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THE NUMERICAL EVALUATION OF DISTRIBUTIONAL TRANSFORMS

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1. Introduction.

In this work t is a one-dimensional real variable, and all distributions are of one dimension. It is a well-known fact that the set of all finite linear combinations of shifted delta functionals is dense in the space D' of all Schwartz distributions [1; Vol. I, p. 75]. More specifically, let D denote the space of all infinitely differentiable functions of compact support supplied with its customary topology [1; Vol. I, p. 64]. Then, any given distribution f can be approximated by a distribution of the form

$$f_M(t) = \sum_{m=1}^{M} a_m \, \delta(t - t_m), \tag{1}$$

where the a_m are constants and $\delta(t - t_m)$ denotes the delta functional concentrated on the point $t = t_m$. In particular, given an $\epsilon > 0$ and a bounded set B of D, the integer M and the a_m and t_m can be so chosen that

$$\sup_{\varphi \in B} |< f - f_M, \varphi >| < \epsilon.$$

(See [1; Vol. I, pp. 71 and 75].)

It is also a fact that, if f has a compact support, it can be approximated by

$$f_{M,q}(t) = \sum_{m=1}^{M} a_m \, \delta^{(q-2)}(t - t_m), \tag{2}$$

where $\delta^{(q-2)}$ denotes the (q-2)th-order distributional derivative of δ and the integer $q \geq 2$ is such that the q th-order primitive of f is a continuous function [2; p. 145].

The objectives of this work are to develop explicit methods for constructing the approximations (1) and (2) and then to apply them to the numerical evaluation of various distributional transforms. There are a number of distributional transforms for which an evaluation can be immediately made once the approximations (1) and (2) have been constructed. Indeed, let us assume that the support of the distribution f is a compact subset of the open interval I. Then, its transform F(s) with respect to some kernel K(s,t) is given by

$$F(s) = \langle f(t), K(s,t) \rangle. \tag{3}$$

Here, s is in general a complex variable restricted to some domain Ω in the s plane. We shall always assume that, for each $s \in \Omega$, K(s,t) has continuous derivatives of all orders with respect to t on the interval I. Excluding the Fourier and Laplace transformations, we list below a number of integral transformations which possess extensions to generalized functions with inversion formulas and take on the form (3) when the support of f is a compact subset of Ω. In each case, we also indicate the corresponding interval I for t and the domain Ω for s.

The Mellin Transformation [3]:

$$F(s) = \langle f(t), t^{s-1} \rangle$$

$I = (0,\infty)$; Ω is the s-plane.

The Hankel Transformation [4]:

$$F(s) = \langle f(t), \sqrt{st}\, J_\mu(st) \rangle$$

J_μ is the Bessel function of first kind and order μ $(-\frac{1}{2} \leq \mu < \infty)$; $I = (0,\infty)$; Ω is the real positive axis in the s plane.

The K Transformation [5]:

$$F(s) = \langle f(t), \sqrt{st}\, K_\mu(st) \rangle$$

K_μ is the modified Bessel function of third kind and order μ $(-\frac{1}{2} \leq \mathrm{Re}\,\mu \leq \frac{1}{2})$; $I = (0,\infty)$; Ω is the s plane.

The Stieltjes Transformation [6; p. 43]:

$$F(s) = < f(t), \frac{1}{s+t} >$$

$I = (0,\infty)$; Ω is the s plane excluding the real nonpositive axis.

The Laguerre Transformation [7]:

$$F(s) = < f(t), \left[\frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}\right]^{1/2} t^{\alpha/2} e^{-t/2} L_s^{\alpha}(t) >$$

L_s^{α} are the generalized Laguerre polynomials of order α ($\alpha > -1$) [8; p. 188]; $I = (0,\infty)$; Γ is the gamma function; Ω is the set of all nonnegative integers.

The Hermite Transformation [7]:

$$F(s) = < f(t), \frac{e^{-t^2/2} H_s(t)}{[2^s(s!)\sqrt{\pi}]^{1/2}} >$$

$H_s(t)$ are the Hermite polynomials [8; p. 192]; $I = (-\infty,\infty)$; Ω is the set of all nonnegative integers.

The Jacobi Transformation [7]:

$$F(s) = < f(t), \sqrt{\frac{w(t)}{h(s)}}\, P_s^{(\alpha,\beta)}(t) >$$

Here, α and β are real and greater than -1, and

$$w(t) = (1-t)^{\alpha}(1+t)^{\beta},$$

$$h(s) = \frac{2^{\alpha+\beta+1}\,\Gamma(s+\alpha+1)\,\Gamma(s+\beta+1)}{s!\,(2s+\alpha+\beta+1)\,\Gamma(s+\alpha+\beta+1)}.$$

$P_s^{(\alpha,\beta)}$ denotes the Jacobi polynomials [8; p. 168]; $I = (-1,1)$; Ω is the set of all nonnegative integers. The Jacobi transformation contains a number of special cases: the Legendre transformation ($\alpha = \beta = 0$), the Chebyshev transformation ($\alpha = \beta = -\frac{1}{2}$), and the Gegenbauer (or ultraspherical) transformation ($\alpha = \beta$).

