

UNIVERSITY AT STONY BROOK

CEAS Technical Report 790

ORDINAL DISTANCES IN TRANSFINITE GRAPHS

A.H. Zemanian

March 21, 2001

ORDINAL DISTANCES IN TRANSFINITE GRAPHS

A. H. Zemanian

Abstract — An ordinal-valued metric can be assigned to a metrizable set \mathcal{M} of nodes in any transfinite graph. \mathcal{M} contains all the nonsingleton nodes, as well as certain singleton nodes. The metric takes its values in the set \aleph_1 of all countable ordinals. Moreover, this construct yields a graphical realization of Cantor's countable ordinals, as well as of the Aristotelian ideas of "potential" and "actual" infinities, the former being represented by the arrow ranks and the latter by the ordinal ranks of transfiniteness. This construct also extends transfinitely the ideas of nodal eccentricities, radius, diameter, center, and periphery for graphs. Several results concerning these transfinitely generalized ideas are proven.

Key Words: Distance in graphs, transfinite distances, centrality in graphs, transfinite eccentricities, radii, diameters, centers, and peripheries of transfinite graphs.

1 Introduction

The idea of distances in connected finite graphs has been quite fruitful, with much research directed toward both theory and applications. See, for example, [2], [3], [5], and the references therein. Such distances are measured by a metric that assigns to each pair of nodes the minimum number of branches among all paths connecting those two nodes. Thus, the metric takes its values in the set \aleph_0 of natural numbers. That distance considerations can be so fruitful in the theory of finite graphs inspires the question of whether distance constructs can be devised for transfinite graphs. Transfinite graphs [7], [8] represent a generalization of graphs that is roughly analogous to Cantor's extension of the natural numbers to the transfinite ordinals.

An affirmative answer to that question was achieved in [4], wherein a metric was devised for the purpose of ascertaining limit points at infinite extremities of a conventionally infinite,

electrical, resistive network, through which points electrical current could flow into other such networks. This construct was extended to higher ranks of transfiniteness [8] with an infinite hierarchy of metrics, one for each rank of transfiniteness. These metrics take their values in the nonnegative real line, are quite different from the standard branch-count metric mentioned above, require a variety of restriction such as local finiteness, and do not reduce to the branch-count metric for finite graphs.

The problem attacked in this work is the invention of a single metric that extends the standard branch count metric to transfinite graphs, one that holds for all ranks of transfiniteness, and reduces to the standard branch-count metric for finite graphs. In closer analogy to Cantor's work, the metric proposed in this paper assigns countable ordinals to pairs of nodes in a connected transfinite graph; that is, it takes its values in the set \aleph_1 of all countable ordinals. Moreover, the metric is applicable even when the graph is not locally finite and may even have uncountably many branches.

As a consequence, several results concerning branch-count distances can be lifted to the transfinite case, some directly and others with various modifications. In particular, the ideas of nodal eccentricity, radius, diameter, center, and periphery that hold for finite graphs are herein extended to transfinite graphs. However, to do so, the set \aleph_1 has to be enlarged by inserting an "arrow rank" [7, page 4], [8, page 4] immediately preceding each limit ordinal. These arrow ranks reflect the Aristotelian idea of a "potential infinity" as distinct from the other Aristotelian idea of an "actual infinity" typified by the ordinals. Several theorems concerning these ideas are proven. Furthermore, this new metric, upon which our results are based, opens up a new area of research.

Various results concerning transfinite graphs are used in this work. These can be found in the book [7]. A simplified but more restrictive rendition of the subject is given in [8]. We will work in the generality of [7] and will refer to specific pages in that book when invoking various concepts and results.

As is conventional, ω denotes the first transfinite ordinal; we use the standard notations for ordinals and cardinals [1].

Furthermore, any transfinite node x^α may (but need not) contain exactly one node of

lower rank x^β ($\beta < \alpha$); x^β in turn may contain exactly one other node x^γ ($\gamma < \beta$), and so forth through finitely many decreasing ranks. We say that x^α *embraces* itself and x^β , x^γ , \dots , as well. On the other hand, if x^α is not embraced by a node of higher rank, we call x^α a *maximal* node. It is the maximal nodes we will be primarily concerned with because connectedness to x^α implies connectedness to x^β , x^γ , \dots , as well. Rather than repeating the adjective “maximal,” we let it be understood throughout that any node discussed is maximal unless the opposite is explicitly stated. This means that different (maximal) nodes must be “totally disjoint,” that is, they embrace no common elements.

In this paper, we do not allow any branch to be a self-loop; thus, every branch is incident to two different nodes.

2 Lengths of Paths

0-Paths:

A (nontrivial) 0-path P^0 is an alternating sequence

$$P^0 = \{\dots, x_m^0, b_m, x_{m+1}^0, b_{m+1}, \dots\} \quad (1)$$

of branches b_m and conventional nodes x_m^0 (also called “0-nodes”) in which no term repeats and each branch is incident to the two 0-nodes adjacent to it in the sequence. If the sequence terminates on either side, it terminates at a 0-node. This is the conventional definition of a path. (The 0-nodes of (1) need not be maximal when P^0 occurs within a transfinite graph.) When P^0 is one-ended (i.e., one-way infinite), its length is defined to be $|P^0| = \omega$. When P^0 is endless (i.e., two-way infinite), its length is taken to be $|P^0| = \omega \cdot 2$. If P^0 is two-ended (i.e., has only finitely many 0-nodes), we set $|P^0| = \tau_0$, where τ_0 is the number of branches in P^0 . We might motivate these definitions by noting that we are using ω to denote the infinity of branches in a one-ended 0-path and using $\omega \cdot 2$ to represent to fact that an endless 0-path is the union of two one-ended paths. Equivalently, we can identify ω with each 0-tip traversed; a one-ended 0-path has one 0-tip, and an endless 0-path has two 0-tips—hence, the length $\omega \cdot 2$.

1-Paths:

A (nontrivial) 1-path P^1 is an alternating sequence

$$P^1 = \{\dots, x_m^1, P_m^0, x_{m+1}^1, P_{m+1}^0, \dots\} \quad (2)$$

of 1-nodes x_m^1 and 0-paths P_m^0 that represents a tracing through a transfinite graph of rank 1 or greater in which no node of the path repeats. If the sequence terminates on either side, it terminates at a 0-node or 1-node. See [7, page 28] for the full definition of a 1-path. The length $|P^1|$ of P^1 is defined as follows. When P^1 is one-ended, $|P^1| = \omega^2$, and, when P^1 is endless, $|P^1| = \omega^2 \cdot 2$. When P^1 is two-ended (i.e., when it has only finitely many 1-nodes), we set $|P^1| = \sum_m |P_m^0|$, where the sum is over the finitely many 0-paths in P^1 ; thus, in this case, $|P^1| = \omega \cdot \tau_1 + \tau_0$, where τ_1 is the number of 0-tips P^1 traverses, and τ_0 is the number of branches in all the 0-paths in (2) that are two-ended. It is important here to write $|P^1|$ as indicated and not as $\tau_0 + \omega \cdot \tau_1$ because ordinal addition is not commutative [1, page 327]. Thus, $\omega \cdot \tau_1 + \tau_0$ takes into account the lengths of all the 0-paths in (2), but $\tau_0 + \omega \cdot \tau_1$ fails to do so.

