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A Superintegrable Time-Dependent System with Kac–Moody Symmetry*

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Abstract—We investigate the Hamiltonian $H_{\rm KL}$ with a time-dependent potential in N-dimensional space that is a special combination of a Kepler and a harmonic-oscillator potential. The corresponding classical system has an angular-momentum tensor and a time-dependent analog of the Laplace–Runge–Lenz vector, which commute with the "quasi-Hamiltonian" H_c. These quantities are conserved on the orbits of $H_{\rm KL}$, and their Poisson brackets yield a realization of twisted or untwisted centerless Kac–Moody algebras of $so(N+1)$. The corresponding quantum-mechanical operators and their commutators yield a representation of the positive subalgebras of the above Kac–Moody algebras. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

A classical N-dimensional Hamiltonian system described by a time-independent Hamiltonian $H(x_1,\ldots,x_N;p_1,\ldots,p_N)\equiv H(x;p)$ is called integrable if it has N integrals of motion $\{X_1,\ldots,X_N\}$ that are in involution; i.e., their Poisson brackets vanish:

$$\{X_i,X_j\}=0 \quad \text{for} \quad i,j=1,\ldots,N. \tag{1}$$

One of X_i can be chosen to be the Hamiltonian H. The system is called superintegrable if it has more than N integrals of the motion. It is called maximally superintegrable if it has $2N-1$ functionally independent quantities that Poisson commute with the Hamiltonian. Both Kepler [1–3] and harmonic-oscillator [4, 5] systems are maximally superintegrable. Superintegrable systems have been studied systematically since 1965 [6].

The importance of the integrability condition (1) becomes clearer when we quantize the Hamiltonian. The integrals of the motion become commuting operators $\{\hat{X}_1,\ldots,\hat{X}_N\}$, with

$$[\hat{X}_i,\hat{X}_j]=0 \quad \text{for} \quad i,j=1,\ldots,N. \tag{2}$$

The commutation relations (2) tell us that we can diagonalize all the N operators $\hat{X}_i$ simultaneously. Moreover, if one of $\hat{X}_i$ is the Hamiltonian, then (2) would enable us to obtain wave functions that are simultaneous eigenfunctions of all operators $\hat{X}_i$. For example, all three-dimensional Hamiltonians with central potentials are integrable, since the three operators $\{H,L^2,L_3\}$ commute with each other. This fact is the basis for the familiar partial-wave decomposition, where $\psi_{nlm}(\mathbf{r})=R_{nl}(r)Y_{lm}(\hat{\mathbf{r}})$.

There arises the question of whether the extension of the above definitions to time-dependent Hamiltonians $H(x;p;t)$ is useful and appropriate. In this case, the Hamiltonian itself will not be an integral of motion, so that, even if we can find N conserved quantities $\{X_1,\ldots,X_N\}$ that satisfy (2), the Hamiltonian will not be one of them. Consequently, the simultaneous eigenfunctions of the corresponding quantum operators $\{\hat{X}_1,\ldots,\hat{X}_N\}$ will not be wave functions (solutions to the Schrödinger equation). To gain familiarity with time-dependent integrable systems, we shall first review, in this paper, the classical time-dependent Katzin–Levine (KL) system [7] and then study its quantization. The KL system can be described by the time-dependent Hamiltonian

$$H_{\rm KL}=\frac{\mathbf{p}^2}{2m}+V_{\rm KL}(r,t)\,,\quad r\equiv\sqrt{x_1^2+\ \ldots\ +x_N^2}, \tag{3}$$

where

$$V_{\rm KL}(r,t)=-\frac{m}{2}\frac{\ddot{U}}{U}r^2-\frac{\alpha}{Ur}\quad [U(t)\neq 0] \tag{4}$$

is a central potential, which is a combination of Coulomb and isotropic harmonic-oscillator potentials. In Section 2, we review the KL model and show that it has the integrals of motion

$$L_{ij}=x_ip_j-x_jp_i,\quad i,j=1,\ldots,N, \tag{5}$$

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$$\boldsymbol{\xi}_{\rm KL} = \mathbf{K} - m\alpha\hat{\mathbf{r}}, \quad K_i := \sum_{j=1}^{N} L_{ij}\pi_j,$$

$$H_c(t) = \frac{\boldsymbol{\pi}^2(t)}{2m} - \frac{U(t)\alpha}{r}, \tag{6}$$

where

$$\boldsymbol{\pi}(t) = U(t)\mathbf{p} - m\dot{U}(t)\mathbf{r}. \tag{7}$$

Note that the scalar quantity H_c has dimensions of energy. We then show that the classical orbits of the KL model are "modulated conic sections." In Section 3, we demonstrate that the KL system is maximally superintegrable; this is done by showing that the Poisson brackets [see Eq. (21) below] among the above integrals (5) of motion are isomorphic to the corresponding quantities of the standard time-independent Coulomb Hamiltonian

$$H_1 = \frac{\mathbf{p}_1^2}{2m} - \frac{\alpha}{r_1}, \tag{8}$$

provided we map H_c into H_1. Note that, in (8), we use $\mathbf{r}_1$ and $\mathbf{p}_1$ to denote the canonical variables of the above standard Hamiltonian H_1 to distinguish them from the $\mathbf{r}$ and $\mathbf{p}$ describing the KL orbits. In Section 4, we quantize the KL system. In Section 5, we show that the quantized integrals of motion yield a time-dependent realization of the H algebras HI_N, which are positive subalgebras of infinite-dimensional affine Kac–Moody algebras of $so(N+1)$ [8, 9]. Finally, we give a summary in Section 6. In the Appendices, we give details of calculations involving classical and quantum quantities.

2. REVIEW OF THE KATZIN–LEVINE MODEL

2.1. The KL Force $\mathbf{F}_{KL}$

Applying Hamilton's equations to the KL Hamiltonian $H_{\rm KL}$, we get

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial H_{\rm KL}}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m}, \tag{9}$$

$$m\ddot{\mathbf{r}} = \dot{\mathbf{p}} = -\frac{\partial H_{\rm KL}}{\partial \mathbf{r}} = -{\rm grad}V \tag{10}$$

$$= \left(m\frac{\ddot{U}}{U} - \frac{1}{U}\frac{\alpha}{r^3}\right)\mathbf{r} =: \mathbf{F}_{\rm KL},$$

where $\mathbf{F}_{\rm KL}$ is the KL force [7].

2.2. Conservation of $\boldsymbol{\xi}_{KL}$

In this subsection, we show that the quantities $\{L_{ij},\ \boldsymbol{\xi}_{\rm KL},\ H_c\}$ in (5) are conserved on the orbits of the KL Hamiltonian $H_{\rm KL}$: clearly, the angular-momentum tensor L_{ij} is conserved, since the KL potential $V_{\rm KL}$ in (4) is central.

To prove that $\boldsymbol{\xi}_{\rm KL}$ is conserved, we use $\dot{L}_{ij} = 0$, so that

$$\dot{\xi}_i^{\rm KL} = \sum_{j=1}^{N} L_{ij}\dot{\pi}_j - m\alpha\frac{d}{dt}\left(\frac{x_i}{r}\right) \tag{11}$$

$$= \frac{\alpha}{r^3}(r^2 p_i - \mathbf{r}\cdot\mathbf{p}x_i) - m\alpha\left(\frac{\dot{x}_i}{r} - \frac{\dot{r}x_i}{r^2}\right) = 0,$$

where we have used the relation

$$\dot{\boldsymbol{\pi}} = \dot{U}(t)\mathbf{p} + U(t)\dot{\mathbf{p}} - m\ddot{U}(t)\mathbf{r} - m\dot{U}\mathbf{v} \tag{12}$$

$$= U(t)\dot{\mathbf{p}} - m\ddot{U}(t)\mathbf{r} = U\mathbf{F}_{\rm KL} - m\ddot{U}\mathbf{r} = -\frac{\alpha}{r^2}\,\hat{\mathbf{r}}.$$

Here, the last equality follows from (10). Note that, for the clarity of notation, we indicate "KL" as subscripts in general (for example, such as $\boldsymbol{\xi}_{\rm KL}$) and as superscripts on the components of vectors (for example, $\xi_i^{\rm KL}$).

Similarly, the conservation of H_c follows from (5), (7), and (12):

$$\frac{d}{dt}H_c(t) = \frac{\boldsymbol{\pi}\cdot\dot{\boldsymbol{\pi}}}{m} - \alpha\left(\frac{\dot{U}}{r} - \frac{U\dot{r}}{r^2}\right) = 0. \tag{13}$$

2.3. Classical Orbits as Modulated Conic Sections

The orbits for central forces in classical physics are always confined to a plane, irrespective of the dimension $N \geq 2$ of the space. This plane can be chosen to be the xy plane as follows: since $\boldsymbol{\xi}_{\rm KL}$ is conserved, we can choose the x axis in the direction of $\boldsymbol{\xi}_{\rm KL}$, i.e., $\hat{\mathbf{x}} = \hat{\boldsymbol{\xi}}_{\rm KL}$, and choose the y axis to lie in the plane of the motion. It follows that $L_{12} = x_1p_2 - x_2p_1$ is the only nonvanishing component of the angular-momentum tensor L_{ij}.

We now determine the orbit by evaluating the scalar product of the position vector $\mathbf{r}_{\rm KL}$ with $\boldsymbol{\xi}_{\rm KL}$; that is,

$$r_{\rm KL}\xi_{\rm KL}\cos\varphi_{\rm KL} \tag{14}$$

$$= \mathbf{r}_{\rm KL}\cdot\boldsymbol{\xi}_{\rm KL} = \sum_{i,j=1}^{N} x_i L_{ij}\pi_j - m\alpha r_{\rm KL}$$

$$= U(t)L^2 - m\alpha r_{\rm KL},$$

where $\varphi_{\rm KL}$ is the azimuthal angle ($\cos\varphi_{\rm KL} := \hat{\mathbf{r}}_{\rm KL}\cdot\hat{\mathbf{x}}$),

$$L^2 \equiv \sum_{i<j}^{N} L_{ij}^2 = \frac{1}{2}\sum_{i,j=1}^{N} L_{ij}^2 \tag{15}$$

$$= \sum_{i,j=1}^{N} L_{ij}x_i p_j = U^{-1}\sum_{i,j=1}^{N} L_{ij}x_i\pi_j,$$

and

$$\xi_{\rm KL} \equiv |\boldsymbol{\xi}_{\rm KL}| = \sqrt{\boldsymbol{\xi}_{\rm KL}\cdot\boldsymbol{\xi}_{\rm KL}} \tag{16}$$
$$= \sqrt{m^2\alpha^2 + \mathbf{K}^2 - 2m\alpha\hat{\mathbf{r}}\cdot\mathbf{K}}$$
$$= \sqrt{m^2\alpha^2 + 2m\left(\frac{\boldsymbol{\pi}^2}{2m} - \frac{U\alpha}{r}\right)L^2}$$
$$= \sqrt{m^2\alpha^2 + 2mH_cL^2}.$$

Here, we have used the equalities $\mathbf{K}^2 = \boldsymbol{\pi}^2L^2$ and $\mathbf{K}\cdot\mathbf{r} = -UL^2$, which are derived in Eqs. (A.6) and (A.7) of Appendix A. Equation (14) yields

$$r_{\rm KL}(t) = r_{\rm KL}(\varphi_{\rm KL}(t), t) \tag{17}$$
$$= \frac{U(t)L^2}{m\alpha + \xi_{\rm KL}\cos\varphi_{\rm KL}} =: U(t)r_1(\varphi_{\rm KL}(t)),$$

where

$$r_1(\varphi) = \frac{l}{\eta + \varepsilon\cos\varphi}, \tag{18}$$

$$l \equiv \frac{L^2}{m|\alpha|}, \quad \eta \equiv \frac{\alpha}{|\alpha|}, \tag{19}$$

$$\varepsilon \equiv \frac{\xi}{m|\alpha|} = \sqrt{1 + \frac{2L^2}{m\alpha^2}H_c} = \sqrt{1 + \frac{2l}{|\alpha|}H_c}. \tag{20}$$

We shall call the orbits (17) "modulated conic sections," since they are simple products of $U(t)$ and the standard expression $r_1 = r_1(\varphi)$ for conic sections, which yields ellipses, parabolas, and hyperbolas for $\varepsilon < 1, \varepsilon = 1$, and $\varepsilon > 1$ (corresponding to $H_c < 0$, $H_c = 0$, and $H_c > 0$), respectively [10, 11]. Note that the eccentricity ε in (20) is determined by the value of the constant of the motion H_c and not by the value of the Hamiltonian $H_{\rm KL}$ in (3)!

Note that the shape of the conic section, namely, $r_1 = r_1(\varphi_{\rm KL})$, is independent of the choice of $U(t)$. Nevertheless, $r_1^{\rm KL}(t) := r_1(\varphi_{\rm KL}(t))$ does depend on $U(t)$—indirectly via $\varphi_{\rm KL}(t)$.

3. SUPERINTEGRABILITY OF THE KATZIN–LEVINE SYSTEM

3.1. The Poisson Brackets among the Integrals of the Motion

In Subsection 2.2, we showed that L_{ij}, H_c, and $\boldsymbol{\xi}_{\rm KL}$ are integrals of the motion on the orbits of $H_{\rm KL}$. The Poisson brackets among these quantities are

$$\{H_c, L_{ij}\} = \{H_c, \xi_i^{\rm KL}\} = 0, \tag{21}$$
$$\{L_{ij}, L_{kl}\} = \delta_{ik}L_{jl} - \delta_{jk}L_{il} + \delta_{jl}L_{ik} - \delta_{il}L_{jk},$$
$$\{L_{ij}, \xi_k^{\rm KL}\} = \delta_{ik}\xi_j^{\rm KL} - \delta_{jk}\xi_i^{\rm KL},$$
$$\{\xi_i^{\rm KL}, \xi_j^{\rm KL}\} = -2mH_cL_{ij}.$$

The above Poisson commutation relations can be checked directly by using the definitions in (5) and (6). We see that the KL integrals of the motion in (5) have exactly the same Poisson brackets as the corresponding quantities of the standard system H_1, provided we map H_c (and not $H_{\rm KL}$) to H_1.

This isomorphism allows us to conclude that the KL system is also maximally superintegrable, since the standard N-dimensional Kepler system defined by H_1 in (8) is maximally superintegrable [3, 12].

In [13], we showed that the above Poisson brackets correspond to twisted or standard loop algebras of $so(N+1)$. In the next section, we shall use the corresponding commutators (27) to construct positive subalgebras of the above loop algebras.

4. QUANTIZING THE KL SYSTEM

If we quantize KL Hamiltonian (3), we get a time-dependent Schrödinger equation. Since the angular momentum is conserved, we can decompose the Schrödinger equation into partial-wave equations. Since the potential $V_{\rm KL}(r,t)$ depends on time, the partial-wave equations will be partial differential equations in two variables, r and t. These differential equations can be solved only numerically for the majority of the functions $U(t)$. Nevertheless, we can quantize the integrals of motion in (5) and calculate their commutators.

By quantizing L_{ij} and H_c in (5) (by replacing the p_i by $-i\hbar\partial_{x_i}$), we automatically get Hermitian operators. In contrast, $\mathbf{K}$ is not Hermitian, as we can see from

$$K_i^\dagger - K_i = \pi_k L_{ik} - L_{ik}\pi_k = [L_{ki}, \pi_k] \tag{22}$$
$$= i\hbar(\delta_{kk}\pi_i - \delta_{ik}\pi_k) = i\hbar(N-1)\pi_i =: 2\gamma\pi_i \neq 0,$$

where

$$\gamma \equiv i\hbar(N-1)/2. \tag{23}$$

Following Pauli, we then define the quantized Runge–Lenz vector as

$$\tilde{\xi}_i^{\rm KL} = \frac{1}{2}(\xi_i^{\rm KL} + \xi_i^{KL\dagger}) = \tilde{K}_i - 2m\alpha\frac{x_i}{r}, \tag{24}$$

where

$$\tilde{K}_i := \frac{1}{2}(K_i + K_i^\dagger) \tag{25}$$
$$= K_i + \frac{1}{2}(K_i^\dagger - K_i) = K_i + \gamma\pi_i = (L_{ik} + \gamma\delta_{ik})\pi_k.$$

4.1. Conservation of Quantized Integrals of Motion

The quantization of the classical integrals of motion in (5) led to the operators $\{L_{ij}, H_c, \tilde{\xi}_i^{\mathrm{KL}}\}$. These operators are conserved, because they satisfy the conservation condition [14]

$$\frac{dA(\mathbf{r},\mathbf{p},t)}{dt} = \frac{\partial A(\mathbf{r},\mathbf{p},t)}{dt} + \frac{1}{i\hbar}[A, H_{\mathrm{KL}}] = 0. \quad (26)$$

Clearly, L_{ij} are conserved. We check the conservation of $\tilde{\xi}_{\mathrm{KL}}$ explicitly in Appendix D.

4.2. Commutators among Integrals of the Motion

We now show that the quantized integrals of motion in (24) have the following commutators:

$$[H_c, L_{ij}] = [H_c, \xi_i^{\mathrm{KL}}] = 0, \quad (27)$$
$$[L_{ij}, L_{kl}] = i\hbar(\delta_{ik}L_{jl} - \delta_{jk}L_{il} + \delta_{jl}L_{ik} - \delta_{il}L_{jk}),$$
$$[L_{ij}, \tilde{\xi}_k^{\mathrm{KL}}] = i\hbar(\delta_{ik}\tilde{\xi}_j^{\mathrm{KL}} - \delta_{jk}\tilde{\xi}_i^{\mathrm{KL}}),$$
$$[\tilde{\xi}_i^{\mathrm{KL}}, \tilde{\xi}_j^{\mathrm{KL}}] = -2i\hbar m H_c L_{ij}.$$

The commutators among L_{ij} are standard. Also, the commutators $[L_{ij}, \tilde{\xi}_k^{\mathrm{KL}}]$ are obvious, since $\tilde{\xi}_i$ must transform like the components of a vector under rotation. The only nontrivial commutators are $[H_c, \xi_i^{\mathrm{KL}}]$ and (27). We shall prove the latter in Appendix C.

5. TIME-DEPENDENT REALIZATIONS OF AFFINE KAC–MOODY ALGEBRAS

Following [8, 9], we define an infinite set of operators by

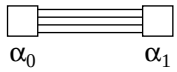
$$L_{ij}^{2n} := (-2mH_c)^n L_{ij} \quad (28)$$
$$\text{for} \quad 1 \le i,j \le N, \quad n \in N,$$
$$\tilde{\xi}_i^{(KL)2n+1} := (-2mH_c)^n \tilde{\xi}_i^{\mathrm{KL}}.$$

Since H_c commutes with L_{ij} and $\tilde{\xi}_i^{\mathrm{KL}}$, we immediately obtain the commutators

$$[L_{ij}^{2m}, L_{kl}^{2n}] = i\hbar(\delta_{ik}L_{jl}^{2m+2n} - \delta_{jk}L_{il}^{2m+2n} \quad (29)$$
$$+ \delta_{jl}L_{ik}^{2m+2n} - \delta_{il}L_{jk}^{2m+2n}),$$
$$[L_{ij}^{2m}, \tilde{\xi}_k^{(\mathrm{KL})2n+1}]$$
$$= i\hbar\left(\delta_{ik}\tilde{\xi}_j^{(\mathrm{KL})2m+2n+1} - \delta_{jk}\tilde{\xi}_i^{(\mathrm{KL})2m+2n+1}\right),$$
$$[\tilde{\xi}_i^{(\mathrm{KL})2m+1}, \tilde{\xi}_j^{(\mathrm{KL})2n+1}] = i\hbar L_{ij}^{2m+2n+2},$$
$$i,j = 1,\ldots,N, \quad m,n \ge 0.$$

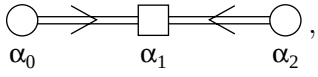
The set $\{L_{ij}^{2n}, \tilde{\xi}_i^{(\mathrm{KL})2n+1}\}$ in (28), with the commutation relations (29), constitutes an infinite-dimensional Lie algebra isomorphic to the H algebra HI_N [8, 9], which is the symmetry algebra of the ordinary hydrogen atom, as defined by the standard Hamiltonian (8). This algebra was identified as the positive subalgebra (since $n \ge 0$) of the "twisted subalgebra" of the affine Kac–Moody algebra of $so(N+1)$. We define this twisted subalgebra to be the subalgebra that contains about half of the elements of the standard Kac–Moody algebra, where the "generation indices" of L_{ij}^{2n} and $\tilde{\xi}_i^{(\mathrm{KL})\ 2n+1}$ are restricted to be even and odd, respectively. However, it turns out that, for even $N = 2l$, i.e., for $B_l \simeq so(2l+1)$, the twisted subalgebra is isomorphic to the original standard algebra, and the relevant map is called the untwist map. (For more details, see [8, 9] and [15–18].)

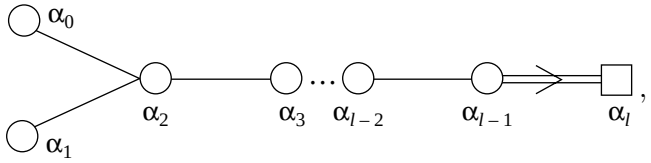
The identification of the algebras HI_N can be summarized by the generalized Dynkin diagrams, where a box denotes a vertex with only a positive root $\mathbf{b}(\mathcal{L})$ denotes the Borel subalgebra of $\mathcal{L}$ while $\mathcal{P}(\mathcal{L})$ denotes the parabolic subalgebra of $\mathcal{L}$ (more details can be found in [8, 9]).

$HI_2 = \boldsymbol{b}(A_1^{(1)})$: 

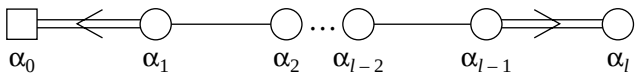
(30)

$HI_3 = A_1^{(1)+}$: 

$HI_4 = \mathcal{P}_1(C_2^{(1)})$: 

$HI_{2l} = \mathcal{P}_l(B_l^{(1)})(l \ge 3)$: 

$HI_{2l+1} = D_{l+1}^{(2)+}(l \geq 2)$:



Note that, for low dimensions ($N \leq 4$), we obtain interesting special cases. In particular, for $D = 3$, we have $HI_3 = A_1^{(1)+}$, which is the positive subalgebra of a standard Kac–Moody algebra. This is achieved by the isomorphism

$$\psi(\hat{H}^n L_i) = t^{2n} \otimes T_i, \qquad (31)$$
$$\psi(\hat{H}^n A_i) = t^{2n+1} \otimes T_i, \quad \text{for} \quad n \geq 0,$$

and $\{T_1, T_2, T_3\}$ are the generators of $A_1 \simeq su(2)$.

In [13] we used the Poisson brackets among classical integrals of the motion to define Lie products. We obtained a similar realization of Kac–Moody algebras of $so(N+1)$, except that, in [13], we took $-\infty < n < +\infty$ instead of $n \geq 0$ in (28). Thus, we obtained a realization of the relevant full loop algebras and not just of the corresponding positive subalgebras.

6. SUMMARY AND OUTLOOK

We have given unusual realizations in terms of time-dependent operators of the positive twisted (for odd N) or standard (untwisted) (for even N) affine Kac–Moody algebras of $so(N+1)$ for $\alpha \neq 0$. The generators are the angular-momentum operators L_{ij} and the formally time-dependent Runge–Lenz vector operator $\tilde{\boldsymbol{\xi}}_{\mathrm{KL}}(t)$ and $H_c(t)$, which acts as a "quasi-Hamiltonian." A feature peculiar to this representation is that $\tilde{\boldsymbol{\xi}}_{\mathrm{KL}}$ and H_c do not commute with the Hamiltonian H_{KL}, since it is time-dependent.

In the classical case, we could say that we made use of superintegrability by employing the conservation of $\boldsymbol{\xi}$ to determine the orbit in (27). However, it is not clear how to make use of the conservation of $\tilde{\boldsymbol{\xi}}$ in the quantum case.

There are many open questions. For example, it would be interesting to investigate how the well-known degeneracy of the standard Coulomb problem is lifted for $U(t)$. Does, for example, the splitting of the energy levels for various l's of the same n follow a certain pattern, say, $E_{nl_1} \geq E_{nl_2}$ for $l_1 \geq l_2$ (or $l_1 \leq l_2$), for all the l's with $0 \leq l \leq n-1$?

However, even the existence of normalized solutions is not clear. If $\ddot{U} > 0$ in the Hamiltonian H_{KL}, then the "effective spring constant" of the oscillator potential will be negative, so that $V_{\mathrm{KL}} \to -\infty$ for $r \to \infty$. Hence, normalized solutions are not expected to exist for such a potential. This conclusion would be true in the static case, where $i\hbar\partial_t$ is replaced by E_n. However, perhaps normalized solutions could exist in the time-dependent case.

The case of $\ddot{U} > 0$ must arise if we start, for example, with a constant Kepler potential $-\alpha/r$ for $t \leq 0$ and end up with a constant potential for $t > 1$ by choosing $U(t)$ to be a function of t that grows from $U(0) = 1$ to $U(1) = 2$ and then remains constant for $t > 1$. In fact, $U(t)$ possessing these properties can be chosen with continuous first and second derivatives at $t = 0$ and $t = 1$, so that $U(t)$ satisfies the conditions

$$\dot{U}(0) = \ddot{U}(0) = \dot{U}(1) = \ddot{U}(1) = 0. \qquad (32)$$

From (32), we can conclude that $\ddot{U}(t)$ must become positive somewhere in the time interval $0 \leq t \leq 1$.

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APPENDIX A

Calculating Classical Quantities and Poisson Brackets

In this appendix, we derive some Poisson brackets and other relations that are needed to calculate (16). We define the Poisson brackets by

$$\{A, B\} = \sum_{k=1}^{N} \left(\frac{\partial A}{\partial x_k}\frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial x_k} \right), \qquad (A.1)$$

so that

$$\{x_i, p_j\} = \delta_{ij}, \quad \{x_i, x_j\} = \{p_i, p_j\} = 0. \qquad (A.2)$$

From (A.2), we first derive some basic relations:

$$\{\pi_i, \pi_j\} = \{Up_i - m\dot{U}x_i, Up_j - m\dot{U}x_j\} = 0, \qquad (A.3)$$
$$\{x_i, \pi_j\} = \{x_i, Up_j - m\dot{U}x_j\}$$
$$= U\{x_i, p_j\} = U(t)\delta_{ij},$$
$$\{L_{ij}, \pi_k\} = \delta_{ik}\pi_j - \delta_{jk}\pi_i.$$

To calculate (16), it is useful first to define the decomposition

$$K_i \equiv \sum_{j=1}^{N} L_{ij}\pi_j = UK_i^p - m\dot{U}K_i^r, \qquad (A.4)$$

where

$$K_i^p \equiv \sum_{j=1}^{N} L_{ij}p_j, \quad K_i^r \equiv \sum_{j=1}^{N} L_{ij}x_j,$$

and then to calculate

$$\sum_{k=1}^{N} L_{ki}L_{kj} = \sum_{k=1}^{N}(x_kp_i - x_ip_k)(x_kp_j - x_jp_k) \tag{A.5}$$
$$= p^2x_ix_j + r^2p_ip_j - (\mathbf{r}\cdot\mathbf{p})(x_ip_j + x_jp_i),$$

which yields

$$\sum_{k=1}^{N} L_{ki}L_{kj}x_ix_j = \mathbf{K}_r^2 = r^2(r^2p^2 - (\mathbf{r}\cdot\mathbf{p})^2) = r^2L^2, \tag{A.6}$$

$$\sum_{k=1}^{N} L_{ki}L_{kj}p_ip_j = \mathbf{K}_p^2 = p^2L^2,$$

$$\sum_{k=1}^{N} L_{ki}L_{kj}x_ip_j = \mathbf{K}_r\cdot\mathbf{K}_p = (\mathbf{r}\cdot\mathbf{p})^2L^2,$$

$$\sum_{k=1}^{N} L_{ki}L_{kj}\pi_i\pi_j = \mathbf{K}^2 = \boldsymbol{\pi}^2L^2.$$

We also need

$$\mathbf{K}\cdot\mathbf{r} = \sum_{k=1}^{N} L_{kj}\pi_jx_k \tag{A.7}$$
$$= \sum_{k=1}^{N} L_{kj}(Up_j - m\dot{U}x_j)x_k$$
$$= U\sum_{k=1}^{N} L_{kj}p_jx_k = -UL^2.$$

APPENDIX B

Calculating Commutators

The commutation relations (27) among the KL integrals of motion are obvious generalizations of the relations among the corresponding quantities of the standard Hamiltonian H_1. Checking some of these commutation relations can be quite complicated. To achieve some order and simplification, we perform calculations in stages. First, we calculate relevant commutators for a free particle ($\alpha = 0$) and then add α-dependent terms. Second, instead of calculating commutators involving $\tilde{K}_i$ directly, it is much simpler first to calculate such commutators by using K_i and then to add the correction terms $\gamma\pi_i$. As we shall see, the majority of these correction terms vanish.

Prior to illustrating the above technique, we first derive some useful relations that we shall need later on:

$$[L_{ik}, x_k] = i\hbar(\delta_{ik}x_k - \delta_{kk}x_i) \tag{A.8}$$
$$= -i\hbar(N-1)x_i =: -2\gamma x_i,$$

$$x_kL_{ki} = x_k(x_kp_i - x_ip_k) = r^2p_i - x_i(\mathbf{r}\cdot\mathbf{p}), \tag{A.9}$$

$$\left[p_i, \frac{x_j}{r}\right] = -i\hbar\left(\frac{\delta_{ij}}{r} - \frac{x_ix_j}{r^3}\right) =: -i\hbar q_{ij}, \tag{A.10}$$

$$[K_j, \pi_k] = [L_{jn}\pi_n, \pi_k] = [L_{jn}, \pi_k]\pi_n \tag{A.11}$$
$$= i\hbar(\delta_{jk}\pi_n - \delta_{nk}\pi_j)\pi_n = i\hbar(\boldsymbol{\pi}^2\delta_{jk} - \pi_j\pi_k),$$

$$[p^2, x_j] = -2i\hbar p_j, [r^2, p_j] = 2i\hbar x_k. \tag{A.12}$$

APPENDIX C

Calculating $[\tilde{\xi}_i^{KL}, \tilde{\xi}_j^{KL}]$ by Stages

We illustrate the above technique by calculating commutators (A.13) in two stages. First, we shall prove (A.15) to obtain the commutators among $\tilde{K}_i$, which are the components of the Laplace–Runge–Lenz vector for the free case ($\alpha = 0$). Then we calculate (A.19), which yields the term in (A.13) that is proportional to α. Also, we prove (A.15) and (A.19) in two stages, as we shall see below. These two equations immediately yield

$$[\tilde{\xi}_i^{\mathrm{KL}}, \tilde{\xi}_j^{\mathrm{KL}}] = \left[\tilde{K}_i - m\alpha\frac{x_i}{r}, \tilde{K}_j - m\alpha\frac{x_j}{r}\right] \tag{A.13}$$
$$= [\tilde{K}_i, \tilde{K}_j] - m\alpha\left(\left[\tilde{K}_i, \frac{x_j}{r}\right] + \left[\frac{x_i}{r}, \tilde{K}_j\right]\right)$$
$$= -i\hbar\boldsymbol{\pi}^2L_{ij} + m\alpha\frac{2i\hbar U}{r}L_{ij}$$
$$= -i\hbar 2m\left(\frac{\boldsymbol{\pi}^2}{2m} - \frac{\alpha U}{r}\right)L_{ij} = -2mH_cL_{ij}.$$

To prove (A.15), we first calculate

$$[K_i, K_j] = [L_{ik}\pi_k, K_j] = [L_{ik}, K_j]\pi_k \tag{A.14}$$
$$+ L_{ik}[\pi_k, K_j] = i\hbar(\delta_{ij}K_k - \delta_{kj}K_i)\pi_k$$
$$+ i\hbar L_{ik}(\pi_k\pi_j - \delta_{jk}\boldsymbol{\pi}^2)$$
$$= i\hbar(\delta_{ij}\mathbf{K}\cdot\boldsymbol{\pi} - K_i\pi_j + K_i\pi_j - L_{ij}\boldsymbol{\pi}^2)$$
$$= -i\hbar L_{ij}\boldsymbol{\pi}^2 = -i\hbar\boldsymbol{\pi}^2L_{ij},$$

where we first used (A.11) and then $\mathbf{K}\cdot\boldsymbol{\pi} = L_{ik}\pi_k\pi_i = 0$. By noticing that the right-hand side of (A.11) is symmetric under the interchange of i and j, we then see that the correction term $\gamma\pi_i$ does not change the commutation relations (A.14):

$$[\tilde{K}_i, \tilde{K}_j] = [K_i, K_j] + \gamma([\pi_i, K_j] + [K_i, \pi_j]) \tag{A.15}$$
$$= [K_i, K_j] = -i\hbar\boldsymbol{\pi}^2L_{ij}.$$

Similarly, we now prove (A.19), again in two stages, as follows. By using (A.10), we get

$$\left[K_i, \frac{x_j}{r}\right] = \left[L_{ik}\pi_k, \frac{x_j}{r}\right] = \left[L_{ik}, \frac{x_j}{r}\right]\pi_k \quad \text{(A.16)}$$
$$+ UL_{ik}\left[p_k, \frac{x_j}{r}\right] = \frac{i\hbar}{r}(\delta_{ij}x_k - \delta_{kj}x_i)\pi_k$$
$$- i\hbar UL_{ik}\left(\frac{\delta_{jk}}{r} - \frac{x_j x_k}{r^3}\right)$$
$$= \frac{i\hbar}{r}(\delta_{ij}\mathbf{r}\cdot\boldsymbol{\pi} - x_i\pi_j - UL_{ij}) + \frac{i\hbar U}{r^3}L_{ik}x_j x_k.$$

By noticing that $x_i\pi_j - x_j\pi_i = UL_{ij}$, we then find that Eq. (A.16) yields

$$\left[K_i, \frac{x_j}{r}\right] + \left[\frac{x_i}{r}, K_j\right] \quad \text{(A.17)}$$
$$= -\frac{i\hbar U}{r}[3L_{ij} - r^{-2}(L_{ik}x_j - L_{jk}x_i)x_k]$$
$$= -\frac{i\hbar U}{r}[3L_{ij} - r^{-2}L_{ij}x_k x_k] = -\frac{2i\hbar U}{r}L_{ij},$$

where we have used the interesting identity

$$L_{jk}x_i + L_{ki}x_j + L_{ij}x_k \quad \text{(A.18)}$$
$$= x_iL_{jk} + x_jL_{ki} + x_kL_{ij} + [L_{jk}, x_i] + [L_{ki}, x_j]$$
$$+ [L_{ij}, x_k] = x_iL_{jk} + x_jL_{ki} + x_kL_{ij} = 0.$$

Taking into account noting the symmetry of (A.10), we finally see that the correction term $\gamma\pi_i$ does not change the commutation relations (A.17):

$$\left[\tilde{K}_i, \frac{x_j}{r}\right] + \left[\frac{x_i}{r}, \tilde{K}_j\right] = \left[K_i, \frac{x_j}{r}\right] + \left[\frac{x_i}{r}, K_j\right] \quad \text{(A.19)}$$
$$+ \gamma\left(\left[\pi_i, \frac{x_j}{r}\right] + \left[\frac{x_i}{r}, \pi_j\right]\right) = -\frac{2i\hbar U}{r}L_{ij}.$$

APPENDIX D

Calculating $[\tilde{\xi}_i^{KL}, H_{KL}]$ *by Stages*

In this appendix, we prove in (A.24) that $\tilde{\xi}_{\mathrm{KL}}$ is conserved, in the sense of (26). For this, we decompose the KL Hamiltonian as

$$H_{\mathrm{KL}} = H^0_{\mathrm{KL}} - \frac{\alpha}{Ur}, \quad \text{(A.20)}$$

where

$$H^0_{\mathrm{KL}} \equiv \frac{p^2}{2m} - \frac{\ddot{U}}{2mU}r^2 \quad \text{(A.21)}$$

is the KL Hamiltonian for the free case ($\alpha = 0$). We shall first prove (A.23), which confirms the above conservation of the Runge–Lenz vector $\tilde{K}_i$ for $\alpha = 0$.

By using (A.12), we get

$$[\pi_i, H^0_{\mathrm{KL}}] = \left[Up_i - m\dot{U}x_i, \frac{p^2}{2m} - \frac{m\ddot{U}}{2U}r^2\right] \quad \text{(A.22)}$$
$$= -\frac{\dot{U}}{2}[x_i, p^2] + \frac{m\ddot{U}}{2}[p_i, r^2]$$
$$= -i\hbar(\dot{U}p_i - m\ddot{U}x_i) = -i\hbar\frac{\partial\pi_i}{\partial t},$$

which shows that $\boldsymbol{\pi}$ is conserved in the free case, as expected from (12). Equation (A.22) yields

$$[\tilde{K}_i, H^0_{\mathrm{KL}}] = (L_{ik} + \gamma\delta_{ik})[\pi_k, H^0_{\mathrm{KL}}] \quad \text{(A.23)}$$
$$= -i\hbar\,(L_{ik} + \gamma\delta_{ik})\,\frac{\partial\pi_k}{\partial t} = -i\hbar\frac{\partial\tilde{K}_i}{\partial t}.$$

Note that (A.23) holds identically for any value of γ! In contrast, as we shall see below, γ must be identically equal to that in (23) for $\tilde{\xi}_{\mathrm{KL}}$ to be conserved for $\alpha \neq 0$.

From (A.23), we get

$$[\tilde{\xi}_i^{\mathrm{KL}}, H_{\mathrm{KL}}] = \left[\tilde{K}_i - m\alpha\frac{x_i}{r}, \frac{p^2}{2m} - \frac{\ddot{U}}{2mU}r^2 - \frac{\alpha}{Ur}\right] \quad \text{(A.24)}$$
$$= [\tilde{K}_i, H^0_{\mathrm{KL}}] - \alpha\left(\left[\frac{x_i}{2r}, p^2\right] + \left[\tilde{K}_i, \frac{1}{Ur}\right]\right)$$
$$= [\tilde{K}_i, H^0_{\mathrm{KL}}] = -i\hbar\frac{\partial\tilde{K}_i}{\partial t} = -i\hbar\frac{\partial\tilde{\xi}_i^{\mathrm{KL}}}{\partial t},$$

using the identity

$$\left[\frac{x_i}{2r}, p^2\right] + \left[\tilde{K}_i, \frac{1}{Ur}\right] = \left[\frac{x_i}{2r}, p^2\right] \quad \text{(A.25)}$$
$$+ \left[K_i^p + \gamma p_i, \frac{1}{r}\right] = i\hbar\left[\left(\frac{1}{r^3}x_kL_{ki} + \gamma\frac{x_i}{r}\right)\right.$$
$$\left. - \left(\frac{1}{r^3}x_kL_{ki} + (2-1)\gamma\frac{x_i}{r^3}\right)\right] = 0,$$

which follows from (A.26) and (A.29). We now prove this:

$$\left[p^2, \frac{x_i}{r}\right] = \left[p_k, \frac{x_i}{r}\right]p_k + p_k\left[p_k, \frac{x_i}{r}\right] \quad \text{(A.26)}$$
$$= 2\left[p_k, \frac{x_i}{r}\right]p_k + \left[p_k, \left[p_k, \frac{x_i}{r}\right]\right]$$
$$= -2i\hbar\left(\frac{x_kL_{ki}}{r^3} + \gamma\frac{x_i}{r}\right),$$

where we have used (A.27) and (A.28). With the aid of (A.10), we get

$$\left[p_k, \frac{x_i}{r}\right]p_k = -i\hbar q_{ik}p_k \quad \text{(A.27)}$$
$$= -i\hbar\left(\frac{1}{r}p_i - \frac{x_i}{r^3}\mathbf{r}\cdot\mathbf{p}\right) = -i\hbar\frac{1}{r^3}x_kL_{ki},$$

$$\left[p_k, \left[p_k, \frac{x_i}{r}\right]\right] = (-i\hbar)^2\frac{\partial}{\partial x_k}\left(\frac{\delta_{ik}}{r} - \frac{x_ix_k}{r^3}\right) \quad \text{(A.28)}$$
$$= \hbar^2(N-1)\frac{x_i}{r^3} = -i\hbar 2\gamma\frac{x_i}{r^3}.$$

We also need

$$\left[K_i^p, \frac{1}{r}\right] = \left[L_{ik}p_k, \frac{1}{r}\right] = L_{ik}\left[p_k, \frac{1}{r}\right] \qquad \text{(A.29)}$$
$$= i\hbar L_{ik}\frac{x_k}{r^3} = i\hbar\frac{x_k}{r^3}L_{ik} + i\hbar\left[L_{ik}, \frac{x_k}{r^3}\right] = i\hbar\frac{x_k}{r^3}L_{ik}$$
$$+ \frac{i\hbar}{r^3}[L_{ik}, x_k] = -i\hbar\left(\frac{1}{r^3}x_k L_{ki} + 2\gamma\frac{x_i}{r^3}\right),$$

where we have used (A.8).

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Polynomial Associative Algebras of Quantum Superintegrable Systems*

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Abstract—The integrals of motion of classical two-dimensional superintegrable systems, with polynomial integrals of motion, close in a restrained polynomial Poisson algebra; the general form of the quadratic case is investigated. The polynomial Poisson algebra of the classical system is deformed into a quantum associative algebra of the corresponding quantum system, and the finite-dimensional representations of this algebra are calculated by using a deformed parafermion oscillator technique. The finite-dimensional representations of the algebra are determined by the energy eigenvalues of the superintegrable system. The calculation of energy eigenvalues is reduced to the roots of algebraic equations in the quadratic case.

1. INTRODUCTION

In classical mechanics, an integrable system is a system possessing a number of constants of motion equal to the dimensionality of the space. A comprehensive review of two-dimensional integrable classical systems is given by Hietarinta [1], who assumed the space to be a flat real one. The case of a nonflat space is under current investigation [2–7]. An interesting subset of the totality of integrable systems is the set of systems that possess a maximum number of integrals; these systems are referred to as superintegrable ones.

The Hamiltonian of a classical system is a quadratic function of momenta. All "nondegenerate" superintegrable systems with quadratic integrals of motion in a complex flat space were classified by Kalnins, Miller, and Pogosyan [8]. In that paper, the term "nondegenerate" means that the potential depends on four independent parameters. These potentials are simultaneously separable in more than two orthogonal coordinate systems [9]. The notions of the multiseparability and superintegrability do not coincide. The most illustrative example is that of an anisotropic harmonic oscillator with a rational ratio of frequencies. The integrals of motion of a two-dimensional superintegrable system in flat space close in a restrained classical Poisson algebra [4, 8, 10–12]. The general form of the Poisson algebra was studied in [8, 12]. In the case of potentials with two quadratic integrals of motion, the Poisson algebra is a quadratic Poisson algebra. In [8], these quadratic Poisson algebras are listed for all superintegrable systems in a complex flat space. In [7], the quadratic algebras for systems superintegrable on a sphere are given for all classified cases. The general form of this algebra is given in [12]. The deformation of the classical Poisson algebra to a polynomial associative algebra with three generators implies a deformation of the parameters of the quadratic algebra [8, 12]. In [13], a three-generator polynomial algebra can be realized by nonlinear combinations of the generators of the $sl(3,R)$ algebra. In [10–19], it was conjectured that the energy eigenvalues correspond to finite-dimensional representations of latent quadratic algebras. Granovskii *et al.* [14] studied the representations of the quadratic Askey–Wilson algebras $QAW(3)$. Using the ladder representation proposed there, they calculated finite-dimensional representations. This method was applied to several superintegrable systems in [15, 17, 19]. Another method [10–12] for calculating finite-dimensional representations consists in the use of the deformed oscillator algebra and their finite-dimensional version, which are referred to as "generalized deformed parafermionic algebras" [20]. The main task of this paper is to reduce the calculations of eigenvalues to a system of two algebraic equations with two parameters to be determined. These equations are universal equations, which are valid for all superintegrable systems, with quadratic integrals of the motion.

2. QUADRATIC POISSON ALGEBRA

Let us consider a two-dimensional superintegrable system. The general form of the Hamiltonian

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is

$$H = a(q_1, q_2)p_1^2 + 2b(q_1, q_2)p_1p_2 + c(q_1, q_2)p_2^2 + V(q_1, q_2); \quad (1)$$

this Hamiltonian is a quadratic form of momenta. The system is superintegrable; therefore, there are two additional integrals of the motion, A and B. In this section, we assume that these integrals of motion are quadratic functions of momenta; i.e., they are given by

$$A = A(q_1, q_2, p_1, p_2) = c(q_1, q_2)p_1^2 + 2d(q_1, q_2)p_1p_2 + e(q_1, q_2)p_2^2 + Q(q_1, q_2).$$

The integral B of the motion is indeed assumed to be a quadratic form that is analogous to the above one:

$$B = B(q_1, q_2, p_1, p_2) = h(q_1, q_2)p_1^2 + 2k(q_1, q_2)p_1p_2 + l(q_1, q_2)p_2^2 + S(q_1, q_2).$$

By definition, the following relations are satisfied:

$$\{H, A\}_{\mathrm{P}} = \{H, B\}_{\mathrm{P}} = 0, \quad (2)$$

where $\{\ .\ ,\ .\ \}_{\mathrm{P}}$ is the usual Poisson bracket.

From the integrals A and B of the motion, we can construct the integral of motion

$$C = \{A, B\}_{\mathrm{P}}. \quad (3)$$

The integral C of motion is not a new independent integral of motion that is a cubic function of the momenta. As will be shown later, the integral C is not independent of the integrals H, A, and B. Starting from the integral of motion C, we can construct the (nonindependent) integrals $\{A, C\}_{\mathrm{P}}$ and $\{B, C\}_{\mathrm{P}}$. These integrals are quartic functions of momenta, i.e., functions of fourth order. Therefore, these integrals could be expressed as quadratic combinations of the integrals H, A, and B. After translations and rotations, the integrals A, B, and C satisfy the quadratic Poisson algebra:

$$\{A, B\}_{\mathrm{P}} = C, \quad (4)$$
$$\{A, C\}_{\mathrm{P}} = \alpha A^2 + 2\gamma AB + \delta A + \epsilon B + \zeta,$$
$$\{B, C\}_{\mathrm{P}} = aA^2 - \gamma B^2 - 2\alpha AB + dA - \delta B + z,$$

where α, γ, and a are constants and

$$\delta = \delta(H) = \delta_0 + \delta_1 H,$$
$$\epsilon = \epsilon(H) = \epsilon_0 + \epsilon_1 H,$$
$$\zeta = \zeta(H) = \zeta_0 + \zeta_1 H + \zeta_2 H^2,$$
$$d = d(H) = d_0 + d_1 H,$$
$$z = z(H) = z_0 + z_1 H + z_2 H^2,$$

with δ_i, ϵ_i, ζ_i, d_i, and z_i being constants. The associative algebra whose generators satisfy Eqs. (4) is a general form of the closed Poisson algebra of the integrals of superintegrable systems with integrals quadratic in momenta.

The quadratic Poisson algebra (4) possesses a Casimir operator that is a function of momenta of degree six and which is given by

$$K = C^2 - 2\alpha A^2 B - 2\gamma AB^2 - 2\delta AB - \epsilon B^2 - 2\zeta B + \frac{2}{3}aA^3 + dA^2 + 2zA = k_0 + k_1 H + k_2 H^2 + k_3 H^3. \quad (5)$$

Obviously, we have

$$\{K, A\}_{\mathrm{P}} = \{K, B\}_{\mathrm{P}} = \{K, C\}_{\mathrm{P}} = 0.$$

Therefore, the integrals of motion of a superintegrable two-dimensional system, with quadratic integrals of motion, close a constrained classical quadratic Poisson algebra (4), corresponding to a Casimir operator equal at most to a cubic function of the Hamiltonian in (5).

In the general case of a superintegrable system, the integrals are not necessarily quadratic functions of the momenta, but they are rather polynomial functions of the momenta. The case of systems with a quadratic and cubic integral of motion were studied by Tsiganov [21]. The general form of the Poisson algebra of the generators A, B, and C is characterized by a polynomial function $h(A, B)$:

$$\{A, B\}_{\mathrm{P}} = C, \quad \{A, C\}_{\mathrm{P}} = \partial h/\partial B, \quad \{C, B\}_{\mathrm{P}} = \partial h/\partial A. \quad (6)$$

The above general forms of the Poisson algebra were introduced by Kalnins, Miller, and Pogosyan [8]. The Casimir operator of the algebra is given by

$$K = K(H) = C^2 - 2h(A, B), \quad \{K, A\}_{\mathrm{P}} = \{K, B\}_{\mathrm{P}} = 0, \quad (7)$$

where $h(A, B)$ is a polynomial function of the integrals A and B of the motion. These relations were also discussed in [8] in a slightly different context.

In the general case of a two-dimensional superintegrable system with a quadratic Hamiltonian, one integral A of order m in momenta, and one integral B of order n ($n \geq m$), the function $h(A, B)$, in most cases, can be represented as

$$h(A, B) = h_0(A) + h_1(A)B + h_2(A)B^2,$$

where $h_i(A)$ are polynomials of the integrals A and H.

3. QUADRATIC ASSOCIATIVE ALGEBRA

The quantum counterparts of classical systems that have been studied in Section 2 are quantum superintegrable systems. The quadratic classical Poisson algebra (4) possesses a quantum counterpart that is a quadratic associative algebra of operators. The form of the quadratic algebra is similar to that of the classical Poisson algebra, the constants involved are generally functions of $\hbar$, and they should coincide with the classical constants in the case of $\hbar \to 0$:

$$[A,B] = C, \tag{8}$$

$$[A,C] = \alpha A^2 + \gamma \{A,B\} + \delta A + \epsilon B + \zeta, \tag{9}$$

$$[B,C] = aA^2 - \gamma B^2 - \alpha \{A,B\} + dA - \delta B + z. \tag{10}$$

The Casimir operator of this algebra is given by

$$K = C^2 - \alpha \left\{A^2, B\right\} - \gamma \left\{A, B^2\right\} + (\alpha\gamma - \delta)\{A,B\} + (\gamma^2 - \epsilon)B^2 + (\gamma\delta - 2\zeta)B + \frac{2a}{3}A^3 + \left(d + \frac{a\gamma}{3} + \alpha^2\right)A^2 + \left(\frac{a\epsilon}{3} + \alpha\delta + 2z\right)A. \tag{11}$$

This quadratic algebra has many similarities to the Racah algebra $QR(3)$, which is a special case of the Askey–Wilson algebra $QAW(3)$. The algebra specified by Eqs. (8)–(10) does not coincide with the Racah algebra $QR(3)$ if $a \neq 0$ in relation (10). A representation theory can be constructed by following the same procedures as those described by Granovskii, Lutzenko, and Zhedanov in [14, 15]. In this paper, we shall give another realization of this algebra using the deformed-oscillator techniques [22]. The finite-dimensional representations of the algebra given by (8)–(10) will be constructed by constructing a realization of the algebra with the generalized parafermionic algebra introduced by Quesne [20].

Let us now consider a realization of the algebra given by (8)–(10) by using the deformed-oscillator technique, i.e., by using a deformed-oscillator algebra [22] $\left\{b^\dagger, b, \mathcal{N}\right\}$, which satisfies

$$\left[\mathcal{N}, b^\dagger\right] = b^\dagger, \quad [\mathcal{N}, b] = -b, \quad b^\dagger b = \Phi(\mathcal{N}), \quad bb^\dagger = \Phi(\mathcal{N}+1), \tag{12}$$

where the function $\Phi(x)$ is a "well-behaved" real function that satisfies the boundary condition

$$\Phi(0) = 0 \quad \text{and} \quad \Phi(x) > 0 \quad \text{for} \quad x > 0. \tag{13}$$

As is well known [22], this constraint entails the existence of a Fock-type representation of the deformed-oscillator algebra; i.e., there is a Fock basis $|n\rangle$, $n = 0, 1, \ldots$, such that

$$\mathcal{N}|n\rangle = n|n\rangle, \tag{14}$$

$$b^\dagger|n\rangle = \sqrt{\Phi(n+1)}|n+1\rangle, \quad n = 0, 1, \ldots,$$
$$b|0\rangle = 0,$$
$$b|n\rangle = \sqrt{\Phi(n)}|n-1\rangle, \quad n = 1, 2, \ldots.$$

In the case of nilpotent deformed-oscillator algebras, there is a positive integer p such that

$$b^{p+1} = 0, \quad (b^\dagger)^{p+1} = 0.$$

The above equations imply that

$$\Phi(p+1) = 0. \tag{15}$$

In that case, the deformed oscillator (12) has a finite-dimensional representation of dimension equal to $p+1$. This kind of oscillator is called a deformed parafermion oscillator of order p. The structure function $\Phi(\mathcal{N})$ has the general form [20]

$$\Phi(\mathcal{N}) = \mathcal{N}(p+1-\mathcal{N})(a_0 + a_1\mathcal{N} + a_2\mathcal{N}^2 + \cdots + a_{p-1}\mathcal{N}^{p-1}).$$

A systematic study and applications of the parafermionic oscillator are given in [20, 23–25].

We shall show that there is a realization of the quadratic algebra such that

$$A = A(\mathcal{N}), \tag{16}$$

$$B = b(\mathcal{N}) + b^\dagger \rho(\mathcal{N}) + \rho(\mathcal{N})\, b, \tag{17}$$

where $A(x)$, $b(x)$, and $\rho(x)$ are functions that will be determined. In this case, (8) implies that

$$C = [A,B] \;\Rightarrow\; C = b^\dagger \Delta A(\mathcal{N})\, \rho(\mathcal{N}) - \rho(\mathcal{N})\, \Delta A(\mathcal{N})\, b, \tag{18}$$

where

$$\Delta A(\mathcal{N}) = A(\mathcal{N}+1) - A(\mathcal{N}).$$

Using Eqs. (16), (17), and (9) we find

$$(\Delta A(\mathcal{N}))^2 = \gamma(A(\mathcal{N}+1) + A(\mathcal{N})) + \epsilon, \tag{19}$$

$$\alpha A(\mathcal{N})^2 + 2\gamma A(\mathcal{N})\, b(\mathcal{N}) + \delta A(\mathcal{N}) + \epsilon b(\mathcal{N}) + \zeta = 0, \tag{20}$$

while the function $\rho(\mathcal{N})$ can be arbitrarily determined. In fact, this function can be fixed in order to have a polynomial structure function $\Phi(x)$ for the deformed-oscillator algebra (12). Solutions to Eqs. (19) depend on the value of the parameter γ, while the function $b(\mathcal{N})$ is uniquely determined by Eq. (20) (provided that at most one of the parameters γ or ϵ is not zero). At this stage, the cases of $\gamma \neq 0$ or $\gamma = 0$ should be treated separately.

Case 1: $\gamma \neq 0$. In this case, solutions to Eqs. (19) and (20) are given by

$$A(\mathcal{N}) = \frac{\gamma}{2}\left((\mathcal{N}+u)^2 - 1/4 - \frac{\epsilon}{\gamma^2}\right), \tag{21}$$

$$b(\mathcal{N}) = -\frac{\alpha\left((\mathcal{N}+u)^2-1/4\right)}{4} + \frac{\alpha\,\epsilon-\delta\,\gamma}{2\,\gamma^2} \qquad (22)$$
$$-\frac{\alpha\,\epsilon^2-2\,\delta\,\epsilon\,\gamma+4\,\gamma^2\,\zeta}{4\,\gamma^4}\frac{1}{((\mathcal{N}+u)^2-1/4)}.$$

Case 2: $\gamma = 0,\ \epsilon \neq 0$. Solutions to Eqs. (19) and (20) are given by

$$A(\mathcal{N}) = \sqrt{\epsilon}\,(\mathcal{N}+u), \qquad (23)$$

$$b(\mathcal{N}) = -\alpha\,(\mathcal{N}+u)^2 - \frac{\delta}{\sqrt{\epsilon}}\,(\mathcal{N}+u) - \frac{\zeta}{\epsilon}. \qquad (24)$$

The constant u will be determined later.

Using the above definitions of $A(\mathcal{N})$ and $b(\mathcal{N})$, we find that the left-hand side and the right-hand side of Eq. (10) give the equation

$$2\,\Phi(\mathcal{N}+1)\left(\Delta A\,(\mathcal{N}) + \frac{\gamma}{2}\right)\rho(\mathcal{N}) \qquad (25)$$
$$-\,2\,\Phi(\mathcal{N})\left(\Delta A\,(\mathcal{N}-1) - \frac{\gamma}{2}\right)\rho(\mathcal{N}-1)$$
$$= aA^2\,(\mathcal{N}) - \gamma b^2(\mathcal{N}) - 2\alpha A\,(\mathcal{N})\,b(\mathcal{N})$$
$$+\,dA\,(\mathcal{N}) - \delta b(\mathcal{N}) + z.$$

Equation (11) gives the relation

$$K = \Phi(\mathcal{N}+1)\big(\gamma^2 - \epsilon - 2\gamma A\,(\mathcal{N}) \qquad (26)$$
$$-\,\Delta A^2\,(\mathcal{N})\big)\rho(\mathcal{N}) + \Phi(\mathcal{N})\big(\gamma^2-\epsilon$$
$$-\,2\gamma A\,(\mathcal{N}) - \Delta A^2\,(\mathcal{N}-1)\big)\rho(\mathcal{N}-1)$$
$$-\,2\alpha A^2\,(\mathcal{N})\,b(\mathcal{N}) + \left(\gamma^2-\epsilon-2\gamma A\,(\mathcal{N})\right)b^2(\mathcal{N})$$
$$+\,2\,(\alpha\gamma-\delta)\,A\,(\mathcal{N})\,b(\mathcal{N}) + (\gamma\delta-2\zeta)\,b(\mathcal{N})$$
$$+\,\frac{2}{3}aA^3\,(\mathcal{N}) + \left(d+\frac{1}{3}a\gamma+\alpha^2\right)A^2\,(\mathcal{N})$$
$$+\left(\frac{1}{3}a\epsilon+\alpha\delta+2z\right)A\,(\mathcal{N}).$$

Equations (25) and (26) are linear functions of the expressions $\Phi(\mathcal{N})$ and $\Phi(\mathcal{N}+1)$. Then, the function $\Phi\,(\mathcal{N})$ can be determined if the function $\rho(\mathcal{N})$ is given. A solution of this system, i.e., the function $\Phi\,(\mathcal{N})$, depends on two parameters, u and K, and is given by the following formulas:

Case 1: $\gamma \neq 0$.

$$\rho(\mathcal{N}) = \frac{1}{3\cdot 2^{12}\cdot\gamma^8(\mathcal{N}+u)(1+\mathcal{N}+u)(1+2(\mathcal{N}+u))^2}$$

and

$$\Phi(\mathcal{N}) = -3072\gamma^6 K(-1+2(\mathcal{N}+u))^2 \qquad (27)$$
$$-\,48\gamma^6(\alpha^2\epsilon-\alpha\delta\gamma+a\epsilon\gamma-d\gamma^2)$$
$$\times(-3+2(\mathcal{N}+u))(-1+2(\mathcal{N}+u))^4$$
$$\times(1+2(\mathcal{N}+u))$$
$$+\,\gamma^8(3\alpha^2+4a\gamma)(-3+2(\mathcal{N}+u))^2(-1+2(\mathcal{N}+u))^4$$
$$\times(1+2(\mathcal{N}+u))^2 + 768(\alpha\epsilon^2-2\delta\epsilon\gamma+4\gamma^2\zeta)^2$$
$$+\,32\gamma^4(-1+2(\mathcal{N}+u))^2(-1-12(\mathcal{N}+u)$$
$$+\,12(\mathcal{N}+u)^2)(3\alpha^2\epsilon^2-6\alpha\delta\epsilon\gamma+2a\epsilon^2\gamma+2\delta^2\gamma^2$$
$$-\,4d\epsilon\gamma^2+8\gamma^3 z+4\alpha\gamma^2\zeta)-256\gamma^2(-1+2(\mathcal{N}+u))^2$$
$$\times(3\alpha^2\epsilon^3-9\alpha\delta\epsilon^2\gamma+a\epsilon^3\gamma+6\delta^2\epsilon\gamma^2-3d\epsilon^2\gamma^2+2\delta^2\gamma^4$$
$$+\,2d\epsilon\gamma^4+12\epsilon\gamma^3 z-4\gamma^5 z$$
$$+\,12\alpha\epsilon\gamma^2\zeta-12\delta\gamma^3\zeta+4\alpha\gamma^4\zeta).$$

Case 2: $\gamma = 0,\ \epsilon \neq 0$.

$$\rho(\mathcal{N}) = 1,$$

$$\Phi(\mathcal{N}) = \frac{1}{4}\left(-\frac{K}{\epsilon} - \frac{z}{\sqrt{\epsilon}} - \frac{\delta}{\sqrt{\epsilon}}\frac{\zeta}{\epsilon} + \frac{\zeta^2}{\epsilon^2}\right) \qquad (28)$$
$$-\,\frac{1}{12}\Big(3d - a\sqrt{\epsilon} - 3\alpha\frac{\delta}{\sqrt{\epsilon}} + 3\left(\frac{\delta}{\sqrt{\epsilon}}\right)^2$$
$$-\,6\frac{z}{\sqrt{\epsilon}} + 6\alpha\frac{\zeta}{\epsilon} - 6\frac{\delta}{\sqrt{\epsilon}}\frac{\zeta}{\epsilon}\Big)(\mathcal{N}+u)$$
$$+\,\frac{1}{4}\left(\alpha^2 + d - a\sqrt{\epsilon} - 3\alpha\frac{\delta}{\sqrt{\epsilon}} + \left(\frac{\delta}{\sqrt{\epsilon}}\right)^2 + 2\alpha\frac{\zeta}{\epsilon}\right)$$
$$\times(\mathcal{N}+u)^2 - \frac{1}{6}\left(3\alpha^2 - a\sqrt{\epsilon} - 3\alpha\frac{\delta}{\sqrt{\epsilon}}\right)$$
$$\times(\mathcal{N}+u)^3 + \frac{1}{4}\alpha^2(\mathcal{N}+u)^4.$$

The above formula is valid for $\epsilon > 0$.

Let us consider a representation of the quadratic algebra that is diagonal in the generator A and the Casimir operator K. Using the parafermionic realization defined by Eqs. (16) and (17), we see that this is a representation diagonal in the parafermionic number operator $\mathcal{N}$ and the Casimir operator K. The basis of this representation corresponds to the Fock basis of the parafermionic oscillator; i.e., the vectors $|k,\,n\rangle,\ n = 0, 1, \ldots$, of the carrier Fock space satisfy the equations

$$\mathcal{N}|k,\,n\rangle = n|k,\,n\rangle, \quad K|k,\,n\rangle = k|k,\,n\rangle.$$

The structure function (27) [or, respectively, (28)] depends on the eigenvalues of the parafermionic number operator $\mathcal{N}$ and the Casimir operator K. If the deformed oscillator corresponds to a deformed parafermionic oscillator of order p, then the two parameters of the calculation, k and u, should satisfy the constraints (13) and (15) of the system:

$$\Phi(0,u,k) = 0 \quad \text{and} \quad \Phi(p+1,u,k) = 0. \qquad (29)$$

Then, the parameter $u = u(k,p)$ is a solution to the set of Eqs. (29). Generally, there are many solutions

to the above set of equations, but a unitary representation of the deformed parafermionic oscillator entails the additional restriction

$$\Phi(x) > 0 \quad \text{for} \quad x = 1, 2, \ldots, p.$$

We must indicate that the set of Eqs. (29) corresponds to a representation of dimension equal to $p+1$. The proposed method for calculating the representation of the quadratic algebra is an alternative to the method given by Granovskii *et al.* [14, 15] and reduces the search for the representations to solving the set of polynomial Eqs. (29). Also, it is applied to an algebra not included in the cases of the algebras that are treated in the above references.

4. QUADRATIC ALGEBRAS FOR QUANTUM SUPERINTEGRABLE SYSTEMS

In this section, we shall give two examples of the calculation of eigenvalues for a superintegrable two-dimensional system using the methods of the preceding section. The calculation by an empirical method was performed in [11], and solving the same problem by a separation of variables was studied in [4]. In order to show the effects of the quantization procedure, we do not use here units in which $\hbar = 1$.

4.1. Potential (i)

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + \frac{k}{r} + \frac{1}{r}\left(\frac{\mu_1}{r+x} + \frac{\mu_2}{r-x}\right)\right).$$

In [4], the parabolic coordinates were used:

$$x = \frac{1}{2}\left(\xi^2 - \eta^2\right), \quad y = \xi\eta,$$
$$[\xi, p_\xi] = i\hbar, \quad [\eta, p_\eta] = i\hbar,$$
$$H = \frac{1}{\xi^2+\eta^2}\left(\frac{1}{2}\left(p_\xi^2 + p_\eta^2\right) + k + \frac{\mu_1}{\xi^2} + \frac{\mu_2}{\eta^2}\right).$$

This potential has the following independent integrals of motion:

$$A = \frac{1}{2}\left(\frac{1}{2}\left(\eta p_\xi - \xi p_\eta\right)^2 + \left(\xi^2+\eta^2\right)\left(\frac{\mu_1}{\xi^2} + \frac{\mu_2}{\eta^2}\right)\right),$$

$$B = \frac{1}{\xi^2+\eta^2}\left(\frac{1}{2}\left(\xi^2 p_\eta^2 - \eta^2 p_\xi^2\right) + \mu_2\frac{\xi^2}{\eta^2} - \mu_1\frac{\eta^2}{\xi^2} + \frac{k}{2}\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right).$$

The constants of the corresponding quadratic algebra (8)–(10) are given by

$$\alpha = 0, \quad \gamma = 2\hbar^2, \quad \delta = 0, \quad \epsilon = -\hbar^4,$$
$$\zeta = -\hbar^2 k(\mu_1 - \mu_2), \quad a = 0, \quad d = 8\hbar^2 H,$$
$$z = -\hbar^2\left(4(\mu_1+\mu_2)H - k^2/2\right) + \hbar^4 H.$$

The Casimir operator (11) has the form

$$K = -\hbar^2\left(2(\mu_1-\mu_2)^2 H - k^2(\mu_1+\mu_2)\right) - 2\hbar^4\left((\mu_1+\mu_2)H - \frac{k^2}{4}\right) + \hbar^6 H.$$

For the sake of simplicity, we introduce the positive parameters k_1 and k_1:

$$\mu_1 = \frac{\hbar^2}{2}\left(k_1^2 - \frac{1}{4}\right), \quad \mu_2 = \frac{\hbar^2}{2}\left(k_2^2 - \frac{1}{4}\right).$$

The structure function (27) of the deformed parafermionic algebra can be given by the simple form

$$\Phi(x) = 3\cdot 2^{14}\hbar^{16}\left(2x - 1 + k_1 + k_2\right) \times \left(2x - 1 + k_1 - k_2\right)\left(2x - 1 - k_1 + k_2\right) \times \left(2x-1-k_1-k_2\right)\left(8\hbar^2 H x^2 - 8\hbar^2 H x + 2\hbar^2 H + k^2\right).$$

where E is the eigenvalue of the energy. The values of the parameters u and E are determined by the restrictions in (29). There are four acceptable solutions, which correspond to the following values of the parameters u and E:

$$u = \frac{1}{2}\left(2 + \epsilon_1 k_1 + \epsilon_2 k_2\right),$$
$$E = -\frac{k^2}{2\hbar^2\left(2(p+1) + \epsilon_1 k_1 + \epsilon_2 k_2\right)^2},$$

where $\epsilon_i = \pm 1$. The positive sign of the structure function for $x = 1, 2, \ldots, p$ is obtained when

$$\epsilon_1 k_1 > -1, \quad \epsilon_2 k_2 > -1, \quad \text{and} \quad \epsilon_1 k_1 + \epsilon_2 k_2 > -1.$$

4.2. Potential (ii)

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + \frac{k}{r} + \mu_1\frac{\sqrt{r+x}}{r} + \mu_2\frac{\sqrt{r-x}}{r}\right) = \frac{1}{\xi^2+\eta^2}\left(\frac{1}{2}\left(p_\xi^2 + p_\eta^2\right) + k + \mu_1\xi + \mu_2\eta\right).$$

This potential has the following independent integrals of motion:

$$A = \frac{1}{2\left(\xi^2+\eta^2\right)}\left(\eta^2 p_\xi^2 - \xi^2 p_\eta^2 + k\left(\eta^2 - \xi^2\right) + 2\xi\eta\left(\mu_1\eta - \mu_2\xi\right)\right),$$

$$B = -\frac{1}{2\left(\xi^2+\eta^2\right)}\left(\xi\eta\left(p_\xi^2 + p_\eta^2\right) - \left(\xi^2+\eta^2\right)p_\xi p_\eta + 2k\xi\eta + \left(\mu_2\xi - \mu_1\eta\right)\left(\eta^2 - \xi^2\right)\right).$$

The constants of the corresponding quadratic algebra (8)–(10) are given by

$$\alpha = 0, \quad \gamma = 0, \quad \delta = 0, \quad \epsilon = -2\hbar^2 H,$$
$$\zeta = \hbar^2 \mu_1 \mu_2 / 2,$$
$$a = 0, \quad d = 2\hbar^2 H,$$
$$z = -\hbar^2 (\mu_1^2 - \mu_2^2)/4.$$

The Casimir operator (11) has the form

$$K = \hbar^2 k^2 H/2 + \hbar^2 k(\mu_1^2 + \mu_2^2)/4 + \hbar^4 H^2.$$

For the sake of simplicity, we introduce the parameters

$$\varepsilon = \sqrt{-2E}/\hbar, \quad \lambda = k/\hbar^2,$$
$$\nu_1 = \mu_1/\hbar^2, \quad \nu_2 = \mu_2/\hbar^2, \quad \nu^2 = \nu_1^2 + \nu_2^2.$$

The structure function (28) of the deformed parafermionic algebra can be given by the form

$$\Phi(x) = \frac{\hbar^4}{16\varepsilon^4}\left(\nu_1^2 - \lambda\varepsilon^2 + 2\left(x + u - \frac{1}{2}\right)\varepsilon^3\right) \times \left(\nu_2^2 - \lambda\varepsilon^2 - 2\left(x + u - \frac{1}{2}\right)\varepsilon^3\right),$$

where the parameter ε is related to the eigenvalue E of the energy. The values of the parameters u and ε are determined by the restrictions in (29), which become

$$\Phi(0) = 0, \quad \Phi(p+1) = 0.$$

The first condition can be used to determine the acceptable values of the parameter u. Two possible solutions are found to be

$$u = u_1 = \frac{\nu_2^2 - \lambda\varepsilon^2 + \varepsilon^3}{2\varepsilon^3}, \tag{30}$$

$$u = u_2 = -\frac{\nu_1^2 - \lambda\varepsilon^2 - \varepsilon^3}{2\varepsilon^3}. \tag{31}$$

Using these solutions and the condition $\Phi(p+1) = 0$, we find that ε must satisfy two possible cubic equations:

$$u_1 \longrightarrow 2(p+1)\varepsilon^3 - 2\lambda\varepsilon^2 + \nu^2 = 0, \tag{32}$$

$$u_2 \longrightarrow 2(p+1)\varepsilon^3 + 2\lambda\varepsilon^2 - \nu^2 = 0. \tag{33}$$

If ε is a solution to Eq. (32), then $-\varepsilon$ is a solution to Eq. (33); therefore, there is at least one positive solution. This solution leads to the structure function

$$\Phi(x) = \frac{\varepsilon^2}{4} x\,(p+1-x)$$

which is positive for $x = 1, 2, \ldots, p$.

5. DISCUSSION

The energy eigenvalues corroborate the results of [4, 13]. The calculation of the energy eigenvalues in [4] was performed by solving the corresponding Schrödinger differential equations, while, in this paper and in [13], the energy eigenvalues are obtained by algebraic methods. The advantage of the proposed method is that the energy eigenvalues are reduced to simple algebraic calculations of the roots of polynomial equations whose form is universally determined by the structure functions (27) and the set of Eqs. (29). These equations are valid for any two-dimensional superintegrable system with integrals of motion that are quadratic functions of momenta. The same equations should be valid in the case of two-dimensional superintegrable systems in a curved space [26]. Superintegrable systems bring up the open problem of the quantization of a Poisson algebra in a well-determined context, because these systems and their quantum counterparts are explicitly known.

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Exact Solution to the Cauchy Problem for a Generalized "Linear" Vectorial Fokker–Planck Equation: Algebraic Approach*

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Abstract—The exact solution to the Cauchy problem for a generalized "linear" vectorial Fokker–Planck equation is found by using the disentangling techniques of Feynman and algebraic (operational) methods.