The Convolution Transformation [9],[10]:

$$F(s) = < f(t), G(s-t) >$$

G is an entire function in the Hirschman-Widder classes I, II, or III; [11; p. 120]; $I = (-\infty,\infty)$; Ω is the s plane. A number of classical integral transformations are, through changes of variables, special cases of the convolution transformation [11].

The Weierstrass Transformation [12]:

$$F(s) = < f(t), \frac{1}{\sqrt{4\pi}} e^{-(s-\tau)^2/4} >$$

$I = (-\infty,\infty)$; Ω is the s plane.

The reason for excluding the Fourier and Laplace transformations in the foregoing list is that our approximation techniques will be based on the assumption that the Fourier or Laplace transform of f can be determined. Our methods use these transforms to construct the approximations (1) and (2), which when substituted into (3) immediately yield the desired approximate expressions for any of the above transforms of f. The justification for this procedure lies on the fact that fairly extensive tables of Fourier and Laplace transforms of distributions already exist [13], [14], whereas very few distributional transform formulas, if any at all, have been developed for the other transformations. (It should be noted that a table containing 56 formulas has been compiled for the distributional Mellin transformation [15].)

The assumption that the distribution f has a compact support with respect to the open interval $I = (\alpha,\beta)$ can be relaxed in the following way. (Here, α may be $-\infty$ and β may be $+\infty$.) Assume that the distribution f on I is such that f(t) is a Lebesque integrable function on finite subintervals of $\alpha < t < a_1$ and $b_1 < t < \beta$ $(a_1 < b_1)$, and, for each $s \in \Omega$, $f(t)K(s,t)$ is absolutely integrable on $\alpha < t < a_1$ and $b_1 < t < \beta$. Then, (3) can be decomposed into

$$F(s) = \left(\int_{\alpha}^{a_2} + \int_{b_2}^{\beta}\right) f(t)K(s,t)dt + < f_1(t),K(s,t) >, \qquad (4)$$

where $\alpha < a_2 < a_1$, $b_1 < b_2 < \beta$, $f_1(t) = f(t)$ for $a_2 \leq t \leq b_2$ and $f_1(t) = 0$ elsewhere. The last term in (4) can be evaluated by the methods described herein. The first term on the right-hand side of (4) can be evaluated by methods designed for such ordinary transforms (see, for example, [16]); or, better still, it can be made small enough to be neglected by choosing a_2 sufficiently close to α and b_2 sufficiently close to β.

By a smooth function we shall mean a function that possesses ordinary derivatives of all orders at all points of its domain. D_I denotes the space of smooth functions whose supports are compact subsets of I. D_I has the topology that makes its dual D_I' the space of Schwartz distributions on I [1; Vol. I, p. 65]. E_I and E_I' are respectively the space of smooth functions on I and the dual space of distributions having compact supports with respect to I [1; Vol. I, pp. 88-90]. Let T be the length of some finite open interval (a,b) (i.e., T = b - a). P_T is the space of smooth periodic functions on $(-\infty,\infty)$ having a period T; P_T' is the dual space of periodic distributions on $(-\infty,\infty)$ of period T [2; Chapter 11]. The number that $f \in E_I'$ or D_I' assigns to $\varphi \in E_I$ or D_I, respectively, will be denoted by $< f,\varphi >$, whereas the number that $p \in P_T'$ assigns to $\theta \in P_T$ will be denoted by $p \cdot \theta$.

Let Ω be an open set on either the real line or the complex plane. By saying that Ξ is a compact subset of Ω, we mean that Ξ is compact in itself (i.e., Ξ is a closed bounded subset of Ω). We denote the nth distributional derivative of a function or distribution f by $f^{(n)}$. The support of f is denoted by supp f.

2. Approximation by Sums of Delta Functionals.

Our first method of approximation is summarized by

Theorem 1: *Let $f \in E_I'$, where now I = (a,b) is chosen as a finite open interval. With T = b - a, $w = 2\pi/T$, and r being some number in the interval $0 < r < 1$, set*

$$p_{n,r}(t) = \frac{1}{T} \sum_{k=-n}^{n} r^{|k|} e^{iwk\,t}\, \tilde{f}(wk), \tag{5}$$

where $\tilde{f}$ *is the Fourier transform of* f,

$$\tilde{f}(wk) = \langle f(t), e^{-iwkt} \rangle.$$

Also, set

$$f_{n,r,m}(t) = \frac{T}{m} \sum_{\nu=0}^{m-1} p_{n,r}(t_\nu)\, \lambda(t_\nu)\, \delta(t - t_\nu), \tag{6}$$

where

$$t_\nu = a + \frac{T}{m}(\nu + \tfrac{1}{2}).$$

and $\lambda(t) \in D_I$ *is such that* $0 \leq \lambda(t) \leq 1$ *on* I *and* $\lambda(t) = 1$ *on a neighborhood of the support of* f.

Then, in the sense of weak convergence in E_I',

$$\lim_{m\to\infty} \lim_{r\to 1-} \lim_{n\to\infty} f_{n,r,m}(t) = f(t).$$

Note that the Fourier transform $\tilde{f}$ is a smooth function (indeed, an entire function) because f has a bounded support [2; p. 227].