μ -Paths:

Now, let μ be any positive natural number. A μ -path an alternating sequence

$$P^\mu = \{\dots, x_m^\mu, P_m^{\alpha_m}, x_{m+1}^\mu, P_{m+1}^{\alpha_{m+1}}, \dots\} \quad (3)$$

of μ -nodes x_m^μ and α_m -paths $P_m^{\alpha_m}$, where $0 \leq \alpha_m < \mu$. (The natural numbers α_m may vary with m , and the μ -nodes need not be maximal.) As before, P^μ represents a tracing through a transfinite graph of rank μ or larger in which no node is met more than once in the tracing. Termination on either side of (3) occurs at a node of rank μ or less. When P^μ is one-ended, its length $|P^\mu|$ is defined to be $\omega^{\mu+1}$, and, when P^μ is endless, we set $|P^\mu| = \omega^{\mu+1} \cdot 2$. When, however, P^μ is two-ended (i.e., has only finitely many μ -nodes), we set $|P^\mu| = \sum_m |P_m^{\alpha_m}|$, where the summation denotes a normal expansion of an ordinal [1, pages 354-355]. Recursively, this gives

$$|P^\mu| = \omega^\mu \cdot \tau_\mu + \omega^{\mu-1} \cdot \tau_{\mu-1} + \dots + \omega \cdot \tau_1 + \tau_0, \quad (4)$$

where $\tau_\mu, \tau_{\mu-1}, \dots, \tau_0$ are natural numbers. τ_μ is the number of $(\mu - 1)$ -tips among all the one-ended and endless $(\mu - 1)$ -paths (i.e., when $\alpha_m = \mu - 1$) appearing in (3); τ_μ is not 0. For $k = \mu - 1, \mu - 2, \dots, 1$, we set τ_k equal to the number of $k - 1$ -tips generated by these recursive definitions. Finally, τ_0 is one-half the number of elementary tips generated recursively by these definitions. Thus, τ_0 is a number of branches because each branch has exactly two elementary tips. Any τ_k ($k < \mu$) can be 0. Here, too, in order to conform with the standard definition of ordinal summation, it is important to write the sum (4) in its normal-expansion form [1, pages 354-355] as shown because of the noncommutativity of ordinal summation.

Example 2.1. Let P^3 be the two-ended 3-path:

$$P^3 = \{x_1^2, P_1^2, x_2^3, P_2^2, x_3^3, P_3^2, x_4^3\}$$

Here, P_1^2 is assumed to be a one-ended 2-path terminating on the left with x_1^2 and reaching x_2^3 through a 2-tip. Hence, $|P_1^2| = \omega^3$. We take P_2^2 to be the two-ended 2-path

$$P_2^2 = \{y_1^2, Q_1^1, y_2^2, Q_2^0, y_3^2\},$$

where y_1^2 and y_2^2 are members of x_2^3 and x_3^3 respectively. Q_1^1 is an endless 1-path reaching the 2-nodes y_1^2 and y_2^2 with 1-tips, and Q_2^0 is a finite 0-path with four branches, whose terminal 0-nodes are members of y_2^2 and y_3^2 . Hence, $|Q_1^1| = \omega^2 \cdot 2$ and $|Q_2^0| = 4$. Finally, we take P_3^2 to be an endless 2-path reaching x_3^3 and x_4^3 through 2-tips. Hence, $|P_3^2| = \omega^3 \cdot 2$.

Altogether then, with understanding that the following ordinal sums should always be rearranged if need be to get the normal-expansion form, we may write

$$\begin{aligned} |P^3| &= |P_1^2| + |P_2^2| + |P_3^2| \\ &= \omega^3 + |Q_1^1| + |Q_2^0| + \omega^3 \cdot 2 \\ &= \omega^3 + \omega^2 \cdot 2 + 4 + \omega^3 \cdot 2 \\ &= \omega^3 \cdot 3 + \omega^2 \cdot 2 + 4 \end{aligned}$$

□

$\bar{\omega}$ -paths:

These occur within paths of ranks ω and higher, but they are never two-ended [7, pages 40-41]. The length of an $\bar{\omega}$ -path $P^{\bar{\omega}}$ is defined to be $|P^{\bar{\omega}}| = \omega^\omega$ when $P^{\bar{\omega}}$ is one-ended, and $|P^{\bar{\omega}}| = \omega^\omega \cdot 2$ when $P^{\bar{\omega}}$ is endless.

ω -paths:

A (nontrivial) ω -path P^ω [7, page 44]

$$P^\omega = \{\dots, x_m^\omega, P_m^{\alpha_m}, x_{m+1}^\omega, P_{m+1}^{\alpha_{m+1}}, \dots\} \quad (5)$$

is an alternating sequence of (not necessarily maximal) ω -nodes x_m^ω and α_m -paths $P_m^{\alpha_m}$ ($0 \leq \alpha_m \leq \bar{\omega}$) that represents a tracing through a graph of rank ω (or larger) in which no node repeats and a termination on either side is at a node of rank ω or less. By definition, when P^ω is one-ended, $|P^\omega| = \omega^{\omega+1}$; also, when P^ω is endless, $|P^\omega| = \omega^{\omega+1} \cdot 2$. When P^ω is two-ended (i.e., has only finitely many ω -nodes), we set

$$|P^\omega| = \sum_m |P_m^{\alpha_m}| = \omega^\omega \cdot \tau_\omega + \sum_{k=0}^{\infty} \omega^k \cdot \tau_k \quad (6)$$

with the proper order of the terms in the summation being understood. Here, τ_ω is the number of $\bar{\omega}$ -tips among all the one-ended and endless $\bar{\omega}$ -paths appearing as elements of $P_m^{\alpha_m}$ in (5) (i.e., when $\alpha_m = \bar{\omega}$); τ_ω is not 0. On the other hand, the τ_k are determined recursively, as they are in (4). There are only finitely many nonzero terms in the summation within (6) because there are only finitely many paths $P_m^{\alpha_m}$ in a two-ended ω -path and each $|P_m^{\alpha_m}|$ is a finite sum as in (4).

An immediate result of all these definitions is the following.

Lemma 2.2. *If Q^β is a subpath of a γ -path P^γ ($0 \leq \beta \leq \gamma$), then $|Q^\beta| \leq |P^\gamma|$.*

Paths of higher ranks:

The above definitions can be extended to paths of ranks higher than ω . It is simply a matter of repeating the recursions through a maximal consecutive set $\{\omega + 1, \omega + 2, \dots, \omega \cdot 2, \omega \cdot 2\}$ of successor-ordinal ranks $\omega + k$ followed by the next arrow rank $\omega \cdot 2$ and then the limit-ordinal rank $\omega \cdot 2$; this pattern continues on with still higher ranks. How much further can these cycles of definitions through countable-ordinals be continued? Is

there some obstacle the prevents these recursions from reaching arbitrarily large countable ordinals? These two questions are open.

Henceforth, we will restrict our attention to the first cycle, wherein the said ranks increase through all the natural numbers and then reach ω .

Let us also note the following: For every transfinite ordinal, there is a transfinite path having that ordinal as its length. This certainly is true for all ordinals up to and including $\omega^\omega \cdot \tau_\omega$, where τ_ω is a natural number, and for many larger ordinals, too. This is easy to show.

As has been asserted, the right-hand sides of (4) and (6) are the normal expansions [1, pages 354-355] for the left-hand-side ordinals. Our contribution to normal expansions is the interpretation of them as lengths of transfinite paths.

It is easy to add ordinals when they are in normal-expansion form—simply add their corresponding coefficients. Thus, the length of the union of two paths that are totally disjoint except for incidence at a terminal node (a “series connection”) is obtained by adding their lengths in normal expansion form. Similarly, if Q is a proper subpath of P , the part of $P \setminus Q$ of P that is not in Q has the total length $|P| - |Q|$, which is obtained by subtracting the coefficients of $|Q|$ from the corresponding coefficients of $|P|$. Lemma 2.2 above also follows readily from these observations.

3 Metrizable Sets of Nodes

In a connected finite graph, for every two nodes there is at least one path terminating at them. This is not in general true for transfinite graphs; see Examples 3.1-5 and 3.1-6 in [7].

