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NONSTANDARD GRAPHS

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Abstract — From any given sequence of finite or infinite graphs, a nonstandard graph is constructed. The procedure is similar to an ultrapower construction of an internal set from a sequence of subsets of the real line, but now the individual entities are the vertices of the graphs instead of real numbers. The transfer principle is then invoked to extend several graph-theoretic results to the nonstandard case. After incidences and adjacencies between nonstandard vertices and edges are defined, several formulas regarding numbers of vertices and edges, and nonstandard versions of Eulerian graphs, Hamiltonian graphs, and a coloring theorem are established for these nonstandard graphs.

1 Introduction

In the book [5] (se Sec. 19.1), R. Goldblatt constructed a nonstandard graph by applying the transfer principle to a set V of vertices with the set E of edges defined by a symmetric irreflexive binary relationship on V. Thus, the conventional graph $\langle V, E\rangle$ is transferred to the nonstandard graph ${}^*G = \langle {}^*V, {}^*E\rangle$. This result was used to establish in a nonstandard way the standard theorem that, if every finite subgraph of a conventional infinite graph G has a coloring with finitely many colors, then G itself has such a finite coloring.

On the other hand, in several recent works (see [7], [8], and the references therein), the idea of nonstandard transfinite graphs and networks was introduced and investigated. The basic idea in those works was to start with a given transfinite graph, to reduce it to a finite graph by shorting and opening edges, and then to obtain an expanding sequence of finite graphs by restoring edges sequentially. If all this is done in an appropriate fashion, it may happen that the sequence of finite graphs fills out and restores the original transfinite graph once the restoration process is completed. If in addition there is an assignment of electrical parameters to the edges, the final result may be sequences of edge currents and edge voltages, from which nonstandard hyperreal currents and voltages can be derived. The latter hyperreal quantities will then automatically satisfy Kirchhoff's laws, even though Kirchhoff's laws may on occasion be violated in the original transfinite network when standard real numbers are used—an important advantage of this nonstandard approach.

However, this is only a partial construction of a nonstandard graph in the sense that the completion of the restoration process—if successful—results in the original standard transfinite graph. The sequence of restorations only provides a means of constructing hyperreal currents and voltages satisfying Kirchhoff's laws. Another approach might start with an arbitrary sequence of conventional graphs and construct from that a nonstandard graph in much the same way as an internal set in the hyperreal line $^*\boldsymbol{R}$ is constructed from a given sequence of subsets of the real line $\boldsymbol{R}$, that is, by means of an ultrapower construction [5, page 36]. In this case, the resulting nonstandard graph has nonstandard edges and nonstandard vertices, and it thereby is much different from those of the prior works [7], [8]. Also, the present approach differs from Goldblatt's result in that the graphs of the sequence need not be vertex-induced subgraphs of a given graph.

Our objective in this work is to develop this latter approach to nonstandard graphs. The individual elements are chosen to be the vertices of the graphs along with all the natural numbers. These individuals are not sets by assumption and therefore have no members. All the other standard and nonstandard entities are derived from these individuals. For example, an edge is defined to be a pair of vertices, which is one of the conventional ways of defining a graph. Then, certain equivalence classes of sequences of standard vertices are defined to be nonstandard vertices, and these then yield nonstandard edges as certain equivalence classes of sequences of standard edges. In this approach, there are no multiedges (i.e., no parallel edges) and no self-loops (i.e., no edge consisting of just a single vertex).

After setting up our nonstandard graphs using an ultrapower approach, we invoke the transfer principle to lift several standard graph-theoretic results into a nonstandard setting. For example, the relationship between the number of vertices, the number of edges, and the cyclomatic number of a standard finite connected graph continues to hold for nonstandard graphs except that these numbers are replaced by hypernatural numbers. Similarly, standard theorems concerning Eulerian graphs, Hamiltonian graphs, and a coloring theorem are also extended to the nonstandard setting. By virtue of the transfer principle, this only requires that the standard theorems be stated as sentences in symbolic logic, which are then transferred to appropriate sentences for nonstandard graphs.

In the following, $|A|$ will denote the cardinality of a set A. Also, $\boldsymbol{N} = \{0, 1, 2, \ldots\}$ is the set of all natural numbers, and $\boldsymbol{R}$ is the set of real numbers. Thus, $^*\boldsymbol{N}$ is the set of hypernaturals, and $^*\boldsymbol{R}$ is the set of hyperreals.

2 Nonstandard Graphs

A standard graph G is a conventional (finite or infinite) graph $G = \{X, B\}$, where X is the set of its vertices and B is the set of its edges. Each edge $b \in B$ is a two-element set $b = \{x, y\}$ with $x, y \in X$ and $x \neq y$; b and x are said to be incident and so, too, are b and y. Also, x and y are said to be adjacent through b.

Next, let $\langle G_n : n \in \boldsymbol{N} \rangle$ be a given sequence of graphs. For each n, we have $G_n = \{X_n, B_n\}$, where X_n is the set of edges and B_n is the set of vertices. We allow $X_n \cap X_m \neq \emptyset$ so that G_n and G_m may be subgraphs of a larger graph. In fact, we may have $X_n = X_m$ and $B_n = B_m$ for all $n, m \in \boldsymbol{N}$ so that G_n may be the same graph for all $n \in \boldsymbol{N}$. Furthermore, let $\mathcal{F}$ be a chosen nonprincipal ultrafilter on $\boldsymbol{N}$ [5, pages 18-19].

