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PERMISSIVELY STRUCTURED TRANSFINITE ELECTRICAL NETWORKS

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Abstract — Transfinite electrical networks of ranks larger than 1 have previously been defined by arbitrarily joining together various infinite extremities through transfinite nodes that are independent of the networks' resistance values. Thus, some or all of those transfinite nodes may remain ineffective in transmitting current "through infinity." In this paper, transfinite nodes are defined in terms of the paths that permit currents to "reach infinity." This is accomplished by defining a suitable metric d^ν on the node set $\mathcal{N}_{\mathbf{S}^\nu}$ of each ν-section $\mathbf{S}^\nu$, a ν-section being a maximal subnetwork whose nodes are connected by two-ended paths of ranks no larger than ν. Upon taking the completion of $\mathcal{N}_{\mathbf{S}^\nu}$ under that metric d^ν, we identify those extremities (now called ν-terminals) that are accessible to current flows. These are used to define transfinite nodes that combine such extremities. The construction is recursive and is carried out through all the natural-number ranks, and then through the first arrow rank $\vec{\omega}$ and the first limit-ordinal rank ω. The recursion can be carried still further.

1 Introduction

The idea of a transfinite graph or network was introduced in [2] and more thoroughly investigated in [3]. The transfinite structure was obtained by joining together various infinite extremities of infinite graphs, but the choices of those connections were quite arbitrary. With regard to transfinite electrical networks, those choices did not reflect the ways in which currents could or could not flow toward infinite extremities and through transfinite

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nodes. A more recent work [1] showed (among other results) that the permissive paths, through which currents might flow toward the infinite extremities of an electrical network, could serve as a guide for joining various choices of those extremities in order to construct transfinite nodes.

However, this was accomplished only for the first rank of transfiniteness. The aim of this paper is to extend this way of constructing transfinite networks to higher ranks of transfiniteness. This is achieved through a recursive development, which is carried through all the natural-number ranks, the first arrow rank $\vec{\omega}$, and the first transfinite rank ω. This paper ends by pointing out how this recursion can be carried forward to ranks larger than ω. The resulting networks are herein called "permissively structured."

The basic idea is to construct metrics on maximal sets of transfinite nodes that are ν-connected, one metric d^{ν} for each rank ν of transfiniteness, and then to complete each such set of nodes under the metric d^{ν}. The new limit points obtained thereby are called "ν-terminals" and are the "permissive extremities." The set of ν-terminals are then partitioned arbitrarily, and the sets of the partition are then joined into transfinite nodes of the next higher rank $\nu+1$. All this works for the natural-number ranks; modifications are needed for the arrow rank $\vec{\omega}$ and the limit rank ω. However, we impose some restrictions to enable the recursion — as, for example, the conditions that no node embraces a node of lower rank, that each ν-section has only finitely many ν-terminals, and that each $(\nu+1)$-node consists of only finitely many ν-terminals. (A ν-section is a maximal subnetwork whose branches are ν-connected [3].)

This paper is written in such a fashion that it can be read independently from [1]. Our notation and terminology conforms with that used in [3]. In addition, we use the following. The notation $x \dashv Q$ denotes that the entity x is embraced by the entity Q. The *resistive length* $|P|$ of a path P is the sum of all the resistances of its embraced branches; in symbols, $|P| = \sum_{b \dashv P} r_b$, where r_b is the resistance of the branch b embraced by P. We say that P is *permissive* if $|P| < \infty$. The *resistive size* $|\mathbf{N}|$ of a network $\mathbf{N}$ is defined similarly: $|\mathbf{N}| = \sum_{b \dashv \mathbf{N}} r_b$. We shall say that a ν-node, where $\nu \geq 1$, is *permissive* if every one of its embraced tips has a permissive representative (in which case all its representatives are permissive). Every 0-node will be called *permissive.* $\boldsymbol{R}^n$ will denote n-dimensional Euclidean space with the usual Euclidean norm.

2 Permissively Structured 1-Networks

We start with a 0-network $\mathbf{N}^0$. The components of $\mathbf{N}^0$ will be called 0-*sections.* We assume the following two conditions.

Condition 2.1. $\mathbf{N}^0$ *is locally finite (i.e., every 0-node has only finitely many incident branches).*

Condition 2.2. $\mathbf{N}^0$ *has at least one 0-section having infinitely many 0-nodes.*

We will be dealing in general with a nonlinear monotone network but will assign a single positive linear resistance value to each branch. We will get from the nonlinear network to the network with assigned linear branch resistances by adopting the construction given in [1].

First of all, we assume that there is no electrical coupling between branches; this will be understood henceforth. Next, having assigned an orientation to each branch b with respect to which we measure the branch voltages v_b and the branch currents i_b, we take it that for each b we have a maximal monotone function $M_b\colon \boldsymbol{R} \rightsquigarrow 2^{\boldsymbol{R}}$ that maps each $i_b \in \boldsymbol{R}$ into a subset of $\boldsymbol{R}$ consisting of corresponding voltages v_b. We recall that a function $f\colon \boldsymbol{R} \rightsquigarrow 2^{\boldsymbol{R}}$ is monotone when

$$(f(x)^* - f(y)^*)(x-y) \ \geq 0$$

for every number $f(x)^*$ in $f(x)$ and for every number $f(y)^*$ in $f(y)$. We may write $(x,y) \in f$ to mean $y \in f(x)$. We define $f^{-1}\colon \boldsymbol{R} \rightsquigarrow 2^{\boldsymbol{R}}$ by the statement: For $(x,y) \in \boldsymbol{R}^2$, $(y,x) \in f^{-1}$ if and only if $(x,y) \in f$. We say that $f\colon \boldsymbol{R} \rightsquigarrow 2^{\boldsymbol{R}}$ is maximal monotone if the two conditions $(x_1,y_1) \in \boldsymbol{R}^2$ and $(y-y_1)(x-x_1) \ \geq \ 0$ for all $(x,y) \in f$ imply that $(x_1,y_1) \in f$.

We will assume that the following three conditions hold.

Conditions A.

(1) For each $b \in \mathcal{B}$, there is at least one y with $(0,y) \in M_b$ (i.e., $0 \ \in \mathrm{Domain}(M_b)$), in which case we set

$$\delta_b''(0,0,M_b) \ = \ \min\{|y|\colon (0,y) \in M_b\}.$$

(2) For each $b \in \mathcal{B}$, there is at least one x with $(x, 0) \in M_b$ (i.e., $0 \in \text{Range}(M_b)$), in which case we set

$$\delta'_b(0, 0, M_b) = \min\{|x| : (x, 0) \in M_b\}.$$

(3)

$$\boldsymbol{I} \equiv \sum_{b \in \mathcal{B}} \delta'_b(0, 0, M_b) < \infty \tag{1}$$

and

$$\boldsymbol{V} \equiv \sum_{b \in \mathcal{B}} \delta''_b(0, 0, M_b) < \infty. \tag{2}$$

Next, to each branch b we assign $R_b \in [0, \infty]$ and $G_b \in [0, \infty]$ as follows:

(i) R_b is the Lipschitz constant of M_b on $[-\boldsymbol{I}, \boldsymbol{I}]$ if M_b is a Lipschitz continuous function on $[-\boldsymbol{I}, \boldsymbol{I}]$; otherwise, $R_b = 1$.

(ii) G_b is the Lipschitz constant of M_b^{-1} on $[-\boldsymbol{V}, \boldsymbol{V}]$ if M_b^{-1} is a Lipschitz continuous function on $[-\boldsymbol{V}, \boldsymbol{V}]$; otherwise, $G_b = 1$.

Finally, we set $r_b = R_b$ if $R_b \in (0, \infty)$ and $r_b > 0$ if $R_b = 0$, but in the latter case we choose the r_b values such that $\sum_{b:r_b=0} r_b < \infty$. This is how we assign a positive linear resistance value r_b to each branch of our nonlinear network satisfying Conditions A. (In this paper we will not make any use of G_b, but G_b will be employed in subsequent works.)

Note that, had we started with a linear network $\mathbf{N}_L$ in which every branch b consists of a positive resistance along with possibly a parallel current source or a series voltage source, then r_b would simply be that positive resistance. If, on the other hand, a branch b in $\mathbf{N}_L$ was a pure voltage source, we would set $r_b = 1$, and, if that branch was a pure current source, we would choose $r_b > 0$ arbitrarily except that the r_b's for all the pure current sources would have a finite sum.