1. INTRODUCTION

The Fokker–Planck equations (FPE), the one-dimensional FPE

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}\left[a(t,x)W\right] + \frac{\partial^2}{\partial x^2}\left[D(t,x)W(t,x)\right], \quad t \geq 0, \quad x \in \mathrm{R}, \tag{1}$$

and the "vectorial" FPE

$$\frac{\partial w}{\partial t} = -\nabla \cdot [\mathbf{a}(t,\mathbf{x})w] + \nabla\nabla : \left\{\hat{D}(t,\mathbf{x})w(t,\mathbf{x})\right\}, \quad t \geq 0, \quad \mathbf{x} \in \mathrm{R}^n, \tag{2}$$

where $\mathbf{a}(t,\mathbf{x}) = (a_1(t,\mathbf{x}), a_2(t,\mathbf{x}), \ldots, a_n(t,\mathbf{x}))^T$ is the "drift vector," $\hat{D}(t,\mathbf{x})$ is a symmetric nonnegative definite "diffusion" tensor field of rank II, and $\nabla\nabla : \hat{D} = \partial^2 D_{ij}/\partial x_i \partial x_j$ (Einstein summation convention accepted), are widely used [1–19] as a tool in modeling various processes in many areas of theoretical and mathematical physics, chemistry, and biology, as well as in pure and applied mathematics and in engineering: for example, in nonequilibrium statistical mechanics (in particular, in the theory of Brownian motion and similar phenomena, such as random walks, fluctuations of liquid surfaces, local density fluctuations in fluids and solids, and fluctuations of currents), metrology (Josephson voltage standards), laser physics, turbulence theory, cellular behavior, neurophysiology, population genetics, and mathematical theory and applications of stochastic processes.

Because of its importance, there have been many attempts at solving FPE exactly or approximately (for a review, see [4, 6–11, 14, 19]). Among recent investigations into this problem, noteworthy for us is the method of Suzuki [18].

In this paper, we find the exact solution to the Cauchy problem

$$\frac{\partial u}{\partial t} = a_1(t)u(t,\mathbf{x}) + \mathbf{a}_2(t)\cdot\nabla u + a_3(t)\mathbf{x}\cdot\nabla u + \hat{a}_4(t) : \nabla\nabla u, \quad u(0,\mathbf{x}) = \phi(\mathbf{x}), \tag{3}$$

where $\hat{a}_4(t)$ is a rank-2 symmetric nonnegative definite tensor function of the scalar parameter t.

It is easy to see that Eq. (3) is related to the "linear" vectorial FPE (2) with the "drift vector" $\mathbf{a}(t,\mathbf{x}) = \mathbf{b}_1 + b_2\mathbf{x}$, which is linear in $\mathbf{x}$, and a diffusion tensor $\hat{D}$ independent of $\mathbf{x}$. (Here, $\mathbf{b}_1$, b_2, and $\hat{D}$ are functions of t.) Therefore, Eq. (3) is a slight generalization of the "linear" vectorial FPE (2) with t-dependent coefficients.

In [20], the "isotropic" problems

$$\frac{\partial u}{\partial t} = a_1 u(t,\mathbf{x}) + \mathbf{a}_2\cdot\nabla u + a_3\mathbf{x}\cdot\nabla u + a_4\Delta u, \quad u(0,\mathbf{x}) = \phi(\mathbf{x}) \tag{4}$$

and

$$\frac{\partial u}{\partial t} = a_1(t)u(t,\mathbf{x}) + \mathbf{a}_2(t)\cdot\nabla u + a_3(t)\mathbf{x}\cdot\nabla u + a_4(t)\Delta u, \quad u(0,\mathbf{x}) = \phi(\mathbf{x}) \tag{5}$$

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were exactly solved [here, a_4 and $a_4(t)$ are an arbitrary nonnegative constant and a function of t, respectively].

In [21], we found the exact solutions to the Cauchy problems

$$\frac{\partial u}{\partial t} = a_1 u(t,\mathbf{x}) + \mathbf{a}_2\cdot\nabla u + a_3\mathbf{x}\cdot\nabla u + \hat{a}_4 : \nabla\nabla u, \quad u(0,\mathbf{x}) = \phi(\mathbf{x}) \tag{6}$$

and

$$\frac{\partial u}{\partial t} = a_1(t)u(t,\mathbf{x}) + \mathbf{a}_2(t)\cdot\nabla u + a_3(t)\mathbf{x}\cdot\nabla u + a_4(t)\hat{a} : \nabla\nabla u, \quad u(0,\mathbf{x}) = \phi(\mathbf{x}), \tag{7}$$

where $\hat{a}_4$ and $\hat{a}$ are rank-2 symmetric nonnegative definite tensors and $a_4(t)$ is a scalar function, $a_4(t) > 0$. [It is obvious that the problem specified by (3) is more general than the problem specified by (7): in (3), $\hat{a}_4(t)$ is an arbitrary rank-2 symmetric nonnegative tensor function, while, in (7), $\hat{a}_4(t)$ has a special form, $\hat{a}_4(t) = a_4(t)\hat{a}$.]

Our method may be regarded as a combination of the disentangling techniques of Feynman [22] with the operational methods developed in functional analysis—in particular, in the theory of pseudodifferential equations with partial derivatives [23–27]. As we noted in [20, 21], this approach is an extension and generalization of Suzuki's method [18] for solving the one-dimensional linear FPE (1).

2. EXACT SOLUTION TO THE CAUCHY PROBLEM (3)

In view of the t dependence of the coefficients in Eq. (3), we formally have, for the solution to the initial-value problem (3), the ordered exponential

$$u(t,\mathbf{x}) = \left(\exp_+\int_0^t [a_1(s) + \mathbf{a}_2(s)\cdot\nabla + a_3(s)\mathbf{x}\cdot\nabla + \hat{a}_4(s) : \nabla\nabla]\, ds\right)\phi(\mathbf{x}), \tag{8}$$

where

$$\exp_+\int_0^t \hat{C}(s)ds \equiv T - \exp\int_0^t \hat{C}(s)ds = \hat{1} + \lim_{k\to\infty}\sum_{n=1}^{k}\int_0^t dt_1\int_0^{t_1} dt_2\ldots \int_0^{t_{n-1}} dt_n \hat{C}(t_1)\hat{C}(t_2)\ldots\hat{C}(t_n). \tag{9}$$

If we introduce the operators

$$\hat{A}(t) = \mathbf{a}_2(t)\cdot\nabla + a_3(t)\mathbf{x}\cdot\nabla \quad \text{and} \quad \hat{B}(t) = \hat{a}_4(t) : \nabla\nabla, \tag{10}$$

we may write (8) in the form

$$u(t,\mathbf{x}) = e^{\int_0^t a_1(s)ds} \times \left(\exp_+\int_0^t \left[\hat{A}(s) + \hat{B}(s)\right] ds\right)\phi(\mathbf{x}), \tag{11}$$

since the first term in the exponent commutes with all others.

To proceed with the pseudodifferential operator in Eq. (11), we shall use the theorem of Suzuki [18]: if

$$\left[\hat{A}(t), \hat{B}(t)\right] = \alpha(t,s)\hat{B}(s),$$

then

$$\exp_+\int_0^t \left[\hat{A}(s) + \hat{B}(s)\right] ds = \left(\exp_+\int_0^t \hat{A}(s)ds\right) \times \left(\exp_+\int_0^t \hat{B}(s)e^{-\int_0^s \alpha(u,s)du}ds\right). \tag{12}$$

In our case, we have

$$\left[\hat{A}(s), \hat{B}(s')\right] \equiv [\mathbf{a}_2(s)\cdot\nabla + a_3(s)\mathbf{x}\cdot\nabla, \hat{a}_4(s') : \nabla\nabla] = -2a_3(s)\hat{a}_4(s') : \nabla\nabla \equiv -2a_3(s)\hat{B}(s'). \tag{13}$$

From (13), we therefore obtain

$$\exp_+\int_0^t \left[\hat{A}(s) + \hat{B}(s)\right] ds = \left(\exp_+\int_0^t \hat{A}(s)ds\right) \times \left(\exp_+\int_0^t \hat{B}(s)e^{2\int_0^s a_3(u)du}ds\right). \tag{14}$$

The linearity of the integral and the explicit form of $\hat{A}$ [see Eq. (10)] permit us to express the first factor

in (14) in terms of the usual (not ordered) operator-valued exponent

$$\exp_+ \int\limits_0^t \hat{A}(s)ds \equiv \exp_+ \int\limits_0^t [\mathbf{a}_2(s)\cdot\nabla + a_3(s)\mathbf{x}\cdot\nabla]\,ds = e^{\boldsymbol{\alpha}_2(t)\cdot\nabla+\alpha_3(t)\mathbf{x}\cdot\nabla}. \quad (15)$$

For the sake of convenience, we introduce the notation

$$\alpha_1(t) = \int\limits_0^t a_1(s)ds, \quad \boldsymbol{\alpha}_2(t) = \int\limits_0^t \mathbf{a}_2(s)ds, \quad (16)$$
$$\alpha_3(t) = \int\limits_0^t a_3(s)ds.$$

Consequently (from now on a prime stands for d/dt),

$$\alpha_1'(t) = a_1(t), \quad \boldsymbol{\alpha}_2'(t) = \mathbf{a}_2(t), \quad \alpha_3'(t) = a_3(t), \quad (17)$$
$$\alpha_1(0) = 0, \quad \boldsymbol{\alpha}_2(0) = \mathbf{0}, \quad \alpha_3(0) = 0.$$

Thus, we find from Eq. (11) that

$$u(t,\mathbf{x}) = e^{\alpha_1(t)}e^{[\boldsymbol{\alpha}_2(t)+\alpha_3(t)\mathbf{x}]\cdot\nabla} \times \left(\exp_+ \int\limits_0^t \hat{a}_4(s)e^{2\alpha_3(s)} : \nabla\nabla ds\right)\phi(\mathbf{x}). \quad (18)$$

Using the formulas (see [20] and [21])

$$\left[\exp_+\left(\int\limits_0^t \hat{\Psi}(s) : \nabla\nabla ds\right)\right]\phi(\mathbf{x}) = \frac{1}{\sqrt{\det(4\pi\hat{\tau}(t))}} \times \int\limits_{\mathrm{R}^n}\left\{\exp\left[-(\mathbf{x}-\mathbf{y})\cdot\frac{\hat{\tau}^{-1}(t)}{4}\cdot(\mathbf{x}-\mathbf{y})\right]\right\}\phi(\mathbf{y})dy, \quad (19)$$

where

$$dy = dy_1 dy_2 \ldots dy_n, \quad \hat{\tau}(t) = \int\limits_0^t \hat{\Psi}(s)ds$$

and

$$e^{\boldsymbol{\alpha}_2(t)\cdot\nabla+\alpha_3(t)\mathbf{x}\cdot\nabla}g(\mathbf{x}) = g\left(\mathbf{x}e^{\alpha_3(t)} + \int\limits_0^t \mathbf{a}_2(s)e^{\alpha_3(s)}ds\right) \equiv g(\mathbf{z}), \quad (20)$$

we finally find from Eq. (18) that the exact solution to the Cauchy problem (3) $\left(\hat{\Psi}(s) = \hat{a}_4(s)\exp[2a_3(s)]\right)$ is given by

$$u(t,\mathbf{x}) = \frac{e^{\alpha_1(t)}}{\sqrt{\det(4\pi\hat{\tau}(t))}} \times \int\limits_{\mathrm{R}^n}\left\{\exp\left[-(\mathbf{z}-\mathbf{y})\cdot\frac{\hat{\tau}^{-1}(t)}{4}\cdot(\mathbf{z}-\mathbf{y})\right]\right\}\phi(\mathbf{y})dy, \quad (21)$$

where

$$\hat{\tau}(t) = \int\limits_0^t \hat{a}_4(s)e^{2\alpha_3(s)}ds$$

is a symmetric nonnegative rank-two tensor function of t, $dy = dy_1 \ldots dy_n$, and $\mathbf{z}$ is defined in (20).

Substituting expression (21) into Eq. (3), we immediately see that the function $u(t,\mathbf{x})$ is a solution to the problem specified by (3); according to the Cauchy theorem, it is the only classical solution to this problem.

3. CONCLUDING REMARKS

The exact solution to the Cauchy problem (3) has been obtained by using the algebraic method described above.

When $\hat{a}_4(t)$ is scalar, $\hat{a}_4(t) = a_4(t)\hat{1}$ (in this case, $\hat{a}_4 : \nabla\nabla = a_4\Delta$), the "anisotropic" problem (3) reduces to an "isotropic" one, with the exact solution found in [20]. It is easy to check that the solution in (21) reduces to the solution obtained in [20] {there is an error in [20]: the sign before $\mathbf{a}_2$ in Eqs. (17) and (34) there should be positive}.

In the case of $\hat{a}_4(t) = a_4(t)\hat{a}$, the Cauchy problem (3) reduces to the problem specified by (7) and treated in [21]. In this case, the solution in (21) reduces to the solution obtained in [21].

For different choices of the coefficients a_j and $\mathbf{a}_2$, Eq. (3) may also be regarded as a set of different diffusion equations. From (21), we therefore obtain the exact solutions to the Cauchy problems for this set of diffusion equations.

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Exact Solution to the Classical Equation of Motion for a Charged Particle in External Electric and Magnetic Fields*

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Abstract—An exact solution to the equation of classical motion of a charged particle in external uniform time-dependent electric and magnetic fields is obtained in two forms by two methods. An exact solution of a more general initial-value problem is found as well. © *2002 MAIK "Nauka/Interperiodica".*

The problem of the movement of a charged particle in external electric (**E**) and magnetic (**B**) fields is of fundamental importance in physics, especially in vacuum electronics, accelerator physics, the theory of magnetism, plasma physics, and astrophysics to mention only a few of them (see [1–3]).

In this paper, we find an exact solution to the problem specified by the equations

$$m\frac{d^2\mathbf{r}(t)}{dt^2} = \frac{q}{c}\frac{d\mathbf{r}(t)}{dt}\times\mathbf{B}f_1(t) + q\mathbf{E}f_2(t), \quad (1)$$
$$\mathbf{r}(0) = \mathbf{r}_0, \qquad \frac{d\mathbf{r}(0)}{dt} = \mathbf{v}_0,$$

where $f_1(t)$ and $f_2(t)$ are arbitrary integrable functions of t. Setting

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \qquad \frac{q\mathbf{B}}{mc} = \boldsymbol{\omega}, \qquad \frac{q\mathbf{E}}{m} = \mathbf{e}, \quad (2)$$

we arrive at

$$\frac{d\mathbf{v}}{dt} + \boldsymbol{\omega}\times\mathbf{v}(t)f_1(t) = \mathbf{e}f_2(t), \qquad \mathbf{v}(0) = \mathbf{v}_0. \quad (3)$$

Thus, solving the problem specified by (1) reduces to solving the problem specified by (3).

According to the theory of differential equations, a general solution to Eq. (3) is given by

$$\mathbf{v}(t) = \mathbf{v}_h(t) + \tilde{\mathbf{v}}(t), \quad (4)$$

where $\tilde{\mathbf{v}}(t)$ is a particular solution to Eq. (3) and $\mathbf{v}_h(t)$ is a general solution to the corresponding homogeneous equation

$$\frac{d\mathbf{v}_h}{dt} + \boldsymbol{\omega}\times\mathbf{v}_h(t)f_1(t) = 0. \quad (5)$$

It is easy to check that $\mathbf{v}_h(t)$ is given by

$$\mathbf{v}_h(t) = \left[e^{-\boldsymbol{\omega}\times\hat{\mathbf{1}}\phi_1(t)}\right]\cdot\mathbf{C}, \quad (6)$$

where $\mathbf{C}$ is a constant vector,

$$\phi_1(t) = \int f_1(t)dt, \quad (7)$$

and

$$e^{\mathbf{a}(t)\times\hat{\mathbf{1}}} = \frac{\mathbf{a}\cdot\mathbf{a}}{|\mathbf{a}|^2} + \left(\hat{\mathbf{1}} - \frac{\mathbf{a}\cdot\mathbf{a}}{\mathbf{a}^2}\right)\cos|\mathbf{a}| + \hat{\mathbf{1}}\times\frac{\mathbf{a}}{|\mathbf{a}|}\sin|\mathbf{a}| \quad (8)$$

is the tensor of the direct rotation through the angle $|\mathbf{a}|$ about the axis $\mathbf{a}$; here, $\mathbf{a}\cdot\mathbf{b}$ is the tensor product of the vectors $\mathbf{a}$ and $\mathbf{b}$: $(\mathbf{a}\cdot\mathbf{b})_{jk} = a_jb_k$; $\hat{\mathbf{1}}$ is the rank-2 unit tensor: $(\hat{\mathbf{1}})_{jk} = \delta_{jk}$; and $(\mathbf{a}\times\hat{\mathbf{1}})_{jk} = (\hat{\mathbf{1}}\times\mathbf{a})_{jk} = \varepsilon_{kjl}a_l$, where δ_{jk} and ε_{ijk} are the Kronecker delta and the Levi-Civita symbol, respectively.

To find a particular solution $\tilde{\mathbf{v}}(t)$ to Eq. (3), we will proceed in two ways.

(A) First method. As the linear independent vectors in our problem are $\boldsymbol{\omega}$, $\mathbf{e}$, and $\boldsymbol{\omega}\times\mathbf{e}$, we will seek a solution to Eq. (3) in the form

$$\tilde{\mathbf{v}}(t) = \alpha(t)\boldsymbol{\omega} + \beta(t)\mathbf{e} + \gamma(t)\boldsymbol{\omega}\times\mathbf{e}, \quad (9)$$

where $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are unknown scalar functions of t. Inserting (9) into Eq. (3), we find that these coefficient functions satisfy the set of three ordinary differential equations

$$\frac{d\alpha}{dt} = -(\boldsymbol{\omega}\cdot\mathbf{e})\gamma(t)f_1(t), \quad (10)$$

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$$\frac{d\beta}{dt} = \omega^2\gamma(t)f_1(t) + f_2(t),$$
$$\frac{d\gamma}{dt} = -\beta(t)f_1(t),$$

where $\omega = |\boldsymbol{\omega}|$. Since we seek a particular solution to Eq. (3) in the form (9), it is sufficient to find one particular solution to the set of Eqs. (10).

From (10), we deduce that $\gamma(t)$ satisfies the equation

$$\frac{d^2\gamma}{dt^2} - \frac{f_1'(t)}{f_1(t)}\frac{d\gamma}{dt} + [\omega f_1(t)]^2\gamma(t) = -f_1(t)f_2(t); \tag{11}$$

introducing the dimensionless variable

$$x = \Omega t, \tag{12}$$

we have

$$\frac{d^2\gamma}{dx^2} - \frac{f_1'(x)}{f_1(x)}\frac{d\gamma}{dx} + \left[\frac{\omega}{\Omega}f_1(x)\right]^2\gamma(x) = -\frac{f_1(x)f_2(x)}{\Omega^2} \equiv h(x). \tag{13}$$

One particular solution to this equation is given by (see [4])

$$\gamma(x) = \varphi_2(x)\int\frac{\varphi_1(x)h(x)}{w(x)}dx - \varphi_1(x)\int\frac{\varphi_2(x)h(x)}{w(x)}dx, \tag{14}$$

where $w(x)$ is the Wronskian of the fundamental solutions $\varphi_1(x)$ and $\varphi_2(x)$ to the corresponding homogeneous equation (see [5])

$$\frac{d^2\gamma_h}{dx^2} - \frac{f_1'(x)}{f_1(x)}\frac{d\gamma_h}{dx} + \left[\frac{\omega}{\Omega}f_1(x)\right]^2\gamma_h(x) = 0, \tag{15}$$

$$\gamma_h(x) = c_1\varphi_1(x) + c_2\varphi_2(x). \tag{16}$$

In our case, we have

$$\varphi_1(x) = \sin\left[\frac{\omega}{\Omega}\phi_1(x)\right], \tag{17}$$
$$\varphi_2(x) = \cos\left[\frac{\omega}{\Omega}\phi_1(x)\right],$$

$$w(x) = \begin{vmatrix}\varphi_1(x) & \varphi_2(x)\\ \varphi_1'(x) & \varphi_2'(x)\end{vmatrix} = -\frac{\omega}{\Omega}f_1(x),$$

where

$$\phi_1(x) = \int f_1(x)dx. \tag{18}$$

Thus, from (13), (14), (17), and (18), we obtain

$$\gamma(x) = -\frac{1}{\omega\Omega}\left\{\sin\left[\frac{\omega}{\Omega}\phi_1(x)\right]\times\int\cos\left[\frac{\omega}{\Omega}\phi_1(x)\right]f_2(x)dx - \cos\left[\frac{\omega}{\Omega}\phi_1(x)\right]\times\int\sin\left[\frac{\omega}{\Omega}\phi_1(x)\right]f_2(x)dx\right\};$$

taking into account (12) and (18), we find one particular solution for the coefficient function $\gamma(t)$; that is,

$$\gamma(t) = -\frac{1}{\omega}\{c(t)\sin[\omega\phi_1(t)] - s(t)\cos[\omega\phi_1(t)]\}, \tag{19}$$

where

$$c(t) = \int\cos[\omega\phi_1(t)]f_2(t)dt, \tag{20}$$
$$s(t) = \int\sin[\omega\phi_1(t)]f_2(t)dt.$$

From (10) and (19), we then obtain particular solutions for $\beta(t)$ and $\alpha(t)$; that is,

$$\beta(t) = c(t)\cos[\omega\phi_1(t)] + s(t)\sin[\omega\phi_1(t)], \tag{21}$$

$$\alpha(t) = -\frac{(\boldsymbol{\omega}\cdot\mathbf{e})}{\omega^2}\{c(t)\cos[\omega\phi_1(t)] + s(t)\sin[\omega\phi_1(t)] - \phi_2(t)\}, \tag{22}$$

where

$$\phi_2(t) = \int f_2(t)dt. \tag{23}$$

Using (7), (8), and (19)–(23), we now have one solution $\tilde{\mathbf{v}}(t)$ to Eq. (3):

$$\tilde{\mathbf{v}}(t) = \frac{(\boldsymbol{\omega}\cdot\mathbf{e})\boldsymbol{\omega}}{\omega^2}\phi_2(t) + \frac{(\boldsymbol{\omega}\times\mathbf{e})\times\boldsymbol{\omega}}{\omega^2}\times\{s(t)\sin[\omega\phi_1(t)] + c(t)\cos[\omega\phi_1(t)]\} - \frac{\boldsymbol{\omega}\times\mathbf{e}}{\omega}\{c(t)\sin[\omega\phi_1(t)] - s(t)\cos[\omega\phi_1(t)]\}. \tag{24}$$

From (4) and (6), we therefore find a general solution to Eq. (3) in the form

$$\mathbf{v}(t) = \left[e^{-\boldsymbol{\omega}\times\hat{\mathbf{i}}\phi_1(t)}\right]\cdot\mathbf{C} + \tilde{\mathbf{v}}(t), \tag{25}$$

where $\tilde{\mathbf{v}}(t)$ is given by (24).

Taking into account the initial condition

$$\mathbf{v}_0 = \mathbf{v}(0) = \left[e^{-\boldsymbol{\omega}\times\hat{\mathbf{i}}\phi_1(0)}\right]\cdot\mathbf{C} + \frac{(\boldsymbol{\omega}\cdot\mathbf{e})\boldsymbol{\omega}}{\omega^2}\phi_2(0) + \frac{(\boldsymbol{\omega}\times\mathbf{e})\times\boldsymbol{\omega}}{\omega^2}\times\{s(0)\sin[\omega\phi_1(0)] + c(0)\cos[\omega\phi_1(0)]\} - \frac{\boldsymbol{\omega}\times\mathbf{e}}{\omega}\{c(0)\sin[\omega\phi_1(0)] - s(0)\cos[\omega\phi_1(0)]\} \tag{26}$$

and using (25), we finally obtain

$$\mathbf{v}(t) = e^{-\boldsymbol{\omega}\times\hat{\mathbf{i}}[\phi_1(t)-\phi_1(0)]}\cdot\left\{\mathbf{v}_0\right. \tag{27}$$

$$
\begin{aligned}
&- \frac{(\boldsymbol{\omega}\cdot\mathbf{e})\boldsymbol{\omega}}{\omega^2}\phi_2(0) - \frac{(\boldsymbol{\omega}\times\mathbf{e})\times\boldsymbol{\omega}}{\omega^2} \\
&\times \{s(0)\sin[\omega\phi_1(0)] + c(0)\cos[\omega\phi_1(0)]\} \\
&+ \frac{\boldsymbol{\omega}\times\mathbf{e}}{\omega}\{c(0)\sin[\omega\phi_1(0)] - s(0)\cos[\omega\phi_1(0)]\}\Bigg\} \\
&+ \frac{(\boldsymbol{\omega}\cdot\mathbf{e})\boldsymbol{\omega}}{\omega^2}\phi_2(t) + \frac{(\boldsymbol{\omega}\times\mathbf{e})\times\boldsymbol{\omega}}{\omega^2} \\
&\times \{s(t)\sin[\omega\phi_1(t)] + c(t)\cos[\omega\phi_1(t)]\} \\
&- \frac{\boldsymbol{\omega}\times\mathbf{e}}{\omega}\{c(t)\sin[\omega\phi_1(t)] - s(t)\cos[\omega\phi_1(t)]\},
\end{aligned}
$$

where $\phi_1(t)$ is from (7), $\phi_2(t)$ is from (23), $c(t)$ and $s(t)$ are from (20), and the tensor $\exp(\mathbf{a}\times\hat{\mathbf{1}})$ is from (8).

By means of a substitution, one can check that expression (27) for the vector $\mathbf{v}(t)$ is the exact solution to the initial-value problem (3).

(B) Second method. This method is based on a "variation of the constants." Having formula (6) for a general solution to the homogeneous Eq. (5), we try to find one particular solution $\tilde{\mathbf{v}}$ to the nonhomogeneous Eq. (3) by varying the constant vector $\mathbf{C}$ in

$$\tilde{\mathbf{v}}(t) = \left[e^{-\boldsymbol{\omega}\times\hat{\mathbf{1}}\int_0^t f_1(s)ds}\right]\cdot\mathbf{C}(t). \tag{28}$$

Inserting this expression into Eq. (3), we obtain

$$\frac{d\mathbf{C}(t)}{dt} = \left[e^{\boldsymbol{\omega}\times\hat{\mathbf{1}}\int_0^t f_1(s)ds} f_2(t)\right]\cdot\mathbf{e};$$

therefore, we have

$$\mathbf{C}(t) = \left[\int_0^t e^{\boldsymbol{\omega}\times\hat{\mathbf{1}}\int_0^\tau f_1(s)ds} f_2(\tau)d\tau\right]\cdot\mathbf{e} + \mathbf{D}, \tag{29}$$

where $\mathbf{D}$ is a constant vector. From (28), we thus finally have

$$
\begin{aligned}
\tilde{\mathbf{v}}(t) &= e^{-\boldsymbol{\omega}\times\hat{\mathbf{1}}\int_0^t f_1(s)ds} \\
&\cdot\left\{\mathbf{D} + \left[\int_0^t e^{\boldsymbol{\omega}\times\hat{\mathbf{1}}\int_0^\tau f_1(s)ds} f_2(\tau)d\tau\right]\cdot\mathbf{e}\right\}.
\end{aligned} \tag{30}
$$

$$\exp_+\int_0^t \hat{\mathbf{A}}(\tau)d\tau = \hat{\mathbf{1}} + \lim_{k\to\infty}\sum_{n=1}^{k}\int_0^t dt_1\int_0^{t_1} dt_2\cdots\int_0^{t_n} dt_n\hat{\mathbf{A}}(t_1)\hat{\mathbf{A}}(t_2)\cdots\hat{\mathbf{A}}(t_n) \tag{35}$$

is the Volterra ordered exponential.

We seek a particular solution to the nonhomogeneous Eq. (32) by varying the constant vector $\mathbf{D}$ in (34):

$$\tilde{\mathbf{v}}(t) = \left[\exp_+\int_0^t \mathbf{a}(\tau)\times\hat{\mathbf{1}}d\tau\right]\cdot\mathbf{D}(t). \tag{36}$$

From (4), (6), and (28), we now obtain

$$
\begin{aligned}
\mathbf{v}(t) &= e^{-\boldsymbol{\omega}\times\hat{\mathbf{1}}\int_0^t f_1(\tau)d\tau}\cdot(\mathbf{C}+\mathbf{D}) \\
&+ e^{-\boldsymbol{\omega}\times\hat{\mathbf{1}}\int_0^t f_1(s)ds}\cdot\left[\int_0^t e^{\boldsymbol{\omega}\times\hat{\mathbf{1}}\int_0^\tau f_2(\tau)d\tau}\right]\cdot\mathbf{e};
\end{aligned}
$$

using the initial condition $\mathbf{v}_0 = \mathbf{v}(0) = \mathbf{C}+\mathbf{D}$, we find another form for the exact solution to the initial-value problem (3):

$$
\begin{aligned}
\mathbf{v}(t) &= e^{-\boldsymbol{\omega}\times\hat{\mathbf{1}}\int_0^t f_1(\tau)d\tau} \\
&\cdot\left\{\mathbf{v}_0 + \left[\int_0^t e^{\boldsymbol{\omega}\times\hat{\mathbf{1}}\int_0^\tau f_1(s)ds} f_2(\tau)d\tau\right]\cdot\mathbf{e}\right\}.
\end{aligned} \tag{31}
$$

The verification of (31) as a solution to the problem specified by (3) is straightforward. By means of simple but cumbersome calculations, one can also derive (27) from (31).

(C) Using the second method—variation of the constants—we can find the exact solution to the initial-value problem

$$\frac{d\mathbf{v}}{dt} = \mathbf{a}(t)\times\mathbf{v}(t) + \mathbf{b}(t), \quad \mathbf{v}(0) = \mathbf{v}_0, \tag{32}$$

which is more general than that specified by (3),

Now, the homogeneous equation is

$$\frac{d\mathbf{v}_h}{dt} = \mathbf{a}(t)\times\mathbf{v}_h(t), \tag{33}$$

with a general solution given by the formula

$$\mathbf{v}_h(t) = \left[\exp_+\int_0^t \mathbf{a}(\tau)\times\hat{\mathbf{1}}d\tau\right]\cdot\mathbf{D}, \tag{34}$$

where

Inserting (36) into (32), we now obtain the following equation for $\mathbf{D}(t)$:

$$\frac{d\mathbf{D}(t)}{dt} = \left[\exp_-\left(-\int_0^t \mathbf{a}(s)\times\hat{\mathbf{1}}ds\right)\right]\cdot\mathbf{b}(t). \tag{37}$$

In the above calculation, we have considered that the ordered exponential possesses the properties

$$\frac{d}{dt}\left[\exp_+\int_0^t \hat{\mathbf{A}}(s)ds\right] = \hat{\mathbf{A}}(t)\cdot\left[\exp_+\int_0^t \hat{\mathbf{A}}(s)ds\right], \tag{38}$$

$$\left[\exp_+\int_0^t \hat{\mathbf{A}}(s)ds\right]^{-1} = \exp_-\left(-\int_0^t \hat{\mathbf{A}}(s)ds\right), \tag{39}$$

where

$$\exp_-\int_0^t \hat{\mathbf{B}}(s)ds = \hat{\mathbf{1}} + \lim_{k\to\infty}\sum_{n=1}^{k}\int_0^t dt_1\int_0^{t_1} dt_2\cdots\int_0^{t_n} dt_n\hat{\mathbf{B}}(t_n)\hat{\mathbf{B}}(t_{n-1})\cdots\hat{\mathbf{B}}(t_1), \tag{40}$$

$$\frac{d}{dt}\left[\exp_-\int_0^t \hat{\mathbf{B}}(s)ds\right] = \left[\exp_-\int_0^t \hat{\mathbf{B}}(s)ds\right]\cdot\hat{\mathbf{B}}(t). \tag{41}$$

From (37), we find

$$\mathbf{D}(t) = \int_0^t\left[\exp_-\left(-\int_0^\tau \mathbf{a}(s)\times\hat{\mathbf{1}}ds\right)\right]\cdot\mathbf{b}(\tau)d\tau + \mathbf{C};$$

from (36), we deduce a particular solution to Eq. (32) in the form

$$\tilde{\mathbf{v}}(t) = \left[\exp_+\int_0^t \mathbf{a}(s)\times\hat{\mathbf{1}}ds\right]\cdot\left\{\mathbf{C} + \int_0^t\left[\exp_-\left(-\int_0^\tau \mathbf{a}(s)\times\hat{\mathbf{1}}ds\right)\right]\cdot\mathbf{b}(\tau)d\tau\right\}. \tag{42}$$

For the general solution of Eq. (32), we then find

$$\mathbf{v}(t) = \mathbf{v}_h(t) + \tilde{\mathbf{v}}(t), \tag{43}$$

where $\mathbf{v}_h(t)$ and $\tilde{\mathbf{v}}(t)$ are from (34) and (42), respectively.

Under the initial condition $\mathbf{v}(0) = \mathbf{v}_0$, we have $\mathbf{v}_0 = \mathbf{D} + \mathbf{C}$. From (43), we thus obtain

$$\mathbf{v} = \left[\exp_+\int_0^t \mathbf{a}(s)\times\hat{\mathbf{1}}ds\right]\cdot\left\{\mathbf{v}_0 + \int_0^t\left[\exp_-\left(-\int_0^\tau \mathbf{a}(s)\times\hat{\mathbf{1}}ds\right)\right]\cdot\mathbf{b}(\tau)d\tau\right\}. \tag{44}$$

Using (38), (39), and (41), one can check that expression (44) is an exact solution to the initial-value problem (32).

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Dynamical *r* Matrices and Chiral WZNW Phase Space*

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Abstract—A dynamical generalization of the classical Yang–Baxter equation that governs the possible Poisson structures on the space of chiral WZNW fields with a generic monodromy is reviewed. It is explained that, for particular choices of chiral WZNW Poisson brackets, this equation reduces to the CDYB equation recently studied by Etingof and Varchenko and by others. Interesting dynamical r matrices are obtained for a generic monodromy, as well as by imposing Dirac constraints on the monodromy.

1. INTRODUCTION

The classical and quantum Yang–Baxter equations occupy a central position in the modern theory of integrable systems. Recently, dynamical generalizations of these structures attracted considerable attention. Of particular interest to us is the classical dynamical Yang–Baxter (CDYB) equation

$$\big[\hat{r}_{12}(\omega),\hat{r}_{23}(\omega)\big]+H_1^i\frac{\partial}{\partial\omega^i}\hat{r}_{12}(\omega)+\text{cycl. perm.}=C\hat{f}, \tag{1}$$

where the variable ω lies in a Cartan subalgebra $\mathcal{H}$ of a simple Lie algebra $\mathcal{G}$, $\{H^i\}$ is a basis of $\mathcal{H}$, C is some constant, and $\hat{f}$ is the canonical $\mathcal{G}$-invariant element in $\mathcal{G}^{3\wedge}$. It is usually assumed that $\hat{r}(\omega)$ is $\mathcal{H}$-invariant and that its symmetric part is proportional to the "tensor Casimir" operator. The CDYB equation is the classical limit of the Gervais–Neveu–Felder equation

$$R_{12}(\omega+\hbar H_3)R_{13}(\omega)R_{23}(\omega+\hbar H_1)=R_{23}(\omega)R_{13}(\omega+\hbar H_2)R_{12}(\omega). \tag{2}$$

These equations govern the classical and quantum exchange algebras of chiral Bloch waves in conformal Toda and WZNW field theories on a cylinder [1–3]. They also appear in describing conformal blocks of the WZNW model on a torus [4] and in studying Calogero–Moser models [5]. Solutions to these equations and the underlying abstract algebraic structures, the so-called dynamical Poisson–Lie (PL) groupoids and dynamical quantum groups, were studied recently in detail by Etingof and Varchenko (see the review article of Etingof and Schiffmann [6] and references therein, where further applications are described too). In [7], generalizations of the CDYB equation were introduced, which are obtained from (1) by replacing the Cartan subalgebra by an arbitrary subalgebra of $\mathcal{G}$. Here, we call this generalization the $\mathcal{H}$-CDYB equation allowing $\mathcal{H}\subseteq\mathcal{G}$ to be any subalgebra.

The possible chiral extensions of the WZNW phase space were investigated in [8], where it was found that a new generalization of the CDYB equation naturally arises in this context. This equation will be called the G-CDYB equation, since its dynamical variable lies in the group G associated with $\mathcal{G}$. The G-CDYB equation encodes the most general Poisson brackets (PB) of the chiral WZNW fields with a generic monodromy. Any solution to this equation also defines a PL groupoid. Under some special circumstances, when natural gauge transformations act on chiral WZNW phase space as a classical $\mathcal{G}$-symmetry, our G-CDYB equation reduces to the $\mathcal{G}$-CDYB equation (i.e., the $\mathcal{H}$-CDYB equation for $\mathcal{H}=\mathcal{G}$). Then, Dirac reductions of chiral WZNW phase space result in dynamical r matrices that solve the $\mathcal{H}$-CDYB equation for self-dual subalgebras $\mathcal{H}\subset\mathcal{G}$. For instance, we recover, in this way, a fundamental solution to the original CDYB equation that was first obtained in [3] by a different method.

It will be illustrated in this report that chiral WZNW phase space serves as an effective source of dynamical r matrices. The quantization of these r matrices and their associated PL groupoids should contribute to a better understanding of the quantum-group properties of the WZNW model, but this issue is not yet properly understood.

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2. G-CDYB EQUATION AND PL GROUPOIDS FROM CHIRAL WZNW

The WZNW model [9] as a classical field theory on a cylinder can be defined for any (real or complex) Lie group G whose Lie algebra $\mathcal{G}$ carries an invariant, nondegenerate bilinear form $\langle\ ,\ \rangle$. The scalar product is proportional to $\mathrm{tr}(XY)$ if $\mathcal{G}$ is a simple Lie algebra; to simplify the notation, we shall denote $\langle X,Y\rangle$ $(\forall X,Y\in\mathcal{G})$ by $\mathrm{tr}(XY)$, in general. A solution to the classical field equation for the G-valued WZNW field that is 2π-periodic in the space variable proves to be the product of left- and right-moving factors. Chiral WZNW fields are quasiperiodic; i.e., they are elements in

$$\mathcal{M}_G := \{g\in C^\infty(\mathbb{R},G)|g(x+2\pi) = g(x)M\ M\in G\}. \tag{3}$$

Since the chiral factors of the full WZNW field are determined only apart from a gauge freedom, the symplectic structure of the WZNW model does not yield a unique PB on $\mathcal{M}_G$. In fact, as explained in [10], $\mathcal{M}_G$ is equipped canonically only with a quasi-Poisson structure in the sense of [11]. To describe the system in terms of genuine PBs for chiral fields and an associated chiral symplectic form [12], in general, one needs to restrict oneself to a submanifold of $\mathcal{M}_G$, where the monodromy matrix M belongs to some submanifold $\check{G}\subset G$. A condition on $\check{G}$ is that the canonical closed 3-form of $G=\{M\}$, given by

$$\chi = \frac{1}{6}\mathrm{tr}\left(M^{-1}dM\wedge M^{-1}dM\wedge M^{-1}dM\right),$$

must become exact upon restriction to $\check{G}\subset G$. One may then choose a 2-form ρ on $\check{G}$ for which $d\rho=\chi_{\check{G}}$, where $\chi_{\check{G}}$ is the restriction of χ to $\check{G}$. For any such ρ, one can define a closed 2-form Ω^ρ on

$$\mathcal{M}_{\check{G}} := \{g\in C^\infty(\mathbb{R},G)|g(x+2\pi) = g(x)M\quad M\in\check{G}\} \tag{4}$$

by the formula

$$\frac{1}{\kappa}\Omega^\rho = -\frac{1}{2}\int_0^{2\pi}dx\mathrm{tr}\left(g^{-1}dg\right)\wedge\left(g^{-1}dg\right)' \tag{5}$$
$$-\frac{1}{2}\mathrm{tr}\left((g^{-1}dg)(0)\wedge dMM^{-1}\right)+\rho(M),$$

where κ is a constant. If a further condition is satisfied, which we shall state below, then Ω^ρ is (weakly) nondegenerate; therefore, it can be inverted to define PBs on a set of "admissible" functions of the chiral WZNW field. The derivation of Ω^ρ from the symplectic structure of the full WZNW model is due to Gawędzki [12].

One may ensure the exactness of $\chi_{\check{G}}$ by choosing $\check{G}\subset G$ to be a topologically trivial open submanifold. In this case, the following description of chiral PBs was obtained in [8] by extending the results of [13]. In fact, the PBs of all admissible functions [8] are encoded by the "distribution-valued" PBs of the matrix elements of $g(x)$, which have the form

$$\left\{g(x)\overset{\otimes}{,}g(y)\right\} = \frac{1}{\kappa}\Big(g(x)\otimes g(y)\Big) \tag{6}$$
$$\times\left(\frac{1}{2}\hat{I}\mathrm{sgn}(y-x)+\hat{r}(M)\right),\quad 0<x,y<2\pi.$$

Here, an interesting object is the "exchange r matrix" $\hat{r}(M)=r^{ab}(M)T_a\otimes T_b\in\mathcal{G}\wedge\mathcal{G}$; $\hat{I}=T_a\otimes T^a$, where $\{T_a\}$ and $\{T^a\}$ denote dual bases of $\mathcal{G}$, $\mathrm{tr}(T_aT^b)=\delta_a^b$, and summation over coinciding indices is implied. The Jacobi identity of the PB is equivalent to a dynamical generalization of the CYB equation, which we call the G-CDYB equation. To write it down, we introduce, on functions ψ on G, the derivations $\mathcal{D}_a^\pm=\mathcal{R}_a\pm\mathcal{L}_a$ by

$$(\mathcal{R}_a\psi)(M) := \frac{d}{dt}\psi(Me^{tT_a})\Big|_{t=0}, \tag{7}$$
$$(\mathcal{L}_a\psi)(M) := \frac{d}{dt}\psi(e^{tT_a}M)\Big|_{t=0}.$$

The G-CDYB equation [8] reads as

$$\Big[\hat{r}_{12}(M),\hat{r}_{23}(M)\Big]+T_1^a\Big(\frac{1}{2}\mathcal{D}_a^+ \tag{8}$$
$$+r_a{}^b(M)\mathcal{D}_b^-\Big)\hat{r}_{23}(M)+\text{cycl. perm.} = -\frac{1}{4}\hat{f},$$

where $\hat{f}:=f_{ab}{}^cT^a\otimes T^b\otimes T_c$ with $[T_a,T_b]=f_{ab}{}^cT_c$ and the cyclic permutation is over the three tensorial factors with $\hat{r}_{23}=r^{ab}(1\otimes T_a\otimes T_b)$, $T_1^a=T^a\otimes 1\otimes 1$, and so on. This equation becomes the modified classical YB equation if $\hat{r}$ is an M-independent constant; at the same time, it is a generalization of the CDYB equation (1).

The exchange r matrix that results from the inversion of the symplectic form in (5) automatically satisfies (8). To describe its dependence on the 2-form ρ, we expand ρ as

$$\rho(M)=\frac{1}{2}q^{ab}(M)\mathrm{tr}\big(T_aM^{-1}dM\big)\wedge\mathrm{tr}\big(T_bM^{-1}dM\big),$$

where $q^{ab}=-q^{ba}$. Further, we denote by $q(M)$ and $r(M)$ the linear operators on $\mathcal{G}$ whose matrices are $q^{ab}(M)$ and $r^{ab}(M)$, respectively, and also introduce the operators $q_\pm(M):=q(M)\pm\frac{1}{2}I$ and $r_\pm(M):=r(M)\pm\frac{1}{2}I$, where I is the identity operator on $\mathcal{G}$. It was proven in [8] that the inversion of Ω^ρ leads to (6) with

$$r_-(M) = -q_-(M)\circ(q_-(M) \tag{9}$$

$$- \mathrm{Ad}M \circ q_+(M))^{-1}.$$

The condition on the pair $(\check{G}, \rho)$ that guarantees the nondegeneracy of the 2-form Ω^ρ is that $(q_-(M) - \mathrm{Ad}M \circ q_+(M)) \in \mathrm{End}(\mathcal{G})$ must be an invertible operator for any $M \in \check{G}$. This can be ensured by restricting M to be sufficiently close to $e \in G$.

Any solution to (8) on some domain $\check{G}$ gives rise to the PB (6) on $\mathcal{M}_{\check{G}}$, and any such PB implies that $J := \kappa g' g^{-1}$ satisfies the standard current-algebra PBs and that $g(x)$ is a primary field with respect to the current algebra. The exchange r matrix drops out from the PBs with any function of the current J; thus, it encodes the "noncurrent-algebraic" aspects of infinite-dimensional chiral WZNW phase space. Remarkably, the exchange r matrix also defines the PBs of an associated PL groupoid, as described below.

Let $\hat{r}$ be a solution to (8) on $\check{G}$; further, we define $\mathcal{G} \otimes \mathcal{G}$-valued functions on $\check{G}$ by

$$\hat{\Theta}(M) = \hat{r}_+(M) - M_2^{-1}\hat{r}_-(M)M_2, \qquad (10)$$
$$\hat{\Delta}(M) = \hat{\Theta}(M) - M_1^{-1}\hat{\Theta}(M)M_1$$

with $M_1 = M \otimes 1$ and $M_2 = 1 \otimes M$ and then introduce, on the manifold P given by

$$P := \check{G} \times G \times \check{G} := \{(M^F, g, M^I)\}, \qquad (11)$$

a PB $\{\ ,\ \}_P$ by the formulas

$$\kappa\{g_1, g_2\}_P = g_1 g_2 \hat{r}(M^I) - \hat{r}(M^F) g_1 g_2, \qquad (12)$$
$$\kappa\{g_1, M_2^I\}_P = g_1 M_2^I \hat{\Theta}(M^I),$$
$$\kappa\{g_1, M_2^F\}_P = M_2^F \hat{\Theta}(M^F) g_1,$$
$$\kappa\{M_1^I, M_2^I\}_P = M_1^I M_2^I \hat{\Delta}(M^I),$$
$$\kappa\{M_1^F, M_2^F\}_P = -M_1^F M_2^F \hat{\Delta}(M^F),$$
$$\kappa\{M_1^I, M_2^F\}_P = 0.$$

Here, P is an example of the simplest sort of groupoids [14]: the base is $\check{G}$, the source and target projections operate as $s : (M^F, g, M^I) \mapsto M^I$ and $t : (M^F, g, M^I) \mapsto M^F$, and the partial multiplication is defined by $(M^F, g, M^I)(\bar{M}^F, \bar{g}, \bar{M}^I) := (M^F, g\bar{g}, \bar{M}^I)$ for $M^I = \bar{M}^F$. Further, P is a PL groupoid in the sense of [15]. This means that the graph of the partial multiplication, i.e., the subset of

$$P \times P \times P = \{(M^F, g, M^I)\} \qquad (13)$$
$$\times \{(\bar{M}^F, \bar{g}, \bar{M}^I)\} \times \{(\hat{M}^F, \hat{g}, \hat{M}^I)\}$$

defined by the constraints $M^I = \bar{M}^F$, $\hat{M}^F = M^F$, $\hat{M}^I = \bar{M}^I$, and $\hat{g} = g\bar{g}$, is a coisotropic submanifold of $P \times P \times P^-$, where P^- denotes the manifold P endowed with the opposite of the PB on P. In other words, the graph is defined by imposing first-class constraints on the Poisson space $P \times P \times P^-$ equipped with the natural direct-product PB. This would actually hold for any choice of structure functions $\hat{r}(M)$, $\hat{\Theta}(M)$, and $\hat{\Delta}(M)$ in (12), and the choice (10) in terms of a solution to (8) guarantees the Jacobi identity for $\{\ ,\ \}_P$.

We have extracted a PL groupoid from any symplectic structure Ω^ρ on chiral WZNW phase space. If the exchange r matrix is constant, then the PL groupoid P carries the same information as the group G endowed with the corresponding Sklyanin bracket. It is an open problem to study these PL groupoids further in general, to understand their quantization, and to relate them to quantized (chiral) WZNW conformal field theory.

3. $\mathcal{G}$-CDYB EQUATION FROM $\mathcal{G}$ SYMMETRY AND $\mathcal{H}$-CDYB EQUATION FROM DIRAC REDUCTIONS

We next describe an interesting special case of the chiral WZNW symplectic structure Ω^ρ, for which the corresponding exchange r matrix becomes a solution to the $\mathcal{G}$-CDYB equation mentioned in the Introduction, and then consider some Dirac reductions.

Let us suppose that $\check{G}$ is diffeomorphic to a domain $\check{\mathcal{G}} \subset \mathcal{G}$ by the exponential parametrization, whereby we write $\check{G} \ni M = e^\omega$ with $\omega \in \check{\mathcal{G}}$. The chiral WZNW fields whose monodromy lies in $\check{G}$ can be parametrized as

$$g(x) = \eta(x) e^{\bar{\omega}x}, \quad \bar{\omega} := \frac{\omega}{2\pi}, \quad \eta \in \widetilde{G}, \qquad (14)$$

where $\widetilde{G} = \{\eta \in C^\infty(\mathbb{R}, G) | \eta(x + 2\pi) = \eta(x)\}$ and Eq. (14) defines the identification $\mathcal{M}_{\check{G}} = \widetilde{G} \times \check{\mathcal{G}} = \{(\eta, \omega)\}$. If we now choose the 2-form ρ on $\check{G} \simeq \check{\mathcal{G}}$ to be

$$\rho_0(\omega) = -\frac{1}{2}\int_0^{2\pi} dx \mathrm{tr}\left(d\bar{\omega} \wedge de^{x\bar{\omega}} e^{-x\bar{\omega}}\right), \qquad (15)$$

then, in terms of the variables η and ω, we find

$$\frac{1}{\kappa}\Omega^{\rho_0} = -\frac{1}{2}\int_0^{2\pi} dx \mathrm{tr}\left(\eta^{-1}d\eta\right) \wedge \left(\eta^{-1}d\eta\right)' \qquad (16)$$
$$+ d\int_0^{2\pi} dx \mathrm{tr}\left(\bar{\omega}\eta^{-1}d\eta\right).$$

Note that Ω^{ρ_0} is invariant under the natural action of the group G on $\mathcal{M}_{\check{G}}$ given by

$$G \ni h : g(x) \mapsto g(x)h^{-1}, \qquad (17)$$

$$\text{i.e.,}\quad \eta(x)\mapsto\eta(x)h^{-1},\quad \omega\mapsto h\omega h^{-1}.$$

It is assumed that $\check{\mathcal{G}}$ is invariant under the action of G; otherwise, one has to consider the corresponding $\mathcal{G}$ action. Since Ω^{ρ_0} is symplectic, which may be ensured by taking $\check{\mathcal{G}}$ to be a neighborhood of 0, the $\mathcal{G}$ symmetry obtained from (17) is generated by a classical momentum map. The value of this $\mathcal{G}\simeq\mathcal{G}^*$-valued momentum map is proportional to ω. In fact, we can find that the PB $\{\ ,\ \}_0$ corresponding to Ω^{ρ_0} gives

$$\{g(x),\omega_a\}_0=\frac{1}{\kappa}g(x)T_a, \tag{18}$$
$$\{\omega_a,\omega_b\}_0=-\frac{1}{\kappa}f_{ab}{}^c\omega_c.$$

Moreover, we have

$$\Big\{g(x)\overset{\otimes}{,}g(y)\Big\}_0=\frac{1}{\kappa}\Big(g(x)\otimes g(y)\Big) \tag{19}$$
$$\times\Big(\frac{1}{2}\hat{I}\mathrm{sgn}(y-x)+\hat{r}^0(\omega)\Big),\quad 0<x,y<2\pi,$$

where $\hat{r}^0(\omega)$ denotes the exchange r matrix associated with $\rho_0(\omega)$ by (9). Now, the Jacobi identity of the PB for the functions $\omega_a, g(x)$, and $g(y)$ $(x\neq y)$ and the relations in (18) imply that $\hat{r}^0$ is a $\mathcal{G}$-equivariant function on $\check{\mathcal{G}}$:

$$\frac{d}{dt}\hat{r}^0(e^{tT}\omega e^{-tT})|_{t=0} \tag{20}$$
$$=[T\otimes 1+1\otimes T,\hat{r}^0(\omega)],\qquad \forall T\in\mathcal{G}.$$

In the present case, the Jacobi identity for three evaluation functions $g(x_i)$ $(x_i\neq x_j)$ gives a simplified version of the G-CDYB Eq. (8). Namely, the Jacobi identity and (18) imply that

$$\big[\hat{r}^0_{12}(\omega),\hat{r}^0_{23}(\omega)\big]+T^a_1\frac{\partial}{\partial\omega^a}\hat{r}^0_{23}(\omega) \tag{21}$$
$$+\text{ cycl. perm.}=-\frac{1}{4}\hat{f}\quad(\check{\mathcal{G}}\ni\omega=\omega^aT_a).$$

This is nothing but the $\mathcal{G}$-CDYB equation mentioned in the Introduction. We stress that this equation arises as the consequence of the Jacobi identity of the PBs (19) and (18).

We can now determine $r^0(\omega)$ explicitly from (9) and thereby find a solution to (21). The result obtained in [8] is given by $r^0(\omega)=f_0(\mathrm{ad}\,\omega)$, with f_0 being the power series expansion of the complex analytic function

$$f_0(z)=\frac{1}{2}\coth\frac{z}{2}-\frac{1}{z} \tag{22}$$

around $z=0$. In a different context, this solution to (21) was found in [16] too.

In [7], the CDYB equation (1) was generalized by allowing the dynamical variable to belong to the dual of an arbitrary subalgebra $\mathcal{H}\subset\mathcal{G}$. Next, we explain that, if $\mathcal{H}\subset\mathcal{G}$ is a "self-dual" subalgebra, then some solutions to the $\mathcal{H}$-CDYB equation arise from the solutions to (20) and (21) upon applying Dirac reduction to the associated PB on $\mathcal{M}_{\check{G}}$.

We now start by considering a PB of the form (19) on $\mathcal{M}_{\check{G}}$ and also suppose that (18) holds, where $\omega=\log M$ varies in a domain $\check{\mathcal{G}}\subset\mathcal{G}$. As we have seen, the exchange r matrix $\hat{r}^0(\omega)\in\mathcal{G}\wedge\mathcal{G}$ then satisfies (20) and (21). We choose a Lie subalgebra $\mathcal{H}\subset\mathcal{G}$ and assume that the restriction of the scalar product of $\mathcal{G}$ remains nondegenerate on $\mathcal{H}$, which means that $\mathcal{H}$ is "self-dual." We have the linear direct-sum decomposition $\mathcal{G}=\mathcal{H}+\mathcal{H}^\perp$ and can introduce an adapted basis of $\mathcal{G}$ in the form $\{T_a\}=\{H_i\}\cup\{E_\alpha\}$, $H_i\in\mathcal{H}$, $E_\alpha\in\mathcal{H}^\perp$, with the dual basis $\{T^a\}=\{H^i\}\cup\{E^\alpha\}$. (The notation is motivated by the "principal example" for which $\mathcal{H}$ is a Cartan subalgebra of a simple Lie algebra.) Correspondingly, we can write

$$\omega=\omega_{\mathcal{H}}+\omega_{\mathcal{H}^\perp}=\omega^iH_i+\omega^\alpha E_\alpha. \tag{23}$$

We wish to impose the Dirac constraint $\omega_{\mathcal{H}^\perp}=0$ on the PB on $\mathcal{M}_{\check{G}}$. To calculate the resulting Dirac bracket, we need to invert the matrix $\mathcal{C}^{\alpha\beta}(\omega_{\mathcal{H}}):=\{\omega^\alpha,\omega^\beta\}_0|_{\omega_{\mathcal{H}^\perp}=0}$. This is identified from (18) as the matrix of the linear operator $\mathcal{C}(\omega_{\mathcal{H}}):\mathcal{H}^\perp\to\mathcal{H}^\perp$, which is equal to the restriction of a multiple of $\mathrm{ad}\,\omega_{\mathcal{H}}$ to $\mathcal{H}^\perp$,

$$\mathcal{C}(\omega_{\mathcal{H}}):=\frac{1}{\kappa}\mathrm{ad}\omega_{\mathcal{H}}|_{\mathcal{H}^\perp}. \tag{24}$$

Thus, we also have to restrict ourselves to a submanifold of $\mathcal{M}_{\check{G}}$, where $\mathcal{C}(\omega_{\mathcal{H}})$ is invertible. We define the domain $\check{\mathcal{H}}\subset\check{\mathcal{G}}$ to contain the $\mathcal{H}$ projection of those elements $\omega\in\check{\mathcal{G}}$ for which the operator $\mathcal{C}^{-1}(\omega_{\mathcal{H}})$ exists. Then, we can compute the Dirac bracket on the constrained manifold $\mathcal{M}_{\check{\mathcal{H}}}:=\{g\in\mathcal{M}_{\check{G}}|\log M\in\check{\mathcal{H}}\}$ by using the standard formula $\{F_1,F_2\}^*_0=\{F_1,F_2\}_0-\{F_1,\omega^\alpha\}_0\mathcal{C}^{-1}_{\alpha\beta}(\omega)\{\omega^\beta,F_2\}_0$. From (18), we obtain

$$\{g(x),\omega_i\}^*_0=\frac{1}{\kappa}g(x)H_i, \tag{25}$$
$$\{\omega_i,\omega_j\}^*_0=-\frac{1}{\kappa}f_{ij}^{\ k}\ \omega_k\quad([H_i,H_j]=f_{ij}^{\ k}\ H_k),$$

where $g(x+2\pi)=g(x)M$ with $\log M=\omega=\omega^iH_i\in\mathcal{H}$. Furthermore, (19) yields

$$\Big\{g(x)\overset{\otimes}{,}g(y)\Big\}^*_0=\frac{1}{\kappa}\Big(g(x)\otimes g(y)\Big) \tag{26}$$
$$\times\Big(\frac{1}{2}\hat{I}\mathrm{sgn}(y-x)+\hat{r}^*(\omega)\Big),\quad 0<x,y<2\pi,$$

where

$$\hat{r}^*(\omega)=\hat{r}^0(\omega)+\frac{1}{\kappa}\mathcal{C}^{-1}_{\alpha\beta}(\omega)E^\alpha\otimes E^\beta. \tag{27}$$

Now, the point is that—analogously to (20) and (21)—the Jacobi identities of the Dirac bracket imply that the function $\hat{r}^* : \check{\mathcal{H}} \to \mathcal{G} \wedge \mathcal{G}$ given by (27) is $\mathcal{H}$-equivariant in the natural sense and satisfies the $\mathcal{H}$-CDYB equation

$$\left[\hat{r}^*_{12}(\omega), \hat{r}^*_{23}(\omega)\right] + H^i_1 \frac{\partial}{\partial \omega^i} \hat{r}^*_{23}(\omega) \qquad (28)$$
$$+ \text{cycl. perm.} = -\frac{1}{4}\hat{f} \qquad (\omega \in \check{\mathcal{H}}).$$

Examples may be obtained by taking $\mathcal{H}$ to be the grade-zero subalgebra in some integral gradation of $\mathcal{G}$ and taking $r^0(\omega)$ to be $f_0(\mathrm{ad}\,\omega)$. Then, the operator $r^*(\omega)$ $(\omega \in \check{\mathcal{H}})$ associated with (27) is found to be

$$r^*(\omega)(X) = f_0(\mathrm{ad}\,\omega)(X), \qquad \forall X \in \mathcal{H}, \qquad (29)$$
$$r^*(\omega)(Y) = \frac{1}{2}\coth\left(\frac{1}{2}\mathrm{ad}\,\omega\right)(Y), \qquad \forall Y \in \mathcal{H}^\perp.$$

We use here the Laurent series expansion of $\frac{1}{2}\coth(\frac{z}{2})$ in a punctured disk around $z = 0$, and the $1/z$ term in the expansion corresponds to the operator $(\mathrm{ad}\omega)^{-1}$ on $\mathcal{H}^\perp$. In the special case of the principal gradation of a simple Lie algebra $\mathcal{G}$, for which $\mathcal{H}$ is a Cartan subalgebra, this gives

$$\hat{r}^*(\omega) = \frac{1}{2}\sum_{\alpha\in\Phi} \coth\left(\frac{1}{2}\alpha(\omega)\right) E_\alpha \otimes E^\alpha, \qquad (30)$$

where Φ is the set of the roots and E_α are the corresponding root vectors. This solution to the CDYB Eq. (1) was obtained in [3] by determining the PBs of chiral WZNW Bloch waves with the aid of a different method.

4. CONCLUSION

We have reported on our recent results from [8], which concern chiral WZNW phase space, focusing on the dynamical generalizations of the CYB equation that appear naturally in this context. Not only several variants of the CDYB equation but also some of their most interesting solutions have been described. The Dirac reduction of certain solutions of the $\mathcal{G}$-CDYB equation to solutions of the $\mathcal{H}$-CDYB equation has been treated in this report in a general manner for the first time. Other aspects of chiral WZNW phase space that have not been mentioned here for lack of space were analyzed in [8, 10, 17]; for instance, explicit solutions to the G-CDYB equation that realize arbitrary PL symmetries were found there. Open problems that arise from our investigation will hopefully be discussed in future publications.

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On Multivortex Solutions of the Weierstrass Representation*

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Abstract—The connection between the complex sine-Gordon equation on the plane associated with a Weierstrass-type system and the possibility of constructing several classes of multivortex solutions is discussed in detail. It is shown that the amplitudes of these vortex solutions represented in polar coordinates satisfy the fifth Painlevé equation. We perform the analysis using the known relations for the Painlevé equations and construct explicit formulas in terms of the Umemura polynomials, which are τ functions for rational solutions to the third Painlevé equation. New classes of multivortex solutions to the Weierstrass system are obtained through the use of this proposed procedure. © *2002 MAIK "Nauka/Interperiodica".*

1. THE GENERALIZED WEIERSTRASS SYSTEM AND ASSOCIATED COMPLEX SINE-GORDON EQUATION

A generalization of the Weierstrass system for inducing two-dimensional surfaces immersed into $\mathbb{R}^4$ is described by four complex-valued functions ψ_i and φ_i, $i = 1, 2$, satisfying the set of equations [1]

$$\partial\psi_1 = Q_1\left(\psi_1 + \frac{\varphi_1}{2\psi_2\bar{\varphi}_2}\right), \quad \bar{\partial}\psi_2 = Q_1\psi_2, \quad (1)$$
$$\bar{\partial}\varphi_1 = Q_2\left(\varphi_1 - \frac{\psi_1}{2\varphi_2\bar{\psi}_2}\right), \quad \partial\varphi_2 = Q_2\varphi_2,$$
$$Q_1 = |\psi_2|^2 - |\psi_1|^2, \quad Q_2 = |\varphi_2|^2 - |\varphi_1|^2,$$

and its respective complex-conjugate equations. The bar denotes complex conjugation, $\partial = \partial/\partial z$, and $\bar{\partial} = \partial/\partial\bar{z}$. The set of Eqs. (1) is a nonlinear first-order set of eight equations for which eight of the sixteen first-order derivatives with respect to z or $\bar{z}$ are known in terms of the functions ψ_i and φ_i. The set of Eqs. (1) admits several conservation laws:

$$\partial(\ln\bar{\psi}_2) = \bar{\partial}(\ln\psi_2), \quad \partial(\ln\varphi_2) = \bar{\partial}(\ln\bar{\varphi}_2),$$
$$\partial\left(\frac{\psi_1\bar{\varphi}_1}{\bar{\psi}_2\varphi_2}\right) = \bar{\partial}\left(\frac{\bar{\psi}_1\varphi_1}{\psi_2\bar{\varphi}_2}\right).$$

In the present paper, we propose a procedure for constructing explicit multivortex solutions to the Weierstrass system (1), which are obtained through the use of a link between the complex sine-Gordon equation on the plane and system (1). We subject the set of Eqs. (1) to several transformations in order to simplify its structure. We start by defining two new complex-valued functions

$$u = \frac{\psi_1}{\bar{\psi}_2}, \quad v = \frac{\varphi_1}{\bar{\varphi}_2}. \quad (2)$$

It is easy to show that, if the complex functions ψ_i and φ_i are solutions to the first-order set of Eqs. (1), then the rational functions u and v defined by (2) are solutions to the first-order set of two equations

$$\partial u = \frac{1}{2}(1 - |u|^2)v, \quad \bar{\partial}v = -\frac{1}{2}(1 - |v|^2)u, \quad (3)$$

and its respective complex-conjugate equations. The elimination of one of the functions u or v in (3) leads to the complex sine-Gordon (CSG) equation

$$\partial\bar{\partial}u + \frac{\bar{u}}{1 - |u|^2}\partial u\bar{\partial}u + \frac{1}{4}u(1 - |u|^2) = 0. \quad (4)$$

As was shown in [2], Eq. (4) was derived in the context of the reduction of the $O(4)$ nonlinear sigma model and, also, the reduction of the self-dual Yang–Mills equations and relativistic equations [3, 4]. This equation has a geometric interpretation as the Gauss–Codazzi equation describing a two-dimensional surface embedded in a three-dimensional sphere that is itself again embedded in a four-dimensional Euclidean space [5].

Note that, if v tends to one, then Q_2 vanishes and the set of Eqs. (1) takes the form

$$\partial\psi_1 = Q_1\left(\psi_1 + \frac{1}{2\psi_2}\right), \quad \bar{\partial}\psi_2 = Q_1\psi_2.$$

Conversely, if u tends to one, then Q_1 vanishes and the set of Eqs. (1) becomes

$$\bar{\partial}\varphi_1 = Q_2\left(\varphi_1 - \frac{1}{2\varphi_2}\right), \quad \partial\varphi_2 = Q_2\varphi_2.$$

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These limits characterize the properties of solutions to the set of Eqs. (1).

2. MULTIVORTEX SOLUTIONS

At this point, we would like to derive, through the link between the set of Eqs. (1) and the CSG equation (4), a procedure for constructing several classes of solutions in explicit form. Let us now discuss certain classes of multivortex solutions to the CSG equation (4) in polar coordinates (r, θ) on a plane,

$$u = A_n(r)e^{in\theta}, \quad n \in \mathbb{Z}. \tag{5}$$

Under the assumption specified by (5), Eq. (4) is reducible to a second-order ordinary differential equation (ODE) of the form

$$\frac{d^2A_n}{dr^2} + \frac{1}{r}\frac{dA_n}{dr} + \frac{A_n}{1-A_n^2}\left[\left(\frac{dA_n}{dr}\right)^2 - \frac{n^2}{r^2}\right] + (1-A_n^2)A_n = 0. \tag{6}$$

Upon a homographic transformation of the dependent variable,

$$A_n = \frac{1+w(z)}{1-w(z)}, \quad z = r, \tag{7}$$

Eq. (6) takes the form of the fifth Painlevé (P5) equation

$$w'' = \frac{3w-1}{2w(w-1)}w'^2 - \frac{w'}{z} + \frac{(w-1)^2}{z}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma}{z}w + \delta\frac{w(w+1)}{w-1}, \tag{8}$$

with the coefficients

$$\alpha = -\beta = \frac{n^2}{8}, \quad \gamma = 0, \quad \delta = -2. \tag{9}$$

In general, equation P5 is not integrable in terms of known classical transcendental functions. For specific values of the parameters, this equation can be reduced, however, to two types of nontranscendental functions, that is, to solutions of a Riccati equation with one arbitrary parameter or to three types of rational solutions of equation P5 [6, 7]. In our case, Eq. (8) can be written, according to [7], in an equivalent form as a first-order set of ODEs

$$z\frac{dp}{dz} = -\frac{\epsilon n}{2} - \epsilon np - pq - p^2q, \quad \epsilon = \pm 1, \tag{10}$$
$$z\frac{dq}{dz} = -2z^2 + \epsilon nq - 4z^2p + \frac{q^2}{2} + pq^2,$$

where $p = w/(1-w)$. The function $q(z)$ satisfies a Painlevé-type equation of the form

$$q'' = \frac{q}{q^2-4z^2}q'^2 - \frac{q^2+4z^2}{q^2-4z^2}\frac{q'}{z} + \frac{q}{4z^2(q^2-4z^2)}(16nz^2(2\epsilon - n) - (q^2-4z^2)^2).$$

The function $q^2 - 4z^2$ has two poles at $q = 2z$. Using the transformation

$$y(z) = \frac{q+2z}{q-2z}, \quad q \neq 2z,$$

we find that $y(z)$ is also a solution to equation P5 with the parameters

$$\tilde{\alpha} = -\tilde{\beta} = \frac{(1-\epsilon n)^2}{8}, \quad \tilde{\gamma} = 0, \quad \tilde{\delta} = -2. \tag{11}$$

In our case, the use of the results from [7] leads to the following proposition.

Proposition 1. Let $w = w(z)$ be a solution to equation P5 with the parameters given by (9) such that

$$\Phi_1(w) \equiv zw' - \frac{\epsilon n}{2}w^2 + 2zw + \frac{\epsilon n}{2} \neq 0; \tag{12}$$

then, the function

$$w_1 = 1 - \frac{4z}{\Phi_1(w)}, \tag{13}$$

is a solution to equation P5 with the parameters given by (11).

Proposition 1 establishes the auto-Bäcklund transformation (auto-BT) for equation P5 when $\delta = -2$. Let us now find a class of solutions to equation P5 according to [7].

Proposition 2. Equation P5 has 1-solutions that can be expressed in terms of Wittaker functions and its derivatives with parameters given either by $n = 1 - 2N$, $N \in \mathbb{Z}^- \cup \phi$ or by $n = 2N$, $N \in \mathbb{Z}^+$.

Equation P5 has rational solutions of the form

$$w = \lambda z + \mu + \frac{P_{n-1}(z)}{Q_n(z)}, \tag{14}$$

where λ and μ are constants and $P_{n-1}(z)$ and $Q_n(z)$ are polynomials. In particular, if $P_{n-1}(z) = 0$, then equation P5 has the solution $w = 1$. Applying the auto-BT to solution (14), we obtain another proposition.

Proposition 3. For rational solutions to equation P5 of the form

$$w = \frac{P_n(z)}{Q_n(z)} \tag{15}$$

to exist, it is necessary and sufficient that the relation $n = 2l + 1$, $l \in \mathbb{Z}$, holds.

Let us now discuss the link between equations P5 with different values of the parameter δ, namely, $\delta \neq 0$ and $\delta = 0$.

Proposition 4. Let $u(z) \neq 0$ be a solution to equation P5 with the parameters $\alpha = -\beta = n^2/8$, $\gamma = 0$, and $\delta = -2$. Then, the function

$$\tilde{u}(z) = \frac{f^2(\sqrt{z})}{f^2(\sqrt{z}) - 1}, \tag{16}$$

$$f(z) = \frac{d}{dz} \ln u(z) - \frac{n}{4z}\left(u(z) - \frac{1}{u(z)}\right), \quad n \in \mathbb{Z},$$

is a solution to equation P5 with the parameters

$$\tilde{\alpha} = \frac{(1+n^2)^2}{2}, \quad \tilde{\beta} = \tilde{\delta} = 0, \quad \tilde{\gamma} = -1/2. \tag{17}$$

Following [7] and using the result of Proposition 4, we can find in our case the relation between equation P3 with $\gamma\delta \neq 0$,

$$w'' = \frac{w'^2}{w} - \frac{w'}{w} + \frac{1}{z}(\alpha\gamma w^2 + \beta) + \gamma^2 w^3 + \frac{\delta}{w}, \tag{18}$$

and equation P5 with $\delta = 0$ and $\gamma \neq 0$. Indeed, Eq. (18) can be written as the first-order set of ODEs

$$zw' = (\alpha\epsilon - 1)w + \epsilon\gamma z w^2 + zv, \quad \epsilon = \pm 1, \tag{19}$$
$$zwv' = \beta w + \delta z + (\alpha\epsilon - 2)wv + zv^2.$$

Eliminating w from (19), we obtain

$$v'' - \frac{v}{v^2+\delta}v'^2 + \frac{v'}{z} + \frac{\beta^2 - (2-\alpha\epsilon)^2\delta}{z^2(v^2+\delta)}v \tag{20}$$
$$+ \epsilon\gamma(v^2+\delta) - \frac{2\delta\beta}{z^2}\frac{(\alpha\epsilon - 2)}{v^2+\delta} + \frac{\beta}{z^2}(\epsilon\alpha - 2) = 0.$$

Upon a homographic transformation of the dependent variable and a change of the independent variable,

$$v = -i\sqrt{\delta}\frac{u+1}{u-1}, \quad z = \sqrt{2\tau}, \tag{21}$$

we reduce Eq. (20) to equation P5; that is,

$$u'' + \frac{3u-1}{2u(u-1)}u'^2 + \frac{u'}{\tau} \tag{22}$$
$$+ \frac{1}{32\delta\tau^2}\left[(u^2-1)\left(Au + \frac{B}{u}\right)\right]$$
$$+ \frac{\epsilon}{\tau}\gamma(-\delta)^{1/2}u = 0,$$

where A and B are given by

$$A = \beta^2 + 4(-\delta)^{1/2}\beta - \delta\alpha^2 - 4\delta \tag{23}$$
$$- 2(-\delta)^{1/2}\epsilon\alpha\beta + 4\epsilon\delta\alpha,$$
$$B = \delta\alpha^2 - 2(-\delta)^{1/2}\epsilon\alpha\beta + 4(-\delta)^{1/2}\beta$$
$$+ 4\delta - \beta^2 - 4\epsilon\delta\alpha.$$

Proposition 5. Let $u = u(z)$ be a solution to equation P5 (22) with the parameters given by (23) such that

$$r(z) = w' - (\alpha\epsilon - 1)\frac{w}{z} - \epsilon\gamma w^2 - 1 \neq 0$$

holds. The function

$$S(\tau) = 1 - 2r^{-1}(\sqrt{2\tau}) \tag{24}$$

is then a solution to equation P3 (18) with the parameters $\gamma \neq 0$ and $\delta < 0$.