In the following, $\langle x_n \rangle = \langle x_n : n \in \boldsymbol{N} \rangle$ will denote a sequence of vertices with $x_n \in X_n$ for all $n \in \boldsymbol{N}$. A nonstandard vertex $^*x$ is an equivalence class of such sequences of vertices, where two such sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ are taken to be equivalent if $\{n : x_n = y_n\} \in \mathcal{F}$, in which case we write "$\langle x_n \rangle = \langle y_n \rangle$ a.e." or say that $x_n = y_n$ "for almost all n." We also write $^*x = [x_n]$, where it is understood that the x_n are the members of any one sequence in the equivalence class.

That this truly defines an equivalence class can be shown as follows. Reflexivity and symmetry being obvious, consider transitivity: Given that $\langle x_n \rangle = \langle y_n \rangle$ a.e. and that $\langle y_n \rangle = \langle z_n \rangle$ a.e., we have $N_{xy} = \{n : x_n = y_n\} \in \mathcal{F}$ and $N_{yz} = \{n : y_n = z_n\} \in \mathcal{F}$. By the properties of the ultrafilter, $N_{xy} \cap N_{yz} \in \mathcal{F}$. Moreover, $N_{xz} = \{n : x_n = z_n\} \supseteq (N_{xy} \cap N_{yz})$. Therefore, $N_{xz} \in \mathcal{F}$. Hence, $\langle x_n \rangle = \langle z_n \rangle$ a.e.; transitivity holds. We let $^*X$ denote the set of nonstandard vertices.

Next, we define the nonstandard edges: Let $^*x = [x_n]$ and $^*y = [y_n]$ be two nonstandard vertices. This time, let $N_{xy} = \{n : \{x_n, y_n\} \in B_n\}$ and $N_{xy}^c = \{n : \{x_n, y_n\} \notin B_n\}$. Since $\mathcal{F}$ is an ultrafilter, exactly one of N_{xy} and N_{xy}^c is a member of $\mathcal{F}$. If it is N_{xy}, then $^*b = [\{x_n, y_n\}]$ is defined to be a nonstandard edge; that is, $^*b$ is an equivalence class of sequences $\langle b_n \rangle$ of edges $b_n = \{x_n, y_n\} \in B_n$, $n = 0, 1, 2, \ldots$. In this case, we also write $^*x, ^*y \in {}^*b$ and $^*b = \{^*x, ^*y\}$. We let $^*B$ denote the set of nonstandard edges. On the other hand, if $N_{xy}^c \in \mathcal{F}$, then $[\{x_n, y_n\}]$ is not a nonstandard edge.

We shall now show that this definition is independent of the representatives chosen for the vertices. Let $[\{x_n, y_n\}]$ and $[\{v_n, w_n\}]$ represent the same nonstandard edge. We want to show that, if $\langle x_n \rangle = \langle v_n \rangle$ a.e., then $\langle y_n \rangle = \langle w_n \rangle$ a.e. Suppose $\langle y_n \rangle \neq \langle w_n \rangle$ a.e. Then, $\{n : x_n = v_n\} \cap \{n : y_n \neq w_n\} \in \mathcal{F}$. Thus, there is at least one n for which the three vertices $x_n = v_n$, y_n, and w_n are all incident to the same standard edge—in violation of the definition of a edge. Similarly, if all of $\langle x_n \rangle$, $\langle y_n \rangle$, $\langle v_n \rangle$, $\langle w_n \rangle$ are different a.e., then there would be a standard edge having four incident vertices—again a violation.

Next, we show that we truly have an equivalence relationship for the set of all sequences of standard edges. Reflexivity and symmetry being obvious again, consider transitivity: Let $^*b = [\{x_n, y_n\}]$, $^*\tilde{b} = [\{\tilde{x}_n, \tilde{y}_n\}]$, $^*\acute{b} = [\{\acute{x}_n, \acute{y}_n\}]$, and assume that $^*b = {}^*\tilde{b}$ and $^*\tilde{b} = {}^*\acute{b}$. We want to show that $^*b = {}^*\acute{b}$. We have $N_{b\tilde{b}} = \{n : \{x_n, y_n\} = \{\tilde{x}_n, \tilde{y}_n\}\} \in \mathcal{F}$ and $N_{\tilde{b}\acute{b}} = \{n : \{\tilde{x}_n, \tilde{y}_n\} = \{\acute{x}_n, \acute{y}_n\}\} \in \mathcal{F}$. Moreover, $N_{b\acute{b}} = \{n : \{x_n, y_n\} = \{\acute{x}_n, \acute{y}_n\}\} \supseteq (N_{b\tilde{b}} \cap N_{\tilde{b}\acute{b}}) \in \mathcal{F}$. Therefore, $N_{b\acute{b}} \in \mathcal{F}$. Thus, $^*b = {}^*\acute{b}$, as desired.

Finally, we define a nonstandard graph $^*G$ to be the pair $^*G = \{^*X, ^*B\}$.

Let us now take note of a special case that arises when all the G_n are the same standard graph $G = \{X, B\}$. In this case, we call $^*G = \{^*X, ^*B\}$ an *enlargement* of G, in conformity with an "enlargement" $^*A$ of a subset A of $\mathbb{R}$ [5, pages 28-29]. If in addition G is a finite graph, each vertex $x \in {}^*X$ can be identified with a vertex of X because the enlargement of a finite set equals the set itself. Similarly, every branch $b \in {}^*B$ can be identified with a branch in B. In short, ${}^*G = G$. On the other hand, if G is a conventionally infinite graph, X is an infinite set and its enlargement ${}^*X$ has more elements, namely, nonstandard vertices that are not equal to standard vertices, (i.e., ${}^*X \setminus X$ is not empty). Similarly, ${}^*B \setminus B$ is not empty too. In short, ${}^*G$ is a proper enlargement of G.