Example 3.1. The 1-graph of Fig. 1 provides another example. In that graph, x_a^1 (resp. x_b^1) is a 1-node containing the 0-tip t_a^0 (resp. t_b^0) for the one-ended path of a_k branches (resp. b_k branches) and also embracing an elementary tip of branch d (resp. e). There are, in addition, uncountably 0-tips for paths that alternate infinitely often between the a_k and b_k branches by passing through c_k branches; those tips are contained in singleton 1-nodes, one for each. x_{abc}^1 denotes one such singleton 1-node; the others are not shown. Note that there is no path connecting x_{abc}^1 to x_a^1 (or to any other 1-node) because any tracing between x_{abc}^1

and x_a^1 must repeat 0-nodes. Thus, our definition (given in the next section) of the distance between two nodes as the minimum path length for all paths connecting those nodes cannot be applied to x_{abc}^1 and x_a^1 . We seek some means of applying this distance concept to at least some pairs of nodes. \square

To this end, we impose the following condition on the transfinite graph \mathcal{G}^ν .

Condition 3.2. *If two tips (perforce of ranks less than ν and possibly differing) are nondisconnectable,¹ then either they are shorted together (i.e., are embraced by the same node) or at least one of them is open (i.e., is the sole member of a singleton node).*

The 1-graph of Fig. 1 satisfies this condition.

The following results ensue: \mathcal{G}^ν is called ν -connected if, for any two branches, there is a two-ended path P^ρ of some rank ρ ($\rho \leq \nu$) that meets those two branches. Even though \mathcal{G}^ν is ν -connected, there may be two nodes not having any path that meets them (i.e., the two nodes are not ν -connected). For instance, the 1-graph of Fig. 1 is 1-connected, but there is no path that meets x_a^1 and x_{abc}^1 . Now, as will be established by Lemma 3.3 below, if \mathcal{G}^ν satisfies Condition 3.2, then, for any two nonsingleton nodes, there will be at least one two-ended path terminating at them. As a result, we will be able to define distances between nonsingleton nodes. Furthermore, some singleton nodes may be amenable to such distance measurements, as well. To test this, we need merely append a new branch b to a singleton node x^α by adding an elementary tip of b to x^α to get a nonsingleton node \hat{x}^α , with the other elementary tip of b left open (i.e., b is added as an end branch)—and then check to see if Condition 3.2 is maintained. More generally, with \mathcal{G}^ν being ν -connected and satisfying Condition 3.2, let \mathcal{M} be a set consisting of all the nonsingleton (maximal) nodes in \mathcal{G}^ν and possibly other singleton (maximal) nodes having the property that, if end branches are appended to those singleton nodes simultaneously, Condition 3.2 is still satisfied by the resulting network. Any such set \mathcal{M} will be call a *metrizable* set of nodes.

Lemma 3.3. *Assume \mathcal{G}^ν is ν -connected and satisfies Condition 3.2. Let \mathcal{M} be a metrizable set of nodes in \mathcal{G}^ν . Then, for any two nodes of \mathcal{M} , there exists a two-ended path*

¹Two tips are called *nondisconnectable* if their representative (one-ended) paths continue to meet no matter how far along the representative paths one proceeds [7, page 58]. Two tips are called *disconnectable* if they have representative paths that are totally disjoint.

terminating to those nodes.

Proof. Let x_a^α and x_b^β be two different nodes in \mathcal{M} . Since they are maximal, they must be totally disjoint. Then, by Condition 3.2, any tip in x_a^α is disconnectable from every tip in x_b^β ; indeed, if they were nondisconnectable, they would have to be shorted together, making x_a^α and x_b^β the same node. Thus, we can choose a representative path P_a for that tip in x_a^α that is totally disjoint from a representative path P_b for a tip in x_b^β . By the definition of ν -connectedness, there will be path P_{ab} connecting a branch of P_a and a branch of P_b . By [7, Corollary 3.5-4], there is in the subgraph $P_a \cup P_{ab} \cup P_b$ induced by the branches of those three paths a two-ended path terminating at x_a^α and x_b^β . \square

Example 3.4. As an illustration, consider again the 0-connected ladder of Fig. 1, but without branches d and e . In this case, x_a^1 and x_b^1 are singleton 1-nodes. Upon appending d and e as shown, the resulting 1-graph satisfies Condition 3.2. On the other hand, if we in addition append another branch to any other 1-node, say, to x_{abc}^1 , Condition 3.2 will be violated. Thus, in this case, we can take \mathcal{M} to be the set of all 0-nodes along with the singleton 1-nodes x_a^1 and x_b^1 , but then \mathcal{M} cannot contain any other 1-node. Clearly, there are many two-ended 1-paths terminating at x_a^1 and x_b^1 . On the other hand, another metrizable set \mathcal{M}' consists of all the 0-nodes along with x_{abc}^1 , but now \mathcal{M}' cannot contain any other 1-node. There are many two-ended 1-paths terminating at x_{abc}^1 and at any 0-node. \square

4 Distances Between Nodes

Our objective now is to define ordinal distances between nodes whereby the metric axioms are satisfied. We always assume henceforth that \mathcal{G}^ν ($\nu \leq \omega$) is ν -connected and satisfies Condition 3.2. Let \mathcal{M} be a metrizable set of nodes in \mathcal{G}^ν . Thus, Lemma 3.3 holds. We define the distance function $d: \mathcal{N} \times \mathcal{N} \rightsquigarrow \aleph_1$ as follows: If x_a^α and x_b^β are different (maximal) nodes in \mathcal{M} , we set

$$d(x_a^\alpha, x_b^\beta) = \min\{|P_{ab}|: P_{ab} \text{ is a twoended path terminating at } x_a^\alpha \text{ and } x_b^\beta\}. \quad (7)$$

If $x_a^\alpha = x_b^\beta$, we set $d(x_a^\alpha, x_b^\beta) = 0$ (a result that would also arise if we allowed trivial paths). By our constructions in Sec. 2, $|P_{ab}|$ is a countable ordinal no larger than $\omega^\omega \cdot k$, where

k is a natural number. Moreover, any set of ordinals is well-ordered and thus has a least member. Therefore, the minimum indicated in (7) exists, and is a countable ordinal.

Obviously, $d(x_a^\alpha, x_b^\beta) > 0$ if $x_a^\alpha \neq x_b^\beta$. Moreover, $d(x_a^\alpha, x_b^\beta) = d(x_b^\beta, x_a^\alpha)$. It remains to prove the triangle inequality; namely, if x_a^α , x_b^β , and x_c^γ are any three (maximal) nodes in \mathcal{M} , then

$$d(x_a^\alpha, x_b^\beta) \leq d(x_a^\alpha, x_c^\gamma) + d(x_c^\gamma, x_b^\beta) \quad (8)$$

Once again, it is understood that the sum of ordinals on the right-hand side is written with the larger ordinal first. (We will not keep repeating this admonition.)

By Lemma 3.3, there exists a path P_{ac} that terminates at x_a^α and x_c^γ , and there exists another path P_{cb} that terminates at x_c^γ and x_b^β . We now invoke [7, Corollary 3.5-4]: There is in $P_{ac} \cup P_{cb}$ a two-ended path P_{ab} that terminates at x_a^α and x_b^β . In fact, a tracing of P_{ab} proceeds from x_a^α along P_{ac} until it reaches the first node that meets both P_{ac} and P_{cb} , and then it proceeds along P_{cb} until it reaches x_b^β . Since the length of a path can be no less than the length of any of its subpaths (Lemma 2.2), we have $|P_{ab}| \leq |P_{ac}| + |P_{cb}|$. Now, consider all the two-ended paths that terminate at x_a^α and x_b^β and take the minimum of their lengths. We get

$$d(x_a^\alpha, x_b^\beta) \leq |P_{ac}| + |P_{cb}|. \quad (9)$$

This inequality holds whatever be the choices of P_{ac} and P_{cb} . Therefore, we can take minimums on the right-hand side of (9) for all choices of P_{ac} and P_{cb} to get (8). In short, we have

Proposition 4.1. *d satisfies the metric axioms.*

Clearly, d reduces to the standard (branch-count) distance function when \mathcal{G}^ν is replaced by a finite graph. We have achieved one of the objectives of this paper by showing that the branch-count distance function can be extended transfinitely to any metrizable set of nodes in \mathcal{G}^ν .