The τ functions for the rational class of solutions to equation P3 can be constructed [8–10] in terms of the Umemura polynomials $T_n = T_n(z,l)$ determined by a sequence of polynomials in z and defined through the recurrence relation

$$T_{n+1}T_{n-1} = \left(\frac{z}{8} - l + \frac{3}{4}n\right)T_n^2 + \frac{\partial T_n}{\partial z}T_n \tag{25}$$
$$+ z\left[\frac{\partial^2 T_n}{\partial z^2}T_n - \left(\frac{\partial T_n}{\partial z}\right)^2\right],$$

with initial conditions $T_0 = T_1 = 1$. Based on [9], we have, in our case, the following.

Proposition 6. For rational solutions to equation P3 of the form

$$w(z) = \frac{T_{n+1}(z,l-1)T_n(z,l)}{T_{n+1}(z,l)T_n(z,l-1)}, \tag{26}$$

where the Umemura polynomials $T_n = T_n(z,l)$ satisfy the recurrence relation (25), to exist, it is necessary and sufficient that the parameters of equation P5 satisfy

$$\alpha = 4(n+l), \quad \beta = 4(n-l), \quad \gamma = -\delta = 4.$$

Another class of solutions to the CSG equation (4) is provided by the set of Eqs. (3) if we define functions u and v in the form

$$u = A_n(r)e^{in\theta}, \quad v = A_{n-1}(r)e^{i(n-1)\theta}, \quad n \in \mathbb{Z}. \tag{27}$$

We substitute (27) into (3) in order to obtain the first-order set of ODEs

$$\text{(i)} \quad \frac{dA_n}{dr} + \frac{n}{r}A_n = (1 - A_n^2)A_n \tag{28}$$

$$\text{(ii)} \quad \frac{dA_{n-1}}{dr} - \frac{(n-1)}{r}A_{n-1} = (1 - A_{n-1}^2)A_n.$$

Equations (28) represent a set of coupled Emden-type equations [11]. This set of equations can be decoupled by the sequence of transformations

$$y_0 = A_n, \quad y_j = \frac{y_{j+1}}{r}, \quad j = 0, 1, \ldots, 2n \tag{29}$$

applied n times to the first equation in (28). This leads to a nonlinear Bernoulli equation of the form

$$\frac{dy_n}{dr} = (1 - r^{-2n} y_n^2) y_n. \tag{30}$$

Equation (30) can be transformed by the substitution $u = y_n^{-2}$ into the linear Bernoulli equation

$$\frac{du}{dr} + 2u = 2r^{-2n}, \tag{31}$$

which has a general solution in terms of the exponential-type integral

$$\mathrm{Ei}(2r) = \int_{-\infty}^{2r} (e^t/t)dt;$$

that is

$$y_n^{-2} = e^{-2r}\Bigg[- \frac{e^{2r}}{(2n-1)r^{2n-1}} - \frac{2e^{2r}}{(2n-1)(2n-2)r^{2n-2}} - \cdots - \frac{2^{2n-1}}{(2n-1)!}\int_{-\infty}^{2r} \frac{e^t}{t}dt\Bigg]. \tag{32}$$

Substituting (32) into the sequence of transformations (29), we get A_n. Upon inserting A_n into the second equation in (28), the resulting Emden equation can be solved formally in a similar way by the sequence of transformations (29).

The procedure for constructing solutions of the Weierstrass system (1) can be reduced to the following. Deduce any two multivortex solutions that are functionally independent of the CSG equation (4) [or equivalently, u and v representing a solution to the first-order set of Eqs. (3)] by using the approach as described above. Substitute it into transformation (2). From (2), we know the ratios $\psi_1/\bar{\psi}_2$ and $\varphi_1/\bar{\varphi}_2$. Using transformation (2), we can therefore eliminate, for example, ψ_2 and φ_2 in the set of Eqs. (1) and integrate the resulting differential equations.

For example, when $n = 1$, the second equation in (28) is solved by taking $A_0 = \pm 1$, whereupon the first equation in (28) becomes the Riccati equation

$$\frac{dA_1}{dr} = 1 - \frac{A_1}{r} - A_1^2. \tag{33}$$

Equation (33) can be linearized by the Cole–Hopf transformation $A_1 = y_r/y$ and solved in terms of Bessel function I_0 of zero order. The vortex solution (27) takes the form

$$u = \frac{I_0'(r)}{I_0(r)} e^{i\theta}, \quad v = \epsilon = \pm 1. \tag{34}$$

Consequently, from transformation (2), we get

$$\psi_1 = \frac{I_0'(r)}{I_0(r)} e^{i\theta}\bar{\psi}_2, \quad \varphi_1 = \epsilon\bar{\varphi}_2. \tag{35}$$

Substituting (35) into Weierstrass system (1) and solving the resulting equations, we obtain

$$\varphi_2 = F(re^{-i\theta}), \tag{36}$$

where F is an arbitrary function of one variable and the function ψ_2 is determined by the quadrature

$$\int_0^{\psi_2} \frac{d\psi_2'}{|\psi_2'|^2\psi_2'} = C\int_0^{\theta} e^{-2i\theta'} J(ce^{-i\theta'})d\theta' \tag{37}$$

with

$$J = 1 - \frac{I_1^2(ce^{-i\theta})}{I_0^2(ce^{-i\theta})}.$$

3. FINAL REMARKS

We have presented a new approach to studying the Weierstrass system (1). It proved to be particularly efficient in constructing multivortex solutions to (1) from which it is possible to derive explicit formulas for associated constant-mean-curvature surfaces embedded in $\mathbb{R}^4$. It is worth noting that the approach to the Weierstrass system (1) proposed here can be applied, with some necessary modifications, to more general cases of Weierstrass-type systems describing more diverse surfaces immersed in multidimensional Minkowski and pseudo-Riemann spaces. Such a generalization of the Weierstrass system was recently studied by Konopelchenko [12], who derived, among other things, explicit formulas for minimal surfaces immersed in $\mathbb{R}^n$ and S^n. The task of obtaining new types of minimal surfaces described by system (1) will be undertaken in our future work.

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Completeness of Multiseparable Superintegrability in Two Dimensions*

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Abstract—For complex Euclidean 2-space and the complex 2-sphere, we have found all classical and quantum superintegrable systems that a polynomial correspond to nondegenerate potentials. These potentials have the property that a polynomial associated with each of them is a quadratic algebra. Furthermore, each of these superintegrable systems admits separation of variables in more than one coordinate system. For degenerate superintegrable systems, both properties may be violated. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

The Hamilton–Jacobi equation in two dimensions

$$H = p_x^2 + p_y^2 + V(x,y) \tag{1}$$
$$= \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + V(x,y) = E$$

can be solved by the method of separation of variables via the ansatz

$$S = U(u,E,\alpha) + V(v,E,\alpha)$$

for a suitable change of coordinates $u = u(x,y), v = v(x,y)$ and some separation constant α. Here, α can be characterized as the value of a classical constant of motion of the form

$$A = a(x,y)p_x^2 + b(x,y)p_y^2 \tag{2}$$
$$+ c(x,y)p_xp_y + d(x,y).$$

The Hamiltonian H in (1) can admit at most three functionally independent second-order constants of motion (2), of which one may be chosen as H itself. In this case, the system is said to be superintegrable [1–5]. By definition, if a Hamiltonian admits two quadratic constants of motion of the form

$$A_h = a_h(x,y)p_x^2 + b_h(x,y)p_y^2 \tag{3}$$
$$+ c_h(x,y)p_xp_y + d_h(x,y), \quad h = 1,2,$$

then each must satisfy the equation $\{H, A_h\} = 0$ with the usual Poisson bracket $\{\ ,\ \}$. It is also clear that $R = \{A_1, A_2\}$ is a third order constant of the motion, i.e., $\{H, R\} = 0$, and is therefore functionally dependent on $A_0 = H, A_1$ and A_2: $R^2 = F(H, A_1, A_2)$.

Using the identity

$$\{K, G\} = \sum_{h=0}^{2} \{K, A_h\} \frac{\partial G}{\partial A_h},$$

we find the relations

$$\{A_1, R\} = \frac{1}{2}\frac{\partial F}{\partial A_2}, \{A_2, R\} = -\frac{1}{2}\frac{\partial F}{\partial A_1}, \tag{4}$$

which tell us that the constants of motion $\{A_1, R\}$ and $\{A_2, R\}$ are easily computed as functions of H, A_1, and A_2 once F is known. We should also note that if F is polynomial in the invariants H, A_1, and A_2, then so are $\{A_1, R\}$ and $\{A_2, R\}$. Of particular interest is the case when F is cubic in the generators, in which case these constants of motion are quadratic in the basic invariants. The above relations then determine a *quadratic algebra*.

Note that, for any such triple of invariants $A_i, i = 0,1,2$, we could always subject the system to a Euclidean motion, where the coordinates x and y transform according to

$$x \to x\cos\alpha - y\sin\alpha + a,$$
$$y \to x\sin\alpha + y\cos\alpha + b.$$

We regard systems so related as *equivalent*. Note that the invariants A_1 and A_2 can be expressed as

$$A_i = C_i^{\ell m} L_\ell L_m + d_i(x,y),$$

where $L_1 = p_x, L_2 = p_y$, and $L_3 = M = xp_y - yp_x$ are the generators of the Lie algebra of the Euclidean

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group. We have calculated [6] all the inequivalent superintegrable potentials V that are *nondegenerate* in the sense that they depend on four arbitrary parameters; i.e., one can prescribe the values of V, V_x, V_y, V_{yy} arbitrarily at any regular point (x_0, y_0). In so doing, we obtained the following list (potential and invariants).

$$1.\ V=\frac{\alpha}{\sqrt{x^2+y^2}}+\frac{1}{\sqrt{x^2+y^2}}\times\left[\frac{\beta}{\sqrt{x^2+y^2}+x}+\frac{\gamma}{\sqrt{x^2+y^2}-x}\right],$$
$$A_1=M^2+d_1,\quad A_2=2Mp_y+d_2.$$

Separation of variables in parabolic and polar coordinates.

$$2.\ V=\frac{\alpha z}{\sqrt{c^2-z^2}}+\frac{\beta}{\sqrt{(c-z)(c+z)}}+\frac{\gamma}{\sqrt{(c+z)(c+\bar z)}},$$

where $z=x+iy, \bar z=x-iy$, and $A_1=M^2+c^2p_x^2+d_1, A_2=M(p_x+ip_y)+d_2$. Separation in elliptic coordinates as well as shifted elliptic coordinates.

$$3.\ V=\frac{\alpha}{\sqrt{\bar z(z+2)}}+\frac{\beta}{\sqrt{\bar z(z-2)}}+\frac{\gamma z}{\sqrt{z^2-4}},$$
$$A_1=M^2+(p_x-ip_y)^2+d_1,$$
$$A_2=M(p_x+ip_y)+d_2.$$

Separation in hyperbolic and shifted elliptic coordinates.

$$4.\ V=\frac{\alpha}{z^2}+\frac{\beta}{\sqrt{z^3(\bar z+2)}}+\frac{\gamma}{\sqrt{z(\bar z+2)}},$$
$$A_1=M^2+(p_x+ip_y)^2+d_1,$$
$$A_2=(M+2i(p_x+ip_y))^2+(p_x+ip_y)^2+d_2.$$

Separation in hyperbolic and displaced hyperbolic coordinates.

$$5.\ V=\alpha(x^2+y^2)+\frac{\beta}{x^2}+\frac{\gamma}{y^2},$$
$$A_1=M^2+d_1,\quad A_2=p_x^2+d_2.$$

Separation in elliptic, shifted elliptic, polar, and Cartesian coordinates.

$$6.\ V=\alpha\frac{z\bar z}{z^4}+\beta\frac{1}{z^2}+\gamma z\bar z,$$
$$A_1=M^2+d_1,\quad A_2=(p_x+ip_y)^2+d_2.$$

Separation in polar and hyperbolic coordinates.

$$7.\ V=\frac{\alpha}{\sqrt{z^2-c^2}}+\frac{\beta\bar z}{\sqrt{z^2-c^2}(z+\sqrt{z^2-c^2})^2}+\gamma z\bar z,$$
$$A_1=M^2+c^2p_x^2+d_1,\quad A_2=(p_x+ip_y)^2+d_2.$$

Separation in elliptic and hyperbolic coordinates

$$8.\ V=\frac{\alpha}{\sqrt{x^2+y^2}}+\beta\frac{(\sqrt{x^2+y^2}+x)^{\frac12}}{\sqrt{x^2+y^2}}+\gamma\frac{(\sqrt{x^2+y^2}-x)^{\frac12}}{\sqrt{x^2+y^2}},$$
$$A_1=2Mp_x+d_1,\quad A_2=2Mp_y+d_2.$$

Separation in two types of parabolic coordinates.

$$9.\ V=\alpha(4x^2+y^2)+\beta x+\frac{\gamma}{y^2},$$
$$A_1=p_x^2+d_1,\quad A_2=2Mp_y+d_2.$$

Separation in Cartesian and parabolic coordinates.

$$10.\ V=\alpha\bar z+\beta\left(z-\frac32\bar z^2\right)+\gamma\left(z\bar z-\frac12\bar z^3\right),$$
$$A_1=2M(p_x-ip_y)+(p_x+ip_y)^2+d_1,$$
$$A_2=(p_x-ip_y)^2+d_2.$$

Separation in semihyperbolic and shifted semihyperbolic coordinates.

$$11.\ V=\frac{\alpha}{\sqrt z}+\beta x+\gamma\frac{x+z}{\sqrt z},$$
$$A_1=p_y(p_x+ip_y)+d_1,\quad A_2=Mp_y+d_2.$$

Separation in parabolic and displaced parabolic coordinates.

$$12.\ V=\alpha z+\frac{\beta z}{\sqrt{\bar z}}+\frac{\gamma}{\sqrt{\bar z}},$$
$$A_1=(p_x-ip_y)^2+d_1,\quad A_2=M(p_x+ip_y)+d_2.$$

Separation in semihyperbolic and shifted semihyperbolic coordinates.

In constructing this list of potentials, we have allowed all variables to be complex. We have accordingly solved our problem in the complex plane. If we consider real forms of the complex plane, then in the case of the real Euclidean plane potentials 1, 5, 8, and 9 are real potentials that have the superintegrability property [1]. In the case of a pseudo-Euclidian plane, there is at least one real form of each potential that has the required property. (We have recently proved similar results for nondegenerate potentials on the complex 2-sphere [7]. See Section 2 for a list of the superintegrable potentials.)

What are the distinguishing features of these 12 potentials and the quadratic invariants that describe

them as superintegrable systems? If we calculate $R = \{A_1, A_2\}$ for each potential, we can directly verify that

$$R^2 = a^{ijk}A_iA_jA_k + b^jA_j + c.$$

As a consequence, a quadratic algebra can be generated from the relations (4). Indeed, if we turn off the potential and look for all quadratic algebras that have these properties up to group equivalence, we find that the equivalence classes are in one-to-one correspondence with the potentials presented above. Furthermore, in each of the cases, solutions via separation of variables are possible in more than one coordinate system.

If we do not have the criterion of nondegeneracy satisfied, we may not have a quadratic algebra or multiseparability. For example, consider the potential $V = \alpha(x+iy)^2$. The corresponding Hamiltonian admits the constants of motion $(p_x+ip_y)^2$ and $(xp_y - yp_x)(p_x+ip_y) - \frac{i}{3}\alpha(x+iy)^3$ and separates in light-cone coordinates only. This degenerate system is superintegrable but not multiseparable.

2. NONDEGENERATE POTENTIALS ON THE COMPLEX 2-SPHERE

Here, $x^2+y^2+z^2 = 1, w = x+iy, \bar{w} = x-iy$.

1. $V = \dfrac{\alpha}{x^2} + \dfrac{\beta}{y^2} + \dfrac{\gamma}{z^2}.$

2. $V = \dfrac{\alpha}{z^2} + \dfrac{\beta}{w^2} + \dfrac{\gamma\bar{w}}{w^3}.$

3. $V = \dfrac{\alpha}{w^2} + \dfrac{\beta z}{\sqrt{x^2+y^2}} + \dfrac{\gamma}{\sqrt{(x^2+y^2)w}}.$

4. $V = \dfrac{\alpha z}{\sqrt{x^2+y^2}} + \dfrac{1}{\sqrt{x^2+y^2}} \times \left[\dfrac{\beta}{\sqrt{x^2+y^2}+x} + \dfrac{\gamma}{\sqrt{x^2+y^2}-x}\right].$

5. $V = \dfrac{\alpha}{w^2} + \dfrac{\beta z}{w^3} + \dfrac{\gamma(1-4z^2)}{w^4}.$

6. $V = \dfrac{\alpha(w+c^2\bar{w})}{\sqrt{(c^2\bar{w}-w^2-4c^2z^2)}} + \dfrac{\beta(w-c^2\bar{w})}{z^2\sqrt{(c^2\bar{w}-w^2-4c^2z^2)}} + \gamma\dfrac{(x^2+y^2)}{z^2}.$

APPENDIX

Separable Coordinates on $E_{2,C}$

(Here, the coordinates are related to Cartesian coordinates and to the operator that characterizes the separation constant in the free Hamiltonian.)

1. Cartesian coordinates

$$x, y, \qquad A = p_x^2.$$

2. Lightlike coordinates

$$x = \xi + i\eta, \quad y = \xi - i\eta, \quad A = (p_x + ip_y)^2.$$

3. Polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad A = M^2 = (xp_y - yp_x)^2.$$

4. Parabolic coordinates

$$x_P = \frac{1}{2}(\xi^2 - \eta^2), \quad y_P = \xi\eta, \quad A = Mp_y.$$

5. Elliptic coordinates

$$x_E^2 = c^2(u-1)(v-1), \quad y_E^2 = -c^2uv,$$
$$A = M^2 + c^2p_x^2.$$

6. Hyperbolic coordinates

$$x_H = c\frac{r^2+s^2+r^2s^2}{2rs}, \quad y_H = ic\frac{r^2+s^2-r^2s^2}{2rs},$$
$$A = M^2 + c^2(p_x+ip_y)^2.$$

7. Semihyperbolic coordinates

$$x_{SH} = -\frac{c}{4}(w-u)^2 + \frac{c}{2}(u+w),$$
$$y_{SH} = -i\left(-\frac{c}{4}(w-u)^2 - \frac{c}{2}(u+w)\right),$$
$$A = 4iM(p_x - ip_y) + c(p_x+ip_y)^2.$$

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Intertwining Operators and *S* Matrix*

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Abstract—The class of one-dimensional many-body systems related to semisimple Lie groups G is studied. The Hamiltonians of these systems are expressed in terms of Casimir operators (or, equivalently, Laplace–Beltrami operators on symmetric spaces) of underlying symmetry groups G. It turns out that the S matrix for all these problems is related to the intertwining operators for groups G. This connection provides immediately the functional form of the S matrix. Moreover, this connection allows one to prove that multiparticle S-matrix elements can be expressed in terms of two-particle ones. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

The algebraic studies of dynamical systems in nonrelativistic quantum mechanics have a long history. The first step in this direction had been made by Pauli [1], before the Schrödinger equation was published. In that pioneering study, Pauli showed that the bound-state spectrum of the nonrelativistic Coulomb problem could be obtained by studying the abstract algebra generated by the angular momentum and the Runge–Lenz vector. Following this, Fock and Bargmann [2, 3] recognized that this algebra is isomorphic to the $\mathfrak{so}(4)$ Lie algebra. Moreover, it was realized that "accidental" degeneracies, i.e., degeneracies not associated with the geometric $SO(3)$ symmetries of the Hamiltonian, are due to dynamical invariance group $SO(4)$. Ever since, dynamical invariance groups have been determined for many quantum-mechanical systems. This is a situation in which the Hamiltonian $\mathcal{H}$ of the system belongs to the center of the enveloping algebra of some group G, i.e., $\mathcal{H} = f(\mathcal{C})$, where $\mathcal{C}$ is the Casimir operator of the dynamical symmetry group G. For example, in the Coulomb bound-state problem, $\mathcal{H} = \alpha/2(\mathcal{C} - 1)$, where $\mathcal{C}$ is the second-order Casimir operator of symmetric tensor representations of $SO(4)$.

Since the study of Zwanzinger [4], it has become clear that algebraic methods can be successfully used in solving scattering problems. In that study, Zwanzinger showed how the symmetry group $SO(3,1)$ allows an algebraic determination of the Coulomb S matrix. However, this method, which, at the beginning, was developed only for the Coulomb problem, cannot be generalized to other scattering problems; for this reason, the Coulomb problem was the only known example for a long time.

Important results were obtained in this respect by the Yale group and others [5–10]. It appears that knowledge of the interrelation between the dynamical algebra that describes the scattering problem and a Euclidean algebra that describes the problem in the absence of interactions allows in principle a pure algebraic calculation of S matrices. This technique, which is called a Euclidean connection, essentially uses the theory of group expansions or deformations [11]. However, due to the absence of a general procedure for describing such connection formulas, it is rather difficult to derive the S matrix by using the above-mentioned method. (Note that the general expansion problem has not yet been solved.)

Since knowledge of the dynamical group is sufficient for solving bound-state problems, it is quite suggestive to ask whether (or not) one can use information on the dynamical group directly to obtain stringent restrictions on the structure of the scattering matrices, or even to determine it completely. The answer is in the affirmative [12]. It was found that the S matrices for the systems under consideration are related to intertwining operators between Weyl-equivalent principal series representations of the dynamical group G. In other words, the S matrices for the systems under consideration are constrained to satisfy the equation

$$SU^{\chi}(g) = U^{\tilde{\chi}}(g)S \quad \text{for all} \quad g \in G \qquad (1)$$

or

$$SdU^{\chi}(b) = dU^{\tilde{\chi}}(b)S \quad \text{for all} \quad b \in \mathfrak{g}, \qquad (2)$$

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where U^χ and $U^{\tilde{\chi}}$ are the Weyl-equivalent unitary irreducible representations (UIRs) of principal series of G, while dU^χ and $dU^{\tilde{\chi}}$ are the corresponding representations of the algebra $\mathfrak{g}$ of G. These equations have a great restrictive power, determining the S matrix, apart from a constant depending on χ. Thus, one can in principle evaluate the S matrix from (1) or (2) without ever writing a Schrödinger equation or wave functions, or ever mentioning the concepts of space and time.

In order to determine the S matrix, we can proceed in two ways. If the principal series of the algebra $\mathfrak{g}$ in the scattering basis is known, we can get recurrence relations for the S matrix by applying both sides of Eq. (2) to the basis vectors. By solving the recurrence relations, one can find the explicit form of the S matrix as a function of the parameters specifying the representation of $\mathfrak{g}$. An alternative way employs Eq. (1). By using a realization of the principal series of G on a Hilbert space of some functions, it is possible to derive, from Eq. (1), functional relations for the kernel of the operator S that allow one to determine it. This global approach, which is complementary to the infinitesimal treatment, allows one to obtain an integral expression for the S matrix.

Let us note at this point that the operator S satisfying Eq. (1) or (2) is called the intertwining operator between representations U^χ and $U^{\tilde{\chi}}$. Therefore, the S matrix for the scattering system described by a Hamiltonian expressed in terms of the Casimir operator of some group G is nothing but the intertwining operator between the Weyl-equivalent principal series representations of G. The explicit expressions for the intertwining operators for semisimple Lie groups in terms of kernels are introduced in [13, 14] and were extensively studied in [15, 16] in a different context.

2. ONE-DIMENSIONAL MANY-BODY SCATTERING SYSTEMS ASSOCIATED WITH SEMISIMPLE LIE GROUPS

It follows from Eq. (1) or (2) that, if the matrix of the representation operator U is diagonal in some basis, then the matrix of the intertwining operator is also diagonal. It follows that, in this case, the matrix of the intertwining operator does not mix one-dimensional subspaces of the carrier space of the representation. This fact leads to the suggestion that there might exist a class of one-dimensional problems for which the scattering matrix is determined by the diagonal elements of the intertwining operator. This is precisely what happens in the approach of Olshanetsky and Perelomov [17] to one-dimensional many-body problems associated with Lie algebras, where Hamiltonians of systems are described in terms of the "radial part" of the Casimir operator (see also [5–10]). Hence, the well-developed theory of intertwining operators for semisimple Lie groups [13–16] gives every reason to hope that one may obtain stringent restrictions on the structure of scattering matrices or even determine it explicitly for all one-dimensional many-body systems associated with semisimple Lie groups.

In this report, we show that an important characteristic of all such systems is its factorizable S matrix; the many-body S matrix is completely determined in terms of the two-body one, the latter being related to an intertwining operator of real-rank-one group. Moreover, it follows that asymptotic outgoing momenta characterizing a final state are determined by the longest element of the Weyl group of the symmetry group.

Let us explain the idea of the algebraic construction of the S matrix in very general forms. To describe these matters in greater detail, we need some notation.

Let G be a noncompact connected semisimple Lie group with a finite center, and let $\mathfrak{g}$ be its Lie algebra over the field R of real numbers. Let $\mathfrak{k}$ be a maximal compact subalgebra of $\mathfrak{g}$, and let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. Then, $\mathfrak{g}$ is the direct sum of $\mathfrak{k}$ and $\mathfrak{p}$. Let $\mathfrak{a}$ be a maximal commutative subalgebra in $\mathfrak{p}$, and let $\mathfrak{a}^*$ be its dual. The dimension of $\mathfrak{a}$ is called the real rank (or split rank) of G. The Killing form of $\mathfrak{g}$ induces a positive definite inner product $\langle\ ,\ \rangle$ on $\mathfrak{a}$ and on $\mathfrak{a}^*$. Let Δ_+ (Δ_-) denote a set of positive (negative) restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$. We also set

$$\mathfrak{n} = \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{v} = \sum_{\alpha\in\Delta_-} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ are the root subspaces of $\mathfrak{g}$ corresponding to the restricted root. Then, it follows that $\mathfrak{n}$ and $\mathfrak{v}$ are the nilpotent subalgebras of $\mathfrak{g}$. Moreover, the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is valid, where the sum is direct. Let K, A, N, and V be analytic subgroups in G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$, and $\mathfrak{v}$, respectively; K is compact, N and V are nilpotent, and A is a vector group. Then, any element $g \in G$ can be uniquely decomposed as

$$g = k(g)a(g)n, \tag{3}$$

where $k(g) \in K$, $a(g) \in A$, and $n \in N$ (the so-called Iwasawa decomposition). Another important decomposition of G is due to Gelfand, Naimark, Bruhat; i.e., almost any element of G can be written in a unique way as the product

$$g = m(g)a(g)nv(g), \tag{4}$$

where $m(g) \in M$, $a(g) \in A$, and $v(g) \in V$.

Let $W(\mathfrak{a})$ be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a})$, and let $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ be the set of simple restricted

roots. Then, the Weyl group is generated by the simple reflections $s_i \equiv s_{\alpha_i}$, $i = 1, 2, \ldots, r$, $r = \dim \mathfrak{a}$ (i.e., s_i is the reflection with respect to the hyperplane orthogonal to the simple root α_i). Therefore, each element $w \in W$ of the Weyl group is representable in the form of the a product of simple reflections $w = s_{i_1} s_{i_2} \ldots s_{i_m}$, where s_{i_k} are elements of the generating set $\{s_1, s_2, \ldots, s_l\}$. This representation is not unique; the least number of simple reflections required in such a decomposition of $w \in W(\mathfrak{a})$ is called the length of w and is denoted by $l(w)$. [It is clear that $l(w_1 w_2) \leq l(w_1) + l(w_2)$.] There exists the unique (longest) element w_o in $W(\mathfrak{a})$ such that $w_o \mathfrak{a}^+ = \mathfrak{a}^-$, where $\mathfrak{a}^+$ is the dominant Weyl chamber and $\mathfrak{a}^- = \{H \in \mathfrak{a} \,|\alpha(H) < 0 \text{ for all } \alpha \in \Delta_+\}$. Hence, $w_o \Delta_+ = -\Delta_+$ and $w_o^{-1} = w_o$. Let M and M' denote, respectively, the centralizer and the normalizer of A in K; M is a normal subgroup in M', and the quotient group M'/M is isomorphic to the Weyl group $W(\mathfrak{a})$. Below, we identify $W(\mathfrak{a})$ and M'/M.

Example. Let $G = SL(n, R)$. Then, $A = \{\mathrm{diag}(a_1, a_2, \ldots, a_n) \mid a_1 a_2 \cdots a_n = 1,\ a_i > 0,\ i = 1, 2, \ldots, n\}$ and $N(V)$ consist of the upper (lower) triangular matrices with 1's on the main diagonal. The subgroup M' is generated by the subgroup M and by the matrices $s_i = \mathrm{diag}(1, \ldots, 1, s, 1, \ldots, 1)$, $i = 1, 2, \ldots, n-1$, where the matrix

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is placed in ith and $(i+1)$th rows. The Weyl group W (imbedded into the subgroup M') is generated by the matrices s_i. The action of W on A is defined by the formula $w \cdot a \equiv waw^{-1}$, $w \in W$, $a \in A$. The group W coincides with the symmetric group S_n and therefore has $n!$ elements. The matrix with all zero entries, except for the entries $(w)_{k,n-k+1} = \pm 1$, is the longest element in W. It permutes the entries a_k and a_{n-k+1}, $k = 1, 2, \ldots, n$, of the matrices $a = \mathrm{diag}(a_1, a_2, \ldots, a_n) \in A$. Moreover, we have $w_o N w_o^{-1} = V$.

Now, we give a short description of the (nondegenerate) principal series. It consists of the UIRs of G induced by the finite-dimensional UIRs of the minimal parabolic subgroup $P = MAN$. Any such representation of P has the form

$$man \to \lambda(a)\xi(m),$$

where λ is a unitary character of A and ξ stands for the UIRs of M. The action of the elements of W on the characters of A and on the representation of M is defined by

$$w\lambda(a) = \lambda(w^{-1}aw), \qquad w\xi(m) = \xi(w^{-1}mw), \quad (5)$$

where $a \in A$, $m \in M$, and $w \in W$. It is also well known [13, 14] that the induced representations $U(\xi, \lambda, \cdot)$ and $U(w\xi, w\lambda, \cdot)$ are unitarily equivalent. Thus, for each $w \in W$, there exists a unitary intertwining operator $\mathfrak{A}(w, \xi, \lambda)$ such that

$$\mathfrak{A}(w, \xi, \lambda) U(\xi, \lambda, g) = U(w\xi, w\lambda, g)\mathfrak{A}(w, \xi, \lambda) \quad (6)$$

for all $g \in G$. Moreover, the intertwining operators satisfy the cocycle relations

$$\mathfrak{A}(w_1 w_2, \xi, \lambda) = \mathfrak{A}(w_1, w_2\xi, w_2\lambda)\mathfrak{A}(w_2, \xi, \lambda). \quad (7)$$

Thus, there exists a one-to-one correspondence between the intertwining operators and the Weyl group elements. Hence, the relations of the Weyl group imply similar relations for the intertwining operators.

Let E_ξ be the (finite-dimensional) Hilbert space on which ξ is realized. Then, the functions in the induced representation space are functions on G such that

$$f(xman) = \mu^{-1/2}(a)\lambda^{-1}(a)\xi^{-1}(m)f(x),$$
$$m \in M, \quad a \in A, \quad n \in N,$$

and the group action is given by

$$U(\xi, \lambda, g)f(x) = f(g^{-1}x), \quad (8)$$

where $\mu(a) = e^{2\rho \log a}$ is a positive character on A and

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} (\dim g_\alpha)\alpha.$$

Since $g = kan$, according to the Iwasawa decomposition, f is determined by its value on K. Therefore, the principal series can also be realized on a subspace of $L^2(K, E_\xi)$ of functions f such that $f(km) = \xi^{-1}(m)f(k)$. Another realization ("noncompact picture") of the principal series is obtained by restrictions to V of the functions in the induced picture. The Hilbert space is therefore $L^2(V, E_\xi)$.

The normalized intertwining operator in the induced picture is given by

$$\mathfrak{A}(w, \xi, \lambda)f(x) = \frac{1}{\gamma(w, \xi, \lambda)} \int\limits_{V \cap w^{-1}Nw} f(xwv)dv, \quad (9)$$

with $w \in W$ and f lying in the space of the induced representation. Here, $\gamma(w, \xi, \lambda)$ are normalizing factors. With these factors, the operators $\mathfrak{A}$ become unitary.

It should be emphasized that the integral in (9) is actually divergent; it converges only for some nonunitary characters λ of A. It can be shown that these integrals may be continued analytically to other λ forming meromorphic functions with poles and zeros at real λ.

To gain a better understanding of our approach, we illustrate it for scattering models with the $SL(2,R)$ symmetry group. In this case,

$$K \ni k = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \; A \ni a = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix},$$

$$N \ni n = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \quad V \ni v = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix},$$

$$M \ni m = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad w_o = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\varepsilon = \pm 1$. Let $\xi(m) = 1$ be the trivial character of M (spherical principal series), and let $\lambda(a) = \exp(ipt)$. Then, the normalized intertwining operator is given by

$$\mathfrak{A}(w_o,1,ip)f(x) = \frac{2^{ip-1/2}\Gamma(1/2+ip)}{\sqrt{\pi}\Gamma(ip)} \times \int_{-\infty}^{\infty} |x-y|^{-1-2ip} f(y)dy$$

(in "noncompact picture") or

$$\mathfrak{A}(w_o,1,ip)f(\theta) = \frac{2^{ip-1/2}\Gamma(1/2+ip)}{\sqrt{\pi}\Gamma(ip)} \times \int_0^{2\pi} \left|\sin\frac{\theta-\theta'}{2}\right|^{-1-2ip} f(\theta')d\theta'$$

(in "compact picture"), where λ is identified with an imaginary number ip. This then gives the integral representation of the matrix elements of $\mathfrak{A}$. As a result, we have

(a) the $SL(2,R) \supset SO(2)$ reduction

$$\langle m' | \mathfrak{A}(w_o,1,ip) | m \rangle = \delta_{mm'} \frac{\Gamma(1-ip)\Gamma(1/2+ip+m)}{\Gamma(1+ip)(1/2-ip+m)}$$

and (b) the $SL(2,R) \supset V$ reduction

$$\langle \nu' | \mathfrak{A}(w_o,1,ip | \nu \rangle = \delta(\nu-\nu')\frac{\Gamma(1-ip)}{\Gamma(1+ip)} |\nu|^{2ip}.$$

It is worth noting that the matrix of $\mathfrak{A}$ is also diagonal in the $SO(1,1)$ basis $|\nu\tau\rangle$, $-\infty < \nu < \infty$, $\tau = \pm 1$, where τ is the multiplicity label. [Note that each UIR of $SO(1,1)$ is doubly degenerate in the UIR of $SL(2,R)$.] In this reduction, we have

$$\langle \mu'\tau' | \mathfrak{A}(w_o,1,ip)| \mu\tau \rangle = \delta(\mu-\mu')S_{\tau\tau'},$$

where

$$S_{++} = S_{--} = \cosh\pi\mu\,\Gamma(1-ip) \times \Gamma(1/2+i\rho+i\mu) \times \Gamma(1/2+i\rho-i\mu)/\Gamma(1+ip),$$
$$S_{+-} = S_{-+} = -i\sinh\pi p\,\Gamma(1-ip) \times \Gamma(1/2+i\rho+i\mu) \times \Gamma(1/2+i\rho-i\mu)/\Gamma(1+ip).$$

Thus, we have arrived at a very important conclusion: there exist three classes of one-dimensional scattering systems related to the $SO(2,1)$ group with the following S matrices:

(i) for class 1 [related to reduction $SO(2,1) \supset SO(2)$],

$$S_m = \begin{pmatrix} R_m & 0 \\ 0 & R_m \end{pmatrix}, \qquad (10)$$
$$R_m = \frac{\Gamma(1-ip)\Gamma(1/2+ip+m)}{\Gamma(1+ip)\Gamma(1/2-ip+m)};$$

(ii) for class 2 [related to reduction $SO(2,1) \supset SO(1,1)$],

$$S_\mu = \begin{pmatrix} R_\mu & T_\mu \\ T_\mu & R_\mu \end{pmatrix}, \qquad (11)$$

where

$$R_\mu = \cosh\pi\mu\,\Gamma(1-ip)\Gamma(1/2+i\rho+i\mu) \times \Gamma(1/2+i\rho-i\mu)/\Gamma(1+ip),$$

$$T_\mu = -i\sinh\pi\, p\Gamma(1-ip)\Gamma(1/2+i\rho+i\mu) \times \Gamma(1/2+i\rho-i\mu)/\Gamma(1+ip);$$

(iii) for class 3 [related to reduction $SO(2,1) \supset E(1)$],

$$S_\lambda = \begin{pmatrix} R_\lambda & 0 \\ 0 & R_\lambda \end{pmatrix}, \quad R = \frac{\Gamma(1-ip)}{\Gamma(1+ip)} |\nu|^{2ip}. \qquad (12)$$

It should be noted that the potential functions V of the second class admit a double degeneracy of the wave function for each positive value of E. The double degeneracy corresponds to the fact that one may construct wave packets that are partly transmitted and partly reflected by the potential V. According to (11), the reflection and transmission coefficients are

$$|R_\mu|^2 = \frac{\cosh^2\pi\mu}{\cosh^2\pi\mu + \sinh^2\pi p},$$

$$|T_\mu|^2 = \frac{\sinh^2\pi p}{\cosh^2\pi\mu + \sinh^2\pi p},$$

respectively. It also worth noting that, according to (10) and (12), we have $|R_m|^2 = |R_\lambda|^2 = 1$ for all potentials of class 1 or 3; hence, the reflection is total. This is the result of very general properties shared by all one-dimensional Hamiltonians that have a continuous nondegenerate spectrum.

Suppose that the Hamiltonian $\mathcal{H}$ of a scattering system in one dimension is a linear function of the Casimir operator. In studying such a system, we can, without loss of generality, restrict ourselves to the case where $\mathcal{H} = -(\mathcal{C} + 1/4)/2$ (on a one-dimensional subspace of the representation space). Then, the corresponding S matrix is given by formulas (10)–(12) with $p = \sqrt{2E}$. Moreover, we can extract the corresponding one-dimensional potential from the Casimir operator. To do this, let us consider (a reducible) representation T of $SL(2,R)$ realized in the Hilbert space of square-integrable functions f on $SL(2,R)/SO(2)$. The representation T is defined by

$$T(g)f(\zeta) = f(\zeta g), \quad g \in SL(2,R).$$

The spherical principal series representation can be realized as a subrepresentation of T. (We note that the representation T is decomposed into the direct integral of a spherical principal representation.) Hence, we require the representation space to be irreducible. Such a restriction is obtained if all functions f are eigenfunctions of the Casimir operator; i.e., $\mathcal{C}f = -(p^2 + 1/4)f$. The "group" Hamiltonian is obtained from the Casimir operator after the reduction condition is imposed. As a result, we obtain the following results:

(i) for the $SL(2,R) \supset SO(2)$ reduction,

$$\mathcal{H} = -\frac{d^2}{d\alpha^2} + \frac{m^2 - 1/4}{\sinh^2\alpha}; \tag{13}$$

(ii) for the $SL(2,R) \supset SO(1,1)$ reduction,

$$\mathcal{H} = -\frac{d^2}{d\beta^2} + \frac{m^2 - 1/4}{\cosh^2\beta}; \tag{14}$$

(iii) for the $SL(2,R) \supset V$ reduction,

$$\mathcal{H} = -\frac{d^2}{dt^2} + \nu^2 \exp(-2t). \tag{15}$$

We now proceed to describe one-dimensional scattering systems related to higher real-rank groups. The S matrix for the system under consideration is defined by $\mathfrak{A}(w_o, \xi, \lambda)$. Therefore, we are interested in examining the intertwining operator corresponding to the longest element of the Weyl group.

The study of the intertwining operators of a higher real-rank group can be reduced, to a large extent, to the case of the real-rank one. The main technique is to use a decomposition of a member of the Weyl group into the product of simple reflections. For example, in the case of $SL(n,R)$, the problem is reduced to the $SL(2,R)$ one [18].

Let w be a general element of W and suppose that

$$w = q_1, q_2, \ldots, q_m,$$

where the q_i are elements of the generating set $\{s_1, s_2, \ldots, s_r\}$, with r being the real rank of G. Let

$$q_i' = q_{i+1} q_{i+2} \cdots q_m$$

for $1 \le i \le m-1$. Then, we have [14]

$$\mathfrak{A}(w, \xi, \lambda) = \mathfrak{A}(q_1, q_1'\xi, q_1'\lambda) \times \mathfrak{A}(q_2, q_2'\xi, q'\lambda) \ldots \mathfrak{A}(q_m, \xi, \lambda). \tag{16}$$

The operators $\mathfrak{A}(q_i, q_i'\xi, q_i\lambda)$, $1 \le i \le m-1$, and $\mathfrak{A}(q_m, \lambda)$ are essentially the operators in the real-rank-one case [13, 14].

Since two-body systems are related to the real-rank-one group, we have arrived at a very important conclusion; the multiparticle S matrix is completely determined in terms of the two-particle ones. Moreover, if a Hamiltonian of scattering systems is related to a second-order Casimir operator $\mathcal{C}$ as $\mathcal{H} = -(\mathcal{C} + \langle\rho, \rho\rangle)/2$, then we obtain an extremely simple relation between initial and final momenta; the latter is determined by the longest element of the Weyl group. For example, let $G = SL(n,R)$, and let λ be identified with the n-tuple of imaginary numbers $ip_1, ip_2, \ldots, ip_n$, where $p_1 + p_2 + \ldots + p_n = 0$. There are $n-1$ simple reflections s_i whose action is

$$s_i(\ldots, p_i, p_{i+1}, \ldots) = (\ldots, p_{i+1}, p_i, \ldots).$$

The longest-element-induced action is

$$w_o(p_1, p_2, \ldots, p_{n-1}, p_n) = (p_n, p_{n-1}, \ldots, p_2, p_1). \tag{17}$$

Thus, we arrive at the result of Calogero *et al.* [19]. (Considering a one-dimensional n-body system with interparticle potential $1/\sinh^2 x$, these authors showed that an ingoing scattering configuration characterized by the initial momenta $p_i, i = 1, 2, \ldots, n$ goes over into a unique outgoing configuration characterized by the momenta $p_i' = p_{n+1-i}$.)

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Classical Mechanics on the $GL(n,\mathbb{R})$ Group and Euler–Calogero–Sutherland Model*

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Abstract—Relations between free motion on the $GL^+(n,\mathbb{R})$ group manifold and the dynamics of an n-particle system with spin degrees of freedom on a line interacting with a pairwise $1/\sinh^2 x$ "potential" (Euler–Calogero–Sutherland model) are discussed within a Hamiltonian reduction. Two kinds of reductions of the degrees of freedom are considered: that which is due to continuous invariance and that which is due to discrete symmetry. It is shown that, upon projecting onto the corresponding invariant manifolds, the resulting Hamiltonian system represents the Euler–Calogero–Sutherland model in both cases. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

In this contribution, we deal with two finite-dimensional Hamiltonian systems. The first one is a generalization of the Calogero–Sutherland–Moser model [1] by introducing the internal degrees of freedom [2, 3] described by the Hamiltonian

$$H_{\mathrm{ECS}} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{8}\sum_{i\neq j}^{N} \frac{l_{ij}^2}{\sinh^2(x_i - x_j)}, \qquad (1)$$

where the canonical pairs (x_i, p_i) obey the nonvanishing Poisson brackets

$$\{x_i, p_j\} = \delta_{ij} \qquad (2)$$

and the "internal" variables l_{ab} satisfy the $SO(n,\mathbb{R})$ Poisson bracket algebra

$$\{l_{ab}, l_{cd}\} = \delta_{ac}l_{bd} + \delta_{ad}l_{bc} + \delta_{bd}l_{ac} - \delta_{bc}l_{ad}. \qquad (3)$$

The dynamics of the second system is given in terms of a geodesic motion on the $GL(n,\mathbb{R})$ group manifold. The corresponding Lagrangian based on the bi-invariant metric on $GL(n,\mathbb{R})$ is given by [4, 5]

$$L_{GL} = \frac{1}{2}\mathrm{tr}\left(\dot{g}g^{-1}\right)^2, \qquad (4)$$

where $g \in GL(n,\mathbb{R})$ and the dot over relevant symbols denotes differentiation with respect to time. Below, we shall represent the Hamiltonian system corresponding to this Lagrangian (4) in terms of a special parametrization adapted to the action of the symmetry group of the system. We demonstrate that the resulting Hamiltonian is a generalization of the Euler–Calogero–Sutherland model (1) with two types of internal degrees of freedom. Performing the Hamiltonian reduction owing to two types of symmetry, a continuous and a discrete one, we show how to arrive at the conventional Hamiltonian of the Euler–Calogero–Sutherland model (1).

2. BI-INVARIANT GEODESIC MOTION ON THE GROUP MANIFOLD

2.1. Explicit Integration of the Classical Equation of Motion

The Euler–Lagrange equation following from the Lagrangian in (4) can be represented as

$$\frac{d}{dt}\left(g^{-1}\dot{g}\right) = 0. \qquad (5)$$

This form demonstrates their explicit integrability,

$$g(t) = g(0)\exp(tJ), \qquad (6)$$

with two arbitrary constant matrices $g(0)$ and J.

2.2. Hamiltonian in Terms of Special Coordinates

The canonical Hamiltonian corresponding to the bi-invariant Lagrangian (4) has the form

$$H_{GL} = \frac{1}{2}\mathrm{tr}\left(\pi^T g\right)^2. \qquad (7)$$

The nonvanishing Poisson brackets between the fundamental phase-space variables are

$$\{g_{ab}, \pi_{cd}\} = \delta_{ac}\delta_{bd}. \qquad (8)$$

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To find the relation to the conventional Euler–Calogero–Sutherland model (1), it is convenient to use the polar decomposition [6] for an arbitrary element of $GL(n,\mathbb{R})$. For the sake of technical simplicity, we investigate in detail the group $GL(3,\mathbb{R})$ hereinafter; i.e.,

$$g = OS, \tag{9}$$

where S is a positive definite 3×3 symmetric matrix and $O(\phi_1,\phi_2,\phi_3) = e^{\phi_1 J_3}e^{\phi_2 J_1}e^{\phi_3 J_3}$ is an orthogonal matrix with $SO(3,\mathbb{R})$ generators $(J_a)_{ik} = \varepsilon_{iak}$. Since the matrix g represents an element of the $GL(n,\mathbb{R})$ group, we can treat the polar decomposition (9) as a uniquely invertible transformation from the configuration variables g to a new set of six Lagrangian coordinates S_{ij} and three coordinates ϕ_i. The induced transformation of momenta to the new canonical pairs (S_{ab},P_{ab}) and (ϕ_a,P_a) is

$$\pi = O\left(P - k_a J_a\right), \tag{10}$$

where

$$k_a = \gamma_{ab}^{-1}\left(\eta_b^L - \varepsilon_{bmn}\left(SP\right)_{mn}\right). \tag{11}$$

Here, η_b^L are three left-invariant vector fields on $SO(3,\mathbb{R})$,

$$\eta_1^L = \frac{\sin\phi_3}{\sin\phi_2}P_1 + \cos\phi_3\,P_2 - \cot\phi_2\sin\phi_3 P_3, \tag{12}$$

$$\eta_2^L = \frac{\cos\phi_3}{\sin\phi_2}P_1 - \sin\phi_3 P_2 - \cot\phi_2\cos\phi_3 P_3,$$

$$\eta_3^L = P_3,$$

and $\gamma_{ik} = S_{ik} - \delta_{ik}\mathrm{tr}S$. In terms of the new variables, the canonical Hamiltonian takes the form

$$H_{GL} = \frac{1}{2}\mathrm{tr}\left(PS\right)^2 + \frac{1}{2}\mathrm{tr}\left(J_a S J_b S\right)k_a k_b. \tag{13}$$

2.3. Restriction of the Hamiltonian to the Principal Orbit

The system specified by Eq. (13) is invariant under the orthogonal transformations $S' = R^T S R$, and the orbit space is given as the quotient space $\mathcal{S}/SO(3,\mathbb{R})$. The quotient space $\mathcal{S}/SO(3,\mathbb{R})$ is a stratified manifold; orbits with the same isotropy group are collected into strata and are uniquely parametrized by the set of ordered eigenvalues of the matrix S ($x_1 \le x_2 \le x_3$). The strata are classified according to the isotropy groups that are determined by the degeneracies of the matrix eigenvalues:

(i) Principal orbit-type stratum, where all eigenvalues are unequal $x_1 < x_2 < x_3$, with the smallest isotropy group $Z_2\otimes Z_2$.

(ii) Singular orbit-type strata forming the boundaries of orbit space with (a) two coinciding eigenvalues (e.g., $x_1 = x_2$), where the isotropy group is $SO(2)\otimes Z_2$, and (b) all three coinciding eigenvalues ($x_1 = x_2 = x_3$), in which case the isotropy group coincides with the isometry group $SO(3,\mathbb{R})$.

Now, we shall first restrict ourselves to the investigation of dynamics that takes place on the principal orbits. To write down the Hamiltonian describing motion on the principal orbit stratum, we introduce the coordinates along the slices x_i and along the orbits χ. Namely, since the matrix S is positive definite and symmetric, we use the main-axis decomposition in the form

$$S = R^T(\chi)e^{2X}R(\chi), \tag{14}$$

where $R(\chi)\in SO(3,\mathbb{R})$ is an orthogonal matrix parametrized by three Euler angles $\chi = (\chi_1,\chi_2,\chi_3)$ and the matrix e^{2X} is diagonal,

$$e^{2X} = \mathrm{diag}\|e^{2x_1},e^{2x_2},e^{2x_3}\|.$$

The original physical momenta P_{ik} are expressed in terms of the new canonical pairs (x_i,p_i) and (χ_i,p_{χ_i}) as

$$P = R^T e^{-X}\left(\sum_{a=1}^{3}\bar{\mathcal{P}}_a\bar{\alpha}_a + \sum_{a=1}^{3}\mathcal{P}_a\alpha_a\right)e^{-X}R, \tag{15}$$

where

$$\bar{\mathcal{P}}_a = \frac{1}{2}p_a, \tag{16}$$

$$\mathcal{P}_a = -\frac{1}{4}\frac{\xi_a^R}{\sinh(x_b - x_c)}, \tag{17}$$

(cyclic permutation $a\ne b\ne c$).

In representation (15), we have introduced the orthogonal basis for the symmetric 3×3 matrices $\alpha_A = (\bar{\alpha}_i,\alpha_i)$, $i = 1,2,3$, with the scalar product

$$\mathrm{tr}(\bar{\alpha}_a\bar{\alpha}_b) = \delta_{ab},\quad \mathrm{tr}(\alpha_a\alpha_b) = 2\delta_{ab},$$
$$\mathrm{tr}(\bar{\alpha}_a\alpha_b) = 0$$

and the $SO(3,\mathbb{R})$ right-invariant Killing vectors

$$\xi_1^R = -\sin\chi_1\cot\chi_2\,p_{\chi_1} + \cos\chi_1\,p_{\chi_2} + \frac{\sin\chi_1}{\sin\chi_2}p_{\chi_3}, \tag{18}$$

$$\xi_2^R = \cos\chi_1\cot\chi_2\,p_{\chi_1} + \sin\chi_1\,p_{\chi_2} - \frac{\cos\chi_1}{\sin\chi_2}p_{\chi_3}, \tag{19}$$

$$\xi_3^R = p_{\chi_1}, \tag{20}$$

Upon going over to these main-axis variables, the canonical Hamiltonian takes the form

$$H_{GL} = \frac{1}{8}\sum_{a=1}^{3}p_a^2 + \frac{1}{16}\sum_{(abc)}\frac{(\xi_a^R)^2}{\sinh^2(x_b - x_c)} \tag{21}$$

$$-\frac{1}{4}\sum_{(abc)}\frac{\left(R_{am}\eta^L_m+\frac{1}{2}\xi^R_a\right)^2}{\cosh^2(x_b-x_c)},$$

where (abc) means cyclic permutations ($a\neq b\neq c$). Thus, the integrable dynamical system describing free motion on principal orbits represents, in the adapted basis, the generalized Euler–Calogero–Sutherland model. The generalization consists in the introduction of two types of internal dynamical variables ξ and η ("spin" and "isospin" degrees of freedom) interacting with each other. Below, we demonstrate the relations to the standard Euler–Calogero–Sutherland model (1).

2.4. Restriction of the Hamiltonian to the Singular Orbit

The motion on the singular orbit is modified owing to the presence of a continuous isotropy group. In the case of $GL(3,\mathbb{R})$, it is $SO(2)\otimes Z_2$. Applying the same machinery as for the principal orbits to the two-dimensional orbit ($x_1=x_2=x, x_3=y$), one can derive the Hamiltonian

$$H^{(2)}_{GL}=\frac{1}{4}p_x^2+\frac{1}{8}p_y^2+\frac{g^2}{\sinh^2(x-y)}, \qquad (22)$$

where the constant g^2 is related to the value of the spin ξ. Thus, the Hamiltonian on the singular orbit corresponds to the two-dimensional Calogero–Sutherland model with particles whose mass ratio is 1 : 2. Due to translation invariance, the equations of motion are equivalent to the corresponding equations for a one-dimensional problem; thus, the system specified by Eq. (22) is integrable.

3. REDUCTION TO THE EULER–CALOGERO–SUTHERLAND MODEL

3.1. Reduction Using Discrete Symmetries

We shall now demonstrate how the IIA_3 Euler–Calogero–Sutherland model arises from the canonical Hamiltonian (7) after projection onto a certain invariant submanifold determined by discrete symmetries. Let us impose the condition of symmetry of the matrices $g\in GL(n,\mathbb{R})$,

$$\chi^{(1)}_a=\varepsilon_{abc}g_{bc}=0. \qquad (23)$$

In order to find an invariant submanifold, it is necessary to supplement the constraints in (23) with the new constraints

$$\chi^{(2)}_a=\varepsilon_{abc}\pi_{bc}=0. \qquad (24)$$

One can check that the surface defined by the constraints in (23) and (24) represents an invariant submanifold in the $GL(3,\mathbb{R})$ phase space and that the dynamics of the corresponding induced system is governed by the reduced Hamiltonian

$$H_{GL}\big|_{\chi^{(1)}_a=0,\chi^{(2)}_a=0}=\frac{1}{2}\mathrm{tr}\,(PS)^2. \qquad (25)$$

The matrices S and P are now symmetric nondegenerate matrices, and one can be convinced that this expression leads to the Hamiltonian of the IIA_3 Euler–Calogero–Sutherland model. To verify this statement, it is necessary to note that, after projection onto the invariant submanifold, the canonical Poisson structure changes. We have to deal with the new Dirac brackets

$$\{F,G\}_{\mathrm{D}}=\{F,G\}_{\mathrm{PB}}-\{F,\chi_a\}C^{-1}_{ab}\{\chi_b,G\} \qquad (26)$$

for arbitrary functions on the phase space. In our case, because $C_{ab}=\|\{\chi^{(1)}_a,\chi^{(2)}_b\}\|=2\delta_{ab}$, the fundamental Dirac brackets between the matrices S and P are

$$\{S_{ab},P_{cd}\}_{\mathrm{D}}=\frac{1}{2}\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}.$$

Considering transformation (14) to the main-axis variables, we have the canonical pairs (x_a,p_a) and (χ_a,p_{χ_a}),

$$\{x_a,p_b\}=\delta_{ab},\quad \{\chi_a,p_{\chi_b}\}=\delta_{ab}.$$

The angular variables (χ_a,p_{χ_b}) are gatherred, according to (18)–(20), into the right-invariant vector fields ξ^R_a, which obey the Poisson bracket algebra:

$$\{\xi^R_a,\xi^R_b\}_D=\varepsilon_{abc}\xi^R_c.$$

Since the matrices S and P are now symmetric and nondegenerate, one can be sure that, after rescaling of the canonical variables, the reduction via the discrete symmetry does indeed lead to the IIA_3 Euler–Calogero–Sutherland model.

3.2. Reduction due to the Continuous Symmetry

The integrals of the motion corresponding to geodesic motion with respect to the bi-invariant metric on the $GL(n,\mathbb{R})$ group are

$$J_{ab}=(\pi^T g)_{ab}. \qquad (27)$$

The algebra of these integrals realizes, on the symplectic level, the $GL(n,\mathbb{R})$ algebra

$$\{J_{ab},J_{cd}\}=\delta_{bc}J_{ad}-\delta_{ad}J_{cb}. \qquad (28)$$

After transformation to the scalar and rotational variables, the expressions for J take the form

$$J=\sum_{a=1}^{3}R^T\left(p_a\bar{\alpha}_a-i_a\alpha_a-j_aJ_a\right)R, \qquad (29)$$

where

$$i_a=\frac{1}{2}\xi^R_a\coth(x_b-x_c) \qquad (30)$$

$$+\left(R_{am}\eta_m^L+\frac{1}{2}\xi_a^R\right)\tanh(x_b-x_c)$$

and

$$j_a=R_{am}\eta_m^L+\xi_a^R. \tag{31}$$

When these integrals are used, there appear several ways to choose an invariant manifold and to derive the corresponding reduced system. Let us consider the surface on phase space defined by the constraints

$$\eta_a^R=0. \tag{32}$$

In the Dirac terminology [7, 8], these constraints are first-class constraints $\{\eta_a^R,\eta_b^R\}=\varepsilon_{abc}\eta_c^R$, and the surface given by (32) is invariant under the evolution governed by the Hamiltonian,

$$\{\eta_a^R,H_{GL}\}=0.$$

Using the relation between left- and right-invariant Killing fields $\eta_a^R=O_{ab}\eta_b^L$, we find that, after projection onto the constraint surface (32), the Hamiltonian reduces to

$$H_{GL}(\eta_a^R=0)=\frac{1}{8}\sum_a^3 p_a^2 \tag{33}$$
$$+\frac{1}{4}\sum_{(abc)}\frac{(\xi_a^R)^2}{\sinh^2 2(x_b-x_c)}.$$

After rescaling the variables as $2x_a\to x_a$, one is convinced that the resulting Hamiltonian coincides with the Euler–Calogero–Sutherland Hamiltonian (1), where the intrinsic spin variables are $l_{ij}=\varepsilon_{ijk}\xi_k^R$. Note that, performing the reduction to the surface defined by the vanishing integrals $j_a=0$, we again arrive at the same Euler–Calogero–Sutherland system.

3.3. Lax Pair for Generalized Euler–Calogero–Sutherland Model

Expressions (29) for the integrals of motion allow us to rewrite the classical equation of motion for the generalized Euler–Calogero–Sutherland model in the Lax form

$$\dot{L}=[A,L], \tag{34}$$

where the 3×3 matrices are given explicitly as

$$L=\begin{pmatrix} p_1 & L_3^+ & L_2^- \\ L_3^- & p_2 & L_1^+ \\ L_2^+ & L_1^- & p_3 \end{pmatrix}$$

and

$$A=\frac{1}{4}e^{+X}\begin{pmatrix} p_1 & -A_3 & A_2 \\ A_3 & p_2 & -A_1 \\ -A_2 & A_1 & p_3 \end{pmatrix}e^{-X}$$

with

$$L_1^\pm=-\frac{1}{2}\frac{\xi_1^R}{\sinh(x_2-x_3)}\pm\frac{R_{1m}\eta_m^L+\frac{1}{2}\xi_1^R}{\cosh(x_2-x_3)}, \tag{35}$$

$$L_2^\pm=-\frac{1}{2}\frac{\xi_2^R}{\sinh(x_3-x_1)}\pm\frac{R_{2m}\eta_m^L+\frac{1}{2}\xi_2^R}{\cosh(x_3-x_1)}, \tag{36}$$

$$L_3^\pm=-\frac{1}{2}\frac{\xi_3^R}{\sinh(x_1-x_2)}\pm\frac{R_{3m}\eta_m^L+\frac{1}{2}\xi_3^R}{\cosh(x_1-x_2)} \tag{37}$$

and

$$A_1=\frac{1}{2}\frac{\xi_1^R}{\sinh^2(x_2-x_3)}-\frac{R_{1m}\eta_m^L+\frac{1}{2}\xi_1^R}{\cosh^2(x_2-x_3)}, \tag{38}$$

$$A_2=\frac{1}{2}\frac{\xi_2^R}{\sinh^2(x_3-x_1)}-\frac{R_{2m}\eta_m^L+\frac{1}{2}\xi_2^R}{\cosh^2(x_3-x_1)}, \tag{39}$$

$$A_3=\frac{1}{2}\frac{\xi_3^R}{\sinh^2(x_1-x_2)}-\frac{R_{3m}\eta_m^L+\frac{1}{2}\xi_3^R}{\cosh^2(x_1-x_2)}. \tag{40}$$

4. CONCLUDING REMARKS

In this paper, we have discussed the generalization of the Euler–Calogero–Sutherland model by introducing two internal variables ("spin" and "isospin") using the integrable model based on the general matrix group $GL(n,\mathbb{R})$. We outline its relation to the well-known integrable model. Our consideration confirms once again that the clue to an integrability of a model is often hidden in the possibility of relating it to a known higher dimensional exactly solvable system by its symplectic reduction to its invariant submanifold [4, 5]. A rich spectrum of these types of finite-dimensional models obtained by the generalized "momentum map" is well known (see, e.g., [9]). Over the last decade, it has been recognized that the same happens in the infinite-dimensional case. Integrable two-dimensional field theories have been found from the so-called WZNW theory by applying the Hamiltonian reduction method [10]. An important class of finite-dimensional systems was discovered by the Hamiltonian reduction method from the so-called matrix models (for a recent review, see, e.g., [11]). Interest in this type of models has a long history starting with the Wigner study of the statistical theory of energy levels of complex nuclear systems [12]. Nowadays, we have a revival of interest in matrix models associated with the search for relations between supersymmetric Yang–Mills theory and integrable systems (for a modern

review, see, e.g., [13]). The relation between the Euler–Calogero–Moser model and $SU(2)$ Yang–Mills theory in the long-wave approximation was obtained in [14].

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Multiseparability and Superintegrability in Three Dimensions*

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Abstract—In complex two-dimensional Euclidean space, the Hamilton–Jacobi or Schrödinger equation with a given "nondegenerate" potential is maximally superintegrable if and only if it is separated in more than one coordinate system. A similar statement for three dimensions is not known. In this paper, a start will be made on this problem by investigating the known separable Hamilton–Jacobi and Schrödinger systems to find those that are superintegrable. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

The Hamilton–Jacobi equation with a given potential is integrable in N dimensions if there exist N constants of the motion in involution under the Poisson bracket. If these constants are quadratic in momenta and satisfy a purely algebraic condition, it is possible to obtain a solution by a separation of variables [1]. When the system possesses more than N constants, it is said to be superintegrable. These extra constants might be related to the existence of another separating coordinate system, in which case the system is said to be multiseparable. There can be at most $2N-1$ functionally independent constants, and a system having $2N-1$ constants is said to be maximally superintegrable.

While multiseparable systems must be superintegrable, the converse is not necessarily the case; the relationship between multiseparability and superintegrable systems has been studied by many authors. Evans concluded their equivalence for $E_{3,\mathbb{R}}$ [2], while Kalnins *et al.* [3, 4] gave conditions that are sufficient for guaranteeing a similar result in $E_{2,\mathbb{C}}$.

The lack of a similar understanding of the same problem in $E_{3,\mathbb{C}}$ is the primary motivation for this paper. We have started with the more modest goal of finding all superintegrable potentials that are separable at least once in an elliptic coordinate system, and we present some initial results.

In two dimensions, multiseparable potentials separating in elliptic coordinates are symmetric rational functions of elliptic coordinates. These have natural generalizations in higher dimensions [5]—in three dimensions, for example, there are several potentials of this form that are known to separate in both elliptic and Cartesian coordinates. It is clear that these potentials should be reproduced in this study, and indeed they are; however, one new potential that separates in two distinct degenerate types of elliptical coordinates was not previously known. This potential is presented below along with some of its properties.

Both the polynomial Poisson algebra of constants of the motion of the classical system and the operator algebra of the corresponding quantum system are given. Both have a quadratic subalgebra, and, in the latter case, the results of Daskaloyannis [6, 7] are used to find some energy eigenvalues.

2. GENERIC MULTISEPARABLE POTENTIALS IN ELLIPTIC COORDINATES

The generic elliptic coordinates u, v, and w are related to the Cartesian coordinates x, y, and z as

$$x^2 = \frac{2(u-e_1)(v-e_1)(w-e_1)}{(e_1-e_2)(e_1-e_3)},$$
$$y^2 = \frac{2(u-e_2)(v-e_2)(w-e_2)}{(e_2-e_3)(e_2-e_1)},$$
$$z^2 = \frac{2(u-e_3)(v-e_3)(w-e_3)}{(e_3-e_1)(e_3-e_2)}.$$

In terms of these coordinates, the free Hamiltonian has the form

$$H_0 = \frac{(u-e_1)(u-e_2)(u-e_3)p_u^2}{(u-v)(u-w)} + \frac{(v-e_1)(v-e_2)(v-e_3)p_v^2}{(v-w)(v-u)} + \frac{(w-e_1)(w-e_2)(w-e_3)p_w^2}{(w-u)(w-v)}.$$

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Degenerate elliptic coordinate systems are obtained by considering various ways in which the roots of $(u-e_1)(u-e_2)(u-e_3)$ can approach infinity or coalesce and can be enumerated by listing the multiplicity of each root. The multiplicity of the "root at infinity" is indicated by placing the symbol "∞" above it. In terms of this naming scheme, the most general case, having three distinct finite roots, is called $\{111\}$ and the remaining cases are $\{21\}$, $\{3\}$, $\{\overset{\infty}{1}\,2\}$, $\{\overset{\infty}{1}\,11\}$, $\{\overset{\infty}{2}\,1\}$, and $\{\overset{\infty}{3}\}$.

Looking amongst potentials known to be separable and superintegrable in three dimensions, one can find examples that separate in elliptic coordinates. This gives a basic set of "known" superintegrable and multiseparable potentials for elliptic coordinates. These potentials are symmetric rational functions of elliptic coordinates and were described in [8]. For example, for the $\{111\}$ coordinates, they are

$$V=\frac{f(u)}{(u-v)(u-w)}+\frac{f(v)}{(v-w)(v-u)}+\frac{f(w)}{(w-u)(w-v)},$$

where f depends on four independent constant parameters, α, β, γ, and δ, and has the form

$$f(u)=\frac{\alpha}{u-e_1}+\frac{\beta}{u-e_2}+\frac{\gamma}{u-e_3}+\delta u^3.$$

(Note that constant and linear terms in f would not contribute to V and that a quadratic term would simply add a trivial constant to V.)

3. A NEW MULTISEPARABLE POTENTIAL

Potentials similar to those described in the preceding section include all but one of the superintegrable potentials that separate in a degenerate elliptic coordinate system. The exception is a potential that separates in the $\{\overset{\infty}{3}\}$ coordinates u, v, and w that are related to the Cartesian coordinates by the equations

$$x+iy=u+v+w,$$
$$x-iy=\frac{1}{4}(u-v-w)(w-u-v)(v-w-u),$$
$$z=\frac{1}{4}(u^2+v^2+w^2)-\frac{1}{2}(vw+wu+uv).$$

In terms of these coordinates, the exceptional Hamiltonian is

$$H=\frac{p_u^2+A}{(u-v)(u-w)}+\frac{p_v^2+B}{(v-w)(v-u)}+\frac{p_w^2+C}{(w-u)(w-v)}, \tag{1}$$

where A, B, and C are arbitrary constants. The potential term in this Hamiltonian is not a symmetric function of the coordinates; hence, it cannot be of the type described in Section 2. Note that the identity

$$\frac{1}{(u-v)(u-w)}+\frac{1}{(v-w)(v-u)}+\frac{1}{(w-u)(w-v)}=0$$

allows us to eliminate one of A, B, and C, and we choose $C=-A-B$ in the following.

3.1. Constants of the Motion

The Hamiltonian in (1) has two constants corresponding to separation in the coordinates u, v, and w; that is,

$$L_1=\frac{1}{2}\Big[-i(p_x+ip_y)J_z-(p_x-ip_y)p_z+ip_z(J_x+iJ_y)\Big]+l_1(x,y,z),$$
$$L_2=-i(p_x+ip_y)(J_x-iJ_y)+\frac{1}{4}(p_x-ip_y)^2-\frac{i}{2}(p_x-ip_y)(J_x+iJ_y)-\frac{1}{4}(J_x+iJ_y)^2+l_2(x,y,z).$$

The functions l_1 and l_2 can be more easily expressed in terms of u, v, and w; we have

$$L_1=\frac{1}{2}\left[\frac{(v+w)(p_u^2+A)}{(u-v)(u-w)}+\frac{(w+u)(p_v^2+B)}{(v-w)(v-u)}+\frac{(u+v)(p_w^2-A-B)}{(w-u)(w-v)}\right],$$
$$L_2=\frac{vw(p_u^2+A)}{(u-v)(u-w)}+\frac{wu(p_v^2+B)}{(v-w)(v-u)}+\frac{uv(p_w^2-A-B)}{(w-u)(w-v)}.$$

In addition, there is a first-order constant

$$S=\frac{3}{2}(p_x-ip_y)-\frac{i}{2}(J_x+iJ_y)=p_u+p_v+p_w;$$

there are also some further second-order constants

$$M_1=\frac{2i}{3}p_z(J_x-iJ_y)+\frac{i}{6}(p_x-ip_y)J_z+\frac{1}{6}J_z(J_x+iJ_y)+m_2(x,y,z),$$
$$M_2=\frac{2i}{3}(p_x-ip_y)(J_x-iJ_y)-\frac{1}{6}J_z^2+m_3(x,y,z).$$

For later convenience, we define

$$L_s = \frac{1}{2}L_2 - \frac{1}{6}S^2 .$$

These constants form a basis for all constants of first or second order in the momenta.