Example 2.1. Let G be a one-way infinite path P:

$$P = \langle x_0, b_0, x_1, b_1, x_2, b_2, \ldots \rangle$$

Also, let $G_n = G$ for all $n \in \mathbb{N}$, and let ${}^*G = [G_n] = \{{}^*X, {}^*B\}$. Next, let $\langle k_n : n \in \mathbb{N} \rangle$ be any sequence of natural numbers, and set ${}^*x = [x_{k_n}]$ and ${}^*y = [x_{k_n+1}]$. Then, ${}^*x$ and ${}^*y$ are two vertices in the enlargement ${}^*G$ of G, and ${}^*b = \{{}^*x, {}^*y\} = [\{x_{k_n}, x_{k_n+1}\}]$ is an edge in ${}^*G$. On the other hand, if $\langle m_n : n \in \mathbb{N} \rangle$ is another sequence of natural numbers with $m_n > 1$, then ${}^*z = [x_{k_n+m_n}]$ is another nonstandard vertex in ${}^*G$ different from ${}^*x$ and ${}^*y$ and appearing after ${}^*y$ in the enlarged path. Moreover, $[\{x_{k_n}, x_{k_n+m_n}\}]$ is not a nonstandard edge. In this way, no vertex or edge repeats in the enlarged path. □

Another special case arises when almost all the G_n are (possibly different) finite graphs. Again in conformity with the terminology used for hyperfinite internal subsets of ${}^*R$ [5, page 149], we will refer to the resulting nonstandard graph ${}^*G$ as a hyperfinite graph.[1] As a result, we can lift many theorems concerning finite graphs to hyperfinite graphs. It is just a matter of writing the standard theorem in an appropriate form using symbolic logic and then applying the transfer principle. We let ${}^*G_f$ denote the set of hyperfinite graphs.

3 Incidences and Adjacencies between Vertices and Edges

Let us now define these ideas for nonstandard graphs both in terms of an ultrapower construction and by transfer of appropriate symbolic sentences. In the subsequent sections, we will usually confine ourselves to the transfer principle. We henceforth drop the asterisks when denoting nonstandard vertices and edges. These are specified as members of ${}^*X$ and ${}^*B$ respectively.

Incidence between a vertex and an edge: Given a sequence $\langle G_n : n \in \mathbb{N} \rangle$ of standard graphs $G_n = \{X_n, B_n\}$, a nonstandard vertex $x = [x_n] \in {}^*X$ and a nonstandard edge

[1]This should not be confused with a hypergraph—an entirely different object [2].

$b = [b_n] \in {}^*B$ are said to be *incident* if $\{n\colon x_n \in b_n\} \in \mathcal{F}$, where as always the nonprincipal ultrafilter $\mathcal{F}$ is understood to be chosen and fixed.

On the other hand, we can define incidence between a standard vertex $x \in X$ and a standard edge $b \in B$ for the graph $G = \{X, B\}$ through the symbolic sentence

$$(\exists\, x \in X)\,(\exists\, b \in B)\,(x \in b)$$

By transfer, we have that $x \in {}^*X$ and $b \in {}^*B$ are incident when the following sentence is true.

$$(\exists\, x \in {}^*X)\,(\exists\, b \in {}^*B)\,(x \in b)$$

These are equivalent definitions.

Adjacency between vertices: For a standard graph $G = \{X, B\}$, two vertices $x, y \in X$ are called *adjacent* and we write $x \diamond y$ if the following sentence on the right-hand side of $\leftrightarrow$ is true.

$$x \diamond y \;\leftrightarrow\; (\exists\, x, y \in X)\,(\exists\, b \in B)\,(b = \{x, y\})$$

By transfer, this becomes for a nonstandard graph ${}^*G = \{{}^*X, {}^*B\}$

$$x \diamond y \;\leftrightarrow\; (\exists\, x, y \in {}^*X)\,(\exists\, b \in {}^*B)\,(b = \{x, y\})$$

Alternatively, under an ultrapower construction, we have for ${}^*G = [G_n] = [\{X_n, B_n\}]$ that $x = [x_n] \in {}^*X$ and $y = [y_n] \in {}^*X$ are adjacent (i.e., $x \diamond y$) if there exists a $b = [b_n] \in {}^*B$ such that $\{n\colon b_n = \{x_n, y_n\}\} \in \mathcal{F}$.

Adjacency between edges: For a standard graph, two edges $b, c \in B$ are called *adjacent* and we write $b \bowtie c$ when the following sentence on the right-hand side of $\leftrightarrow$ is true.

$$b \bowtie c \;\leftrightarrow\; (\exists\, b, c \in B)\,(\exists\, x \in X)\,(x \in b \wedge x \in c)$$

By transfer, we have for nonstandard edges b and c

$$b \bowtie c \;\leftrightarrow\; (\exists\, b, c \in {}^*B)\,(\exists\, x \in {}^*X)\,(x \in b \wedge x \in c)$$

Under an ultrapower approach, we would have $b = [b_n]$ and $c = [c_n]$ are adjacent nonstandard edges when there exists a nonstandard vertex $x = [x_n]$ such that

$$\{n\colon x_n \in b_n \text{ and } x_n \in c_n\} \;\in\; \mathcal{F}.$$

4 Nonstandard Hyperfinite Paths and Loops

Again, we start with a standard graph $G = \{X, B\}$. Remember that since B is a set of two-element subsets of X, all edges are distinct (i.e., there are no multiedges) and there are no self-loops. A *finite path* P in G is defined by the sentence

$$(\exists\, k \in \boldsymbol{N} \setminus \{0\})\, (\exists\, x_0, x_1, \ldots, x_k \in X)\, (\exists\, b_0, b_1, \ldots b_{k-1} \in B)$$

$$(x_0 \in b_0 \,\wedge\, b_0 \ni x_1 \,\wedge\, x_1 \in b_1 \,\wedge\, b_1 \ni x_2 \,\wedge\, \ldots \,\wedge\, x_{k-1} \in b_{k-1} \,\wedge\, b_{k-1} \ni x_k) \qquad (1)$$

That all the vertices and edges in P are distinct is implied by the fact that the those k vertices in X and those $k-1$ edges in B are perforce all distinct. The *length* $|P|$ of P is the number of edges in P; thus, $|P| = k$.

