M is a 0-network means that there are no connections at infinity, that is, there are no nodes of ranks larger

than 0 (again in the terminology of [12]). Thus, \mathbf{M} has no infinite loops. In general, \mathbf{M} will have infinite nodes (i.e., nodes with infinitely many incident branches). We may say that \mathbf{M} is “finitely constructible from opens” because of the following construction.

To *open* a branch b will mean that, with respect to a Norton representation (Figure 2(a)), the branch’s conductance g , current source h , and current i are all set equal to 0: $g = h = i = 0$. Since $i + h = gv$, it follows that the branch’s voltage v is indeterminate. We shall arbitrarily set $v = 0$ when b is opened. This will not affect the validity of the following arguments. With regard to the conditions $g = h = i = 0$, the same result can be obtained simply by deleting the branch.

Let us number the branches of \mathbf{M} with the indices $k = 1, 2, 3, \dots$. Also, for each $n = 1, 2, 3, \dots$, let \mathbf{M}_n be the one-port obtained from \mathbf{M} by opening every branch b_k with $k > n$ but leaving b_k unchanged when $1 \leq k \leq n$. We shall think of \mathbf{M}_n as having the same infinite graph as that of \mathbf{M} but with branch values as stated. Thus, \mathbf{M}_n may have one or more *isolated nodes* n_0 whereby all the branches incident to n_0 have been opened. In fact, the terminal nodes n_1 and n_2 may be isolated for some sufficiently small n . Moreover, the finite-subnetwork \mathbf{M}'_n of \mathbf{M} induced by those branches b_k ($k \leq n$) that have not been opened at some step n may have many components.

Now let the port terminals n_1 and n_2 be excited by an external branch b_0 in the Thevenin or Norton form (Figure 2); in this case, we allow b_0 to be a pure voltage source e_0 (with $r_0 = 0$) or a pure current source h_0 (with $g_0 = 0$). Whatever be the source value $e_0 \in \mathcal{R}$ or $h_0 \in \mathcal{R}$ and whatever be $n \in N$, a standard nodal or fundamental-loop analysis can be applied to $\mathbf{M}'_n \cup b_0$ to obtain a unique voltage $v_{k,n}$ and a unique current $i_{k,n}$ for branch b_k for each $k = 0, 1, \dots, n$. For each fixed $n \in N$ and for any (finite or infinite) internal node n_0 in \mathbf{M}_n , Kirchhoff’s current law in the form $\sum_{(n_0)} \pm i_{k,n} = 0$ will be satisfied — even at the isolated nodes of \mathbf{M}_n . The same will be true at the port nodes when the current in b_0 is taken into account. On the other hand, with $n \in N$ still fixed, Kirchhoff’s voltage law will be satisfied around those loops that remain within $\mathbf{M}'_n \cup b_0$ (i.e., those loops that pass only through branches with indices no larger than n) — and trivially around those loops that remain outside of $\mathbf{M}'_n \cup b_0$ (i.e., all branch indices for the loop larger than n). In general,

Kirchhoff's voltage law will not be satisfied for loops in $\mathbf{M}_n \cup b_0$ having branch indices both larger and no larger than n .

For each branch b_k of $\mathbf{M} \cup b_0$, we have hereby assigned a sequence $\{i_{k,n}\}_{n=1}^{\infty}$ of currents $i_{k,n}$ and a sequence $\{v_{k,n}\}_{n=1}^{\infty}$ of voltages $v_{k,n}$ and thereby a hyperreal current $I_k = \langle i_{k,n} \rangle$ and a hyperreal voltage $V_k = \langle v_{k,n} \rangle$. Moreover, $i_{k,n} = 0$ and $v_{k,n} = 0$ whenever $1 \leq n < k$. Therefore, we can invoke (3) to obtain the hyperreal version of Kirchhoff's current law at every (finite or infinite) node n_0 of $\mathbf{M} \cup b_0$:

$$\sum_{(n_0)} \pm I_k = \sum_{(n_0)} \pm \langle i_{k,n} \rangle = \langle 0 \rangle. \quad (23)$$

(The summation symbol and the choices of signs are the same as that for (16).)