3.2. Poisson Algebra

Denoting the Poisson bracket by $\{\ ,\ \}_{\rm PB}$, we can define $R = \{L_s, M_1\}_{\rm PB}$. The constants H, L_1, L_s, S, M_1, M_2, and R form a closed polynomial Poisson algebra defined by

$$\{S, L_1\}_{\rm PB} = H, \quad \{S, L_s\}_{\rm PB} = L_1, \qquad (2)$$
$$\{S, M_1\}_{\rm PB} = L_s, \quad \{S, M_2\}_{\rm PB} = M_1,$$
$$\{L_1, L_s\}_{\rm PB} = \frac{1}{3}SH, \quad \{L_1, M_1\}_{\rm PB} = \frac{1}{3}SL_1,$$
$$\{L_1, M_2\}_{\rm PB} = -R + \frac{2}{3}L_sS,$$
$$\{L_s, M_1\}_{\rm PB} = R, \quad \{L_s, M_2\}_{\rm PB} = \frac{2}{3}SM_1,$$
$$\{M_1, M_2\}_{\rm PB} = \frac{2}{3}SM_2, \quad \{S, R\}_{\rm PB} = \frac{1}{3}SL_1,$$
$$\{L_1, R\}_{\rm PB} = -\frac{1}{9}HS^2 - \frac{1}{3}L_1^2 + \frac{1}{3}HL_s,$$
$$\{L_s, R\}_{\rm PB} = -\frac{2}{3}L_1L_s - \frac{2}{9}L_1S^2 + HM_1,$$
$$\{M_2, R\}_{\rm PB} = -\frac{4}{9}M_1S^2 + \frac{2}{3}L_1M_2 - \frac{2}{3}L_sM_1,$$
$$\{M_1, R\}_{\rm PB} = \frac{1}{3}L_1M_1 - \frac{2}{3}L_s^2$$
$$- \frac{2}{9}L_sS^2 - \frac{1}{3}RS + \frac{2}{3}HM_2$$

and subject to two identities,

$$6L_sS^2 - 18L_1M_1 + 18HM_2 - 18RS \qquad (3)$$
$$+ S^4 + 9L_s^2 = 3(A^2 + B^2 + AB)$$

and

$$0 = H(126S^2M_2 + 243M_1^2 - 432L_sM_2) \qquad (4)$$
$$+ (A^2 + B^2 + AB)(18L_s - 30S^2)$$
$$+ 27AB(A + B) - 126S^2L_1M_1$$
$$+ 270RL_sS + 36S^4L_s + 162L_1^2M_2 - 54L_1L_sM_1$$
$$- 126S^3R + 8S^6 - 243R^2.$$

These identities allow R and M_2 to be given in terms of S, H, L_1, L_2, and M_1, which can be shown to be functionally independent. Hence, the system is maximally superintegrable.

If we were to include S^2 as well, along with S in the algebra's generating set, the closure under the Poisson bracket would be quadratic.

3.3. A Quadratic Poisson Subalgebra in Involution with S

The existence of a first-order constant S allows a reduction of the Hamilton–Jacobi equations to a two-dimensional system on surfaces of constant S. Constants in involution with S are constant on these surfaces and can be used to investigate the properties of the two-dimensional system in question. The inspection of the Poisson algebra above reveals two such constants,

$$T_1 = L_1^2 - 2HL_s \quad \text{and} \quad T_2 = R - \frac{1}{3}SL_s.$$

These satisfy the quadratic algebra given by

$$\{T_1, T_2\}_{\rm PB} = T, \quad \{T, T_1\}_{\rm PB} = -\frac{\partial h}{\partial T_2},$$
$$\{T, T_2\}_{\rm PB} = \frac{\partial h}{\partial T_1},$$

where

$$h = \frac{2}{9}T_1^3 - \frac{4}{3}H^2ST_1T_2 + \frac{2}{27}H^2S^4T_1$$
$$- \frac{2}{9}H^2(A^2 + B^2 + AB)T_1 + 2H^3T_2^2.$$

In this representation, the Casimir operator is

$$C = T^2 - 2h = -\frac{4}{243}H^3(S^2 - 3A)$$
$$\times (S^2 - 3B)(S^2 + 3A + 3B).$$

3.4. Separation in the $\{\overset{\infty}{1}\ 2\}$ Elliptic Coordinates

Since we have a complete list of second-order constants, we can determine all separating coordinates systems. From the Poisson algebra (2), it can be deduced that the only commuting pairs of constants are

$$\{L_1, L_2\}_{\rm PB} = 0 \quad \text{and} \quad \{M_2, L_s - \frac{1}{3}S^2\}_{\rm PB} = 0.$$

Furthermore, M_2 and $L_s - \frac{1}{3}S^2$ satisfy the algebraic conditions necessary for them to describe separable coordinates [1]. This new set of coordinates is related to u, v, and w by the equations

$$u' = \frac{1}{4}\Big(u^2 - (v - w)^2\Big),$$
$$v' = \frac{1}{4}\Big(v^2 - (w - u)^2\Big),$$
$$w' = \frac{1}{4}\Big(w^2 - (u - v)^2\Big),$$

and the Hamiltonian (1) can be rewritten as

$$H = \frac{u'^2p_{u'}^2 + Au'}{(u' - v')(u' - w')} + \frac{v'^2p_{v'}^2 + Bv'}{(v' - w')(v' - u')}$$

$$+\frac{w'^2p_{w'}^2+Cw'}{(w'-u')(w'-v')}.$$

From the form of the Hamiltonian in these new coordinates, it is clear that they form a degenerate elliptical coordinate system with two coincident finite roots, that is, the coordinate system denoted $\{\overset{\infty}{1}\ 2\}$.

3.5. Polynomial Operator Algebra

We can also investigate the corresponding Schrödinger equation in these coordinates, and one obtains similar results when the Poisson bracket of constants is replaced by the commutator of differential operators.

Replacing p_u, p_v, and p_w by ∂_u, ∂_v, and ∂_w, and symmetrizing all products, we obtain the differential operators H, S, L_1, L_2, and M_1. However, we must also add $u\partial_u+v\partial_v+w\partial_w$ for M_2 in order to ensure that it commutes with H. We define $R=[L_s,M_1]$ and denote the symmetric product of two or three operators as $\{A,B\}=\frac{1}{2}(AB+BA)$ and $\{A,B,C\}=\frac{1}{6}(ABC+ACB+BAC+BCA+CAB+CBA)$.

Again, we find that the algebra closes polynomially. The commutators are

$$[S,L_1]=H,\quad [S,L_s]=L_1,\quad [S,M_1]=L_s,$$
$$[S,M_2]=M_1+\frac{1}{6}S,\quad [L_1,L_s]=\frac{1}{3}SH,$$
$$[L_1,M_1]=\frac{1}{3}\{S,L_1\}+\frac{1}{6}H,$$
$$[L_1,M_2]=-R+\frac{2}{3}\{S,L_s\}+\frac{1}{6}L_1,$$
$$[L_1,R]=-\frac{1}{9}HS^2-\frac{1}{3}L_1^2+\frac{1}{3}HL_s,$$
$$[L_s,M_2]=\frac{2}{3}\{S,M_1\}+\frac{1}{9}S^2,$$
$$[M_1,M_2]=\frac{2}{3}\{S,M_2\}-\frac{1}{6}M_1-\frac{1}{18}S,$$
$$[S,R]=\frac{1}{3}\{S,L_1\}+\frac{1}{6}H,$$
$$[L_s,R]=-\frac{2}{3}\{L_1,L_s\}$$
$$-\frac{2}{9}\{L_1,S^2\}+HM_1+\frac{1}{9}HS,$$
$$[M_1,R]=\frac{1}{3}\{L_1,M_1\}-\frac{2}{3}L_s^2$$
$$-\frac{2}{9}\{L_s,S^2\}-\frac{1}{3}\{R,S\}+\frac{2}{3}HM_2,$$
$$[M_2,R]=\frac{2}{3}\{L_1,M_2\}-\frac{4}{9}\{M_1,S^2\}$$
$$-\frac{2}{3}\{L_s,M_1\}-\frac{2}{9}\{S,L_s\}+\frac{1}{18}L_1-\frac{2}{27}S^3+\frac{1}{6}R.$$

The identities in (3) and (4) have their counterparts

$$L_s^2=-\frac{2}{3}\{L_s,S^2\}+2\{L_1,M_1\}$$
$$-2HM_2+2\{R,S\}-\frac{1}{9}S^4$$
$$+\frac{1}{3}(A^2+B^2+AB)-\frac{1}{3}H$$

and

$$0=126H\{S^2,M_2\}+243HM_1^2$$
$$-432H\{L_s,M_2\}-30(A^2+B^2+AB)S^2$$
$$-126\{S^2,L_1,M_2\}+18(A^2+B^2+AB)L_s$$
$$+270\{R,S,L_s\}+36\{S^4,L_s\}$$
$$+162\{L_1,L_1,M_2\}-54\{L_1,L_s,M_1\}$$
$$-126\{S^3,R\}+8S^6+27AB(A+B)$$
$$-243R^2+81\{L_1,R\}-54\{S,L_1,L_s\}$$
$$+9HL_s+81H\{S,M_1\}+\frac{127}{2}HS^2-\frac{3}{2}L_1^2.$$

3.6. Quadratic Subalgebra

By using the results above, it is easily verified that

$$T_1=L_1^2-2HL_s\quad\text{and}\quad T_2=R-\frac{1}{3}\{S,L_s\}-\frac{1}{6}L_1$$

commute with S. Choosing a representation in which S and H are diagonal, we can treat them as constants, and we find that T_1, T_2, and their commutator satisfy the quadratic algebra

$$[T_1,T_2]=T,\ \ [T,T_1]=\frac{4}{3}SH^2T_1-4H^3T_2,$$
$$[T,T_2]=\frac{2}{3}T_1^2-\frac{4}{3}H^2ST_2+\frac{2}{9}H^3+\frac{2}{27}H^2S^4.$$

In this representation, the Casimir operator is

$$C=T^2+\frac{8}{3}SH^2\{T_1,T_2\}-\frac{4}{9}T_1^3$$
$$-\frac{4}{27}(9H^3+H^2S^4-3H^2(A^2+B^2+AB))T_1$$
$$=-\frac{4}{243}H^3(S^2-3A)(S^2-3B)$$
$$\times(S^2+3A+3B)-\frac{4}{9}H^4S^2.$$

Following Daskaloyannis [6, 7], we can use this quadratic algebra to determine some allowed energy eigenvalues of states with a finite degeneracy. Using the conventions and formulas of [6, 7], we define this quadratic algebra as

$$\alpha=0,\quad \gamma=0,\quad \delta=-\frac{4}{3}SH^2,$$

$$\epsilon = 4H^3, \quad \zeta = 0, \quad a = -2/3, \quad d = 0,$$
$$z = -\frac{2}{9}H^3 - \frac{2}{27}H^2S^4 + \frac{2}{9}(A^2 + B^2 + AB).$$

A p-fold degenerate set of states can be found with the energy H given by

$$H = \frac{-\frac{1}{3}Z^2 + A^2 + B^2 + AB}{(p+1)^2},$$

where

$$8Z^3 - 18(A^2 + B^2 + AB)(Z \pm S^2)$$
$$\pm 2S^6 \pm 27AB(A + B) = 0.$$

Note that this energy depends both on p and on the eigenvalue of S.

4. DISCUSSION

With the potential presented here, the Hamilton–Jacobi and Schrödinger equations are both maximally superintegrable and multiseparable. This potential is of particular interest since it is not separable in coordinates other than degenerate forms of elliptic coordinates and does not have the form expected from generalizing two-dimensional results. Whether this potential has natural generalizations in higher dimensions or on a three-dimensional sphere is under investigation.

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An Algebraic Treatment of the MIC–Kepler Problem on S^3 Sphere*

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Abstract—The quantum-mechanical problem of motion in a dual charged Coulomb field modified by a centrifugal term (MIC–Kepler problem) is considered in a three-dimensional space of constant positive curvature, S^3. Conserved operators are found, and their commutation relations are derived. It is shown that, in the MIC–Kepler problem in S^3 space, conserved operators form a cubic algebra similar to that of the Kepler problem in the same space. This symmetry algebra is used to obtain the energy spectrum of the problem. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

The problem of motion in a dual charged Coulomb field with an additional inverse-square potential (MIC–Kepler problem) in a flat space was independently introduced in [1] and [2] and then studied in [3–7].

It is known that the Kepler and MIC–Kepler problems in R^3 are quite similar. In this paper, we consider the MIC–Kepler problem in three-dimensional spaces of constant curvature—in particular, on the S^3 sphere. We show that the MIC–Kepler problem in these spaces possesses similarities to the Kepler problem in spaces of constant curvature. In particular, we show that conserved quantum-mechanical operators of the Runge–Lenz type, together with the generalized angular-momentum operator, form a nonlinear (cubic) algebra similar to that of the Kepler problem on S^3. For this reason, we first give a brief review of this last problem.

The quantum-mechanical Kepler problem in a three-dimensional space S^3 of constant positive curvature was first considered by Schrödinger [8] and, in a space H^3 of constant negative curvature, by Infeld and Schild [9]. Those authors found the energy spectrum to be degenerate, similarly to that in a flat space. An additional constant of motion, an analog of the Runge–Lenz vector, which is the cause of this degeneracy, was found in [10–12] for the problem on the S^3 sphere and in [13] for the Lobachevsky space H^3. As was noted in [12], these operators, together with the angular momentum, generate an algebraic structure that may be considered as a nonlinear extension of a Lie algebra and which was referred to in [14] as a cubic algebra.

Recently, the Kepler problem on S^3 sphere was used as a model to describe quarkonium spectra [15] and excitons in quantum dots [16]. Many aspects of this problem in S^3 and H^3 spaces—in particular, separation of variables and path-integral formulation—were investigated in [14, 17–19].

We write the Schrödinger equation for the Kepler problem on the S^3 sphere as

$$H\psi = E\psi, \quad H = -\frac{1}{4R^2}M_{\mu\nu}M_{\mu\nu} - \frac{\alpha}{R}\frac{x_4}{|\mathbf{x}|}, \quad (1)$$

where

$$\mathbf{x} = \{x_1, x_2, x_3\}, \quad M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu,$$
$$x_\mu x_\mu = \mathbf{x}^2 + x_4^2 = R^2 \quad (\mu, \nu = 1, 2, 3, 4),$$

x_μ are coordinates in the four-dimensional flat space into which the sphere is embedded, and R is the radius of curvature. We use units such that $\hbar = m = 1$. Note that the operator $M_{\mu\nu}M_{\mu\nu}/2R^2$ coincides with the Laplacian operator on S^3 and that $M_{\mu\nu}M_{\mu\nu}/2$ is the Casimir operator of the geometric $O(4)$ group. Three generators $-iM_{ab}$ $(a, b = 1, 2, 3)$ constitute the angular-momentum vector $\mathbf{L}$, and three generators $-iM_{a4} = P_a$ are the boost generators on the sphere. The spectrum of this problem is $E_n = -\alpha^2/2n^2 + (n^2-1)/2R^2$ $(n = 1, 2, 3, \ldots)$. The Hamiltonian H commutes with the angular-momentum operator

$$L_a = -i\epsilon_{abc}x_b\partial_c, \quad a, b, c = 1, 2, 3, \quad (2)$$

and with the analog of the Runge–Lenz operators:

$$A_a = \frac{1}{2R}\epsilon_{abc}(L_bP_c - P_bL_c) + \frac{\alpha x_a}{|\mathbf{x}|}. \quad (3)$$

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These operators form a nonlinear (cubic) algebra with $o(3)$ subalgebra generated by L_a:

$$[A_a, A_b] = -2i\left(H - \frac{L^2}{R^2}\right)\epsilon_{abc}L_c, \qquad (4)$$
$$[L_a, A_b] = i\epsilon_{abc}A_c, \quad [L_a, L_b] = i\epsilon_{abc}L_c.$$

Recently, algebras of this type were intensively studied [20] in the context of symplectic reduction of Lie algebras and are called finite W algebras by analogy with infinite-dimensional W algebras that appeared in conformal field theories. The algebra in (4) is some deformation of the $so(4)$ algebra. The Casimir operators and some unitary irreducible representations for such algebras were constructed in [21]. For the case of the algebra in (4), the first and second Casimir operators in the notation of [21] are

$$C_{1d} = a\mathbf{L}^2 + b\mathbf{L}^4 + \mathbf{A}^2, \quad C_{2d} = \mathbf{L}\cdot\mathbf{A} \qquad (5)$$
$$\left(a = -2H + \frac{2}{R^2}, \quad b = \frac{1}{R^2}\right).$$

But from expression (3), one can find that

$$\mathbf{A}^2 = 2H(\mathbf{L}^2+1) - \frac{1}{R^2}\mathbf{L}^2(\mathbf{L}^2+2) + \alpha^2, \qquad (6)$$
$$\mathbf{A}\cdot\mathbf{L} = \mathbf{L}\cdot\mathbf{A} = 0;$$

therefore, $C_{1d} = 2H + \alpha^2$ and $C_{2d} = 0$. Thus, the Kepler problem on S^3 realizes some degenerate unitary irreducible representation of the cubic algebra (4), just as the flat Kepler problem realizes a degenerate representation of $O(4)$.

It will be shown below that the MIC–Kepler problem on the S^3 sphere realizes more general (nondegenerate) unitary representations of the cubic algebra (4). However, we first construct the Hamiltonian of the problem.

2. THE HAMILTONIAN OF THE MIC–KEPLER PROBLEM ON S^3

The most natural way to define the Dirac-like potential in a three-dimensional space of constant curvature is to solve Maxwell equations in this space with a Coulomb-type magnetic field. The Dirac monopole in curved spaces was considered in [22], where it was shown that the curvature of the background space plays no role in quantization of the magnetic charge of the test particle.

Consider a dual charged test particle of unit mass and charges (e_0, g_0) moving at a nonrelativistic velocity on the S^3 sphere in the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ of a stationary dual charged particle with charges (e, g) situated at the origin. We adopt the following abbreviations: $\alpha = (e_0e + g_0g)$ and $\mu = (e_0g - eg_0)$. Quantization of the component of the angular momentum leads to the condition $\mu = n/2$, where n is an integer. At first, we consider the case where $e = 0$, so that $\mathbf{E} = 0$.

It is convenient to use the four-dimensional spherical coordinates

$$x_1 = R\sin\chi\sin\theta\sin\phi, \quad x_2 = R\sin\chi\sin\theta\cos\phi,$$
$$x_3 = R\sin\chi\cos\theta, \quad x_4 = R\cos\chi, \qquad (7)$$
$$0 \le \chi \le \pi, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi.$$

In terms of these coordinates, the Maxwell equations for the magnetic field $\mathbf{H}$ have a Coulomb-type magnetic-field solution:

$$H_\phi = 0, \quad H_\theta = 0, \quad H_\chi = \mu/(R^2\sin^2\chi). \qquad (8)$$

Integration of the equations for the corresponding potential,

$$\boldsymbol{\nabla}\times\mathbf{A} = \mathbf{H}, \quad \boldsymbol{\nabla}\cdot\mathbf{A} = 0, \qquad (9)$$

leads to a Dirac monopole-like potential as a particular solution; that is,

$$A_\chi = A_\theta = 0, \quad A_\phi = \frac{\mu\tan\theta/2}{R\sin\chi}. \qquad (10)$$

This solution is valid everywhere, with the exception of the singularity line $\theta = \pi$ connecting the points $\chi = 0$ and $\chi = \pi$. In fact, this solution describes the field of two magnetic charges with opposite signs situated at the points $\chi = 0$ and $\chi = \pi$ and connected by the singularity line. It is worth noting that the electric Coulomb field considered by Schrödinger [8] is also created by two electric charges located at opposite points of S^3.

In the coordinates $\mathbf{x} = \{x_1, x_2, x_3\}$, the potential (10) has a form similar to that in R^3:

$$A_1(\mathbf{x}) = \mu\frac{-x_2}{|\mathbf{x}|(|\mathbf{x}| + x_3)}, \qquad (11)$$
$$A_2(\mathbf{x}) = \mu\frac{x_1}{|\mathbf{x}|(|\mathbf{x}| + x_3)}, \quad A_3(\mathbf{x}) = 0.$$

The quantum-mechanical Hamiltonian of the motion of a charged or a dual charged particle in the monopole field is obtained by the substitution $\nabla_a \to \nabla_a + iA_a$ in the Laplacian operator $\Delta = \nabla_a\nabla^a$:

$$H_A = -\frac{1}{2}(\nabla^a + iA^a)(\nabla_a + iA_a). \qquad (12)$$

In order to have a more obvious analogy with the quantum-mechanical Kepler problem in the spaces of constant curvature, we will use four-dimensional notation. We have seen in the Introduction that, when $\alpha = 0$, the Hamiltonian in (1) is proportional to the $O(4)$ Casimir operator, which is $(L^2 + P^2)/2R^2$. Here, $P_a = -i(x_4\partial_a - x_a\partial_4)$ are boost operators on S^3. A natural generalization of this operator on a

sphere in the presence of the Dirac-type potential (11) is

$$N_a = x_4\pi_a - x_a p_4, \tag{13}$$

where the operators $\pi_a = -i\partial_a + A_a$ and $p_4 = -i\partial_4$ obey the commutation relations

$$[\pi_a, x_b] = -i\delta_{ab}, \quad [\pi_a, \pi_b] = i\mu\epsilon_{abc}\frac{x_c}{|\mathbf{x}|^3}, \tag{14}$$
$$[\pi_a, p_4] = 0, \quad [p_4, x_4] = -i.$$

By direct calculations, it can be verified that the Hamiltonian in (12) commutes with the generalized angular-momentum vector:

$$J_a = \epsilon_{abc}x_b\pi_c - \frac{\mu x_a}{|\mathbf{x}|}. \tag{15}$$

We now rewrite the Hamiltonian H_A in a more convenient form. It should be noted that the presence of the Dirac-type potential (11) breaks the $O(4)$ symmetry of the problem. The right-hand side of the commutator of two operators N_a contains a term proportional to the field strength; therefore, operators J_a and N_a do not form an $o(4)$ algebra:

$$[N_a, N_b] = i\epsilon_{abc}J_c + R^2F_{ab}, \tag{16}$$
$$[J_a, N_b] = i\epsilon_{abc}N_c, \quad [J_a, J_b] = i\epsilon_{abc}J_c,$$

where $F_{ab} = [\pi_a, \pi_b]$ is given in (14). Despite this, the Hamiltonian H_A can be represented in the form similar to that of the Hamiltonian of the Kepler problem (1) for $\alpha = 0$:

$$H_A = \frac{J^2 + N^2}{2R^2} - \frac{\mu^2}{2R^2}. \tag{17}$$

The spectrum of this Hamiltonian obtained from the solution of the Schrödinger equation depends on the eigenvalues of J^2. Therefore, it is clear that, apart from J_a (15), there are no additional quantities that commute with the Hamiltonian in (17). For this reason, we consider a modification of this problem.

By analogy with the flat case, we introduce a Hamiltonian with the Zwanziger-like term $\mu^2/2|\mathbf{x}|^2$:

$$H_\mu = H_A + \frac{\mu^2}{2|\mathbf{x}|^2} = \frac{J^2 + N^2}{2R^2} + \frac{\mu^2x_4^2}{2R^2|\mathbf{x}|^2}. \tag{18}$$

Using the commutational relations (14) and (16), we then obtain

$$[H_\mu, N_a] = i\mu\frac{x_4J_a}{|\mathbf{x}|^3}. \tag{19}$$

With the aid of this relation, one can check that the Hamiltonian in (18) commutes with the generalized angular-momentum operator (15) and the additional constant of motion

$$\tilde{A}_a = \frac{1}{2R}\epsilon_{abc}(J_bN_c - N_bJ_c). \tag{20}$$

Operators $\tilde{A}_a$ and J_a obey the commutation relations

$$[\tilde{A}_a, \tilde{A}_b] = -2i\left(H_\mu - \frac{J^2}{R^2} + \frac{\mu^2}{2R^2}\right)\epsilon_{abc}J_c, \tag{21}$$
$$[J_a, \tilde{A}_b] = i\epsilon_{abc}\tilde{A}_c, \quad [J_a, J_b] = i\epsilon_{abc}J_c.$$

We now add the Coulomb term to the Hamiltonian in (18) and finally obtain the Hamiltonian of the MIC–Kepler problem on a sphere:

$$H_\alpha = \frac{J^2 + N^2}{2R^2} + \frac{\mu^2x_4^2}{2R^2|\mathbf{x}|^2} - \frac{\alpha x_4}{R|\mathbf{x}|}. \tag{22}$$

We then verify that this Hamiltonian commutes with the generalized angular-momentum operator (15) and with the analog of the Runge–Lenz vector,

$$A_a = \frac{1}{2R}\epsilon_{abc}(J_bN_c - N_bJ_c) + \frac{\alpha x_a}{|\mathbf{x}|}. \tag{23}$$

These operators satisfy the commutational relations of the cubic algebra:

$$[A_a, A_b] = -2i\left(H_\alpha - \frac{J^2}{R^2} + \frac{\mu^2}{2R^2}\right)\epsilon_{abc}J_c, \tag{24}$$
$$[J_a, A_b] = i\epsilon_{abc}A_c, \quad [J_a, J_b] = i\epsilon_{abc}J_c.$$

Furthermore, the following equalities hold:

$$\mathbf{A}^2 = 2H_\alpha(\mathbf{J}^2 - \mu^2 + 1) \tag{25}$$
$$-\frac{1}{R^2}\mathbf{J}^2(\mathbf{J}^2 - \mu^2 + 2) + \alpha^2,$$
$$\mathbf{A}\cdot\mathbf{J} = \mathbf{J}\cdot\mathbf{A} = -\alpha\mu.$$

The deformed Casimir operators (see Introduction) for the algebra in (24) are

$$C_{1d} = c\mathbf{J}^2 + d\mathbf{J}^4 + \mathbf{A}^2, \quad C_{2d} = \mathbf{J}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{J} \tag{26}$$
$$\left(c = -2H_\alpha + \frac{2-\mu^2}{R^2}, \quad d = \frac{1}{R^2}\right);$$

therefore, $C_{1d} = 2H_\alpha(1-\mu^2) + \alpha^2$ and $C_{2d} = -\alpha\mu$. Thus, the representation of the cubic algebra realized by the MIC–Kepler problem on the S^3 sphere is nondegenerate.

3. THE SPECTRUM OF THE MIC–KEPLER PROBLEM ON A SPHERE

In this section, we show that, by using relations (24) and (25), one can find the spectrum of the MIC–Kepler problem on a sphere. This algebraic treatment of the MIC–Kepler problem in spaces of constant curvature is based on the approach commonly used to obtain infinitesimal operators of the unitary representations of the proper Lorentz group (see [23]). A

similar approach was applied to nonlinear algebras in [11, 12, 19, 21].

Introducing the operators $J_\pm = J_1 \pm iJ_2$ and $A_\pm = A_1 \pm iA_2$, we write the commutation relations of the cubic algebra in the form

$$[A_\pm, A_3] = \pm 2\left(H_\alpha - \frac{J^2}{R^2} + \frac{\mu^2}{2R^2}\right) J_\pm, \qquad (27)$$

$$[A_+, A_-] = -4\left(H_\alpha - \frac{J^2}{R^2} + \frac{\mu^2}{2R^2}\right) J_3,$$

$$[J_\pm, A_3] = \mp A_\pm,\ [J_\pm, A_\mp] = \pm 2A_3,$$

$$[J_3, A_\pm] = \pm A_\pm,$$

$$[J_\pm, A_\pm] = [J_3, A_3] = 0,\ [J_\pm, J_3] = \mp J_\pm,$$

$$[J_+, J_-] = 2J_3.$$

Let ψ_{Ejm} denote the common eigenfunctions of the operators H, J^2, and J_3 with the eigenvalues E, $j(j+1)$, and m, respectively. We then find from (27) that the operators $A_\pm$ and A_3 are given by

$$A_\pm \psi_{Ejm} = \pm C_j \sqrt{(j \mp m)(j \mp m - 1)}\, \psi_{E,j-1,m\pm 1}$$
$$\pm C_{j+1}\sqrt{(j \pm m + 1)(j \pm m + 2)}\, \psi_{E,j+1,m\pm 1}$$
$$- B_j \sqrt{(j \mp m)(j \pm m + 1)}\, \psi_{E,j,m\pm 1},$$
$$A_3 \psi_{Ejm} = C_j \sqrt{j^2 - m^2}\, \psi_{E,j-1,m} \qquad (28)$$
$$-C_{j+1}\sqrt{(j+1)^2 - m^2}\, \psi_{E,j+1,m} - mB_j \psi_{Ejm},$$

where B_j and C_j do not depend on m. For the operators $J_\pm$, we have the conventional relations

$$J_\pm \psi_{Ejm} = \sqrt{(j \mp m)(j \pm m + 1)}\, \psi_{E,j,m\pm 1}. \quad (29)$$

From Eqs. (27), (28), and (29), it follows that B_j and C_j satisfy the recurrence relations

$$[(j+2)B_{j+1} - jB_j]C_{j+1} = 0, \qquad (30)$$

$$(2j-1)C_j^2 - (2j+3)C_{j+1}^2 - B_j^2 \qquad (31)$$
$$= 2\left[E - \frac{j(j+1)}{R^2} + \frac{\mu^2}{2R^2}\right].$$

By analogy with representations of the Lorentz group, we denote by $j_0 \geq 0$ the lowest value of angular momentum belonging to the representation space of the algebra in (24). From Eqs. (28), we can see that this definition is equivalent to

$$C_{j_0} = 0, \quad C_{j_0+1} \neq 0. \qquad (32)$$

From Eq. (30), we can see that $j(j+1)B_j$ does not depend on j. Denoting this constant by $j_0 c$, we obtain

$$B_j = \frac{j_0 c}{j(j+1)}. \qquad (33)$$

Introducing the notation $(2j-1)(2j+1)C_j^2 = \sigma_j$, we now find from Eq. (31) that

$$\sigma_{j_0} - \sigma_j = \sum_{k=j_0}^{j-1}(\sigma_j - \sigma_{j+1}) = (j^2 - j_0^2) \qquad (34)$$
$$\times \left(2E - \frac{j^2 + j_0^2 - 1}{R^2} + \frac{\mu^2}{R^2} + \frac{c^2}{j^2}\right).$$

Since $\sigma_{j_0} = 0$, we arrive at

$$C_j^2 = -\frac{j^2 - j_0^2}{4j^2 - 1} \qquad (35)$$
$$\times \left(2E + \frac{c^2}{j^2} - \frac{j^2 + j_0^2 - 1}{R^2} + \frac{\mu^2}{R^2}\right).$$

Using Eqs. (28) and (29) and taking into account Eq. (33), we find that $(\mathbf{A}\cdot\mathbf{J})\psi_{Ejm} = (-j_0 c)\psi_{Ejm}$; therefore, Eqs. (25) yield

$$cj_0 = \alpha\mu. \qquad (36)$$

Using Eqs. (28) and taking into account Eqs. (33), (35), and (36), we obtain

$$\mathbf{A}^2\psi_{Ejm} = \left[2E(j^2 + j - j_0^2 + 1)\right. \qquad (37)$$
$$- \frac{j(j+1)(j^2 + j - j_0^2 + 2)}{R^2}$$
$$\left. + \frac{(\mu^2 - j_0^2)(j^2 + j - j_0^2 + 1)}{R^2} + c^2\right]\psi_{Ejm}.$$

By comparing Eqs. (37) and (25), we then find that $c^2 = \alpha^2$ and $j_0^2 = \mu^2$. Thus, the final expression for C_j^2 [see (35)] is

$$C_j^2 = -\frac{(j^2 - \mu^2)[2Ej^2R^2 - j^2(j^2-1) + \alpha^2R^2]}{R^2 j^2(4j^2-1)}. \qquad (38)$$

Due to the quantization condition for the magnetic charge, $\mu = \pm 0, \pm 1/2, \pm 1, ...$, one can identify the following: $j_0 = |\mu|$ and $c = \alpha\mu/|\mu|$.

From the condition requiring that conserved operators be Hermitian, it follows that the coefficients C_j must satisfy the conditions

$$C_0 = C_0^\star, \qquad (39)$$

$$C_j = -C_j^\star, \quad j \geq 1. \qquad (40)$$

It can be seen from (38) that the condition in (39) is satisfied identically. In S^3 space, the condition in (40) is satisfied only if (i) $j \geq |j_0| = |\mu|$ and (ii) if, for a fixed value of E, the quantum number j is bounded from above, that is $j \leq j_{\max}$. Denoting $j_{\max} + 1 = N$, we obtain $C_N = 0$, that is [see (38)], $[2E_N - (N^2 -$

$1)/R^2 + \alpha^2/N^2] = 0$ and $N > |\mu|$, whence it follows that the energy levels are given by

$$E_N = -\frac{\alpha^2}{2N^2} + \frac{N^2-1}{2R^2}, \qquad (41)$$
$$N = |\mu|+1, |\mu|+2, |\mu|+3, \ldots .$$

This spectrum coincides with the spectrum of the Kepler problem on the S^3 sphere at $\mu = 0$ and goes over to the spectrum of the flat MIC–Kepler problem for $R \to \infty$.

We note that the spectrum of the MIC–Kepler problem in H^3 space can be obtained by the same method. The expression for the spectrum in this space is obtained by the formal substitution $R \to i\rho$, where ρ is a real number.

The spectra obtained by means of the above algebraic consideration coincide with those derived by solving the Schrödinger equation in these spaces.

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The A_3 Calogero–Sutherland System: Constructing a Separating Kernel*

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Abstract—The method of separation of variables is applied to the A_3 Calogero–Sutherland system. Using conjectures on the form of separated functions, we obtain a system of partial differential equations for the kernel of the separating operator and investigate the analytic properties of its solution. © *2002 MAIK "Nauka/Interperiodica".*

1. THE QUANTUM CALOGERO–SUTHERLAND MODEL

Consider N particles on a circle interacting with a long-range potential with coordinates q_i, $0 \le q_i \le \pi$, $i = 1, \dots, N$. The total momentum of the system and the Hamiltonian that describe the dynamics of the particles are given by [1, 2]

$$P = \sum_{i=1}^{N} \frac{1}{i} \frac{\partial}{\partial q_i}, \tag{1}$$

$$H = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial q_i^2} + \sum_{i<j} \frac{\lambda(\lambda+1)}{\sin^2(q_i - q_j)}.$$

These operators can be included in the family of mutually commutative operators H_k, $k = 1, \dots, N$, which can be written as [3]

$$H_k = \sum_{0 \le l \le \frac{k}{2}} \sum_{\sigma \in S_N} \frac{1}{G(l, k-2l)} \times D_{l,k-2l}(\sigma(q_1), \dots, \sigma(q_N)), \tag{2}$$

where

$$D_{m,n} = \prod_{i=1}^{2m-1} \frac{-\lambda(\lambda+1)}{\sin^2(q_i - q_{i+1})} \times \frac{(-1)^n \partial^n}{\partial q_{2m+1} \partial q_{2m+2} \dots \partial q_{2m+n}} \tag{3}$$

and $G(m, n)$ is the number of permutations $\sigma \in S_N$ such that $D^{\sigma}_{m,n} = D_{m,n}$.

Hereafter, we shall assume that $\lambda > -1$ and define the space of quantum states as the complex Hilbert space of functions Ψ symmetric with respect to q_i with the scalar product

$$\langle \Psi_1, \Psi_2 \rangle = \int_0^{\pi} dq_1 \dots \int_0^{\pi} dq_N \bar{\Psi}_1(\mathbf{q}) \Psi_2(\mathbf{q}). \tag{4}$$

We define the ground state Ω as

$$\Omega(\mathbf{q}) = |\prod_{i<j} \sin(q_i - q_j)|^{\lambda+1}. \tag{5}$$

Then, the complete set of orthogonal eigenvectors for H_k is

$$H_k \Psi_{\mathbf{n}} = h_k \Psi_{\mathbf{n}}, \quad \Psi_{\mathbf{n}}(\mathbf{q}) = \Omega(\mathbf{q}) J_{\mathbf{n}}(\mathbf{q}), \tag{6}$$

where

$$h_k = 2^k \sum_{j_1 < \dots < j_k} m_{j_1} \dots m_{j_k}, \tag{7}$$

$$m_j = n_j + (\lambda+1)\left(j - \frac{N+1}{2}\right),$$

all eigenvectors are parametrized by the sequences $\mathbf{n} = \{n_1 \le \dots \le n_N\}$, and $J_{\mathbf{n}}(\mathbf{q})$ are symmetric trigonometric polynomials that coincide with Jack polynomials related to the root system A_{N-1} [4]. In the new variables $t_i = e^{2iq_i}$, Jack polynomials have the representation

$$J_{\mathbf{n}} = S_{\mathbf{n}} + \sum_{\mathbf{n} \ge \mathbf{m}} u_{\mathbf{m},\mathbf{n}} S_{\mathbf{m}}, \tag{8}$$

where $S_{\mathbf{n}}(\mathbf{t}) = \sum_{\sigma \in S_N} t^{\sigma(n_1)} \dots t^{\sigma(n_N)}$ are elementary symmetric polynomials (the sum is taken over all different permutations) and

$$\mathbf{m} \ge \mathbf{n} \Leftrightarrow \sum_{i=1}^{N} m_i = \sum_{i=1}^{N} n_i \quad \text{and} \tag{9}$$

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$$\sum_{i=1}^{r} m_i \geq \sum_{i=1}^{r} n_i \ \text{ for all } \ r \geq 1.$$

2. CONJECTURES ON A SEPARATION OF VARIABLES

The method of separation of variables developed by Sklyanin and Kuznetsov suggests the following conjectures [5, 6].

Conjecture 1. There exists a linear integral operator K that transforms any eigenfunction $\Psi_{\mathbf{n}}(\mathbf{q})$ into the factorized form

$$K : \Psi_{\mathbf{n}}(\mathbf{q}) \to \tilde{\Psi}_{\mathbf{n}}(y_1, \ldots, y_{N-1}; Q) \qquad (10)$$
$$= \epsilon^{ih_1 Q} \prod_{k=1}^{N-1} \psi_{\mathbf{n}}(y_k), \quad Q \equiv q_N.$$

Conjecture 2. The function $\psi_{\mathbf{n}}(x)$ has the form

$$\psi_{\mathbf{n}}(x) = (\sin x)^{(N-1)(\lambda+1)} \phi_{\mathbf{n}}(y), \qquad (11)$$

where $\phi_{\mathbf{n}}(y)$ is a Laurent polynomial in $t = e^{2iy}$,

$$\phi_{\mathbf{n}}(y) = \sum_{i=n_1}^{n_N} t^k c_k(\mathbf{n}; \lambda). \qquad (12)$$

Here, the coefficients $c_k(\mathbf{n}; \lambda)$ are rational functions of k, n_i, λ and can be determined from the following explicit expression for the function $\phi_{\mathbf{n}}(y)$:

$$\phi_{\mathbf{n}}(y) = t^{n_1}(1-t)^{1-N(\lambda+1)} \qquad (13)$$
$$\times_N F_{N-1}(a_1, \ldots, a_N; b_1, \ldots, b_{N-1}; t),$$

where

$$a_i = n_1 - n_{N-i+1} + 1 - (N-i+1)(\lambda+1), \quad (14)$$
$$b_i = a_i + \lambda + 1,$$

and ${}_N F_{N-1}$ is a hypergeometric function.

These conjectures were suggested and proven for $N = 2, 3$ in [6]. Hereafter, we are going to make the next step and examine the case of $N = 4$.

However, we want to clarify in more detail the structure of the linear operator K.

Conjecture 3. The function $\tilde{\Psi}_{\mathbf{n}}(y_1, \ldots, y_{N-1}; Q)$ in (10) allows the representation

$$\tilde{\Psi}_{\mathbf{n}}(y_1, \ldots, y_{N-1}; Q) \qquad (15)$$
$$= \int dx_1 \ldots \int dx_{N-1} K(\mathbf{y}; \mathbf{x})$$
$$\times \Psi(x_1 + Q, \ldots, x_{N-1} + Q; Q),$$

and $K(\mathbf{y}; \mathbf{x})$ can be chosen as

$$K(\mathbf{y}; \mathbf{x}) = \delta\left(\sum_{i=1}^{N-1} y_i - \sum_{i=1}^{N-1} x_i\right) \qquad (16)$$
$$\times \left\{ \frac{\prod_{i=1}^{N-1} \frac{\sin x_i}{\sin y_i}}{\prod_{i<j} \sin(x_i - x_j)} \right\}^{\lambda} L(\mathbf{y}; \mathbf{x}),$$

where $L(\mathbf{y}; \mathbf{x})$ depends on pairwise differences of y_i and x_j,

$$L(y_1 + t, \ldots, y_{N-1} + t; x_1 + t, \ldots, x_{N-1} + t) \quad (17)$$
$$= L(\mathbf{y}; \mathbf{x}), \quad t \in \mathcal{C}.$$

3. THE A_3 CASE

In this section, we shall follow the strategy of [6] to obtain differential equations for the kernel of the operator K.

We introduce the differential operators

$$\mathcal{D}_y^{\text{sep}} \equiv h_4 + \lambda(\lambda+1)\left[\frac{h_2}{\sin^2 y} + 2ih_1 \frac{(\lambda-1)\cot y}{\sin^2 y}\right. \qquad (18)$$
$$\left. - 3\frac{(\lambda-1)(\lambda-2)}{\sin^4 y} + 4\frac{\lambda^2 - \lambda + 1}{\sin^2 y}\right]$$
$$+ \partial_y^4 - ih_1 \partial_y^3 - \left[h_2 + 6\frac{\lambda(\lambda+1)}{\sin^2 y}\right]\partial_y^2$$
$$+ \left[ih_3 + 3ih_1 \frac{\lambda(\lambda+1)}{\sin^2 y}\right.$$
$$\left. - 8\frac{\lambda(\lambda+1)(\lambda-1)\cot y}{\sin^2 y}\right]\partial_y,$$

where h_i are the eigenvalues of H_i defined by (7). Using the form of separated functions (11)–(13) and the differential equations for the hypergeometric function ${}_N F_{N-1}$ [7], one can show that

$$\mathcal{D}_y^{\text{sep}} \psi_{\mathbf{n}}(y) = 0; \qquad (19)$$

as a result, we have

$$\mathcal{D}_{y_i}^{\text{sep}} \tilde{\Psi}_{\mathbf{n}}(y_1, y_2, y_3; Q) = 0, \quad i = 1, 2, 3. \qquad (20)$$

Substituting (15) into (20), we can replace, due to (6), the eigenvalues h_k by the operators H_k; integrating by parts, we replace H_k by their adjoint H_k^* with the aid of the formula

$$\int d\mathbf{x} K(\mathbf{y}; \mathbf{x})(H_k \Psi(\mathbf{x}; Q)) \qquad (21)$$
$$= \int d\mathbf{x} (H_k^* K)(\mathbf{y}; \mathbf{x}) \Psi(\mathbf{x}; Q).$$

However, some boundary terms can appear while integrating by parts. If we assume that λ is not an integer, then one can always choose the integration path to be a closed curve on the Riemann surface of the integrand, avoiding the problem of boundary terms.

The case of integer λ requires a different consideration since such a choice of contour will not work.

Now, using (2) and (3), one can obtain the following expressions for $\mathcal{H}_k^*$:

$$\mathcal{H}_1^* = i\partial_Q, \tag{22}$$

$$\mathcal{H}_2^* = -\partial_Q \sum_{i=1}^{3} \partial_{x_1} + \sum_{i=1}^{3} \partial_{x_i}^2 + \sum_{i<j} \partial_{x_i}\partial_{x_j} \tag{23}$$

$$- \lambda(\lambda+1)\left\{ \sum_{i<j} \frac{1}{\sin^2(x_i - x_j)} + \sum_{i=1}^{3} \frac{1}{\sin^2 x_i} \right\},$$

$$\mathcal{H}_3^* = -i\partial_Q \sum_{i<j} \partial_{x_i}\partial_{x_j} + i \sum_{i\neq j} \partial_{x_i}^2 \partial_{x_j} \tag{24}$$

$$+ 2i\partial_{x_1}\partial_{x_2}\partial_{x_3} - i\lambda(\lambda+1)\partial_Q$$

$$\times \sum_{i<j} \frac{1}{\sin^2(x_i - x_j)} + i\lambda(\lambda+1)$$

$$\times \sum_{i<j<k} \left\{ \frac{1}{\sin^2(x_i - x_j)} - \frac{1}{\sin^2 x_k} \right\} (\partial_{x_i} + \partial_{x_j}),$$

$$\mathcal{H}_4^* = \left(\partial_Q - \sum_{i=1}^{3} \partial_{x_i} \right) \partial_{x_1}\partial_{x_2}\partial_{x_3} + \lambda(\lambda+1) \tag{25}$$

$$\times \left\{ \sum_{i<j<k} \frac{1}{\sin^2(x_i - x_j)} (\partial_Q - \partial_{x_1} - \partial_{x_2} - \partial_{x_3})\partial_{x_k} \right.$$

$$\left. + \frac{1}{\sin^2 x_k} \partial_{x_i}\partial_{x_j} \right\} + \lambda^2(\lambda+1)^2$$

$$\times \sum_{i<j<k} \frac{1}{\sin^2(x_i - x_j)\sin^2 x_k}.$$

Using formulas (15), (20), and (21), we finally find that the kernel K satisfies the complicated differential equations

$$ih_1 \mathcal{D}_{y_i;\mathbf{x}}^{(1)} + \mathcal{D}_{y_i;\mathbf{x}}^{(0)} K(\mathbf{y},\mathbf{x}) = 0, \quad i = 1,2,3, \tag{26}$$

which are equivalent to

$$\mathcal{D}_{y_i;\mathbf{x}}^{(\alpha)} K(\mathbf{y},\mathbf{x}) = 0, \quad i = 1,2,3, \quad \alpha = 0,1, \tag{27}$$

where

$$\mathcal{D}_{y;\mathbf{x}}^{(1)} \equiv \partial_y^3 + \partial_y^2 \sum_{i=1}^{3} \partial_{x_i} + \partial_y \tag{28}$$

$$\times \left[\sum_{i<j} \partial_{x_i}\partial_{x_j} + \sum_{i<j} \frac{\lambda(\lambda+1)}{\sin^2(x_i - x_j)} - 3\frac{\lambda(\lambda+1)}{\sin^2 y} \right]$$

$$+ \partial_{x_1}\partial_{x_2}\partial_{x_3} + \lambda(\lambda+1)$$

$$\times \left[\sum_{i<j<k} \frac{1}{\sin^2(x_i - x_j)} \partial_{x_k} \right.$$

$$\left. - \frac{1}{\sin^2 y} \sum_{i=1}^{3} \partial_{x_i} - 2\frac{(\lambda-1)\cot y}{\sin^2 y} \right],$$

$$\mathcal{D}_{y;\mathbf{x}}^{(0)} \equiv \partial_y^4 - \partial_y^2 \left[\sum_{i<j<k} (\partial_{x_i}\partial_{x_j} + \partial_{x_k}^2) + \lambda(\lambda+1) \right. \tag{29}$$

$$\left. \times \left[\frac{6}{\sin^2 y} - \sum_{i<j<k} \left[\frac{1}{\sin^2(x_i - x_j)} + \frac{1}{\sin^2 x_k} \right] \right] \right]$$

$$- \partial_y \left[2\partial_{x_1}\partial_{x_2}\partial_{x_3} + \sum_{i\neq j} \partial_{x_i}^2 \partial_{x_j} + \lambda(\lambda+1) \right.$$

$$\times \left\{ \sum_{i<j<k} \left[\frac{1}{\sin^2(x_i - x_j)} - \frac{1}{\sin^2 x_k} \right] (\partial_{x_i} + \partial_{x_j}) \right.$$

$$\left. \left. + 8(\lambda-1)\frac{\cot y}{\sin^2 y} \right\} \right]$$

$$- \partial_{x_1}\partial_{x_2}\partial_{x_3} \sum_{i=1}^{3} \partial_{x_i} + \lambda(\lambda+1)$$

$$\times \left[-3\frac{(\lambda-1)(\lambda-2)}{\sin^4 y} + 4\frac{(\lambda^2-\lambda+1)}{\sin^2 y} \right.$$

$$+ \sum_{i<j<k} \left\{ \frac{1}{\sin^2 y} (\partial_{x_i}\partial_{x_j} + \partial_{x_k}^2) \right.$$

$$\left. \left. - \frac{1}{\sin^2(x_i - x_j)} \partial_{x_k} \sum_{l=1}^{3} \partial_{x_l} \right\} \right]$$

$$+ \lambda^2(\lambda+1)^2 \sum_{i<j<k} \left[\frac{1}{\sin(x_i - x_j)^2 \sin^2 x_k} \right.$$

$$\left. - \frac{1}{\sin^2 y} \left\{ \frac{1}{\sin(x_i - x_j)^2} + \frac{1}{\sin^2 x_k} \right\} \right].$$

Here, we would like to compare these results with the A_2 case [6], where equations similar to (27) can also be obtained. However, there is a significant difference between the A_2 and A_3 cases. A direct analysis of the A_2 analogs of (27) shows that the compatibility conditions for these equations lead to Conjecture 3 with the simple equations for $L(y_1, y_2; x_1, x_2)$:

$$(\partial_{y_\alpha} + \partial_{x_i}) L(y_1, y_2; x_1, x_2) \tag{30}$$
$$= \lambda \cot(y_\alpha - x_j) L(y_1, y_2; x_1, x_2),$$
$$\alpha = 1,2, \quad i \neq j = 1,2.$$

The analysis of the compatibility conditions for (27) is much more complicated and does not lead in general to (16) and (17). However, assuming that $L(\mathbf{y};\mathbf{x})$ in (16) should satisfy (17) and substituting this into

$\mathcal{D}^{(1)}_{y_i;\mathbf{x}}K(\mathbf{y},\mathbf{x})=0$, we obtain very simple equations for the $L(\mathbf{y};\mathbf{x})$:

$$\begin{gathered}\{\sin(x_k-y_\alpha)\sin(x_i-x_j)(\partial_{y_\alpha}+\partial_{x_i})(\partial_{y_\alpha}+\partial_{x_j})\\ +\lambda[\sin(x_i+x_k-x_j-y_\alpha)(\partial_{y_\alpha}+\partial_{x_i})\\ -\sin(x_j+x_k-x_i-y_\alpha)(\partial_{y_\alpha}+\partial_{x_j})]\}\,L(\mathbf{y};\mathbf{x})=0,\\ \alpha=1,2,3;\ i\neq j\neq k=1,2,3.\end{gathered}\tag{31}$$

Then after tedious calculations, one can show that both equations in (27) are satisfied. It would be interesting to understand the following: Are there any other simple compatibility conditions for Eqs. (27) like (31)? For the A_2 case, the answer is negative, and (30) is the only possible solution.

4. THE SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

The function $L(y_1,y_2,y_3;x_1,x_2,x_3)$ depends only on four independent variables [because of the δ function in (16) and (17)] which can be chosen as

$$u_i=x_i-y_1,\quad v=y_2-y_1,\quad i=1,2,3.\tag{32}$$

Let us introduce the variables $t_j=e^{2iu_j}$ and $s=e^{2iv}$; using these, we denote

$$L(t_1,t_2,t_3;s)\equiv L(y_1,y_2,y_3;x_1,x_2,x_3).\tag{33}$$

Later, we will show that, for general values of λ, there exists the only solution of (31) that is symmetric with respect to y_i and x_j. In terms of the variables t_i and s, this symmetry can be written as

$$L(t_1,t_2,t_3;s)=L\left(\frac{t_1}{s},\frac{t_2}{s},\frac{t_3}{s};\frac{1}{s}\right)\tag{34}$$

$$=L\left(t_1,t_2,t_3;\frac{t_1t_2t_3}{s}\right).$$

We introduce the differential operators

$$\begin{gathered}\mathcal{D}_{ij}(t_1,t_2,t_3;s)=(t_i-t_j)(s-t_it_j)t_it_j\partial_{t_i}\partial_{t_j}\\ +\lambda\left[t_j^2(t_i^2-s)\partial_{t_j}-t_i^2(t_j^2-s)\partial_{t_i}\right],\quad i\neq j,\end{gathered}\tag{35}$$

$$\begin{gathered}\tilde{\mathcal{D}}_{ij}(t_1,t_2,t_3;s)\\ =(t_i-t_j)(1-t_k)(s\partial_s+t_i\partial_{t_i}+t_k\partial_{t_k})\\ \times(s\partial_s+t_j\partial_{t_j}+t_k\partial_{t_k})\\ +\lambda((t_it_k-t_j)(s\partial_s+t_j\partial_{t_j}+t_k\partial_{t_k})\\ -(t_jt_k-t_i)(s\partial_s+t_i\partial_{t_i}+t_k\partial_{t_k})),\end{gathered}\tag{36}$$

where $\{i,j,k\}$ is a permutation of $\{1,2,3\}$. Then, Eqs. (31) are equivalent to

$$\begin{cases}\mathcal{D}_{ij}(t_1,t_2,t_3;s)L(t_1,t_2,t_3;s)=0,\\ \tilde{\mathcal{D}}_{ij}(t_1,t_2,t_3;s)L(t_1,t_2,t_3;s)=0,\end{cases}\quad i\neq j=1,2,3.\tag{37}$$

Theorem 1. A general solution of (37) depends on six constants. There are five nontrivial independent integrals of (37) and one obvious integral, which is a constant. For any solution $L(t_1,t_2,t_3;s)$ of (37), the following relations are also valid:

$$\begin{aligned}\partial_{t_i}\partial_s L=\Bigg\{&-\frac{s}{2t_i}\partial_s^2+\left[\lambda\frac{s(s^2-t_jt_k+t_i(t_j+t_k-2s))}{t_i(s-t_i)(s-t_j)(s-t_k)}-\frac{\lambda+1}{2t_i}\right]\partial_s\\ &+\frac{\lambda(t_i-t_k)t_j(t_j-1)(s^2-t_it_jt_k)}{2t_i(s-t_i)(s-t_k)(s-t_it_j)(s-t_jt_k)}\partial_{t_j}\\ &+\frac{\lambda(t_i-t_j)t_k(t_k-1)(s^2-t_it_jt_k)}{2t_i(s-t_i)(s-t_j)(s-t_it_k)(s-t_jt_k)}\partial_{t_k}\\ &-\lambda\frac{(s^2-t_it_jt_k)[2(s^2+t_it_jt_k)-s(1+t_i)(t_j+t_k)]}{2s(s-t_j)(s-t_k)(s-t_it_j)(s-t_it_k)}\partial_{t_i}\Bigg\}L\end{aligned}\tag{38}$$

and

$$\begin{aligned}\partial_{t_i}^2L=\Bigg\{&\left[\lambda\frac{2t_jt_k-s(t_j+t_k)}{t_i(s-t_j)(s-t_k)}-\lambda\frac{t_i^2(1+t_i)+(3-t_i)t_jt_k-2t_i(t_j+t_k)}{t_i(1-t_i)(t_i-t_j)(t_i-t_k)}-\frac{1}{t_i}\right]\partial_{t_i}\\ &+\frac{s^2}{t_i^2}\partial_s^2+\frac{\lambda t_j(1-t_j)(t_it_k-s)(st_i-t_jt_k)}{t_i^2(1-t_i)(t_i-t_j)(s-t_k)(s-t_jt_k)}\partial_{t_j}\\ &+\frac{\lambda t_k(1-t_k)(t_it_j-s)(st_i-t_jt_k)}{t_i^2(1-t_i)(t_i-t_k)(s-t_j)(s-t_jt_k)}\partial_{t_k}\end{aligned}\tag{39}$$

$$+ s\left[\lambda\frac{s^2(t_i-3)+(1-3t_i)t_jt_k+s(1+t_i)(t_j+t_k)}{t_i^2(1-t_i)(s-t_j)(s-t_k)}+\frac{1}{t_i^2}\right]\partial_s\Bigg\}L.$$

Proof: Differentiating (37) with respect to t_i and s and comparing the mixed derivatives, one can show that Eqs. (38) and (39) are satisfied. Then, it follows from (37)–(39) that any mixed derivative of a high order of L can be expressed in terms of $\partial_{t_1}L$, $\partial_{t_2}L$, $\partial_{t_3}L$, ∂_sL, and ∂_s^2L. Hence, any solution K of (37) satisfies a sixth-order differential equation in each variable.

The next theorem gives an explicit solution to (37).

Theorem 2. A general solution to the set of Eqs. (37) can be written in the form

$$L(t_1,t_2,t_3;s)=\oint\limits_{\mathcal{C}}\frac{du}{u}g(t_1,t_2,t_3;s|u), \tag{40}$$

$$g(t_1,t_2,t_3,s|u)=\frac{(1-u)^\lambda(1-su)^\lambda}{s^\lambda u^{2\lambda}(1-su^2)^{2\lambda}} \tag{41}$$
$$\times\prod_{i=1}^{3}(1-ut_i)^\lambda\left(1-u\frac{s}{t_i}\right)^\lambda,$$

and the contour of integration $\mathcal{C}$ is closed on the Riemann surface of the integrand.

Proof: A proof is straightforward and based on two identities. The first one

$$\mathcal{D}_{ij}(t_1,t_2,t_3;s)\left\{g(t_1,t_2,t_3;s|u)\right\}=0, \tag{42}$$
$$i\neq j=1,2,3,$$

is simple and shows that the integrand itself satisfies the first set of equations in (37).

The second identity is much less trivial and requires some calculations to check

$$\tilde{\mathcal{D}}_{ij}(t_1,t_2,t_3;s)\left\{g(t_1,t_2,t_3;s|u)\right\} \tag{43}$$
$$=\lambda u\frac{\partial}{\partial_u}\left\{\frac{(t_i-t_j)u(1-u)(t_k-su)(1-st_ku^2)}{(1-su)(1-su^2)(1-t_ku)}\right.$$
$$\left.\times\, g(t_1,t_2,t_3;s|u)\right\},\quad i\neq j\neq k=1,2,3.$$

It shows that the expression on the right-hand side of (43) is the total derivative of the function with the same singularities as the integrand in (40).

To proceed further, we give the formula

$$\oint\limits_{[0;1]}du\,u^{a-1}(u-1)^{b-1} \tag{44}$$
$$=\frac{(2\pi i)^2e^{i\pi a+2\pi ib}}{\Gamma(1-a)\Gamma(1-b)\Gamma(a+b)},$$

which is valid for any complex a and b; we imply that u^a has to be understood as $\exp(a\log(u))$, with $\log(u)$ having the cut $(-\infty,0]$, and the notation $[x;y]$ stands for the double Pochhammer loop [7] slung around two points x and y (see figure). This type of contour is closed on the Riemann surface of the integrand.

Let us now denote

$$I(t_1,t_2,t_3;s)=\oint\limits_{[0;1]}\frac{du}{u}g(t_1,t_2,t_3;s|u). \tag{45}$$

We shall assume that all the remaining singularities of the function $g(t_1,t_2,t_3;s|u)$ in u are outside the double Pochhammer loop [0; 1]. This implies that, if any one of these singularities encircles another singularity outside the double loop, then the integral in (45) remains unchanged. As a result, the integral in (45) is a symmetric function of t_1, t_2, and t_3.

Let us introduce two maps τ and σ, $\tau^2=1$ and $\sigma^2=1$, acting on the variables $\{t_1,t_2,t_3;s\}$ as

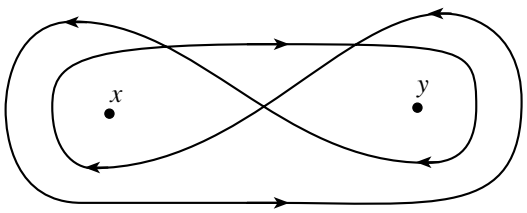
$$\tau\{t_1,t_2,t_3;s\}=\left\{\frac{t_1}{s},\frac{t_2}{s},\frac{t_3}{s};\frac{1}{s}\right\}, \tag{46}$$
$$\sigma\left\{t_1,t_2,t_3;s\right\}=\left\{t_1,t_2,t_3;\frac{t_1t_2t_3}{s}\right\}$$

and denote

$$I^\rho(t_1,t_2,t_3;s)\equiv I(\rho\{t_1,t_2,t_3;s\}),$$

where ρ is any composition of τ and σ. It is easy to see that this gives six integrals of (37): I, I^τ, I^σ, $I^{\tau\sigma}$, $I^{\sigma\tau}$, and $I^{\tau\sigma\tau}$, where

$$I^{\sigma\tau}=I\left(\frac{t_1}{s},\frac{t_2}{s},\frac{t_3}{s};\frac{t_1t_2t_3}{s^2}\right), \tag{47}$$
$$I^{\tau\sigma}=I\left(\frac{s}{t_1t_2},\frac{s}{t_1t_3},\frac{s}{t_2t_3};\frac{s^2}{t_1t_2t_3}\right),$$
$$I^{\tau\sigma\tau}=I\left(\frac{s}{t_1t_2},\frac{s}{t_1t_3},\frac{s}{t_2t_3};\frac{s^2}{t_1t_2t_3}\right).$$



A double Pochhammer loop.

It can be proven that precisely three of them are linearly independent. They give three independent integrals of the set of Eqs. (37), which are symmetric in t_1, t_2, and t_3. Also, let us introduce

$$J_i(t_1,t_2,t_3;s)=\oint\limits_{[0;t_i]}\frac{du}{u}g(t_1,t_2,t_3;s|u). \qquad (48)$$

The functions $J_i(t_1,t_2,t_3;s)$ also solve (37) due to Theorem 2. However, they produce "nonsymmetric" integrals of (37), which are symmetric only with respect to t_j and t_k (which are outside the double loop $[0;t_i]$). We can choose, say, J_1; J_2; and I, I^τ, and I^σ to be five linearly independent integrals (λ is not an integer) of the set of Eqs. (37). Due to Theorem 1, any other solution of (37) is a linear combination of these five integrals. It follows that the set of Eqs. (37) admits the only solution satisfying (34) and symmetric with respect to t_1, t_2, and t_3. Assuming that $0<\arg(s)<\arg(t_1)<\arg(t_2)<\arg(t_3)<\arg\left(\frac{t_1t_2t_3}{s}\right)<\pi$, one can then show that

$$\begin{gathered} I\sin 3\pi\lambda+I^\tau\sin\pi\lambda \qquad (49)\\ -\sin 2\pi\lambda(e^{i\pi\lambda}I^\sigma+e^{-i\pi\lambda}I^{\tau\sigma\tau})=0,\\ I-I^\tau-2\cos\pi\lambda\, e^{3i\pi\lambda}(I^\sigma-I^{\sigma\tau})=0,\\ I-I^{\tau\sigma}+e^{2i\pi\lambda}(I^\tau-I^\sigma)=0. \end{gathered}$$

If we take $L(t_1,t_2,t_3;s)$ to be any of these integrals of (37), then, by construction (there is no boundary terms coming from integration by parts), the integral in (15) will satisfy

$$\mathcal{D}^{\rm sep}_{y_i}\{\text{r.h.s. of (15)}\}=0,\quad i=1,2,3. \qquad (50)$$

To prove Conjecture 3, we have to select a special polynomial solution by choosing the contour of integration. We will publish details of this somewhere else.

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Five-Dimensional $SU(2)$-Monopole: Continuous Spectrum*

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Abstract—Hyperspherical and parabolic wave functions are calculated for the five-dimensional "charge–$SU(2)$ monopole" system in the continuous spectrum. It is shown that the coefficients of parabolic-hyperspherical and hyperspherical-parabolic transformations are proportional to the generalized hypergeometric function ${}_3F_2\{...|1\}$. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

The five-dimensional "charge–dyon" system with the Yang $SU(2)$ monopole [1] is described by the equation [2]

$$\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x_j}-\hbar A_j^a\hat{T}_a\right)^2\psi \qquad (1)$$
$$+\frac{\hbar^2}{2mr^2}\hat{T}^2\psi-\frac{e^2}{r}\psi=\epsilon\psi,$$

where $j=0,1,2,3,4$ and $a=1,2,3$; $\hat{T}_a$ are the generators of the group $SU(2)$,

$$[\hat{T}_a,\hat{T}_b]=i\epsilon_{abc}\hat{T}_c;$$

and a triplet of five-dimensional vectors $\mathbf{A}^a$ has the form

$$\mathbf{A}^1=\frac{1}{r(r+x_0)}(0,-x_4,-x_3,x_2,x_1),$$
$$\mathbf{A}^2=\frac{1}{r(r+x_0)}(0,x_3,-x_4,-x_1,x_2),$$
$$\mathbf{A}^3=\frac{1}{r(r+x_0)}(0,x_2,-x_1,x_4,-x_3).$$

Each term of A_j^a (for fixed a) coincides with the vector potential of the five-dimensional Dirac monopole [1] with the unit charge and the singularity axis directed along the nonpositive x_0 axis. The vectors A_j^a are orthogonal to one another,

$$A_j^aA_j^b=\frac{1}{r^2}\frac{r-x_0}{r+x_0}\delta_{ab},$$

and also to the vector $\mathbf{x}=(x_0,x_1,x_2,x_3,x_4)$.

This article has the following structure. Sections 2 and 3 present the solutions of Eq. (1) in five-dimensional hyperspherical and parabolic coordinates. In Sections 4 and 5, the method developed in [3] is used to calculate the interbasis coefficients for parabolic and hyperspherical wave functions.

2. HYPERSPHERICAL BASIS

We define the hyperspherical coordinates $r\in[0,\infty)$, $\theta\in[0,\pi]$, $\alpha\in[0,2\pi)$, $\beta\in[0,\pi]$, and $\gamma\in[0,4\pi)$ in the space $\mathbb{R}^5$ as

$$x_0=r\cos\theta,\quad x_2+ix_1=r\sin\theta\sin\frac{\beta}{2}e^{i\frac{\alpha-\gamma}{2}},\quad (2)$$
$$x_4+ix_3=r\sin\theta\cos\frac{\beta}{2}e^{i\frac{\alpha+\gamma}{2}}.$$

The differential elements of length and volume and the Laplace operator in terms of the coordinates given by (2) can be written as

$$dl^2=dr^2+r^2d\theta^2$$
$$+\frac{r^2}{4}\sin^2\theta(d\alpha^2+d\beta^2+d\gamma^2+2\cos\theta d\alpha d\gamma),$$
$$dV=\frac{r^4}{8}\sin^3\theta\sin\beta dr d\theta d\alpha d\beta d\gamma,$$
$$\Delta=\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}\right)$$
$$+\frac{1}{r^2\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial}{\partial\theta}\right)-\frac{4\hat{L}^2}{r^2\sin^2\theta},$$

where

$$\hat{L}_1=i\left(\cos\alpha\cot\beta\frac{\partial}{\partial\alpha}+\sin\alpha\frac{\partial}{\partial\beta}-\frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right),$$
$$\hat{L}_2=-i\left(\sin\alpha\cot\beta\frac{\partial}{\partial\alpha}-\cos\alpha\frac{\partial}{\partial\beta}-\frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right),$$

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$$\hat{L}_3 = i\frac{\partial}{\partial\alpha},$$

and

$$\hat{L}^2 = -\left[\frac{\partial^2}{\partial\beta^2} + \cot\beta\frac{\partial}{\partial\beta} + \frac{1}{\sin^2\beta}\left(\frac{\partial^2}{\partial\alpha^2} - 2\cos\beta\frac{\partial^2}{\partial\alpha\partial\gamma} + \frac{\partial^2}{\partial\gamma^2}\right)\right].$$

Using the identity

$$iA_j^a\frac{\partial}{\partial x_j} = \frac{2}{r(r+x_0)}\hat{L}_a,$$

$$\hat{L}_1 = \frac{i}{2}\left(-x_4\frac{\partial}{\partial x_1} - x_3\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4}\right),$$

$$\hat{L}_2 = \frac{i}{2}\left(x_3\frac{\partial}{\partial x_1} - x_4\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial x_4}\right),$$

$$\hat{L}_3 = \frac{i}{2}\left(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4}\right),$$

we can transform Eq. (1) into

$$\left(\Delta_{r\theta} - \frac{\hat{L}^2}{r^2\sin^2(\theta/2)} - \frac{\hat{J}^2}{r^2\cos^2(\theta/2)}\right)\psi + \frac{2m}{\hbar^2}\left(\epsilon + \frac{e^2}{r}\right)\psi = 0, \quad (3)$$

where

$$\Delta_{r\theta} = \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial}{\partial\theta}\right)$$

and $\hat{J}_a = \hat{L}_a + \hat{T}_a$. Note that

$$[\hat{L}_a, \hat{L}_b] = i\epsilon_{abc}\hat{L}_c, \quad [\hat{J}_a, \hat{J}_b] = i\epsilon_{abc}\hat{J}_c.$$

The solution to Eq. (3) can be chosen as the eigenfunction of the operators $\hat{L}^2$, $\hat{T}^2$, and $\hat{J}^2$ with the eigenvalues $L(L+1)$, $T(T+1)$, and $J(J+1)$ [4]:

$$\psi^{\rm hsp} = R_{k\lambda}(r)Z_{\lambda LJ}(\theta) \times \mathcal{D}^{JM}_{LTm't'}(\alpha,\beta,\gamma;\alpha_T,\beta_T,\gamma_T), \quad (4)$$

$$\mathcal{D}^{JM}_{LTm't'}(\alpha,\beta,\gamma;\alpha_T,\beta_T,\gamma_T) = \sqrt{\frac{(2L+1)(2T+1)}{4\pi^4}} \times \sum_{M=m+t} C^{JM}_{L,m;T,t}D^L_{mm'}(\alpha,\beta,\gamma)D^T_{tt'}(\alpha_T,\beta_T,\gamma_T).$$

Here, $C^{JM}_{L,m;T,t}$ are Clebsch–Gordan coefficients; $D^L_{mm'}$ and $D^T_{tt'}$ are Wigner functions; and α_T, β_T, and γ_T are the $SU(2)$-group space coordinates.

The function $Z_{\lambda LJ}(\theta)$ is given by [4]

$$Z_{\lambda LJ}(\theta) = N^\lambda_{JLT}(1-\cos\theta)^L(1+\cos\theta)^J \times P^{(2L+1,2J+1)}_{\lambda-L-J}(\cos\theta),$$

where $P^{(a,b)}_n(x)$ are Jacobi polynomials. The normalization constant N^λ_{JLT} is determined from the condition

$$\int_0^\pi \sin^3\theta Z_{\lambda'LJ}(\theta)Z_{\lambda LJ}(\theta)d\theta = \delta_{\lambda'\lambda} \quad (5)$$

and is equal to

$$N^\lambda_{LJT} = \left[\frac{(2\lambda+3)(\lambda-J-L)!\Gamma(\lambda+J+L+3)}{2^{2J+2L+3}\Gamma(\lambda+J-L+2)\Gamma(\lambda-J+L+2)}\right]^{1/2}.$$

The quantum numbers run through the values $|L-T| \le J \le L+T$ and $\lambda = L+J, L+J+1, ..., \infty$.

The radial wave function for the continuous spectrum has the form

$$R_{k\lambda}(r) = C_{k\lambda}\frac{e^{-ikr}}{(2\lambda+3)!}(2ikr)^\lambda \times F\left(\lambda+2+\frac{i}{kr_0}; 2\lambda+4; 2ikr\right),$$

where $k = \sqrt{2m\epsilon}/\hbar$ and $r_0 = \hbar^2/me^2$ is the Bohr radius.