We may apply transfer to (1) to get the following definition of a *nonstandard path* $^*P$.

$$(\exists\, k \in {}^*\boldsymbol{N} \setminus \{0\})\, (\exists\, x_0, x_1, \ldots, x_k \in {}^*X)\, (\exists\, b_0, b_1, \ldots b_{k-1} \in {}^*B)$$

$$(x_0 \in b_0 \,\wedge\, b_0 \ni x_1 \,\wedge\, x_1 \in b_1 \,\wedge\, b_1 \ni x_2 \,\wedge\, \ldots \,\wedge\, x_{k-1} \in b_{k-1} \,\wedge\, b_{k-1} \ni x_k) \qquad (2)$$

In this case, the *length* $|^*P|$ equals $k \in {}^*\boldsymbol{N} \setminus \{0\}$; in general, k is now a positive hypernatural number. We therefore call $^*P$ a *hyperfinite path.* To view this fact in terms of an ultrapower construction of $^*G = [G_n]$, note that the G_n may be finite graphs growing in size or indeed be conventionally infinite graphs. Thus, $^*P$ may have an unlimited length, that is, its length may be a member of $^*\boldsymbol{N} \setminus \boldsymbol{N}$.

A standard loop is defined as is a standard path except that the first and last vertices are required to be the same. Upon applying transfer, we get the following definition of a *nonstandard loop.*

$$(\exists\, k \in {}^*\boldsymbol{N} \setminus \{0\})\, (\exists\, x_0, x_1, \ldots, x_{k-1} \in {}^*X)\, (\exists\, b_0, b_1, \ldots b_{k-1} \in {}^*B)$$

$$(x_0 \in b_0 \,\wedge\, b_0 \ni x_1 \,\wedge\, x_1 \in b_1 \,\wedge\, b_1 \ni x_2 \,\wedge\, \ldots \,\wedge\, x_{k-1} \in b_{k-1} \,\wedge\, b_{k-1} \ni x_0) \qquad (3)$$

5 Connected Nonstandard Graphs

A standard graph $G = \{X, B\}$ is called *connected* if, for every two vertices x and y in G, there is a finite path (1) terminating at those vertices, that is, $x_0 = x$ and $x_k = y$. Let $\mathcal{C}$

denote the set of connected standard graphs, and let $\mathcal{P}(G)$ be the set of all finite paths in a given standard graph G. Also, for any $P \in \mathcal{P}(G)$, let $x_0(P)$ and $x_k(P)$ denote the first and last nodes of P in accordance with (1); k depends upon P. Then, the connectedness of G is defined symbolically by the truth of the following sentence to the right of $\leftrightarrow$.

$$G \in \mathcal{C} \;\leftrightarrow\; (\forall\, x, y \in X)\, (\exists\, P \in \mathcal{P}(G))\, ((x_0(P) = x) \wedge (x_k(P) = y)) \tag{4}$$

By transfer, we obtain the definition of the set ${}^*\mathcal{C}$ of all *connected* nonstandard graphs: For ${}^*G = \{{}^*X, {}^*B\}$, for ${}^*\mathcal{P}({}^*G)$ being the set of nonstandard paths ${}^*P \in {}^*G$, and for $x_0({}^*P)$ and $x_k({}^*P)$ being the first and last vertices of ${}^*P$, we have

$${}^*G \in {}^*\mathcal{C} \leftrightarrow\; (\forall\, x, y \in {}^*X)\, (\exists\, {}^*P \in {}^*\mathcal{P}({}^*G))\, ((x_0({}^*P) = x) \wedge (x_k({}^*P) = y)) \tag{5}$$

Here, $k \in {}^*\boldsymbol{N} \setminus \{0\}$ as in (2).

Let us explicate this still further in terms of an ultrapower construction of ${}^*G = [G_n]$ from an equivalence class of sequences of (possibly infinite) graphs, $\langle G_n \rangle$ being one of those sequences. With ${}^*P = [P_n]$ denoting a nonstandard path obtained similarly from a representative sequence $\langle P_n \rangle$ of finite paths, P_n being in G_n, we let $x_0(P_n)$ and $x_k(P_n)$ denote the first and last vertices of P_n. (Thus, k also depends on n of course.) Then, $x_0({}^*P) = [x_0(P_n)]$ and $x_{{}^*k}({}^*P) = [x_k(P_n)]$ are the first and last nonstandard vertices of ${}^*P$. (In (5), ${}^*k$ is denoted by $k \in {}^*\boldsymbol{N} \setminus \{0\}$.) Then, ${}^*G$ is called *connected* if and only if, given any nonstandard vertices ${}^*x = [x_n]$ and ${}^*y = [y_n]$ in ${}^*G$, we have that, for almost all n, there exists a finite path P_n terminating at x_n and y_n. This can be restated by saying that there exists a hyperfinite path ${}^*P$ in ${}^*G$ terminating at ${}^*x$ and ${}^*y$.

Later on, we will need a special case of ${}^*\mathcal{C}$: Let $\mathcal{C}_f$ denote the subset of $\mathcal{C}$ consisting of all finite connected standard graphs $G = \{X, B\}$, where $|X| \in \boldsymbol{N} \setminus \{0, 1\}$, $|B| \in \boldsymbol{N} \setminus \{0\}$. Then, ${}^*\mathcal{C}_f$ is the subset of $\mathcal{C}$ obtained by lifting $\mathcal{C}_f$ through transfer to a subset of ${}^*\mathcal{C}$. In this case,

$${}^*G \in {}^*\mathcal{C}_f \;\leftrightarrow\; ({}^*G = \{{}^*X, {}^*B\} \in {}^*\mathcal{C}) \wedge (|{}^*X| \in {}^*\boldsymbol{N} \setminus \{0, 1\} \wedge |{}^*B| \in {}^*\boldsymbol{N} \setminus \{0\}). \tag{6}$$

We call such a ${}^*G$ a nonstandard *hyperfinite* connected graph.