Proof: Let $p(t)$ denote the periodic extension with period T of $f(t)$; that is, $p(t) = f(t)$ on $a < t < b$, $p(t) = 0$ on sufficiently small neighborhoods of $t = a$ and $t = b$, and $p(t) = p(t + nT)$ for every integer n. Also, let $\lambda \in D_I$ be such that $0 \leq \lambda(t) \leq 1$ on I and $\lambda(t) = 1$ on a neighborhood of the support of f. For $\varphi \in E_I$ the smooth periodic extension with period T of $\lambda\varphi$ is denoted by $(\lambda\varphi)_T$. Hence,

$$\langle f,\varphi \rangle = \langle f,\lambda\varphi \rangle = p\cdot(\lambda\varphi)_T. \tag{7}$$

(See [2; Sec. 11.3].)

Next, we shall have need of

Lemma 1: For $0 < r < 1$, set

$$A_{n,r}(t) = \frac{1}{T} \sum_{k=-n}^{n} r^{|k|} e^{iwkt}$$

Then, in the sense of convergence in P_T' [2; p. 320],

$$\lim_{r\to 1-}\ \lim_{n\to\infty}\ A_{n,r}(t) = \delta_T(t) .$$

Proof: $A_{n,r}(t)$ is the r th-order Abel mean of the n th partial sum of the Dirichlet kernel. Hence, this becomes a standard result where we apply $A_{n,r}(t)$ to any $\theta \in P_T$.

Now, $p_{n,r}(t)$ can be written as the T-convolution [2; Sec. 11.4] of p and $A_{n,r}$. Indeed, letting Δ denote T-convolution, we have

$p(t)\ \Delta\ A_{n,r}(t) = p(x)\cdot A_{n,r}(t-x) = \frac{1}{T}\sum_{k=-n}^{n} r^{|k|} e^{iwkt}\ [p(x)\cdot e^{-iwkx}]$, which is the same as (5) in view of (7) and the fact that

$$p(x)\cdot e^{-iwkx} = p(x)\cdot[\lambda(x)\ e^{-iwkx}]_T.$$

Moreover, again in the sense of convergence in P_T' we have from lemma 1 and the continuity of T-convolution [2; theorem 11.4-4] that

$$\lim_{r\to 1-}\ \lim_{n\to\infty}\ p_{n,r}(t) = \lim_{r\to 1-}\ \lim_{n\to\infty}\ p(t)\ \Delta\ A_{n,r}(t) = p(t)\ \Delta\ \delta_T(t) = p(t).$$

Here, δ_T denotes the periodic extension of the delta functional.

Next, let

$$p_{n,r,m}(t) = \frac{T}{m}\sum_{\nu=0}^{m-1} p_{n,r}[a + \frac{T}{m}(\nu + \tfrac{1}{2})]\ \delta_T[t - a - \frac{T}{m}(\nu + \tfrac{1}{2})].$$

Since $p_{n,r}$ is smooth everywhere, it immediately follows that $p_{n,r,m} \to p_{n,r}$ in P_T' as $m \to \infty$. [This technique of getting a delta-sum approximation by forming a Riemann-sum approximation to $p_{n,r}\cdot\theta$ ($\theta \in P_T$) was suggested by Wohlers and Beltrami [17].]

Altogether then, we have shown that for any $\varphi \in E_I$

$$< f_{n,r,m}, \varphi > = p_{n,r,m} \cdot (\lambda\varphi)_T$$

$$\rightarrow p_{n,r} \cdot (\lambda\varphi)_T \qquad (m \rightarrow \infty) \tag{8}$$

$$\rightarrow p \cdot (\lambda\varphi)_T = < f,\varphi > \qquad (n \rightarrow \infty,\ r \rightarrow 1-).$$

Q.E.D.

Our approximation for f is $f_{n,r,m}$. As is indicated in the proof, we developed $f_{n,r,m}$ in two steps: first, by constructing the smooth approximation $p_{n,r}$ to the periodic extension of f and then by forming the delta functional sum $p_{n,r,m}$ arising from the Riemann sum for the integral on I of $p_{n,r}\ \theta$ $(\theta \in P_T)$. It may be interesting to note that if the distribution f is a continuous function on some open subinterval J of I, then $p_{n,r}$ converges uniformly to f on compact subsets of J. This is proven in the appendix.

The proof of theorem 1 also establishes

Corollary 1a: Under the hypothesis of theorem 1 with however r set equal to 1,

$$\lim_{m\to\infty} \lim_{n\to\infty} f_{n,1,m}(t) = f(t)$$

in the sense of weak convergence in E_I'

Actually, the parameter r was introduced in (5) to smooth out any unnecessary Gibbs-phenomena oscillations arising in the approximation. It is usually more convenient to use $f_{n,r,m}$ $(0 < r < 1)$ rather than $f_{n,1,m}$. On the other hand, the use of this r th order Abel mean makes the convergence of $f_{n,r,m}$ to f very slow. However, if the order of f is known, then a Cesaro mean of sufficiently high order could instead be used to accomplish this smoothing and obtain a higher rate of convergence.