As the next step, we define the metric d^0 (this was denoted by d_s in [1, page 5]), which we may now refer to as a 0-*metric* to distinguish it from metrics of higher ranks that we shall consider subsequently. Let $\mathcal{N}_{\mathbf{S}^0}$ be the set of 0-nodes in a given 0-section $\mathbf{S}^0$. Let $\mathcal{P}(\mathbf{S}^0, m, n)$ be the set of all 0-paths in $\mathbf{S}^0$ that terminate at the nodes m and n in $\mathcal{N}_{\mathbf{S}^0}$. Set

$$d^0(m, n) = \inf\{|P| : P \in \mathcal{P}(\mathbf{S}^0, m, n)\}.$$

It is easy to see that d^0 is a metric on d^0. (For example, $d^0(m,n) = 0$ since trivial paths are allowed.)

Now, take the completion $\widehat{\mathcal{N}}_{\mathbf{S}^0}$ of $\mathcal{N}_{\mathbf{S}^0}$ under d^0 and call the resulting limit points *0-terminals.* (Each 0-node of $\mathbf{S}^0$ is an isolated point under d^0.) $\mathcal{B}^0(T^0,\epsilon)$ will denote the open ball in $\widehat{\mathcal{N}}_{\mathbf{S}^0}$ with center at the 0-terminal T^0 and radius ϵ:

$$\mathcal{B}^0(T^0,\epsilon) \;=\; \{x : x \in \widehat{\mathcal{N}}_{\mathbf{S}^0}, d^0(x,T^0) < \epsilon\}$$

We will say that a one-ended 0-path P^0 (whether permissive or not) *converges to a 0-terminal* T^0 if, for each $\epsilon > 0$, the node sequence of P^0 is eventually in the ball $\mathcal{B}^0(T^0,\epsilon)$.

Let us now assume

Conditions 2.3. *Every 0-section has only finitely many 0-terminals. Moreover, infinitely many 0-sections each have at least one 0-terminal.*

We have available the following results.

Proposition 2.4. [1, Theorem 2.3] *For any 0-terminal T^0 of $\mathbf{S}^0$ and any one-ended 0-path P^0 that starts at the 0-node n^0 of $\mathbf{S}^0$ and converges to T^0, we have $d^0(n^0,T^0) \le |P^0|$, where $|P^0|$ denotes the resistive length of P^0, that is, the sum of the resistances for all the branches in P^0. Moreover, for any 0-node n^0 of $\mathbf{S}^0$, any 0-terminal T^0 of $\mathbf{S}^0$, and any $\epsilon > 0$, there exists a one-ended 0-path P^0 within $\mathbf{S}^0$ that starts at n^0 and converges to T^0 and whose resistive length satisfies $|P^0| < d^0(n^0,T^0) + \epsilon$.*

Let $\mathcal{O}^0_{\mathbf{S}^0}$ be the set of all permissive one-ended paths in $\mathbf{S}^0$.

Proposition 2.5. [1, Corollary 2.4] *Each permissive 0-path in a given 0-section $\mathbf{S}^0$ converges to one and only one 0-terminal of $\mathbf{S}^0$. Thus, the 0-terminals determine a partition of $\mathcal{O}^0_{\mathbf{S}^0}$.*

Proposition 2.6. [1, Corollary 2.5] *$\mathcal{B}^0(T^0,\epsilon) \cap \mathcal{N}_{\mathbf{S}^0}$ is 0-connected whatever be the choices of T^0 and ϵ.*

Proposition 2.7. [1, Corollary 2.6] *For any two 0-terminals T^0_1 and T^0_2 of a given 0-section $\mathbf{S}^0$, there exists a permissive endless 0-path P^0 within $\mathbf{S}^0$, one of whose 0-tips is in T^0_1 and the other in T^0_2 (i.e., along each of its infinite extensions P^0 converges to each of those two terminals) and whose resistive length satisfies $|P^0| < d^0(T^0_1,T^0_2) + \epsilon$.*

We now create an entity, which we shall call a *permissive 1-node*, by partitioning the set of all the 0-terminals for all the 0-sections of $\mathbf{N}^0$ in any arbitrary way and then by taking each set of the partition to be a permissive 1-node. But, we shall do so in such a fashion that the following holds.

Condition 2.8(a). *Every permissive 1-node consists of only finitely many 0-terminals and does not embrace any 0-node.*

We can relate these permissive 1-nodes to the 1-nodes in [2] and [3] as follows. Each 0-terminal corresponds to the set of all permissive one-ended 0-paths that converge to that 0-terminal. Therefore, by Condition 2.8(a) each permissive 1-node corresponds to the union of only finitely many such sets of permissive one-ended paths. Furthermore, all the representatives of a permissive 0-tip are perforce in the set of permissive 0-paths for some 0-terminal. Consequently, each permissive 1-node can be identified with some set of permissive 0-tips and is therefore a 1-node by the definition used in [2] and [3].

Finally, each nonpermissive 0-tip (i.e., all its representatives are nonpermissive) will be assigned to a singleton 1-node, which we shall refer to as a *nonpermissive 1-node.* Thus, every nonpermissive 0-tip is open. In other words, we have

Condition 2.8(b). *Every nonpermissive 1-node is a singleton consisting of exactly one nonpermissive 0-tip.*

Thus, all the 0-tips have now been assigned to 1-nodes, and we have a special case of the 1-networks defined in [2] or [3]. We shall call any 1-network constructed in this way a *permissively structured 1-network.* To simplify terminology, we shall say that each 0-tip is *in* some 0-terminal, which in turn is *in* some 1-node.

Here are some direct consequences of this construction.

Note 2a. Every node is maximal (by Condition 2.8(b) and the definition of a nonpermissive 1-node).

Note 2b. 0-sections and 0-subsections coincide (by (i)). Thus, every boundary node of a 0-section is a bordering node of that 0-subsection. Conversely, if a bordering node is incident to two or more 0-subsections, it is a boundary node of that 0-section.

Note 2c. Every 0-section has only finitely many incident permissive 1-nodes (by Condition 2.3). (A 0-section may have infinitely many incident nonpermissive 1-nodes.)

Note 2d. Every permissive 1-node is incident to only finitely many 0-sections (by Condition 2.8(a)). Every nonpermissive 1-node is incident to exactly one 0-section.

Note 2e. Every 0-path is restricted to a 0-subsection, that is, all its nodes are interior nodes of the 0-subsection (by Condition 2.8(b) and the same fact about nonpermissive 1-nodes). Thus, every representative of a 0-tip (whether permissive or not) reaches exactly one 1-node.

Note 2f. Every permissive 1-node is 1-adjacent to only finitely many permissive 1-nodes (by Conditions 2.3 and 2.8(a)).

Proposition 2.9. *If two 0-tips are permissive and nondisconnectable, they are in the same 0-terminal.*

Proof. Let P_1^0 (resp. P_2^0) be a representative of the permissive 0-tip t_1^0 (resp. t_2^0). Then, P_1^0 and P_2^0 meet infinitely often (i.e., they share infinitely many 0-nodes). Let T^0 be the 0-terminal to which the nodes common to P_1^0 and P_2^0 converge. Then, all the nodes of the permissive P_1^0 must converge to the same limit T^0, and similarly for P_2^0 (see Proposition 2.5). This is equivalent to saying that t_1^0 and t_2^0 are in T^0. ♣

3 The 1-Metric

As the next step in our recursive construction, we set up a "1-metric" for the nodes of a 1-section in a permissively structured 1-network $\mathbf{N}^1$. A 1-section of $\mathbf{N}^1$ is a component of $\mathbf{N}^1$. Let $\mathcal{N}_{\mathbf{S}^1}$ be the set of all 0-nodes and permissive 1-nodes in a 1-section $\mathbf{S}^1$ of $\mathbf{N}^1$. For the sake of a convenient terminology, we shall call every 0-node *permissive*. Thus, $\mathcal{N}_{\mathbf{S}^1}$ is the set of all permissive nodes in $\mathbf{S}^1$. (Since we have not yet defined the 2-nodes, $\mathbf{S}^1$ has no bordering nodes at this point.) Let m and n be distinct nodes in $\mathcal{N}_{\mathbf{S}^1}$. Let $\mathcal{P}(\mathbf{S}^1, m, n)$ be the set of all permissive paths (of ranks 0 or 1) in $\mathbf{S}^1$ that terminate at m or n. For any $P \in \mathcal{P}(\mathbf{S}^1, m, n)$, let $|P|$ denote the *resistive length* of P, that is, the sum of the resistance values r_b for all the branches embraced by P. Thus,

$$|P| = \sum_{b \dashv P} r_b,$$

where $b \dashv P$ indicates that branch b is embraced by the path P. Set

$$d^1(m,n) = \inf\{|P|: P \in \mathcal{P}(\mathbf{S}^1, m, n) \tag{3}$$

Proposition 3.1. *If m and n are distinct permissive nodes in the same 1-section of $\mathbf{N}^1$, there exists a permissive two-ended path $P_{m,n}$ terminating at m and n (thus, $|P_{m,n}| < \infty$). Furthermore, if at least one of m and n is of rank 1, then any path terminating at them will be of rank 1.*