6 Nonstandard Subgraphs

If A and C are sets of vertices with $A \subseteq C$, we get upon transfer the following definition for sets of nonstandard vertices.

$$^*A \subseteq {}^*C \leftrightarrow (\forall x \in {}^*A)(x \in {}^*C)$$

Our purpose in this section is to define a nonstandard subgraph $^*G_s$ of a given nonstandard graph $^*G$. We let $\mathcal{G}$ denote the set of all standard graphs. By definition, $G_s = \{X_s, B_s\}$ is a (vertex induced) subgraph of $G = \{X, B\} \in \mathcal{G}$ if $X_s \subseteq X$ and B_s is the set of those edges in B that are each incident to two vertices in X_s. Let us denote the set of all such subgraphs of G by $\mathcal{G}_s(G)$. Then, symbolically G_s is defined by

$$G_s \in \mathcal{G}_s(G) \leftrightarrow$$

$$(\exists\, G_s = \{X_s, B_s\} \in \mathcal{G})\,(\exists\, G = \{X, B\} \in \mathcal{G})$$

$$((X_s \subseteq X) \wedge (B_s = \{b = \{x, y\} \in B : x, y \in X_s\})).$$

By transfer, we get the definition of a *nonstandard subgraph* $^*G_s$ of a given nonstandard graph $^*G$. In this case, $^*\mathcal{G}$ denotes the set of all nonstandard graphs, and $^*\mathcal{G}_s(^*G)$ denotes the set of all nonstandard subgraphs of a given $^*G \in {}^*\mathcal{G}$.

$$^*G_s \in {}^*\mathcal{G}_s(^*G) \leftrightarrow$$

$$(\exists\, {}^*G_s = \{{}^*X_s, {}^*B_s\} \in {}^*\mathcal{G})\,(\exists\, {}^*G = \{{}^*X, {}^*B\} \in {}^*\mathcal{G})$$

$$(({}^*X_s \subseteq {}^*X) \wedge ({}^*B_s = \{b = \{x, y\} \in {}^*B : x, y \in {}^*X_s\})$$

7 Nonstandard Trees

The symbols $\mathcal{G}$, $\mathcal{G}_s(G)$, $\mathcal{C}$, $\mathcal{C}_f$, and their nonstandard counterparts have been defined above. Now, let $\mathcal{L}(G)$ be the set of all loops in the standard graph G, and let $\mathcal{T}$ be the set of all standard trees. Then, a tree T can be defined symbolically by

$$T \in \mathcal{T} \leftrightarrow (\exists\, T \in \mathcal{C})\,(\neg(\exists\, L \in \mathcal{L}(T))) \tag{7}$$

To transfer this, we let ${}^*\mathcal{L}({}^*G)$ be the set of all nonstandard loops in a given nonstandard graph ${}^*G$, as defined by (3), and we let ${}^*\mathcal{T}$ denote the set of *nonstandard trees* ${}^*T$, defined as follows:

$$ {}^*T \in {}^*\mathcal{T} \;\leftrightarrow\; (\exists\, {}^*T \in {}^*\mathcal{C})\,(\neg(\exists\, {}^*L \in {}^*\mathcal{L}({}^*T))) \tag{8} $$

Next, let $\mathcal{T}_{sp}(G)$ be the set of all spanning trees in a given finite connected standard graph $G = \{X, B\}$. That T is such a spanning tree can be expressed symbolically as follows.

$$ T \in \mathcal{T}_{sp}(G) \;\leftrightarrow $$

$$ (\exists\, G = \{X, B\} \in \mathcal{C}_f)\,(\exists\, T = \{X_T, B_T\} \in \mathcal{T})\,((T \in \mathcal{G}_s(G)) \;\wedge\; (|X| = |X_T|)) \tag{9} $$

By transfer, we have the set ${}^*\mathcal{T}_{sp}({}^*G)$ of all *spanning trees* of a given hyperfinite connected nonstandard graph ${}^*G$, defined as follows:

$$ {}^*T \in {}^*\mathcal{T}_{sp}({}^*G) \;\leftrightarrow $$

$$ (\exists\, {}^*G = \{{}^*X, {}^*B\} \in {}^*\mathcal{C}_f)\,(\exists\, {}^*T = \{{}^*X_T, {}^*B_T\} \in {}^*\mathcal{T})\,(({}^*T \in {}^*\mathcal{G}_s({}^*G)) \wedge (|{}^*X| = |{}^*X_T|)) \tag{10} $$

8 Some Numerical Formulas

With these symbolic definition in hand, we can now lift some standard formulas regarding numbers of vertices and edges into a nonstandard setting. For example, if p, q, and r are the number of vertices, the number of edges, and the cyclomatic number respectively of a given connected finite graph, then $r = q - p + 1$. Symbolically, this can be stated as follows. Again, we use the notation $G = \{X, B\}$ for a graph and $T = \{X_T, B_T\}$ for a tree.

$$ (\forall\, p, q, r \in \mathbb{N})\,(\forall\, G \in \mathcal{C}_f)\,(\forall\, T \in \mathcal{T}_{sp}(G)) $$

$$ ((p = |X| \;\wedge\; q = |B| \;\wedge\; r = |B| - |B_T|) \;\rightarrow\; (r = q - p + 1)) $$