3. Numerical Evaluation of Distributional Transforms: First Method.

Theorem 1 and its corollary provide a means for numerically evaluating the distributional transform (3). Indeed, for every s in the domain set Ω, $K(s,t)$ is in E_I, and therefore

$$\lim_{m\to\infty} \lim_{r\to 1-} \lim_{n\to\infty} F_{n,r,m}(s) = F(s), \tag{9}$$

where

$$F_{n,r,m}(s) = < f_{n,r,m}(t),\ K(s,t) > = \frac{T}{m} \sum_{\nu=0}^{m-1} p_{n,r}(t_\nu)\lambda(t_\nu)K(s,t_\nu) \tag{10}$$

and

$$t_\nu = a + \frac{T}{m}(\nu + \tfrac{1}{2}).$$

We shall give an example of the use of (9) subsequently. Let us merely note at this point that the approximate transform (10) can be written down immediately once the Fourier transform $\tilde{f}$ is known.

Theorem 1 asserts merely the pointwise convergence of (9) for $s \in \Omega$. However, more is true:

Theorem 2: *If Ξ is a subset of the open s-domain Ω such that, for each non-negative integer μ,*

$$\sup_{\substack{t\in \operatorname{supp} \lambda \\ s\in \Xi}} \left| \frac{\partial^\mu}{\partial t^\mu} K(s,t)\right| < B_\mu, \tag{11}$$

when B_μ is a constant depending on Ξ and the choice of $\lambda \in D_I$, then

$$\lim_{m\to\infty} \lim_{r\to 1-} \lim_{n\to\infty} F_{n,r,m}(s) = F(s)$$

uniformly on Ξ.

Note that the hypothesis is automatically satisfied if Ξ is a compact subset of Ω and $K(s,t)$ is an analytic function of $s \in \Omega$.

Proof:

$$F_{n,r,m}(s) = < \frac{T}{m} \sum_{\nu=0}^{m-1} p_{n,r}(t_\nu)\ \delta(t - t_\nu),\ \lambda(t)K(s,t) >$$

$$\to < f(t),\ \lambda(t)\ K(s,t) > = F(s) \text{ as } n\to\infty,\ r\to 1-,\ m\to\infty.$$

Now, weak convergence in E_I' implies weak convergence in D_I', which in turn implies strong convergence in D_I' [1; Vol. I, p. 71 and p. 74]. But, if λ is fixed and the parameter s varies throughout Ξ, then the $\lambda(t)\ K(s,t)$ comprise a bounded set in D_I. Thus, theorem 2 follows directly from the definition of strong convergence in D_I'.

Let us now estimate a bound on the error between the exact transform F(s) and the approximate transform (10). First of all, we have that

$$\int_a^b p_{n,r}(t)\ \lambda(t)\ K(s,t)dt$$

$$= \int_a^b < f(\tau),\ \frac{1}{T} \sum_{k=-n}^{n} r^{|k|} e^{iwk(t-\tau)} > \lambda(t)\ K(s,t)dt$$

Since $f \in E_I'$ and $\lambda K \in E_I'$ for fixed s, we may interchange the integration and "inner product" [2; pp. 124-125]. Then, replacing τ by t and t by x, we get

$$< f(t),\ \sum_{k=-n}^{n} r^{|k|} e^{-iwkt} \frac{1}{T} \int_a^b \lambda(x)\ K(s,x) e^{iwkx} dx >.$$

The testing function inside the last expression is the r th-order Abel mean of the n th partial sum of the Fourier series for $\lambda(t)\ K(s,t)$, s being fixed.

Now the approximation (6) to f was obtained in two steps; see (8). Hence, let us first compute a bound on the error $E_1(s)$ between the exact transform F(s) and the transform arising from $p_{n,r}$.

Let

$$c_k(s) = \frac{1}{T} \int_a^b \lambda(x)\ K(s,x)\ e^{iwkx} dx. \tag{12}$$

Then

$$E_1(s) = < f(t),\ \lambda(t)\ K(s,t) > - \int_a^b p_{n,r}(t)\ \lambda(t)\ K(s,t)\ dt$$

$$= < f(t),\ \sum_{k=-n}^{n} (1-r^{|k|}) e^{-iwkt}\ c_k(s) + \sum_{|k|>n} e^{-iwkt}\ c_k(s) > \tag{13}$$

Since $\lambda(x)\ K(s,x)$ is a smooth function with a compact support contained in

$a < x < b$, we may integrate (12) by parts any number of times to get

$$c_k(s) = \frac{1}{T}\int_a^b (iwk)^{-n} e^{iwkx} \frac{\partial^n}{\partial x^n}[\lambda(x)\ K(s,x)]dx \qquad (14)$$
$$(n=0,1,2,\ldots)$$

This shows that the Fourier coefficients $c_k(s)$ are of rapid descent as $|k| \to \infty$ (i.e., $c_k(s) = 0\,(|k|^{-n})$ for every n). This also implies that we may differentiate the series inside the right-hand side of (13) with respect to t term by term as often as we wish.