Example. 4.1. For the 1-graph of Fig. 1 and with \mathcal{M} chosen as in the first part of Example 3.4, $d(x_1^0, x_2^0) = 1$, $d(x_1^0, x_a^1) = d(x_1^0, x_b^1) = \omega$, $d(x_1^0, y_a^0) = d(x_1^0, y_b^0) = \omega + 1$, and $d(y_a^0, y_b^0) = \omega \cdot 2 + 2$. \square

In the rest of this paper, we shall lift transfinitely several standard results concerning distances in graphs, but here, too, various complications arise.

5 Ordinals and Ranks

As we have seen, the distance between any two nodes of \mathcal{M} is a countable ordinal. However, given any $x \in \mathcal{M}$, the set $\{d(x, y) : y \in \mathcal{M}\}$ may have no maximum. For example, this is the case for a one-ended 0-path P^0 where x is any fixed node of P^0 and y ranges through all the 0-nodes of P^0 . On the other hand, for finite graphs the said maximum exists and is the “eccentricity” of x . We will be able to define an “eccentricity” for every node of \mathcal{M} if we expand the set \aleph_1 of countable ordinals into the set \mathcal{R} of ranks [7, page 4], [8, page 4]. This is done by inserting an *arrow rank* $\vec{\rho}$ immediately before each $\rho \in \aleph_1$. \mathcal{R} looks like²

$$\mathcal{R} = \{\vec{0}, 0, 1, 2, \dots, \vec{\omega}, \omega, \omega + 1, \dots, \vec{\omega \cdot 2}, \omega \cdot 2, \omega \cdot 2 + 1, \dots, \vec{\omega \cdot n}, \omega \cdot n, \omega \cdot n + 1, \dots, \\ \vec{\omega^2}, \omega^2, \omega^2 + 1, \dots, \vec{\omega^k}, \omega^k, \omega^k + 1, \dots, \vec{\omega^\omega}, \omega^\omega, \omega^\omega + 1, \dots\}.$$

Note that the set of all ranks is well-ordered. Indeed, there is an order-preserving bijection from \mathcal{R} to \aleph_1 obtained by replacing each rank by its successor rank. Since \aleph_1 is well-ordered, so, too, is \mathcal{R} .

In accordance with two Aristotelian ideas [6, page 3], we can view each transfinite (successor or limit) ordinal as an “actual infinity” because distances between nodes can assume those values, whereas each arrow rank (other than $\vec{0}$) can be viewed as a “potential infinity” because distances can increase toward an arrow rank but will never achieve it.

The arrow ranks served as a notational convenience in the prior works [7] and [8], but in this paper occasions will arise when arrow ranks need to be added. For this reason, we now define arrow ranks in terms of sequences of countable ordinals.

Let \mathcal{A} be any set of countable ordinals having a countable ordinal ζ as an upper bound (i.e., $\zeta \geq \alpha$ for all $\alpha \in \mathcal{A}$). Let \mathcal{B} be the set of countable ordinals, each of which is greater than every member of \mathcal{A} and is no greater than ζ . If \mathcal{B} is empty, \mathcal{A} has a greatest member, namely, ζ . (Because of the upper bound ζ on \mathcal{A} , this is the only way \mathcal{B} can be empty.)

²As was done in the prior works, we treat 0 as the first limit ordinal and $\vec{0}$ as the first arrow rank, but in this paper $\vec{0}$ will never be used. All our arrow ranks will be understood to be other than $\vec{0}$.

So, assume \mathcal{B} is not empty. By well-ordering, \mathcal{B} has a least member λ . If λ is a successor ordinal, then \mathcal{A} has a greatest member, namely, $\lambda - 1$; in this case, $\lambda - 1$ is either a successor ordinal or a limit ordinal. If λ is a limit ordinal, then there exists an increasing sequence $\{\alpha_k\}_{k=0}^{\infty}$ contained in \mathcal{A} such that, for each $\gamma \in \mathcal{A}$, $\alpha_k > \gamma$ for all k sufficiently large (i.e., there exists a k_0 such that $\alpha_k > \gamma$ for all $k \geq k_0$).

With λ being a nonzero limit ordinal, we define the arrow-rank $\bar{\lambda}$ as an equivalence class of such increasing sequences of ordinals, where two such sequences $\{\alpha_k\}_{k=0}^{\infty}$ and $\{\beta_k\}_{k=0}^{\infty}$ are taken to be equivalent if, for each γ less than λ , there exists a natural number k_0 such that $\gamma < \alpha_k, \beta_k < \lambda$ for all $k > k_0$. The axioms of an equivalence relationship are clearly satisfied. Each such sequence $\{\alpha_k\}_{k=0}^{\infty}$ is a representative of $\bar{\lambda}$, and we say that $\{\alpha_k\}_{k=0}^{\infty}$ reaches $\bar{\lambda}$.

Note that this equivalence class of increasing sequences is different from the set of ordinals less than λ . The latter is λ itself by the definition of ordinals.

We have established the following.

Lemma 5.1. *If \mathcal{A} is any set of countable ordinals that are bounded above by a countable ordinal ζ , then $\sup \mathcal{A}$ exists either as a (successor or limit) ordinal or as an arrow rank.*

We define the sum $\bar{\alpha} + \bar{\beta}$ of two arrow ranks $\bar{\alpha}$ and $\bar{\beta}$ as the componentwise sum of two representative sequences for them. That is, if $\{\alpha_k\}$ and $\{\beta_k\}$ are representatives of $\bar{\alpha}$ and $\bar{\beta}$ respectively, then $\{\alpha_k + \beta_k\}$ is a representative³ for $\bar{\alpha}_k + \bar{\beta}_k$.

Lemma 5.2. *Let α and β be two nonzero limit ordinals, and let $\bar{\alpha}$ and $\bar{\beta}$ be respectively the arrow ranks immediately preceding α and β . Then, the sum $\bar{\alpha} + \bar{\beta}$ is equal to $\alpha \bar{+} \beta$, where the latter is the arrow rank immediately preceding the limit ordinal $\alpha + \beta$.*

Proof. First note that the sum of two limit ordinals is a limit ordinal [1, page 330], so our conclusion has a meaning. Let $\{\alpha_k\}$ and $\{\beta_k\}$ be representative sequences for $\bar{\alpha}$ and $\bar{\beta}$ respectively. Then, for any ordinal $\gamma < \alpha$ (resp. $\delta < \beta$), there is a k_0 such that $\gamma < \alpha_k < \alpha$ (resp. $\delta < \beta_k < \beta$) for all $k < k_0$. Now, let ϵ be any ordinal less than $\alpha + \beta$. Choose γ and δ such that $\gamma + \delta = \epsilon$, but otherwise let γ and δ be arbitrary. Then, for all $k > k_0$, $\epsilon < \alpha_k + \beta_k < \alpha + \beta$. Since ϵ is arbitrary except as stated, $\{\alpha_k + \beta_k\}_{k=0}^{\infty}$ is a representative

³Remember the admonition about writing $\alpha_k + \beta_k$ in the proper order so as to get $\alpha_k + \beta_k > \max\{\alpha_k, \beta_k\}$.

of the arrow rank $\alpha \bar{+} \beta$. But, by our definition of the sum of arrow ranks, $\{\alpha_k + \beta_k\}_{k=0}^{\infty}$ is also a representative of $\bar{\alpha} + \bar{\beta}$. Hence, $\alpha \bar{+} \beta = \bar{\alpha} + \bar{\beta}$. \square

(Let us note in passing that the sum of the arrow rank $\bar{\alpha}$ and of the ordinal γ can be defined by adding γ to each term of $\{\alpha_k\}_{k=0}^{\infty}$, in the proper order of course. Then, if $\alpha < \gamma$, the sum $\gamma + \bar{\alpha}$ is the arrow rank immediately preceding the limit ordinal $\gamma + \alpha$. However, if $\gamma < \alpha$, the sum $\bar{\alpha} + \gamma$ is the arrow rank immediately preceding α . We will not need this result.)