By transfer, we obtain the following formula in hypernatural numbers.

$$ (\forall\, p, q, r \in {}^*\mathbb{N})\,(\forall\, {}^*G \in {}^*\mathcal{C}_f)\,(\forall\, {}^*T \in {}^*\mathcal{T}_{sp}({}^*G)) $$

$$ ((p = |{}^*X| \;\wedge\; q = |{}^*B| \;\wedge\; r = |{}^*B| - |{}^*B_T|) \;\rightarrow\; (r = q - p + 1)) $$

Another standard formula for a connected finite graph having no multiedges is that $p-1 \leq q \leq p(p-1)/2$. Symbolically, we have

$$(\forall\, p, q \in \boldsymbol{N})\,(\forall\, G \in \mathcal{C}_f)\,((p = |X| \,\wedge\, q = |B|) \,\rightarrow\, (p-1 \leq q \leq p(p-1)/2)).$$

So, by transfer, we have

$$(\forall\, p, q \in {}^*\boldsymbol{N})\,(\forall\, G \in {}^*\mathcal{C}_f)\,((p = |{}^*X| \,\wedge\, q = |{}^*B|) \,\rightarrow\, (p-1 \leq q \leq p(p-1)/2)).$$

Still another example of such a lifting concerns the radius R and diameter D of a finite connected graph G. It is a fact that $R \leq D \leq 2R$ [3, page 37], [4, pages 20-21]. Again, we need to express ideas symbolically.

Let $A \subset \boldsymbol{N}$ be such that $|A| \in \boldsymbol{N}$ (i.e., A is a finite subset of $\boldsymbol{N}$). We use the symbols $\overline{a} = \max A$ and $\underline{a} = \min A$ as abbreviations for the following sentences.

$$\overline{a} = \max A \;\leftrightarrow\; (\forall\, c \in A)\,(\exists\, \overline{a} \in A)\,(\overline{a} \geq c)$$

$$\underline{a} = \min A \;\leftrightarrow\; (\forall\, c \in A)\,(\exists\, \underline{a} \in A)\,(\underline{a} \leq c)$$

This transfers to

$$\overline{a} = \max {}^*A \;\leftrightarrow\; (\forall\, c \in {}^*A)\,(\exists\, \overline{a} \in {}^*A)\,(\overline{a} \geq c)$$

$$\underline{a} = \min {}^*A \;\leftrightarrow\; (\forall\, c \in^* A)\,(\exists\, \underline{a} \in {}^*A)\,(\underline{a} \leq c),$$

where now ${}^*A$ is a hyperfinite subset of ${}^*\boldsymbol{N}$ and $\overline{a}$ and $\underline{a}$ are hypernatural numbers. ${}^*A$ does have a maximum element and a minimum element so that these definitions are valid [5, pages 149-150].

Now, for a given finite connected graph $G \in \mathcal{C}_f$, let $\mathcal{P}_x$ denote the set of all paths P_x starting at x. The length $|P_x|$ of any $P_x \in \mathcal{P}_x$ is the number of edges in P_x. Also, let $\mathcal{E}(G)$ be the set of eccentricities of the vertices in $G = \{X, B\}$. Symbolically, we have

$$e_x \in \mathcal{E}(G) \;\leftrightarrow\; (\exists\, x \in X)\,(\forall\, P_x \in \mathcal{P}_x)\,(e_x = \max\{|P_x|\colon P_x \in \mathcal{P}_x\}).$$

So, for a hyperfinite connected graph ${}^*G = \{{}^*X, {}^*B\} \in {}^*\mathcal{C}_f$, we have by transfer

$$e_x \in \mathcal{E}({}^*G) \;\leftrightarrow\; (\exists\, x \in {}^*X)\,(\forall\, {}^*P_x \in {}^*\mathcal{P}_x)\,(e_x = \max\{|{}^*P_x|\colon {}^*P_x \in {}^*\mathcal{P}_x\}),$$

where now $\mathcal{E}({}^*G)$ is the set of hypernatural eccentricities in ${}^*G$ and ${}^*P_x$ is any nonstandard hyperfinite path in ${}^*G$ starting at the nonstandard vertex x.

Then, for any $G = \{X, B\} \in \mathcal{C}_f$, the radius $R(G)$ is defined by

$$(\forall\, e_x \in \mathcal{E}(G))\,(\exists\, R(G) \in \boldsymbol{N})\,(R(G) = \min\{e_x : x \in X\}),$$

which by transfer gives the following definition of the hypernatural radius $R({}^*G) \in {}^*\boldsymbol{N}$ of any ${}^*G = \{{}^*X, {}^*B\} \in {}^*\mathcal{C}_f$:

$$(\forall\, e_x \in {}^*\mathcal{E}(G))\,(\exists\, R({}^*G) \in {}^*\boldsymbol{N})\,(R({}^*G) = \min\{e_x : x \in {}^*X\}),$$

Similarly, the diameter $D(G)$ of G is defined by

$$(\forall\, e_x \in \mathcal{E}(G))\,(\exists\, D(G) \in \boldsymbol{N})\,(D(G) = \max\{e_x : x \in X\}),$$

which by transfer gives the hypernatural diameter $D(G) \in {}^*\boldsymbol{N}$ of ${}^*G$.

$$(\forall\, e_x \in \mathcal{E}({}^*G))\,(\exists\, D({}^*G) \in {}^*\boldsymbol{N})\,(D({}^*G) = \max\{e_x : x \in {}^*X\}),$$