Proof. If m and n are incident to the same 0-section, the conclusion follows from [1, Theorem 2.3 and Corollary 2.6]. So, assume m and n are not incident to the same 0-section. Since they are in the same 1-section, there is a 1-path

$$P^1_{m,n} = \{m, P^0_0, n^1_1, P^0_1, \ldots, n^1_K, P^0_K, n\} \tag{4}$$

where K is a finite positive integer. Since there are no embraced 0-nodes, each 0-path is incident to the 1-nodes next to it through a 0-tip. Moreover, each n^k_1 is a nonsingleton 1-node and therefore is permissive. (Remember that each nonpermissive 0-tip was left open.) On the other hand, m and n are permissive by hypothesis. This means that all the 0-tips in m, n, and all the n^1_k are permissive. Consequently, each P^0_k is permissive. Since there are only finitely many P^0_k, $P^1_{m,n}$ is permissive too. ♣

We have shown that $d^1(m,n) < \infty$ whenever m and n are permissive nodes in the same 1-section.

Lemma 3.2. *When m and n are distinct permissive nodes in the same 1-section, $d^1(m,n) > 0$.*

Proof. If either m or n is a 0-node and $P_{m,n}$ is a permissive path connecting m and n, $|P_{m,n}|$ is larger than the smallest r_b for the finitely many branches incident to m and n. So, $d^1(m,n)$ is clearly positive in this case.

Now, let both m and n be permissive 1-nodes. Every 1-path connecting m and n must pass completely through a 0-section incident to m and therefore must have a resistive length no less than $\min_q\{d^1(m,q)\}$, the minimum being taken over all the finitely many 1-nodes q that are 1-adjacent to m. Again, we can conclude that $d^1(m,n) > 0$. ♣

For any $m \in \mathcal{N}_{\mathbf{S}^1}$, we set $d^1(m,m) = 0$. Also, if $m, n \in \mathcal{N}_{\mathbf{S}^1}$ are distinct, we clearly have from (3) that $d^1(m,n) = d^1(n,m)$.

Now, for the triangle inequality for d^1.

Lemma 3.3. *For $m, n, q \in \mathcal{N}_{\mathbf{S}^1}$,*

$$d^1(m,n) \;\le\; d^1(m,q) + d^1(q,n). \tag{5}$$

Proof. We have something to prove only when m, n, and q are distinct. By Proposition 3.1, there exists a permissive path $P_{m,q}$ terminating at m and q and a permissive path $P_{q,n}$ terminating at q and n. By Proposition 2.9 and the fact that each nonpermissive tip is open, we have from [3, Corollary 3.5-4] that there is in $P_{m,q} \cup P_{q,n}$ a path $P_{m,n}$ (perforce permissive) terminating at m and n. Thus,

$$|P_{m,n}| \;\le\; |P_{m,q}| + |P_{q,n}|. \tag{6}$$

Moreover, by the definition (3), for each $\epsilon > 0$ we can choose $P_{m,q}$ and $P_{q,n}$ such that the right-hand side of (6) is no larger than $d^1(m,q) + d^1(q,n) + \epsilon$. Since this is so for all $\epsilon > 0$, the definition (3) now implies that (5) holds. ♣

Altogether then, we have established

Proposition 3.4. *d^1 is a metric on $\mathcal{N}_{\mathbf{S}^1}$.*

Remark 3.5. Although the 0-nodes of $\mathcal{N}_{\mathbf{S}^0}$ for any given 0-section $\mathbf{S}^0$ are all isolated under d^0 and d^1, the 1-nodes of $\mathcal{N}_{\mathbf{S}^1}$ are not isolated from the 0-nodes of $\mathcal{N}_{\mathbf{S}^1}$ under d^1. ♣

Remark 3.6. For $m, n \in \mathcal{N}_{\mathbf{S}^0}$ where $\mathbf{S}^0$ is a 0-section within the 1-section $\mathbf{S}^1$, $d^1(m,n) \le d^0(m,n)$ because there are in general 1-paths in $\mathbf{S}^1$ connecting m and n in addition to the 0-paths in $\mathbf{S}^0$ connecting m and n. We will discuss this fact in more detail in the proof of Lemma 3.7. ♣

We could now complete $\mathcal{N}_{\mathbf{S}^1}$ under d^1, establish what the limit points are, and then construct a permissive 2-network. ($\mathbf{S}^1$ need not have any limit points under d^1 other than those obtained under d^0, but, if it does have such points, it can then be shown to have permissive 1-tips, and our recursive process can continue.) Now, however, we will instead continue our recursive construction of "permissively structured " transfinite networks by passing directly to the more general case of a μ-network where μ is a positive natural number. But, before doing this, there is one more preliminary result to consider.

Given the permissively structured 1-network $\mathbf{N}^1$, consider the set $\mathcal{N}_{\mathbf{S}^0}$ of all 0-nodes in a chosen 0-section $\mathbf{S}^0$ of $\mathbf{N}^1$. We may complete $\mathcal{N}_{\mathbf{S}^0}$ under d^0; in addition to the said 0-nodes, which are all isolated points under d^0 (and under d^1 as well), we now obtain the aforementioned 0-terminals as limit points. On the other hand, we may complete $\mathcal{N}_{\mathbf{S}^0}$ under the weaker metric d^1. Conceivably, distinct limit point under d^0 may now merge into a single limit point under d^1. We will now show that this does not happen.

Lemma 3.7. *Given any 0-section $\mathbf{S}^0$ of a permissively structured 1-network $\mathbf{N}^1$, a sequence of 0-nodes in $\mathcal{N}_{\mathbf{S}^0}$ is a Cauchy sequence under d^0 if and only if it is a Cauchy sequence under d^1.*

Proof. *Only if:* Let $m^0, n^0 \in \mathcal{N}_{\mathbf{S}^0}$. Let $\mathcal{P}^0$ be the set of 0-paths terminating at m^0 and n^0. Perforce, all the nodes of such a 0-path are in $\mathcal{N}_{\mathbf{S}^0}$. Let $\mathcal{P}^{01}$ be the set of 1-paths terminating at m^0 and n^0 and meeting only one 1-node. Perforce, the 1-node for such a 1-path is a bordering node of $\mathbf{S}^0$, and all the other nodes of that 1-path are in $\mathcal{N}_{\mathbf{S}^0}$. Let $\mathcal{P}^1$ be the set of 1-paths terminating at m^0 and n^0 and meeting two or more 1-nodes incident to $\mathbf{S}^0$. Then, by virtue of [1, Theorem 2.3 and Corollary 2.6] and the triangle inequality,

$$d^0(m^0, n^0) \;=\; \inf\{|P| : P \in \mathcal{P}^0 \cup \mathcal{P}^{01}\}$$

and

$$\begin{aligned} d^1(m^0, n^0) \;&=\; \inf\{|P| : P \in \mathcal{P}^0 \cup \mathcal{P}^{01} \cup \mathcal{P}^1\} \\ &=\; \min(d^0(m^0, n^0), \inf\{|P| : P \in \mathcal{P}^1\}). \end{aligned} \tag{7}$$

Hence,

$$0 \;<\; d^1(m^0, n^0) \;\le d^0(m^0, n^0).$$

This establishes the only if part of Lemma 7.1.

If: Set $D = \min\{d^1(l^1, q^1)\}$, where the minimum is taken over all pairs of 1-nodes l^1 and q^1 incident to $\mathbf{S}^0$. There are only finitely many such pairs. Therefore, $D > 0$. It follows that $\inf\{|P| : P \in \mathcal{P}^1\} \geq D$. The "if" part of Lemma 3.7 now follows from (7) because D is independent of the choices of m^0 and n^0 in $\mathbf{S}^0$. ♣

An immediate consequence of Lemma 3.7 is

Proposition 3.8. *The limit points obtained when $\mathcal{N}_{\mathbf{S}^0}$ is completed under d^0 coincide with the limit points obtained when $\mathcal{N}_{\mathbf{S}^0}$ is completed under d^1.*

4 The Recursive Assumptions

In order to treat the more general case of recursively proceeding from a permissively structured μ-network $\mathbf{N}^\mu$ to such a $(\mu+1)$-network $\mathbf{N}^{\mu+1}$, where μ is any positive natural number, we first have to state the conditions and properties that permissively structured networks of lower ranks are assumed to have. All of them have already been assumed or proven for the case where $\mu = 1$. In the next section the conditions stated in this section will be assumed for the rank $\mu + 1$, but the properties stated in this section will be proven. Let it be understood from now on that all of the conditions and properties displayed in this section hold for all ranks $\rho = 1, \ldots, \mu$; we will at times refrain from stating this restriction.