Let $\zeta = \xi + iw$ be the independent variable of the Laplace transform of f(t).

$$\angle(\zeta) = < f(t),\ e^{-\zeta t} > = \tilde{f}(-i\zeta).$$

We have assumed that $\angle(\zeta)$ is a known (entire) function. Choose the integer $q > 0$ such that on some strip $\xi_1 < \mathrm{Re}\ \xi < \xi_2$ $(0 < \xi_1 < \xi_2 < \infty)$ we have $|\zeta^{-q+2}\ \angle(\zeta)| < A$ where A is some constant. (We could also choose this strip inside the left-half ζ plane, or even containing the iw axis. In either case, the following analysis with certain modifications would still apply.) It now follows that $f = h^{(q)}$ where h is the continuous function

$$h(t) = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} \zeta^{-q}\ \angle(\zeta)\ e^{\zeta t} d\zeta \qquad (15)$$

and ξ is fixed with $\xi_1 < \xi < \xi_2$ (see [2; Sec. 8.4]). Hence, (13) becomes

$$E_1(s) = < h(t),\ (-1)^q \frac{\partial^q}{\partial t^q} \{\lambda(t)[\sum_{k=-n}^{n} (1-r^{|k|})\ e^{-iwkt}\ c_k(s) + \sum_{|k|>n} e^{iwkt}\ c_k(s)]\} >$$

$$= < h(t),\ (-1)^q \sum_{\mu=0}^{q} \binom{q}{\mu}\ \lambda^{(q-\mu)}(t)[\sum_{\substack{k=-n\\k\neq 0}}^{n} (1-r^{|k|})(-iwk)^{\mu} e^{-iwkt} c_k(s) +$$

$$\sum_{|k|>n} (-iwk)^{\mu} e^{-iwkt} c_k(s)\ ] > \quad ,$$

Let $n = \mu + 2$ in (14). Then,

$$|c_k(s)| \leq B(s,\mu)\ |wk|^{-\mu-2}$$

where

$$B(s,\mu) = \sup_{x \in I} \left| \frac{\partial^{\mu+2}}{\partial x^{\mu+2}} [\lambda(x)\ K(s,x)] \right| \tag{16}$$

Therefore,

$$|E_1(s)| \leq \sum_{\mu=0}^{q} \binom{q}{\mu} B(s,\mu) \Big[\sum_{\substack{k=-n \\ k \neq 0}}^{n} \frac{1-r^{|k|}}{|wk|^2} + \sum_{|k|>n} |wk|^{-2} \Big] \int_a^b |h(t)\ \lambda^{(q-\mu)}(t)|dt$$

From (15) we have, for $t \in \operatorname{supp} \lambda$,

$$|h(t)| \leq \frac{1}{2\pi} \int_{\xi-i\infty}^{\xi+i\infty} \frac{A}{|\zeta^2|} e^{\xi t} |d\zeta| < \frac{Ae^{\xi b}}{2\xi}$$

Consequently,

$$|E_1(s)| \leq \frac{T^3 Ae^{\xi b}}{4\xi\pi^2} \Big[\sum_{k=1}^{n} \frac{1-r^k}{k^2} + \sum_{k=n+1}^{\infty} k^{-2} \Big] \sum_{\mu=0}^{q} \binom{q}{\mu} B(s,\mu)\ \sup_t |\lambda^{(q-\mu)}(t)|. \tag{17}$$

The right-hand side of (17) is the bound on $|E_1(s)|$ that we seek. Note that it can be made arbitrarily small by first choosing n sufficiently large and then choosing r sufficiently close to 1. Also note that on a subset Ξ' of Ω, as described in theorem 2, $B(s,\mu)$ can be replaced by a constant that is independent of s. Hence, the error $E_1(s)$ converges uniformly to zero on Ξ'.

Next, we compute a bound on the error $E_2(s)$ between the transform arising from $p_{n,r}$ and the approximate transform (10) (see (8)).

$$E_2(s) = \int_a^b p_{n,r}(t)\ \lambda(t) K(s,t) dt - < f_{n,r,m}(t),\ K(s,t) >$$

$$= \sum_{\nu=0}^{m-1} \int_{a+\frac{T\nu}{m}}^{a+\frac{T(\nu+1)}{m}} [p_{n,r}(t)\ \lambda(t)\ K(s,t) - p_{n,r}(t_\nu)\lambda(t_\nu)K(s,t_\nu)]dt$$

$$= \sum_{\nu=1}^{m=1} \int_{a+\frac{T\nu}{m}}^{a+\frac{T(\nu+1)}{m}} dt \int_{t_\nu}^{t} \frac{\partial}{\partial y} [p_{n,r}(y)\ \lambda(y)K(s,y)]dy$$

where, as before,

$$t_\nu = \alpha + \frac{T}{m}(\nu+\tfrac{1}{2}).$$

Let

$$D(s) = \sup_{t\in I} \left|\frac{\partial}{\partial t} [p_{n,r}(t)\lambda(t)K(s,t)]\right|.$$

$$= \sup_{t\in I} \left|\lambda K \frac{\partial}{\partial t} p_{n,r} + p_{n,r} \frac{\partial}{\partial t} (\lambda K)\right|.$$