In the same way, we also have established the hyperreal version of Kirchhoff's voltage law around any loop L of $\mathbf{M} \cup b_0$:

$$\sum_{(L)} \pm V_k = \sum_{(L)} \pm \langle v_{k,n} \rangle = \langle 0 \rangle. \quad (24)$$

(The summation symbol and the choices of signs are the same as that for (17).) This is because there are only finite loops in $\mathbf{M} \cup b_0$ since \mathbf{M} has no nodes of ranks larger than 0 (i.e., has no connections at infinity). Thus, for each fixed loop L , Kirchhoff's voltage law will be violated only for finitely many indices n . The set of those n has measure 0, whence (24).

Finally, we can apply either the standard Thevenin theorem or the standard Norton theorem to \mathbf{M}'_n to get the components of a hyperreal Thevenin representation $V_0 + E_0 = R_0 I_0$ or a hyperreal Norton representation $I_0 + H_0 = G_0 V_0$ at the port terminals n_1 and n_2 of \mathbf{M} ; here, $G_0 = R_0^{-1}$ and $E_0 = -R_0 H_0$. Note that a different numbering of the branches of \mathbf{M} by the indices $k = 1, 2, 3, \dots$ will lead in general to different hyperreal representations. To be sure, there are special cases, such as the infinite parallel circuit of identical resistors, for which the representation is independent of the numbering.

5b. Internally infinite one-ports finitely constructible from shorts: This time let \mathbf{M} be a one-port, which internally is a countably infinite, ν -connected ν -network for some natural number ν and whose ordinary 0-nodes are all of finite degree. Again, n_1 and n_2

will be the port terminals of \mathbf{M} . The precise definition of a ν -connected ν -network is given in [12, Chapter 5]. Let us merely say here that a ν -network is obtained by first connecting 0-networks together at their infinite extremities to get a 1-network, then by connecting 1-networks together at their infinite extremities to get a 2-network, and so forth through the countable ordinals up to ν . The connections at the infinite extremities are made through μ -nodes (i.e., generalized nodes) of ranks up to ν . That \mathbf{M} is ν -connected means that every two branches in \mathbf{M} are connected through possibly transfinite paths passing through the μ -nodes. In general, \mathbf{M} will have infinite loops. That a 0-node is ordinary means that it is not part of a node of higher rank. We will apply Kirchhoff's current law only to the ordinary 0-nodes, and these we assume are all of finite degree. In this case, we may say that \mathbf{M} is "finitely constructible from shorts" again because of the following construction.

To *short* a branch b will mean that, with respect to a Thevenin representation (Figure 9), the branch's resistance r , voltage source e , and voltage v are all set equal to 0: $r = e = v = 0$. Since $e + v = ri$, it follows that the branch's current i is indeterminate. This time we arbitrarily set $i = 0$ when b is shorted. This too will not lead to any difficulties.

After numbering the branches of \mathbf{M} by $k = 1, 2, 3, \dots$ and fixing $n \in N$, let \mathbf{M}_n now be the one-port obtained from \mathbf{M} by shorting every branch b_k with $k > n$ but leaving b_k unchanged when $1 \leq k \leq n$. Here too, we think of \mathbf{M}_n as having the same infinite graph as that of \mathbf{M} but with the stated branch values. This time \mathbf{M}'_n will denote the connected finite network obtained from \mathbf{M}_n by coalescing into a single node every maximal set of nodes that are connected by paths of short circuits in \mathbf{M}_n , a different coalesced node for each maximal set.

Let the port terminals n_1 and n_2 be excited by an external branch b_0 in the Thevenin or Norton form (Figure 2); here too, we allow b_0 to be a pure voltage source e_0 or a pure current source h_0 . Whatever be $e_0 \in \mathcal{R}$ or $h_0 \in \mathcal{R}$ and whatever be $n \in N$, a standard nodal or fundamental-loop analysis can be applied to $\mathbf{M}'_n \cup b_0$ to obtain a unique voltage $v_{k,n}$ and a unique current $i_{k,n}$ for the k th branch b_k , where $k = 1, \dots, n$. For each finite or infinite loop L in $\mathbf{M}_n \cup b_0$, Kirchhoff's voltage law will be satisfied around L . However, Kirchhoff's current law may not be satisfied at any node of $\mathbf{M}_n \cup b_0$ having incident branches with

indices both no larger than and larger than n .