So, we have the following sentence for the standard result:

$$(\forall\, G \in \mathcal{C}_f)\,(R(G) \le D(G) \le 2R(G))\ [illegible]$$

which by transfer yields the nonstandard result

$$(\forall\, G \in {}^*\mathcal{C}_f)\,(R({}^*G) \le D({}^*G) \le 2R({}^*G)).$$

9 Eulerian Graphs

A finite trail is defined much as a finite path is defined except that the condition that all the vertices be distinct is relaxed; however, edges are still required to be distinct. Thus, the truth of the following sentence defines a *trail* T in a finite graph $G = \{X, B\}$, with T having two or more edges. This time we use the notation $b_m = \{x_m, y_m\}$ to display the vertices x_m and y_m that are incident to b_m.

$$(\exists\, k \in \boldsymbol{N} \setminus \{0\})\,(\exists\, b_0, b_1, \ldots, b_k \in B)\,(\forall m \in \{0, \ldots, k-1\})\,(y_m = x_{m+1})$$

That B is a set insures that the edges $b_0, b_1, \ldots, b_k$ are all distinct. On the other hand, this sentence allows vertices to repeat in a trail.

For a *closed trail*, we have the truth of the following sentence as its definition.

$$(\exists\, k \in \boldsymbol{N} \setminus \{0\})\,(\exists\, b_0, b_1, \ldots, b_k \in B)\,(\forall m \in \{0, \ldots, k-1\})\,(y_m = x_{m+1}) \;\wedge\; (y_k = x_0)$$

With Q denoting a trail, we denote the set of edges in Q by $B(Q)$. Also, we let $\mathcal{Q}(G)$ denote the set of closed trails in a given graph $G = \{X, B\}$.

By attaching asterisks as usual, we obtain by transfer the corresponding sentence for trails in a given nonstandard graph ${}^*G = \{{}^*X, {}^*B\}$. Thus, a *nonstandard trail* ${}^*Q$ is defined by the truth of the following sentence; now, $b_m = \{x_m, y_m\}$ is a nonstandard edge with the nonstandard vertices x_m and y_m.

$$(\exists\, k \in {}^*\boldsymbol{N} \setminus \{0\})\,(\exists\, b_0, b_1, \ldots, b_k \in {}^*B)\,(\forall m \in \{0, \ldots, k-1\})\,(y_m = x_{m+1}).$$

A similar expression holds for a *nonstandard closed trail* (just append $\wedge\ (y_k = x_0)$). With ${}^*Q$ denoting a nonstandard trail, we denote the set of nonstandard edges in ${}^*Q$ by ${}^*B({}^*Q)$. Also, we let ${}^*\mathcal{Q}({}^*G)$ denote the set of nonstandard closed trails in a given nonstandard graph ${}^*G = \{{}^*X, {}^*B\}$.

A finite connected graph $G = \{X, B\} \in \mathcal{C}_f$ is called *Eulerian* if it contains a closed trail that meets every vertex of X. The degree d_x of $x \in X$ is the natural number $d_x = |\{b \in B\colon x \in b\}|$. The nonstandard version of this definition is as follows: Given ${}^*G = \{{}^*X, {}^*B\} \in {}^*\mathcal{C}_f$, for any $x \in {}^*X$, the *degree* of x is $d_x = |\{b \in {}^*B : x \in b\}|$. In this case, d_x may be an unlimited hypernatural number when ${}^*G$ is a hyperfinite graph. However, ${}^*G$ might happen to be a finite graph $G \in \mathcal{C}_f$, which from the point of view of an ultrapower construction can occur if all the G_n for ${}^*G = [G_n]$ are the same finite graph $G \in \mathcal{G}_f$; in this case, d_x will be a natural number for all $x \in {}^*X$.

Let $\mathcal{E}_u$ (resp. ${}^*\mathcal{E}_u$) denote the set of all standard Eulerian graphs (resp. nonstandard Eulerian graphs). Then, Eulerian graphs can be defined by asserting the truth of the following sentence to the right of $\leftrightarrow$, where as usual $G = \{X, B\}$.

$$G \in \mathcal{E}_u \;\leftrightarrow\; (\exists\, Q \in \mathcal{Q}(G))\,((\forall\, b \in B)\,(b \in B(Q))$$

By transfer, the truth of the following right-hand side defines *nonstandard Eulerian graphs.* Now, ${}^*G = \{{}^*X, {}^*B\}$.

$$ {}^*G \in {}^*\mathcal{E}_u \leftrightarrow (\exists\, {}^*Q \in {}^*\mathcal{Q}(G))\,((\forall\, b \in {}^*B)\,(b \in {}^*B({}^*Q))) $$

Now an ancient theorem of Euler asserts that a graph G is Eulerian if and only if the degree of every vertex of $G = \{X, B\}$ is an even natural number. Symbolically, this can be stated as follows.

$$ G \in \mathcal{E}_u \leftrightarrow (\forall\, x \in X)\,(d_x/2 \in \mathbb{N}) $$

Transferring this, we get the nonstandard version of this theorem of Euler:

$$ {}^*G \in {}^*\mathcal{E}_u \leftrightarrow (\forall\, x \in^* X)\,(d_x/2 \in {}^*\mathbb{N}) $$

10 Hamiltonian Graphs

In this section, it is assumed that each graph $G = \{X, B\}$ is connected and finite and has at least three vertices (i.e., $|X| \geq 3$). A graph G is called *Hamiltonian* if it contains a loop that meets every vertex in the graph. Let $\mathcal{L}(G)$ denote the set of all loops in G. Also, for any loop $L \in \mathcal{L}(G)$, let $X(L)$ denote the vertex set of L. Then, a Hamiltonian graph $G = \{X, B\} \in \mathcal{C}_f$ is also defined by the truth of the following sentence to the right of $\leftrightarrow$. The set of Hamiltonian graphs will be denoted by $\mathcal{H}$, where $\mathcal{H} \subset \mathcal{C}_f$.