The asymptotic expansion of the confluent hypergeometric function [5]

$$F(a;c;z) = \frac{\Gamma(c)}{\Gamma(c-a)}(-z)^{-a}G(a;a-c+1;-z) + \frac{\Gamma(c)}{\Gamma(a)}e^z z^{a-c}G(c-a;1-a;z), \quad (6)$$

where

$$G(a;c;z) = 1 + \frac{ac}{1!z} + \frac{a(a+1)c(c+1)}{2!z^2} + \ldots,$$

makes it possible to obtain the following expression for $R_{k\lambda}(r)$:

$$R_{k\lambda}(r) = C_{k\lambda}\frac{(-i)^\lambda}{2k^2r^2}e^{-\pi/2kr_0} \times {\rm Re}\left\{\frac{\exp\left[-i\left(kr - \frac{\pi}{2}(\lambda+2) + \frac{\ln 2kr}{kr_0}\right)\right]}{\Gamma\left(\lambda+2-\frac{i}{kr_0}\right)}\right.$$

$$\times G\left(\lambda+2-\frac{i}{kr_0};\frac{i}{kr_0}-\lambda-1;-2ikr\right)\Bigg\}.$$

The normalization constant $C_{k\lambda}$ is

$$C_{k\lambda}=(-i)^{\lambda}\cdot 4k^2e^{\pi/2kr_0}\left|\Gamma\left(\lambda+2-\frac{i}{kr_0}\right)\right|$$

if the normalization condition for the radial wave function is

$$\int_0^\infty r^4R_{k'\lambda}(r)R_{k\lambda}(r)dr=2\pi\delta\left(k-k'\right).$$

3. PARABOLIC BASIS

We define the parabolic coordinates in $\mathbb{R}^5$ as

$$x_0=\frac{1}{2}\left(\mu-\nu\right),\quad x_2+ix_1=\sqrt{\mu\nu}\sin\frac{\beta}{2}e^{i\frac{\alpha-\gamma}{2}},$$
$$x_4+ix_3=\sqrt{\mu\nu}\cos\frac{\beta}{2}e^{i\frac{\alpha+\gamma}{2}},$$

where $\mu,\nu\in[0,\infty)$. The differential elements of length and volume and Laplace operator in terms of these coordinates can be written as

$$dl^2=\frac{\mu+\nu}{4}\left(\frac{d\mu^2}{\mu}+\frac{d\nu^2}{\nu}\right)$$
$$+\frac{\mu\nu}{4}\left(d\beta^2+d\alpha^2+2\cos\beta d\alpha d\gamma+d\gamma^2\right),$$
$$dV=\frac{\mu\nu}{32}(\mu+\nu)\sin\beta d\mu d\nu d\beta d\alpha d\gamma,$$
$$\Delta=\frac{4}{\mu+\nu}\left[\frac{1}{\mu}\frac{\partial}{\partial\mu}\left(\mu^2\frac{\partial}{\partial\mu}\right)\right.$$
$$\left.+\frac{1}{\nu}\frac{\partial}{\partial\nu}\left(\nu^2\frac{\partial}{\partial\nu}\right)\right]-\frac{4}{\mu\nu}\hat{L}^2,$$

whence, instead of Eq. (1), we obtain

$$\left[\Delta_{\mu\nu}-\frac{4\widehat{J}^2}{\mu(\mu+\nu)}-\frac{4\widehat{L}^2}{\nu(\mu+\nu)}\right]\psi^{\rm par}\qquad(7)$$
$$+\frac{2m}{\hbar^2}\left(\epsilon+\frac{2e^2}{\mu+\nu}\right)\psi^{\rm par}=0,$$

where

$$\Delta_{\mu\nu}=\frac{4}{\mu+\nu}\left[\frac{1}{\mu}\frac{\partial}{\partial\mu}\left(\mu^2\frac{\partial}{\partial\mu}\right)+\frac{1}{\nu}\frac{\partial}{\partial\nu}\left(\nu^2\frac{\partial}{\partial\nu}\right)\right].$$

After the substitution

$$\psi^{\rm par}=\Phi(\mu,\nu)\mathrm{D}^{JM}_{LTm't'}(\alpha,\beta,\gamma;\alpha_T,\beta_T,\gamma_T)$$
$$=f_1(\mu)f_2(\nu)\mathcal{D}^{JM}_{LTm't'}(\alpha,\beta,\gamma;\alpha_T,\beta_T,\gamma_T),$$

the variables in Eq. (7) are separated, and we arrive at the set of differential equations

$$\frac{1}{\mu}\frac{d}{d\mu}\left(\mu^2\frac{df_1}{d\mu}\right)+\left[\frac{m\epsilon}{2\hbar^2}\mu-\frac{J(J+1)}{\mu}\right.$$
$$\left.+\frac{\hbar}{2\sqrt{m}}\Omega+\frac{me^2}{2\hbar^2}\right]f_1=0,$$
$$\frac{1}{\nu}\frac{d}{d\nu}\left(\nu^2\frac{df_2}{d\nu}\right)+\left[\frac{m\epsilon}{2\hbar^2}\nu-\frac{L(L+1)}{\nu}\right.$$
$$\left.-\frac{\hbar}{2\sqrt{m}}\Omega+\frac{me^2}{2\hbar^2}\right]f_2=0,$$

where Ω is the separation constant. The function $\Phi(\mu,\nu)$ normalized by the condition

$$\frac{1}{4}\int\Phi^*_{k'\Omega'JL}(\mu,\nu)\Phi_{k\Omega JL}(\mu,\nu)$$
$$\times\mu\nu(\mu+\nu)d\mu d\nu=2\pi\delta\left(k-k'\right)\delta\left(\Omega-\Omega'\right)$$

leads to the basis

$$\psi^{\rm par}=C^{JL}_{k\Omega}f^J_{k\Omega}(\mu)f^L_{k-\Omega}(\nu)\qquad(8)$$
$$\times\mathcal{D}^{JM}_{LTm't'}(\alpha,\beta,\gamma;\alpha_T,\beta_T,\gamma_T),$$

where

$$f^q_{k\Omega}(x)=\frac{(ikx)^q}{(2q+1)!}e^{-ikx/2}$$
$$\times F\left(q+1+\frac{i}{2kr_0}+\frac{i\hbar}{2k\sqrt{m}}\Omega;2q+2;ikx\right),$$

$$C^{JL}_{k\Omega}=(-i)^{J+L}\sqrt{\frac{\hbar k^3}{2\pi\sqrt{m}}}\exp\left(\frac{\pi}{2kr_0}\right)$$
$$\times\left|\Gamma\left(J+1-\frac{i}{2kr_0}-\frac{i\hbar}{2k\sqrt{m}}\Omega\right)\right.$$
$$\left.\times\Gamma\left(L+1-\frac{i}{2kr_0}+\frac{i\hbar}{2k\sqrt{m}}\Omega\right)\right|.$$

We calculate the constant $C^{JL}_{k\Omega}$ using the asymptotic expansion (6).

4. INTERBASIS EXPANSION

According to first principles, any parabolic wave function (8) corresponding to a given value of energy ϵ can be expanded in terms of the hyperspherical wave function (4) associated with the eigenvalue ϵ. Thus, we have

$$\psi^{\rm par}=\sum_{\lambda=T}^{\infty}W^{\lambda}_{k\Omega JL}\psi^{\rm hsp},\qquad(9)$$

where it is implied that the wave functions on the left- and right-hand sides are written in hyperspherical

coordinates $(r, \theta, \alpha, \beta, \gamma)$ owing to $\mu = r(1+\cos\theta)$ and $\nu = r(1-\cos\theta)$. Using the representation

$$F(a;c;z) = \sum_{s=0}^{\infty} \frac{(a)_s z^s}{s!\,(c)_s},$$

where

$$(a)_s = \frac{\Gamma(a+s)}{\Gamma(a)},$$

for the parabolic wave functions and the orthonormality relation (5), we find

$$W^{\lambda}_{k\Omega JL} F\left(\lambda + 2 + \frac{i}{kr_0}; 2\lambda + 4; 2ikr\right)$$
$$= \frac{(2\lambda+3)!}{2^{\lambda}(2J+1)!(2L+1)!}\frac{C^{JL}_{k\Omega}}{C_{k\lambda}}$$
$$\times \sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{(u)_s (v)_t}{(2J+2)_s (2L+2)_t}$$
$$\times \frac{i^{s+t+J+L-\lambda}}{s!t!}(kr)^{J+L-\lambda+s+t} Q^{st}_{\lambda JL},$$

where

$$u = J + 1 + \frac{i}{2kr_0} + \frac{i\hbar}{2k\sqrt{m}}\Omega,$$
$$v = L + 1 + \frac{i}{2kr_0} - \frac{i\hbar}{2k\sqrt{m}}\Omega,$$

and

$$Q^{st}_{\lambda JL} = \int_0^{\pi} \sin^3\theta\,(1+\cos\theta)^{J+s}$$
$$\times (1-\cos\theta)^{L+t} Z_{\lambda LJ}(\theta)\,d\theta.$$

Now, with the aid of the Rodrigues formula for the Jacobi polynomial [6],

$$P_n^{(a,b)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-a}(1+x)^{-b}\frac{d^n}{dx^n}$$
$$\times \left[(1-x)^{a+n}(1+x)^{b+n}\right],$$

and integration by parts, we verify that the integral $Q^{st}_{\lambda JL}$ is nonzero only for $s+t+J+L-\lambda \le 0$; therefore, all the terms of the series contain r raised to a nonnegative power. In the limit $r \to 0$, we therefore obtain

$$W^{\lambda}_{k\Omega JL} = \frac{(2\lambda+3)!}{2^{\lambda}(2J+1)!(2L+1)!}\frac{C^{JL}_{k\Omega}}{C_{k\lambda}} \qquad (10)$$
$$\times \sum_{s=0}^{\lambda-J-L} \frac{(u)_s (v)_{\lambda-J-L-s}}{(2J+2)_s (2L+2)_t}\frac{Q^{s,\lambda-J-L-s}_{\lambda JL}}{s!(\lambda-J-L-s)!}.$$

The integral $Q^{st}_{\lambda JL}$ at $t = \lambda - J - L - s$ reduces to the closed expression

$$Q^{s,\lambda-J-L-s}_{\lambda JL} = (-1)^{\lambda-J-L+s}\frac{2^{\lambda+2}}{(2\lambda+3)!}$$
$$\times \left[\frac{2\lambda+3}{2}(\lambda-J-L)!(\lambda+J+L+2)!\right.$$
$$\left.\times\,(\lambda+J-L+1)!(\lambda-J+L+1)!\right]^{1/2}.$$

The last expression, formula (10), and the auxiliary equalities

$$(v)_{\lambda-J-L-s} = (-1)^s \frac{(v)_{\lambda-J-L}}{(1-\lambda+J+L-v)_s},$$
$$(2L+2)_{\lambda-J-L-s}$$
$$= (-1)^s \frac{(\lambda-J+L+1)!}{(2L+1)!(-\lambda+J-L-1)_s},$$
$$(\lambda-J-L-s)! = (-1)^s \frac{(\lambda-J-L)!}{(-\lambda+J+L)_s}$$

lead to

$$W^{\lambda}_{k\Omega JL} = \frac{(-1)^{\lambda-J-L}}{(2J+1)!}\frac{C^{JL}_{k\Omega}}{C_{k\lambda}}\left[8(2\lambda+3)\frac{(\lambda+J+L+2)!(\lambda+J-L+1)!}{(\lambda-J-L)!(\lambda-J+L+1)!}\right]^{1/2}$$
$$\times (v)_{\lambda-J-L}\,{}_3F_2\left\{\begin{matrix} -\lambda+J+L, -\lambda+J-L-1, u \\ 2J+2, 1-\lambda+J-L-v\end{matrix}\,\middle|\, 1\right\}.$$

Further, by using the formula [6]

$${}_3F_2\left\{\begin{matrix} s, s', -N \\ t', 1-N-t\end{matrix}\,\middle|\, 1\right\} = \frac{(t+s)_N}{(t)_N}\,{}_3F_2\left\{\begin{matrix} s, t'-s', -N \\ t', t+s\end{matrix}\,\middle|\, 1\right\}, \qquad (11)$$

we can recast the last relation into the form

$$W^{\lambda}_{k\Omega JL} = \frac{(-1)^{\lambda-J-L}}{(2J+1)!}\frac{C^{JL}_{k\Omega}}{C_{k\lambda}}\left[8(2\lambda+3)\frac{(\lambda+J+L+2)!(\lambda+J-L+1)!}{(\lambda-J-L)!(\lambda-J+L+1)!}\right]^{1/2} \qquad (12)$$

$$\times \frac{\Gamma\left(\lambda+2+\frac{i}{kr_0}\right)}{\Gamma\left(J+L+2+\frac{i}{kr_0}\right)} {}_3F_2\left\{ \begin{array}{l} -\lambda+J+L, \lambda+J+L+3, J+1+\frac{i}{2kr_0}+\frac{i\hbar}{2k\sqrt{m}}\Omega \\ 2J+2, J+L+2+\frac{i}{kr_0} \end{array} \middle| 1 \right\}.$$

Let us note that, up to now, we have not used an explicit form of the normalization constants $C_{k\lambda}$ and $C^{JL}_{k\Omega}$. In this sense, formula (12) is correct for an arbitrary method for normalizing the wave function. In our case, we have

$$W^{\lambda}_{k\Omega JL} = \frac{(-i)^{\lambda-J-L}}{(2J+1)!}\left[\frac{(2\lambda+3)(\lambda+J+L+2)!(\lambda+J-L+1)!}{4(\lambda-J-L)!(\lambda-J+L+1)!}\right]^{1/2} \tag{13}$$
$$\times e^{-\delta_\lambda}\sqrt{\frac{\hbar}{\pi k\sqrt{m}}}\frac{\left|\Gamma\left(J+1-\frac{i}{2kr_0}-\frac{i\hbar}{2k\sqrt{m}}\Omega\right)\Gamma\left(L+1-\frac{i}{2kr_0}+\frac{i\hbar}{2k\sqrt{m}}\Omega\right)\right|}{\Gamma\left(J+L+2+\frac{i}{kr_0}\right)}$$
$$\times {}_3F_2\left\{ \begin{array}{l} -\lambda+J+L, \lambda+J+L+3, J+1+\frac{i}{2kr_0}+\frac{i\hbar}{2k\sqrt{m}}\Omega \\ 2J+2, J+L+2+\frac{i}{kr_0} \end{array} \middle| 1 \right\},$$

where

$$\delta_\lambda = \arg\Gamma\left(\lambda+2-\frac{i}{kr_0}\right).$$

Using formula (11), we can verify that the coefficients $W^{\lambda}_{k\Omega JL}$ are real.

5. INVERSE EXPANSION

Let us consider the integral

$$Q_{\lambda\lambda'} = \int\limits_{-\infty}^{\infty} W^{\lambda}_{k\Omega JL} W^{\lambda'*}_{k\Omega JL} d\Omega.$$

Using expression (13), going over from the generalized hypergeometric function ${}_3F_2$ to a finite sum by the formula

$${}_3F_2\left\{ \begin{array}{l} -n, a, b \\ c, d \end{array} \middle| z \right\} = \sum_{s=0}^{n} \frac{(-n)_s (a)_s (b)_s}{(c)_s (d)_s} z^s,$$

and changing Ω by $z = \frac{i\hbar}{2k\sqrt{m}}\Omega$, we find

$$Q_{\lambda\lambda'} = \frac{(-i)^{\lambda} i^{\lambda'}}{[(2J+1)!]^2}\frac{e^{i(\delta_{\lambda'}-\delta_\lambda)}}{\left|\Gamma\left(J+L+2-\frac{i}{kr_0}\right)\right|^2}\sqrt{(2\lambda+3)(2\lambda'+3)}$$
$$\times\left[\frac{(\lambda+J+L+2)!(\lambda+J-L+1)!(\lambda'+J+L+2)!(\lambda'+J-L+1)!}{(\lambda-J-L)!(\lambda'-J-L)!(\lambda-J+L+1)!(\lambda'-J+L+1)!}\right]^{1/2}$$
$$\times \sum_{s=0}^{\lambda-J-L}\frac{(-\lambda+J+L)_s(\lambda+J+L+3)_s}{s!(2J+2)_s\left(J+L+2+\frac{i}{kr_0}\right)_s}\sum_{t=0}^{\lambda'-J-L}\frac{(-\lambda'+J+L)_t(\lambda'+J+L+3)_t}{t!(2J+2)_t\left(J+L+2-\frac{i}{kr_0}\right)_t}B_{st},$$

where

$$B_{st} = \frac{1}{2\pi i}\times\int\limits_{-i\infty}^{i\infty}\Gamma\left(J+1+s+\frac{i}{2kr_0}+z\right)$$
$$\times\Gamma\left(J+1+t-\frac{i}{2kr_0}-z\right)$$
$$\times\Gamma\left(L+1-\frac{i}{2kr_0}+z\right)\Gamma\left(L+1+\frac{i}{2kr_0}-z\right)dz.$$

Now, we are going to use the Barnes formula [7]

$$\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty}\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s)ds \tag{14}$$

$$= \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}.$$

It is supposed in (14) that the poles of the expressions $\Gamma(\gamma-s)\Gamma(\delta-s)$ and $\Gamma(\alpha+s)\Gamma(\beta+s)$ do not coincide, lying on the right and on the left of the path of integration. In our case, the conditions of this lemma are fulfilled; therefore, we have

$$Q_{\lambda\lambda'} = (-i)^{\lambda} i^{\lambda'} \frac{(2L+1)!}{(2J+1)!} \left[(2\lambda+3)(2\lambda'+3)\frac{(\lambda+J+L+2)!(\lambda+J-L+1)!}{(\lambda-J-L)!(\lambda'-J-L)!}\right]^{1/2}$$
$$\times \sqrt{\frac{(\lambda'+J+L+2)!(\lambda'+J-L+1)!}{(\lambda-J+L+1)!(\lambda'-J+L+1)!}}$$
$$\times \sum_{s=0}^{\lambda-J-L} \frac{(-\lambda+J+L)_s(\lambda+J+L+3)_s}{s!\Gamma(2J+2L+s+4)}$$
$$\times e^{i(\delta_{\lambda'}-\delta_{\lambda})} {}_3F_2\left\{ \begin{matrix} -\lambda'+J+L, \lambda'+J+L+3, 2J+s+2 \\ 2J+2, 2J+2L+s+4 \end{matrix} \middle| 1 \right\}.$$

Now, using the Saalshütz theorem [6]

$${}_3F_2\left\{ \begin{matrix} a, b, -n \\ c, 1+a+b-c-n \end{matrix} \middle| 1 \right\} = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}, \tag{15}$$

we obtain

$$Q_{\lambda\lambda'} = (-1)^{\lambda'-J-L} \left[\frac{(\lambda+J+L+2)!(\lambda+J-L+1)!(\lambda'+J+L+2)!(\lambda'-J+L+1)!}{(\lambda-J-L)!(\lambda'-J-L)!(\lambda-J+L+1)!(\lambda'+J-L+1)!}\right]^{1/2}$$
$$\times \frac{(-i)^{\lambda} i^{\lambda'} e^{i(\delta_{\lambda'}-\delta_{\lambda})}\sqrt{(2\lambda+3)(2\lambda'+3)}}{(\lambda'+J+L+3)!\Gamma(-\lambda'+J+L+1)} {}_3F_2\left\{ \begin{matrix} -\lambda+J+L, \lambda+J+L+3, 1 \\ -\lambda'+J+L+1, \lambda'+J+L+4 \end{matrix} \middle| 1 \right\}.$$

Using the Saalshütz theorem (15) once again, we finally have

$$Q_{\lambda\lambda'} = (-i)^{\lambda} i^{\lambda'} \frac{e^{i(\delta_{\lambda'}-\delta_{\lambda})}}{\Gamma(\lambda-\lambda'+1)\Gamma(\lambda'-\lambda+1)} \frac{\sqrt{(2\lambda+3)(2\lambda'+3)}}{\lambda+\lambda'+3}$$
$$\times \left[\frac{(\lambda'-J-L)!(\lambda+J+L+2)!(\lambda+J-L+1)!(\lambda'-J+L+1)!}{(\lambda-J-L)!(\lambda'+J+L+2)!(\lambda-J+L+1)!(\lambda'+J-L+1)!}\right]^{1/2}.$$

Since the numbers λ and λ' are simultaneously integer or half-integer, the last expression vanishes for $\lambda \neq \lambda'$ because of the product of the gamma functions $\Gamma(\lambda-\lambda'+1)$ and $\Gamma(\lambda'-\lambda+1)$ and is equal to unity for $\lambda = \lambda'$; that is,

$$\int_{-\infty}^{\infty} W^{\lambda}_{k\Omega JL} W^{\lambda'*}_{k\Omega JL} d\Omega = \delta_{\lambda\lambda'}. \tag{16}$$

Now, taking into account (16), we find for the inverse expansions that

$$\psi^{\mathrm{hsp}} = \int_{-\infty}^{\infty} W^{\lambda}_{k\Omega JL} \psi^{\mathrm{par}} d\Omega,$$

where integration is performed along the real axis.

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How to Relate the Oscillator and Coulomb Systems on Spheres and Pseudospheres?*

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Abstract—We show that oscillators on a sphere and a pseudosphere are related, by the so-called Bohlin transformation, with Coulomb systems on a pseudosphere: even states of an oscillator yield a conventional Coulomb system on a pseudosphere, while odd states yield a Coulomb system on a pseudosphere in the presence of a magnetic flux tube generating half-spin. In higher dimensions, oscillator and Coulomb(-like) systems are connected in similar way. In particular, applying the Kustaanheimo–Stiefel transformation to oscillators on a sphere and a pseudosphere, we obtained a pseudospherical generalization of the MIC–Kepler problem describing a three-dimensional charge–dyon system. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

A (d-dimensional) oscillator and Coulomb systems are the most known representatives of mechanical systems possessing hidden symmetries that define the $su(d)$ symmetry algebra for the oscillator and $so(d+1)$ for the Coulomb system. The hidden symmetry has a very transparent meaning in the case of an oscillator, while, in the case of a Coulomb system, it has a more complicated interpretation in terms of geodesic flows of a d-dimensional sphere. On the other hand, the transformation $r = R^2$ converts the $(p+1)$-dimensional radial Coulomb problem into the $2p$-dimensional radial oscillator one, both in classical and quantum cases, where r and R denote the radial coordinates of, respectively, Coulomb and oscillator systems (see, e.g., [1]). In three distinguished cases of $p = 1, 2$, and 4, one can establish the complete correspondence between the Coulomb and oscillator systems by using the so-called Bohlin (or Levi-Civita) [2], Kustaanheimo–Stiefel [3], and Hurwitz [4] transformations, respectively. These transformations assume the reduction of the oscillator system by the action of Z_2, $U(1)$, and $SU(2)$ groups, respectively, and yield Coulomb-like systems specified by the presence of monopoles [5–7]. On the other hand, oscillator and Coulomb systems admit generalizations to a d-dimensional sphere and a two-sheet hyperboloid (pseudosphere) of radius R_0 given by the potentials [8, 9]

$$V_{\rm osc} = \frac{\alpha^2 R_0^2}{2}\frac{\mathbf{x}^2}{x_{d+1}^2}, \quad V_{\rm C} = -\frac{\gamma}{R_0}\frac{x_{d+1}}{|\mathbf{x}|}, \qquad (1)$$

where $\mathbf{x}$ and x_{d+1} are the (pseudo)Euclidean coordinates of ambient space $\mathbb{R}^{d+1}(\mathbb{R}^{d.1})$: $\epsilon\mathbf{x}^2 + x_{d+1}^2 = R_0^2, \epsilon = \pm 1$. The case of $\epsilon = 1$ corresponds to a sphere, while the case of $\epsilon = -1$ corresponds to the pseudosphere. These systems, which possess nonlinear hidden symmetries providing them with properties similar to those of conventional oscillator and Coulomb systems, have been investigated from many points of view (see, e.g., [10] and references therein).

How to relate oscillator and Coulomb systems on a sphere and a pseudosphere? Recently, this problem was considered in [11], where oscillator and Coulomb systems on spheres were related by some complicated mappings containing transitions to imaginary coordinates. The geometric origin of this mapping was not clarified there, and the reductions to Coulomb-like systems with monopoles and the relations of the constants of the motion that are responsible for hidden symmetries were not considered there either. In our recent study with Pogosyan [12], we established a transparent correspondence between oscillator and Coulomb systems on (pseudo)spheres for the simplest, two-dimensional, case ($p = 1$). We showed that, in the stereographic projection, the conventional Bohlin transformation relates a two-dimensional oscillator on a (pseudo)sphere to Coulomb systems on a pseudosphere, as well as those interacting with specific external magnetic fields. This simple construction immediately allows one to connect the con-

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stants of the motion that define the hidden symmetry of the systems under consideration and to clarify the mappings suggested in [11]. This construction can be straightforwardly used in higher dimensional cases ($p = 2, 4$), a subject to obtain the pseudospherical analogs of the known Coulomb-like systems, specified by the presence of monopoles: the so-called MIC–Kepler [6, 13] and $SU(2)$ Kepler [7] problems. In the present paper, we give a detailed description of this construction for the $p = 1$ case corresponding to the Bohlin transformation (Section 2) and for the $p = 2$ case corresponding to the Kustaanheimo–Stiefel one (Section 3) and discuss the $p = 4$ case corresponding to the Hurwitz transformation (Section 4).

2. THE BOHLIN TRANSFORMATION

Let us introduce a complex coordinate z that parametrizes a sphere by the complex projective plane $\mathbb{C}\mathrm{P}^1$ and a two-sheeted hyperboloid by the Poincaré disks $\mathcal{L}$:

$$\mathbf{x} \equiv x_1 + ix_2 = R_0 \frac{2z}{1+\epsilon z\bar{z}}, \qquad (2)$$
$$x_3 = R_0 \frac{1-\epsilon z\bar{z}}{1+\epsilon z\bar{z}}.$$

In these terms, the metric takes the Kähler form

$$ds^2 = R_0^2 \frac{4dzd\bar{z}}{(1+\epsilon z\bar{z})^2}, \qquad (3)$$

while $R_0 x_k$ define the isometries of the Kähler structure [$su(2)$ if $\epsilon = 1$ and $su(1.1)$ if $\epsilon = -1$]. The lower hemisphere and the lower sheet of the hyperboloid are parametrized by the unit disk $|z| < 1$, while the upper hemisphere and the upper sheet of the hyperboloid are parametrized by its outside; they transform into each other by the inversion $z \to 1/z$. Since, in the limit $R_0 \to \infty$, the lower hemisphere (the lower sheet of the hyperboloid) converts into the whole two-dimensional plane, we must restrict ourselves, for the correspondence with conventional oscillator and Coulomb problems, to those defined on the lower hemisphere and the lower sheet of the hyperboloid (pseudosphere).

Let us equip the oscillator phase space $T^*\mathbb{C}\mathrm{P}^1$ ($T^*\mathcal{L}$) with the symplectic structure

$$\omega = d\pi \wedge dz + d\bar{\pi} \wedge d\bar{z} \qquad (4)$$

and the rotation generators [defining $su(2)$ algebra if $\epsilon = 1$ and $su(1.1)$ if $\epsilon = -1$]

$$\mathbf{J} \equiv \frac{iJ_1 - J_2}{2} = \pi + \epsilon\bar{z}^2\bar{\pi}, \qquad (5)$$
$$J \equiv \frac{\epsilon J_3}{2} = i(z\pi - \bar{z}\bar{\pi}).$$

In these terms, the oscillator Hamiltonian is given by the expression

$$H^\epsilon_{\mathrm{osc}}(\pi, \bar{\pi}, z, \bar{z}) = \frac{\mathbf{J}\cdot\bar{\mathbf{J}} + \epsilon J^2}{2R_0^2} + \frac{\alpha^2 R_0^2}{2}\frac{\mathbf{x}^2}{x_3^2} \qquad (6)$$
$$= \frac{(1+\epsilon z\bar{z})^2 \pi\bar{\pi}}{2R_0^2} + \frac{2\alpha^2 R_0^2 z\bar{z}}{(1-\epsilon z\bar{z})^2}.$$

The hidden symmetry is given by the complex (or vectorial) constant of motion [9]

$$\mathbf{I} = I_1 + iI_2 = \frac{\mathbf{J}^2}{2R_0^2} + \frac{\alpha^2 R_0^2}{2}\frac{\bar{\mathbf{x}}^2}{x_3^2}, \qquad (7)$$

which defines, together with J and H_{osc}, the cubic algebra

$$\{\mathbf{I}, J\} = 2i\mathbf{I}, \qquad (8)$$
$$\{\bar{\mathbf{I}}, \mathbf{I}\} = 4i\left(\alpha^2 J + \frac{\epsilon J H_{\mathrm{osc}}}{R_0^2} - \frac{J^3}{2R_0^4}\right).$$

The energy surface of the oscillator on the (pseudo)sphere $H^\epsilon_{\mathrm{osc}} = E$ reads

$$\frac{\left(1-(z\bar{z})^2\right)^2 \pi\bar{\pi}}{2R_0^4} + 2\left(\alpha^2 + \epsilon\frac{E}{R_0^2}\right) z\bar{z} \qquad (9)$$
$$= \frac{E}{R_0^2}\left(1+(z\bar{z})^2\right).$$

Now, performing the canonical Bohlin transformation [2]

$$w = z^2, \quad p = \frac{\pi}{2z}, \qquad (10)$$

we convert the energy surface of the oscillator specified by (9) onto the energy surface of the Coulomb system on the pseudosphere:

$$\frac{(1-w\bar{w})^2 p\bar{p}}{2r_0^2} - \frac{\gamma}{r_0}\frac{1+w\bar{w}}{2|w|} = \mathcal{E}_{\mathrm{C}}, \qquad (11)$$

where

$$r_0 = R_0^2, \quad \gamma = \frac{E}{2}, \quad -2\mathcal{E}_{\mathrm{C}} = \alpha^2 + \epsilon\frac{E}{r_0}. \qquad (12)$$

The constants of motion of the oscillators, J and $\mathbf{I}$ [which are equal on the energy surfaces (9)] are converted into, respectively, the doubled angular momentum and the doubled Runge–Lenz vector of the Coulomb system

$$J \to 2J_{\mathrm{C}}, \quad \mathbf{I} \to 2\mathbf{A}, \qquad (13)$$
$$\mathbf{A} = -\frac{iJ_{\mathrm{C}}\mathbf{J}_{\mathrm{C}}}{r_0} + \gamma\frac{\bar{\mathbf{x}}_{\mathrm{C}}}{|\mathbf{x}_{\mathrm{C}}|},$$

where $\mathbf{J}_{\mathrm{C}}$, J_{C}, and $\mathbf{x}_{\mathrm{C}}$ denote the rotation generators and the pseudo-Euclidean coordinates of the Coulomb system.

It is easy to obtain from (8) the symmetry algebra of the reduced system,

$$\{\mathbf{A}, J\} = i\mathbf{A}, \qquad (14)$$
$$\{\bar{\mathbf{A}}, \mathbf{A}\} = -4i\left(\mathcal{H}_{\mathrm{C}} + \frac{J_{\mathrm{C}}^2}{r_0^2}\right) J_{\mathrm{C}}.$$

Hence, the Bohlin transformation of a classical isotropic oscillator on a (pseudo)sphere yields the classical Coulomb problem on a pseudosphere.

The quantum-mechanical counterpart of the energy surface (9) is the Schrödinger equation

$$\mathcal{H}^{\epsilon}_{\mathrm{osc}}(\alpha, R_0|\pi, \bar{\pi}, z, \bar{z})\Psi(z, \bar{z}) = E\Psi(z, \bar{z}), \qquad (15)$$

with the quantum Hamiltonian defined (owing to the two-dimensional origin of the system) by expression (6), where π and $\bar{\pi}$ are the momentum operators (hereafter, we set $\hbar = 1$)

$$\pi = -i\frac{\partial}{\partial z}, \quad \bar{\pi} = -i\frac{\partial}{\partial \bar{z}}. \qquad (16)$$

The energy spectrum of this system is given by the expression (see, e.g., [10] and references therein)

$$E = \tilde{\alpha}(N+1) + \epsilon\frac{(N+1)^2}{2R_0^2}, \qquad (17)$$
$$N = 2n_r + |M|, \quad n_r = 0, 1, \ldots$$

where $\tilde{\alpha} = \sqrt{\alpha^2 + 1/(4R_0^4)}$, M is the eigenvalue of J, N is the principal quantum number, and n_r is the radial quantum number; we have

$$|M|,\ N = 1, \ldots, N_{\max}, \qquad (18)$$
$$N_{\max} = \begin{cases} \infty & \text{if } \epsilon = 1 \\ [2\tilde{\alpha}R_0^2] - 1 & \text{if } \epsilon = -1. \end{cases}$$

Thus, the number of levels in the energy spectrum of the oscillator is infinite on the sphere and finite on the pseudosphere.

The quantum-mechanical correspondence between oscillator and Coulomb systems is more complicated, because the Bohlin transformation (10) maps the z plane into the two-sheeted Riemann surface, since $\arg w \in [0, 4\pi)$. Thus, we have to supply the quantum-mechanical Bohlin transformation with the reduction by the Z_2 group action, choosing either even ($\sigma = 0$) or odd ($\sigma = 1/2$) wave functions

$$\Psi_\sigma(z, \bar{z}) = \psi_\sigma(z^2, \bar{z}^2)\left(\frac{z}{\bar{z}}\right)^{2\sigma}: \qquad (19)$$
$$\psi_\sigma(|w|, \arg w + 2\pi) = \psi_\sigma(|w|, \arg w).$$

This implies that the range of definition of w can be restricted, without loss of generality, to $\arg w \in [0, 2\pi)$. In that case, the resulting system is the Coulomb problem on a hyperboloid given by the Schrödinger equation

$$H^{-}_{\mathrm{C}}(\gamma, r|p_\sigma, \bar{p}_\sigma, w, \bar{w})\psi_\sigma = \mathcal{E}_{\mathrm{C}}\psi_\sigma, \qquad (20)$$

where γ, $\mathcal{E}_{\mathrm{C}}$, and r are given by (12) and the momenta operators are of the form

$$p_\sigma = -i\frac{\partial}{\partial w} - \frac{\sigma}{iw}, \quad \bar{p}_\sigma = -i\frac{\partial}{\partial \bar{w}} + \frac{\sigma}{i\bar{w}}. \qquad (21)$$

Hence, the resulting Coulomb system includes the interaction with a magnetic vortex (an infinitely thin solenoid) with the magnetic flux $\pi\sigma$ and zero strength, $\mathrm{curl}\sigma/w = 0$. Such composites are typical representatives of anionic systems with spin σ. Thus, we get a conventional $2d$ Coulomb problem on a hyperboloid at $\sigma = 0$ and that with half-spin generated by the magnetic flux at $\sigma = 1/2$. Taking into account the relations in (12), one can rewrite the oscillator energy spectrum (17) as follows:

$$\sqrt{\frac{1}{4r_0^2} - \epsilon\frac{2\gamma}{r_0} - 2\mathcal{E}_{\mathrm{C}}} = \frac{2\gamma}{N+1} - \epsilon\frac{N+1}{2r_0}. \qquad (22)$$

From this expression, one can easily obtain the energy spectrum of the reduced system on the pseudosphere. The result is

$$\mathcal{E}_{\mathrm{C}} = -\frac{N_\sigma(N_\sigma+1)}{2r_0^2} - \frac{\gamma^2}{2(N_\sigma+1/2)^2}, \qquad (23)$$

where

$$N_\sigma = n_r + m_\sigma, \quad m_\sigma = M/2, \qquad (24)$$
$$n_r, m_\sigma - \sigma, N_\sigma - \sigma = 0, 1, \ldots, N_\sigma^{\max} - \sigma.$$

Here, m_σ denotes the eigenvalue of the angular momentum of the reduced system, and n_r is the radial quantum number of the initial (and reduced) system. Notice that the magnetic vortex shifts the energy levels of the two-dimensional Coulomb system, which is nothing else but the reflection of the Aharonov–Bohm effect.

It is seen that the whole spectrum of the oscillator on the pseudosphere ($\epsilon = -1$) transforms into the spectra of the constructed Coulomb systems on the pseudosphere, while, for the oscillator on the sphere ($\epsilon = 1$), the positivity of the left-hand side of (22) restricts the admissible values of N_σ. Thus, only part of the spectrum of the oscillator on the sphere transforms into the spectrum of the Coulomb system. Hence, we get the same result in both cases:

$$N_\sigma^{\max} = [\sqrt{r_0\gamma} - (1/2 + \sigma)]. \qquad (25)$$

3. KUSTAANHEIMO–STIEFEL TRANSFORMATION

It is easy to see that the $2p$-dimensional oscillator on a (pseudo)sphere can be related to $(p+1)$-dimensional Coulomb-like systems on a pseudosphere in higher dimensions ($p = 2, 4$) as well. In stereographic coordinates, the oscillator on $2p$-dimensional (pseudo)sphere is indeed described by the Hamiltonian system given by (4) and (6), where the following substitution is performed: $(z, \pi) \to (z^a, \pi_a)$, $a = 1, \ldots, p$, with the summation over these indices. Consequently, the oscillator energy surfaces are of the form (9). A further reduction to the $(p+1)$-dimensional Coulomb-like system on a pseudosphere must be similarly followed in the corresponding reduction in the flat case [6, 7]. Since $|\mathbf{u}| = z\bar{z}$ in all three cases, we can interpret $\mathbf{u}$ as the stereographic coordinates of the reduced system, consequently interpreting the last one as the Coulomb-like system on a $(p+1)$-dimensional pseudosphere.

For example, if $p = 2$, we should reduce the four-dimensional oscillator by the Hamiltonian action of $U(1)$ group given by the generator

$$J = i(z\pi - \bar{z}\bar{\pi}).$$

For this purpose, we have to fix the level surface

$$J = 2s \tag{26}$$

and factorize it by the $U(1)$-Hamiltonian flow, choosing six $U(1)$-invariant stereographic coordinates in the form of conventional Kustaanheimo–Stiefel transformation [3, 6]

$$\mathbf{u} = z\boldsymbol{\sigma}\bar{z}, \quad \mathbf{p} = \frac{z\boldsymbol{\sigma}\pi + \bar{\pi}\boldsymbol{\sigma}\bar{z}}{2(z\bar{z})}, \tag{27}$$

where $\boldsymbol{\sigma}$ are Pauli matrices.

As a result, the reduced symplectic structure reads

$$d\mathbf{p} \wedge d\mathbf{u} + s\frac{(\mathbf{u} \times d\mathbf{u}) \wedge d\mathbf{u}}{|\mathbf{u}|^3}, \tag{28}$$

and the oscillator energy surface takes the form

$$\frac{(1-\mathbf{u}^2)^2}{8r_0^2}\left(\mathbf{p}^2 + \frac{s^2}{\mathbf{u}^2}\right) - \frac{\gamma}{r_0}\frac{1+\mathbf{u}^2}{2|\mathbf{u}|} = \mathcal{E}_{\mathrm{C}}, \tag{29}$$

where $\mathbf{u}$ stands for the stereographic coordinates of the three-dimensional pseudosphere and r_0, γ, and $\mathcal{E}_{\mathrm{C}}$ are defined by expressions (12).

Thus, we get the energy surface of the pseudospherical analog of a Coulomb-like system describing the interaction of two nonrelativistic dyons, which was proposed in [13] and which is known as the MIC–Kepler system.

In the coordinates of ambient space, the potential of the pseudospherical MIC–Kepler system has the form

$$V_{\mathrm{MIC}} = \frac{s^2}{r_0^2}\left(\frac{x_4^2}{2|\mathbf{x}|^2} - 2\right) - \frac{\gamma}{r_0}\frac{x_4}{|\mathbf{x}|}. \tag{30}$$

To quantize the system, we should replace Eqs. (9) and (26) by the spectral problem

$$\hat{\mathcal{H}}_{\mathrm{osc}}(\pi, \bar{\pi}, z, \bar{z})\Psi(z, \bar{z}) = E_{\mathrm{osc}}\Psi(z, \bar{z}), \tag{31}$$
$$\hat{J}_0(\pi, \bar{\pi}, z, \bar{z})\Psi(z, \bar{z}) = 2s\Psi(z, \bar{z}),$$

where the momenta π_α and $\bar{\pi}_\alpha$ are replaced by the operators

$$\pi_\alpha = -i\frac{\partial}{\partial z^\alpha}, \quad \bar{\pi}_\alpha = -i\frac{\partial}{\partial \bar{z}^\alpha} \tag{32}$$

and the appropriate ordering in the Hamiltonian is assumed.

The second equation in (31) can be resolved by the substitution of the ansatz

$$\Psi_s(z, \bar{z}) = \psi_s(\mathbf{u})\mathrm{e}^{is\lambda}: \tag{33}$$
$$[\hat{J}_0\lambda] = i, \quad \lambda = is\log\frac{z^1}{\bar{z}^1},$$

which reduces the first equation in (31) (i.e., the oscillator Schrödinger equation) to those corresponding to the generalized MIC–Kepler system (29), where

$$\hat{\mathbf{p}}_s = \mathrm{e}^{-is\lambda}\hat{\mathbf{p}}\mathrm{e}^{is\lambda} = -i\frac{\partial}{\partial \mathbf{u}} - s\mathbf{A}(\mathbf{u}), \tag{34}$$

with $\mathbf{A}(\mathbf{u})$ being the vector potential of the Dirac monopole with a singularity directed along the u_3 axes and $\hat{\mathbf{p}}$ being defined by the second expression in (27), where π and $\bar{\pi}$ are given by operators (32) placed at the right.

The requirement that Ψ be a single-valued wave function causes s to be integer or half-integer; i.e., it leads to the Dirac quantization condition.

Solving the Schrödinger equation, one gets the oscillator energy spectrum

$$E = \tilde{\alpha}(N+2) + \epsilon\frac{(N+2)^2 - 2}{2R_0^2}, \tag{35}$$
$$N = 2n_r + |L|,$$

where $\tilde{\alpha} = \sqrt{\alpha^2 + 1/(4R_0^4)}$, L is the eigenvalue of the total angular momentum, N is the principal quantum number, n_r is the radial quantum number, and

$$2|s| = 0, 1, \ldots, L, \quad |L|, N = 1, \ldots, N_{\max}, \tag{36}$$
$$N_{\max}$$
$$= \begin{cases} \infty, & \text{if } \epsilon = 1, \\ \left[\tilde{\alpha}R_0^2\left(1 + \sqrt{1 + 2/(\tilde{\alpha}R_0^2)^2}\right)\right] - 2, & \text{if } \epsilon = -1. \end{cases}$$

Following the same line of reasoning as in the preceding case, we can get, from this expression, the energy spectrum of the MIC–Kepler system on a pseudosphere. The result is

$$\mathcal{E}_{\rm C} = -\frac{(n_r+|L_s|)(n_r+|L_s|+2)}{2r_0^2} - \frac{\gamma^2}{2(n_r+|L_s|+1)^2}, \tag{37}$$

where

$$L_s = L/2, \qquad |L_s|, n_r+|l_s| = |s|, |s|+1, \ldots, N_s^{\max}. \tag{38}$$

Here, l_s denotes the eigenvalue of the total angular momentum of the reduced system, and n_r is the radial quantum number of the initial (and reduced) system. As in the $p=1$ case, one gets

$$N_s^{\max}+1 = \left[\sqrt{r_0\gamma - \frac{1}{2r_0^2}}\,\right].$$

It is convenient to introduce new quantum number

$$k \equiv n_r + |L_s| - |s|, \quad k = 0, 1, \ldots, N_s^{\max} - |s|$$

and rewrite expression (37) as

$$\mathcal{E}_{\rm C} = -\frac{(k+|s|)(k+|s|+2)}{2r_0^2} - \frac{\gamma^2}{2(k+|s|+1)^2}. \tag{39}$$

It is seen that the degeneracy of the reduced system is the same as in the usual MIC–Kepler problem [6], viz., $k(k+|s|-1)$. It is a pleasure to notice that the spherical generalization of the MIC–Kepler system has also been presented at this colloquium; it was constructed by Gritsev, Kurochkin, and Otchik [14].

4. DISCUSSION: THE HURWITZ TRANSFORMATION

We have shown that applying the standard Bohlin/Kustaanheimo–Stiefel transformations to the stereographic (conformally flat) coordinates of the two- and four-dimensional oscillators on a sphere and a pseudosphere yields the pseudospherical two-dimensional Coulomb and (three-dimensional) MIC–Kepler systems, respectively. It is obvious from the above consideration that the relation of an eight-dimensional oscillator on a (pseudo)sphere and of the pseudospherical analog of the so-called $SU(2)$ Kepler (or Yang–Coulomb) system [7] would be completely similar to the aforementioned cases. In order to establish such a connection [and to construct the pseudospherical $SU(2)$ Kepler system], we should perform the Hamiltonian reduction of the eight-dimensional oscillator by the $SU(2)$-group action

$$z^a \to z^a g, \;\; g\bar{g} = 1, \quad g \in \mathbb{H},\; z^a \in \mathbb{H}^2, \tag{40}$$

where z^1 and z^2 are quaternions parametrizing stereographic coordinates of the eight-dimensional (pseudo)sphere. The spatial stereographic coordinates of the reduced system should be chosen in the form of standard Hurwitz transformation [4, 7]

$$u = 2z_1\bar{z}_2, \quad u_5 = z_1\bar{z}_1 - z_2\bar{z}_2, \qquad u \in \mathbb{H},\; u_5 \in \mathbb{R}, \tag{41}$$

and completed with the conjugate momenta and isospin coordinates as well. The potential of the pseudospherical $SU(2)$ Kepler system would be of the form similar to the MIC–Kepler one,

$$V_{SU(2)\text{-Kepler}} = \frac{j(j+1)}{r_0^2}\left(\frac{x_6^2}{2\mathbf{x}^2} - 2\right) - \frac{\gamma}{r_0}\frac{x_6}{2|\mathbf{x}|}, \tag{42}$$

where $(\mathbf{x}, x_6)$ denote the coordinates of the ambient space of the five-dimensional pseudosphere, $j(j+1)$ is the eigenvalue of operator $\mathcal{J}_i^2$ defining the $SU(2)$-group action (40), and r_0 and γ are given by expressions (12). The kinetic term of the Hamiltonian would include the interaction with the vector potential of the five-dimensional $SU(2)$ monopole [15].

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RPA Equation Embedded into Infinite-Dimensional Fock Space F_∞*

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Abstract—To clear up both algebraic and geometric structures for integrable systems derived from self-consistent field theory, in particular, geometric aspect of the random-phase-approximation (RPA) equation is exhibited on the basis of the viewpoint of symmetry of the evolution equation. The RPA equation for an infinite-dimensional Grassmannian is constructed. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

A conventional standard description of fermion many-body systems starts with the most basic approximation that is founded on the independent-particle picture, i.e., the self-consistent field (SCF) for the motion of fermions. Hartree–Fock (HF) theory is a typical one of such an approximation for ground states of fermion systems. Excited states are treated within the well-known random-phase approximation (RPA) if only a small fluctuation in the time-dependent HF (TDHF) mean field is taken into account around a stationary HF ground-state solution [1]. The TDHF equation is a nonlinear equation owing to its SCF character and may have no unique solution. A set of particle–hole-type pair operators of fermions with n single-particle states is closed under the Lie multiplication and forms a basis of a Lie algebra u_n [2]. The Lie algebra u_n of the pair operators generates a set of canonical transformations to a Slater determinant (S-det), i.e., the Thouless transformation [3], which induces a representation of the corresponding $U(n)$ group. It provides an exact generator coordinate representation of fermion state vectors. The RPA is a standard method for describing collective excitations in a fermion system with small quantum fluctuations.

In [4, 5], we studied the relation between TDHF theory [6] and the τ-functional method in soliton theory [7]. To go beyond a perturbative method with respect to periodic collective variables [8], we aimed at constructing TDHF theory on the associative affine Kac–Moody algebra along the soliton theory on the infinite-dimensional fermions. They are introduced through the Laurent expansion of finite-dimensional fermion operators with respect to degrees of freedom of fermions related to the mean-field potential. We attempted to embed the HF Lie algebra u_n into an infinite-dimensional Lie algebra gl_∞ with the aid of the Laurent expansion of fermion operators with respect to parameter z. Thus, the TDHF equation on the finite-dimensional Grassmannian Gr_m is embedded into the infinite-dimensional Grassmannian. We gave an expression for TDHF theory on the τ-functional space. We also showed that the TDHF equation on F_∞ under level one is nothing else but the Laurent expansion of the TDHF equation on Gr_m. The construction of the TDHF equation on F_∞ presents us explicit algebraic structures as a gauge theory inherent in SCF theory. From these facts, the SCF theory can be regarded as a method for determining self-consistently both quasiparticle energies and boson energies of collective motions which are unified into a gauge phase. Thus, we could obtain a common language, the infinite-dimensional Grassmannian and the Lie algebra, together with the associative affine Kac–Moody algebra.

2. GEOMETRIC ASPECT OF THE RPA EQUATION

First, we recapitulate the fundamental idea in a series of papers [9]. In viewing symmetries of time-evolution equations, let us consider an abstract evolution equation $\partial_t u(t) = K(u(t))$ for a function u depending only on time t. Suppose that there exists a transformation that converts a solution for u to another solution. Introducing a parameter s, we assume another kind of evolution equation with respect to s, i.e., $\partial_s u(t,s) = \bar{K}(u(t,s))$. Then, an integrability condition for the existence of the transformation is given by $\partial_s K(u(t,s)) = \partial_t \bar{K}(u(t,s))$.

In a differential-geometry approach to nonlinear problems, the above integrability condition is transcribed into zero curvature of the connection on the corresponding Lie groups of systems. Nonlinear evolution equations, e.g., famous soliton equations, such as KdV, KP, and sine/sinh-Gordon equations, originate from the well-known Lax equation [10],

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which arises as zero curvature of connection [11]. These soliton equations appear as evolution equations for the tangent vector fields of local gauge fields depending on time (t) and space (x) coordinates. On the contrary, in TDHF SCF theory, the corresponding Lie groups are unitary transformation groups of orthonormal bases that are dependent on t but not on x.

Our basic idea is to introduce a sort of Lagrange manner familiar to fluid dynamics to describe collective coordinates. This manner enables us to take a one-form Ω linearly composed of the TDHF Hamiltonian and the infinitesimal generators induced by collective-variable differentials of a canonical transformation $U(n)$. The curvature C can be defined as $C \stackrel{d}{=} d\Omega - \Omega \wedge \Omega$. Then, the integrability condition of our system reads $C = 0$. The condition in the quasiparticle frame (QPF) is nothing but the formal RPA equation imposed by weak orthogonal conditions among the infinitesimal generators, i.e., an equation for tangent vector fields on the group submanifold with respect to the collective variables.

Relative vector fields made of the SCF Hamiltonian around each point on an integral curve also constitute solutions for the formal RPA equation around the same point, which is in turn a fixed point in the QPF. This means that the formal RPA equation is a natural extension of the usual RPA equation for small-amplitude quantal fluctuations around the ground state to that at any point on the collective submanifold.

The problem of extracting a certain collective submanifold out of the fully parameterized TDHF manifold may be reduced to the search for the corresponding sphere on which the top of arrow of spatial generators around a fixed point exists. Then, it is interpreted that the formal RPA is just an extension of the usual RPA form on a flat surface (linear) to that on a curved surface (nonlinear). We notice that the starting point selected by us on the moving frame becomes a standard point (new fixed point). This fact presents a geometric interpretation for symmetry breaking and recovery. The former is brought as a choice of spontaneous symmetry breaking, and the latter causes the motion, which has already been running, owing to a recovery of the symmetry.

3. CONSTRUCTION OF THE FORMAL RPA EQUATION ON F_∞

Following [4, 5], we sketch briefly the TDHF method on F_∞. For fermion operators of n single-particle states in a time-periodic self-consistent mean-field potential with a normal mode ω_c, we introduce infinite-dimensional fermion operators $\psi_{nr+\alpha}$ and $\psi^*_{nr+\alpha}$ $(\alpha = 1, \ldots, n,\ r \in \mathbb{Z})$, the normalized perfect vacuum $\{\psi_{nr+\alpha}|\mathrm{Vac}\rangle = 0, \langle \mathrm{Vac}|\psi^*_{nr+\alpha} = 0 \ \ (r \le -1); \psi^*_{nr+\alpha}|\mathrm{Vac}\rangle = 0, \langle \mathrm{Vac}|\psi^*_{nr+\alpha} = 0 \ \ (r \ge 0)\}$ with $\langle \mathrm{Vac}|\mathrm{Vac}\rangle = 1$, and the reference vacuum $\{|m\rangle = \psi_m \ldots \psi_1|\mathrm{Vac}\rangle, \langle m|m\rangle = 1, \ \ m = 1, \ldots, n\}$. The normal-ordered pair operators $: \psi_{nr+\alpha}\psi^*_{ns+\beta} := \psi_{nr+\alpha}\psi^*_{ns+\beta} - \delta_{\alpha\beta}\delta_{rs}$ $(s < 0)$ generate an affine Kac–Moody algebra [12]. We define the following $\widehat{su_n}$ $(\subset \widehat{sl_n})$ Lie algebra:

$$\left.\begin{aligned} X_\gamma &= \bar{X}_\gamma + \mathbb{C}\cdot c,\ \mathbb{C}^* = -\mathbb{C}\ \text{(pure imaginary)},\\ \bar{X}_\gamma &= \sum_{r=-N}^{N}\sum_{s\in\mathbb{Z}} (\gamma_r)_{\alpha\beta} : \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta} :, \gamma_r^\dagger = -\gamma_{-r}, \quad \mathrm{tr}\gamma_r = 0, \quad [X_\gamma\ ,\ c] = 0,\\ [X_\gamma\ ,\ X_{\gamma'}] &= \bar{X}_{[\gamma\ ,\ \gamma']} + \alpha(\gamma,\gamma')\cdot c, c|m\rangle = 1\cdot|m\rangle \end{aligned}\right\} \quad (1)$$

The expression for the matrix γ is given in [4, 5]. The infinite-dimensional fermion operator is transformed by a canonical transformation $U(\hat{g})$ $(\hat{g} = e^\gamma)$, which satisfies the relations $U^{-1}(\hat{g}) = U(\hat{g}^{-1}) = U(\hat{g}^\dagger)$ and $U(\hat{g}\hat{g}') = U(\hat{g})U(\hat{g}')$ with $\hat{g}^\dagger\hat{g} = \hat{g}\hat{g}^\dagger = I_\infty$, into the forms

$$\left.\begin{aligned} \psi_{nr+\alpha}(\hat{g}) &\stackrel{d}{=} U(\hat{g})\psi_{nr+\alpha}U^{-1}(\hat{g})\\ &= \sum_{s\in\mathbb{Z}} \psi_{n(r-s)+\beta}(g_s)_{\beta\alpha},\\ \psi^*_{nr+\alpha}(\hat{g}) &\stackrel{d}{=} U(\hat{g})\psi^*_{nr+\alpha}U^{-1}(\hat{g})\\ &= \sum_{s\in\mathbb{Z}} \psi^*_{n(r-s)+\beta}(g^*_s)_{\beta\alpha} \end{aligned}\right\} \quad (2)$$

where I_∞ is the infinite-dimensional identity matrix and

$$\begin{aligned} \hat{g}_{nr+\alpha,ns+\beta} &= (g_{s-r})_{\alpha\beta},\\ \hat{g}^\dagger_{nr+\alpha,ns+\beta} &= (g^\dagger_{r-s})_{\alpha\beta}, \end{aligned} \quad (3)$$

$$\left.\begin{aligned} \delta_{rs}\delta_{\alpha\beta} &= (\hat{g}\hat{g}^\dagger)_{nr+\alpha,ns+\beta}\\ &= \sum_{t\in\mathbb{Z}} (g_t g^\dagger_{t+(r-s)})_{\alpha\beta},\\ \delta_{rs}\delta_{\alpha\beta} &= (\hat{g}^\dagger\hat{g})_{nr+\alpha,ns+\beta}\\ &= \sum_{t\in\mathbb{Z}} (g^\dagger_t g_{t-(r-s)})_{\alpha\beta} \end{aligned}\right\} \quad (4)$$

The elements of the density matrix corresponding to the formal Laurent expansion of the usual one on the finite-dimensional Grassmannian G_m can be defined as

$$(W_r)_{\alpha\beta} \stackrel{d}{=} \sum_{s\in\mathbb{Z}} \langle m|U(\hat{g}^\dagger) : \psi_{n(s+r)+\beta}\psi^*_{ns+\alpha} :, \quad (5)$$

$$U(\hat{g})|m\rangle = \sum_{s\in\mathbb{Z}}\sum_{\gamma=1}^{m} (g_s)_{\alpha\gamma}(g^\dagger_{s-r})_{\gamma\beta}.$$

Following [4, 5], we can obtain the SCF Hamiltonian on F_∞ as

$$H_{F_\infty;\mathrm{HF}} = \sum_{k\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} (\mathcal{F}_r)_{\alpha\beta} : \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta} :, \quad (6)$$

$$(\mathcal{F}_r)_{\alpha\beta} = h_{\alpha\beta}\delta_{r,0} + [\alpha\beta|\gamma\delta](W_r)_{\delta\gamma}.$$

Using the covariant differential operator $D_r = i\partial_t + r\omega_c$, one can express the TDHF equation as

$$\left.\begin{aligned} D_t\hat{g} = \mathcal{F}(\hat{g})\hat{g}, \;\; D_t\hat{g} &\stackrel{d}{=} \begin{bmatrix} \ddots & & & & & \ddots \\ & D_{-1}g_{-1} & D_0g_0 & D_1g_1 & & \\ & & D_{-1}g_{-1} & D_0g_0 & D_1g_1 & \\ & & & D_{-1}g_{-1} & D_0g_0 & D_1g_1 \\ \ddots & & & & & \ddots \end{bmatrix}, \\ \mathcal{F}(\hat{g}) &\stackrel{d}{=} \begin{bmatrix} \ddots & & & & & \ddots \\ & \mathcal{F}_{-1} & \mathcal{F}_0 & \mathcal{F}_1 & & \\ & & \mathcal{F}_{-1} & \mathcal{F}_0 & \mathcal{F}_1 & \\ & & & \mathcal{F}_{-1} & \mathcal{F}_0 & \mathcal{F}_1 \\ \ddots & & & & & \ddots \end{bmatrix}, \quad \hat{g} \stackrel{d}{=} \begin{bmatrix} \ddots & & & & & \ddots \\ & g_{-1} & g_0 & g_1 & & \\ & & g_{-1} & g_0 & g_1 & \\ & & & g_{-1} & g_0 & g_1 \\ \ddots & & & & & \ddots \end{bmatrix} \end{aligned}\right\}. \quad (7)$$

Upon the introduction of $(\mathcal{F}^c_r)_{\alpha\beta}(\hat{g},\omega_c) = \omega_c \sum\limits_{s\in\mathbb{Z}} s(g_sg^\dagger_{s-r})_{\alpha\beta}$, the matrix $\mathcal{F}^c(\hat{g},\omega_c)$ takes the form

$$\mathcal{F}^c(\hat{g},\omega_c) = \omega_c \begin{bmatrix} \ddots & & & & & \ddots \\ & -g_{-1} & 0 & g_1 & & \\ & & -g_{-1} & 0 & g_1 & \\ & & & -g_{-1} & 0 & g_1 \\ \ddots & & & & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & & & & \ddots \\ & g^\dagger_1 & g^\dagger_0 & g^\dagger_{-1} & & \\ & & g^\dagger_1 & g^\dagger_0 & g^\dagger_{-1} & \\ & & & g^\dagger_1 & g^\dagger_0 & g^\dagger_{-1} \\ \ddots & & & & & \ddots \end{bmatrix}. \quad (8)$$

Then, Eq. (7) transforms into

$$\left.\begin{aligned} i\partial_t\hat{g} = \mathcal{F}^p(\hat{g})\hat{g}, \quad \mathcal{F}^p(\hat{g}) = \mathcal{F}(\hat{g}) - \mathcal{F}^c(\hat{g}), \\ (\mathcal{F}^p_r)_{\alpha\beta} = (\mathcal{F}_r - \mathcal{F}^c_r)_{\alpha\beta} = h_{\alpha\beta}\delta_{r,0} \\ + [\alpha\beta|\gamma\delta]\,(W_r)_{\delta\gamma} - \omega_c \sum_{s\in\mathbb{Z}} s(g_sg^\dagger_{s-r})_{\alpha\beta} \end{aligned}\right\}, \quad (9)$$

introducing $\widehat{D}_t = i\partial_t + H^c_{F_\infty;\mathrm{HF}}$, which is recast into that on the state vector $U(\hat{g})|m\rangle$ as

$$\left.\begin{aligned} \widehat{D}_tU(\hat{g})|m\rangle = H_{F_\infty;\mathrm{HF}}U(\hat{g})|m\rangle, \\ H^c_{F_\infty;\mathrm{HF}} = \sum_{r,s\in\mathbb{Z}} (\mathcal{F}^c_r)_{\alpha\beta} : \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta} :, \\ i\partial_tU(\hat{g})|m\rangle = H^p_{F_\infty;\mathrm{HF}}U(\hat{g})|m\rangle, \\ H^p_{F_\infty;\mathrm{HF}} = \sum_{r,s\in\mathbb{Z}} (\mathcal{F}^p_r)_{\alpha\beta} : \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta} : \end{aligned}\right\}. \quad (10)$$

Let ϵ and ϵ^* be parameters that specify a two-dimensional surface. Further, we set $z = e^{i\varphi}$ and $\varphi =$

$-\omega_c t$ and let the parameters be independent of the angle φ on each *loop*, i.e., $\gamma = \sum_{r=-N}^{N} \gamma_r(\epsilon, \epsilon^*) z^r$ and $\partial\gamma_r/\partial\varphi = 0$ for all r. The canonicity conditions must guarantee (ϵ, ϵ^*) to be an orthogonal canonical coordinate system,

$$\langle \hat{g}|\partial_\epsilon|\hat{g}\rangle = \langle m|U(\hat{g}^\dagger)\partial_\epsilon U(\hat{g})|m\rangle = \frac{1}{2}\epsilon^*, \qquad (11)$$

$$\langle \hat{g}|\partial_{\epsilon^*}|\hat{g}\rangle = \langle m|U(\hat{g}^\dagger)\partial_{\epsilon^*} U(\hat{g})|m\rangle = -\frac{1}{2}\epsilon.$$

We define infinitesimal generators on the collective submanifold as

$$\left.\begin{aligned} X_{\theta^\dagger} &= i\partial_\epsilon U(\hat{g})\cdot U(\hat{g})^\dagger = \bar{X}_{\theta^\dagger} + \mathbb{C}(i\partial_\epsilon \hat{g}\cdot\hat{g}^\dagger),\ \ \theta^\dagger = i\partial_\epsilon \hat{g}\cdot\hat{g}^\dagger, \\ X_\theta &= i\partial_{\epsilon^*} U(\hat{g})\cdot U(\hat{g})^\dagger = \bar{X}_\theta + \mathbb{C}(i\partial_{\epsilon^*}\hat{g}\cdot\hat{g}^\dagger),\ \ \theta = i\partial_{\epsilon^*}\hat{g}\cdot\hat{g}^\dagger, \end{aligned}\right\} \qquad (12)$$

$$\theta^\dagger = \begin{bmatrix} \ddots & & & & \ddots \\ \theta_1^\dagger & \theta_0^\dagger & \theta_{-1}^\dagger & & \\ & \theta_1^\dagger & \theta_0^\dagger & \theta_{-1}^\dagger & \\ & & \theta_1^\dagger & \theta_0^\dagger & \theta_{-1}^\dagger \\ \ddots & & & & \ddots \end{bmatrix}, \quad \theta = \begin{bmatrix} \ddots & & & & \ddots \\ \theta_{-1} & \theta_0 & \theta_1 & & \\ & \theta_{-1} & \theta_0 & \theta_1 & \\ & & \theta_{-1} & \theta_0 & \theta_1 \\ \ddots & & & & \ddots \end{bmatrix}$$

where $\mathbb{C}(\ldots)$ vanishes. From $\partial_{\epsilon^*}\langle\hat{g}|\partial_\epsilon|\hat{g}\rangle - \partial_\epsilon\langle\hat{g}|\partial_{\epsilon^*}|\hat{g}\rangle$, we have the *weak* orthogonality condition

$$\begin{aligned} 1 &= \langle\hat{g}|[X_\theta, X_{\theta^\dagger}]|\hat{g}\rangle \qquad (13)\\ &= \sum_{\alpha=1}^{m}\sum_{\gamma=1}^{n}\sum_{r\in\mathbb{Z}}([\theta,\theta^\dagger]_r)_{\alpha\gamma}(W_{-r})_{\gamma\alpha} \\ &\quad - \frac{1}{2}\mathrm{tr}\begin{bmatrix} -I & \\ & I \end{bmatrix}[\bar{\theta},\bar{\theta}^\dagger](\hat{I} = I_\infty). \end{aligned}$$

Using the idea of Lax pairs [10], we can recast Eqs. (7) and (12) into

$$\left.\begin{aligned} &D_t\hat{g} = \mathcal{F}(\hat{g})\hat{g},\ \ \partial_t\hat{g}^0 = 0,\ \ \mathcal{F}(\hat{g}) = \mathcal{F}(\hat{g}^0), \\ &i\partial_\epsilon\hat{g} = \theta^\dagger(\hat{g})\hat{g},\ \ \theta^\dagger(\hat{g}) = \theta^\dagger(\hat{g}^0) + \hat{g}^0(\partial_\epsilon\hat{\epsilon})\hat{g}^{0\dagger}\cdot t, \\ &i\partial_{\epsilon^*}\hat{g} = \theta(\hat{g})\hat{g},\ \ \theta(\hat{g}) = \theta(\hat{g}^0) + \hat{g}^0(\partial_{\epsilon^*}\hat{\epsilon})\hat{g}^{0\dagger}\cdot t \end{aligned}\right\}. \qquad (14)$$

Upon the introduction of $E = \sum_{\alpha=1}^{m}\epsilon_\alpha(\epsilon,\epsilon^*)$, the canonicity condition (11) transforms into

$$\left.\begin{aligned} \langle\hat{g}|\partial_\epsilon|\hat{g}\rangle &= \langle\hat{g}^0|\partial_\epsilon|\hat{g}^0\rangle \\ -i\partial_\epsilon E\cdot t &= \frac{1}{2}\epsilon^* - i\partial_\epsilon E\cdot t, \\ \langle\hat{g}|\partial_{\epsilon^*}|\hat{g}\rangle &= \langle\hat{g}^0|\partial_{\epsilon^*}|\hat{g}^0\rangle \\ -i\partial_{\epsilon^*}E\cdot t &= -\frac{1}{2}\epsilon - i\partial_{\epsilon^*}E\cdot t \end{aligned}\right\}. \qquad (15)$$

From Eq. (15), the weak orthogonality condition (13) is expressed as

$$\begin{aligned} 1 &= \partial_{\epsilon^*}\langle\hat{g}|\partial_\epsilon|\hat{g}\rangle - \partial_\epsilon\langle\hat{g}|\partial_{\epsilon^*}|\hat{g}\rangle \qquad (16)\\ &= \partial_{\epsilon^*}\langle\hat{g}^0|\partial_\epsilon|\hat{g}^0\rangle - \partial_\epsilon\langle\hat{g}^0|\partial_{\epsilon^*}|\hat{g}^0\rangle \\ &= \langle\hat{g}^0|[X_{\theta(\hat{g}^0)}, X_{\theta^\dagger(\hat{g}^0)}]|\hat{g}^0\rangle. \end{aligned}$$

To satisfy the integrability conditions for ϵ, ϵ^*, and t, curvatures obtained from (14) should vanish; that is,

$$\left.\begin{aligned} \mathcal{C}_{t,\epsilon} &= D_t\theta^\dagger(\hat{g}) - i\partial_\epsilon\mathcal{F}(\hat{g}) + [\theta^\dagger(\hat{g})\ ,\ \mathcal{F}(\hat{g})] = 0, \\ \mathcal{C}_{t,\epsilon^*} &= D_t\theta(\hat{g}) - i\partial_{\epsilon^*}\mathcal{F}(\hat{g}) + [\theta(\hat{g})\ ,\ \mathcal{F}(\hat{g})] = 0, \\ \mathcal{C}_{\epsilon,\epsilon^*} &= i\partial_\epsilon\theta(\hat{g}) - i\partial_{\epsilon^*}\theta^\dagger(\hat{g}) + [\theta(\hat{g})\ ,\ \theta^\dagger(\hat{g})] = 0 \end{aligned}\right\}, \qquad (17)$$

and $\partial_t\hat{g}^0 = 0$. Here, $D_t\theta$ and $D_t\theta^\dagger$ are defined as

$$\begin{aligned} (D_t\theta)_r &= D_{r;t}\theta_r = (i\partial_t + r\omega_c)\theta_r, \qquad (18)\\ (D_t\theta^\dagger)_r &= D_{r;t}\theta^\dagger_{-r} = (i\partial_t + r\omega_c)\theta^\dagger_{-r}. \end{aligned}$$

The expressions for the curvatures on the QPF are the same forms as those of the RPA equations in the finite Fock space [9]. The TDHF equation on F_∞ leads to the RPA equation if we take into account only a small fluctuation around a stationary ground-state solution. Thus, the curvature equation in the QPF is regarded as the formal RPA equation on the infinite-dimensional Grassmannian.

According to [9], Eq. (14) is rewritten on the above QPF as

$$\left.\begin{aligned} -D_t\hat{g}^\dagger &= \mathcal{F}(\hat{g}^\dagger)|_{\rm QPF}\hat{g}^\dagger, \mathcal{F}(\hat{g}^\dagger)|_{\rm QPF} \stackrel{d}{=} \hat{g}^\dagger\mathcal{F}(\hat{g})\hat{g}, \\ -i\partial_\epsilon\hat{g}^\dagger &= \theta^\dagger(\hat{g}^\dagger)|_{\rm QPF}\hat{g}^\dagger, \theta^\dagger(\hat{g}^\dagger)|_{\rm QPF} \stackrel{d}{=} \hat{g}^\dagger\theta^\dagger(\hat{g})\hat{g}, \\ -i\partial_{\epsilon^*}\hat{g}^\dagger &= \theta(\hat{g}^\dagger)|_{\rm QPF}\hat{g}^\dagger, \theta(\hat{g}^\dagger)|_{\rm QPF} \stackrel{d}{=} \hat{g}^\dagger\theta(\hat{g})\hat{g} \end{aligned}\right\}. \tag{19}$$

For Eq. (17), we also obtain another expression on this QPF:

$$\left.\begin{aligned} (D_t\theta^\dagger - i\partial_\epsilon\mathcal{F} - [\theta^\dagger,\ \mathcal{F}])|_{\rm QPF} = 0, \\ (D_t\theta - i\partial_{\epsilon^*}\mathcal{F} - [\theta\ ,\ \mathcal{F}])|_{\rm QPF} = 0, \\ (i\partial_\epsilon\theta - i\partial_{\epsilon^*}\theta^\dagger - [\theta\ ,\ \theta^\dagger])|_{\rm QPF} = 0 \end{aligned}\right\}. \tag{20}$$

Further, using (19) and the relation

$$\begin{aligned} i\partial_\epsilon\mathcal{F}|_{\rm QPF} &= i\partial_\epsilon(\hat{g}^\dagger\mathcal{F}(\hat{g})\hat{g}) \\ &= -[\theta^\dagger,\ \mathcal{F}]|_{\rm QPF} + \hat{g}^\dagger i\partial_\epsilon\mathcal{F}\hat{g}, \end{aligned}$$

one can rewrite equations in the first line of (20) as

$$\begin{aligned} D_t\theta^\dagger|_{\rm QPF} - \hat{g}^\dagger i\partial_\epsilon\mathcal{F}(\hat{g})\hat{g} = 0, \\ D_t\theta|_{\rm QPF} - \hat{g}^\dagger i\partial_{\epsilon^*}\mathcal{F}(\hat{g})\hat{g} = 0. \end{aligned} \tag{21}$$

From Eqs. (19) and (14), the infinitesimal operators are expressed as

$$\left.\begin{aligned} \theta^\dagger(\hat{g}^\dagger)|_{\rm QPF} &= -i\partial_\epsilon\hat{g}^\dagger\cdot\hat{g} \\ &= e^{i\hat{\epsilon}t}\left\{\partial_\epsilon\hat{\epsilon}\cdot t + \theta^\dagger(\hat{g}^{0\dagger})|_{\rm QPF}\right\}e^{-i\hat{\epsilon}t}, \\ \theta(\hat{g}^\dagger)|_{\rm QPF} &= -i\partial_{\epsilon^*}\hat{g}^\dagger\cdot\hat{g} \\ &= e^{i\hat{\epsilon}t}\{\partial_{\epsilon^*}\hat{\epsilon}\cdot t + \theta(\hat{g}^{0\dagger})|_{\rm QPF}\}e^{-i\hat{\epsilon}t} \end{aligned}\right\}, \tag{22}$$

where $\theta^\dagger(\hat{g}^{0\dagger})|_{\rm QPF} = -i\partial_\epsilon\hat{g}^{0\dagger}\cdot\hat{g}^0$ and $\theta(\hat{g}^{0\dagger})|_{\rm QPF} = -i\partial_{\epsilon^*}\hat{g}^{0\dagger}\cdot\hat{g}^0$. Then, from (21), we can derive the formal RPA equation on the infinite-dimensional Grassmannian in the form

$$\left.\begin{aligned} \omega_c\Gamma\left\{\theta^\dagger(\hat{g}^{0\dagger})|_{\rm QPF}\right\} + i\partial_\epsilon\hat{\epsilon} - [\hat{\epsilon}\ ,\ \theta^\dagger(\hat{g}^{0\dagger})|_{\rm QPF}] - i\hat{g}^{0\dagger}\partial_\epsilon\mathcal{F}(\hat{g}^0)\hat{g}^0 = 0, \\ \Gamma\left\{\theta^\dagger(\hat{g}^{0\dagger})|_{\rm QPF}\right\} \stackrel{d}{=} \begin{bmatrix} \ddots & & & & & \ddots \\ & -\theta_1^{0\dagger} & 0 & \theta_{-1}^{0\dagger} & & \\ & & -\theta_1^{0\dagger} & 0 & \theta_{-1}^{0\dagger} & \\ & & & -\theta_1^{0\dagger} & 0 & \theta_{-1}^{0\dagger} \\ \ddots & & & & & \ddots \end{bmatrix}_{\rm QPF} \end{aligned}\right\}, \tag{23}$$

where $\hat{\epsilon}$ is a diagonal QP energy matrix. We introduce the auxiliary density matrix $\widehat{R} = \hat{g}^0\widehat{I}_{m\otimes(n-m)}\hat{g}^{0\dagger}$. Let $\widehat{I}$ be an infinite-dimensional identity matrix. The matrix $\widehat{R}$ is related to the density matrix W_r as $\widehat{R} = \widehat{I} - 2\widehat{W}$, where

$$\widehat{W} = W(\hat{g}) \stackrel{d}{=} \begin{bmatrix} \ddots & & & & & \ddots \\ & W_{-1} & W_0 & W_1 & & \\ & & W_{-1} & W_0 & W_1 & \\ & & & W_{-1} & W_0 & W_1 \\ \ddots & & & & & \ddots \end{bmatrix}. \tag{24}$$

Then, we get

$$i\partial_\epsilon\widehat{W} = -\frac{1}{2}\hat{g}^0\Big\{-i\partial_\epsilon\hat{g}^{0\dagger}\cdot\hat{g}^0\widehat{I}_{m\otimes(n-m)} - \widehat{I}_{m\otimes(n-m)}(-i\partial_\epsilon\hat{g}^{0\dagger}\cdot\hat{g}^0)\Big\}\hat{g}^{0\dagger} - \frac{1}{2}\hat{g}^0[\,\theta^\dagger(\hat{g}^{0\dagger})|_{\rm QPF},\ \widehat{I}_{m\otimes(n-m)}]\,\hat{g}^{0\dagger}. \tag{25}$$

Further, we introduce the quantities

$$\theta_r^{0\dagger}|_{\rm QPF} \stackrel{d}{=} \begin{bmatrix} \xi_r^0 & \phi_r^0 \\ \psi_r^0 & \bar{\xi}_r^0 \end{bmatrix}_{\rm QPF}, \tag{26}$$

$$B_r^\dagger \stackrel{d}{=} -\frac{1}{2}[\,\theta_r^{0\dagger}|_{\rm QPF}\ ,$$

$$I_{m\otimes(n-m)}] = \begin{bmatrix} 0 & -\phi_r^0 \\ \psi_r^0 & 0 \end{bmatrix}.$$

Using these, we rewrite Eq. (25) as

$$\left.\begin{aligned} i\partial_\epsilon \widehat{W} &= \hat{g}^0 \widehat{B}^\dagger|_{\mathrm{QPF}} \hat{g}^{0\dagger}, \\ i\partial_\epsilon W_r &= \sum_{k,l\in\mathbb{Z}} g^0_k B^\dagger_{k-l-r}|_{\mathrm{QPF}} g^{0\dagger}_l \\ &= \sum_{k,l\in\mathbb{Z}} g^0_k \begin{bmatrix} 0 & -\phi^0_{k-l-r} \\ \psi^0_{k-l-r} & 0 \end{bmatrix} g^{0\dagger}_l \end{aligned}\right\}, \qquad (27)$$

where $\widehat{B}^\dagger|_{\mathrm{QPF}}$ is defined in the same form as $\widehat{W}$, but it has the components $\widehat{B}^\dagger_r|_{\mathrm{QPF}}$ instead of the components W_r.