Once again, we have a sequence $\{i_{k,n}\}_{n=1}^{\infty}$ of currents and a sequence $\{v_{k,n}\}_{n=1}^{\infty}$ of voltages for each branch b_k of $\mathbf{M} \cup b_0$ and thereby a hyperreal current $I_k = \langle i_{k,n} \rangle$ and a hyperreal voltage $V_k = \langle v_{k,n} \rangle$. Here too, $i_{k,n} = 0$ and $v_{k,n} = 0$ whenever $1 \leq n < k$. So, we can invoke (3) to get the hyperreal form of Kirchhoff's voltage law (24) around every (finite or infinite) loop L of $\mathbf{M} \cup b_0$. Similarly, we have established the hyperreal form of Kirchhoff's current law (23) at every ordinary 0-node n_0 of $\mathbf{M} \cup b_0$. This is because we are now assuming that all ordinary 0-nodes of \mathbf{M} are of finite degree. Thus, for each node n_0 , Kirchhoff's current law can be violated at n_0 for only finitely many n .

As before, by the standard Thevenin or Norton theorem applied to \mathbf{M}'_n , we obtain the components of the hyperreal Thevenin representation $V_0 + E_0 = R_0 I_0$ or equivalently the hyperreal Norton representation $I_0 + H_0 = G_0 V_0$ for \mathbf{M} at its port terminals n_1 and n_2 . Here again, these representations may depend upon the order in which the branches of \mathbf{M} are numbered.

5c. Internally infinite one-ports with no infinite nodes and with no specifications at infinity: Finally, let us note that some internally infinite one-ports are both finitely constructible from opens and finitely constructible from shorts. For example, this is the case for a one-way infinite ladder network whose connections at infinity (i.e., whether there is an open or a short at infinity) has been left unspecified. In this case, the procedure of Subsection 5a will imply an open at infinity and that of Subsection 5b will imply a short at infinity. Two different hyperreal representations for the one-port may thus be obtained depending upon which procedure is followed. (For certain branch values, this is even so under a standard analysis [12, Example 1.6-4].) In general, no infinite electrical network is completely specified until its connections at infinity are stipulated.

6 Conclusions

By combining the results of Sections 3, 4, and 5, we can draw the following conclusions. Given any standard infinite network \mathbf{N} , whose every branch has a positive resistance, and having chosen a measure m according to Conditions 2.1, we may be able to partition \mathbf{N} into

finitely many one-ports and then analyze the interior circuitry of each one-port by applying Kirchhoff's laws and Ohm's law to a sequence of finite networks arising from either opens or shorts in order to find hyperreal solutions to that circuitry. In particular, we can obtain a hyperreal Thevenin or Norton representation for each one-port at its port terminals. Then, a nonstandard nodal or fundamental-loop analysis applied to the finite circuit of hyperreal one-ports will yield the hyperreal port voltages and port currents at all the one-ports of our partition of N . These in turn determine all the hyperreal currents and voltages within the one-ports, and the latter will satisfy nonstandard versions of Kirchhoff's laws at all (finite and infinite) ordinary nodes and around all (finite and infinite) loops.

Actually, the separate analyses of the individual one-ports followed by the analysis of Section 3 can be combined into a single analysis as follows. After having partitioned N into one-ports each of which is either internally finite or of the type discussed in Subsection 5a or 5b, we can for each $n \in N$ make a standard analysis of the finite network obtained from N by opening or shorting branches in accordance with the types of one-ports the branches are in. This will yield the n th component of a representative for each hyperreal voltage or current in N , and thereby the hyperreal voltages and currents throughout N . The latter will satisfy nonstandard versions of Kirchhoff's laws and Ohm's law. Furthermore, those laws determine in this way the hyperreal voltage-current regime for N . This is our conclusion. Let us restate it as follows:

Let N be a standard countable infinite network whose every branch has a positive resistance. Assume that N can be partitioned into a finite number of one-ports such that each one-port is either internally finite, or internally infinite and finitely constructible from opens, or internally infinite and finitely constructible from shorts. (The one-ports need not be all of the same kind.) Then, for each choice of such a partitioning, for each choice of a branch numbering within each one-port, and for each choice of the measure m , N will have a nonstandard image in which a unique set of hyperreal branch voltages and currents satisfy nonstandard versions of Kirchhoff's laws and Ohm's law. The nonstandard image of each internally infinite, finitely constructible one-port is obtained from a sequence of internally finite one-ports as stated in Section 5. Finally, the standard Kirchhoff's laws and Ohm's

law determine a unique set of currents and voltages throughout each finite truncation of N obtained at each stage of the replacements of internally infinite one-ports by internally finite ones and in this way determine the hyperreal voltages and currents throughout the nonstandard image of N .

It is perhaps disconcerting that the hyperreal voltage-current regime depends upon our choices of the partitioning of N into one-ports, of the numbering of the branches within each one-port, and of the measure m . One might say that an infinite electrical network is an abstraction that defies comprehension. On the one hand, standard analysis is simply not delicate enough to force the satisfaction of Kirchhoff's laws at all infinite nodes and around all infinite loops. On the other hand, it appears that nonstandard analysis is overly delicate, for it requires more specifications than is provided by just the graph and the element values of an infinite network. The constructive approach to nonstandard analysis does resolve the paradoxes about Kirchhoff's laws, at least for certain infinite networks, but it may also produce many different nonstandard images of a given infinite network. The circuit theorist is free to choose among them.

Finally, let us comment on what has not been achieved. The sentences of symbolic logic are of finite length and therefore cannot encompass an infinity of terms. As a result, assertions concerning an infinity of real terms cannot in general be transferred into the hyperreal realm. However, there are exceptions. The infinite series (2) and (3) do deal with infinitely many hyperreals, but they do so through sequences of finite partial sums (add columnwise in (3)). This is why we partitioned N into a finite network of internally infinite one-ports and then dealt with each one-port through a sequence of finite circuits obtained either by opening branches or by shorting them. Opening branches insures the satisfaction of Kirchhoff's current law at each finite stage of an infinite node, but it can fail for his voltage law at finite stages of an infinite loop; hence, infinite loops were disallowed when resorting to opens. Similarly, shorting branches works for the voltage law around finite stages of infinite loops but not in general for the current law at finite stages of infinite nodes, and so infinite nodes were disallowed in this case. How to proceed when every finite partitioning of the network N results in at least one one-port having both infinite nodes

and infinite loops remains an open problem.

Another open problem is the restoration of Kirchhoff's laws for uncountable networks like that of Subsection 4e.

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Legends for Figures

Figure 1. An infinite parallel circuit of 1Ω resistors fed by a source branch consisting of a 1 V voltage source in series with another 1Ω resistor. The input resistance of the infinite parallel circuit is r .

Figure 2. (a) A standard Norton branch containing both a conductance g_j and an independent current source h_j . The orientation of the branch is from left to right, and v_j , i_j , and h_j are measured accordingly with the polarities shown. u_k and u_l are node voltages.

(b) A standard Thevenin branch containing a resistance r_j and an independent voltage source e_j . If $0 < r_j < \infty$, if $r_j = g_j^{-1}$, and if $e_j = -r_j h_j$, then this Thevenin branch is equivalent to the Norton branch so far as terminal conditions are concerned.

Figure 3. (a) The synthesis of an arbitrary $G \in {}^*\mathcal{R}$ by means of an infinite parallel circuit of hyperreal images $\langle a_k \rangle$ of real conductances a_k . The order of insertion of the conductances is taken to be the same as that of the indices $k = 1, 2, 3, \dots$

(b) The corresponding infinite parallel circuit of real conductances a_k . Now, there may be no real input conductance.