$$ G \in \mathcal{H} \leftrightarrow (\exists\, L \in \mathcal{L}(G))\,((\forall\, x \in X)\,(x \in X(L))) $$

By transfer of this sentence, we define a *nonstandard Hamiltonian graph* as follows, where now ${}^*\mathcal{H}$ is the set of nonstandard hyperfinite Hamiltonian graphs, $\mathcal{L}({}^*G)$ is the set of all nonstandard loops in ${}^*G$, ${}^*X((L)$ is the set of nonstandard vertices in $L \in \mathcal{L}({}^*G)$, and ${}^*G = \{{}^*X, {}^*B\}$.

$$ {}^*G \in {}^*\mathcal{H} \leftrightarrow (\exists\, L \in \mathcal{L}({}^*G))\,((\forall\, x \in {}^*X)\,(x \in {}^*X(L))). $$

A simple criterion for a graph $G = \{X, B\}$ to be Hamiltonian is that the degree d_x of each of its vertices x be no less than one half of $|X|$ [1, page 134], [3, page 79]. Symbolically, this condition is expressed as follows:

$$ ((\forall\, x \in X)\,(d_x \geq |X|/2)) \rightarrow \mathcal{G} \in \mathcal{H}. $$

By transfer, we get the following criterion for a nonstandard Hamiltonian graph.

$$((\forall\, x \in {}^*X)\,(d_x \geq |{}^*X|/2)) \;\rightarrow\; {}^*\mathcal{G} \in {}^*\mathcal{H}.$$

A more general criterion due to Ore asserts that $G = \{X, B\}$ is Hamiltonian if, for every pair of nonadjacent vertices x and y, $d_x + d_y \geq |X|$ [1, page 134], [3, page 79]. Symbolically, we have

$$((\forall\, x, y \in X)\,((\neg(x \diamond y)) \rightarrow (d_x + d_y \geq |X|))) \;\rightarrow\; G \in \mathcal{H},$$

which by transfer becomes

$$((\forall\, x, y \in^* X)\,((\neg(x \diamond y)) \rightarrow (d_x + d_y \geq |{}^*X|))) \;\rightarrow\; {}^*G \in {}^*\mathcal{H}.$$

Still more general is Posa's theorem [1, page 132], [3, page 79]: If, for every $j \in \boldsymbol{N}$ satisfying $1 \leq j < |X|/2$, the number of vertices of degree no larger than j is less than j, then the graph $G = \{X, B\}$ is Hamiltonian. The following symbolic sentence states this criterion.

$$((\forall\, j \in \boldsymbol{N})\,(\forall\, x \in X)\,((1 \leq j < |X|/2) \rightarrow (|\{x \in X : d_x \leq j\}| < j))) \;\rightarrow\; G \in \mathcal{H}$$

By transfer the following criterion holds for nonstandard graphs ${}^*G = \{{}^*X, {}^*B\}$.

$$((\forall\, j \in {}^*\boldsymbol{N})\,(\forall\, x \in {}^*X)\,((1 \leq j < |{}^*X|/2) \rightarrow (|\{x \in {}^*X : d_x \leq j\}| < j))) \;\rightarrow\; {}^*G \in {}^*\mathcal{H}$$

11 A Coloring Theorem

A simple graph-coloring theorem that is not restricted to planar graphs asserts that, if the largest of the degrees for the vertices of a graph $G = \{X, B\}$ is k, then the graph is $(k+1)$-colorable [6, page 82].[2] To express this symbolically, first let $M(X, \boldsymbol{N}_{k+1})$ denote the set of all functions that map a set X into the set $\boldsymbol{N}_{k+1}$ of those natural numbers j satisfying $1 \leq j \leq k+1$. Then, the following restates this theorem for the given graph G.

$$(\exists\, k \in \boldsymbol{N})\,(\forall\, x, y \in X)$$

[2] There exists a function f that assigns to each vertex one of $k+1$ colors such that no two adjacent vertices have the same color.

$$((d_x \leq k) \;\rightarrow\; ((\exists\, f \in M(X, \boldsymbol{N}_{k+1}))\,((x \diamond y) \rightarrow (f(x) \neq f(y)))))$$

To transfer this, we first let $^*M(^*X, \boldsymbol{N}_{k+1})$ be the set of all internal functions mapping the enlargement $^*X$ into $\boldsymbol{N}_{k+1}$. Then, this theorem is transferred to nonstandard graphs simply by appending asterisks, as usual:

$$(\exists\, k \in \boldsymbol{N})\,(\forall\, x, y \in {}^*X)$$

$$((d_x \leq k) \;\rightarrow\; ((\exists\, {}^*f \in {}^*M(^*X, \boldsymbol{N}_{k+1}))\,((x \diamond y) \rightarrow ({}^*f(x) \neq {}^*f(y)))))$$

Note here that the assumption of a natural-number bound k on the degrees of all the nonstandard vertices has been maintained. This conforms to the fact that the enlargement of the finite set $\boldsymbol{N}_{k+1}$ is $\boldsymbol{N}_{k+1}$. As a consequence, the conclusion remains strong.

On the other hand, we could generalize this transferred theorem as follows: In terms of an ultrapower construction, we could replace $\boldsymbol{N}_{k+1}$ by an internal set $^*\boldsymbol{N}_{k+1}$ obtained from a sequence $\langle \boldsymbol{N}_{k_n+1} : n \in \boldsymbol{N} \rangle$ of finite sets $\boldsymbol{N}_{k_n+1}$, one set for each G_n with regard to $^*G = [G_n]$ and with k_n being the maximum vertex degree in G_n. But then, our conclusion would be weakened to a coloring with a hypernatural number $^*k = [k_n]$ of colors.

12 A Final Comment

Undoubtedly, other standard results for graphs can be lifted in this way to nonstandard settings.

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