Let a ($\bar{a}$) and i ($\bar{i}$) be, respectively, $1,\ldots,m$ and $m+1,\ldots,n$ states of the QPF. Substituting the second equation of (27) into (6), we get the following equation for $r \neq 0$:

$$\begin{aligned} i\partial_\epsilon(\mathcal{F}_r)_{\alpha\beta} &= [\alpha\beta|\gamma\delta] \qquad (28) \\ &\times \sum_{k,l\in\mathbb{Z}} \Big\{ (g^0_k)_{\delta i} (g^{0\dagger}_l)_{a\gamma} (\psi^0_{k-l-r})_{ia} \\ &- (g^0_k)_{\delta a} (g^{0\dagger}_l)_{i\gamma} (\phi^0_{k-l-r})_{ai} \Big\}. \end{aligned}$$

Thus, we can obtain the desired form of the equation, part of the formal RPA equation on the infinite-dimensional Grassmannian (23),

$$i(\hat{g}^{0\dagger}\cdot\partial_\epsilon\mathcal{F}\cdot\hat{g}^0)_r = \sum_{k,l\in\mathbb{Z}} g^{0\dagger}_k \cdot i\partial_\epsilon \mathcal{F}_{k-l+r} \cdot g^0_l \qquad (29)$$

$$\begin{aligned} &= \sum_{k,l\in\mathbb{Z},\ \bar{k},\bar{l}\in\mathbb{Z}} \left[\begin{array}{l} \left[\begin{matrix} kl & |\mathbf{F}| & \bar{k}\bar{l} \\ ab & & \bar{i}\bar{a} \end{matrix} \right] (\psi^0_{(\bar{k}-\bar{l})-(k-l)-r})_{\bar{i}\bar{a}} - \left[\begin{matrix} kl & |\overline{\mathbf{F}}| & \bar{k}\bar{l} \\ ab & & \bar{a}\bar{i} \end{matrix} \right] (\phi^0_{(\bar{k}-\bar{l})-(k-l)-r})_{\bar{a}\bar{i}}, \\ \left[\begin{matrix} kl & |\mathbf{D}| & \bar{k}\bar{l} \\ ia & & \bar{i}\bar{a} \end{matrix} \right] (\psi^0_{(\bar{k}-\bar{l})-(k-l)-r})_{\bar{i}\bar{a}} - \left[\begin{matrix} kl & |\overline{\mathbf{D}}| & \bar{k}\bar{l} \\ ia & & \bar{a}\bar{i} \end{matrix} \right] (\phi^0_{(\bar{k}-\bar{l})-(k-l)-r})_{\bar{a}\bar{i}}, \\ \left[\begin{matrix} kl & |\mathbf{D}| & \bar{k}\bar{l} \\ ai & & \bar{i}\bar{a} \end{matrix} \right] (\psi^0_{(\bar{k}-\bar{l})-(k-l)-r})_{\bar{i}\bar{a}} - \left[\begin{matrix} kl & |\overline{\mathbf{D}}| & \bar{k}\bar{l} \\ ai & & \bar{a}\bar{i} \end{matrix} \right] (\phi^0_{(\bar{k}-\bar{l})-(k-l)-r})_{\bar{a}\bar{i}} \\ \left[\begin{matrix} kl & |\mathbf{F}| & \bar{k}\bar{l} \\ ij & & \bar{i}\bar{a} \end{matrix} \right] (\psi^0_{(\bar{k}-\bar{l})-(k-l)-r})_{\bar{i}\bar{a}} - \left[\begin{matrix} kl & |\overline{\mathbf{F}}| & \bar{k}\bar{l} \\ ij & & \bar{a}\bar{i} \end{matrix} \right] (\phi^0_{(\bar{k}-\bar{l})-(k-l)-r})_{\bar{a}\bar{i}} \end{array} \right]. \end{aligned}$$

Substituting the above result into (23), we can derive the formal RPA equation on F_∞.

4. SUMMARY AND CONCLUDING REMARKS

The formal RPA equation has been provided as a tool for truncating a collective submanifold with only one normal mode out of the infinite-dimensional Grassmannian. We have given a simple geometric interpretation for the formal RPA equation. The collective submanifold is interpreted as a rotator on a curved surface in the infinite-dimensional Grassmannian. In F_∞, to study motions of finite fermion systems, it is manifestly natural and useful to introduce the infinite-dimensional Lie algebra arising from the anticommutation relation between fermions. In order to discuss the relation between TDHF theory and soliton theory, we have given expressions for TDHF theory on the τ-functional space along soliton theory. From the loop-group viewpoint and with a clearer physical picture, we have proposed descriptions of particle and collective motions in SCF theory on F_∞ in relation to the isospectral equation in soliton theory. Then, SCF theory on F_∞ may be regarded as soliton theory in the sense that it is based on the infinite-dimensional Grassmannian and may describe dynamics on an infinite set of real fermion-harmonic oscillators.

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Classical and Algebraic Hamiltonian with $su(3)$ Symmetry*

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Abstract—We use Gelfand–Zetlin patterns to obtain the coherent state for an arbitrary symmetric irreducible representation of $su(3)$. The semiclassical evolution of a dynamical system whose Hamiltonian contains the Casimir operators of both $su(2)$ and $so(3)$ subalgebras is investigated, and it is concluded that the presence of a common operator in the subalgebras induces integrability despite the absence of dynamical symmetry. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

An interesting way to approach quantum chaos is the coherent-state method, which allows one to write a classical Hamiltonian from an algebraic one. The coherent-state calculus requires three inputs: a Lie group G and its algebra g; a Hilbert space h, which is an irreducible representation of g; and a ground state $|0\rangle$, which is the lowest state of h. In this paper, we are concerned with the $su(3)$ Lie algebra, which can be generated, according to the Cartan–Weyl scheme, by eight operators

$$\{H_1, H_2, E_{12}, E_{21}, E_{23}, E_{32}, E_{13}, E_{31}\},$$

$$H_1 = \frac{1}{2}(E_{11} - E_{22}),$$
$$H_2 = \frac{1}{2}(E_{22} - E_{33}),$$
$$E_{i,j} = E_{j,i}^{\dagger}.$$

Knowing that $(E_{i,j})_{A,B} = \delta_{i,A}\delta_{j,B}$, we have the commutation relations

$$[E_{i,j}, E_{k,l}] = \delta_{j,k}E_{i,l} - \delta_{i,l}E_{k,j}.$$

Each irreducible representation of $su(3)$ can be labeled with two integer numbers j and k, and a general element $|m\rangle$ of this vector space is given by the so-called Gelfand–Zetlin pattern,

$$|m\rangle = \left| \begin{array}{ccc} m_{13} & m_{23} & 0 \\ & m_{12} \quad m_{22} & \\ & m_{11} & \end{array} \right\rangle,$$

the ground state being

$$|0\rangle = \left| \begin{array}{ccc} m_{13} & m_{23} & 0 \\ & m_{12} \quad 0 & \\ & 0 & \end{array} \right\rangle.$$

The action of the algebra generators is given by [1]

$$E_{k,k}\,|m\rangle = (r_k - r_{k-1})\,|m\rangle\,,$$
$$E_{k,k-1}\,|m\rangle = \sum_{j=1}^{k-1} a_{k-1}^{j}(m)\,|m_{j,k-1}\rangle\,,$$
$$E_{k-1,k}\,|m\rangle = \sum_{j=1}^{k-1} b_{k-1}^{j}(m)\,|\hat{m}_{j,k-1}\rangle\,,$$
$$E_{k,k+2} = [E_{k,k+1}, E_{k+1,k+2}],$$

where

$$a_{k-1}^{j}(m) = \left[-\frac{\prod_{i=1}^{k}(l_{i,k} - l_{j,k-1} + 1)\prod_{i=1}^{k-2}(l_{i,k-2} - l_{j,k-1})}{\prod_{i\neq j}(l_{i,k-1} - l_{j,k-1} + 1)(l_{i,k-1} - l_{j,k-1})} \right]^{1/2},$$

$$b_{k-1}^{j}(m) = \left[-\frac{\prod_{i=1}^{k}(l_{i,k} - l_{j,k-1})\prod_{i=1}^{k-2}(l_{i,k-2} - l_{j,k-1} - 1)}{\prod_{i\neq j}(l_{i,k-1} - l_{j,k-1})(l_{i,k-1} - l_{j,k-1} - 1)} \right]^{1/2},$$

$$r_0 = 0, \quad r_k = \sum_{i=1}^{k} m_{j,i},$$
$$l_{i,k} = m_{i,k} - i,$$

and $|m_{j,k-1}\rangle$ ($|\hat{m}_{j,k-1}\rangle$) is obtained from $|m\rangle$ by replacing $m_{j,k-1}$ by $m_{j,k-1} - 1$ ($m_{j,k-1} + 1$).

To calculate the coherent state, we must first distinguish between the symmetric ($m_{23} = 0$ or $m_{23} =$

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m_{13}) and nonsymmetric ($0 < m_{23} < m_{13}$) representations of $su(3)$. In the first case, the coherent state is obtained from the action of an element of the coset space $SU(3)/U(2)$ on the vacuum state. Such an element can be generally written as

$$\exp\{\eta_1 E_{13} + \eta_2 E_{23} - \eta_1^* E_{31} - \eta_2^* E_{32}\}, \quad (1)$$

so that the coherent state $|\eta^s\rangle$ will be simply

$$|\eta^s\rangle = \exp\{\eta_1 E_{13} + \eta_2 E_{23} - \eta_1^* E_{31} - \eta_2^* E_{32}\}|0\rangle. \quad (2)$$

For the nonsymmetric representations, the relevant coset space is $SU(3)/[U(1)\otimes U(1)]$ and the coherent state is given by

$$|\eta^n\rangle = \exp\{\eta_1 E_{12} + \eta_2 E_{13} + \eta_3 E_{23} - \eta_1^* E_{21} - \eta_2^* E_{31} - \eta_3^* E_{32}\}|0\rangle. \quad (3)$$

In this paper, we are just concerned with the symmetric case, and the calculations for the nonsymmetric one will be presented elsewhere.

2. COSET SPACE

For the symmetric representations (we consider just $m_{23} = k = 0$ and write $m_{13} = j$), the relevant coset space is $SU(3)/U(2)$. Its general element is $C = \exp\{\eta_1 E_{13} + \eta_2 E_{23} - \eta_1^* E_{31} - \eta_2^* E_{32}\}$.

To calculate the coherent state of an arbitrary representation, we must perform the Gaussian decomposition

$$C = \exp\{\alpha E_{12} + \beta E_{13} + \delta E_{23}\} \times \exp\{2h_1 H_1 + 2h_2 H_2\} \times \exp\{\tilde{\alpha} E_{21} + \tilde{\beta} E_{31} + \tilde{\delta} E_{32}\}, \quad (4)$$

where $\tilde{\alpha}(\eta_1, \eta_2) = \alpha(-\eta_1^*, -\eta_2^*)$ and the analogous relations hold for β and δ. The coherent state

$$|\eta; j\rangle = C\,|0(j)\rangle$$

can then be written as

$$|\eta; j\rangle = e^{\{\alpha E_{12} + \beta E_{13} + \delta E_{23}\}}\,|0(j)\rangle\, e^{-jh_1}. \quad (5)$$

We now use the Baker–Campbell–Hausdorff formula

$$e^{A+B} = e^A e^B e^{-[A,B]/2},$$

which is valid if and only if $[A,[A,B]] = [B,[A,B]] = 0$, to decompose the exponential in (5):

$$\exp\{\alpha E_{12} + \beta E_{13} + \delta E_{23}\} = \exp\left\{\left(\beta - \frac{\alpha\delta}{2}\right) E_{13}\right\} \exp\{\alpha E_{12}\} \exp\{\delta E_{23}\}. \quad (6)$$

This way, it is easy to see that

$$|\eta; j\rangle = \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{p=0}^{\min(j-k,k-l)} \delta^k \alpha^l \times \left(\beta - \frac{\alpha\delta}{2}\right)^p (\cos|\xi|)^j \times \sqrt{\binom{j}{k}\binom{k}{l}\binom{j-k}{p}\binom{k-l}{p}} \left| \begin{matrix} j & 0 & 0 \\ k+p & 0 & \\ & l+p & \end{matrix} \right\rangle.$$

The coefficients α, β, and δ depend only on η_1 and η_2 and not on j [2]. Therefore, to determine them, we just have to write C in the fundamental representation ($j = 1$), a three-dimensional vector space, in which case we have

$$C = \exp\left\{\begin{pmatrix} 0 & 0 & \eta_1 \\ 0 & 0 & \eta_2 \\ -\eta_1^* & -\eta_2^* & 0 \end{pmatrix}\right\} = \frac{1}{|\eta|^2} \begin{pmatrix} |\eta_1|^2 \cos|\eta| + |\eta_2|^2 & \eta_2^*\eta_1(\cos|\eta| - 1) & \eta_1 |\eta| \sin|\eta| \\ \eta_2\eta_1^*(\cos|\eta| - 1) & |\eta_2|^2 \cos|\eta| + |\eta_1|^2 & \eta_2 |\eta| \sin|\eta| \\ -\eta_1^* |\eta| \sin|\eta| & -\eta_2^* |\eta| \sin|\eta| & |\eta|^2 \cos|\eta| \end{pmatrix},$$

where $|\eta|^2 = |\eta_1|^2 + |\eta_2|^2$. On the other hand, the Gauss decomposition gives

$$C = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \delta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{h_1} & 0 & 0 \\ 0 & e^{h_2 - h_1} & 0 \\ 0 & 0 & e^{-h_2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \tilde{\alpha} & 1 & 0 \\ \tilde{\beta} & \tilde{\delta} & 1 \end{pmatrix};$$

hence, we get

$$h_2 = -\ln\{\cos|\xi|\},$$

$$h_1 = -\ln\left\{\frac{|\sigma|^2 \cos|\xi| + |\eta|^2}{|\xi|^2}\right\},$$

$$\delta = \frac{\eta \sin|\xi|}{|\xi| \cos|\xi|},$$

$$\beta + \frac{\alpha\delta}{2} = \frac{\sigma \sin|\xi|}{|\xi| \cos|\xi|},$$

$$\alpha = -\frac{\eta^* \sigma(\cos|\xi| - 1)}{|\sigma|^2 \cos|\xi| + |\eta|^2}.$$

3. CLASSICAL DYNAMICS

Once we have the coherent state of a symmetric representation, we consider the mean value of some algebraic Hamiltonian H in this state to be its classical counterpart $H_{\rm cl}$; that is,

$$H_{\rm cl}(q_1,p_1,q_2,p_2) = \langle\eta|\, H\,|\eta\rangle\,, \tag{7}$$

the relation between the classical degrees of freedom q_i and p_i and the parameters η_i being [2]

$$\frac{q_i + ip_i}{\sqrt{2j}} = \frac{\eta_i \sin|\eta|}{|\eta|}. \tag{8}$$

We now follow [3] and consider two symmetry-breaking chains for the $su(3)$ algebra,

$$su(3) \supset su(2), \tag{9}$$

$$su(3) \supset so(3), \tag{10}$$

and build a Hamiltonian involving the Casimir operators of the two chains:

$$H = (1-\varepsilon)L^2 + \varepsilon X^2. \tag{11}$$

This Hamiltonian has the dynamical symmetry of chain (9) when $\varepsilon = 0$ and the dynamical symmetry of chain (10) when $\varepsilon = 1$. When $0 < \varepsilon < 1$, the system has no dynamical symmetry.

Two subalgebras are generated by the operators

$$E_{12} \equiv L_+, \quad E_{21} \equiv L_-, \quad (E_{11} - E_{22})/2 \equiv L_z,$$
$$i(E_{12} - E_{21}) \equiv X_z, \quad i(E_{13} - E_{31}) \equiv X_x,$$
$$i(E_{32} - E_{23}) \equiv X_y,$$

and the operators involved in (11), which do not commute, are

$$L^2 = \frac{1}{4}(E_{11} - E_{22})^2 + \frac{1}{2}(E_{12}E_{21} + E_{21}E_{12}) \tag{12}$$

for the $su(2)$ algebra and

$$X^2 = (E_{12} - E_{21})^2 + (E_{23} - E_{32})^2 + (E_{13} - E_{31})^2 \tag{13}$$

for the $so(3)$ algebra. For the smallest irreducible representations, the classical counterparts of these Casimir operators are

$$\langle\eta;1|\, L^2\,|\eta;1\rangle = \frac{3}{16}(q_1^2 + p_1^2 + q_2^2 + p_2^2),$$
$$\langle\eta;1|\, X^2\,|\eta;1\rangle = 2,$$
$$\langle\eta;2|\, L^2\,|\eta;2\rangle = \frac{1}{32}(q_1^2 + p_1^2 + q_2^2 + p_2^2 + 12) \times (q_1^2 + p_1^2 + q_2^2 + p_2^2),$$
$$\langle\eta;2|\, X^2\,|\eta;2\rangle = \frac{1}{2}\left[(p_1^2 + p_2^2)(p_1^2 + p_2^2 - 4) + (q_1p_1 + q_2p_2)^2 - 8\right].$$

We see that, in the fundamental representation, our system is nothing but a two-dimensional harmonic oscillator and is therefore always integrable, irrespective of the value of ε. In fact, integrability will hold for any symmetric irreducible representation and for any value of ε. To understand why this is so, we must recall that, under the Poisson bracket $\{,\}$ operation, the mean values of the algebra generators in the coherent state have the same structure constants as the algebra itself [4], for instance,

$$\{\langle L_+\rangle_\eta, \langle L_-\rangle_\eta\} = 2\langle L_z\rangle_\eta,$$
$$\{\langle L_z\rangle_\eta, \langle L_\pm\rangle_\eta\} = \pm\langle L_\pm\rangle_\eta,$$
$$\{\langle X_j\rangle_\eta, \langle X_k\rangle_\eta\} = \epsilon_{jkl}\langle X_l\rangle_\eta.$$

Since the operator $L_y = X_z = i(E_{12} - E_{21})$ belongs to both subalgebras, it has the vanishing Poisson bracket with the total Hamiltonian for any value of ε:

$$\langle L_y\rangle_\eta = \langle X_x\rangle_\eta = F(\mathbf{q},\mathbf{p}),$$
$$\{F(\mathbf{q},\mathbf{p}), H\} = (1-\varepsilon)\{F(\mathbf{q},\mathbf{p}), \langle L^2\rangle_\eta\} + \varepsilon\{F(\mathbf{q},\mathbf{p}), \langle X^2\rangle_\eta\} = 0;$$

as a second constant of the motion, it ensures integrability.

We conclude that the presence of a common operator in both subalgebras induces integrability, although our Hamiltonian has no dynamical symmetry, since we have a situation where two noncommuting matrices (L^2 and X^2) both commute with a third one. This is in disagreement with [3], where the authors claim to have established a correspondence between integrability and dynamical symmetry. Since, in $su(3)$, it is not possible to find two maximal subalgebras that are both disjoint and nonisomorphic, it is not possible to find a Hamiltonian with a chaotic behavior in this context, at least for the symmetric representations. The analysis for the nonsymmetric ones is still under way.

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The Coulomb–Oscillator Relation on n-Dimensional Spheres and Hyperboloids*

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Abstract—We establish a relation between Coulomb and oscillator systems on n-dimensional spheres and hyperboloids for $n \geq 2$. We show that, as in Euclidean space, the quasiradial equation for the $(n+1)$-dimensional Coulomb problem coincides with the $2n$-dimensional quasiradial oscillator equation on spheres and hyperboloids. Using the solution of the Schrödinger equation for the oscillator system, we construct the energy spectrum and wave functions for the Coulomb problem. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

It has long been known that the Coulomb and oscillator potentials are two paradigms in quantum mechanics that possess dynamical or hidden symmetries: $O(n+1)$ for motion in a Coulomb field [1] and $SU(n)$ for an oscillator. On the other hand, the connections with these two Lie groups of dynamical symmetries provide relations between the Coulomb and oscillator systems. In particular, the $(n+1)$-dimensional radial Schrödinger equation for the Coulomb system is identical to the oscillator equation for $2n$ dimensions by the duality transformation [2]. It is also known that the complete relation (not only for the radial part) is possible only for special dimensions of (2, 2), (3, 4), and (5, 8). The dual mappings in these cases are so-called Levi-Civita, Kustaanheimo–Stiefel, and Hurwitz transformations [3–5].

The generalization of the Coulomb problem to a three-sphere was performed in the famous article of Schrödinger [6]; for the n-dimensional hyperboloid, this problem was solved in [7]. Later, the Coulomb and the oscillator problem on spheres and pseudospheres were discussed from many points of view in [8–19].

In [20], we constructed a series of complex mappings $S_{2C} \to S_2$, $S_{4C} \to S_3$, and $S_{8C} \to S_5$, which extend to spherical geometry the Levi-Civita, Kustaanheimo–Stiefel, and Hurwitz transformations, which are well known for Euclidean space. We showed that these transformations establish a correspondence between Coulomb and oscillator problems in classical and quantum mechanics for dimensions of (2, 2), (3, 4), and (5, 8) on the spheres. A detailed analysis of the real mapping on a curved space was performed in [21]. It was shown that, in the stereographic projection (see also [22]), the relation between Coulomb and oscillator problems functionally coincide with the flat-space Levi-Civita and Kustaanheimo–Stiefel relations.

In the present paper, we find the relation between the quasiradial Schrödinger equations for Coulomb and oscillator problems on an n-dimensional sphere and one- and two-sheeted hyperboloids for $n \geq 2$.

2. COULOMB–OSCILLATOR RELATION ON n-SPHERE

The Schrödinger equation describing a nonrelativistic quantum motion on the n-dimensional sphere $s_0^2 + s_1^2 + \cdots + s_n^2 = R^2$, where s_i are Cartesian coordinates in the ambient $(n+1)$-dimensional Euclidean space, has the form ($\hbar = \mu = 1$)

$$\mathcal{H}\Psi = \left[-\frac{1}{2}\Delta_{\mathrm{LB}} + V(\mathbf{s})\right]\Psi = E\Psi, \qquad (1)$$

where the Laplace–Beltrami operator in arbitrary curvilinear coordinates ξ_μ is

$$\Delta_{\mathrm{LB}} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_\mu}g^{\mu\nu}\sqrt{g}\frac{\partial}{\partial\xi_\nu}, \qquad (2)$$
$$g = \det||g_{\mu\nu}||, \quad g_{\alpha\mu}g^{\mu\nu} = \delta_\alpha^\nu.$$

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For any central potential $V(\chi)$, the Schrödinger equation admits separation of variables in the hyperspherical coordinates that are specified as

$$\begin{aligned} s_0 &= R\cos\chi, \\ s_1 &= R\sin\chi\cos\vartheta_1, \\ s_2 &= R\sin\chi\sin\vartheta_1\cos\vartheta_2, \\ &\dots \\ s_{n-1} &= R\sin\chi\sin\vartheta_1\sin\vartheta_2\dots\sin\vartheta_{n-2}\cos\varphi, \\ s_n &= R\sin\chi\sin\vartheta_1\sin\vartheta_2\dots\sin\vartheta_{n-2}\sin\varphi, \end{aligned}$$

where $\chi, \vartheta_1, \dots, \vartheta_{n-2} \in [0, \pi]$ and $\varphi \in [0, 2\pi)$. We can separate the angular part of the wave function using the ansatz

$$\begin{aligned} &\Psi(\chi, \vartheta_1, \dots, \vartheta_{n-2}, \varphi) \\ &= \mathcal{R}(\chi)\, Y_{L,l_1,l_2,l_{n-2}}(\vartheta_1, \dots, \vartheta_{n-2}, \varphi), \end{aligned} \tag{3}$$

where l_i are the angular hypermomenta, L is the total angular momentum, and the hyperspherical function $Y_{L,l_1,l_2,l_{n-2}}(\vartheta_1, \dots, \vartheta_{n-2}, \varphi)$ is a solution of the Laplace–Beltrami eigenvalue equation on an $(n-1)$-dimensional sphere. After the separation of variables in (1), we obtain the quasiradial equation

$$\begin{aligned} &\frac{1}{\sin^{n-1}\chi}\frac{d}{d\chi}\sin^{n-1}\chi\frac{d\mathcal{R}(\chi)}{d\chi} \\ &+\left[2R^2E - \frac{L(L+n-2)}{\sin^2\chi} - 2R^2V(\chi)\right]\mathcal{R}(\chi) = 0. \end{aligned} \tag{4}$$

Using the substitution

$$Z(\chi) = (\sin\chi)^{(n-1)/2}\mathcal{R}(\chi), \tag{5}$$

we find

$$\begin{aligned} &\frac{d^2Z}{d\chi^2} + \Big[\tilde{E} - \frac{(2L+n-1)(2L+n-3)}{4\sin^2\chi} \\ &- 2R^2V(\chi)\Big]Z = 0, \end{aligned} \tag{6}$$

where $\tilde{E} = 2R^2E + (n-1)^2/4$ and the quasiradial wave function $Z(\chi)$ satisfies the normalization condition

$$\int_0^\pi Z(\chi)Z^*(\chi)\, R^n\, d\chi = 1. \tag{7}$$

(i) Let us now consider the n-dimensional oscillator potential [8, 9]

$$\begin{aligned} V(\chi) &= \frac{\omega^2R^2}{2}\frac{s_1^2 + s_2^2 + \dots + s_n^2}{s_0^2} \\ &= \frac{\omega^2R^2}{2}\tan^2\chi. \end{aligned} \tag{8}$$

Substituting the oscillator potential into Eq. (6), we obtain the Pöschl–Teller-type equation

$$\begin{aligned} &\frac{d^2Z}{d\chi^2} + \Big[\epsilon - \frac{\nu^2 - 1/4}{\cos^2\chi} \\ &- \frac{(L + (n-2)/2)^2 - 1/4}{\sin^2\chi}\Big]Z = 0, \end{aligned} \tag{9}$$

where $\nu = \sqrt{\omega^2R^4 + 1/4}$ and $\epsilon = \tilde{E} + \omega^2R^4$. The solution of the above equation that is regular for $\chi \in [0, \pi/2]$ and which is expressed in terms of the hypergeometric function is [23]

$$\begin{aligned} Z(\chi) \equiv Z^n_{n_rL\nu}(\chi) &= \sqrt{\frac{2(2n_r + L + \nu + n/2)\Gamma(n_r + L + \nu + n/2)\Gamma(n_r + L + n/2)}{R^n[\Gamma(L + n/2)]^2\Gamma(n_r + \nu + 1)(n_r)!}}, \\ &\times(\sin\chi)^{L+(n-1)/2}(\cos\chi)^{\nu+1/2}\,{}_2F_1\Big(-n_r, n_r + L + \nu + \frac{n}{2};\, L + \frac{n}{2};\, \sin^2\chi\Big), \end{aligned} \tag{10}$$

and the ϵ is quantized as

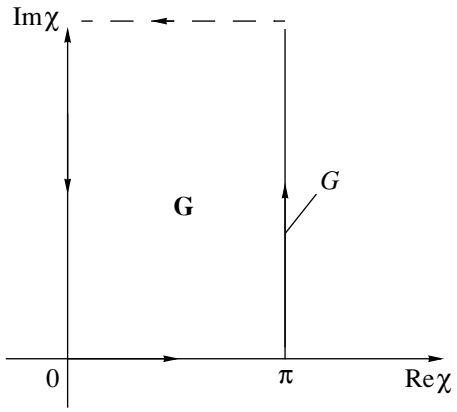
$$\epsilon = \Big(2n_r + L + \nu + \frac{n}{2}\Big)^2, \tag{11}$$

where $n_r + L = 0, 1, 2, \dots$ is a "quasiradial" quantum number. The energy spectrum of the n-dimensional oscillator is given by

$$\begin{aligned} E^n_N(R) = \frac{1}{2R^2}\Big[&(N+1)(N+n) \\ &+ (2\nu - 1)\Big(N + \frac{n}{2}\Big)\Big], \end{aligned} \tag{12}$$

where $N = 2n_r + L = 0, 1, \dots$ is a principal quantum number. In the contraction limit where $R \to \infty$, $\chi \to 0$, and $R\chi \sim r$ is fixed and for $\nu \sim \omega R^2$, we see that

$$\lim_{R\to\infty} E^n_N(R) = \omega\Big(N + \frac{n}{2}\Big) \tag{13}$$



Domain $\mathbf{G} = \{0 \le \mathrm{Re}\chi \le \pi;\ 0 \le \mathrm{Im}\ \chi < \infty\}$ in the complex plane of χ.

and

$$\lim_{R\to\infty} (R)^{(n-1)/2}\, Z^n_{NL\nu}(\chi) \tag{14}$$
$$= \frac{(\omega)^{L/2+n/4}}{\Gamma(L+n/2)} \sqrt{2\Gamma\left(\frac{N+L+n}{2}\right)\Big/\left(\frac{N-L}{2}\right)!}$$
$$\times r^{L+(n-1)/2} e^{-\omega r^2/2}\, {}_1F_1\left(-\frac{N-L}{2},\, L+\frac{n}{2};\, \omega r^2\right).$$

Formula (14) coincides with the known formula for n-dimensional flat radial wave functions [24].

(ii) The potential that is the analog of the Coulomb potential on the n-dimensional sphere has the form [6, 8, 9]

$$V(\chi) = -\frac{\alpha}{R}\frac{s_0}{\sqrt{s_1^2+s_2^2+\ldots+s_n^2}} \tag{15}$$
$$= -\frac{\alpha}{R}\cot\chi.$$

The Schrödinger equation (6) for this potential is

$$\frac{d^2Z}{d\chi^2} + \Big[\tilde{E} - \frac{(2L+n-1)(2L+n-3)}{4\sin^2\chi} \tag{16}$$
$$+ 2\alpha R\cot\chi\Big] Z = 0.$$

We now go over to the new variable $\theta \in [0, \pi/2]$ defined as

$$e^{i\chi} = \cos\theta. \tag{17}$$

This is possible if we continue the variable χ in the complex domain $\mathbf{G}$: Re $\chi = 0$, $0 \le$ Im $\chi < \infty$ (see figure). We also complexify the coupling constant α by introducing $k = i\alpha$ in a such a way that

$$\alpha\cot\chi = k(1 - 2\sin^{-2}\theta). \tag{18}$$

As a result, we obtain the equation

$$\frac{d^2W}{d\theta^2} + \Big[\epsilon - \frac{\nu^2-1/4}{\cos^2\theta} \tag{19}$$
$$- \frac{(2L+n-2)^2-1/4}{\sin^2\theta}\Big] W = 0,$$

where $W(\theta) = (\cot\theta)^{1/2} Z(\theta)$ and

$$\epsilon = \tilde{E} + 2kR, \quad \nu^2 = \tilde{E} - 2kR. \tag{20}$$

From the above equation, we see that, apart from the substitution in (20) and the transformation $L \to 2L$, the quasiradial equation (19) for the $n^{\mathrm{Coul}} = (d+1)$-dimensional Coulomb problem coincides with the $n^{\mathrm{osc}} = 2d$-dimensional quasiradial oscillator Eq. (9). This means that relations between these two systems are possible only for oscillators in even dimensions: $n^{\mathrm{osc}} = 2, 4, 6, 8 \ldots$.

Thus, Eq. (19) describes the $2(n-1)$-dimensional oscillator quasiradial functions with even angular momentum $2L$. According to (10), the regular (for $\theta \in [0, \pi/2]$ and $\nu \le 1/4$) solution to this equation has the form

$$Z(\theta) = \frac{W(\theta)}{\sqrt{\cot\theta}} \equiv Z_{n_rL}(\theta) \tag{21}$$
$$= C^n_{n_rL}(\nu)\,(\sin\theta)^{2L+n-1}\,(\cos\theta)^{\nu}$$
$$\times {}_2F_1(-n_r, n_r+2L+\nu$$
$$+ n-1;\, 2L+n-1;\, \sin^2\theta),$$

where $C^n_{n_rL}(\nu)$ is the normalization constant. To compute the constant $C^n_{n_rL}(\nu)$ for the corresponding Coulomb quasiradial function, we require that the wave function (21) satisfy the normalization condition

$$R^n \int_0^\pi Z_{n_rL}\, Z^\diamond_{n_rL}\, d\chi = 1, \tag{22}$$

where the symbol "$\diamond$" means the complex conjugate together with the inversion $\chi \to -\chi$; i.e., $Z^\diamond(\chi) = Z^*(-\chi)$. [We choose the scalar product as $Z^\diamond$ because, for $\chi \in \mathbf{G}$ and real α and $\tilde{E}$, the function $Z^\diamond(\chi)$ also belongs to the solution space of (16).] By analogy with what was done in [20], we consider the integral along the contour G in the complex plane of variable χ (see figure):

$$\oint Z_{n_rL}(\chi)\, Z^\diamond_{n_rL}(\chi) d\chi \tag{23}$$
$$= \int_0^\pi Z_{n_rL}(\chi)\, Z^\diamond_{n_rL}(\chi) d\chi$$
$$+ \int_\pi^{\pi+i\infty} Z_{n_rL}(\chi)\, Z^\diamond_{n_rL}(\chi) d\chi$$
$$+ \int_{\pi+i\infty}^{i\infty} Z_{n_rL}(\chi)\, Z^\diamond_{n_rL}(\chi) d\chi$$

$$+\int_{i\infty}^{0} Z_{n_rL}(\chi)\, Z^{\diamond}_{n_rL}(\chi)d\chi.$$

Considering that the integrand vanishes in proportion to $e^{2i\nu\chi}$ and that $Z_{n_rL}(\chi)$ is regular in the domain **G** (see figure), we then find, with the aid of the Cauchy theorem, that

$$\int_0^{\pi} Z_{n_rL}(\chi)\, Z^{\diamond}_{n_rL}(\chi)d\chi \qquad (24)$$

$$= \left(1 - e^{2i\pi\nu}\right) \int_0^{i\infty} Z_{n_rL}(\chi)\, Z^{\diamond}_{n_rL}(\chi)d\chi.$$

Making the change of variable according to (17) in the integral on the right-hand side of Eq. (24), we obtain

$$\int_0^{\pi} Z_{n_rL}(\chi)\, Z^{\diamond}_{n_rL}(\chi)d\chi \qquad (25)$$

$$= i\left(1 - e^{2i\pi\nu}\right) \int_0^{\pi/2} [Z_{n_rL}] \tan\theta\, d\theta.$$

After integration with respect to the angle θ, we finally get [24]

$$C^n_{n_rm}(\nu) = \sqrt{\frac{(-2i\nu)(\nu + 2n_r + 2L + n - 1)\,(n_r)!\Gamma(2L + n_r + \nu + n - 1)}{R^n[1 - e^{2i\pi\nu}](2n_r + 2L + n - 1)\,(n_r + 2L + n - 2)!\Gamma(n_r + \nu + 1)}}. \qquad (26)$$

Comparing now Eq. (11) with (20) and setting $k = i\alpha$, we get

$$\nu = -\left(n_r + L + \frac{n-1}{2}\right) + i\sigma, \qquad (27)$$

$$\sigma = \frac{\alpha R}{n_r + L + (n-1)/2},$$

and obtain the energy spectrum for the Coulomb problem,

$$E_n = \frac{N(N+n-1)}{2R^2} - \frac{\alpha^2}{2(N + (n-1)/2)^2}, \quad (28)$$

$$N = n_r + L = 0, 1, 2, \ldots .$$

Returning to the variable χ, we see that the Coulomb quasiradial wave function has the form

$$Z_{NL}(\chi) = C_{NL}(\sigma)\,(\sin\chi)^{L+(n-1)/2} \qquad (29)$$

$$\times \exp[-i\chi(N - L - i\sigma)]$$

$$\times\, {}_2F_1\Big(-N + L, L + \frac{n-1}{2}$$

$$+ i\sigma; 2L + n - 1; 1 - e^{2i\chi}\Big),$$

where the normalization constant $C_{NL}(\sigma)$ is

$$C^n_{NL}(\sigma) = 2^{L+(n-1)/2}\, e^{\pi\sigma/2} \qquad (30)$$

$$\times \frac{|\,\Gamma(L + (n-1)/2 - i\sigma)\,|}{\Gamma(2L + n - 1)}$$

$$\times \sqrt{\frac{[(N + (n-1)/2)^2 + \sigma^2](N + L + n - 2)!}{2R^n\pi(N + (n-1)/2)(N - L)!}}.$$

Thus, by using the relation between Coulomb and oscillator systems, we have constructed the quasiradial wave functions and the energy spectrum for a Coulomb system on an n-dimensional sphere.

Finally, we note that, in the contraction limit $R \to \infty$ (for details, see [14]), it is easy to recover the well-known formulas for the flat-space n-dimensional Coulomb problem both for the discrete and for the continuous spectrum [1].

3. COULOMB–OSCILLATOR RELATION ON AN n-DIMENSIONAL TWO-SHEETED HYPERBOLOID

The pseudospherical coordinates on the n-dimensional two-sheeted hyperboloid $s_0^2 - s_1^2 - s_2^2 - \ldots - s_n^2 = R^2$, $s_0 \geq R$, are

$$s_0 = R\cosh\tau,$$

$$s_1 = R\sinh\tau\cos\vartheta_1,$$

$$s_2 = R\sinh\tau\sin\vartheta_1\cos\vartheta_2,$$

$$\ldots$$

$$s_{n-1} = R\sinh\tau\sin\vartheta_1\sin\vartheta_2\ldots\sin\vartheta_{n-2}\cos\varphi,$$

$$s_n = R\sinh\tau\sin\vartheta_1\sin\vartheta_2\ldots\sin\vartheta_{n-2}\sin\varphi,$$

where $\tau \in [0, \infty)$. The variables in the Schrödinger Eq. (1) may be separated for any central potential $V(\tau)$ by the ansatz

$$\Psi(\tau, \vartheta_1, \ldots, \vartheta_{n-2}, \varphi) \qquad (31)$$

$$= \mathcal{R}(\tau)\, Y_{L,l_1,l_2,l_{n-2}}(\vartheta_1, \ldots, \vartheta_{n-2}, \varphi),$$

where, as in the preceding case, l_i are the angular hypermomenta and L is the total angular momentum; the hyperspherical function $Y_{L,l_1,l_2,l_{n-2}}(\vartheta_1,\ldots,\vartheta_{n-2},\varphi)$ is a solution of the Laplace–Beltrami equation on the $(n-1)$-dimensional sphere. After the separation of variables, we obtain the quasiradial equation

$$\frac{1}{\sinh^{n-1}\tau}\frac{d}{d\tau}\sinh^{n-1}\tau\frac{d\mathcal{R}}{d\tau} + \left[2R^2E - \frac{L(L+n-2)}{\sinh^2\tau} - 2R^2V(\tau)\right]\mathcal{R} = 0. \tag{32}$$

Using now the substitution

$$Z(\tau) = (\sinh\tau)^{(n-1)/2}\mathcal{R}(\tau), \tag{33}$$

we arrive at the equation

$$\frac{d^2Z}{d\tau^2} + \Big[\tilde{E} - \frac{(2L+n-1)(2L+n-3)}{4\sinh^2\tau} - 2R^2V(\tau)\Big]Z = 0, \tag{34}$$

where $\tilde{E} = 2R^2E - (n-1)^2/4$ and the quasiradial wave function $Z(\tau)$ satisfies the normalization condition

$$\int_0^\infty Z(\tau)Z^*(\tau)\,R^n\,d\tau = 1. \tag{35}$$

(i) The oscillator potential on the two-sheeted n-dimensional hyperboloid is given by

$$V(\tau) = \frac{\omega^2R^2}{2}\frac{s_1^2+s_2^2+\ldots+s_n^2}{s_0^2} = \frac{\omega^2R^2}{2}\tanh^2\tau. \tag{36}$$

From Eq. (34), we obtain

$$\frac{d^2Z}{d\tau^2} + \Big[\epsilon + \frac{\nu^2-1/4}{\cosh^2\tau} - \frac{(L+(n-2)/2)^2-1/4}{\sinh^2\tau}\Big]Z = 0, \tag{37}$$

where $\nu = \sqrt{\omega^2R^4+1/4}$ and $\epsilon = \tilde{E} - \omega^2R^4$. Thus, the oscillator problem on a hyperboloid is described by the modified Pöschl–Teller equation; in contrast to the oscillator equation on a sphere which has only bound states, Eq. (37) possesses both bound and unbound states.

The discrete-spectrum wave functions regular on the line $\tau\in[0,\infty)$ have the form [16, 19, 25]

$$Z(\tau) \equiv Z_{n_rL}(\tau) = \frac{1}{\Gamma(L+n/2)}\times\sqrt{\frac{2(\nu-L-2n_r-n/2)\Gamma(\nu-n_r)\Gamma(n_r+L+n/2)}{R^n(n_r)!\Gamma(\nu-L-n_r-n/2+1)}} \times(\sinh\tau)^{L+(n-1)/2}(\cosh\tau)^{2n_r-\nu+1/2} \times {}_2F_1(-n_r,-n_r+\nu;\,L+\frac{n}{2};\,\tanh^2\tau), \tag{38}$$

with $n_r = 0,1,\ldots,\ n_r^{\max} = [(\nu-L-n/2)/2]$. The quantity ϵ is quantized as

$$\epsilon = -(2n_r+L-\nu+n/2)^2, \tag{39}$$

and the energy spectrum for a quantum oscillator on an n-dimensional two-sheeted hyperboloid is

$$E_N^n(R) = \frac{1}{2R^2}\Big[-N(N+n-1) + (2\nu-1)\Big(N+\frac{n}{2}\Big)\Big]. \tag{40}$$

Here, $N = 2n_r+L$ is the principal quantum number, and the bound-state solution is possible only for

$$0 \le N \le \Big[\nu - \frac{n}{2}\Big]. \tag{41}$$

In the contraction limit where $R\to\infty$, $\tau\sim r/R$, and $\nu\sim\omega R^2$, we see that the continuous spectrum vanishes while the discrete spectrum is infinite, and it is easy to reproduce the oscillator energy spectrum (13) and wave function (14).

(ii) The Coulomb potential on the two-sheeted n-dimensional hyperboloid has the form [7, 12]

$$V(\tau) = -\frac{\alpha}{R}\left(\frac{s_0}{\sqrt{s_1^2+s_2^2+\ldots+s_N^2}} - 1\right) = -\frac{\alpha}{R}(\coth\tau - 1). \tag{42}$$

Substituting the potential (42) into the Schrödinger Eq. (34), we arrive at the equation

$$\frac{d^2Z}{d\tau^2} + \Big[(\tilde{E}-2\alpha R) - \frac{(2L+n-1)(2L+n-3)}{4\sinh^2\tau} + 2\alpha R\coth\tau\Big]Z = 0, \tag{43}$$

which is known to represent the problem of the Manning–Rosen potential [26].

Making the transformation from variable τ ($0\le\tau<\infty$) to the new variable $\mu\in[0,\infty)$,

$$e^\tau = \cosh\mu, \tag{44}$$

and setting $Z(\mu) = W(\mu)/\sqrt{\coth\mu}$, we arrive at the modified Pöschl–Teller equation

$$\frac{d^2W}{d\mu^2} + \Big[\tilde{E} + \frac{(-\tilde{E}+4\alpha R)-1/4}{\cosh^2\mu} \tag{45}$$

$$-\frac{(2L+n-2)^2-1/4}{\sinh^2\mu}\Big]W=0.$$

As can be seen from Eq. (45) with the substitution

$$\epsilon=\tilde{E},\quad \nu^2=-\tilde{E}+4\alpha R \tag{46}$$

and the transformation $L\to 2L$, the quasiradial equation (19) for $n^{\mathrm{Coul}}=(2d+1)$-dimensional Coulomb problem coincides with the $n^{\mathrm{osc}}=2d$-dimensional quasiradial oscillator Eq, (37).

Thus, the regular {for $\mu\in[0,\infty)$} solution of (43) or (45) has the form

$$Z(\mu)=\frac{W(\mu)}{\sqrt{\coth\mu}}\equiv Z^n_{n_rL}(\mu) \tag{47}$$
$$=A^n_{n_rL}(\nu)\,(\sinh\mu)^{L+n/2}\,(\cosh\mu)^{2n_r-\nu}$$
$$\times\,{}_2F_1\left(-n_r,-n_r+\nu;\,L+\frac{n}{2};\,\tanh^2\mu\right),$$

where $A^n_{n_rL}(\nu)$ is the normalization constant. The constant $A^n_{n_rL}(\nu)$ is computed from the requirement that the wave function (47) satisfy the normalization condition

$$R^n\int\limits_0^\infty |Z^n_{n_rL}(\tau)|^2\,d\tau \tag{48}$$
$$=R^n\int\limits_0^\infty |Z^n_{n_rL}(\mu)|^2\,\tanh\mu\,d\mu=1$$

and has the form

$$A^n_{n_rL}(\nu)=\frac{1}{\Gamma(L+n/2)} \tag{49}$$
$$\times\sqrt{\frac{2\nu(\nu-L-2n_r-n/2)\Gamma(\nu-n_r)\Gamma(n_r+L+n/2)}{R^n(L+2n_r+n/2)(n_r)!\Gamma(\nu-L-n_r-n/2+1)}}.$$

Comparing now Eq. (46) with (39) and passing from the oscillator to the Coulomb angular quantum number, $L\to 2L$, with the substitution $n\to 2(n-1)$ of dimensions, we get

$$\nu=\left(n_r+L+\sigma+\frac{n-1}{2}\right), \tag{50}$$
$$\sigma=\frac{\alpha R}{n_r+L+(n-1)/2}.$$

Thus, the discrete energy spectrum of the Coulomb problem on an n-dimensional two-sheeted hyperboloid is described by the formula

$$E^n_N(R)=-\frac{N(N+n-1)}{2R^2} \tag{51}$$
$$-\frac{\alpha^2}{2(N+(n-1)/2)^2}+\frac{\alpha}{R},$$

where $N=n_r+L$ is the principal quantum number, and the bound states occur for

$$0\le N\le\left[\sigma-\frac{n-1}{2}\right]. \tag{52}$$

The discrete-spectrum wave function has the form

$$Z^n_{NL}(\tau)=A^n_{NL}(\sigma)\,(\sinh\tau)^{L+(n-1)/2}\times \tag{53}$$
$$\times e^{\tau(N-L-\sigma)}\,{}_2F_1\Big(-N+L,L+\frac{n-1}{2}$$
$$+\sigma;\,2L+n-1;\,1-e^{-2\tau}\Big),$$

where the normalization constant $A^n_{NL}(\sigma)$ is

$$A^n_{NL}(\sigma)=\frac{2^{L+(n-1)/2}}{\Gamma(2L+n-1)}\sqrt{\frac{[\sigma^2-(N+(n-1)/2)^2]\Gamma(N+L+n-1)\Gamma(\sigma+L+(n-1)/2)}{R^n(N+(n-1)/2)(N-L)!\Gamma(\sigma-L-(n-1)/2+1)}}. \tag{54}$$

The solution for the Coulomb quasiradial equation, for both the energy spectrum and the wave functions, is identical to that obtained in [12] by applying the path-integral approach. We do not consider here the contraction limit $R\to\infty$ to a flat Euclidean space E_n for the Coulomb problem because this was done in [12].

It should be noted that, instead of substitution (44), it is possible to use the trigonometric transformation

$$e^{-\tau}=\cos\varphi,\quad \varphi\in[0,\pi/2]. \tag{55}$$

It is easy to see that, apart from the permutation

$$\epsilon=-\tilde{E}+4\alpha R,\quad \nu^2=-\tilde{E} \tag{56}$$

and the transformation $L\to 2L$, the quasiradial Eq. (43) for the $n^{\mathrm{Coul}}=(d+1)$-dimensional Coulomb problem goes over to the $n^{\mathrm{osc}}=2d$-dimensional quasiradial oscillator Eq. (9). Thus, the Coulomb problem on a two-sheeted hyperboloid is related to the oscillator problem on a sphere or a two-sheeted hyperboloid.

4. COULOMB–OSCILLATOR RELATION ON AN n-DIMENSIONAL ONE-SHEETED HYPERBOLOID

The pseudospherical coordinates on the n-dimensional one-sheeted hyperboloid $s_0^2-s_1^2-s_2^2-\ldots$

$-s_n^2 = -R^2$ are

$$\begin{aligned} s_0 &= R\sinh\tau, \\ s_1 &= R\cosh\tau\cos\vartheta_1, \\ s_2 &= R\cosh\tau\sin\vartheta_1\cos\vartheta_2, \\ &\cdots \\ s_{n-1} &= R\cosh\tau\sin\vartheta_1\sin\vartheta_2\cdots\sin\vartheta_{n-2}\cos\varphi, \\ s_n &= R\cosh\tau\sin\vartheta_1\sin\vartheta_2\cdots\sin\vartheta_{n-2}\sin\varphi, \end{aligned}$$

where $\tau \in (-\infty, \infty)$. The variables in the Schrödinger Eq. (1) may be separated by using the ansatz (31)

$$\begin{aligned} &\Psi(\tau, \vartheta_1, \ldots, \vartheta_{n-2}, \varphi) \\ &= \mathcal{R}(\tau)\, Y_{L,l_1,l_2,l_{n-2}}(\vartheta_1, \ldots, \vartheta_{n-2}, \varphi), \end{aligned}$$

where, as in the preceding case, l_i are the angular hypermomenta, L is total angular momentum, and the hyperspherical function $Y_{L,l_1,l_2,l_{n-2}}(\vartheta_1, \ldots, \vartheta_{n-2}, \varphi)$ is a solution of the Laplace–Beltrami equation on the $(n-1)$-dimensional sphere. After the separation of variables, we obtain the quasiradial equation

$$\begin{aligned} &\frac{1}{\cosh^{n-1}\tau}\frac{d}{d\tau}\cosh^{n-1}\tau\frac{d\mathcal{R}}{d\tau} \qquad (57) \\ &+\left[2R^2E + \frac{L(L+n-2)}{\cosh^2\tau} - 2R^2V(\tau)\right]\mathcal{R} = 0. \end{aligned}$$

Using now the substitution

$$Z(\tau) = (\cosh\tau)^{(n-1)/2}\mathcal{R}(\tau), \qquad (58)$$

we arrive at the equation

$$\begin{aligned} &\frac{d^2Z}{d\tau^2} + \Big[\tilde{E} + \frac{(2L+n-1)(2L+n-3)}{4\cosh^2\tau} \qquad (59) \\ &\qquad - 2R^2V(\tau)\Big]Z = 0, \end{aligned}$$

where $\tilde{E} = 2R^2E - (n-1)^2/4$ and the quasiradial wave function $Z(\tau)$ satisfies the normalization condition

$$\int_{-\infty}^{\infty} Z(\tau)Z^*(\tau)\,R^n\,d\tau = 1. \qquad (60)$$

(i) The oscillator potential on an n-dimensional one-sheeted hyperboloid is given by

$$\begin{aligned} V(\tau) &= \frac{\omega^2R^2}{2}\,\frac{s_1^2 + s_2^2 + \ldots + s_n^2}{s_0^2} \qquad (61) \\ &= \frac{\omega^2R^2}{2}\coth^2\tau. \end{aligned}$$

For Eq. (59), we then have

$$\begin{aligned} &\frac{d^2Z}{d\tau^2} + \left[\epsilon + \frac{(L+(n-2)/2)^2 - 1/4}{\cosh^2\tau}\right. \qquad (62) \\ &\qquad \left. - \frac{\nu^2 - 1/4}{\sinh^2\tau}\right]Z = 0, \end{aligned}$$

where $\nu = \sqrt{\omega^2R^4 + 1/4}$ and $\epsilon = \tilde{E} - \omega^2R^4$. As in the preceding case, the oscillator system is described by the modified Pöschl–Teller equation and possesses a discrete and a continuous spectrum. Since, however, the situation here is different from that in the case of motion on a two-sheeted hyperboloid, the number of bound states depends on the total angular momentum. The discrete-state wave functions regular on the line $\tau \in (-\infty, \infty)$ are

$$\begin{aligned} Z(\tau) \equiv Z_{n_rL}(\tau) &= \sqrt{\frac{(L-\nu-2n_r+n/2-2)\Gamma(L-n_r+n/2-1)\Gamma(n_r+\nu+1)}{R^n(n_r)![\Gamma(\nu+1)]^2\Gamma(L-\nu-n_r+n/2-1)}} \qquad (63) \\ &\times(\sinh\tau)^{\nu+1/2}(\cosh\tau)^{2n_r-L-n/2+3/2}\,{}_2F_1\left(-n_r, -n_r+L+\frac{n}{2}-1;\, \nu+1;\, \tanh^2\tau\right) \end{aligned}$$

and

$$\epsilon = -\left(2n_r - L + \nu - \frac{n}{2} + 2\right)^2, \qquad (64)$$

where the bound states occur for $n_r = 0, 1, \ldots, n_r^{\max} = [(L-\nu+n/2-2)/2]$. The last formula means that the discrete spectrum depends on the quantum number L, and the energy spectrum of the oscillator system takes the form

$$\begin{aligned} E_{n_rL}(R) = -\frac{1}{2R^2}\Big[&(2n_r - L + 2)(2n_r - L - n + 3) \qquad (65) \\ &+ (2\nu - 1)\left(2n_r - L - \frac{n}{2} + 2\right)\Big]. \end{aligned}$$

(ii) The Coulomb potential on the n-dimensional hyperboloid has the form [7, 12]

$$\begin{aligned} V(\tau) &= -\frac{\alpha}{R}\left(\frac{s_0}{\sqrt{s_1^2 + s_2^2 + \ldots + s_n^2}} + 1\right) \qquad (66) \\ &= -\frac{\alpha}{R}(\tanh\tau + 1). \end{aligned}$$

The Schrödinger equation for this potential is

$$\frac{d^2Z}{d\tau^2} + \Big[(\tilde{E} + 2\alpha R) \qquad (67)$$

$$+\frac{(2L+n-1)(2L+n-3)}{4\cosh^2\tau}+2\alpha R\tanh\tau\Big]Z=0,$$

which coincides with the Rosen–Morse equation [25].

Making the transformation from the variable τ $(-\infty<\tau<\infty)$ to the new variable $\mu\in[0,\infty)$,

$$e^{\tau}=\sinh\mu, \tag{68}$$

we arrive at the equation

$$\frac{d^2W}{d\mu^2}+\Big[(\tilde{E}+4\alpha R) \tag{69}$$
$$+\frac{(2L+n-2)^2-1/4}{\cosh^2\mu}-\frac{(-\tilde{E})-1/4}{\sinh^2\mu}\Big]W=0,$$

where $W(\mu)=(\tanh\mu)^{1/2}Z(\mu)$. From this equation, we see that, apart from the substitution

$$\tilde{E}\to\tilde{E}+4\alpha R,\quad \nu^2=-\tilde{E}, \tag{70}$$

and the simultaneous transformation $L\to 2L$ for total angular momentum, the quasiradial Eq. (69) for the Coulomb problem on a $n^{\mathrm{Coul}}=(d+1)$-dimensional one-sheeted hyperboloid coincides with the $n^{\mathrm{osc}}=2d$-dimensional quasiradial oscillator Eq. (62).

Comparing now Eq. (69) with (62) and taking into account Eqs. (64) and (70), we see that the discrete-spectrum wave function satisfying the normalization condition

$$R^n\int\limits_{-\infty}^{\infty}|Z^n_{n_rL}(\tau)|^2\,d\tau \tag{71}$$
$$=R^n\int\limits_{-\infty}^{\infty}|Z^n_{n_rL}(\mu)|^2\,\coth\mu\,d\mu=1$$

has the form

$$Z^n_{n_rL}(\tau)=\frac{2^{n_r-L-n/2}}{\Gamma(L-n_r+\dfrac{n-1}{2})}\sqrt{\frac{[(L-n_r+(n-3)/2)^2-\sigma^2]\Gamma(2L-n_r+n-2)\Gamma(L+(n-1)/2)}{R^n(L-n_r+(n-3)/2)(n_r)!\Gamma(L-\sigma+1/2)}} \tag{72}$$
$$\times(\cosh\tau)^{n_r-L-(n-1)/2}\,e^{\tau(\sigma-1)}\,{}_2F_1\left(-n_r,-n_r+L+n-2;\,L-n_r+\frac{n-3}{2}+\sigma;\,\frac{1}{1+e^{-2\tau}}\right),$$

with the discrete energy spectrum of the Coulomb problem being described by the formula

$$E_n=-\frac{(L-n_r-1)(L-n_r+n-2)}{2R^2} \tag{73}$$
$$-\frac{\alpha^2}{2(L-n_r+(n-3)/2)^2}-\frac{\alpha}{R}.$$

Bound states occur for $n_r=0,1,\ldots,n_r^{\max}=[(L+(n-3)/2+\sigma)]$.

Finally, we note that, in contrast to a sphere and a two-sheeted hyperboloid, the contraction limit $R\to\infty$ on one-sheeted hyperboloids is meaningless for the oscillator and Coulomb problems.

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Q Operators for the Simple Quantum Relativistic Toda Chain*

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Abstract—We investigate the simple quantum relativistic Toda chain. The ultralocal simple Weyl algebra pair is associated with each site of the chain. Weyl's q is considered to be inside a unit circle. Both independent Baxter operators Q are constructed explicitly as series in local Weyl generators. The operator-valued Wronskian of Q-s is also calculated. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

Long ago, in his famous papers [1], R.J. Baxter introduced the object known now as a Q operator. The Q operator and an auxiliary transfer matrix T satisfy the so-called Baxter equation (or $TQ = Q' + Q''$ equation). Usually, the TQ equation is considered as an equation for the eigenvalues of the transfer matrix and the function Q, and, in this sense, an investigation of the Baxter equation is called the algebraic Bethe ansatz.

Baxter introduced his operator implicitly. The method of algebraic Bethe ansatz needs only the theorem of the existence of a Q operator and its analytic properties; further, one solves $TQ = Q' + Q''$ as a functional equation.

In contrast to the algebraic Bethe ansatz machinery, the subject of our interest is Q as an operator. The possibility of constructing matrix elements of the operator Q explicitly was found originally by Bazhanov and Stroganov [2]. Intensive studies of operators Q began from [3]. Recently, the operators Q were discussed in [4] in connection with continuous quantum field theory. A relation between the operator Q and quantum Bäklund transformations was indicated in [5, 6]. A relation between Q and Bloch solutions of the quantum linear problem was discovered in [7]. Explicit constructions of operators Q were recently obtained for several models, like the isotropic Heisenberg spin chain [8], the periodic Toda chain, and other models with a rational R matrix [7]. It is known that, with free boundary conditions for Q, the TQ equation provides a one-parametric family of solutions, so that one may extract two independent solutions with a nonzero discrete Wronskian (see [4, 9]). In [7, 8], both independent Q operators were obtained for the models considered.

In this paper, we investigate the exactly integrable model known as the quantum simple relativistic Toda chain (see, for example, [10–12]). The local Lax matrix for the model is constructed with the aid of Weyl algebra elements, commuting on q, and we deal with the case of $|q| < 1$. In this paper, we do not consider the Jacobi partners to the Weyl algebra, dealing thus with compact q dilogarithms (the word "simple" in the title of the paper means the absence of the modular dualization, as is explained in [13–15]). The quantum space of our model is a formal module of an enveloping of the tensor product of several copies of Weyl algebras. For Weyl elements, we only suggest, first, their invertibility and, second, a q-equidistant spectrum for one of them. Both independent operators Q_+ and Q_- and their Wronskian are calculated locally as operators that act in the ultralocal Weyl algebra. Actually all our results are to be understood as the well-defined series expansions for functions from the enveloping mentioned.

2. FORMULATION OF THE MODEL

First, let us define the Lax matrix for the simple quantum relativistic Toda chain at fth site as

$$L_f(x) = \begin{pmatrix} x\mathbf{u}_f - (x\mathbf{u}_f)^{-1}, & \mathbf{v}_f \\ q^{-1/2}\lambda\mathbf{v}_f^{-1}, & 0 \end{pmatrix}, \qquad (1)$$

where $\{\mathbf{u}_f, \mathbf{v}_f\}$ form the "half-integer" ultralocal Weyl algebra,

$$\mathbf{u}_f \cdot \mathbf{v}_f = q^{1/2}\mathbf{v}_f \cdot \mathbf{u}_f, \qquad (2)$$

and the elements of different f's commute. As usual, the whole quantum space is the tensor product of

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some copies of Weyl modules, and f labels the "number" of a given Weyl algebra in this tensor product. Recall that we will always imply $|q| < 1$.

The correspondence between the quantum relativistic Toda chain and usual quantum Toda chain may be established, for example, in the parametrization

$$q = \mathbf{e}^{-i\epsilon}, \quad \lambda = -\epsilon^2, \quad x = \mathbf{e}^{\epsilon\theta/2}, \tag{3}$$
$$\mathbf{u}_f = \mathbf{e}^{-\epsilon\mathbf{p}_f/2}, \quad \mathbf{v}_f = \epsilon\mathbf{e}^{\mathbf{q}_f},$$

where

$$[\mathbf{p}, \mathbf{q}] = i, \tag{4}$$

in the limit

$$\lim_{\epsilon=0} \frac{1}{\epsilon} L_f(x) = \begin{pmatrix} \theta - \mathbf{p}_f, & \mathbf{e}^{\mathbf{q}_f} \\ -\mathbf{e}^{-\mathbf{q}_f}, & 0 \end{pmatrix}. \tag{5}$$

The right-hand side of this relation is known as the Lax matrix of the usual quantum Toda chain.

In the L matrix, as well as in all other objects, the spectral parameter x will always be coupled to $\mathbf{u}_f$. We introduce the useful notation

$$x^2\mathbf{u}_f^2 \stackrel{\text{def}}{=} q^{\mathbf{s}_f}, \tag{6}$$

so that, for any formal function $g(\mathbf{s}_f)$,

$$g(\mathbf{s}_f) \cdot \mathbf{v}_f^n = \mathbf{v}_f^n \cdot g(\mathbf{s}_f + n), \quad \forall \quad n. \tag{7}$$

Actually, we deal with the pairs $\mathbf{s}_f, \mathbf{v}_f$; it is more restrictive than the Weyl algebra pair as a matter of fact. Further, we define the transfer matrix for the chain with F sites, $f = 1, \ldots, F$, as

$$T(x^2) \tag{8}$$
$$= \left((-x)^F \prod_f \mathbf{u}_f \right) \cdot \mathrm{tr}(L_1(x) \cdot L_2(x) \cdots L_F(x)).$$

The matrix $T(x^2)$ becomes a polynomial of x^2 with commutative coefficients:

$$T(x^2) = \sum_{j=0}^{F} (-x^2)^{F-j} t_j. \tag{9}$$

Here, it is implied that $t_F = 1$ and

$$t_0 = \prod_f \mathbf{u}_f^2. \tag{10}$$

Note that, apart from the trivial relation $t_F = 1$, all other coefficients F are independent. For the set $\{t_j\}$ given, one can define another set $\{\overline{t_j}\}$ by

$$\overline{t_j} = t_0^{-1} t_{F-j}. \tag{11}$$

This merely means that

$$T(x^2) = (-x^2)^F t_0 \overline{T}(x^{-2}). \tag{12}$$

We shall now fix the coefficients in the Baxter equation

$$T(x^2)Q(x^2) \tag{13}$$
$$= \left((-\lambda x^2)^F t_0\right) Q(qx^2) + Q(q^{-1}x^2),$$

where t_0 is given by (10). With this normalization of the coefficients in (13), the Baxter equation has a solution that is entire in x^2. We shall call this solution

$$Q_+(x^2) = J(x^2, \lambda, \{t\}). \tag{14}$$

Proposition 1. The series expansion of the solution of (13) entire in x^2 in terms of λ^F is

$$J(x^2, \lambda, \{t\}) = \left(\prod_{k=1}^{\infty} T(q^k x^2) \right) \tag{15}$$
$$\times \left(\sum_{k=0}^{\infty} (-\lambda^F)^k c_k(x^2) \right),$$

where $c_{-1} \equiv 0$, $c_0 \equiv 1$, and recursively

$$c_k(x^2) = \sum_{j=1}^{\infty} \frac{(q^j x^2)^F c_{k-1}(q^{1+j}x^2)}{T(q^j x^2) T(q^{1+j} x^2)}. \tag{16}$$

Note that $J(x^2, \lambda, \{t\})$ is the entire function of all of its arguments. The proof of this proposition is a rather simple exercise.

The other solution $Q_-(x^2)$ must contain a cut in x, and, up to this cut, we guess that $Q_-(x^2)$ is entire in x^{-2}. More precisely, with $\mathbf{s}_f$ according to the notation introduced in (6), we have

$$Q_-(x^2) = \lambda^{-\sum_f \mathbf{s}_f} \cdot J(x^{-2}, \lambda, \{\overline{t}\}). \tag{17}$$

The last definition we need here is the definition of the q-Wronskian of these two solutions:

$$W(x^2) \stackrel{\text{def}}{=} Q_+(q^{-1}x^2)Q_-(x^2) \tag{18}$$
$$- Q_+(x^2)Q_-(q^{-1}x^2).$$

3. INTERTWINERS

The commutativity of the transfer matrices $T(x^2)$ [see Eq. (8)],

$$T(x^2) \cdot T(y^2) = T(y^2) \cdot T(x^2), \tag{19}$$

is ensured by the intertwining relation

$$R_{1,2}(x/y) \cdot L_{1,f}(x) \cdot L_{2,f}(y) \tag{20}$$
$$= L_{2,f}(y) \cdot L_{1,f}(x) \cdot R_{1,2}(x/y),$$

where $L_{1,f}(x) = L_f(x) \otimes 1$, $L_{2,f}(y) = 1 \otimes L_f(y)$, etc.; the cross product implies the tensor product of 2×2 matrices; and the six-vertex R matrix has the form

$$R(x) = \begin{pmatrix} 1-x^{-2}q & 0 & 0 & 0 \\ 0 & q^{1/2}(1-x^{-2}) & x^{-1}(1-q) & 0 \\ 0 & x^{-1}(1-q) & q^{1/2}(1-x^{-2}) & 0 \\ 0 & 0 & 0 & 1-x^{-2}q \end{pmatrix}. \quad (21)$$

The appearance of the six-vertex R matrix is the criterion of the existence of the Baxter $TQ = Q' + Q''$ relation for our transfer matrix. Let $M_{h,f}(x^2)$ be an operator acting in the tensor product of the fth quantum Weyl algebra and its auxiliary space h such that the trace over this auxiliary space of the monodromy of M operators gives a Q operator:

$$Q(x^2) = \mathrm{tr}_h \left(M_{h,1}(x) \cdot M_{h,2}(x) \cdots M_{h,F}(x) \right). \quad (22)$$

The commutativity of Q with T must be ensured by the intertwining relation for $M_{h,f}$ and L_f:

$$\widetilde{L}_h(x/y) * L_f(x) \cdot M_{h,f}(y) = M_{h,f}(y) \cdot L_f(x) * \widetilde{L}_h(x/y). \quad (23)$$

Here, asterisks denote multiplication of 2×2 matrices, and $\widetilde{L}_h(z)$ is an auxiliary L matrix.

Proposition 2. Equation (23) is consistent for

$$\widetilde{L}(x) = \begin{pmatrix} xq^{\mathbf{N}/2} - x^{-1}q^{-\mathbf{N}/2}, & \lambda \mathbf{a}^+ q^{\mathbf{N}/2} \\ \lambda q^{\mathbf{N}/2}\mathbf{a}, & -\lambda x^{-1} q^{\mathbf{N}/2} \end{pmatrix}, \quad (24)$$

where, for $\forall g$,

$$\mathbf{a} \cdot g(\mathbf{N}) = g(\mathbf{N}+1) \cdot \mathbf{a} \quad (25)$$

and

$$\mathbf{a}^+ \cdot \mathbf{a} = -q^{1/2}\lambda^{-1}(1 - q^{-\mathbf{N}}) \stackrel{\mathrm{def}}{=} [\mathbf{N}]. \quad (26)$$

The quantum Lax matrix (24) corresponds to the dimer self-trapping L matrix [6], in the same way as (1) corresponds to (5). There exists a more general expression for (24), namely, the complete massive sine-Gordon L matrix. But it would lead to an inessential complication.

Equation (23) is now to be solved with respect to operator $M_{h,f}$; M may be written out in two forms. The first one is a series of positive integer powers of the quantum oscillator elements $\mathbf{a}$ and $\mathbf{a}^+$ up to some function of $\mathbf{N}$ containing the term $q^{\mathbf{N}^2}$. The second form contains the permutation operator $P_{f,h}$, implying the identification

$$q^{\mathbf{N}/2} = \mathbf{u}_h, \quad \mathbf{a} = \mathbf{v}_h. \quad (27)$$

Actually, $P_{f,h}$ is a way to absorb $q^{\mathbf{N}^2/2}$. The two forms of $M_{h,f}$ coincide, apart from the definition of $P_{h,f}$ in terms of $\mathbf{N}$ and in terms of $\mathbf{a}$ and $\mathbf{v}_f$. Explicitly, the forms of M are rather complicated, and they are not of importance in this brief note. However, it is important to indicate that the two possible ways to interpret $\mathbf{a}$, $\mathbf{a}^+$, and $q^{\mathbf{N}}$ lead to two possible ways to define tr_h. Namely, in terms of the q oscillator, the natural definition of the trace is

$$\mathrm{tr}_h G(\mathbf{N}, \mathbf{a}, \mathbf{a}^+) = \sum_{n=0}^{\infty} G(n, 0, 0), \quad (28)$$

while, with the permutation operator $P_{h,f}$ extracted, the notion of the trace is invariant:

$$\mathrm{tr}_h P_{h,f} = 1. \quad (29)$$

The auxiliary Lax matrix $\widetilde{L}(x)$ [see Eq. (24)] degenerates at $x = 1$. As usual, Eq. (23) at this point becomes the so-called triangle equations, leading to the Baxter Eq. (13). This procedure is well known; therefore, we will not discuss it here.