Figure 4. The synthesis of any hyperreal Norton branch, for which $I = VG - H$, as an infinite parallel circuit of images of reals. Again, the order of insertion of the elements is taken to be the same as that of the indices $k = 1, 2, 3, \dots$

Figure 5. The synthesis of an arbitrary $R \in {}^*\mathcal{R}$ by means of an infinite series circuit of hyperreal images $\langle x_k \rangle$ of real resistors x_k . The order of insertion of resistors is that of the indices $k = 1, 2, 3, \dots$

Figure 6. The synthesis of any hyperreal Thevenin branch, for which $V = IR - E$, as an infinite series circuit of images of real. Elements are inserted in the order of their indices $k = 1, 2, 3, \dots$

Figure 7. An infinite binary tree connected at its uncountably many infinite extremities to resistors. At each horizontal level, the resistance values are all the same. The entire circuit is fed by a current source h .

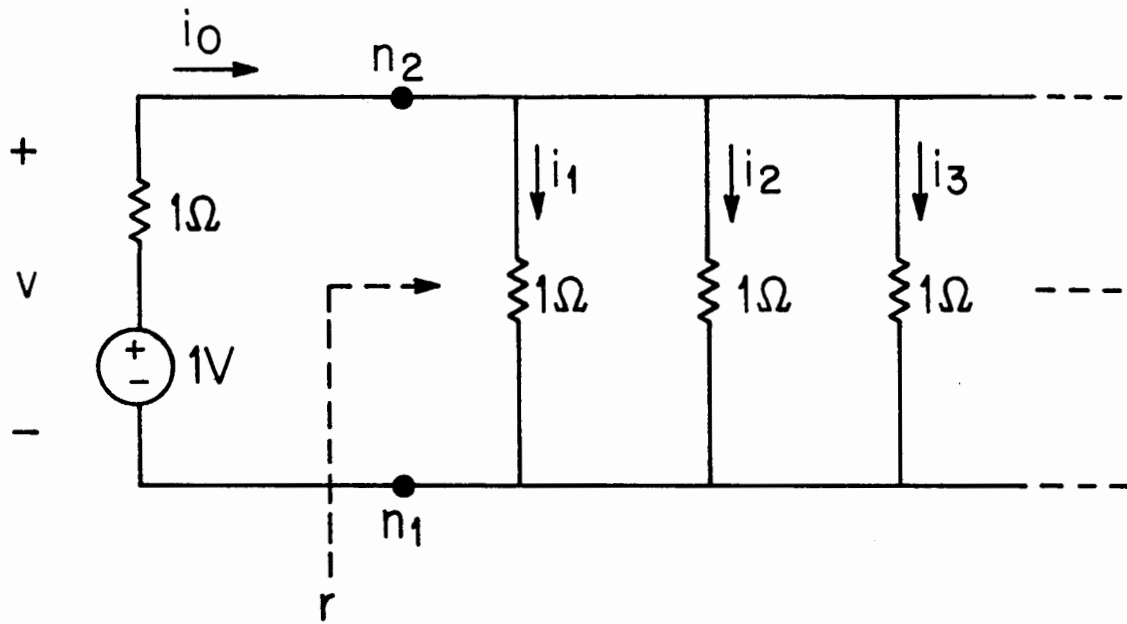
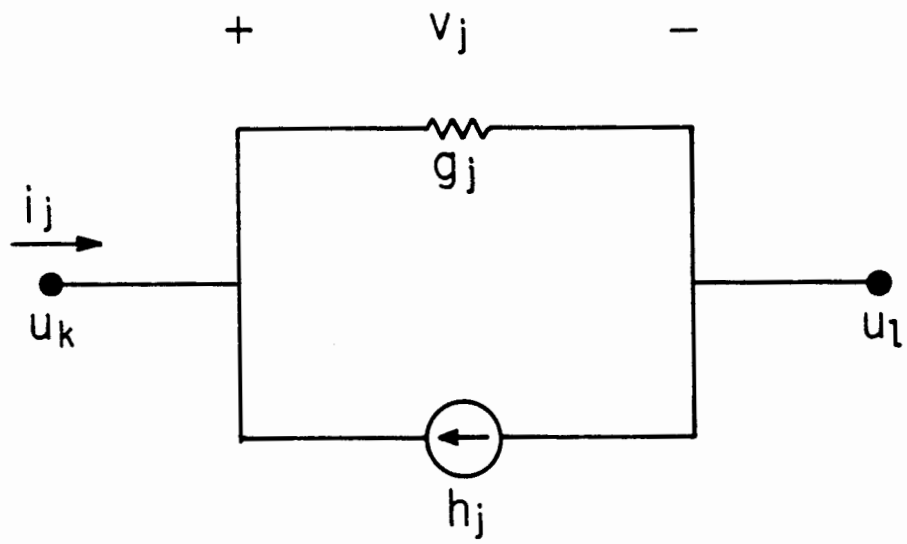
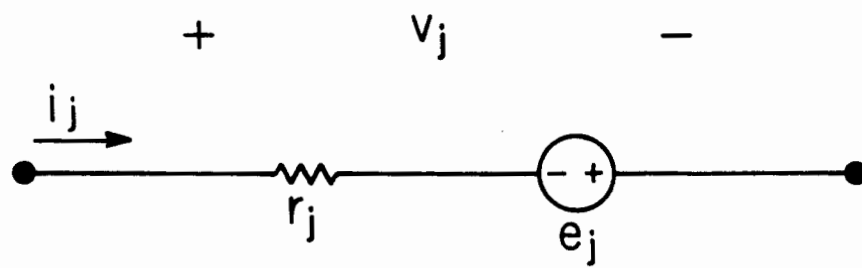