We start by assuming that, for each $\rho = 1, \ldots, \mu$, the permissively structured ρ-networks $\mathbf{N}^\rho$ have already been constructed; this includes the specifications of the permissive and nonpermissive ρ-nodes. The assumption that a ρ-network exists implies that there is at least one ρ-node, therefore at least one $(\rho-1)$-tip, therefore again one-ended $(\rho-1)$-paths — each having of course infinitely many $(\rho-1)$-nodes. Therefore, we are led to

Condition 4.1. *For each $\rho = 1, \ldots, \mu$, there is at least one $(\rho-1)$-section having infinitely many $(\rho-1)$-nodes.*

To simplify matters, we also assume

Condition 4.2. *No node embraces any node of lower rank.* (Thus, every ρ-node is maximal and simply a set of $(\rho-1)$-tips.)

We also impose

Property 4.3. *If m and n are distinct permissive nodes of ranks no larger than $\rho-1$ in the same $(\rho-1)$-section $\mathbf{S}^{\rho-1}$, then there exists a permissive η-path ($\eta \leq \rho-1$) terminating at m and n and lying within $\mathbf{S}^{\rho-1}$.*

Next, we take it that for each $\rho - 1$ there is a metric $d^{\rho-1}$ defined on the set $\mathcal{N}_{\mathbf{S}^{\rho-1}}$ of all permissive nodes of all ranks from 0 to $\rho - 1$ in each $(\rho - 1)$-section $\mathbf{S}^{\rho-1}$ of $\mathbf{N}^{\mu}$. To be specific, for $m, n \in \mathcal{N}_{\mathbf{S}^{\rho-1}}$, let $\mathcal{P}(\mathbf{S}^{\rho-1}, m, n)$ be the set of all two-ended permissive paths of ranks no larger than $\rho - 1$ that terminate at m and n. Perforce, each such path lies within $\mathbf{S}^{\rho-1}$. Indeed, by virtue of Condition 4.2, no path of $\mathcal{P}(\mathbf{S}^{\rho-1}, m, n)$ meets a bordering node of $\mathbf{S}^{\rho-1}$. We define a mapping $d^{\rho-1}\colon \mathcal{N}_{\mathbf{S}^{\rho-1}} \times \mathcal{N}_{\mathbf{S}^{\rho-1}} \rightsquigarrow \boldsymbol{R}^1$ by

$$d^{\rho-1}(m,n) \;=\; \inf\{|P|\colon P \in \mathcal{P}(\mathbf{S}^{\rho-1}, m, n)\},$$

where as before $|P|$ denotes the resistive length of P, that is, the sum of resistances for all the branches embraced by P.

Property 4.4. *$d^{\rho-1}$ is a metric on $\mathcal{N}_{\mathbf{S}^{\rho-1}}$.*

Furthermore, let us assume that the following holds.

Property 4.5. *For each $\rho = 1, \ldots, \mu$, a sequence of permissive nodes of ranks $\rho - 2$ or less in a $(\rho-2)$-section $\mathbf{S}^{\rho-2}$ is a Cauchy sequence under $d^{\rho-2}$ if and only if it is a Cauchy sequence under $d^{\rho-1}$.* (This property is trivially satisfied when $\rho = 1$.)

By virtue of this property, the completion $\widehat{\mathcal{N}}_{\mathbf{S}^{\rho-1}}$ of $\mathcal{N}_{\mathbf{S}^{\rho-1}}$ under $d^{\rho-1}$ yields two disjoint sets of limit points, the first being the limit points of all the $\mathcal{N}_{\mathbf{S}^{\rho-2}}$ for all $(\rho-2)$-sections $\mathbf{S}^{\rho-2}$ in $\mathbf{S}^{\rho-1}$ and the second being new limit points different from (i.e., additional to) those former limit points. We call the latter limit points $(\rho - 1)$-*terminals.*

Condition 4.6. *For each $\rho = 1, \ldots, \mu$, each $(\rho-1)$-section $\mathbf{S}^{\rho-1}$ has only finitely many $(\rho-1)$-terminals.*

We shall say that a one-ended $(\rho-1)$-path $P^{\rho-1}$ in $\mathbf{S}^{\rho-1}$ *converges* to a $(\rho-1)$-terminal $T^{\rho-1}$ of $\mathbf{S}^{\rho-1}$ if the nodes embraced by $P^{\rho-1}$ converge to $T^{\rho-1}$ of under $d^{\rho-1}$. To be more specific, let us observe that the nodes of all ranks embraced by $P^{\rho-1}$ form a totally ordered set, that ordering being determined by a tracing of $P^{\rho-1}$ starting at its initial node and progressing toward its $(\rho-1)$-tip. If m and n are distinct nodes embraced by $P^{\rho-1}$, we say that n is *beyond* m or m is *before* n and we write $n \succ m$ or $m \prec n$ if that tracing meets m before n. We say that $P^{\rho-1}$ *converges* to $T^{\rho-1}$ if, given any $\epsilon > 0$, there exists a node m embraced by $P^{\rho-1}$ such that for all nodes n embraced by $P^{\rho-1}$ beyond m we have $d^{\rho-1}(n, T^{\rho-1}) < \epsilon$.

Let $\mathcal{O}^{\rho-1}_{\mathbf{S}^{\rho-1}}$ be the set of all permissive one-ended $(\rho-1)$-paths in the $(\rho-1)$-section $\mathbf{S}^{\rho-1}$.

Property 4.7. *Every member of $\mathcal{O}^{\rho-1}_{\mathbf{S}^{\rho-1}}$ converges to a unique $(\rho-1)$-terminal of $\mathbf{S}^{\rho-1}$.*

Thus, the $(\rho-1)$-terminals of $\mathbf{S}^{\rho-1}$ partition $\mathcal{O}^{\rho-1}_{\mathbf{S}^{\rho-1}}$ in accordance with which $(\rho-1)$-terminals the members of $\mathcal{O}^{\rho-1}_{\mathbf{S}^{\rho-1}}$ converge. Furthermore, all the representatives of a permissive $(\rho-1)$-tip clearly converge to the same $(\rho-1)$-terminal. In fact, upon identifying $T^{\rho-1}$ with its corresponding subset in $\mathcal{O}^{\rho-1}_{\mathbf{S}^{\rho-1}}$, we can say that each $(\rho-1)$-terminal $T^{\rho-1}$ is partitioned by the permissive $(\rho-1)$-tips whose representatives converge to $T^{\rho-1}$. Thus, we can identify each $(\rho-1)$-terminal as a union of permissive $(\rho-1)$-tips, and every permissive $(\rho-1)$-tip is a subset of some $(\rho-1)$-terminal. We will say that such a $(\rho-1)$-tip is *in* that $(\rho-1)$-terminal.

Property 4.8. *If two $(\rho-1)$-tips are permissive and nondisconnectable, they are subsets of the same $(\rho-1)$-terminal.*

Property 4.9. *For any $(\rho-1)$-terminal $T^{\rho-1}$ of the $(\rho-1)$-section $\mathbf{S}^{\rho-1}$ and for any one-ended $(\rho-1)$-path $P^{\rho-1}$ that starts at an interior node n of $\mathbf{S}^{\rho-1}$ and converges to $T^{\rho-1}$ within $\mathbf{S}^{\rho-1}$, we have that $d^{\rho-1}(n, T^{\rho-1}) \leq |P^{\rho-1}|$. Moreover, for any interior node n of $\mathbf{S}^{\rho-1}$, any $(\rho-1)$-terminal of $\mathbf{S}^{\rho-1}$ of $T^{\rho-1}$, and any $\epsilon > 0$, there exists a permissive one-ended $(\rho-1)$-path $P^{\rho-1}$ within $\mathbf{S}^{\rho-1}$ that starts at n and converges to $T^{\rho-1}$ and whose resistive length satisfies $|P^{\rho-1}| < d^{\rho-1}(n^0, T^{\rho-1}) + \epsilon$.*