The way to calculate the Wronskian of two Q operators is much more complicated. It needs the investigation of the intertwining relation between two Lax matrices (24). Here, we will give only the answer. The reader may find the details in [16].

4. SOLUTION

Now, we give explicit expressions for both functions $Q_\pm$, i.e., the results of the tracing. There arises the following natural question: We already have the form (15) and (16)—what else can one do? In the QUISM approach, we construct $Q_\pm(x^2)$ not as functions of $\{t\}$, but as functions of local $\mathbf{u}_f$ and $\mathbf{v}_f$. This is in a sense a factorization; the simplest analog of this is the well-known q-exponential formula

$$(\mathbf{x} + \mathbf{y}; q)_\infty = (\mathbf{x}; q)_\infty \cdot (\mathbf{y}; q)_\infty, \quad \mathbf{x} \cdot \mathbf{y} = q\mathbf{y} \cdot \mathbf{x}, \quad (30)$$

where conventionally

$$(x; q)_n \stackrel{\mathrm{def}}{=} \prod_{k=0}^{n-1} (1 - q^k x), \quad (31)$$

$$(x; q)_\infty \stackrel{\mathrm{def}}{=} \prod_{n=0}^{\infty} (1 - q^n x),$$

or, in the form of series expansions, we have

$$(x;q)_\infty = \sum_{n=0}^{\infty} q^{n(n-1)/2}\frac{(-x)^n}{(q;q)_n}, \qquad (32)$$
$$(x;q)_\infty^{-1} = \sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}.$$

The right-hand side of Eq. (30) is called the local form of its "global" left-hand side.

Now, we describe the local form of all solutions. First of all, we introduce the function

$$\mathbf{g}_{\alpha,\beta}(n,m) \stackrel{\text{def}}{=} q^{nm}\alpha^n\beta^m \qquad (33)$$
$$\times \frac{(q^{1+n};q)_\infty (q^{1+m};q)_\infty}{(q;q)_\infty},$$

where α and β are complex numbers and the elements q^n and q^m commute.

Proposition 3. The local form of the operator $Q_+(x^2)$ defined by Eqs. (14)–(16) is

$$Q_+(x^2) = \sum_{\{n_f\geq 0\}}\left(\prod_f \mathbf{g}_{1,\lambda}(n_f+\mathbf{s}_f, n_f)\right) \qquad (34)$$
$$\times\left(\prod_f (\mathbf{u}\cdot\mathbf{v})_f^{n_{f+1}-n_f}\right),$$

and the local form of the operator $Q_-(x^2)$ defined by Eqs. (17), (15), and (16) is

$$Q_-(x^2) = \sum_{\{n_f\geq 0\}}\left(\prod_f \mathbf{g}_{1,\lambda}(n_f, n_f-\mathbf{s}_f)\right) \qquad (35)$$
$$\times\left(\prod_f (\mathbf{u}\cdot\mathbf{v})_f^{n_f-n_{f-1}}\right).$$

Their Wronskian defined by Eq. (18) is

$$W(x^2) = \left(\prod_f (q^{\mathbf{s}_f};q)_\infty (q^{1-\mathbf{s}_f};q)_\infty \lambda^{-\mathbf{s}_f}\right) \qquad (36)$$
$$\times\left(\prod_f \left(\frac{\lambda(\mathbf{u}\cdot\mathbf{v})_f}{(\mathbf{u}\cdot\mathbf{v})_{f+1}};q\right)_\infty\right).$$

5. DISCUSSION

The technique and the results given in this paper are rather formal. We have dealt with the single Weyl pair at each site of the lattice, and q is an arbitrary complex number inside a unit circle. It is well known that this regime is absolutely unphysical; thus, the results presented here are to be considered as just an exercise in the field of q-combinatorial analysis. Nevertheless, the results and the technique presented may find some applications.

Speaking about the Weyl algebra, people usually keep in mind two aspects: the first one implies the dualization and $q = \exp\{i\pi\mathbf{e}^{i\theta}\}$ (see [13–15]), while the second one is a finite-state representation of the Weyl algebra at $q = \mathbf{e}^{2\pi i/N}$. Actually, this paper suggests a third aspect, applied in the backward direction yet: several Toda-chain-type models, physical as well, may be obtained from a model with arbitrary q in the limit $q \mapsto 1+\hbar$ regarded in a special way such that a rational Weyl algebra mapping is linearized with respect to one of the Weyl generators in the first order in $\hbar$.

Our experience in Weyl algebra exercises says that the majority of our results, especially those containing q dilogarithms and permutations, may be immediately rewritten in a dualized form. In this way, the results may be applied to the physical relativistic Toda chain [17]. This will be done in a separate publication.

The second aspect is also valid, especially in the part of the technique derived. Preliminary considerations show that, at the root of unity, the model contains the Baxter curve for the chiral Potts model (CPM), so that the point on the Baxter curve is the spectral parameter of the Q operator, and our constant parameter λ is connected with the modulus of the Baxter curve. Remarkable is that, in the relativistic Toda chain, there appears only one point on the Baxter curve at the root of unity, while, in the CPM, each site of the spin chain contains such point separately. This fact makes the relativistic Toda chain much simpler than the CPM itself. This model will be considered in the forthcoming publication.

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Quantum Bound States Embedded in a Continuum and Their Classical Analogs*

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Abstract—We show that both a rigid and a nonrigid dipole can be trapped by an external uniform magnetic field in classical mechanics. The trapped states of a dipole present a nontrivial example of classical bound states embedded in a continuum (BSEC) that can be treated as analogs of quantum BSECs. For example, the classical motion of a dipole is confined to a finite region in space, though there are no classical turning points. We also examine the quantum motion of a dipole in a magnetic field and show that, for the most natural choices of the parameters (the rigid rotating dipole or the one bound by oscillator potential, uniform time-independent magnetic field, etc.), there are no quantum BSEC solutions.

1. INTRODUCTION

A quantum system can have bound states with normalizable wave functions and continuum-spectrum (scattering) states with nonnormalizable wave functions. It is usually supposed that the energies of all of the bound states are negative ($E_i < 0$), while continuum-spectrum states have positive energies ($E > 0$); i.e., continuum-spectrum states are well separated in energy from bound ones. However, as early as 1929, von Neumann and Wigner [1] showed that a quantum system can have bound states embedded in a continuum (BSEC), i.e., bound states with energies $E > 0$. Recently, BSECs were observed experimentally by Capasso *et al.* [2]. Interest in BSECs was also stimulated by the reported narrow e^+e^--coincidence peaks in heavy-ion collisions at GSI, which were interpreted by Arbuzov *et al.* [3] and independently by Spence and Vary [4] as those that are due to bound states of the electron–positron system embedded in a continuum. Thus, BSECs were studied in a number of recent papers [5]. Shirokov, Smirnov, and Zaytsev [6] also proposed so-called isolated states that are, by definition, bound states that do not correspond to the S-matrix poles and which are a generalization of BSECs.

Bound states embedded in a continuum are conventionally regarded as exotic quantum states that do not have classical analogs. However, BSECs are very natural in classical mechanics. If the potential energy of a classical system has two barriers, then any state of positive energy trapped by the barriers is a BSEC. Nevertheless, the corresponding quantum system does not have a BSEC: a double-barrier potential generates resonances instead of BSECs in quantum mechanics.

To support a BSEC in quantum mechanics, a local potential should have an infinite number of oscillations with amplitudes decreasing at large distances [1]. The wells of the long-range oscillating tail of the von Neumann–Wigner type potential are capable of trapping a classical particle of positive energy. Pursey and Weber [7] analyzed this phenomenon in some detail for the three-dimensional case with zero angular momentum. Note, however, that the quantum BSEC wave function in this case decreases very slowly as the distance r tends to infinity. As a result, the probability of detecting a particle outside the trapping potential well is much larger than the probability of detecting it within the well. Moreover, the quantum BSEC in the von Neumann–Wigner type of potential can have an energy larger than the maximum of the potential energy, so that the corresponding classical motion does not have classical turning points and is not trapped. If the angular momentum is positive ($L > 0$), these potentials can also support classical BSECs of energies greater than the height of the potential barrier, provided that the angular momentum is sufficiently large. However, this is possible only because almost all of the energy is associated with

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rotational motion—the energy associated with radial motion is still less than the height of the potential barrier. Therefore, classical BSECs supported by the von Neumann–Wigner type of potential differ essentially from quantum BSECs and cannot serve as analogs of quantum BSECs—in particular, such potentials do not provide a classical analog of a quantum BSEC with no classical turning points.

In quantum mechanics, BSECs are also natural when the interaction is nonlocal [6, 8] or in the case of multichannel scattering (see, e.g., [9, 10]). However, such quantum systems do not have explicit classical analogs.

In general, it seems that, when a quantum system has a BSEC, the corresponding classical system does not, and vice versa. The reason for this is that the mechanisms of the formation of BSECs in quantum and classical mechanics are very different: in the classical case, the motion of a particle with a positive energy is confined to a finite region of space because of potential barriers that give rise to classical turning points, while, in the quantum case, the BSEC wave function decreases at large distances because of the interference between the waves reflected by an infinite number of potential barriers in the case of the von Neumann–Wigner type potential or because of the features peculiar to the quantum nonlocal interaction or multichannel scattering. Thus, it would be interesting to find a classical system that supports BSECs with no classical turning points, i.e., where the trapping of a particle is not associated with potential barriers. Such classical BSECs can serve as analogs of quantum BSECs and may indicate the possible formation of quantum BSECs.

Classical BSECs were discussed by Pollak and Pechukas [11] in the context of reactive scattering theory in physical chemistry. Pollak and Pechukas showed that a classical system can have a BSEC solution when the potential surface has a saddle point. There is a close analogy between the Pollak–Pechukas BSEC solutions and the position of unstable equilibrium of a physical pendulum. The unstable equilibrium of a classical system when the particle involved is stopped for an infinite time just at the maximum point of the potential barrier represents some kind of a classical BSEC. However, such BSEC is uninteresting because (i) it is unstable and any small perturbation destroys it and (ii) the corresponding quantum system does not have a BSEC solution at the same or at a close energy.

Recently, we suggested [12] another example of a classical BSEC that seems to have some features of quantum BSECs. In particular, we demonstrated [12] that a classical rigid electric dipole can be trapped by an external magnetic field. The trapping of the dipole is not associated with any potential; thus, this system presents a nontrivial example of a classical BSEC without classical turning points. However, we did not examine quantum solutions for the dipole in a magnetic field. Thus, it is not clear whether an electric dipole in a magnetic field presents the first example of a system that has BSEC solutions both in the quantum and in the classical case. We discuss this problem in this paper. First, we present the classical equations for the dipole in a magnetic field in the form that can be easily generalized to the quantum case and prove the existence of classical BSEC solutions. In contrast to [12], we do not discuss the general case of the classical motion of a rigid dipole in a magnetic field, but we restrict the discussion to a particular case that can be solved analytically. In addition, we present the equations of classical motion and demonstrate the existence of classical BSEC solutions for a particular solvable case of a nonrigid dipole in a magnetic field. Next, we perform a quantum consideration of the above examples.

Some of the results discussed below were presented in [13].

2. CLASSICAL TRAPPING OF AN ELECTRIC DIPOLE BY A MAGNETIC FIELD

2.1. Rigid Rotating Dipole

Our model of the rigid dipole is a massless rod fixing the distance $2a$ between two particles, each having a mass m. One of the particles carries a charge $+e$, while the charge of the other is $-e$. The y axis is chosen to be in the direction of a uniform magnetic field $\mathbf{B}$. We restrict the center-of-mass motion of the dipole to the z direction and the rotational degrees of freedom of the dipole to the xy plane. These restrictions are consistent with, and therefore provide a special case of, the general classical motion in three dimensions of a rigid dipole in a magnetic field as discussed in [12].

In our case, the Hamiltonian has the form

$$H = (P + \beta a\cos\varphi)^2/(4m) + (\pi + \beta R a\sin\varphi)^2/(4ma^2). \quad (1)$$

Here, R is the center-of-mass coordinate; P is the canonically conjugate momentum; φ is the angle between the x axis and $\mathbf{r}$, where $\mathbf{r}$ is the relative coordinate, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, with $\mathbf{r}_1$ and $\mathbf{r}_2$ being the coordinates of the particles; π is the momentum canonically conjugated to φ; and $\beta = eB/c$.

It is easy to check that there is an integral of motion,

$$J = P - \beta a\cos\varphi. \quad (2)$$

It is important to note that J is also an integral of motion in the quantum case, because $[\hat{H}, \hat{J}] = 0$, as is

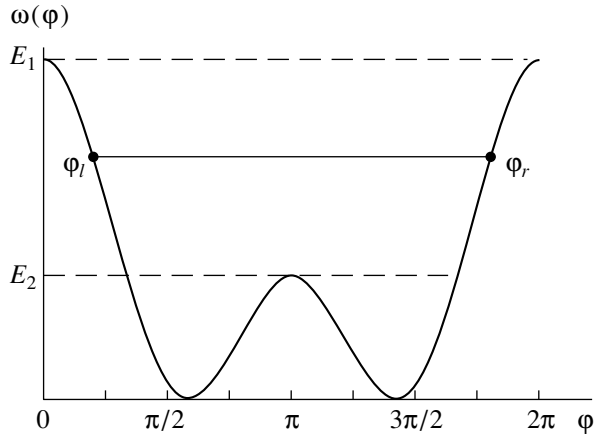

Fig. 1. Effective potential $w(\varphi)$ versus φ for $0 < J < 2\beta a$.

seen from Eqs. (1) and (2) (hereafter, we use the "hat" symbol to denote quantum operators). Of course, the corresponding Poisson bracket vanishes ($\{H, J\} = 0$) in the classical case as well.

The canonical classical equations of motion are

$$\dot{R} = (P + \beta a \cos\varphi)/(2m), \tag{3}$$

$$\dot{\varphi} = (\pi + \beta R a \sin\varphi)/(2ma^2), \tag{4}$$

$$\dot{P} = -\beta(\pi + \beta R a \sin\varphi)\sin\varphi/(2ma), \tag{5}$$

$$\dot{\pi} = \beta\left[aP\sin\varphi - \beta(R^2 - a^2)\sin\varphi\cos\varphi - (R/a)\pi\cos\varphi\right]/(2m). \tag{6}$$

From Eqs. (3)–(6), it is easy to derive the equations of motion in the form

$$2m\ddot{R} = -2\beta a\dot{\varphi}\sin\varphi, \tag{7}$$

$$2ma^2\ddot{\varphi} = \beta a\dot{R}\sin\varphi. \tag{8}$$

The right-hand sides of Eqs. (7) and (8) obviously represent the net force and the net torque, respectively, applied to the dipole. With the aid of (2), Eq. (3) can be rewritten as

$$\dot{R} = (J + 2\beta a\cos\varphi)/(2m). \tag{9}$$

The total energy E of the system is the kinetic energy. Using Eq. (9), the total energy can thus be expressed as

$$ma^2\dot{\varphi}^2 + (J + 2\beta a\cos\varphi)^2/(4m) = E. \tag{10}$$

The classical bound (or trapped) state corresponds to the case where the mean value is

$$\langle\dot{R}\rangle_t = 0. \tag{11}$$

Therefore, classical BSECs are solutions to Eq. (10) that fit the equation

$$\langle\dot{R}\rangle_t = \frac{1}{2m}\langle J + 2\beta a\cos\varphi\rangle_t \tag{12}$$

$$= \frac{1}{2mT}\int_0^T (J + 2\beta a\cos\varphi)dt = 0,$$

where T is a period of the motion.

If $J < -2\beta a$ or $J > 2\beta a$, then it is seen from (12) that $\langle\dot{R}\rangle_t \neq 0$, and the trapping is impossible. Thus, let us examine the case of $|J| \leq 2\beta a$. Suppose that $0 < J \leq 2\beta a$. From Eq. (10), we have

$$dt = \frac{\sqrt{m}\,a d\varphi}{\sqrt{E - \frac{1}{4m}(J + 2\beta a\cos\varphi)^2}}. \tag{13}$$

Equation (10) describes the motion of a particle in the effective potential $w(\varphi) = \frac{1}{4m}(J + 2\beta a\cos\varphi)^2$ (see Fig. 1). For $|J| < 2\beta a$, the effective potential $w(\varphi)$ has two maxima, $E_1 = \frac{1}{4m}(J + 2\beta a)^2$ at $\varphi = 0$ and $E_2 = \frac{1}{4m}(J - 2\beta a)^2$ at $\varphi = \pi$. When $E > E_1$, the dipole rotates unidirectionally. When $E < E_1$, the rotational motion is oscillatory between the left, φ_l, and the right, φ_r, classical turning points. If $E_2 < E < E_1$, then $\varphi_l + \varphi_r = 2\pi$. If $E < E_2$, then the oscillation is confined to one or the other of two wells, as is clear from Fig. 1. The energy $E = E_1$ corresponds to the position of unstable equilibrium with $\varphi = 0$, while $E = E_2$ corresponds to the position of unstable equilibrium with $\varphi = \pi$. With the aid of (13), Eq. (12) is reduced to

$$\langle\dot{R}\rangle_t = \frac{a}{\sqrt{m}\,T}\int_{\varphi_l}^{\varphi_r} \frac{(J + 2\beta a\cos\varphi)d\varphi}{\sqrt{E - \dfrac{1}{4m}(J + 2\beta a\cos\varphi)^2}}, \tag{14}$$

where T is the period of a complete oscillation if $E < E_1$; if $E > E_1$, then T is the period of one complete rotation, while φ_l and φ_r are to be replaced by 0 and π, respectively. If $E > E_1$, the numerator of the integrand in Eq. (14) is the largest (and positive) and the denominator is the smallest when $\varphi = 0$. Therefore, we conclude that $\langle\dot{R}\rangle_t > 0$ for $E > E_1$. Indeed, $\langle\dot{R}\rangle_t \to +\infty$ as E approaches E_1 from either above or below. By a similar argument, $\langle\dot{R}\rangle_t \to -\infty$ as $E \to E_2$ from either above or below, since the integral is then dominated by the region near $\varphi = \pi$, where $J + 2\beta a\cos\varphi < 0$. Hence, $\langle\dot{R}\rangle_t$ is a continuous function of E that is negative for $E = E_2 + \varepsilon$ and positive for $E = E_1 - \varepsilon$. Thus, $\langle\dot{R}\rangle_t = 0$ for some unique $E\,(J) \in (E_2, E_1)$, and the motion is trapped. Alternatively, for any energy $E \in (0, 2\beta a)$, there exists a unique value $J\,(E)$ of J for which the motion is trapped. Omitting further details, we merely indicate that, according to our analysis of (14), the motion is untrapped for any $E > E_1$ and for any $E < E_2$, the only exception being

the trivial case of $E = 0$ in which the dipole is at rest. The case of $-2\beta a \le J < 0$ can be analyzed in the same manner; this leads to a trapped state when $J = -J(E)$.

The remaining case of $J = 0$ is trivial. The integrand in Eq. (14) is antisymmetric about $\varphi = \pi/2$ if $J = 0$, while $\varphi_r = \pi - \varphi_l$ in this case. Hence, $\langle \dot{R} \rangle_t = 0$, and the motion is trapped for any E with the only exception of $E = E_1$, which corresponds to unstable static equilibrium.

Our conclusions are the following:

(i) If $J = 0$, then the classical motion is trapped for any energy $E > 0$, with the exception of $E = \beta^2 a^2/m$.

(ii) For any $J \neq 0$ and $E > \beta^2 a^2/m$, there are no trapped states.

(iii) For any E from the interval $0 < E < \beta^2 a^2/m$, there exists $J(E) \in (0, 2\beta a)$ such that, for $J = \pm J(E)$, the classical motion is trapped; $J(E) \to 0$ when $E \to \beta^2 a^2/m$, and $J(E) \to 2\beta a$ when $E \to 0$.

A qualitative plot of the curve $J(E)$ showing the combinations of values of the energy E and of the integral of motion J corresponding to classical BSECs is presented in Fig. 2.

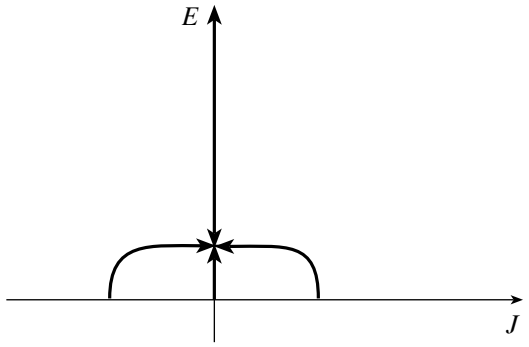

Fig. 2. A qualitative plot of $J(E)$ in the plane E versus J for the rigid dipole. Thick lines show the values of J that, for a given E, give rise to the trapped motion.

2.2. Nonrigid Dipole

Our model of a nonrigid dipole is two particles of mass m bound together by the potential $W(\mathbf{r})$, where $\mathbf{r}$ is the relative coordinate, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. One of the particles carries a charge $+e$, while the charge of the other is $-e$. The y axis is chosen to be parallel to a uniform magnetic field $\mathbf{B}$. In the same manner as for a rigid dipole, we restrict the motion of the center of mass to the z axis and the relative motion of the particles to the x axis. These restrictions are consistent with, and therefore provide a special case of, the full classical motion of a nonrigid dipole in three dimensions.

The Hamiltonian is

$$H = (P + \beta x/2)^2/(4m) + (p_x - \beta R/2)^2/m + V(x), \tag{15}$$

where x is the projection of the relative coordinate $\mathbf{r}$ onto the x axis, p_x is the canonically conjugate momentum, and $V(x)$ is the potential energy of the relative motion of the particles. We note that the Hamiltonian in (15) can be used in a more general case of the relative motion constrained to the xy plane, provided that the potential $V(\mathbf{r})$ can be expressed as $V(\mathbf{r}) = V_x(x) + V_y(y)$. This is the case, for example, of the harmonic oscillator potential $V(\mathbf{r}) = V_x(x) + V_y(y)$. With this assumption, the motion in the y direction can be treated independently.

There is an integral of motion J given by

$$J = P - \beta x/2, \tag{16}$$

which is the nonrigid equivalent of Eq. (2). As in the case of the rigid dipole, J is an integral of motion both in the classic and in the quantum case.

The canonical equations of motion associated with Hamiltonian (15) are

$$\dot{R} = (P + \beta x/2)/(2m), \tag{17}$$

$$\dot{x} = 2(p_x - \beta R/2)/m, \tag{18}$$

$$\dot{P} = \beta(p_x - \beta R/2)/m, \tag{19}$$

$$\dot{p}_x = -\beta(P + \beta x/2)/(4m) - \frac{d}{dx}V(x). \tag{20}$$

From Eqs. (17)–(20), it is easy to derive the equations of motion in the form

$$2m\ddot{R} = \beta\dot{x}\,, \tag{21}$$

$$m\ddot{x}/2 = -\beta\dot{R} - \frac{d}{dx}V(x). \tag{22}$$

Equations (21) and (22) are just Newton's equations of motion for the center of mass and for the relative motion of the particles. With the aid of (16), Eq. (17) can be rewritten as

$$\dot{R} = (J + \beta x)/(2m). \tag{23}$$

The total energy of the system E is the kinetic energy together with $V(x)$, the potential energy of binding. Thus, using Eq. (23), we can express the total energy as

$$m\dot{x}^2/4 + (J + \beta x)^2/(4m) + V(x) = E. \tag{24}$$

For the trapped motion, $\langle \dot{R} \rangle_t = 0$, which, in view of (23), is equivalent to

$$J + \beta\langle x \rangle_t = 0\,. \tag{25}$$

The average $\langle x \rangle_t$, to be derived from Eq. (24), is a function of E and J. For an arbitrary potential $V(x)$,

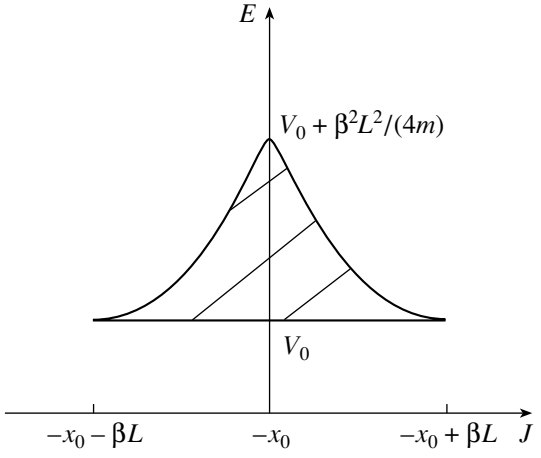

Fig. 3. Qualitative picture of the region of E and J where the motion is trapped in the case of a potential with a plateau (shaded area).

Eq. (25) may have solutions or have none. We shall discuss a few particular choices of $V(x)$.

First, we note that, if $J = 0$, then, for any symmetric potential $V(x) = V(-x)$ such that the effective potential $w(x) = V(x) + \beta^2 x^2/(4m)$ is a monotonically increasing function for $x > 0$, a trapped state will exist at any energy E.

Next, we discuss the case of the oscillator potential $V(x) = \dfrac{m\omega^2}{4}(x - x_0)^2$. Although this potential does not satisfy the physically necessary condition $V(x) = V(-x)$, it may be a good approximation if the motion confines x to a sufficiently small interval $x \in [x_0 - \delta x_0, x_0 + \delta x_0]$. For this case, we find from (24) that

$$\frac{m}{4}\,\dot{x}^2 + \left(\frac{m\omega^2}{4} + \frac{1}{4m}\,\beta^2\right) \times \left(x + \frac{\beta J - m^2\omega^2 x_0}{m^2\omega^2 + \beta^2}\right)^2 + \frac{m\omega^2}{4(m\omega^2 + \beta^2)}(J + \beta x_0)^2 = E; \tag{26}$$

obviously, we then have

$$\langle x\rangle_t = (m^2\omega^2 x_0 - \beta J)/(\beta^2 + m^2\omega^2). \tag{27}$$

From Eqs. (25) and (27), we find that, for any energy E, the motion is trapped if

$$J = -\beta x_0. \tag{28}$$

If this condition is met, then $\langle x\rangle_t = x_0$, the energy $E = \dfrac{m}{4}\,\dot{x}^2 + \dfrac{1}{4m}\left(\beta^2 + m^2\omega^2\right)(x - x_0)^2$, and the amplitude of the oscillation in x is $\sqrt{4mE\,/(\beta^2 + m^2\omega^2)}$. Hence, if the assumed form of $V(x)$ is an approximation valid only for $|x - x_0| \le \delta x_0$, our conclusion that Eq. (28) leads to a trapped motion is valid only for energies $E \le \dfrac{1}{4m}\left(\beta^2 + m^2\omega^2\right)\delta x_0^2$.

We next consider the case of a potential that has a plateau for some interval of x; that is, we suppose that $V(x) = V_0 = \text{const}$ for $x \in [x_0 - L, x_0 + L]$, where $L > 0$. Within this range, the effective potential is $w(x) = V_0 + \dfrac{(J + \beta x)^2}{4m}$, which has a minimum at $x_1 = -J/\beta$. If $x_1 \equiv -J/\beta \in [x_0 - L, x_0 + L]$, that is, if $J \in [-\beta x_0 - \beta L,\ -\beta x_0 + \beta L]$, and if the motion in x is confined to the range $[x_0 - L, x_0 + L]$, then $\langle x\rangle_t = -J/\beta = x_1$, so that $\langle \dot{R}\rangle_t = 0$ by virtue of Eq. (23) and the classical motion is confined. The amplitude of the x oscillation is $\sqrt{4m\,(E - V_0)\,/\beta^2}$. Hence, the trapped classical motion is possible for any energy $E \in \left[V_0,\ V_0 + \beta^2 L^2/(4m)\right]$, provided that J takes any value from the interval

$$-\beta(L + x_0) + \sqrt{4m(E - V_0)} \le J \le \beta(L - x_0) - \sqrt{4m(E - V_0)} \tag{29}$$

(see Fig. 3).

This example is particularly interesting, because it shows that the dipole can be trapped for a wide and continuous range of values of the energy E and of the integral of motion J, or, in other words, in a wide and continuous range of initial conditions.

3. QUANTIZATION

For both problems, we have a Hamiltonian and an additional integral of motion commuting with the Hamiltonian. Thus, in the quantum case, we should seek a simultaneous solution of the Schrödinger equation

$$\hat{H}\Psi = E\Psi, \tag{30}$$

where $\hat{H}$ is given by (1) or (15), and of the equation for the integral of motion J,

$$\hat{J}\Psi = J\Psi, \tag{31}$$

where the operator $\hat{J}$ is given by (2) or (16). We shall use units in which $\hbar = 1$.

3.1. Rigid Dipole

We use the following trick to simplify the Hamiltonian in (1). We introduce a wave function ψ that is related to Ψ by the equation

$$\Psi\,(R, \varphi) = e^{i\beta Ra\cos\varphi}\psi\,(R, \varphi)\,. \tag{32}$$

Clearly, $\Psi(R,\varphi)$ will represent a confined state (i.e., a quantum BSEC) if and only if $\int |\psi(R,\varphi)|^2\, dR < \infty$. By using Eqs. (1), (2), and (32), we reduce (30) and (31) to the form

$$\left[(\hat{P} + 2\beta a\cos\varphi)^2/(4m) + \hat{\pi}^2/(4ma^2)\right]\psi = E\psi, \tag{33}$$

$$\hat{P}\psi = J\psi. \tag{34}$$

Equation (34) has a continuous eigenvalue spectrum, with plane-wave eigenfunctions $\exp(iJR)$. In Eq. (33), $\hat{P}$ can be replaced by its eigenvalue J, and this equation can be solved with respect to the single variable φ. Equation (33) is subjected to periodic boundary conditions, $\psi(\varphi) = \psi(\varphi + 2\pi)$; thus, the spectrum of E for any given J is discrete. An infinite number of discrete energy eigenvalues $E_n(J)$ will be associated with any particular value of J. However, the lowest energy eigenvalue is $E_0(J) \sim J^2$ for large J. Thus, any finite energy E can be realized only by a finite number of plane waves with a given momentum J. It is impossible to construct a normalizable wave function as a superposition of a finite number of plane waves. Therefore, a rigid dipole in a uniform magnetic field does not have quantum BSEC solutions.

3.2. Nonrigid Dipole

We apply, to Eqs. (30) and (31) with $\hat{H}$ given by (15) and $\hat{J}$ given by (16), a trick similar to that used in the case of a rigid dipole; namely, we set

$$\Psi = e^{i\beta Rx/2}\psi \tag{35}$$

in order to obtain the equations

$$\left[(\hat{P} + \beta x)^2/(4m) + \hat{p}_x^2/m + V(x)\right]\psi = E\psi, \tag{36}$$

$$\hat{P}\psi = J\psi. \tag{37}$$

By virtue of (37), the operator $\hat{P}$ can be replaced by its eigenvalue J in Eq. (36).

The most natural choice for the potential $V(x)$ is the oscillator potential $V(x) = \frac{1}{4}m\omega^2(x - x_0)^2$. In this case, the Schrödinger Eq. (36) can be easily solved. As a result, we obtain an infinite number of discrete energy eigenvalues

$$E_n = \sqrt{\omega^2 + \beta^2/m^2}(n + 1/2) + [m\omega^2(J + \beta x_0)^2]/[4(\beta^2 + m^2\omega^2)] \tag{38}$$

for any value of the integral J. For any particular energy E, Eq. (38) and $E = E_n$ together define a functional relation between J and n. However, Eq. (38) shows that there is a maximal n consistent with any particular energy E. Hence, only a finite number of eigenvalues J of $\hat{J}$ will be consistent with any fixed energy E; this implies that there can be only a finite number of plane waves e^{iJR} associated with this energy. Thus, there are also no quantum BSEC solutions for a nonrigid dipole with an oscillator binding potential.

As in the classical case, if the oscillator form for $V(x)$ is valid only for $|x - x_0| < \delta x_0$, then we should require that $\left\langle (x - x_0)^2 \right\rangle^{1/2} < \delta x_0$. This will imply a condition for the validity of Eq. (38) of the form $n < n_{\max}$, with the upper limit $n_{\max}$ deduced from $\langle (x - x_0)^2\rangle^{1/2} < \delta x_0$.

Lastly, we consider the case where $V(x)$ has a plateau, $V(x) = V_0$ for $x_0 - L \le x \le x_0 + L$. In the simple but unphysical case of $L = \infty$, the Schrödinger equation (36) has the discrete energy eigenvalues

$$E_n = V_0 + \beta(n + 1/2)/m \tag{39}$$

independent of J, while there are no constraints on J. More realistically, if L is finite, then a solution leading to Eq. (39) might give a good approximation, provided that the wave function is nonzero only within the interval $x_1 \in [x_0 - L, x_0 + L]$. The rms radius of the state provides an estimate of the wave-function range. The standard quantum theory of a harmonic oscillator yields $\langle (x - x_1)^2\rangle^{1/2} = \sqrt{2m(E_n - V_0)/\beta^2}$ with $x_1 = -J/\beta$. This leads to the same constraints (29) as in the classical case. In particular, these constraints can be satisfied only if $E_n - V_0 < \beta^2 L^2/(4m)$. Because a continuous range of eigenvalues J is allowed, we may construct infinitely many BSEC solutions, that is, normalizable wave packets, all of them being energy eigenfunctions corresponding to the energy eigenvalue E_n.

Note, however, that the wave functions corresponding to the energy levels (39) have small but nonzero amplitudes for $x \notin [x_0 - L, x_0 + L]$, where $V(x) \ne V_0$. It is easy to show that, if the potential $V(x)$ is a rectangular well, then the spectrum of J is discrete for any given energy E. Thus, due to the constraints in (29), only a finite number of plane waves e^{iJR} is associated with any given energy E and the quantum BSEC is destroyed. Nevertheless, for energies that are sufficiently low, the number of possible values of J can be very large. As a result, a BSEC is expected to transform into a very sharp resonant state that may appear to be indistinguishable from a BSEC in applications. At the same time, we believe that it is possible to find a reasonable potential $V(x)$ binding the particles that will result in BSEC solutions for a dipole in a uniform magnetic field.

4. DISCUSSION

We have shown that a classical dipole can be trapped by a uniform time-independent magnetic field. The trapping is possible in all cases that we have considered, provided that the integrals of the motion E and J satisfy particular constraints. The trapping of the dipole presents an interesting and nontrivial example of a classical BSEC. This classical BSEC shares some characteristics of quantum BSECs—in particular, the trapping of the dipole is not associated with potential barriers and classical turning points.

We have also investigated the corresponding quantum problem. We have found that there are no quantum BSECs for the most natural choices of the parameters of the system, namely, a rigid rotating dipole or a nonrigid dipole bound by an oscillator potential. This conclusion looks strange and intriguing. We have already mentioned in the Introduction that, for all known quantum systems supporting BSECs, either there is no classical analog (e.g., systems with nonlocal interaction), or the corresponding classical system does not support a classical BSEC. On the other hand, we might conclude from all previously known examples of classical systems supporting BSECs that the corresponding quantum system does not have BSEC solutions, and our study seems to confirm this conclusion even for some classical systems that support BSECs without classical turning points.

One can suspect that, in the quantum case, the dipole has no BSEC solutions because of the uncertainty principle. For a rigid dipole or that bound by an oscillator potential, a classical BSEC is possible only if J has a precise value (which depends on E for the rigid case). However, Eqs. (2) and (16) show that $[\hat{J}, \hat{R}] = [\hat{P}, \hat{R}] = -i$. Therefore, the uncertainty principle requires that $\Delta J \Delta R \geq 1/2$. Hence, if J must have a precise value, then $\Delta J = 0$, so that $\Delta R = \infty$ and the quantum state cannot be confined.

However, the actual situation is somewhat different from this. Quantum mechanics for the dipole allows a quantum state with energy E associated with more than one eigenvalue J of $\hat{J}$, and the corresponding wave functions may be superposed to form a wave packet that is still an eigenfunction of $\hat{H}$ with the same energy eigenvalue E. However, for the rigid dipole or the dipole confined by an oscillator potential, only a finite number of eigenvalues J can be associated with any energy eigenvalue E, and this is not sufficient for producing a normalizable wave packet. Thus, the reason for the absence of quantum BSECs is more subtle than what is suggested by the uncertainty principle.

Note also the example of a classical potential with a plateau discussed above (see Fig. 3). In this case, the classical dipole is trapped in a wide and continuous range of E and J, so that the initial conditions need not violate the uncertainty principle.

Quantum BSEC solutions may be possible if the dipole is bound by some potential other than the harmonic-oscillator potential. Our experience encourages us to seek a realistic potential $V(x)$ that will allow an infinite number of continuous J eigenvalues associated with the same energy eigenvalue. We derive a limiting case for such a potential directly from (38), namely, the limit $\beta \to 0$, or, more accurately, $\beta \ll m\omega$. This is the limit of a weak magnetic field. Note that this limit is opposite to the limit of a potential with a plateau: the potential $V(x) = 0$ is obtained from the oscillator potential in the limit $\omega \to 0$ and corresponds to $\beta \gg m\omega$. One might expect the limit $\beta \ll m\omega$ to be just the classical limit. However, even in the limit $m \to \infty$, the motion in the lowest oscillator states remains essentially quantum. The true classical limit, in which quantum BSECs should become classical BSECs, is $n \to \infty$ and $m \to \infty$. If n is small, only the center-of-mass motion is classical, while the internal oscillations are quantum. If the internal oscillation is frozen in the ground state (the rms radius of the $n = 0$ oscillator state is $\sqrt{2/(m\omega)}$, so that the limit $m\omega \to \infty$ guarantees that the motion is frozen in the ground state), as would be true for a typical dipole gas molecule at normal room temperatures, we get a different classical limit. In this case, if we restore the possibility of rotational motion (ignored in Subsections 2.2 and 3.2), we should be able to find classical BSECs associated with the rigid dipole in Subsection 2.1.

It would be interesting to find additional nontrivial and physically reasonable potentials $V(x)$ binding the particles that will provide quantum BSEC solutions for a dipole in a magnetic field. Note that a potential $V(x)$ binding a dipole molecule (to take a specific example) can be modified to some extend by an external electric field in the x direction.

The rigid dipoles that we have studied in this paper may also be interesting for experimental investigation. We expect that a rigid-dipole molecule should behave classically in a weak magnetic field, but that, in a stronger field, quantum effects should become important. In a gas of dipole molecules at normal temperatures, vibrational degrees of freedom are frozen in the ground state, so that the dipole is essentially rigid. In these circumstances, some of them may be trapped by a weak magnetic field. When the field is increased or when the temperature is lowered so that angular-momentum quantization is important enough to be affect the specific heat, the trapped state should be destroyed by quantum effects. A classical BSEC when destroyed by a magnetic field

sufficiently strong for giving rise to quantum effects should transform into a very narrow long-lived resonance. Such resonances should affect strongly the density of states in the system and give rise to some pecularities in the dispersion of some excitations.

The BSECs discussed in this paper may also be interesting for studying chaotic motion. The general classical Hamiltonian of a rigid dipole is nonintegrable [12]. The same is true for a nonrigid dipole. Therefore, the motion of a dipole can be chaotic. Classical BSEC solutions correspond to islands of integrability in the phase space of the system.

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Lie Algebras on Hyperelliptic Curves and Finite-Dimensional Integrable Systems*

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Abstract—We construct a new family of infinite-dimensional Lie algebras on hyperelliptic curves. Using them, we find new integrable Hamiltonian systems, which are direct higher rank generalizations of the Steklov–Liapunov integrable systems associated with the $e(3)$ algebra and the Steklov–Veselov integrable systems associated with the $so(4)$ algebra. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

The main purpose of the present paper is to introduce new integrable Hamiltonian systems admitting Lax pair representations with spectral parameters [1]. Usually, the dependence of the Lax operator on the spectral parameter is rational or elliptic [2, 3]. In the present paper, we find Lax pairs with the hyperelliptic dependence on the spectral parameter. They will make it possible to construct new integrable Hamiltonian systems on finite-dimensional Lie algebras.

Our approach is based on the use of infinite-dimensional Lie algebras. It is known [2, 3] that the group-theoretical explanation of the integrability of Lax equations on finite-dimensional Lie algebras with rational spectral parameters is based on the Kostant–Adler scheme [4] and loop algebras. In [5, 6], it was shown that, in a similar way, the Lax equation with elliptic spectral parameters on the $so(3)$ algebra and some of its extensions could be obtained from the infinite-dimensional Lie algebras of special elliptic matrix-valued functions with values in $so(3)$.

We generalize the construction of [6] to the case of classical matrix algebras of higher ranks. Increasing the rank of an algebra requires automatically increasing the genus of the curve. As a result, we obtain the algebras of $gl(n)$-, $so(n)$-, and $sp(n)$-valued functions on hyperelliptic curves of genus g, where $n = 2g + 2$ or $n = 2g + 1$. The most important property of the discovered algebras is that they admit the Kostant–Adler scheme and, hence, could be used to construct new integrable systems.

As an example of this construction, we obtain new integrable systems on the direct and semidirect sums of two simple Lie algebras that are higher rank generalizations of the Steklov–Veselov integrable case [7] on $so(4) = so(3) \oplus so(3)$ and of the Steklov–Liapunov integrable case on $e(3) = so(3) + \mathbb{R}^3$ [8].

2. QUASIGRADED ALGEBRAS ON HYPERELLIPTIC CURVES

2.1. Construction

1. *Hyperelliptic curve embedded in* $\mathbb{C}^n$. In $\mathbb{C}^n$ space with coordinates $w_1, w_2, \ldots, w_n$, we consider the system of quadrics

$$w_i^2 - w_j^2 = a_j - a_i, \quad i, j = 1, n, \qquad (1)$$

where a_i are arbitrary complex numbers. The rank of this system is $n - 1$; therefore, the substitution

$$w_i^2 = w - a_i, \quad y = \prod_{i=1}^{n} w_i$$

solves these equations and defines the equation of the hyperelliptic curve $\mathcal{H}$. Hence, Eqs. (1) define the embedding of the hyperelliptic curve $\mathcal{H}$ in the linear space $\mathbb{C}^n$.

2. *Classical Lie algebras.* Let $\mathfrak{g}$ denote one of the classical matrix Lie algebras $gl(n)$, $so(n)$, and $sp(n)$ over the field of complex numbers. We will need the explicit form of their bases. Let $I_{i,j} \in \mathrm{Mat}(n, C)$ be the matrix defined as

$$(I_{ij})_{ab} = \delta_{ia}\delta_{jb}.$$

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Evidently, a basis in the algebra $gl(n)$ could be built from the matrices $X_{ij} \equiv I_{ij}, i,j \in 1,\dots,n$. The commutation relations in $gl(n)$ will have the standard form

$$[X_{i,j}, X_{k,l}] = \delta_{k,j}X_{i,l} - \delta_{i,l}X_{k,j}.$$

A basis in the algebra $so(n)$ could be chosen as $X_{ij} \equiv I_{ij} - I_{i,j}$, $i,j \in 1,\dots,n$, with the "skew-symmetry" property $X_{ij} = -X_{ji}$ and the commutation relations

$$[X_{i,j}, X_{k,l}] = \delta_{k,j}X_{i,l} - \delta_{i,l}X_{k,j} + \delta_{j,l}X_{k,i} - \delta_{k,i}X_{j,l}.$$

The basis in the algebra $sp(n)$ is chosen here as $X_{ij} = I_{ij} - \epsilon_i\epsilon_j I_{-i,-j}$, $|i|,|j| \in 1,\dots,n$, with the property $X_{i,j} = -\epsilon_i\epsilon_j X_{-j,-i}$, where $\epsilon_j = \mathrm{sgn} j$, and the commutation relations

$$[X_{i,j}, X_{k,l}] = \delta_{k,j}X_{i,l} - \delta_{i,l}X_{k,j} + \epsilon_i\epsilon_j(\delta_{j,-l}X_{k,-i} - \delta_{k,-i}X_{-j,l}).$$

3. *Algebras on the curve.* For the basic elements X_{ij} of all three algebras $gl(n)$, $so(n)$, and $sp(n)$ and arbitrary $n \in \mathbb{Z}$, we introduce the following algebra-valued functions on the curve $\mathcal{H}$, or, to be more precise, on its double covering:

$$X_{ij}^n = X_{ij} \otimes w^n w_i w_j.$$

The following theorem holds:

Theorem 1. (i) Elements X_{ij}^n form $n \in \mathbb{Z}$ quasi-graded Lie algebra $\tilde{\mathfrak{g}}_{\mathcal{H}}$ with the commutation relations

$$[X_{ij}^n, X_{kl}^m] = \delta_{kj}X_{il}^{n+m+1} - \delta_{il}X_{kj}^{n+m+1} + a_i\delta_{il}X_{kj}^{n+m} - a_j\delta_{kj}X_{il}^{n+m} \quad \text{for} \quad gl(n), \tag{2a}$$

$$[X_{ij}^n, X_{kl}^m] = \delta_{kj}X_{il}^{n+m+1} - \delta_{il}X_{kj}^{n+m+1} + \delta_{jl}X_{ki}^{n+m+1} - \delta_{ik}X_{jl}^{n+m+1} + a_i\delta_{il}X_{kj}^{n+m} - a_j\delta_{kj}X_{il}^{n+m} + a_i\delta_{ik}X_{jl}^{n+m} - a_j\delta_{jl}X_{ki}^{n+m} \quad \text{for} \quad so(n), \tag{2b}$$

$$[X_{ij}^n, X_{kl}^m] = \delta_{kj}X_{il}^{n+m+1} - \delta_{il}X_{kj}^{n+m+1} + \epsilon_i\epsilon_j(\delta_{j-l}X_{k-i}^{n+m+1} - \delta_{i-k}X_{j-l}^{n+m+1}) + a_i\delta_{il}X_{kj}^{n+m} - a_j\delta_{kj}X_{il}^{n+m} + a_i\epsilon_i\epsilon_j(a_i\delta_{i-k}X_{j-l}^{n+m} - a_j\delta_{j-l}X_{k-i}^{n+m}) \quad \text{for} \quad sp(n). \tag{2c}$$

(ii) The algebra $\tilde{\mathfrak{g}}_{\mathcal{H}}$ as a linear space admits a decomposition into the direct sum of two subalgebras: $\tilde{\mathfrak{g}}_{\mathcal{H}} = \tilde{\mathfrak{g}}_{\mathcal{H}}^+ + \tilde{\mathfrak{g}}_{\mathcal{H}}^-$, where the subalgebras $\tilde{\mathfrak{g}}_{\mathcal{H}}^+$ and $\tilde{\mathfrak{g}}_{\mathcal{H}}^-$ are generated by the elements X_{ij}^0 and X_{ij}^{-1}, respectively.

Example. Let $\mathfrak{g} = so(3)$. In this case, the constructed algebra will coincide with the "even" subalgebra of the algebra of hidden symmetry of Landau–Lifschitz equations. Indeed, setting $X_k \equiv \epsilon_{ijk}X_{ij}$, we obtain the commutation relations

$$[X_i^n, X_j^m] = \epsilon_{ijk}X_k^{n+m+1} + \epsilon_{ijk}a_kX_k^{n+m}.$$

Remark. From item (i) of Theorem (1), it follows that, in the case of rational degeneracy of the curve $\mathcal{H}$, i.e., when $a_i = 0$, $\tilde{\mathfrak{g}}_{\mathcal{H}} = \tilde{\mathfrak{g}}$, where $\tilde{\mathfrak{g}}$ is an ordinary loop algebra.

2.2. Coadjoint Representation

To define the coadjoint representation, we have to define $\tilde{\mathfrak{g}}_{\mathcal{H}}^*$. We assume that $\tilde{\mathfrak{g}}_{\mathcal{H}}^* \subset \mathfrak{g} \otimes A$, where A is an algebra of functions on the double covering of the curve $\mathcal{H}$. Let us define pairing between $L(w) \in \tilde{\mathfrak{g}}_{\mathcal{H}}^*$ and $X(w) \in \tilde{\mathfrak{g}}_{\mathcal{H}}$ as

$$\langle X(w), L(w)\rangle_f = c_n \mathrm{res}_{w=0} f^{-1}(w)y^{-1}(w)(X(w)|Y(w)), \tag{3}$$

where $f(w)$ is an arbitrary function on the curve $\mathcal{H}$. It is easy to show that the element dual to X_{ij}^{-m} with respect to this pairing is $Y_{ij}^m \equiv (X_{ij}^{-m})^* = \dfrac{w^{m-1}f(w)y(w)}{w_iw_j}X_{ij}^*$. Hence, the general element of the dual space has the form

$$L(w) = \sum_{m\in\mathbb{Z}}\sum_{i,j=1}^{n} l_{ij}^m \frac{w^{m-1}f(w)y(w)}{w_iw_j}X_{ij}^*. \tag{4}$$

The coadjoint action of the algebra $\tilde{\mathfrak{g}}_{\mathcal{H}}$ on its dual space $\tilde{\mathfrak{g}}_{\mathcal{H}}^*$ coincides with the commutator:

$$ad^*_{X(w)}L(w) = [L(w), X(w)]. \tag{5}$$

From the explicit form of the coadjoint action (5), we obtain the following statement:

Proposition 1. The functions $I_m^k(L(w)) = \mathrm{res}_{w=0}w^{-m-1}\mathrm{tr}L(w)^k$, where $m \in \mathbb{Z}$, are invariants of the coadjoint representation.

Hence, not only do the constructed Lie algebras admit decomposition into the direct sum of two subalgebras, but they also possess an infinite number of invariant functions. This permits us to use them in constructing integrable systems.

3. INTEGRABLE SYSTEMS FROM HYPERELLIPTIC ALGEBRAS

3.1. Poisson Structures and Poisson Subspaces

1. *First Lie–Poisson structure.* In the space $\tilde{\mathfrak{g}}_{\mathcal{H}}^*$, one can define many Lie–Poisson structures using different pairings. We will use the pairing in (3) with $f(w) = w$:

$$\langle X(w), L(w)\rangle_{-1} \tag{6}$$

$$= c_n \mathrm{res}_{w=0} w^{-1} y^{-1}(w)(X(w)|L(w)).$$

It defines brackets on $P(\tilde{\mathfrak{g}}^*_{\mathcal{H}})$ as

$$\{F(L), G(L)\} \tag{7}$$
$$= \sum_{l,m\in\mathbb{Z}} \sum_{i,j,p,s=1}^{n} \langle L(w), [X_{ij}^{-l}, X_{ps}^{-m}]\rangle_{-1} \frac{\partial G}{\partial l_{ij}^{l}} \frac{\partial F}{\partial l_{ps}^{m}}.$$

Proposition 1 entails the following statement:

Proposition 2. The functions $I_m^k(L(w))$ are central for brackets $\{\ ,\ \}$.

Let us explicitly calculate the Poisson brackets (7). Considering that $l_{ij}^m = \langle L(w), X_{ij}^{-m}\rangle_{-1}$, we can easily show that, for the coordinate functions l_{ij}^m, these brackets will have the form

$$\{l_{ij}^n, l_{kl}^m\} = \delta_{kj} l_{il}^{n+m-1} - \delta_{il} l_{kj}^{n+m-1} \tag{8a}$$
$$+ a_i \delta_{il} l_{kj}^{n+m} - a_j \delta_{kj} l_{il}^{n+m} \quad \text{for} \quad gl(n),$$
$$\{l_{ij}^n, l_{kl}^m\} = \delta_{kj} l_{il}^{n+m-1} - \delta_{il} l_{kj}^{n+m-1} \tag{8b}$$
$$+ \delta_{jl} l_{ki}^{n+m-1} - \delta_{ik} l_{jl}^{n+m-1} + a_i \delta_{il} l_{kj}^{n+m} - a_j \delta_{kj} l_{il}^{n+m}$$
$$+ a_i \delta_{ik} l_{jl}^{n+m} - a_j \delta_{jl} l_{ki}^{n+m} \quad \text{for} \quad so(n),$$
$$\{l_{ij}^n, l_{kl}^m\} = \delta_{kj} l_{il}^{n+m-1} - \delta_{il} l_{kj}^{n+m-1} \tag{8c}$$
$$+ \epsilon_i \epsilon_j (\delta_{j-l} l_{k-i}^{n+m-1} - \delta_{i-k} l_{j-l}^{n+m-1}) + a_i \delta_{il} l_{kj}^{n+m}$$
$$- a_j \delta_{kj} l_{il}^{n+m} + \epsilon_i \epsilon_j (a_i \delta_{i-k} l_{j-l}^{n+m} - a_j \delta_{j-l} l_{k-i}^{n+m})$$
$$\text{for} \quad sp(n).$$

2. *Second Lie–Poisson structure.* In the space $\tilde{\mathfrak{g}}^*_{\mathcal{H}}$, we introduce new Poisson brackets $\{\ ,\ \}_0$, which are Lie–Poisson brackets for the algebra $\tilde{\mathfrak{g}}^0_{\mathcal{H}}$, where $\tilde{\mathfrak{g}}^0_{\mathcal{H}} = \tilde{\mathfrak{g}}^-_{\mathcal{H}} \ominus \tilde{\mathfrak{g}}^+_{\mathcal{H}}$. Explicitly, these brackets have the form

$$\{l_{ij}^n, l_{kl}^m\}_0 = -\{l_{ij}^n, l_{kl}^m\}, \quad n, m \in \mathbb{Z}_+,$$
$$\{l_{ij}^n, l_{kl}^m\}_0 = \{l_{ij}^n, l_{kl}^m\}, \quad n, m \in \mathbb{Z}_- \cup 0,$$
$$\{l_{ij}^n, l_{kl}^m\}_0 = 0, \quad m \in \mathbb{Z}_- \cup 0,$$
$$n \in \mathbb{Z}_+ \quad \text{or} \quad n \in \mathbb{Z}_- \cup 0, \quad m \in \mathbb{Z}_+.$$

Let subspace $\mathcal{M}_{s,p} \subset \tilde{\mathfrak{g}}^*_{\mathcal{H}}$ be defined as

$$\mathcal{M}_{s,p} = \sum_{m=-s+1}^{p} (\tilde{\mathfrak{g}}^*_{\mathcal{H}})_m.$$

Brackets $\{\ ,\ \}_0$ could be correctly restricted to $\mathcal{M}_{s,p}$. This follows from Proposition 3 formulated immediately below.

Proposition 3. The subspaces $\mathcal{J}_{p,s} = \sum_{m=-\infty}^{-p-1} (\tilde{\mathfrak{g}}_{\mathcal{H}})_m + \sum_{m=s}^{\infty} (\tilde{\mathfrak{g}}_{\mathcal{H}})_m$ are ideals in $\tilde{\mathfrak{g}}^0_{\mathcal{H}}$.

3.2. Algebras of Integrals and Hamilton's Equations

To construct integrable Hamiltonian systems, we need a large family of mutually commuting functions (integrals of motion). This is provided by the following theorem:

Theorem 2. Let functions $\{I_m^k(L)\}$ be defined as in Proposition 1. They generate a commutative algebra with respect to the restriction of the brackets $\{\ ,\ \}_0$ on $\mathcal{M}_{s,p}$.

The dynamical equations considered here are Hamilton's equations of the form

$$\frac{dl_{ij}^k}{dt} = \{l_{ij}^k, H(l_{kl}^m)\}_0, \tag{9}$$

where the Hamiltonian H is one of the functions I_m^k or their linear combination. These equations can be written in the Lax form [3]

$$\frac{dL(w)}{dt} = [L(w), M(w)], \tag{10}$$

where $L(w) \in \mathcal{M}_{s,p}$, and the second operator is defined as $M(w) = (P_- - P_+) \times \nabla H(L)_{L=L(w)}$. Here, $P_\pm$ are projection operators on the subalgebra $\tilde{\mathfrak{g}}^\pm_{\mathcal{H}}$ and

$$\nabla H(L) = \sum_{k\in\mathbb{Z}} \sum_{i,j=1}^{s-1} \sum_{n} \frac{\partial H}{\partial l_{ij}^k} X_{ij}^{-k} \tag{11}$$

is an algebra-valued gradient of H.

Thus, we have constructed Hamiltonian systems admitting the Lax pair representation with the hyperelliptic spectral parameter and possessing (Theorem 2) a lot of mutually commuting integrals of motion. In the next section, we will consider several examples.

4. INTEGRABLE SYSTEMS IN FINITE-DIMENSIONAL QUOTIENTS

From the physical point of view, the most interesting examples usually arise in $\mathcal{M}_{s,p}$ spaces with small s and p. We will assume that the curve $\mathcal{H}$ is nondegenerate; i.e., $a_i \neq a_j$ for $i \neq j$. The basic algebra in all examples will be $\mathfrak{g} = so(n)$.

4.1. Generalized Interacting Tops

Let us consider the subspace $\mathcal{M}_{1,1}$. In the case of $a_i \neq 0$, it follows from the explicit form of the brackets given below that $\mathcal{M}_{1,1} = (\mathfrak{g} \oplus \mathfrak{g})^*$. The corresponding Lax operator $L(w) \in \mathcal{M}_{0,1}$ has the form

$$L(w) = \sum_{i,j=1}^{n} (l_{ij}^{(0)} + w l_{ij}^{(1)}) \frac{y(w)}{w_i w_j} X_{ij}^*.$$

In the $so(n)$ case, we may set $X^*_{ij} = X_{ij}$. The Lie–Poisson brackets between the coordinate functions $l^{(1)}_{ij}$ are the following:

$$\{l^{(0)}_{ij}, l^{(0)}_{kl}\} = -a_i\delta_{il}l^{(0)}_{kj} + a_j\delta_{kj}l^{(0)}_{il} - a_i\delta_{ik}l^{(0)}_{jl} + a_j\delta_{jl}l^{(0)}_{ki},$$
$$\{l^{(1)}_{i,j}, l^{(1)}_{k,l}\} = \delta_{k,j}l^{(1)}_{i,l} - \delta_{i,l}l^{(1)}_{k,j} + \delta_{j,l}l^{(1)}_{k,i} - \delta_{k,i}l^{(1)}_{j,l},$$
$$\{l^{(0)}_{ij}, l^{(1)}_{kl}\} = 0.$$

Setting $b_i = a_i^{1/2}$ and making the change of variables $l_{ij} = l^{(1)}_{i,j}$ and $m_{ij} = l^{(0)}_{ij}/(b_ib_j)$, we obtain the canonical coordinates of the direct sum of two algebras $so(n)$:

$$\{m_{i,j}, m_{k,l}\} = \delta_{k,j}m_{i,l} - \delta_{i,l}m_{k,j} + \delta_{j,l}m_{k,i} - \delta_{k,i}m_{j,l},$$
$$\{l_{i,j}, l_{k,l}\} = \delta_{k,j}l_{i,l} - \delta_{i,l}l_{k,j} + \delta_{j,l}l_{k,i} - \delta_{k,i}l_{j,l},$$
$$\{l_{ij}, m_{kl}\} = 0.$$

Commuting integrals are constructed by using the expansion of the functions $I_k(w) = \mathrm{tr}(L(w))^k$ in powers of w. We are interested in the quadratic integrals

$$h(w) \equiv I_2(w) = \sum_{s=0}^{n} h_s(l^{(1)}_{ij})w^s = \sum_{ij}\left(\prod_{k\neq i,j}(w-a_k)\right)(l^{(0)}_{ij} + wl^{(1)}_{ij})^2.$$

By performing direct calculations and making the above change of variables, we obtain

$$h_0 = (-1)^{n-2}(b_1^2b_2^2\ldots b_n^2)\sum_{i,j=1}^{n} m_{ij}^2,$$
$$h_1 = (-1)^{n-1}\sum_{i,j=1}^{n}\left(\sum_{k\neq i,j}\frac{b_1^2b_2^2\ldots b_n^2}{b_k^2}\right)(m_{ij})^2 - 2\frac{b_1^2b_2^2\ldots b_n^2}{b_ib_j}m_{ij}l_{ij},$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$h_{n-1} = -\sum_{i,j=1}^{n}\left(\sum_{k=1}^{n}b_k^2 - (b_i^2+b_j^2)\right)l_{ij}^2 - 2b_ib_jm_{ij}l_{ij}, \quad h_n = \sum_{i,j=1}^{n}(l_{ij})^2.$$

It is evident that the functions h_0 and h_n are invariants. For the Hamiltonian of the generalized interacting rigid bodies, we can take either h_{n-1} or h_1. The operator M and the Lax equations for these Hamiltonians are calculated straightforwardly.

Remark. In the $n = 3$ case, the functions h_2 and h_1 coincide with two independent integrals for the Steklov system on $so(4)$ that were discovered by Veselov [7].

4.2. Generalized Steklov–Liapunov System

Let us consider the subspace $\mathcal{M}_{0,2} = (\tilde{\mathfrak{g}}^+_{\mathcal{H}}/\mathcal{J}_{2,0})^*$. It is easy to show that $\mathcal{M}_{0,2} = (\mathfrak{g}+\mathfrak{g})^*$. The corresponding Lax operator $L(w) \in \mathcal{M}_{0,2}$ has the form

$$L(w) = w\left(\sum_{i,j=1}^{n}(l^{(1)}_{ij} + wl^{(2)}_{ij})\frac{y(w)}{w_iw_j}X^*_{ij}\right).$$

We will again concentrate on the $\mathfrak{g} = so(n)$ case and set $X^*_{ij} = X_{ij}$. The Lie–Poisson brackets between the coordinate functions are following:

$$\{l^{(1)}_{ij}, l^{(1)}_{kl}\} = \delta_{kj}l^{(1)}_{il} - \delta_{il}l^{(1)}_{kj} + \delta_{jl}l^{(1)}_{ki} - \delta_{ik}l^{(1)}_{jl} + a_i\delta_{il}l^{(2)}_{kj} - a_j\delta_{kj}l^{(2)}_{il} + a_i\delta_{ik}l^{(2)}_{jl} - a_j\delta_{jl}l^{(2)}_{ki},$$
$$\{l^{(1)}_{ij}, l^{(2)}_{kl}\} = \delta_{kj}l^{(2)}_{il} - \delta_{il}l^{(2)}_{kj} + \delta_{jl}l^{(2)}_{ki} - \delta_{ik}l^{(2)}_{jl},$$
$$\{l^{(2)}_{ij}, l^{(2)}_{kl}\} = 0.$$

The change of variables $l^{(1)}_{ij} = l_{ij} - 1/2(a_i + a_j)p_{ij}$ and $l^{(2)}_{ij} = p_{ij}$ transforms the above brackets into the standard brackets on the half-direct sum $so(n) + so(n)$:

$$\{l_{ij}, l_{kl}\} = \delta_{kj}l_{il} - \delta_{il}l_{kj} + \delta_{jl}l_{ki} - \delta_{ik}l_{jl},$$
$$\{l_{ij}, p_{kl}\} = \delta_{kj}p_{il} - \delta_{il}p_{kj} + \delta_{jl}p_{ki} - \delta_{ik}p_{jl},$$
$$\{p_{ij}, p_{kl}\} = 0.$$

Commuting integrals are constructed by using the expansion of the functions $I_k(w) = \mathrm{tr}(L(w))^k$ in powers of w. We are again interested predominantly in the quadratic integrals

$$h(w) \equiv I_2(w) = w^2\sum_{s=0}^{n} h_{s+2}(l^{(1)}_{ij})w^s = w^2\sum_{ij}\left(\prod_{k\neq i,j}(w-a_k)\right)(l^{(1)}_{ij} + wl^{(2)}_{ij})^2.$$

By performing direct calculations and making the above change of variables, we obtain the following set of Hamiltonians:

$$h_2 = (-1)^{n-2}\sum_{i,j=1}^{n}\frac{a_1a_2\ldots a_n}{a_ia_j}(l_{ij} - 1/2(a_i+a_j)p_{ij})^2,$$
$$\ldots\ldots\ldots\ldots\ldots$$

$$h_{n+1} = (-1)\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{i,j=1}^{n} p_{ij}^2\right) - 2\left(\sum_{i,j=1}^{n} l_{ij}p_{ij}\right),$$

$$h_{n+2} = \sum_{i,j=1}^{n} p_{ij}^2.$$

The last two functions are invariant functions. We choose the function $H = h_2$ for the Hamiltonian. The corresponding operator M is

$$M(w) = 2\sum_{i,j=1}^{n} \frac{a_1 a_2 \ldots a_n}{a_i a_j}$$

$$\times\, (l_{ij} - 1/2(a_i + a_j)p_{ij})w^{-1}w_i w_j X_{ij}.$$

The Lax equation has the standard form (10).

Remark. In the $n = 3$ case, the Hamiltonian H coincides, apart from the rescaling of momenta, with the Hamiltonian of the Steklov–Liapunov system in the form of Kotter [9].

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$N = 4$ Toda Chain (KdV) Hierarchy in $N = 4$ Superspace*

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Abstract—The Lax pair and Hamiltonian formulations are presented for the $N = 4$ supersymmetric Toda chain (KdV) hierarchy in $N = 4$ superspace. The general formulas for an infinite tower of its bosonic flows in terms of the Lax operator in $N = 4$ superspace are derived, and five real forms of the hierarchy are presented. New $N = 4$ superfield bases in which the flows are local are discussed. A relation between the two descriptions of the hierarchy in $N = 4$ superspace used in the literature is established.

1. INTRODUCTION

Recently, the Lax pair representation of the $N = 4$ supersymmetric Toda chain hierarchy in $N = 2$ superspace was constructed in [1]. The explicit relationship between the $N = 4$ supersymmetric Toda chain [1] and KdV [2, 3] hierarchies was established in [4], and this is the reason why we call both these hierarchies the $N = 4$ Toda chain (KdV) hierarchy. This relationship induces a new Lax pair representation of the $N = 4$ KdV hierarchy in $N = 2$ superspace. Despite knowledge of the Lax pair representations of the $N = 4$ Toda (KdV) hierarchy in $N = 2$ superspace [1, 4–6], as well as of its first few flows both in harmonic [2] and in ordinary [3, 4] $N = 4$ superspace, the Lax pair formulation in $N = 4$ superspace was not known. Quite recently, this problem was solved in [7], and the content of this talk is based on [7]. Thus, we present the Lax pair and Hamiltonian formulations for the $N = 4$ Toda (KdV) hierarchy in $N = 4$ superspace, as well as a simple relation between the two descriptions of the hierarchy in $N = 4$ superspace used in [3, 4]. We also present general formulas for its bosonic flows in terms of the Lax operator in $N = 4$ superspace, its five real forms, and new $N = 4$ superfield bases in which the flows are local.

2. LAX PAIR FORMULATION IN $N = 4$ SUPERSPACE

Our starting point is a manifestly $N = 2$ supersymmetric Lax pair representation of bosonic flows of the $N = 4$ Toda chain (KdV) hierarchy [1],

$$L = D_- + vD_+^{-1}u, \tag{1}$$

$$\frac{\partial}{\partial t_l}L = [(L^{2l})_{\geq 0}, L], \tag{2}$$

$$\frac{\partial}{\partial t_l}v = [(L^{2l})_{\geq 0}v], \tag{3}$$

$$\frac{\partial}{\partial t_l}u = (-1)^{l+1}\left[\left((L^T)^{2l}\right)_{\geq 0}u\right],$$

where $v \equiv v(z, \theta^+, \theta^-)$ and $u \equiv u(z, \theta^+, \theta^-)$ are unconstrained bosonic $N = 2$ superfields; $D_\pm$ are fermionic covariant derivatives,

$$D_\pm = \frac{\partial}{\partial\theta^\pm} + \theta^\pm\partial, \quad \{D_\pm, D_\pm\} = +2\partial; \tag{4}$$

the subscript ≥ 0 denotes the differential part of the operator; and L^T is the operator-conjugate Lax operator. Let us recall the operator conjugation rules: $D_\pm^T = -D_\pm$ and $(OP)^T = (-1)^{d_O d_P}P^T O^T$, where $O(P)$ is an arbitrary operator of Grassmann parity d_O (d_P) and $d_O = 0$ ($d_O = 1$) for bosonic (fermionic) operators O. Hereafter, we use the notation $[Of]$ for an operator O acting only on a function f inside the brackets (only nonzero algebra brackets are present explicitly) and use the notation $v' \equiv \partial v \equiv \frac{\partial}{\partial z}v$.

It is instructive to rewrite the Lax operator (1) in another superfield basis [4],

$$J \equiv uv + D_-D_+\ln u, \quad \bar{J} \equiv -uv, \tag{5}$$

where $J \equiv J(z, \theta^+, \theta^-)$ and $\bar{J} \equiv \bar{J}(z, \theta^+, \theta^-)$ are new unconstrained bosonic $N = 2$ superfields. It becomes

$$L \equiv D_- - \bar{J}\,\frac{1}{D_+ + [D_-^{-1}(\bar{J} + J)]}. \tag{6}$$

The conjecture [7] is that, if one replaces the $N = 2$ superfields J and $\bar{J}$ in this very special basis by

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one chiral $\mathcal{J}(z,\theta^+,\theta^-,\eta^+,\eta^-)$ and one antichiral $\bar{\mathcal{J}}(z,\theta^+,\theta^-,\eta^+,\eta^-)$ bosonic $N=4$ superfield,

$$\mathcal{D}_\pm \mathcal{J} = 0, \quad \bar{\mathcal{D}}^\pm \bar{\mathcal{J}} = 0, \tag{7}$$

then the Lax pair representation (2) with the new $N=4$ Lax operator L_1,

$$L_1 = D_- - \bar{\mathcal{J}} \frac{1}{D_+ + [D_-^{-1}(\bar{\mathcal{J}} + \mathcal{J})]}, \tag{8}$$
$$D_\pm \equiv \mathcal{D}_\pm + \bar{\mathcal{D}}^\pm,$$

gives consistent $N=4$ supersymmetric flows $\partial/\partial t_l$. Here, $\mathcal{D}_\pm$ and $\bar{\mathcal{D}}^\pm$ are $N=4$ fermionic covariant derivatives,

$$\mathcal{D}_\pm = \frac{1}{2}\left(\frac{\partial}{\partial\theta^\pm} + i\frac{\partial}{\partial\eta^\pm} + (\theta^\pm + i\eta^\pm)\partial\right), \tag{9}$$
$$\bar{\mathcal{D}}^\pm = \frac{1}{2}\left(\frac{\partial}{\partial\theta^\pm} - i\frac{\partial}{\partial\eta^\pm} + (\theta^\pm - i\eta^\pm)\partial\right),$$
$$\{\mathcal{D}_k, \bar{\mathcal{D}}^m\} = \delta_k{}^m\partial, \quad \{\mathcal{D}_k, \mathcal{D}_m\} = \{\bar{\mathcal{D}}^k, \bar{\mathcal{D}}^m\} = 0,$$
$$k, m = \pm,$$

$\eta^\pm$ being two additional fermionic coordinates. We would like to emphasize that such a prescription of a supersymmetrization leads to inconsistent Lax pair representations in general, with the exception of some cases—of these, one is considered here.

In order to prove this rather nontrivial conjecture, it is sufficient to show that the flow in Eqs. (3), being rewritten in terms of the basis in (5), admit the chirality constraints (7). Below, we present the proof in a few steps.