FIG. 1

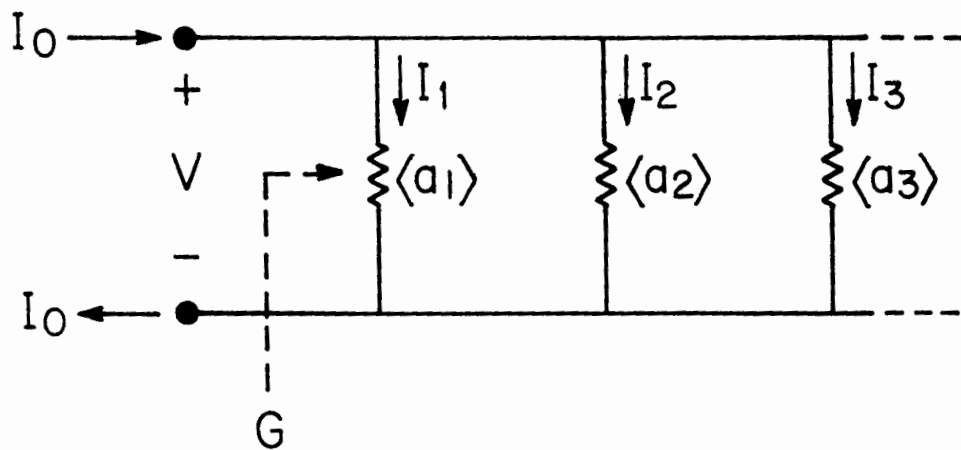


(a)

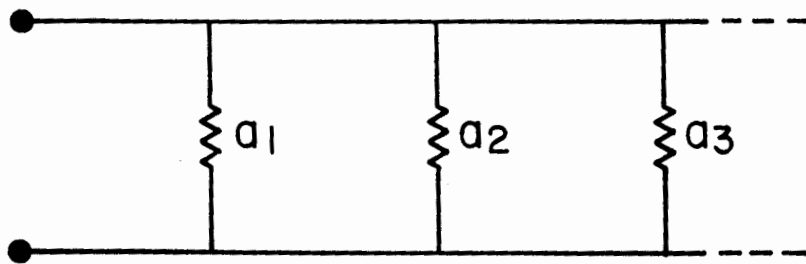


(b)

FIG. 2



(a)



(b)