$\mathcal{B}^{\rho-1}(T^{\rho-1}, \epsilon)$ denotes the open ball in $\widehat{\mathcal{N}}_{\mathbf{S}^{\rho-1}}$ centered at $T^{\rho-1}$ and of radius ϵ:

$$\mathcal{B}^{\rho-1}(T^{\rho-1}, \epsilon) \;=\; \{x : x \in \widehat{\mathcal{N}}_{\mathbf{S}^{\rho-1}}, d^{\rho-1}(x, T^{\rho-1} < \epsilon\}$$

Property 4.10. *$\mathcal{B}^{\rho-1}(T^{\rho-1}, \epsilon) \cap \mathcal{N}_{\mathbf{S}^{\rho-1}}$ is $(\rho-1)$-connected whatever be the choices of $T^{\rho-1}$ and ϵ.*

Property 4.11. *For any two $(\rho-1)$-terminals $T_1^{\rho-1}$ and $T_2^{\rho-1}$ of a given $(\rho-1)$-section $\mathbf{S}^{\rho-1}$, there exists a permissive endless $(\rho-1)$-path $P^{\rho-1}$ within $\mathbf{S}^{\rho-1}$, one of whose $(\rho-1)$-tips is in $T_1^{\rho-1}$ and the other in $T_2^{\rho-1}$; moreover the resistive length of $P^{\rho-1}$ satisfies $|P^{\rho-1}| < d^{\rho-1}(T_1^{\rho-1}, T_2^{\rho-1}) + \epsilon$.*

With regard to the ρ-nodes we assume

Condition 4.12. *Every $(\rho-1)$-tip is assigned to some ρ-node as follows: Every ρ-node is either a singleton with a nonpermissive $(\rho-1)$-tip or consists of all the $(\rho-1)$-tips in only finitely many $(\rho-1)$-terminals (chosen from among all the $(\rho-1)$-terminals of all the $(\rho-1)$-sections in $\mathbf{N}^\mu$).*

The former ρ-nodes will be called *nonpermissive*, and the latter ρ-nodes will be called *permissive*. In order to save words, we shall simply say that each permissive ρ-node *consists of* only finitely many $(\rho-1)$-terminals and that the permissive ρ-nodes *partition* the set of $(\rho-1)$-terminals. We do allow a permissive ρ-node to be a singleton (i.e., to have exactly one $(\rho-1)$-terminal having exactly one $(\rho-1)$-tip). By Condition 4.6, every $(\rho-1)$-section has only finitely many incident ρ-nodes.

Here are some direct consequences of the above conditions. As always, $\rho = 1, \ldots, \mu$ except in Note 4i.

Note 4a. Every node is maximal — by Condition 4.2.

Note 4b. $(\rho-1)$-sections and $(\rho-1)$-subsections coincide — by Condition 4.2 again. Thus, every boundary node of a $(\rho-1)$-section is a bordering node of that $(\rho-1)$-subsection. Conversely, if a bordering node of a $(\rho-1)$-subsection is incident to two or more $(\rho-1)$-subsections, then it is a boundary node of that $(\rho-1)$-section.

Note 4c. Every bordering node of a $(\rho-1)$-section is of rank ρ — by Condition 4.2.

Note 4d. Every $(\rho-1)$-path (whether permissive or not) is restricted to a single $(\rho-1)$-section $\mathbf{S}^{\rho-1}$ and will not meet any bordering node of $\mathbf{S}^{\rho-1}$ — by Condition 4.2. (However, it will reach one bordering ρ-node or two bordering nodes if it is one-ended or endless respectively. Thus, Condition 4.2-2(c) of [3, page 87] is fulfilled.)

Note 4e. Every permissive ρ-node is incident to only finitely many $(\rho-1)$-sections — by Condition 4.12 and the fact that each $(\rho-1)$-terminal belongs to exactly one $(\rho-1)$-section. Every nonpermissive ρ-node is incident to exactly one $(\rho-1)$-section.

Note 4f. Every $(\rho-1)$-section has only finitely many permissive bordering ρ-nodes, and they are all of rank ρ — by Conditions 4.2, 4.6, and 4.12. On the other hand, a

$(\rho - 1)$-section may have infinitely many nonpermissive bordering ρ-nodes.

Note 4g. Every ρ-node is ρ-adjacent to only finitely many permissive ρ-nodes and is not ρ-adjacent to any θ-node $(\theta > \rho)$ — by Notes 4e and 4f and Condition 4.2.

Note 4h. All nodes of any path — except perhaps for the first and last nodes — are permissive because a nonpermissive node is a singleton (Condition 4.12).

Note 4i. For each $\rho = 2, \ldots, \mu$, there are infinitely many $(\rho - 2)$-sections having infinitely many $(\rho - 2)$-nodes. This follows from Conditions 4.1, 4.6, and 4.12.

5 Permissively Structured $(\mu + 1)$-Networks

With the permissively structured μ-network $\mathbf{N}^\mu$ in hand, we now set about constructing a permissively structured $(\mu + 1)$-network $\mathbf{N}^{\mu+1}$. In doing so, we shall assume for $\mathbf{N}^\mu$ the conditions corresponding to the conditions displayed in the last section but will prove all the properties displayed therein; the latter will now become propositions. All the conditions displayed herein must hold if this recursive process is to continue. If any one of them fail, our recursive process ceases.

Corresponding to Condition 4.1, we assume the following.

Condition 5.1. *There is in* $\mathbf{N}^\mu$ *at least one* μ*-section having infinitely many* μ*-nodes.*

Let us restate Condition 4.2 as

Condition 5.2. *No node embraces any node of lower rank.*

On the other hand, corresponding to Property 4.3, we have

Proposition 5.3. *If* m *and* n *are distinct permissive nodes in the same* μ*-section* $\mathbf{S}^\mu$ *of* $\mathbf{N}^\mu$*, there exists a permissive path* P_{mn} *in* $\mathbf{S}^\mu$ *terminating at* m *and* n*. Furthermore, if at least one of* m *and* n *are of rank* μ*, any path terminating at them will be of rank* μ*.*

Note. The ranks of m and n can be no larger than μ because these nodes reside in the μ-network $\mathbf{N}^\mu$.

Proof. The proof is simply an adaptation of the proof of Proposition 3.1. Just replace 0 by $\mu - 1$, 1 by μ, and Theorem 2.3 and Corollary 2.6 of [1] by Properties 4.9 and 4.11. ♣

Let $\mathcal{N}_{\mathbf{S}^\mu}$ be the set of all permissive nodes of all ranks from 0 to μ in $\mathbf{S}^\mu$. With $m, n \in \mathcal{N}_{\mathbf{S}^\mu}$, let $\mathcal{P}(\mathbf{S}^\mu, m, n)$ be the set of all permissive paths (of any ranks) in $\mathbf{S}^\mu$ that terminate at m and n. (We allow a trivial path when $m = n$.) Define the mapping $d^\mu : \mathcal{N}_{\mathbf{S}^\mu} \times \mathcal{N}_{\mathbf{S}^\mu} \rightsquigarrow \boldsymbol{R}^1$ by

$$d^\mu(m,n) \ = \ \inf\{|P| \colon P \in \mathcal{P}(\mathbf{S}^\mu, m, n)\}$$

where $|P|$ denotes the resistive length of P as before. We now set about showing that d^μ is a metric on $\mathcal{N}_{\mathbf{S}^\mu}$.

By Proposition 5.3, $d^\mu(m,n) < \infty$ whenever m and n are in the same μ-section. Clearly, $d^\mu(m,n) \geq 0$ and $d^\mu(m,n) = d^\mu(n,m)$. Also, $d^\mu(m,m) = 0$ since trivial paths are allowed.

Lemma 5.4. *When m and n are distinct permissive nodes in the same μ-section, $d^\mu(m,n) > 0$.*

Proof. It is no restriction to assume that the rank ρ_m of m is no larger than the rank of n. Then, every permissive path that terminates at m and n must pass completely through a $(\rho_m - 1)$-section to which m is incident. (If $\rho_m = 0$, that ρ_m-section is simply a branch.) So, the resistive length of any such path can be no less than $\min_q\{d^\mu(m,q)\}$ where the minimum is taken over the permissive ρ_m-nodes that are ρ_m-adjacent to m. There are only finitely many such ρ_m-nodes according to Note 4g. ♣

Lemma 5.5. *For $m, n, q \in \mathcal{N}_{\mathbf{S}^\mu}$,*

$$d^\mu(m,n) \ \leq \ d^\mu(m,q) + d^\mu(q,n).$$

Proof. This proof is obtained from the proof of Lemma 3.3 by replacing 1 by μ, Lemma 3.1 by Lemma 5.3, and Lemma 2.9 by Property 4.8. ♣

Altogether we have extended Property 4.4 to

Proposition 5.6. *d^μ is a metric on $\mathcal{N}_{\mathbf{S}^\mu}$.*

Let us also extend Property 4.5.