6 Eccentricities and Related Ideas

First of all, note that the lengths of all paths in a ν -graph \mathcal{G}^ν are bounded by $\omega^{\nu+1} \cdot 2$ because the longest possible paths in \mathcal{G}^ν are the endless paths of rank ν . Therefore, all distances in \mathcal{G}^ν are also bounded above by $\omega^{\nu+1} \cdot 2$. As before, \mathcal{M} will always denote a metrizable set of nodes in \mathcal{G}^ν .

The eccentricity $e(x)$ of any node $x \in \mathcal{M}$ is defined by

$$e(x) = \sup\{d(x, y) : y \in \mathcal{M}\}. \quad (10)$$

Two cases arise: First, the supremum is achieved at some node $\hat{y} \in \mathcal{M}$. In this case, $e(x)$ is an ordinal; so, we can replace “sup” by “max” in (10) and write $e(x) = d(x, \hat{y})$. Second, the supremum is not achieved at any node in \mathcal{M} . In this case, $e(x)$ is an arrow rank.

The ideas of radii and diameters for finite graphs [3, page 32], [5, page 21] can also be extended transfinitely. Given \mathcal{G}^ν and \mathcal{M} , the *radius* $\text{rad}(\mathcal{G}^\nu, \mathcal{M})$ is the least eccentricity among the nodes of \mathcal{M} :

$$\text{rad}(\mathcal{G}^\nu, \mathcal{M}) = \min\{e(x) : x \in \mathcal{M}\} \quad (11)$$

We also denote this simply by rad with the understanding that \mathcal{G}^ν and \mathcal{M} are given. The minimum exists as a rank (either as an ordinal or as an arrow rank) because the set of ranks is well-ordered. Thus, there will be at least one $x \in \mathcal{M}$ with $e(x) = \text{rad}$.

Furthermore, the *diameter* $\text{diam}(\mathcal{G}^\nu, \mathcal{M})$ is defined by

$$\text{diam}(\mathcal{G}^\nu, \mathcal{M}) = \sup\{e(x) : x \in \mathcal{M}\}. \quad (12)$$

With \mathcal{G}^ν and \mathcal{M} understood, we denote the diameter simply by diam . The right-hand side of (12) can be rewritten as

$$\sup_{x \in \mathcal{M}} \sup_{y \in \mathcal{M}} \{d(x, y)\} = \sup\{d(x, y) : x, y \in \mathcal{M}\}.$$

As we have noted before, each $d(x, y)$ is no greater than $\omega^{\nu+1} \cdot 2$. Let \mathcal{D} be the set of all ranks greater than every member of $\mathcal{C} = \{d(x, y) : x, y \in \mathcal{M}\}$ but less than $\omega^{\nu+1} \cdot 3$. \mathcal{D} is not empty. Since the set of all ranks is well-ordered, \mathcal{D} has a least member λ . If λ is a successor ordinal, diam is a (successor or limit) ordinal. If λ is a limit ordinal, diam is an arrow rank. If λ is an arrow rank, \mathcal{C} must contain increasing sequences of ordinal less than λ but reaching λ ; this implies that diam is an arrow rank again. We conclude that diam exists either as an ordinal or as an arrow rank.

The ideas of the center and periphery of finite graphs can also be extended. The *center* of $(\mathcal{G}^\nu, \mathcal{M})$ is the set of nodes in \mathcal{M} having the least eccentricity, namely, rad . As noted before, there will be at least one node in \mathcal{M} having rad as its eccentricity. Thus, the center is never empty. The *periphery* of $(\mathcal{G}^\nu, \mathcal{M})$ is the set of nodes in \mathcal{M} having the greatest eccentricity, namely, diam . If diam is an ordinal, there will be at least two nodes of \mathcal{M} in the periphery. It seems that, if diam is an arrow rank, the periphery will have infinitely many nodes of \mathcal{M} , but presently this is only a conjecture.

Example 6.1. Let \mathcal{G}^0 be a one-ended 0-path with \mathcal{M} being the set of all 0-nodes. (We do not assign a 1-node at the path's infinite extremity.) Then, every 0-node has an eccentricity of $\bar{\omega}$. Thus, $\text{rad} = \text{diam} = \bar{\omega}$, and \mathcal{M} is both the center and the periphery of $(\mathcal{G}^0, \mathcal{M})$. \square

Example 6.2. Consider the 1-graph of Fig. 1 with \mathcal{M} being the set of all 0-nodes along with x_a^1 and x_b^1 . (Ignore x_{abc}^1 and all other 1-nodes.) The 0-nodes to the left of the 1-nodes all have the eccentricity $\omega + 1$. Also, $e(x_a^1) = e(x_b^1) = \omega \cdot 2 + 1$, and $e(y_a^0) = e(y_b^0) = \omega \cdot 2 + 2$. Thus, $\text{rad} = \omega + 1$ and $\text{diam} = \omega \cdot 2 + 2$. The center consists of all the 0-nodes to the left of the 1-nodes, and the periphery is $\{y_a^0, y_b^0\}$. \square

Example 6.3. Now, consider the 1-graph obtained from Fig. 1 by deleting the branches d and e and the 0-nodes y_a^0 and y_b^0 , but append a new branch to x_1^0 and x_a^1 . Let \mathcal{M} be all the nodes. Then, the eccentricity of every node is ω . Thus, $\text{rad} = \text{diam} = \omega$, and the center

and periphery are the same, namely, \mathcal{M} . \square

Example 6.4. This time, let \mathcal{G}^ν consist of a one-ended 0-path P_a^0 and an endless 0-path P_b^0 forming a 1-loop, as shown in Fig. 2. \mathcal{M} is now the set of all (maximal) nodes. P_a^0 starts at the nonmaximal 0-node z^0 embraced by the 1-node x^1 and reaches the 1-node y^1 . P_b^0 reaches both x^1 and y^1 . Let v^0 be a 0-node of P_a^0 at a distance of k from x^1 . (For z^0 , $k = 0$.) The shortest distance between v^0 and any node w^0 of P_b^0 is provided by a path that passes through x^1 ; it has the length $\omega + k$. (The path passing through y^1 and terminating at v^0 and w^0 has length $\omega \cdot 2$.) Thus, $e(v^0) = \omega + k$. On the other hand, $e(w^0) = \omega + \bar{\omega} = \omega \cdot 2$; indeed, $d(w^0, v^0) = \omega + k$, which increases indefinitely but never achieves $\omega \cdot 2$ as v^0 approaches y^1 . Furthermore, $e(x^1) = e(y^1) = \omega$. Thus, $\text{rad} = \omega$, $\text{diam} = \omega \cdot 2$, the center is $\{x^1, y^1\}$, and the periphery is the set of all the 0-nodes of P_b^0 . \square

These examples can immediately be converted into examples for graphs of higher ranks by replacing branches by endless paths of the same rank. For instance, if every branch is replaced by an endless path of rank $\nu - 2$, then every 0-node becomes a $(\nu - 1)$ -node, and every 1-node becomes a ν -node. To get the new eccentricities, replace ω by ω^ν , $\bar{\omega}$ by $\bar{\omega}^\nu$, and k by $\omega^{\nu-1} \cdot 2k$. Of course, there are far more complicated ν -graphs.