FIG. 3

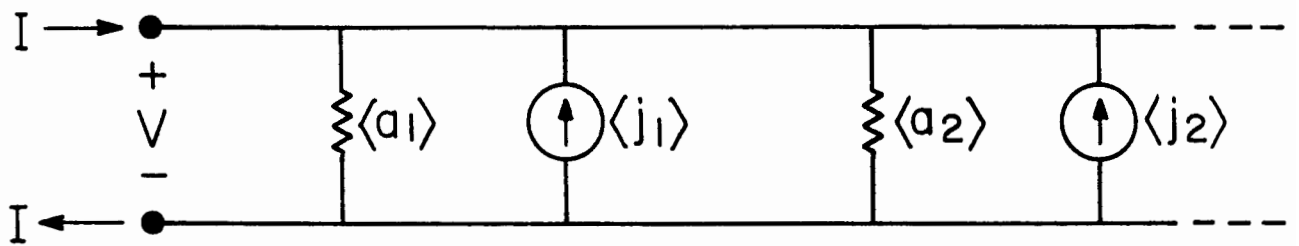


FIG. 4

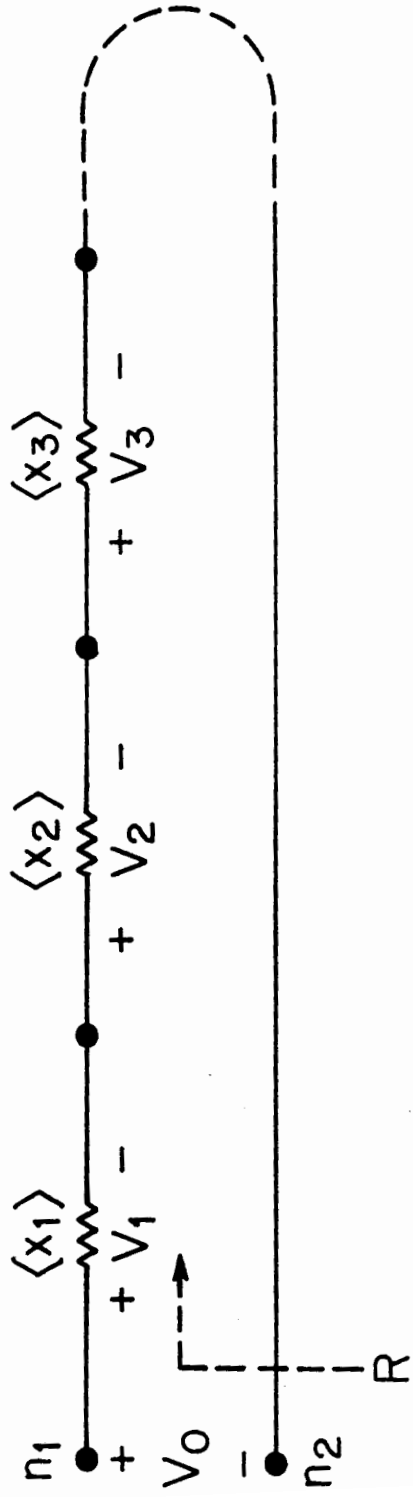