Proposition 5.7. *A sequence of permissive nodes of ranks $\mu - 1$ or less in a $(\mu - 1)$-section $\mathbf{S}^{\mu-1}$ is a Cauchy sequence under $d^{\mu-1}$ if and only if it is a Cauchy sequence under d^μ.*

Proof. This proof is an adaptation of the proof of Lemma 3.7 by means of the following replacements: 0 is replaced by $\mu - 1$, 1 by μ, Theorem 2.3 and Corollary 2.6 of [1] by Properties 4.9 and 4.11, and Note 2c by Note 4f. ♣

An easily obtained consequence of Lemma 5.7 and Property 4.5 is

Proposition 5.8. *For each $\rho = 1, \ldots, \mu$, the limit points obtained when $\mathcal{N}_{\mathbf{S}^{\rho-1}}$ is completed under $d^{\rho-1}$ is the same as the limit points obtained when $\mathcal{N}_{\mathbf{S}^{\rho-1}}$ is completed under d^{μ}.*

We now take the completion $\widehat{\mathcal{N}}_{\mathbf{S}^\mu}$ of $\mathcal{N}_{\mathbf{S}^\mu}$ under d^μ. This yields two disjoint sets of limit points, the first set being the limit points of all the $\mathcal{N}_{\mathbf{S}^{\rho-1}}$ for all the $(\rho-1)$-sections in $\mathbf{N}^\mu$ $(\rho = 1, \ldots, \mu)$ and the second set being new limit points, those that do not belong to any $\widehat{\mathcal{N}}_{\mathbf{S}^{\rho-1}}$. We call the latter limit points *μ-terminals.*

To continue, we now assume

Condition 5.9. *There is at least one μ-section having one or more μ-terminals. Moreover, each μ-section has only finitely many μ-terminals.*

(If Condition 5.9 fails, our recursive procedure terminates at this rank μ.)

Next, using the terminology and notation specified after Condition 11.2, we say that a one-ended μ-path *converges* to a μ-terminal T^μ if, given any $\epsilon > 0$, there exists a node $m \dashv P^\mu$ such that, for all nodes $n \dashv P^\mu$ beyond m, we have $d^\mu(n, T^\mu) < \epsilon$.

As before, let $\mathcal{O}^\mu_{\mathbf{S}^\mu}$ denote the set of permissive one-ended μ-paths in the given μ-section $\mathbf{S}^\mu$.

Proposition 5.10. *Each member of $\mathcal{O}^\mu_{\mathbf{S}^\mu}$ converges to a unique μ-terminal of* $\mathbf{S}^\mu$.

Proof. Let $P^\mu \in \mathcal{O}^\mu_{\mathbf{S}^\mu}$. The embraced nodes of P^μ form a totally ordered set, the ordering being given by a tracing of P^μ starting at its first node and progressing toward its μ-tip. Because P^μ is permissive, $\{r_b\}_{b \dashv P^\mu}$ is a summable set, that is $\sum_{b \dashv P^\mu} < \infty$. This means that, given any $\epsilon > 0$, there is a node $m \dashv P^\mu$ such that, for all $n, q \dashv P^\mu$ with $m \prec n \prec q$, that part P_{nq} of P^μ between n and q satisfies $|P_{nq}| < \epsilon$. Thus, the nodes embraced by P^μ form a Cauchy net. Therefore, there is a unique μ-terminal T^μ in the completion $\widehat{\mathcal{N}}_{\mathbf{S}^\mu}$ of $\mathcal{N}_{\mathbf{S}^\mu}$ under d^μ to which P^μ converges. ♣

As was done in the preceding section, we can partition $\mathcal{O}^\mu_{\mathbf{S}^\mu}$ in accordance with the μ-

terminals to which the members of $\mathcal{O}^{\mu}_{\mathbf{S}^\mu}$ converge. This allows us to identify each μ-terminal with a set of permissive μ-tips of $\mathbf{S}^\mu$, and then each permissive μ-tip becomes a subset of a μ-terminal under that identification. To save words, we will speak of any μ-terminal as being a limit point of $\mathcal{N}_{\mathbf{S}^\mu}$ or alternatively as being a subset of $\mathcal{O}^{\mu}_{\mathbf{S}^\mu}$, as well as being a union of some permissive μ-tips.

Proposition 5.11. *If two tips of ranks no larger than μ are nondisconnectable, they must be of the same rank.*

Proof. Let t^ζ and t^η be two tips of ranks ζ and η respectively with $\zeta < \eta$. Because no node embraces a node of lower rank, all representatives of t^ζ must be restricted to some ζ-section. On the other hand, every representative of t^η cannot be restricted to any single ζ-section; that is, it must leave every ζ-section it enters. Thus, t^ζ and t^η cannot be nondisconnectable. (If however $\zeta = \eta$, it is possible for t^ζ and t^η to be nondisconnectable.) ♣

Proposition 5.12. *If two μ-tips are permissive and nondisconnectable, they are subsets of the same μ-terminal.*

Proof. The proof of this is just like that of Proposition 2.9 but with 0 replaced by μ. (Now, we deal with the embraced nodes of permissive paths.) ♣

Proposition 5.13. *If T^μ is the μ-terminal to which a given $P^\mu \in \mathcal{O}^{\mu}_{\mathbf{S}^\mu}$ converges, then, for any node $m \dashv P^\mu$, $d^\mu(m, T^\mu) \leq |P^\mu|$. On the other hand, for any permissive node $m \dashv \mathbf{S}^\mu$, any μ-terminal T^μ of $\mathbf{S}^\mu$, and any $\epsilon > 0$, there is a $P^\mu \in \mathcal{O}^{\mu}_{\mathbf{S}^\mu}$ such that m is its initial node, P^μ converges to T^μ, and*

$$|P^\mu| \; < \; d^\mu(m, T^\mu) + \epsilon. \tag{8}$$

Proof. The first statement is obvious. As for the second statement, we start by taking a sequence $\{n^\mu_k\}_{k=1}^{\infty}$ of permissive μ-nodes with $d^\mu(n^\mu_k, T^\mu) < \epsilon/2^k 3$. We can choose a μ-path P_0 terminating at m and n^μ_1 with $|P_0| < d^\mu(m, T^\mu) + \epsilon/2$. Also, for each $k = 1, 2, \ldots$, we can choose a μ-path P_k terminating at n^μ_k and n^μ_{k+1} with $|P_k| < \epsilon/2^{k+1}$; this can be done because

$$d^\mu(n^\mu_k, n^\mu_{k+1}) \; \leq \; d^\mu(n^\mu_k, T^\mu) \, + \, d^\mu(n^\mu_{k+1}, T^\mu) \; < \; \epsilon/2^{k+1}.$$

Now, $Q = \cup_{k=0}^{\infty} P_k$ is a μ-graph that satisfies Condition 4.2-2 of [3, page 87]. Indeed, part (a) of those conditions is clear, part b is asserted by Property 4.8, part (c) is asserted by Note 4d, and part (d) is asserted by Note 4g. (Regarding Condition 4.2-2(b), we need that condition only for permissive tips of ranks $\mu - 1$ or less. Since the nonpermissive tips are all open, the arguments of [3, Section 4.2] carry over to this present case. Similarly, Condition 4.2-2(d) is needed only for the permissive nodes. In general, we may restrict the arguments to permissive tips, nodes, and paths and still arrive at Theorem 4.2-4 — but this time with permissive one-ended paths.) So, by [3, Theorem 4.2-4], there is in Q a one-ended μ-path P^μ starting at m. Furthermore,

$$|P^\mu| \;\le\; \sum_{b \dashv Q} r_b \;\le\; \sum_{k=0}^{\infty} |P_k|$$

$$<\; d^\mu(m, T^\mu) + 2^{-1}\epsilon + \sum_{k=1}^{\infty} 2^{-k-1}\epsilon \;=\; d^\mu(m, T^\mu) + \epsilon.$$

Thus, we have (8).