Then,

$$|E_2(s)| \le D(s) \sum_{\nu=0}^{m-1} \int_{a+\frac{T\nu}{m}}^{a+\frac{T(\nu+1)}{m}} |t - t_\nu| dt = \frac{T^2}{4m} D(s).$$

A bound on $D(s)$ can be obtained as follows. From (5) we have that

$$|p_{n,r}(t)| \le \frac{1}{T} \sum_{k=-n}^{n} r^{|k|} |\tilde{f}(wk)| = N(r,n)$$

$$\left|\frac{d}{dt} p_{n,r}(t)\right| \le \frac{2\pi}{T^2} \sum_{k=-n}^{n} |k|\ r^{|k|} |\tilde{f}(wk)| = M(r,n)$$

Therefore, using (16) as the definition of $B(s,\mu)$ when $\mu = -1$ and $\mu = -2$, we obtain

$$D(s) \le M(r,n)\ B(s,-2) + N(r,n)\ B(s,-1).$$

Consequently,

$$|E_2(s)| \le \frac{T^2}{4m} [M(r,n)\ B(s,-2) + N(r,n)\ B(s,-1)] \tag{18}$$

Here again, on a subset Ξ, as specified in theorem 2, $B(s,\mu)$ can be replaced by a constant that is independent of s, so that, for fixed r and n, $E_2(s)$ converges to zero as $m \to \infty$ uniformly on Ξ.

By adding the right-hand sides of (17) and (18), we obtain the desired bound on the error between the exact transform (9) and the opposite transform (10). This bound is very weak, and the actual error between (9) and (10) will in general be much less. Moreover, because of the uniform convergence of $E_1(s)$ to zero on Ξ, we have here another proof of theorem 2.

4. Approximations by Sums of Derivatives of Delta Functionals.

The second method of approximation is based upon a technique, which was developed for ordinary functions [16]. Assuming once again that the Laplace transform

$$\angle(\zeta) = < f(t), e^{-\zeta t} > = \tilde{f}(-i\zeta) \qquad (\zeta = \xi + iw)$$

is known, we can represent $f \in E_I'$ by $f = h^{(q)}$; h is a continuous function given by (15), where ξ is some fixed positive number. The positive integer q is so chosen that, on some strip $\xi_1 < \mathrm{Re}\,\zeta < \xi_2$ with $0 < \xi_1 < \xi_2 < \infty$, we have as before

$$|\zeta^{-q+2} \angle(\zeta)| < A. \tag{19}$$

where A is some constant.

Now, if h(t) cannot be exactly determined as the inverse Laplace transform of $\zeta^{-q+2} \angle(\zeta)$ approximate it by the method of [16]. Assume that f is a real distribution. (If it isn't, we can apply the following analysis to its real and imaginary parts separately.) Let

$$(\xi + iw)^{-q} \angle(\xi + iw) = R(w) + iX(w).$$

Then, on the axis $0 < w < \infty$, approximate R(w) and X(w) by two polygonal arcs $R_A(w) + X_A(w)$, respectively, such that $R_A(w) + X_A(w)$ are identically zero for all sufficiently large w, and neither $R_A(w)$ now $X_A(w)$ have any vertical segments. (Since $X(0) = 0$, choose $X_A(0) = 0$ also.) Let $S_{R,k}$ ($k=1,\ldots,n,n+1$) be the slopes of the straight line segments in R_A, where the $(n+1)$th segment is a section of the w axis extending out to $w = \infty$. (Number the segments consecutively.) In a similar fashion, let $S_{X,k}$ $(k=1,\ldots,m,m+1)$ be the slopes of the straight line segments in X_A. Furthermore, let $w_{R,k}$ be the w coordinate of the vertex between the segments whose slopes are $S_{R,k}$ and $S_{R,k+1}$, and denote the corresponding w coordinates for X_A by $w_{X,k}$. Finally, set

$$A_k = S_{R,k+1} - S_{R,k}$$

$$B_k = S_{X,k+1} - S_{X,k}$$

Then, the approximation $h_1(t)$ to $h(t)$ is given by [16]

$$h_1(t) = \frac{e^{\xi t}}{\pi t^2} \left(- S_{R,1} - \sum_{k=1}^{n} A_k \cos w_k t + \sum_{k=1}^{m} B_k \sin w_k t\right) \tag{20}$$

and

$$h_1(0) = \frac{1}{2\pi} \sum_{k=1}^{n} A_k \, w_{R,k}^2 . \tag{21}$$

Moreover, the error between $h(t)$ and $h_1(t)$ is bounded as follows.

$$|h(t) - h_1(t)| \leq \frac{e^{\zeta t}}{\pi} \left[\int_0^\infty |R(w) - R_A(w)|\, dw + \int_0^\infty |X(w) - X_A(w)|\, dw\right] \tag{22}$$

In view of (19), $|h(t) - h_1(t)|$ can be made arbitrarily small uniformly on any finite t interval by choosing R_A sufficiently close to R and X_A sufficiently close to X.