FIG. 5

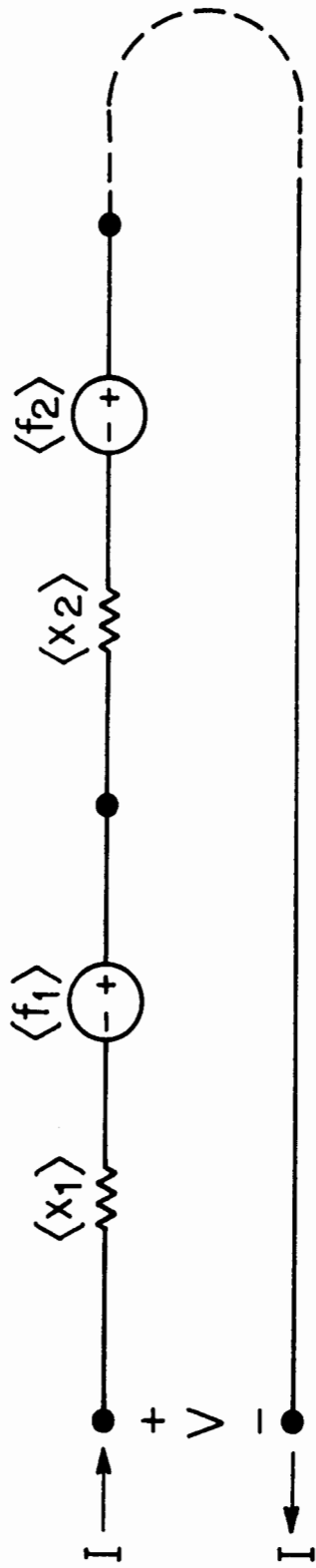


FIG. 6

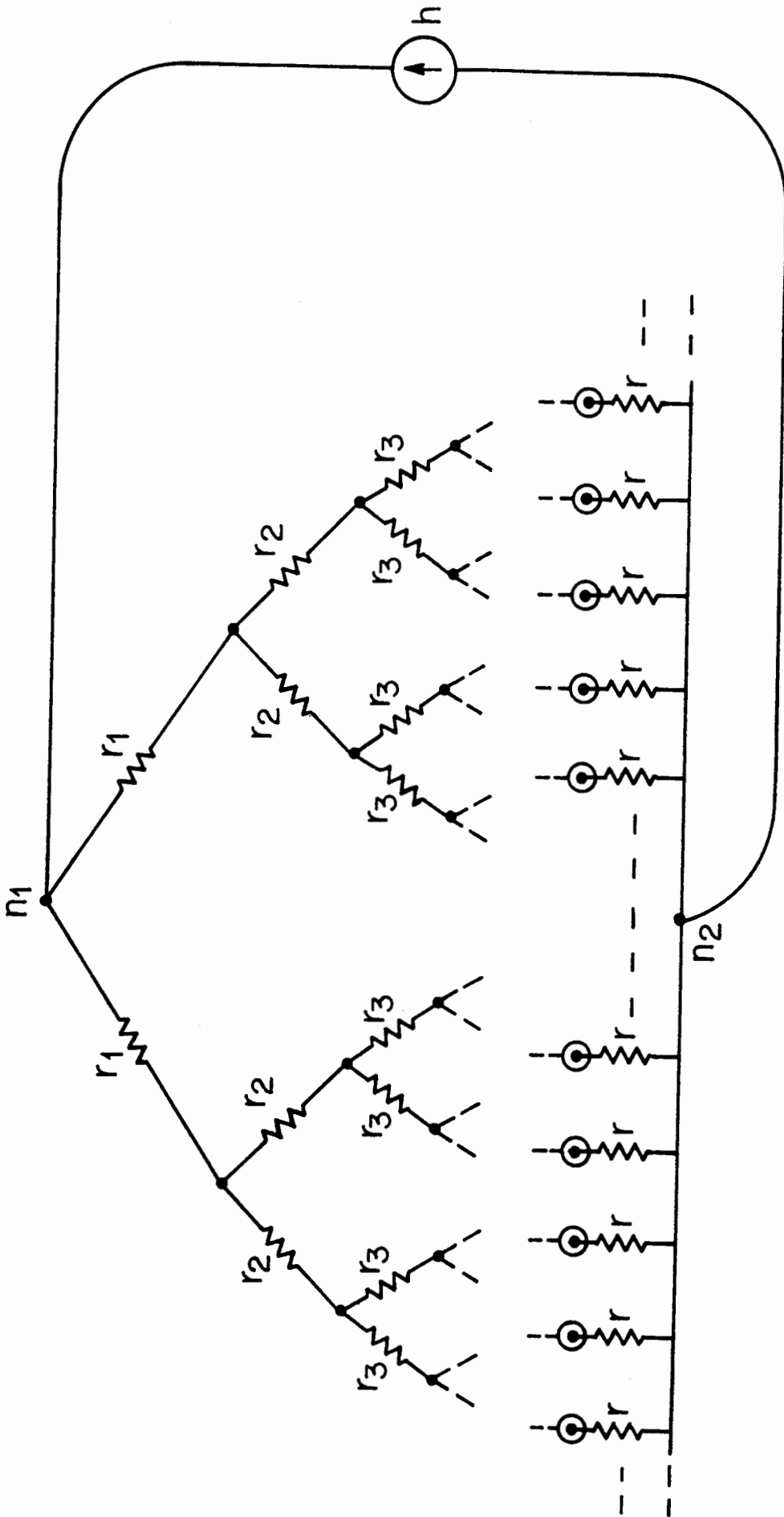


FIG. 7