7 Some General Results

For any rank ρ with $0 \leq \rho < \nu$, let us define a ρ -section \mathcal{S}^ρ of \mathcal{G}^ν as the subgraph of the ρ -graph of \mathcal{G}^ν induced by a maximal set of branches that are ρ -connected.⁴ Thus, all the (maximal and nonmaximal) nodes of \mathcal{S}^ρ have ranks ρ or less, and all the (maximal) bordering nodes of \mathcal{S}^ρ have ranks greater than ρ . In the next theorem, \mathcal{S}^ρ is any ρ -section whose bordering nodes are incident to \mathcal{S}^ρ only through ρ -tips. It follows that all the (maximal and nonmaximal) nodes of \mathcal{S}^ρ are internal nodes [7, page 81], [8, page 37]; that is, they are not embraced by any bordering nodes of \mathcal{S}^ρ . In Fig. 1, x_a^1 and x_b^1 are bordering nodes of the 0-section to the left of those nodes, and the condition is satisfied, that is, those 1-nodes are incident to that 0-section only through 0-tips. However, branch d induces a 0-section by itself, and the condition is not satisfied because d reaches x_a^1 through a (-1) -tip; similarly

⁴This is the same definition of a ρ -section as that given in [8, page 36] but is slightly stronger than that of [7, page 49].

for e and x_b^1 . In Fig. 2, P_a^0 and P_b^0 are different 0-sections; P_b^0 satisfies the condition, but P_a^0 does not.

Theorem 7.1. *Let S^ρ be a ρ -section in \mathcal{G}^ν ($0 \leq \rho < \nu$) all of whose bordering nodes are incident to S^ρ only through ρ -tips. Then, all the nodes of S^ρ that are in \mathcal{M} have the same eccentricity.*

Proof. By virtue of our hypothesis and the ρ -connectedness of S^ρ , for any internal node x^α ($\alpha \leq \rho$) and any bordering node z^γ ($\gamma > \rho$) of S^ρ in \mathcal{M} , there is a representative ρ -path P^ρ for a ρ -tip embraced by z^γ and lying in S^ρ , and there also is a two-ended path Q lying in S^ρ and terminating at x^α and a node of P^ρ by virtue of Lemma 3.3. So, by Condition 3.2 and [7, Corollaries 3.5-4 and 3.5-5], there is in $P \cup Q$ a one-ended ρ -path R^ρ that terminates at x^α and reaches z^γ through a ρ -tip. Moreover, all paths that terminate at x^α , that lie in S^ρ , and that reach z^γ must be one-ended ρ -paths. Therefore, $d(x^\alpha, z^\gamma) = \omega^{\rho+1}$. For any other node y^β ($\beta \leq \rho$) of \mathcal{M} in S^ρ , we have $d(x^\alpha, y^\beta) < \omega^{\rho+1}$ by Lemma 3.3 and the ρ -connectedness of S^ρ . So, if \mathcal{G}^ν consists only of S^ρ and its bordering nodes (so that $\nu = \rho + 1$), we can conclude that $c(x^\alpha) = \omega^{\rho+1}$, whatever be the choice of x^α in S^ρ and in \mathcal{M} .

Next, assume that there is a node v^δ of \mathcal{G}^ν in \mathcal{M} lying outside of S^ρ and different from all the bordering nodes of S^ρ in \mathcal{M} . By the ν -connectedness of \mathcal{G}^ν , there is a path P_{xv} terminating at x^α and v^δ . Let z^γ now be the last bordering node of S^ρ that P_{xv} meets. Let P_{zv} be that part of P_{xv} lying outside of S^ρ . Then, by what we have shown above, there is a one-ended ρ -path Q_{xz}^ρ that terminates at x^α , lies in S^ρ , and reaches z^γ through a ρ -tip. Then, $R_{xv} = Q_{xz}^\rho \cup P_{zv}$ is a two-ended path that terminates at x^α and v^δ . Moreover, $|R_{xv}| \leq |P_{xv}|$.

Now, let y^β ($\beta \leq \rho$) be any other node of S^ρ in \mathcal{M} (i.e., different from x^α). Again, there is a one-ended ρ -path Q_{yz}^ρ satisfying the same conditions as Q_{xz}^ρ . We have $d(x^\alpha, z^\gamma) = d(y^\beta, z^\gamma) = \omega^{\rho+1}$. Let $R_{yv} = Q_{yz}^\rho \cup P_{zv}$. Thus, $|R_{xv}| = |R_{yv}|$. We have shown that, for each one-ended path R_{xv} terminating at x^α and v^δ and passing through exactly one bordering node of z^γ of S^ρ , there is another path R_{yv} of the same length terminating at y^β and v^δ and identical to R_{xv} outside S^ρ . It follows that $d(x^\alpha, v^\delta) = d(y^\beta, v^\delta)$. We can conclude that

$e(x^\alpha) = e(y^\beta)$ whatever be the choices of x^α and y^β in S^ρ and \mathcal{M} . \square

Figs. 1 and 2 provide examples for Theorem 7.1. In Fig. 1, all the 0-nodes to the left of the 1-nodes have the same eccentricity $\omega + 1$ in accordance with the theorem. In Fig. 2, all the nodes of P_b^0 have the same eccentricity $\omega \cdot 2$, whereas the eccentricities of the nodes of P_a^0 vary; this, too, conforms with Theorem 7.1.

A standard result [5, page 21] extends readily to the transfinite case. In the following, $\text{rad} \cdot 2$ denotes $\text{rad} + \text{rad}$, which has a meaning not only when rad is an ordinal but also when rad is an arrow rank; the latter sum was defined in Sec. 5.

Theorem 7.2. *With \mathcal{G}^ν and \mathcal{M} specified, $\text{rad} \leq \text{diam} \leq \text{rad} \cdot 2$.*

Proof. When rad and diam are ordinals, the proof is the same as that for finite graphs. So, consider the case when either or both of rad and diam are arrow ranks. That $\text{rad} \leq \text{diam}$ follows directly from the definitions. Next, by the definition of the diameter (12) we can choose two sequences $\{y_k\}_{k=0}^\infty$ and $\{z_k\}_{k=0}^\infty$ of nodes contained in \mathcal{M} such that the sequence $\{d(y_k, z_k)\}_{k=0}^\infty$ reaches or achieves diam . Let x be any node in the center. By the triangle inequality,

$$d(y_k, z_k) \leq d(y_k, x) + d(x, z_k).$$

Since $d(y_k, x) \leq \text{rad}$ and $d(x, z_k) \leq \text{rad}$, we have $d(y_k, z_k) \leq \text{rad} + \text{rad} = \text{rad} \cdot 2$. Here, we invoke Lemma 5.2 in the event that rad is an arrow rank. \square

Another standard result is that the nodes of any finite graph comprise the center of some finite connected graph [5, page 22]. This, too, can be extended transfinitely—in fact, in several ways, but the proofs are more complicated than that for finite graphs. Nonetheless, the scheme of the proofs remains the same. First, we need the following lemma. It continues to be understood that every mentioned node is maximal unless otherwise noted and is a member of the chosen metrizable set \mathcal{M} .