Finally, note that each P_k is a two-ended μ-path (i.e., has only finitely many μ-nodes) because it terminates at two μ-nodes and is in the interior of a μ-section. Since P^μ is a one-ended μ-path, it is eventually in $\cup_{k \ge K} P_k$ whatever be the choice of the positive natural number K. But, $\cup_{k \ge K} |P_k| \le \sum_{k=K} \epsilon/2^{k+1} = \epsilon/2^K$. Therefore, given any $\epsilon' > 0$, we can choose K so large that, for each node $q \dashv P^\mu$ lying in $\cup_{k \ge K} P_k$, we have $d^\mu(q, T^\mu) \le \cup_{k \ge K} |P_k| \le \epsilon/2^K < \epsilon'$. So truly, P^μ converges to T^μ. ♣

Given any μ-terminal T^μ of $\mathbf{S}^\mu$ and any $\epsilon > 0$, the open ball $\mathcal{B}_{\mathbf{S}^\mu}(T^\mu, \epsilon)$ is defined by

$$\mathcal{B}_{\mathbf{S}^\mu}(T^\mu, \epsilon) \;=\; \{x \colon x \in \widehat{\mathcal{N}}_{\mathbf{S}^\mu}, d^\mu(x, T^\mu) < \epsilon\}.$$

Proposition 5.14. *$\mathcal{B}_{\mathbf{S}^\mu}(T^\mu, \epsilon) \cap \mathcal{N}_{\mathbf{S}^\mu}$ is μ-connected whatever be the choices of T^μ and ϵ.*

Proof. Let $m, n \in \mathcal{B}_{\mathbf{S}^\mu}(T^\mu, \epsilon)$. By Proposition 5.13, there is a one-ended μ-path P_1^μ starting at m and converging to T^μ and also a one-ended μ-path P_2^μ starting at n and also converging to T^μ such that $|P_1^\mu| < \epsilon$ and $|P_2^\mu| < \epsilon$. Thus, all nodes of P_1^μ and P_2^μ lie in $\mathcal{B}_{\mathbf{S}^\mu}(T^\mu, \epsilon)$. Moreover, there is a node m^* of P_1^μ such that $d^\mu(m^*, T^\mu) < \epsilon/2$, and there is a node n^* of P_2^μ such that $d^\mu(n^*, T^\mu) < \epsilon/2$. Hence, $d^\mu(m^*, n^*) \le d^\mu(m^*, T^\mu) + d^\mu(n^*, T^\mu) < \epsilon$.

So, there is a path P_3 terminating at m^* and n^* with $|P_3| < \epsilon$. Now, for any node $q \dashv P_3$, either the subpath P_{qm^*} terminating at q and m^* has $|P_{qm^*}| < \epsilon/2$ or the subpath P_{qn^*} of P_3 terminating at q and n^* has $|P_{qn^*}| < \epsilon/2$. Therefore, by the triangle inequality, $d^\mu(q, T^\mu) < \epsilon$. Hence, all the nodes of P_3 lie in $\mathcal{B}_{\mathbf{S}^\mu}(T^\mu, \epsilon)$. Let P_{mm^*} be the subpath of P_1^μ terminating at m and m^*, and let P_{nn^*} be the subpath of P_2^μ terminating at n and n^*. As in the proof of Proposition 5.13, we are free to invoke [3, Theorem 4.2-4] to conclude that there is a path in $P_{mm^*} \cup P_3 \cup P_{nn^*}$ connecting m and n. All the embraced nodes of that path lie in $\mathcal{N}_{\mathbf{S}^\mu}$, and therefore that path is of rank no larger than μ. ♣

An endless μ-path P^μ will be said to *converge* to two μ-terminals T_1^μ and T_2^μ if any representative of one of its μ-tips converges to T_1^μ and any representative of its other μ-tip converges to T_2^μ.

Proposition 5.15. *Let T_1^μ and T_2^μ be two μ-terminals of $\mathbf{S}^\mu$. Choose any $\epsilon > 0$. Then, there is an endless μ-path P^μ in $\mathbf{S}^\mu$ that converges to T_1^μ and T_2^μ and is such that $|P^\mu| \leq d^\mu(T_1^\mu, T_2^\mu) + \epsilon$.*

Proof. Choose a node n of $\mathbf{S}^\mu$ such that $d^\mu(n, T_1^\mu) < \epsilon/2$. By Proposition 5.13, there is a one-ended μ-path Q^μ starting at n, converging to T_1^μ, and such that $|Q^\mu| < \epsilon/2$. We have

$$d^\mu(n, T_2^\mu) \;\leq\; d^\mu(n, T_1^\mu) \,+\, d^\mu(T_1^\mu, T_2^\mu) \;<\; d^\mu(T_1^\mu, T_2^\mu) \,+\, \epsilon/2.$$

So, by Proposition 5.13 again, there is a one-ended μ-path L^μ starting at n, converging to T_2^μ, and such that $|L^\mu| < d^\mu(T_1^\mu, T_2^\mu) + \epsilon/2$. We are free to construct a $(\mu+1)$-node $n_1^{\mu+1}$ consisting of the μ-tips in T_1^μ and another $(\mu+1)$-node $n_2^{\mu+1}$ consisting of the μ-tips in T_2^μ. Then, by virtue of Property 4.8 and Proposition 5.12, we can invoke [3, Corollary 3.5-4] to conclude that there exists a two-ended $(\mu+1)$-path P^μ terminating at $n_1^{\mu+1}$ and $n_2^{\mu+1}$ with all its branches embraced by $Q^\mu \cup L^\mu$. (The hypothesis of Corollary 3.5-4 requires that $n_1^{\mu+1}$ and $n_2^{\mu+2}$ be nonsingletons. If need be, we can add a μ-tip of an otherwise isolated one-ended μ-path to $n_1^{\mu+1}$ to make it a nonsingleton, and similarly for $n_2^{\mu+1}$.) Thus, P^μ has one of its μ-tips in T_1^μ and its other μ-tip in T_2^μ because $Q^\mu \cup L^\mu$ reaches T_1^μ only with the μ-tip of Q^μ and reaches T_2^μ only with the μ-tip of L^μ. Now,

$$|P^\mu| \;\leq\; |Q^\mu| + |L^\mu| \;<\; \epsilon/2 + d^\mu(T_1^\mu, T_2^\mu) + \epsilon/2.$$

Since P^μ is the union of two members of $\mathcal{O}^\mu_{\mathbf{S}^\mu}$, Proposition 5.10 now implies that P^μ converges to T_1^μ and T_2^μ. ♣

So far, for this rank μ, we have assumed all the conditions listed in the preceding section. We have also proven all the properties therein — this time stated as propositions. We are at last ready to construct our "permissively structured" $(\mu+1)$-network $\mathbf{N}^{\mu+1}$, given a permissively structured μ-network $\mathbf{N}^\mu$. $\mathbf{N}^\mu$ has at least one and possibly infinitely many μ-sections, that is, components. Some or all of them may have no μ-terminals. But, there may be at least one μ-section $\mathbf{S}^\mu$ having μ-terminals (hence, our assumption of Condition 5.9); for this to be so, it is necessary that $\mathbf{S}^\mu$ have infinitely many permissive μ-nodes. (On the other hand, by Condition 5.9, $\mathbf{S}^\mu$ has no more than finitely many μ-terminals.)

A *permissive* $(\mu+1)$-node $n^{\mu+1}$ is by definition a finite set of μ-terminals chosen from one or more μ-sections. As before, we may identify each μ-terminal T^μ as the set of all permissive one-ended μ-paths that converge to T^μ. Moreover, every permissive μ-tip is a set of permissive μ-paths, all of which reside in a single μ-terminal T^μ under that identification. Whence, T^μ can be viewed as the union of a set of permissive μ-tips. Thus, a permissive $(\mu+1)$-node is thereby a set of permissive μ-tips. On the other hand, every nonpermissive μ-tip (i.e., all its representatives are nonpermissive) of $\mathbf{N}^\mu$ is taken to be the sole member of a singleton $(\mu+1)$-node, which is then called *nonpermissive*. In this way, our definitions of the permissive and nonpermissive $(\mu+1)$-nodes conform with the definitions given in [2] and [3] for transfinite nodes in general. Let us restate our definition of the $(\mu+1)$-nodes as follows.

Condition 5.16. *Every μ-tip is assigned to some $(\mu+1)$-node as follows:*

(a) *Every permissive $(\mu+1)$-node consists of all the μ-tips in only finitely many μ-terminals and does not embrace any node of lower rank.*

(b) *Every nonpermissive $(\mu+1)$-node consists of exactly one nonpermissive μ-tip.*

With this specification of all the $(\mu+1)$-nodes, we have defined the *permissively structured $(\mu+1)$-network* $\mathbf{N}^{\mu+1}$. Indeed, we have completed a recursive cycle, going from $\mathbf{N}^\mu$ to $\mathbf{N}^{\mu+1}$. Thus, permissively structured transfinite networks have hereby been established recursively for all natural numbers μ.