Next, let $\lambda(t)$ be that member of D_I which was defined in the first paragraph of theorem 1. Let I_1 be the finite open interval whose endpoints are inf supp λ and sup supp λ. Approximate $h_1(t)$ on I_1 by a polygonal arx $h_2(t)$, where now $c_k = S_{k+1} - S_k$ $(k=1,2,\ldots,\ell)$, S_k being the slopes of the straight-line segments of h_2, and t_k $(k=1,2,\ldots,\ell)$ denote the t coordinates of the vertices, numbered consecutively. (In this case, $h_2(t)$ need not be identically zero on any interval of positive length.) Then the approximation to the distribution f is

$$f_A(t) = \sum_{k=1}^{\ell} c_k \lambda(t)\, \delta^{(q-2)}(t - t_k) \qquad (t_k \in I_1). \tag{23}$$

Note that

$$\lambda(t)\, \delta^{(q-2)}(t - t_k) = \sum_{\nu=0}^{q-2} (-1)^{\nu} \binom{q-2}{\nu} \lambda^{(\nu)}(t_k)\, \delta^{(q-2-\nu)}(t - t_k)$$

and that, if the point t_k is contained in that neighborhood of supp f on which $\lambda \equiv 1$, then

$$\lambda(t)\, \delta^{(q-2)}(t - t_k) = \delta^{(q-2)}(t - t_k). \tag{24}$$

Hence, if all the t_k are chosen in this neighborhood, the approximation (23) takes on a simpler form.

<u>Theorem 3</u>: f_A converges in the weak sense in E_I' to f as the quantity

$$\int_{I_1} |h(t) - h_1(t)|dt + \int_{I_1} |h_1(t) - h_2(t)|dt \tag{25}$$

converges to zero.

Note that (25) can be made as small as one wishes simply by choosing the straight line approximations R_A, X_A, and h_2 sufficiently close to R, X, and h_1, respectively.

<u>Proof</u>: For any $\varphi \in E_I$,

$$< f - f_A, \varphi > = < h^{(q)}(t) - \sum_{k=1}^{\ell} c_k\, \delta^{(q-2)}(t - t_k), \lambda(t)\varphi(t) >$$

$$= \int_{I_1} [h(t) - h_2(t)]\, (-1)^q \frac{d^q}{dt^q} [\lambda(t)\varphi(t)]dt$$

Therefore,

$$|< f - f_A, \varphi >| \leq \sup_{t \in I_1} \left| \frac{d^q}{dt^q} [\lambda(t)\varphi(t)] \right| \left[\int_{I_1} |h(t) - h_1(t)| dt + \int_{I_1} |h_1(t) - h_2(t)| dt \right]. \tag{26}$$

Q.E.D.

5. Numerical Evaluation of Distributional Transforms: Second Method.

In view of theorem 3, we can contruct the following approximate transform.

$$F_A(s) = < f_A(t), K(s,t) > = \sum_{k=1}^{\ell} c_k (-1)^q \frac{\partial^{q-2}}{\partial t^{q-2}} [\lambda(t)K(s,t)]\Big|_{t = t_k} \tag{27}$$

We are still assuming here that f is a real distribution. Otherwise, two sums such as (27) would arise, one from the real part and the other from the imaginary part of f.

Theorem 4: Let Ξ be a subset of Ω as described in theorem 2. Then, the approximate transform $F_A(s)$ converges to the exact transform $F(s)$ uniformly on Ξ as the term (25) converges to zero.

Proof: This theorem follows from (26) by replacing $\varphi(t)$ by $K(s,t)$ and invoking (11).

Also note that with $\varphi(t)$ replaced by $K(s,t)$ the right-hand side of (26) becomes an error bound for $F(s) - F_A(s)$.

6. Comparison of the Two Methods.

The approximate transform (10) of the first method can be written down immediately once the Fourier transform $\tilde{f}$ of f is determined. In the second method however additional computations must be performed, after the Laplace transform $F(s)$ of f has been obtained, in order to determine the coefficients c_k and thereby the approximate transform (27). Indeed, if (15) cannot be determined exactly, then three separate straight line approximations have to be performed when computing the c_k.

On the other hand, the bound (16) on the error in the first method, which is the sum of (17) and (18), is quite a complicated expression and hence difficult to estimate. In contrast to this, the error in the second method is readily controlled by calculating the right-hand side of (22) and the first term in (25); these computations are easily made once the straight-line approximations have been constructed. This determines a bound on (26), which is our error bound when $\varphi(t)$ is replaced by $K(s,t)$.

Appendix

In this appendix we shall prove the following result.

If $f \in E_I'$ and if f is a continuous function on $x < t < y$ where $a < x < y < b$, then

$$\lim_{r \to 1-} \lim_{n \to \infty} p_{n,r}(t) = f(t)$$

where the convergence is uniform on every compact subset of (x,y).

The proof proceeds by means of several lemmas.

Lemma 1: For any fixed r in the range $0 < r < 1$, $A_{n,r}(t)$ converges in P_T to $A_{\infty,r}(t)$ as $n \to \infty$.

Proof: Denoting the j th derivative of $A_{n,r}$ by $A_{n,r}^{(j)}$, we may write

$$|A_{\infty,r}^{(j)}(t) - A_{n,r}^{(j)}(t)| \leq \frac{2}{T} \sum_{k=n+1}^{\infty} r^k (wk)^j$$

The right-hand side is independent of t and converges to zero as $n \to \infty$. Hence, $A_{n,r}^{(j)} \to A_{\infty,r}^{(j)}$ uniformly for all t. Q.E.D.