Lemma 7.3. *Let $S^{\nu-1}$ be a $(\nu - 1)$ -section of \mathcal{G}^ν . Let $u^\nu \in \mathcal{M}$ be a ν -node incident to $S^{\nu-1}$ (thus, a bordering node of $S^{\nu-1}$ [7, page 81]), and let $x^\alpha \in \mathcal{M}$ ($\alpha < \nu$) be an α -node in $S^{\nu-1}$ (thus, an internal node of $S^{\nu-1}$ [7, page 81]). Then, there exists a two-ended path of length no larger than ω^ν connecting x^α and u^ν .*

Proof. That u^ν is incident to $S^{\nu-1}$ means that there is in $S^{\nu-1}$ a one-ended β -path P^β

with $\beta \leq \nu - 1$ whose β -tip is embraced by u^ν . Let $P^{\beta+1}$ be the two-ended path obtained by appending to P^β the $(\beta + 1)$ -node $y^{\beta+1}$ embraced by u^ν and reached by P^β . ($y^{\beta+1}$ will not be maximal if $\beta + 1 < \nu$; otherwise, $y^{\beta+1} = u^\nu$.) The length $|P^{\beta+1}|$ of $P^{\beta+1}$ is no larger than $\omega^{\beta+1}$. Moreover, $P^{\beta+1}$ traverses only one β -tip; all other tips traversed by $P^{\beta+1}$ are of lesser rank. Let z^γ be any node of P^β ; thus, $\gamma \leq \beta$. By the $(\nu - 1)$ -connectedness of $S^{\nu-1}$, there is in $S^{\nu-1}$ a two-ended λ -path Q^λ ($0 \leq \lambda \leq \nu - 1$) terminating at x^α and z^γ . The tips traversed by Q^λ have ranks no greater than $\lambda - 1$, hence, no greater than $\nu - 2$. By [7, Corollaries 3.5-4 and 3.5-5], there is a two-ended path R^δ in $P^{\beta+1} \cup Q^\lambda$ terminating at x^α and $y^{\beta+1}$. All the tips traversed by R^δ are of ranks no greater than $\nu - 1$, and there is at most one traversed tip of rank $\nu - 1$. Hence, the length of R^δ satisfies $|R^\delta| \leq \omega^\nu$. \square

Given any ν -graph \mathcal{G}^ν , let us construct a larger ν -graph \mathcal{H}^ν by appending six additional ν -nodes p_i^ν and q_i^ν ($i = 1, 2, 3$) and also appending isolated endless $(\nu - 1)$ -paths⁵ that reach ν -nodes as shown in Fig. 3. Such paths connect p_1^ν to p_2^ν , p_2^ν to p_3^ν , p_3^ν to every ν -node in \mathcal{G}^ν , and similarly for p_i^ν replaced by q_i^ν . Note that the singleton end-nodes p_1^ν and q_1^ν can be included in the chosen metrizable set \mathcal{M} for \mathcal{H}^ν . All the other ν -nodes of \mathcal{H}^ν are nonsingletons and therefore are in \mathcal{M} , too.

Theorem 7.4. *The ν -nodes of \mathcal{G}^ν comprise the center of \mathcal{H}^ν , and the periphery of \mathcal{H}^ν is $\{p_1^\nu, q_1^\nu\}$.*

Proof. We look for bounds on the eccentricities of all the nodes in \mathcal{M} . Let x^α and y^β be any two (maximal) nodes in \mathcal{M} whose ranks satisfy $0 \leq \alpha, \beta < \nu$. It follows that x^α (resp. y^β) is an internal node of a $(\nu - 1)$ -section in \mathcal{G}^ν , and that section has at least one ν -node u^ν (resp. v^ν) as a bordering node because \mathcal{G}^ν is ν -connected. By the triangle inequality,

$$d(x^\alpha, y^\beta) \leq d(x^\alpha, u^\nu) + d(u^\nu, p_3^\nu) + d(p_3^\nu, v^\nu) + d(v^\nu, y^\beta).$$

By Lemma 7.3, $d(x^\alpha, u^\nu) \leq \omega^\nu$ and $d(v^\nu, y^\beta) \leq \omega^\nu$. Clearly, $d(u^\nu, p_3^\nu) = d(p_3^\nu, v^\nu) = \omega^\nu \cdot 2$. Thus, $d(x^\alpha, y^\beta) \leq \omega^\nu \cdot 6$. This also shows that, for any ν -node v^ν in \mathcal{G}^ν , $d(x^\alpha, v^\nu) \leq \omega^\nu \cdot 5$. Since $d(u^\nu, p_1^\nu) = \omega^\nu \cdot 6$, we have $d(x^\alpha, p_1^\nu) = d(x^\alpha, u^\nu) + d(u^\nu, p_1^\nu) \leq \omega^\nu + \omega^\nu \cdot 6 = \omega^\nu \cdot 7$.

⁵An isolated endless path embraces no tips other than the ones it traverses. Thus, to reach any other part of a graph in which the isolated path is a subgraph, one must proceed through a terminal tip of that path.

Now, $d(x^\alpha, u^\nu) \geq 1$ because there is at least one branch in any path connecting x^α and u^ν . Thus, we also have $d(x^\alpha, p_1^\nu) \geq \omega^\nu \cdot 6 + 1$. Note also that the distance from x^α to any node of the appended endless paths is strictly less than $\omega^\nu \cdot 7$. All these results hold for p_i^ν replaced by q_i^ν . Altogether then, we can conclude the following: For any node in \mathcal{G}^ν of rank less than ν , say, x^α , the eccentricity $e(x^\alpha)$ of x^α is bounded as follows:

$$\omega^\nu \cdot 6 + 1 \leq e(x^\alpha) \leq \omega^\nu \cdot 7$$

Next, consider any two ν -nodes of \mathcal{G}^ν , say, u^ν and v^ν again. By what we have already shown, $d(u^\nu, v^\nu) \leq \omega^\nu \cdot 4$, and $d(u^\nu, p_1^\nu) = d(u^\nu, q_1^\nu) = \omega^\nu \cdot 6$. The distance from u^ν to any node of the appended endless $(\nu - 1)$ -paths is less than $\omega^\nu \cdot 6$. Also, for any node y^β in \mathcal{G}^ν of rank less than ν , $d(u^\nu, y^\beta) \leq \omega^\nu \cdot 5$. So, the largest distance between u^ν and any other node in \mathcal{H}^ν is equal to $\omega^\nu \cdot 6$; that is, $e(u^\nu) = \omega^\nu \cdot 6$.

Finally, we have $e(p_3^\nu) = e(q_3^\nu) = \omega^\nu \cdot 8$, $e(p_2^\nu) = e(q_2^\nu) = \omega^\nu \cdot 10$, and $e(p_1^\nu) = e(q_1^\nu) = \omega^\nu \cdot 12$. The eccentricities of the nodes of the appended endless paths lie between these values.

We have considered all cases. Comparing these equalities and inequalities for all the eccentricities, we can draw the conclusion of the theorem. \square

As an immediate corollary, we have the following generalization of a result for finite graphs.

Corollary 7.5. *The ν -nodes of \mathcal{G}^ν comprise the center of some ν -graph \mathcal{H}^ν .*

Variations of Corollary 7.5 can also be established through much the same proofs. For instance, all the (maximal) nodes of \mathcal{G}^ν of one or more specified ranks can be made to comprise the center of some ν -graph. This is because the $(\nu - 1)$ -sections of \mathcal{G}^ν partition \mathcal{G}^ν . Still more generally, if \mathcal{G}^ν has only finitely many ν -nodes, any arbitrary set of nodes of \mathcal{G}^ν in \mathcal{M} can be made the center simply by appending enough endless $(\nu - 1)$ -paths in series.

It appears that still other results concerning distances in finite graphs can be extended transfinitely as well.