As the last task for this section, we can check that all the Notes 4a through 4i continue to hold when ρ is replace by $\mu + 1$. This is straightforward.

6 Permissively Structured $\vec{\omega}$-Networks, ω-Networks, and Networks of Still Higher Ranks

Our objective now is to obtain a permissively structured network whose rank is the first transfinite limit ordinal ω. To do so, we must first consider the arrow rank $\vec{\omega}$ [3, pages 4 and 5]. In short, we now extend our recursive procedure from $\vec{\omega}$ to ω. So, let us now assume that our recursive procedure has been extended through all the positive natural numbers $\mu = 1, 2, 3, \ldots$ and has thereby yielded an $\vec{\omega}$-network $\mathbf{N}^{\vec{\omega}}$ [3, Section 2.3 and page 122]. The following is assumed to hold.

Condition 6.1. *All the conditions and properties displayed in Section 4 hold for each and all positive natural numbers μ.*

In this case, $\mathbf{N}^{\vec{\omega}}$ will be called a *permissively structured $\vec{\omega}$-network.* Consequently, $\mathbf{N}^{\vec{\omega}}$ has infinitely many nodes for each rank μ, but it does not have any $\vec{\omega}$-node. Also, note that an $\vec{\omega}$-network need not have any $\vec{\omega}$-tip — permissive or nonpermissive [3, Section 2.3].

Let $\mathcal{N}_{\mathbf{S}^{\vec{\omega}}}$ be the set of all permissive nodes of all natural-number ranks in a given $\vec{\omega}$-section $\mathbf{S}^{\vec{\omega}}$ of $\mathbf{N}^{\vec{\omega}}$. Given $m, n \in \mathcal{N}_{\mathbf{S}^{\vec{\omega}}}$, we now let $\mathcal{P}(\mathbf{S}^{\vec{\omega}}, m, n)$ be the set of all permissive paths of any natural-number ranks in $\mathbf{S}^{\vec{\omega}}$ that terminate at m and n. In this case, we define the mapping $d^{\vec{\omega}}: \mathcal{N}_{\mathbf{S}^{\vec{\omega}}} \times \mathcal{N}_{\mathbf{S}^{\vec{\omega}}} \leadsto \boldsymbol{R}^1$ by

$$d^{\vec{\omega}}(m, n) \;=\; \inf\{|P|: P \in \mathcal{P}(\mathbf{S}^{\vec{\omega}}, m, n)\}.$$

The same adaptation that leads from Propositin 3.1 to Proposition 5.3 also leads to the fact that, if $m, n \in \mathcal{N}_{\mathbf{S}^{\vec{\omega}}}$, then there exists a permissive path in $\mathbf{S}^{\vec{\omega}}$ terminating at m and n. Consequently, $d^{\vec{\omega}}(m, n) < \infty$. Obviously, $d^{\vec{\omega}}(m, n) \geq 0$ and $d^{\vec{\omega}}(m, n) = d^{\vec{\omega}}(n, m)$. Also, $d^{\vec{\omega}}(m, m) = 0$. Since the chosen nodes m and n will be in the same μ-section for a sufficiently large μ, Lemma 5.4 shows that $d^{\vec{\omega}}(m, n) > 0$ when m and n are different nodes. Similarly, any three chosen nodes will be in the same μ-section if μ is large enough, and therefore, by Lemma 5.5, $d^{\vec{\omega}}$ satisfies the triangle inequality. Thus, $d^{\vec{\omega}}$ is a metric on $\mathcal{N}_{\mathbf{S}^{\vec{\omega}}}$. For the same reason, we can restate Proposition 5.7 as follows. (Again adapt the proof of Proposition 3.7.)

Proposition 6.2. *Let $\mathbf{S}^\mu$ be a μ-section in $\mathbf{S}^{\vec{\omega}}$. Then, a sequence of permissive nodes in $\mathbf{S}^\mu$ is a Cauchy sequence under the metric d^μ for $\mathcal{N}_{\mathbf{S}^\mu}$ if and only if it is a Cauchy sequence under $d^{\vec{\omega}}$.*

So, upon taking the completion $\widehat{\mathcal{N}}_{\mathbf{S}^{\vec{\omega}}}$ of $\mathcal{N}_{\mathbf{S}^{\vec{\omega}}}$ under $d^{\vec{\omega}}$, we obtain two disjoint sets of limit points. The first consists of all the limit points of all the μ-sections in $\mathbf{S}^{\vec{\omega}}$ for all μ; this set is certainly not void. The second set will consist of all the additional limit points; this set may be void, but we shall henceforth assume that it is nonvoid. Each limit point in the latter set will be called an *$\vec{\omega}$-terminal.* We now assume the following.

Condition 6.3. *There is at least one $\vec{\omega}$-section having one or more $\vec{\omega}$-terminals. Moreover, every $\vec{\omega}$-section has only finitely many $\vec{\omega}$-terminals.*

A one-ended $\vec{\omega}$-path $P^{\vec{\omega}}$ is said to converge to an $\vec{\omega}$-terminal $T^{\vec{\omega}}$ if, given any $\epsilon > 0$, there exists a node $m \dashv P^{\vec{\omega}}$ such that, for all nodes $n \dashv P^{\vec{\omega}}$ beyond m, we have $d^{\vec{\omega}}(n, T^{\vec{\omega}}) < \epsilon$. Let $\mathcal{O}^{\vec{\omega}}_{\mathbf{S}^{\vec{\omega}}}$ denote the set of all permissive one-ended $\vec{\omega}$-paths in $\mathbf{S}^{\vec{\omega}}$. Replacing μ by $\vec{\omega}$ in the proof of Proposition 5.10, we get

Proposition 6.4. *Every member of $\mathcal{O}^{\vec{\omega}}_{\mathbf{S}^{\vec{\omega}}}$ converges to a unique $\vec{\omega}$-terminal of $\mathbf{S}^{\vec{\omega}}$.*

As before, we can partition $\mathcal{O}^{\vec{\omega}}_{\mathbf{S}^{\vec{\omega}}}$ according to which terminal each member of $\mathcal{O}^{\vec{\omega}}_{\mathbf{S}^{\vec{\omega}}}$ converges. Then, each $\vec{\omega}$-tip is a subset of some set in that partitioning, and each $\vec{\omega}$-terminal can be identified as the union of some $\vec{\omega}$-tips. Now, the following propositions can be extended to the present case; just replace μ by $\vec{\omega}$ in their statements: Propositions 5.11, 5.12, 5.13, 5.14, and 5.15. The proofs of these results are easily adapted from their corresponding versions at lower ranks. For example, in the proof of Proposition 5.13, we need merely replace the fixed μ by an increasing sequence of ranks $\mu_1, \mu_2, \mu_3, \ldots$ so that n_k^μ is replaced by $n_k^{\mu_k}$. Similar alterations are used for the proofs of Propositions 5.14 and 5.15.

By virtue of the first sentence of Condition 6.3, we can now define ω-nodes as follows. Each *ω-node* is either a singleton containing a nonpermissive $\vec{\omega}$-tip or is the union of only finitely many $\vec{\omega}$-terminals chosen from one or more $\vec{\omega}$-sections, where in the second case each $\vec{\omega}$-terminal is identified as a set of permissive $\vec{\omega}$-tips (note Proposition 6.4). As before, the first kind of ω-node will be called *nonpermissive*, and the second kind *permissive*. Every $\vec{\omega}$-terminal is assigned to some ω-node. This defines the permissively structured ω-network $\mathbf{N}^{\omega}$. It satisfies

Conditions 6.5. *Every $\vec{\omega}$-tip is assigned to some ω-node as follows:*

(a) *Every permissive ω-node consists of all the $\vec{\omega}$-tips in only finitely many $\vec{\omega}$-terminals and does not embrace any node of natural-number rank.*

(b) *Every nonpermissive ω-node is a singleton consisting of exactly one nonpermissive $\vec{\omega}$-tip.*

We have now attained permissively structured transfinite networks for all ranks up to the first limit ordinal ω. We can continue our recursive procedure to the successor-ordinal ranks $\omega + \mu$ by using the techniques explicated in Sections 2 through 5 and then can continue on to the arrow rank $\omega + \vec{\omega} = \omega \vec{\cdot} 2$ followed by the second limit-ordinal rank $\omega \cdot 2$ by repeating the development of this section. And then again, we can continue still further to even higher successor-ordinal, arrow, and limit-ordinal ranks. How far can this procedure be taken? It is tempting to say— through all the countable and arrow ranks, but we have not explicitly done this.

References

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