Lemma 2: For any fixed r in the range $0 < r < 1$, $p_{n,r}(t)$ converges to

$$p_r(t) = p(t) \Delta A_{\infty,r}(t) \in P_T$$

as $n \to \infty$ uniformly for all t.

Proof:

$$p_r - p_{nr} = p \Delta (A_{\infty,r} - A_{n,r}) = p(x) \cdot \frac{1}{T} \sum_{|k|>n} r^{|k|} e^{iwk(t-x)}$$

By lemma 1 we may interchange the summation with the "dot product" to get

$$p_r(t) - p_{n,r}(t) = \frac{1}{T} \sum_{|k|>n} r^{|k|} e^{iwkt}[p(x) \cdot e^{-iwkx}] = \sum_{|k|>n} r^{|k|} e^{iwkt} \tilde{p}(k)$$

where $\tilde{p}(k)$ denote the Fourier coefficients of p. Since the $\tilde{p}(k)$ comprise a sequence of slow growth [2; theorem 11.6-2],

$$|p_r(t) - p_{n,r}(t)| \leq \sum_{|k|>n} r^{|k|} |\widetilde{p}(k)|$$

and the right-hand side, which is independent of t, converges to zero as $n \to \infty$. Q.E.D.

Next, let μ and ζ be elements of D_T with the following properties. The support of ζ is contained in $x < t < y$ and $\zeta(t) = 1$ on a neighborhood of some (arbitrarily chosen) closed subinterval $c \leq t \leq d$ of (x,y) (i.e., $x < c < d < y$). Also, $\mu(t) + \zeta(t) = 1$ on some neighborhood of the support of f. Hence, we may write $p_r(t)$ in the form

$$p_r(t) = \int_x^y f(x)\, \zeta(x)\, A_{\infty,r}(t-x)dx + p(x) \cdot [\mu_T(x)\, A_{\infty,r}(t - x)], \qquad \text{(A-1)}$$

where μ_T denotes the periodic extension of μ. It is a standard result [18; p. 632] that

$$T\, A_{\infty,r}(t - x) = \frac{1-r^2}{1+r^2-2r \cos w(t-x)} \in P_T.$$

It is also a fact [18; p. 633] that the first term on the right-hand side of (A-1) converges to $f(t)\, \zeta(t)$ uniformly on $a \leq t \leq b$ as $r \to 1-$. (Here, $f(t)\, \zeta(t)$ is taken to be the identically zero function on $a \leq t \leq x$ and $y \leq t \leq b$.) Lemma 2 and this result indicate that our proof will be complete when we show that the second term on the right-hand side of (A-1) converges uniformly to zero on $c \leq t \leq d$ as $r \to 1-$.

Lemma 3: For each nonnegative interger ν, $A^{(\nu)}_{\infty,r}(x)$ converges to zero as $r \to 1-$ uniformly on every x-interval such that $\eta < |x| < T - \eta$ where $0 < \eta < T/4$.

Proof: Some computation shows that for $\nu \geq 1$

$$T\, A^{(\nu)}_{\infty,r}(x) = (1 - r^2)\left\{\frac{a_{\nu,\nu+1}(x)}{[1+r^2-2r \cos wx]^{\nu+1}} + \frac{a_{\nu,\nu}(x)}{[1+r^2-2r \cos wx]^{\nu}} + \cdots + \frac{a_{\nu,2}(x)}{[1+r^2-2r \cos wx]^2}\right. \qquad \text{(A-2)}$$

where each $a_{\nu,k}(x)$ a constant times some derivatives of $1 + r^2 - 2r \cos wx$. Since for $\frac{1}{2} < r \leq 1$ and $\eta \leq |x| \leq T - \eta$

$$1 - \cos w\eta \leq 2r(1 - \cos wx) \leq 1 + r^2 - 2r \cos wx = (1 - r)^2 + 2r(1 - \cos wx) < 2,$$

it follows that the quantity inside the braces in (A-2) is a smooth bounded function for all x, which implies lemma 3.

Finally, by a standard result [2; theorem 3.3-1], there exist a positive constant C and a nonnegative integer q such that

$$\begin{aligned} |p(x)\cdot[\mu_T(x)\, A_{\infty,r}(t - x)]| &= | < f(x), \mu(x)\, A_{\infty,r}(t - x) > | \\ &\leq C \sup_{x\in \operatorname{supp}\mu} \left| \frac{\partial^q}{\partial x^q} [\mu(x)\, A_{\infty,r}(t - x)]\right| . \end{aligned} \tag{A-3}$$

where supp μ denotes the support of μ. If x varies through supp μ and t varies through (c,d), there will be some fixed positive number $\eta < \frac{T}{4}$ such that $\eta < |t - x| < T - \eta$. Lemma 3 and (A - 3) now show that the second term on the right-hand side of (A-1) converges uniformly to zero on $c \leq t \leq d$ as $r \to 1-$. Hence, our original assertion has been completely established.

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