References

- [1] A. Abian, *The Theory of Sets and Transfinite Arithmetic*, W.B. Saunders Co., Philadelphia, 1965.
- [2] F. Buckley (Editor), Centrality Concepts in Network Location, *Networks*, **34** (1999), Issue No. 4.
- [3] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley Publishing Co., Redwood City, CA, 1990.
- [4] B.D. Calvert and A.H. Zemanian, Operating points in infinite nonlinear networks approximated by finite networks, *Trans. Amer. Math. Soc.*, **352** (2000), 753-780.
- [5] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Third Edition, Chapman & Hall, New York, 1996.
- [6] R. Rucker, *Infinity and the Mind*, Birkhauser, Boston, 1982.
- [7] A.H. Zemanian, *Transfiniteness - for Graphs, Electrical Networks, and Random Walks*, Birkhauser, Boston, 1996.
- [8] A.H. Zemanian, *Pristine Transfinite Graphs and Permissive Electrical Networks*, Birkhauser, Boston, 2001.

Figure Captions

Fig. 1. A 1-graph consisting of a one-way infinite ladder along with two branches, d and e , connected to infinite extremities of the ladder. x_a^1 and x_b^1 are the only nonsingleton 1-nodes; all the other 1-nodes are singletons.

Fig. 2. A 1-loop having two 0-sections.

Fig. 3. The ν -graph \mathcal{H}'' .

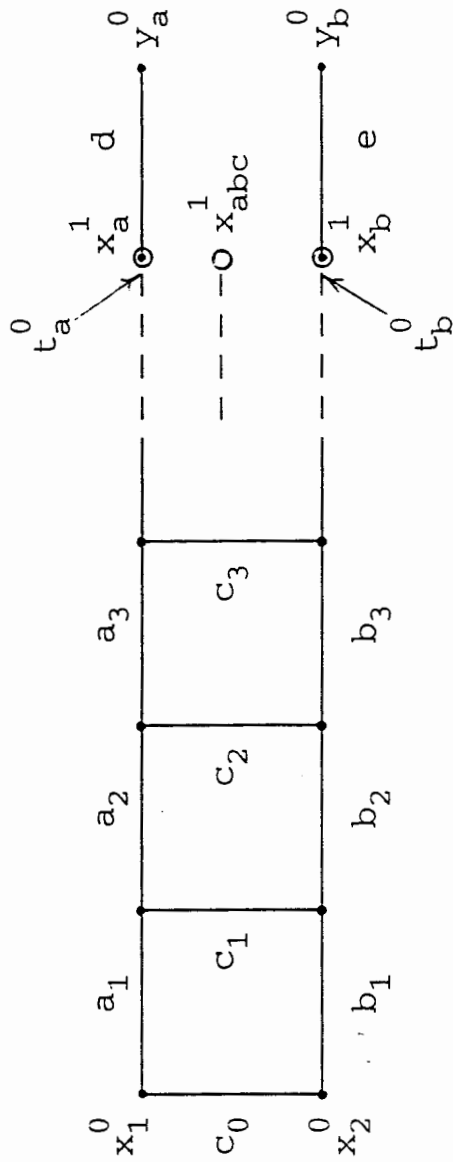
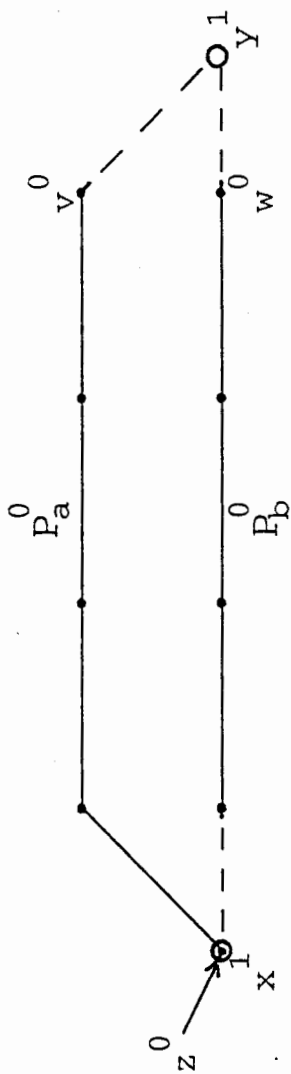


Fig. 1 Zamestian

Fig. 1



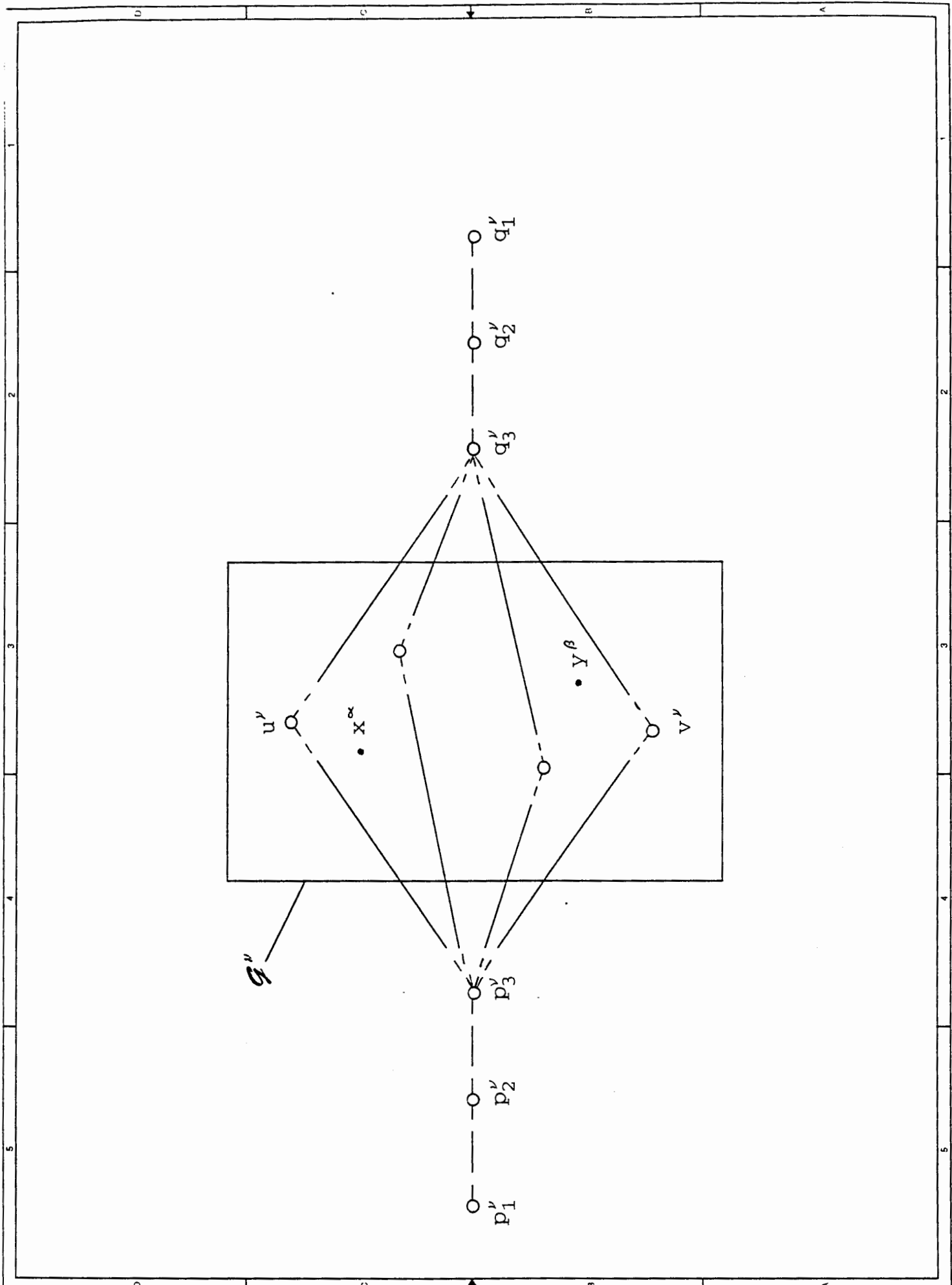


Fig. 3. *Зона видна*