First, let us simplify the Lax operator L_1 (8) by applying a gauge transformation to it and by requiring that the gauge-transformed Lax operator

$$\widetilde{L}_1 = e^{-\xi} L_1 e^{\xi} \tag{10}$$

possess the following two properties: It should anticommute with the supersymmetric covariant derivative $\bar{\mathcal{D}}^+$, $\{\bar{\mathcal{D}}^+, \widetilde{L}_1\} = 0$, and it should comprise only a first-order pole in the fermionic derivative D_+. It turns out that these two requirements fix completely the gauge-transformation function

$$\xi \equiv [(\mathcal{D}_- + \bar{\mathcal{D}}^-)^{-1}(\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}(\bar{\mathcal{J}} + \mathcal{J})], \tag{11}$$

and the gauge-transformed Lax operator is

$$\widetilde{L}_1 = \mathcal{D}_- + \bar{\mathcal{D}}^- \tag{12}$$
$$+ [(\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}(\bar{\mathcal{J}} + \mathcal{J})] - \bar{\mathcal{J}}(\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}.$$

Then, the Lax pair representation (2) and flows (3) become

$$\frac{\partial}{\partial t_l}\widetilde{L}_1 = \left[(\widetilde{L}_1^{2l})_{\geq 0} - \frac{\partial}{\partial t_l}\xi, \widetilde{L}_1\right], \tag{13}$$

$$(-1)^l \frac{\partial}{\partial t_l}\bar{\mathcal{J}} = \left[\left(\left((\widetilde{L}_1^T)^{2l}\right)_{\geq 1}\right)^T \bar{\mathcal{J}}\right], \tag{14}$$
$$(-1)^l \frac{\partial}{\partial t_l}\xi = \left((\widetilde{L}_1^T)^{2l}\right)_0,$$

respectively, where the subscripts ≥ 1 and 0 denote, respectively, the pure differential part of the operator without the constant part and the constant part. After substituting $\frac{\partial}{\partial t_l}\xi$ from Eqs. (14) into (13), the latter becomes

$$(-1)^l \frac{\partial}{\partial t_l}\widetilde{L}_1 = \left[\left(\left((\widetilde{L}_1^T)^{2l}\right)_{\geq 1}\right)^T, \widetilde{L}_1\right], \tag{15}$$

where the useful identity

$$(-1)^l (\widetilde{L}_1^{2l})_+ \equiv \left(\left((\widetilde{L}_1^T)^{2l}\right)_{\geq 1}\right)^T + \left((\widetilde{L}_1^T)^{2l}\right)_0 \tag{16}$$

has been employed. Finally, Eq. (15), being transposed, takes the form

$$(-1)^{l+1} \frac{\partial}{\partial t_l}\widetilde{L}_1^T = \left[\left((\widetilde{L}_1^T)^{2l}\right)_{\geq 1}, \widetilde{L}_1^T\right], \tag{17}$$

which we use in what follows, where

$$\widetilde{L}_1^T \equiv -\mathcal{D}_- - \bar{\mathcal{D}}^- + [(\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}(\bar{\mathcal{J}} + \mathcal{J})] \tag{18}$$
$$- (\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}\bar{\mathcal{J}}, \quad \{\bar{\mathcal{D}}^+, \widetilde{L}_1^T\} = 0.$$

Second, let us describe the important properties of the Lax operator $\widetilde{L}_1^T$ (18). Thus, it can identically be represented as the sum of the operators M and $\bar{M}$; that is,

$$\widetilde{L}_1^T = M + \bar{M}, \tag{19}$$
$$M \equiv -\mathcal{D}_- + [\bar{\mathcal{D}}^+ \partial^{-1} \mathcal{J}],$$
$$\bar{M} \equiv (\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}\left(\bar{\mathcal{D}}^- - [\mathcal{D}_+ \partial^{-1}\bar{\mathcal{J}}]\right)(\mathcal{D}_+ + \bar{\mathcal{D}}^+),$$

with the properties

$$M^2 = \bar{M}^2 = 0, \ \{\bar{\mathcal{D}}^+, M\} = \{\bar{\mathcal{D}}^+, \bar{M}\} = 0, \tag{20}$$
$$\{\mathcal{D}_-, M\} = 0, \ \{\bar{\mathcal{D}}^-, \bar{M}\} = 0,$$
$$\{\mathcal{D}_+, (\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1} M (\mathcal{D}_+ + \bar{\mathcal{D}}^+)\}$$
$$= \{\mathcal{D}_+, (\mathcal{D}_+ + \bar{\mathcal{D}}^+)\bar{M}(\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}\} = 0,$$

$$M \equiv (\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}(\bar{M}^T)^{\#}(\mathcal{D}_+ + \bar{\mathcal{D}}^+), \tag{21}$$
$$\bar{M} \equiv (\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}(M^T)^{\#}(\mathcal{D}_+ + \bar{\mathcal{D}}^+),$$

where the symbol # denotes the substitution

$$\{\mathcal{D}_\pm, \bar{\mathcal{D}}^\pm, \mathcal{J}, \bar{\mathcal{J}}\}^{\#} = \{\bar{\mathcal{D}}^\pm, \mathcal{D}_\pm, -\bar{\mathcal{J}}, -\mathcal{J}\},$$

which respects the chirality constraints (7) and, being applied twice, gives the identity. The obvious consequences of Eqs. (20)–(22) are

$$(\widetilde{L}_1^T)^2 = \{M, \bar{M}\} \tag{22}$$

and

$$[M, (\widetilde{L}_1^T)^{2l}] = 0, \quad [\bar{M}, (\widetilde{L}_1^T)^{2l}] = 0, \tag{23}$$

as well as

$$(\widetilde{L}_1^T)^{\#} = (\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}\widetilde{L}_1(\mathcal{D}_+ + \bar{\mathcal{D}}^+). \tag{24}$$

We will also need the important relation

$$\left(\left((\widetilde{L}_1^T)^{2l}\right)_0\right)^{\#} = (-1)^l\left((\widetilde{L}_1^T)^{2l}\right)_0, \tag{25}$$

which results from (24), and the identity

$$(D_+^{-1}OD_+)_0 = (-1)^{d_O}(O^T)_0 \tag{26}$$

for a pseudodifferential operator O of Grassmann parity d_O.

Third, using these properties of the Lax operator $\widetilde{L}_1^T$ (18), we are ready to prove three identities

$$\left[\left(\left((\widetilde{L}_1^T)^{2l}\right)_{\geq 1}\right)^T\bar{\mathcal{J}}\right] = \left[\bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\left((\widetilde{L}_1^T)^{2l}\right)_0\right], \tag{27}$$

$$\left[\mathcal{D}_-\bar{\mathcal{D}}^+\left((\widetilde{L}_1^T)^{2l}\right)_0\right] = 0, \tag{28}$$

$$\left[\mathcal{D}_+\bar{\mathcal{D}}^-\left((\widetilde{L}_1^T)^{2l}\right)_0\right] = 0, \tag{29}$$

which are crucial for proving the required conjecture. Indeed, if they are satisfied, then one can easily recast the flows in (14) into the form

$$(-1)^l\frac{\partial}{\partial t_l}\bar{\mathcal{J}} = \left[\bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\left((\widetilde{L}_1^T)^{2l}\right)_0\right], \tag{30}$$

$$(-1)^l\frac{\partial}{\partial t_l}\mathcal{J} = \left[\mathcal{D}_+\mathcal{D}_-\left((\widetilde{L}_1^T)^{2l}\right)_0\right],$$

which respects manifestly the chirality constraints (7), and this proves the conjecture.

In order to prove relations (27)–(29), let us extract the equations resulting from the order 0 over D_+ of the identity

$$[\bar{\mathcal{D}}^+, (\widetilde{L}_1^T)^{2l}] = 0 \tag{31}$$

and from the order -1 over D_+ of identities (23). They are

$$\left((\widetilde{L}_1^T)^{2l}\right)_{-1} = \left[\bar{\mathcal{D}}^+\left((\widetilde{L}_1^T)^{2l}\right)_0\right], \tag{32}$$

$$\left[\mathcal{D}_-\left((\widetilde{L}_1^T)^{2l}\right)_{-1}\right] = 0, \tag{33}$$

$$-\left[\bar{\mathcal{D}}^-\left((\widetilde{L}_1^T)^{2l}\right)_{-1}\right] = \left[\left(\left((\widetilde{L}_1^T)^{2l}\right)_{\geq 1}\right)^T\bar{\mathcal{J}}\right], \tag{34}$$

respectively, where the subscript -1 denotes the coefficient of the derivative D_+^{-1} of the pseudodifferential operator $(\widetilde{L}_1^T)^{2l}$. Now, the quantity $\left((\widetilde{L}_1^T)^{2l}\right)_{-1}$, being substituted from Eq. (32) into Eqs. (33) and (34), just leads to relations (27) and (28). As concerns the remaining relation (29), it can easily be derived from relation (28), if substitution (22) is applied to it and if identity (25) is whereupon used. Alternatively, it can be obtained if one extracts equations of the order 0 over D_+,

$$\left(\left((\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}\widetilde{L}_1(\mathcal{D}_+ + \bar{\mathcal{D}}^+)\right)^{2l}\right)_{-1} \tag{35}$$

$$= \left[\mathcal{D}_+\left(\left((\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}\widetilde{L}_1(\mathcal{D}_+ + \bar{\mathcal{D}}^+)\right)^{2l}\right)_0\right],$$

and the order -1 over D_+,

$$\left[\bar{\mathcal{D}}^-\left(\left((\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}\widetilde{L}_1(\mathcal{D}_+ + \bar{\mathcal{D}}^+)\right)^{2l}\right)_{-1}\right] = 0, \tag{36}$$

from the identities

$$\left\{\mathcal{D}_+, \left((\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}\widetilde{L}_1(\mathcal{D}_+ + \bar{\mathcal{D}}^+)\right)^{2l}\right\} = 0 \tag{37}$$

and

$$\left\{(\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}\bar{M}^T(\mathcal{D}_+ + \bar{\mathcal{D}}^+),\right. \tag{38}$$

$$\left.\left((\mathcal{D}^+ + \bar{\mathcal{D}}^+)^{-1}\ \widetilde{L}_1(\mathcal{D}_+ + \bar{\mathcal{D}}_+)\right)^{2l}\right\} = 0,$$

respectively, related obviously to identities (31) and (23); then substitutes the quantity $(((\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}\widetilde{L}_1(\mathcal{D}_+ + \bar{\mathcal{D}}^+))^{2l})_{-1}$ from (36) into (37),

$$\left[\mathcal{D}_+\bar{\mathcal{D}}^-\left((\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}\widetilde{L}_1^{2l}(\mathcal{D}_+ + \bar{\mathcal{D}}^+)\right)_0\right] = 0; \tag{39}$$

and, at last, uses identity (26). This ends the proof of the conjecture.

3. LOCAL FLOWS IN $N = 4$ SUPERSPACE

The $N = 4$ Toda (KdV) hierarchy flows (30) in the $N = 4$ superfield basis $\{\mathcal{J}, \bar{\mathcal{J}}\}$ are obviously nonlocal because of the nonlocal dependence of the Lax operator $\widetilde{L}_1^T$ (18) on the superfields $\mathcal{J}$ and $\bar{\mathcal{J}}$. Nevertheless, it is possible to localize them. Indeed, let us introduce a new superfield basis $\{\Omega, \bar{\Omega}\}$ defined by the invertible transformations

$$\mathcal{J} \equiv \mathcal{D}_+\bar{\Omega}, \quad \bar{\mathcal{J}} \equiv \bar{\mathcal{D}}^+\Omega, \tag{40}$$

$$\bar{\Omega} \equiv \bar{\mathcal{D}}^+\partial^{-1}\mathcal{J}, \quad \Omega \equiv \mathcal{D}_+\partial^{-1}\bar{\mathcal{J}},$$

where Ω and $\bar{\Omega}$ are new constrained fermionic $N = 4$ superfields,

$$\mathcal{D}_+\Omega = \bar{\mathcal{D}}^-\Omega = 0, \quad \mathcal{D}_-\bar{\Omega} = \bar{\mathcal{D}}^+\bar{\Omega} = 0. \tag{41}$$

Then, in terms of the superfields $\{\Omega, \bar{\Omega}\}$, the Lax operator $\widetilde{L}_1^T$ (18) becomes local,

$$\widetilde{L}_1^T \equiv -\mathcal{D}_- - \bar{\mathcal{D}}^- + \Omega + \bar{\Omega} - (\mathcal{D}_+ + \bar{\mathcal{D}}^+)^{-1}[\bar{\mathcal{D}}^+\Omega]. \quad (42)$$

Using identities (28) and (29), one can easily rewrite the flows in (30) in terms of this basis; that is,

$$(-1)^l \frac{\partial}{\partial t_l}\Omega = \left[\bar{\mathcal{D}}^- \left((\widetilde{L}_1^T)^{2l}\right)_0\right], \quad (43)$$
$$(-1)^l \frac{\partial}{\partial t_l}\bar{\Omega} = \left[\mathcal{D}_- \left((\widetilde{L}_1^T)^{2l}\right)_0\right],$$

where they are obviously local because of the locality of the Lax operator $\widetilde{L}_1^T$ (42).

Actually, apart from the basis in (40), there are at least three other superfield bases with the constrained fermionic $N = 4$ superfields $\{\Psi, \bar{\Psi}\}$, $\{\Sigma, \bar{\Sigma}\}$, and $\{\Xi, \bar{\Xi}\}$ that ensure the locality of the flows for a reason that is less evident than that for the basis in (40). The corresponding formulas are

$$\bar{\Omega} = \bar{\Psi}, \quad \Omega \equiv \mathcal{D}_+\bar{\mathcal{D}}^-\partial^{-1}\Psi, \quad (44)$$
$$\mathcal{D}_-\Psi = \bar{\mathcal{D}}^+\Psi = 0, \quad \mathcal{D}_-\bar{\Psi} = \bar{\mathcal{D}}^+\bar{\Psi} = 0,$$
$$\widetilde{L}_1^T = e^{\psi}\left(-\mathcal{D}_- - \bar{\mathcal{D}}^- + \bar{\Psi} - \frac{1}{\mathcal{D}_+ + \bar{\mathcal{D}}^+ - \Psi}[\bar{\mathcal{D}}^-\Psi]\right)e^{-\psi},$$
$$\psi \equiv -[\mathcal{D}_+\partial^{-1}\Psi],$$
$$(-1)^{l+1}\frac{\partial}{\partial t_l}\Psi = \left[\bar{\mathcal{D}}^+\left((\widetilde{L}_1^T)^{2l}\right)_0\right],$$
$$(-1)^l \frac{\partial}{\partial t_l}\bar{\Psi} = \left[\mathcal{D}_-\left((\widetilde{L}_1^T)^{2l}\right)_0\right]$$

and

$$\bar{\Omega} = \mathcal{D}_-\bar{\mathcal{D}}^+\partial^{-1}\bar{\Sigma}, \quad \Omega \equiv \Sigma, \quad (45)$$
$$\mathcal{D}_+\Sigma = \bar{\mathcal{D}}^-\Sigma = 0, \quad \mathcal{D}_+\bar{\Sigma} = \bar{\mathcal{D}}^-\bar{\Sigma} = 0,$$
$$\widetilde{L}_1^T = e^{\sigma}\left(-\mathcal{D}_- - \bar{\mathcal{D}}^- + \Xi - \frac{1}{\mathcal{D}_+ + \bar{\mathcal{D}}^+ + \bar{\Sigma}}[\bar{\mathcal{D}}^+\Sigma]\right)e^{-\sigma},$$
$$\sigma \equiv [\bar{\mathcal{D}}^+\partial^{-1}\bar{\Sigma}],$$
$$(-1)^l\frac{\partial}{\partial t_l}\Sigma = \left[\bar{\mathcal{D}}^-\left((\widetilde{L}_1^T)^{2l}\right)_0\right],$$
$$(-1)^l\frac{\partial}{\partial t_l}\bar{\Sigma} = \left[\mathcal{D}_+\left((\widetilde{L}_1^T)^{2l}\right)_0\right],$$

as well as

$$\bar{\Omega} = \mathcal{D}_-\bar{\mathcal{D}}^+\partial^{-1}\bar{\Xi}, \quad \Omega \equiv \mathcal{D}_+\bar{\mathcal{D}}^-\partial^{-1}\Xi, \quad (46)$$
$$\mathcal{D}_-\Xi = \bar{\mathcal{D}}^+\Xi = 0, \quad \mathcal{D}_+\bar{\Xi} = \bar{\mathcal{D}}^-\bar{\Xi} = 0,$$
$$\widetilde{L}_1^T = e^{\xi}L_1^T e^{-\xi},$$
$$(-1)^{l+1}\frac{\partial}{\partial t_l}\Xi = \left[\bar{\mathcal{D}}^+\left((\widetilde{L}_1^T)^{2l}\right)_0\right],$$
$$(-1)^l\frac{\partial}{\partial t_l}\bar{\Xi} = \left[\mathcal{D}_+\left((\widetilde{L}_1^T)^{2l}\right)_0\right],$$

where L_1^T and ξ in Eqs. (46) are defined by (8) and (11), respectively, rewritten in terms of this basis as

$$L_1^T = -\mathcal{D}_- - \bar{\mathcal{D}}^- - \frac{1}{\mathcal{D}_+ + \bar{\mathcal{D}}^+ + \bar{\Xi} - \Xi}[\bar{\mathcal{D}}^-\Xi], \quad (47)$$
$$\xi \equiv [\bar{\mathcal{D}}^+\partial^{-1}\bar{\Xi}] - [\mathcal{D}_+\partial^{-1}\Xi].$$

Despite the nonlocality of the Lax operator $\widetilde{L}_1^T$ for each of these three bases, the quantity $\left((\widetilde{L}_1^T)^{2l}\right)_0$ entering into the corresponding flow equations (44)–(46) is in fact local, and this ensures the locality of the flows. Indeed, let us demonstrate this remarkable fact, e.g., for the basis in (46). In this case, we have

$$\left((\widetilde{L}_1^T)^{2l}\right)_0 \equiv \left((L_1^T)^{2l}\right)_0 + \left(e^{\xi}\left((L_1^T)^{2l}\right)_{\geq 1}e^{-\xi}\right)_0. \quad (48)$$

The first term on the right-hand side of Eq. (48) is of course local because of the locality of the Lax operator L_1^T (48) with respect to the superfields Ξ and $\bar{\Xi}$. Nonlocality could come only from the last, second, term because of the nonlocality of ξ (48). However, this term is actually a polynomial in the derivatives of ξ, but a derivative being applied to ξ makes it local owing to the chirality properties (46) of the superfields Ξ and $\bar{\Xi}$. Therefore, all potential nonlocalities in fact disappear. Completely the same argument is valid for each basis from the set in (44)–(46).

As an example, we present a few first nontrivial manifestly $N = 4$ supersymmetric flows resulting from Eqs. (30) and (44),

$$\frac{\partial}{\partial t_2}\mathcal{J} = -\mathcal{J}'' - \mathcal{D}_+\mathcal{D}_-[2(\mathcal{J}\partial^{-1}\bar{\mathcal{J}})' - (\bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\partial^{-1}\mathcal{J})^2], \quad (49)$$
$$\frac{\partial}{\partial t_2}\bar{\mathcal{J}} = +\bar{\mathcal{J}}'' - \bar{\mathcal{D}}^+\bar{\mathcal{D}}^-[2(\bar{\mathcal{J}}\partial^{-1}\mathcal{J})' - (\mathcal{D}_+\mathcal{D}_-\partial^{-1}\bar{\mathcal{J}})^2],$$
$$\frac{\partial}{\partial t_3}\mathcal{J} = \mathcal{J}''' + \mathcal{D}_+\mathcal{D}_-\{3[\mathcal{J}'\partial^{-1}\bar{\mathcal{J}} + (\mathcal{J}\partial^{-1}\bar{\mathcal{J}})\mathcal{D}_+\mathcal{D}_-\partial^{-1}\bar{\mathcal{J}} - 1/2(\bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\partial^{-1}\mathcal{J})^2]' - (\bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\partial^{-1}\mathcal{J})^3 - 3(\bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\partial^{-1}\mathcal{J})^2\mathcal{D}_+\mathcal{D}_-\partial^{-1}\bar{\mathcal{J}} \quad (50)$$

$$+ 6\mathcal{J}\bar{\mathcal{J}}\bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\partial^{-1}\mathcal{J}\big\},$$

$$\frac{\partial}{\partial t_3}\bar{\mathcal{J}} = \bar{\mathcal{J}}'''$$
$$+ \bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\big\{3\big[-\bar{\mathcal{J}}'\partial^{-1}\mathcal{J} + (\bar{\mathcal{J}}\partial^{-1}\mathcal{J})\bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\partial^{-1}\mathcal{J}$$
$$+ 1/2(\mathcal{D}_+\mathcal{D}_-\partial^{-1}\bar{\mathcal{J}})^2]' - (\mathcal{D}_+\mathcal{D}_-\partial^{-1}\bar{\mathcal{J}})^3$$
$$- 3(\mathcal{D}_+\mathcal{D}_-\partial^{-1}\bar{\mathcal{J}})^2\bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\partial^{-1}\mathcal{J}$$
$$+ 6\mathcal{J}\bar{\mathcal{J}}\ \mathcal{D}_+\mathcal{D}_-\partial^{-1}\bar{\mathcal{J}}\big\}$$

and

$$\frac{\partial}{\partial t_2}\Psi = +\Psi'' + 2\mathcal{D}_-\bar{\mathcal{D}}^-(\bar{\Psi}\bar{\mathcal{D}}^-\Psi) - \bar{\mathcal{D}}^+(\mathcal{D}_+\Psi)^2, \tag{51}$$

$$\frac{\partial}{\partial t_2}\bar{\Psi} = -\bar{\Psi}'' - 2\bar{\mathcal{D}}^+\mathcal{D}_+(\Psi\mathcal{D}_+\bar{\Psi}) + \mathcal{D}_-(\bar{\mathcal{D}}^-\bar{\Psi})^2,$$

$$\frac{\partial}{\partial t_3}\Psi = \Psi''' + 3\mathcal{D}_-\big[(\bar{\mathcal{D}}^-\Psi)'\bar{\mathcal{D}}^-\bar{\Psi} \tag{52}$$
$$+ (\bar{\mathcal{D}}^-\Psi)(\bar{\mathcal{D}}^-\bar{\Psi})^2 + 1/2\bar{\mathcal{D}}^+\bar{\mathcal{D}}^-(\mathcal{D}_+\Psi)^2]$$
$$+ \bar{\mathcal{D}}^+\big[(\mathcal{D}_+\Psi)^3 - 3(\mathcal{D}_+\Psi)^2\ \bar{\mathcal{D}}^-\bar{\Psi}$$
$$- 6(\mathcal{D}_+\bar{\Psi})(\bar{\mathcal{D}}^-\Psi)\mathcal{D}_+\Psi],$$
$$\frac{\partial}{\partial t_3}\bar{\Psi} = \bar{\Psi}''' + 3\bar{\mathcal{D}}^+\big[(\mathcal{D}_+\bar{\Psi})'\mathcal{D}_+\Psi$$
$$+ (\mathcal{D}_+\bar{\Psi})(\mathcal{D}_+\Psi)^2 - 1/2\mathcal{D}_+\mathcal{D}_-(\bar{\mathcal{D}}^-\bar{\Psi})^2]$$
$$+ \mathcal{D}_-\big[(\bar{\mathcal{D}}^-\bar{\Psi})^3 - 3(\bar{\mathcal{D}}^-\bar{\Psi})^2\ \mathcal{D}_+\Psi$$
$$- 6(\mathcal{D}_+\bar{\Psi})(\bar{\mathcal{D}}^-\Psi)\bar{\mathcal{D}}^-\bar{\Psi}],$$

respectively.

4. HAMILTONIAN STRUCTURE IN $N = 4$ SUPERSPACE

Now, we would like to discuss the Hamiltonian structure of the $N = 4$ Toda (KdV) hierarchy in $N = 4$ superspace. Let us first present the general formulas for the conserved quantities (Hamiltonians) H_l^t of the $N = 4$ flows (30) and (43)–(46) in $N = 4$ superspace,

$$H_l^t = \int dzd\theta^+d\eta^+d\theta^-d\eta^-\partial^{-1}\left((\widetilde{L}_1^T)^{2l}\right)_0. \tag{53}$$

Hereafter, we use the following definitions of the $N = 2$ and $N = 4$ superspace integrals for an arbitrary superfield functional $f(\theta^+, \eta^+, \theta^-, \eta^-)$:

$$\int dzd\theta^+d\eta^+d\theta^-d\eta^- f(\theta^+, \eta^+, \theta^-, \eta^-) \tag{54}$$
$$\equiv \int dzd\theta^+d\eta^+\left(\mathcal{D}_-\bar{\mathcal{D}}^-\ f\right)\big|_{\theta^-=\eta^-=0}$$
$$\equiv \int dz\left(\mathcal{D}_+\bar{\mathcal{D}}^+\mathcal{D}_-\bar{\mathcal{D}}^-\ f\right)\big|_{\theta^\pm=\eta^\pm=0},$$

respectively, as well as the following realization of the inverse derivative:

$$\partial_z^{-1} \equiv \frac{1}{2}\int_{-\infty}^{+\infty} dx\epsilon(z-x), \tag{55}$$
$$\epsilon(z-x) = -\epsilon(x-z) \equiv 1,$$
$$\text{if} \quad z > x.$$

Using these definitions and the identities in (28) and (29), one can equivalently rewrite H_l^t (53) in the form of the $N = 2$ superfield integral,

$$H_l^t = \int dzd\theta^+d\eta^+\left((\widetilde{L}_1^T)^{2l}\right)_0\Big|_{\theta^-=\eta^-=0}. \tag{56}$$

Then, using identity (32) and relation (10), one can easily find that the Hamiltonians H_l^t (56) reproduce the corresponding bosonic Hamiltonians discussed in [1]. Recalling that the quantities $\left((\widetilde{L}_1^T)^{2l}\right)_0$ are local for all the local flows (43)–(46) [see the discussion after Eqs. (48)], one can conclude that the corresponding Hamiltonians and their $N = 2$ densities (56) are local quantities as well, while the Hamiltonian densities (53) in $N = 4$ superspace are nonlocal even when the flows corresponding to them are local.

We have verified that the formula

$$\int dzd\theta^+d\eta^+d\theta^-d\eta^-\partial^{-1}\frac{\delta}{\delta\mathcal{J}}\left((\widetilde{L}_1^T)^{2l}\right)_0 \tag{57}$$
$$= l\bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\partial^{-2}\left((\widetilde{L}_1^T)^{2(l-1)}\right)_0$$

is valid for a few first values of l [in this derivation, we have performed integration by parts and extensively used the realization in (55) for the inverse derivative]. The variation formula with respect to $\bar{\mathcal{J}}$ can be obtained if one applies substitution (22) to Eq. (57) and uses identity (25),

$$\int dzd\theta^+d\eta^+d\theta^-d\eta^-\partial^{-1}\frac{\delta}{\delta\bar{\mathcal{J}}}\left((\widetilde{L}_1^T)^{2l}\right)_0 \tag{58}$$
$$= -l\mathcal{D}_+\mathcal{D}_-\partial^{-2}\left((\widetilde{L}_1^T)^{2(l-1)}\right)_0.$$

It is plausible to suppose that formulas (57) and (58) are valid for all values of l as well, but we cannot present the proof here. Then, using them and the Hamiltonians H_l^t (53), one can represent the flows given by (30) in the Hamiltonian form

$$(-1)^{l+1}\frac{\partial}{\partial t_{l-1}}\begin{pmatrix}\bar{\mathcal{J}}\\ \mathcal{J}\end{pmatrix} = \frac{1}{l}J_1\begin{pmatrix}\delta/\delta\bar{\mathcal{J}}\\ \delta/\delta\mathcal{J}\end{pmatrix}H_l^t \tag{59}$$
$$= \frac{1}{l-1}J_2\begin{pmatrix}\delta/\delta\bar{\mathcal{J}}\\ \delta/\delta\mathcal{J}\end{pmatrix}H_{l-1}^t,$$

where J_1,

$$J_1 = \begin{pmatrix} 0 & -\bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\mathcal{D}_+\mathcal{D}_- \\ \mathcal{D}_+\mathcal{D}_-\bar{\mathcal{D}}^+\bar{\mathcal{D}}^- & 0 \end{pmatrix}, \quad (60)$$

is the first Hamiltonian structure in $N = 4$ superspace. Using the flows in (50) and (51), we have also found the second Hamiltonian structure J_2,

$$J_2 = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}, \quad (61)$$

$$J_{11} = -(\partial\bar{\mathcal{J}} + \bar{\mathcal{J}}\partial)\bar{\mathcal{D}}^+\bar{\mathcal{D}}^-,$$

$$J_{22} = (\partial\mathcal{J} + \mathcal{J}\partial)\mathcal{D}_+\mathcal{D}_-,$$

$$J_{12} = \bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\left(\partial + [\mathcal{D}_-\mathcal{D}_+\partial^{-1}\bar{\mathcal{J}}] + [\bar{\mathcal{D}}^-\,\bar{\mathcal{D}}^+\partial^{-1}\mathcal{J}]\right)\mathcal{D}_+\mathcal{D}_-,$$

$$J_{21} = \mathcal{D}_+\mathcal{D}_-\left(\partial - [\mathcal{D}_-\mathcal{D}_+\partial^{-1}\bar{\mathcal{J}}] - [\bar{\mathcal{D}}^-\,\bar{\mathcal{D}}^+\partial^{-1}\mathcal{J}]\right)\bar{\mathcal{D}}^+\bar{\mathcal{D}}^-.$$

In terms of these two Hamiltonian structures, the Poisson brackets of the superfields $\bar{\mathcal{J}}$ and $\mathcal{J}$ are given by

$$\left\{\begin{pmatrix}\bar{\mathcal{J}}(Z_1)\\ \mathcal{J}(Z_1)\end{pmatrix}, \begin{pmatrix}\bar{\mathcal{J}}(Z_2), \mathcal{J}(Z_2)\end{pmatrix}\right\}_k = J_k(Z_1)\delta^{N=4}(Z_1 - Z_2), \quad (62)$$

where $\delta^{N=4}(Z) \equiv \delta(z)\theta^+\eta^+\theta^-\eta^-$ is the delta function in $N = 4$ superspace with the coordinates $Z \equiv \{z, \theta^+, \eta^+, \theta^-, \eta^-\}$. It is interesting to note that the second Hamiltonian structure J_2 (61) can be identically rewritten in terms of the single $N = 4$ superfield ξ (11),

$$J_2 = \begin{pmatrix} -\bar{\mathcal{D}}^+\bar{\mathcal{D}}^-(\partial\xi + \xi\partial)\bar{\mathcal{D}}^+\bar{\mathcal{D}}^- & \bar{\mathcal{D}}^+\bar{\mathcal{D}}^-(\partial + \xi')\mathcal{D}_+\mathcal{D}_- \\ \mathcal{D}_+\mathcal{D}_-(\partial - \xi')\bar{\mathcal{D}}^+\bar{\mathcal{D}}^- & \mathcal{D}_+\mathcal{D}_-(\partial\xi + \xi\partial)\mathcal{D}_+\mathcal{D}_- \end{pmatrix}. \quad (63)$$

Knowledge of the first and second Hamiltonian structures allows us to construct the recursion operator of the hierarchy in $N = 4$ superspace by using the general rule

$$R = J_2J_1^{-1} \equiv \begin{pmatrix} J_{12} & -J_{11} \\ J_{22} & -J_{21} \end{pmatrix}\partial^{-2}, \quad (64)$$

$$\frac{\partial}{\partial t_{l+1}}\begin{pmatrix}\bar{\mathcal{J}}\\ \mathcal{J}\end{pmatrix} = R\frac{\partial}{\partial t_l}\begin{pmatrix}\bar{\mathcal{J}}\\ \mathcal{J}\end{pmatrix},$$

$$J_{l+1} = R^lJ_1,$$

where the matrix J_1^{-1} is

$$J_1^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\partial^{-2}, \quad (65)$$

$$J_1^{-1}J_1\begin{pmatrix}\bar{\mathcal{J}}\\ \mathcal{J}\end{pmatrix} = J_1J_1^{-1}\begin{pmatrix}\bar{\mathcal{J}}\\ \mathcal{J}\end{pmatrix} = \begin{pmatrix}\bar{\mathcal{J}}\\ \mathcal{J}\end{pmatrix}.$$

The Hamiltonian structures J_1 and J_2 [see Eqs. (60) and (61), respectively] are obviously compatible e.g., the deformation $[\mathcal{D}_-\mathcal{D}_+\partial^{-1}\bar{\mathcal{J}}] \to [\mathcal{D}_-\mathcal{D}_+\partial^{-1}\bar{\mathcal{J}}] + \alpha$, where α is an arbitrary parameter, transforms J_2 into their algebraic sum $J_2 - \alpha J_1$. Thus, one concludes that the recursion operator R (65) is hereditary as the operator obtained from the compatible pair of Hamiltonian structures.

Applying formulas (65), we obtain the recurrence relations for the flows (30) in $N = 4$ superspace,

$$\frac{\partial}{\partial t_{l+1}}\bar{\mathcal{J}} = \bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\left((+\partial + \xi')\mathcal{D}_+\mathcal{D}_- + (\partial\xi + \xi\partial)\,\bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\right)\partial^{-2}\frac{\partial}{\partial t_l}(\bar{\mathcal{J}} + \mathcal{J}), \quad (66)$$

$$\frac{\partial}{\partial t_{l+1}}\mathcal{J} = \mathcal{D}_+\mathcal{D}_-\left((-\partial + \xi')\,\bar{\mathcal{D}}^+\bar{\mathcal{D}}^- + (\partial\xi + \xi\partial)\mathcal{D}_+\mathcal{D}_-\right)\partial^{-2}\frac{\partial}{\partial t_l}(\bar{\mathcal{J}} + \mathcal{J}).$$

5. REAL FORMS IN $N = 4$ SUPERSPACE

It is well known that different real forms derived from the same complex integrable hierarchy are nonequivalent in general. In view of this, it seems important to find as many real forms of the $N = 4$ Toda (KdV) hierarchy in $N = 4$ superspace as is possible.

Let us prove that the $N = 4$ flows $\partial/\partial t_l$ (30) admit the following five complex conjugations:

$$(\mathcal{J}, \bar{\mathcal{J}})^* = -(\mathcal{J}, \bar{\mathcal{J}}), \quad (67)$$

$$(z, \theta^\pm, \eta^\pm)^* = (-z, \theta^\pm, -\eta^\pm),$$

$$t_l^* = (-1)^l t_l,$$

$$(\mathcal{J}, \bar{\mathcal{J}})^\bullet = (\mathcal{J} - \mathcal{D}_-\mathcal{D}_+ \ln \bar{\mathcal{J}}, \bar{\mathcal{J}}), \quad (68)$$
$$(z, \theta^\pm, \eta^\pm)^\bullet = (-z, \theta^\pm, -\eta^\pm), \quad t_l^\bullet = -t_l,$$

$$(\mathcal{J}, \bar{\mathcal{J}})^\star = (\bar{\mathcal{J}}, \mathcal{J}), \quad (69)$$
$$(z, \theta^\pm, \eta^\pm)^\star = (-z, \theta^\pm, \eta^\pm),$$
$$t_l^\star = -t_l,$$

$$(\mathcal{J}, \bar{\mathcal{J}})^\dagger = -(\mathcal{J}, \bar{\mathcal{J}}), \quad (70)$$
$$(z, \theta^\pm, \eta^\pm)^\dagger = (-z, i\eta^\pm, i\theta^\pm),$$
$$t_l^\dagger = (-1)^l t_l,$$

$$(\mathcal{J}, \bar{\mathcal{J}})^\ddagger = (\mathcal{J}, \bar{\mathcal{J}}), \quad (71)$$
$$(z, \theta^\pm, \eta^\pm)^\ddagger = (-z, \theta^\mp, -\eta^\mp),$$
$$t_l^\ddagger = (-1)^l t_l.$$

With this aim, let us elaborate the corresponding involutive properties of the Lax operator $\widetilde{L}_1^T$ (18) and the quantity $\left((\widetilde{L}_1^T)^{2l}\right)_0$ entering into the flow Eq. (30). They are

$$(\widetilde{L}_1^T)^* = \widetilde{L}_1^T, \quad (72)$$
$$\left((\widetilde{L}_1^T)^{2l}\right)_0^* = (-1)^l \left((\widetilde{L}_1^T)^{2l}\right)_0,$$

$$(\widetilde{L}_1^T)^\bullet = -\frac{1}{\bar{\mathcal{J}}}\widetilde{L}_1\bar{\mathcal{J}}, \quad (73)$$
$$\left((\widetilde{L}_1^T)^{2l}\right)_0^\bullet = \left((\widetilde{L}_1^T)^{2l}\right)_0 + \frac{1}{\bar{\mathcal{J}}}\left[\left(\left((\widetilde{L}_1^T)^{2l}\right)_{\geq 1}\right)^T \bar{\mathcal{J}}\right],$$

$$(\widetilde{L}_1^T)^\star = (\mathcal{D}^+ + \bar{\mathcal{D}}_+)^{-1}\widetilde{L}_1(\mathcal{D}^+ + \bar{\mathcal{D}}_+), \quad (74)$$
$$\left((\widetilde{L}_1^T)^{2l}\right)_0^\star = \left((\widetilde{L}_1^T)^{2l}\right)_0,$$

where identities (16) and (26) have been used in deriving Eqs. (74) and (75) (see below), respectively. Now, with these relations and the identity in (27), it is a simple exercise to verify that the flows in (30) do indeed possess the complex conjugations (67)–(69). As concerns the remaining complex conjugations (70) and (71), a direct verification shows that the second and third flows (50) and (51), as well as the recursion relations (66), admit them; therefore all the flows (30) of the hierarchy possess these complex conjugations as well.

The complex conjugations (67)–(71) extract two different real forms of the algebra specified by (9). Thus, the real forms of this algebra, with the involutions in (67)–(69) and (70) and (71), correspond to a twisted real $N = 4$ supersymmetry, while the real form corresponding to the involution in (69) reproduces the algebra of real $N = 4$ supersymmetry.

6. $N = 4$ TODA (KdV) HIERARCHY IN $N = 2$ SUPERSPACE

Let us discuss the relationship between the description of the $N = 4$ Toda (KdV) hierarchy in $N = 4$ superspace developed in the preceding sections and its description in the three $N = 2$ superfield bases (a), (b), and (c) from [3] [see Eqs. (4.5) and (4.3a)–(4.3c) therein] characterized by local flows at the level of the second flow (50).

Basis (a). Let us introduce the $N = 2$ superfield basis $\{\widetilde{V}, F, \bar{F}\}$ defined by

(a)

$$\widetilde{V} \equiv \frac{1}{2}[D_-D_+^{-1}(\bar{\mathcal{J}} + \mathcal{J})] \times (z, \theta^+, \theta^- = 0, \eta^+, \eta^- = 0), \quad (75)$$
$$F \equiv \mathcal{J}(z, \theta^+, \theta^- = 0, \eta^+, \eta^- = 0),$$
$$\bar{F} \equiv \bar{\mathcal{J}}(z, \theta^+, \theta^- = 0, \eta^+, \eta^- = 0),$$

where $\widetilde{V} \equiv \widetilde{V}(z, \theta^+, \eta^+)$, $F \equiv F(z, \theta^+, \eta^+)$, and $\bar{F} \equiv \bar{F}(z, \theta^+, \eta^+)$ are new unconstrained chiral and antichiral ($\mathcal{D}_+ F = \bar{\mathcal{D}}^+\bar{F} = 0$) bosonic $N = 2$ superfields, respectively, and $D_\pm$ is defined in Eq. (8). Using the chirality constraints (7) and the definition of $\widetilde{V}$ (75), one can express $\mathcal{D}_-\bar{\mathcal{J}}$ and $\bar{\mathcal{D}}^-\mathcal{J}$ in terms of $\mathcal{D}^+\widetilde{V}$ and $\bar{\mathcal{D}}^+\widetilde{V}$ as $\mathcal{D}_-\bar{\mathcal{J}} = -2\bar{\mathcal{D}}^+\widetilde{V}$ and $\bar{\mathcal{D}}^-\mathcal{J} = -2\mathcal{D}_+\widetilde{V}$. Using these inputs, one can rewrite (50) in the basis $\{\widetilde{V}, F, \bar{F}\}$,

$$\frac{\partial}{\partial t_2}\widetilde{V} = ([\mathcal{D}_+, \bar{\mathcal{D}}^+]\widetilde{V} + 2\widetilde{V}^2 - F\bar{F})', \quad (76)$$
$$\frac{\partial}{\partial t_2}F = -F'' + 4\mathcal{D}_+\bar{\mathcal{D}}^+(F\widetilde{V}),$$
$$\frac{\partial}{\partial t_2}\bar{F} = +\bar{F}'' + 4\bar{\mathcal{D}}^+\mathcal{D}_+(\bar{F}\widetilde{V}).$$

Basis (c). Let us introduce the $N = 2$ superfield basis $\{\widetilde{J}, \Phi, \bar{\Phi}\}$ defined by

(c)

$$\frac{1}{2}(\Phi + \bar{\Phi}) - i\widetilde{J} \equiv \mathcal{J}(z, \theta^+, \theta^-, \eta^+ = 0, \eta^- = 0), \quad (77)$$
$$\frac{1}{2}(\Phi + \bar{\Phi}) + i\widetilde{J} \equiv \bar{\mathcal{J}}(z, \theta^+, \theta^-, \eta^+ = 0, \eta^- = 0),$$

$$\widetilde{J} \equiv \frac{i}{2}(\mathcal{J} - \bar{\mathcal{J}})\Big|_{\eta_\pm = 0}, \quad (78)$$
$$\Phi \equiv \left[D\bar{D}\partial^{-1}(\bar{\mathcal{J}} + \mathcal{J})\right]\Big|_{\eta_\pm = 0},$$
$$\bar{\Phi} \equiv \left[\bar{D}D\partial^{-1}(\bar{\mathcal{J}} + \mathcal{J})\right]\Big|_{\eta_\pm = 0},$$

where $\widetilde{J} \equiv \widetilde{J}(z,\theta^+,\theta^-), \Phi \equiv \Phi(z,\theta^+,\theta^-)$, and $\bar{\Phi} \equiv \bar{\Phi}(z,\theta^+,\theta^-)$ are new unconstrained, chiral, and antichiral ($D\Phi = \bar{D}\bar{\Phi} = 0$) bosonic $N = 2$ superfields, respectively, and D and $\bar{D}$ are $N = 2$ fermionic covariant derivatives,

$$D \equiv \frac{1}{2}(D_+ + iD_-), \quad \bar{D} \equiv \frac{1}{2}(D_+ - iD_-), \quad (79)$$
$$\{D, \bar{D}\} = \partial, \quad D^2 = \bar{D}^2 = 0.$$

Using the explicit realization of the derivatives $D_\pm$ (4) and $\mathcal{D}_\pm$, $\bar{\mathcal{D}}^\pm$ (9), as well as the chirality constraints (7), one can express $\mathcal{D}_\pm\bar{\mathcal{J}}$ and $\bar{\mathcal{D}}^\pm\mathcal{J}$ in terms of $D_\pm\bar{\mathcal{J}}$ and $D_\pm\mathcal{J}$ as

$$\bar{\mathcal{D}}^\pm\mathcal{J} = D_\pm\mathcal{J}, \quad \bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\mathcal{J} = D_+D_-\mathcal{J}, \quad (80)$$
$$\equiv \mathcal{D}_\pm\bar{\mathcal{J}} = D_\pm\bar{\mathcal{J}}, \quad \mathcal{D}_+\mathcal{D}_-\bar{\mathcal{J}} = D_+D_-\bar{\mathcal{J}}.$$

Using these inputs, one can rewrite (50) in the basis $\{\widetilde{J}, \Phi, \bar{\Phi}\}$,

$$-i\frac{\partial}{\partial t_2}\widetilde{J} = -\frac{1}{2}(\Phi + \bar{\Phi})'' - 2(\widetilde{J}(\Phi - \bar{\Phi}))' \quad (81)$$
$$+ [D, \bar{D}](\widetilde{J}(\Phi + \bar{\Phi})),$$
$$-i\frac{\partial}{\partial t_2}\Phi = 2D\bar{D}\left(\widetilde{J}' - \widetilde{J}^2 - \frac{3}{4}\Phi^2 + \frac{1}{2}\Phi\bar{\Phi}\right),$$
$$-i\frac{\partial}{\partial t_2}\bar{\Phi} = 2\bar{D}D\left(\widetilde{J}' + \widetilde{J}^2 + \frac{3}{4}\bar{\Phi}^2 - \frac{1}{2}\Phi\bar{\Phi}\right).$$

Basis (b). The $N = 2$ basis (b) and the corresponding second flow equations can be obtained from Eqs. (77)–(79) and (81) of the basis (c) by means of the substitution

$$\text{(b)} \quad \{\widetilde{J}, \Phi, \bar{\Phi}\} \to \{\widetilde{J}, i\Phi, -i\bar{\Phi}\}, \quad t_2 \to it_2. \quad (82)$$

Equations (76), (81), and (82) reproduce the corresponding Eqs. (4.5) and (4.3a)–(4.3c) from [3]. Therefore, we arrive at the conclusion that the $N = 4$ Toda (KdV) hierarchy in $N = 4$ superspace unifies the three bases (a), (b), and (c) [particular "$SU(2)$ frames"] of the hierarchy in $N = 2$ superspace with a local realization of its flows.

As a by-product of this consideration, one can establish the precise correspondence with [2, 3], where a formalism that differs from ours was developed to describe the $N = 4$ KdV equation in $N = 4$ superspace. Thus, if one introduces auxiliary $N = 4$ superfields V^{12} and V^{21} as

$$V^{12} \equiv V^{21} \equiv -\frac{i}{2}D_-D_+^{-1}(\bar{\mathcal{J}} + \mathcal{J}) \quad (83)$$

and the notation

$$V^{11} \equiv -i\,\bar{\mathcal{J}}, \quad V^{22} \equiv i\,\mathcal{J}, \quad \mathcal{D}_1 \equiv \mathcal{D}_+, \quad (84)$$
$$\mathcal{D}_2 \equiv \bar{\mathcal{D}}^-, \quad \bar{\mathcal{D}}^1 \equiv \bar{\mathcal{D}}^+, \quad \bar{\mathcal{D}}^2 \equiv \mathcal{D}_-$$

[compare Eqs. (83) and (85) with Eqs. (75)], then the constraints in (7) and (83) can equivalently be rewritten in the form

$$V^{ij} = V^{ji}, \quad \mathcal{D}^{(i}V^{jk)} = 0, \quad (85)$$
$$\bar{\mathcal{D}}^{(i}V^{jk)} = 0, \quad i, j, k = 1, 2,$$

where the indices i, j, and k are raised and lowered by the antisymmetric tensors ϵ^{ij} and ϵ_{ij}, respectively ($\epsilon^{ij}\epsilon_{jk} = \delta^i{}_k$, $\epsilon_{12} = -\epsilon^{12} = 1$), and (i, j, k) means symmetrization. The $SU(2)$ spin-1 $N = 4$ supercurrent V^{ij} (85) was initially introduced in [2, 3] to describe the $N = 4$ KdV equation. Thus, formulas (83)–(85) establish the precise correspondence with [2, 3] if one additionally replaces the bosonic coordinate z by iz. Using this correspondence, one can calculate, e.g., the matrices a^{ij} from [3],

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (86)$$

which correspond to Eqs. (76) and (81), respectively.

7. CONCLUSION

In this talk, we have described the consistent Lax pair formulation of the $N = 4$ supersymmetric Toda chain (KdV) hierarchy in $N = 4$ superspace. The explicit general formulas (30) for its bosonic flows in terms of the Lax operator in $N = 4$ superspace and its five real forms (67)–(71) have been derived. Then, a basis change in $N = 4$ superspace has allowed us to eliminate all nonlocalities in the flows given by (43)–(46). We have also presented the formulas for the corresponding Hamiltonians [(53)], the first two Hamiltonian structures [(60), (61)], and the recursion operator [(65), (66)] in $N = 4$ superspace. Finally, the explicit formulas (83)–(85) relating the two descriptions of the flows in $N = 4$ superspace used in [3] and [4] have been established. It is obvious that there remains a lot of work to do in order to improve our understanding of the hierarchy in $N = 4$ superspace, but the construction of the $N = 4$ Lax pair formulation is a crucial step toward a complete description.

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Supersymmetry and Solvable Periodic Potentials*

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Abstract—We use the formalism of supersymmetric quantum mechanics to enlarge considerably the limited class of analytically solvable one-dimensional periodic potentials. In particular, we derive and discuss the energy-band structure of the Lamé potentials $pm\,\mathrm{sn}^2(x,m)$ and associated Lamé potentials $pm\,\mathrm{sn}^2(x,m) + qm\,\mathrm{cn}^2(x,m)/\mathrm{dn}^2(x,m)$, both of which involve Jacobi elliptic functions with modulus parameter m. We find several new analytic expressions for band-edge energies and wave functions. The supersymmetric partners of Lamé and associated Lamé potentials constitute even more new solvable potentials with exactly the same energy-band structure. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

The aim of this article is to show how supersymmetry advances our understanding of the Schrödinger differential equation for periodic potentials, often called Hill's equation [1]. To set the stage, let us begin with a few simple statements.

One-dimensional potential wells have bound states. They are solutions of the Schrödinger equation that satisfy appropriate boundary conditions. When two identical potential wells are very far apart, then each potential has the same energy levels, and each eigenstate is doubly degenerate. As the wells are brought closer together, there is communication between them, and each level is split into two. Similarly, if one has an array of many identical wells forming a periodic potential, then one gets energy bands, which play, for example, a crucial role in determining the electronic properties of crystalline solids. To illustrate this band structure quantitatively, condensed-matter-physics texts usually treat the problem of a one-dimensional periodic array of delta functions, called the Kronig–Penney model. Here, one gets a transcendental equation for computing band edges. Another well-studied class of periodic potentials are the Lamé potentials $V(x) = pm\,\mathrm{sn}^2(x,m)$ [1, 2]. They have a simple oscillatory structure produced by the Jacobi elliptic function $\mathrm{sn}(x,m)$ [3]. There are explicit analytic expressions for the band edges. However, this is an exceptional example. The bottom line is that, even in one dimension, there are very few solvable periodic potentials, and it would be nice to have more, especially some with a richer spatial structure.

We will describe how to obtain new solvable periodic potentials via two approaches. First, we will further expand our knowledge of Lamé potentials to the wider class of associated Lamé potentials $V(x) = pm\,\mathrm{sn}^2(x,m) + qm\,\mathrm{cn}^2(x,m)/\mathrm{dn}^2(x,m)$. Although the associated Lamé equation has been studied for numerous years, many of the results that we obtain are new. Second, we will further expand the class of solvable potentials using the techniques of supersymmetry [4]. Let us recall that, in the context of quantum mechanics, supersymmetry relates two partner Hamiltonians with the same energy spectra. Indeed, supersymmetric quantum mechanics has proved to be useful in discovering many new analytically solvable potentials on both the full and the half, line [4]. It is then natural to inquire whether one can also use similar techniques to discover new solvable periodic potentials. In this article, we demonstrate that this is indeed possible. In fact, given any periodic potential, supersymmetry can be used to get a new solvable potential, and this is precisely the technique we plan to exploit.

The outline of this article is as follows. In Section 2, we review some general properties of (a) periodic potentials, (b) supersymmetric quantum mechanics, and (c) Jacobi elliptic functions. Then, in Section 3, we discuss the energies and wave functions for the Lamé potentials. We also describe how the Lamé potential results can be vastly expanded to get solutions for the associated Lamé potentials. Many key new results are summarized in Table 2 and Fig. 4. Finally, in Section 4, we obtain the supersymmetric partners of both the Lamé and the associated Lamé potentials, which are new solvable periodic potentials.

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Details of the work and results described here can be found in [5].

2. GENERAL PROPERTIES

(a) Periodic potentials. For a potential with period L, one seeks solutions of the Schrödinger equation subject to the Bloch condition $\psi(x) = e^{ikL}\,\psi(x+L)$, where k denotes the crystal momentum. The spectrum shows energy bands whose edges correspond to $kL = 0, \pi$; that is, the wave functions at the band edges satisfy the condition $\psi(x) = \pm\psi(x+L)$. For periodic potentials, the band-edge energies and wave functions are often called eigenvalues and eigenfunctions, and we will also use this terminology.

A general property of eigenstates for a potential with period L is the oscillation theorem. It states that band-edge wave functions, when arranged in order of increasing energy, $E_0 \le E_1 \le E_2 \le E_3 \le E_4 \le E_5 \le E_6 \le \ldots$, have periods $L,\ 2L,\ 2L,\ L,\ L,\ 2L,\ 2L,\ \ldots$, respectively. The corresponding number of wave-function nodes in the interval L are $0,\ 1,\ 1,\ 2,\ 2, 3,\ 3,\ \ldots$, and the energy-band gaps are given by $\Delta_1 \equiv E_2 - E_1,\ \Delta_2 \equiv E_4 - E_3,\ \Delta_3 \equiv E_6 - E_5,\ \ldots$. We shall see that the oscillation theorem is very useful in identifying whether all band-edge eigenstates have been properly determined or whether some have been missed.

(b) Supersymmetric quantum mechanics. The supersymmetric partner potentials $V_\pm(x)$ are defined in terms of the superpotential $W(x)$ by $V_\pm(x) = W^2(x) \pm W'(x)$. The corresponding Hamiltonians $H_\pm$ can be factorized as $H_- = A^+A,\ H_+ = AA^+$, where

$$A = \frac{d}{dx} + W(x), \quad A^+ = -\frac{d}{dx} + W(x), \qquad (1)$$

so that the spectra of $H_\pm$ are nonnegative. It is also clear that, on the full line, both $H_\pm$ cannot have zero energy modes, since both $\psi_0^{(\pm)}$ given by

$$\psi_0^{(\pm)}(x) = \exp\left(\pm\int^x W(y)dy\right) \qquad (2)$$

cannot be simultaneously normalized.

On the other hand, when the superpotential $W(x)$ is periodic [$W(x+L) = W(x)$], the potentials $V_-(x)$ and $V_+(x)$ are isospectral—their spectra match completely, including the zero modes, and one has unbroken supersymmetry, provided that

$$\int_0^L W(y)dy = 0. \qquad (3)$$

It is worth noting that, in this case, both $\psi_0^{(\pm)}$ belong to the Hilbert space. The condition in (3) is trivially satisfied when $W(x)$ is an odd function of x, and, throughout this article, we shall only consider superpotentials W that are odd functions of x. Further, using the known eigenfunctions $\psi_n^{(-)}(x)$ of $V_-(x)$, one can immediately write down the corresponding eigenfunctions $\psi_n^{(+)}(x)$ of $V_+(x)$. In particular, from Eq. (2), it follows that the ground state of $V_+(x)$ is given by

$$\psi_0^{(+)}(x) = \frac{1}{\psi_0^{(-)}(x)}, \qquad (4)$$

while the unnormalized excited states $\psi_n^{(+)}(x)$ are obtained from $\psi_n^{(-)}(x)$ by using the relation

$$\psi_n^{(+)}(x) = \left[\frac{d}{dx} + W(x)\right]\psi_n^{(-)}(x) \quad (n \ge 1). \qquad (5)$$

Thus, by starting from an exactly solvable periodic potential $V_-(x)$, one gets a new isospectral periodic potential $V_+(x)$.

The concept of self-isospectral periodic potentials was defined and developed in detail in [6]. A one-dimensional potential $V_-(x)$ of period L is said to be self-isospectral if its supersymmetric-partner potential $V_+(x)$ is just the original potential, apart from a discrete transformation—a translation by any constant amount, a reflection, or both. A common example is translation by half a period, in which case the condition for self-isospectrality is $V_+(x) = V_-(x - L/2)$. In this sense, any self-isospectral potential is rather uninteresting, since the application of supersymmetry just yields a discrete transformation and basically nothing new.

(c) Jacobi elliptic functions. Since our examples involve Jacobi elliptic functions, it is convenient to state some of the main properties. The three functions $\mathrm{sn}(x,m)$, $\mathrm{cn}(x,m)$, and $\mathrm{dn}(x,m)$ are all defined in terms of elliptic integrals and involve a real elliptic-modulus parameter $m (0 \le m \le 1)$. For the sake of simplicity, from now on, we will not explicitly display the parameter m as an argument. The functions $\mathrm{sn}(x)$ and $\mathrm{cn}(x)$ have a period $4K(m)$, whereas $\mathrm{dn}(x)$ has a period $2K(m)$. The quantity $K(m)$ is defined by

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{(1 - m\sin^2\theta)^{1/2}}.$$

It is a monotonically increasing function of m with limiting values of $K(0) = \pi/2$ and $K(1) \to \infty$. The elliptic functions $\mathrm{sn}(x)$ and $\mathrm{cn}(x)$ have zeros at $x = 0, 2K(m), 4K(m), \ldots$ and $x = K(m), 3K(m),$

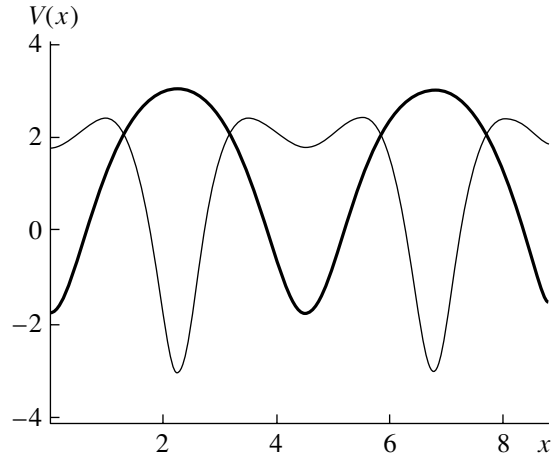

Fig. 1. The (6, 0) Lamé potential $V_-(x)$ corresponding to $a = 2$ (thick curve) as given by Eq. (8) and its supersymmetric-partner potential $V_+(x)$ (thin curve) as given by Eq. (9) for $m = 0.8$.

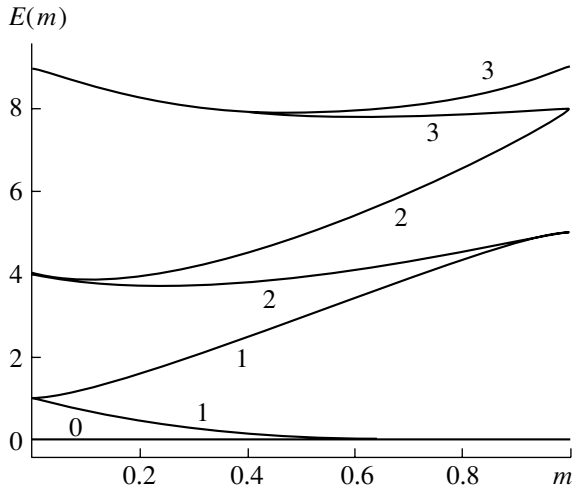

Fig. 2. Band-edge energies for the (12, 0) Lamé potential corresponding to $a = 3$ as a function of the elliptic-modulus parameter m. This figure is drawn by using the eigenvalues given in Table 2.

$5K(m), \ldots$, respectively, whereas $\mathrm{dn}(x)$ has no zeros. When $m = 0$, the functions $\mathrm{sn}(x)$, $\mathrm{cn}(x)$, and $\mathrm{dn}(x)$ reduce to the familiar functions $\sin x$, $\cos x$, and 1, whereas, when $m = 1$, they reduce to $\tanh x$, $\mathrm{sech}x$, and $\mathrm{sech}x$, respectively. Two useful identities are $\mathrm{sn}^2(x) + \mathrm{cn}^2(x) = 1$ and $\mathrm{dn}^2(x) + m\mathrm{sn}^2(x) = 1$.

3. SOLVABLE AND QUASI EXACTLY SOLVABLE POTENTIALS

(a) Lamé potentials. The potentials $V(x) = pm\,\mathrm{sn}^2(x, m)$, $p = a(a+1)$, have a period $L = 2K(m)$. They are called Lamé potentials, since the corresponding Schrödinger equation (with $\hbar = 2m = 1$)

$$-\frac{d^2\psi}{dx^2} + a(a+1)m\,\mathrm{sn}^2(x,m)\psi = E\psi$$

is called Lamé's equation [1, 2]. It is well known that, for any integer value $a = 1, 2, 3, \ldots$, the corresponding Lamé potential has a bound bands followed by a continuum band [1, 2]. All band-edge energies and wave functions are analytically known.

The $a = 1$ Lamé potential $V_- = 2m\,\mathrm{sn}^2(x) - m$ (shifted by a constant $-m$ so that the ground state is at zero energy) has one energy band ranging from energy 0 to energy $1 - m$, with a continuum starting at energy 1 [2]. The corresponding eigenfunctions are $\mathrm{dn}(x)$, $\mathrm{cn}(x)$, and $\mathrm{sn}(x)$, respectively.

For the $a = 2$ case, the Lamé potential has two bound bands and a continuum band. The potential is shown in Fig. 1. The energies and wave functions of the five band edges are well known [1, 2]. The lowest energy band ranges from $2 + 2m - 2\delta$ ($\delta = \sqrt{1 - m + m^2}$) to $1 + m$, the second energy band ranges from $1 + 4m$ to $4 + m$, and the continuum starts at energy $2 + 2m + 2\delta$. The wave functions of all the band edges are given in Table 1. Note that, in the interval $2K(m)$ corresponding to the period of the Lamé potential, the number of nodes increases with energy, in agreement with the oscillation theorem.

Similarly, one can discuss the band structure for all integer values of a. For the $a = 3$ case, the band edges are shown in Fig. 2.

(b) Associated Lamé potentials. We now expand our discussion to the band edges and wave functions of a much richer class of periodic potentials given by

$$V(x) = pm\,\mathrm{sn}^2(x) + qm\,\frac{\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)}, \qquad (6)$$
$$p \equiv a(a+1), \quad q \equiv b(b+1).$$

These potentials are called associated Lamé potentials, since the corresponding Schrödinger equation is called the associated Lamé's equation [1]. More precisely, we often refer to the associated Lamé potential of Eq. (6) as the (p, q) potential and note that $(p, 0)$ potentials are just the ordinary Lamé potentials. Although some results for (p, q) potentials are available in scattered form in the mathematical literature, many of our results are new.

The associated Lamé potentials can also be rewritten in the alternative form $V(x) = pm\,\mathrm{sn}^2(x) + qm\,\mathrm{sn}^2(x + K(m))$ [3, 5]. Clearly, the potentials (p, q) and (q, p) have the same energy spectra with wave functions shifted by $K(m)$. Therefore, it is sufficient to restrict our attention to $p \geq q$.

Table 1. Eigenvalues and eigenfunctions for five band edges corresponding to the $a = 2$ Lamé potential V_- [Eq. (8)] which gives $(p, q) = (6, 0)$ and its SUSY partner V_+ [Eq. (9)]

E	$\psi^{(-)}$	$[B - 3m\,\mathrm{sn}^2(x)]\psi^{(+)}$	Period	Node
0	$m + 1 + \delta - 3m\,\mathrm{sn}^2(x)$	1	$2K$	0
$2\delta - 1 - m$	$\mathrm{cn}(x)\mathrm{dn}(x)$	$\mathrm{sn}(x)[6m - (m+1)B + m\,\mathrm{sn}^2(x)(2B - 3 - 3m)]$	$4K$	1
$2\delta - 1 + 2m$	$\mathrm{sn}(x)\mathrm{dn}(x)$	$\mathrm{cn}(x)[B + m\,\mathrm{sn}^2(x)(3 - 2B)]$	$4K$	1
$2\delta + 2 - m$	$\mathrm{sn}(x)\mathrm{cn}(x)$	$\mathrm{dn}(x)[B + \mathrm{sn}^2(x)(3m - 2B)]$	$2K$	2
4δ	$m + 1 - \delta - 3m\,\mathrm{sn}^2(x)$	$\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}(x)$	$2K$	2

Note: Here, $B \equiv 1 + m + \delta$ and $\delta \equiv \sqrt{1 - m + m^2}$. The periods of various eigenfunctions and the number of nodes in the interval $2K(m)$ are tabulated.

Table 2. Eigenvalues and eigenfunctions for various associated Lamé potentials (p, q) with $p = a(a + 1)$ and $q = (a - n + 1)(a - n)$ for $n = 1, 2, 3$

q	E	$\mathrm{dn}^{-a}(x)\psi$	Period	Node
$a(a-1)$	ma^2	1	$2K$	0
$(a-1)(a-2)$	$1 + m(a-1)^2$	$\dfrac{\mathrm{cn}(x)}{\mathrm{dn}(x)}$	$4K$	1
$(a-1)(a-2)$	$1 + ma^2$	$\dfrac{\mathrm{sn}(x)}{\mathrm{dn}(x)}$	$4K$	1
$(a-2)(a-3)$	$2 + m(a^2 - 2a + 2) \pm 2\delta'$	$\dfrac{[m(2a-1)\mathrm{sn}^2(x) - 1 + m - ma \pm \delta']}{\mathrm{dn}^2(x)}$	$2K$	2, 0
$(a-2)(a-3)$	$4 + m(a-1)^2$	$\dfrac{\mathrm{sn}(x)\mathrm{cn}(x)}{\mathrm{dn}^2(x)}$	$2K$	2

Note: The periods of various eigenfunctions and the number of nodes in the interval $2K(m)$ are tabulated. Here, $\delta' \equiv \sqrt{1 - m + m^2(a-1)^2}$. The table can be extended to any integer $n \geq 4$.

In general, for any values of p and q, the associated Lamé potentials have a period $2K(m)$, but, for the special case of $p = q$, the period is $K(m)$. From the physical viewpoint, if one thinks of a Lamé potential $(p, 0)$ as that which is due to a one-dimensional regular array of atoms with spacing $2K(m)$ and "strength" p, then the associated Lamé potential (p, q) results from two alternating types of atoms spaced by $K(m)$ with "strengths" p and q, respectively. If the two types of atoms are identical [in which case $p = q$], one expects a potential of period $K(m)$.

Extrema (defined for this discussion as either local or global maxima and minima) of associated Lamé potentials are easily found by setting $dV(x)/dx = 0$. Extrema occur when $\mathrm{sn}(x) = 0$ or $\mathrm{cn}(x) = 0$. Also, for fixed values of q and m, there are additional extrema if p lies in the critical range

$$q(1 - m) \leq p \leq q/(1 - m).$$

The associated Lamé potentials for $q = 2$, $m = 0.5$, and several values of p are plotted in Fig. 3. In the critical range of p values, $1 \leq p \leq 4$, one expects additional extrema, and these are clearly seen.

(c) Parabolas of solvability. The associated Lamé's equation has the form

$$-\frac{d^2\psi}{dx^2} + \left[pm\,\mathrm{sn}^2(x) + qm\frac{\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)} - E\right]\psi = 0. \quad (7)$$

On substituting $\psi(x) = [\mathrm{dn}(x)]^{-b}y(x)$, it is easily shown that $y(x)$ satisfies the Hermite elliptic equation [1]. On further substituting $\mathrm{sn}(x) = \sin t$ and $y(x) \equiv z(t)$, one obtains Ince's equation which is a well-known quasi exactly solvable equation [1]. In particular, if $a + b + 1 = n$ ($n = 1,\ 2,\ 3,\ \ldots$), then one obtains n solutions, which are given in Table 2. In particular, for any given choice of $p = a(a + 1)$, Table 2 lists the eigenstates of the associated Lamé's equation for various values of q.

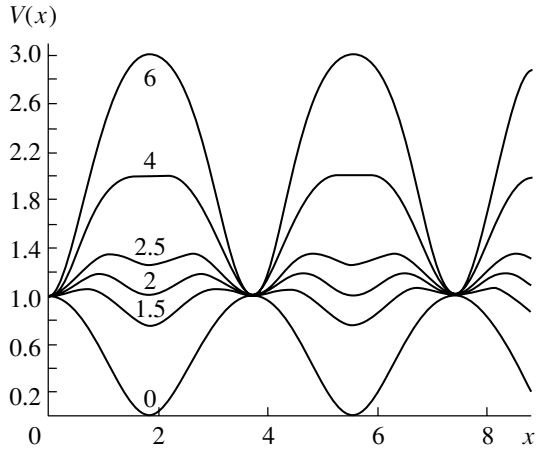

Fig. 3. Plots of the (p, q) associated Lamé potentials for $q = 2$, $m = 0.5$, and several values of p.

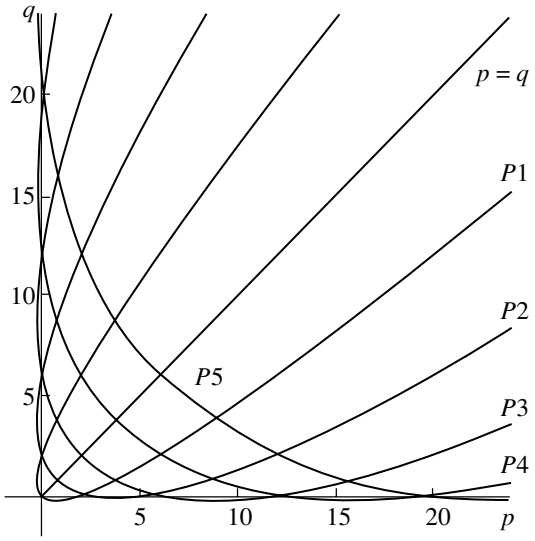

Fig. 4. The illustration of the parabolas of solvability for all associated Lamé potentials (p, q), which are quasi-solvable. Each parabola corresponds to a choice of q in Table 2. The parabola Pn is for $q = (a - n + 1)(a - n)$, and, for any point on it, one knows n eigenstates from Table 2.

For $q = a(a - 1)$, there is just one eigenstate with energy ma^2 and wave function $\psi = dn^a(x)$. Since the wave function has a period $2K(m)$ and is nodeless, this is clearly the ground-state wave function of the $(a(a + 1), a(a - 1))$ potential for any real choice of the parameter a. The equations $p = a(a + 1)$ and $q = a(a - 1)$ are the parametric forms of the equation of the parabola $(p - q)^2 = 2(p + q)$, which is plotted in Fig. 4 and which is denoted by $P1$. For any point on the parabola, one knows the ground-state wave function and energy $E_0 = ma^2$. The parabola $P1$ includes the points (2, 0) and (6, 2).

For $q = (a - 1)(a - 2)$, we see from Table 2 that two eigenstates at energies $1 + m(a - 1)^2$ and $1 + ma^2$ are known. Since they have a period $4K(m)$ and just one node in the interval $L = 2K(m)$, they must correspond to the first and second band-edge energies E_1 and E_2 of the $(a(a + 1), (a - 1)(a - 2))$ potential. Eliminating a from the equations $p = a(a + 1)$ and $q = (a - 1)(a - 2)$ gives the "parabola of solvability" $(p - q)^2 = 8(p + q) - 12$, which is plotted in Fig. 4 and which is denoted by $P2$. This parabola includes the points (2, 0) and (6, 0), which correspond to Lamé potentials. Similarly, the parabolas of solvability $Pn(n = 0, 1, 2, ...)$ corresponding to $q = (a - n + 1)(a - n)$ in Table 2 are plotted; n eigenstates are known for any point on the parabola of solvability Pn.

Note that all (2, 0), (6, 0), (12, 0), ... Lamé potentials have two parabolas of solvability passing through. This provides a good understanding of why they are completely solvable. For instance, the (2, 0) potential is at the intersection of parabolas $P1$ (one known state) and $P2$ (two known states), thus giving three known band edges. Other fully solvable examples correspond to $p = q$. For example, the (2, 2) potential has a period $K(m)$. It lies on parabola $P3$ (three known states), and the band-edge periods are $K(m)$, $2K(m)$, and $2K(m)$, as given in Table 3. From the oscillation theorem, one knows that this is the full solution consisting of one energy band and the continuum. Let us now consider the (6, 2) Lamé potential. It lies on parabola $P1$ [one known nodeless state of period $2K(m)$] and parabola $P4$ [four known states of period $4K(m)$, two with one node and two with three nodes]. Since we know from the oscillation theorem that two states of period $2K(m)$ are missing, this is necessarily an example of a quasi exactly solvable potential [7].

Table 3. Three eigenvalues and eigenfunctions for the (2, 2) associated Lamé potential that has a period $K(m)$ [the number of nodes in the interval $K(m)$ is tabulated]

E	$\mathrm{dn}(x)\psi^{(-)}$	Period	Node
0	$\mathrm{dn}^2(x) + \sqrt{1-m}$	K	0
$4\sqrt{1-m}$	$\mathrm{dn}^2(x) - \sqrt{1-m}$	$2K$	1
$2 - m + 2\sqrt{1-m}$	$\mathrm{sn}(x)\mathrm{cn}(x)$	$2K$	1

4. SUPERSYMMETRIC PARTNERS

(a) Lamé potentials. The supersymmetric-quantum-mechanics formalism of the preceding section will now be applied to the Lamé potentials $ma(a + 1)\mathrm{sn}^2(x, m)$. Analytic solutions are known

for integer values of a [2], and the supersymmetric-partner potentials can be readily computed. We first discuss the results for small integer values of a and then present some eigenstate results for arbitrary integer values of a.

In order to use the supersymmetry formalism, we must shift the Lamé potential by a constant to ensure that the ground state (i.e., the lower edge of the lowest band) has energy $E = 0$. For $a = 1$, one has $V_-(x) = 2m\,\mathrm{sn}^2(x) - m$ and $\psi_0^{(-)} = \mathrm{dn}(x)$ and the superpotential is $W = m\,\mathrm{sn}(x)\mathrm{cn}(x)/\mathrm{dn}(x)$. The partner $V_+(x)$ proves to be just $V_-(x - K(m))$, so that this is an example of self-isospectrality. For $a = 2$, the potential is

$$V_-(x) = -2 - 2m + 2\delta + 6m\,\mathrm{sn}^2(x) \qquad (8)$$

with a corresponding unnormalized wave function $\psi_0^{(-)}(x) = 1 + m + \delta - 3m\,\mathrm{sn}^2(x)$ [2]. The corresponding superpotential is $W = 6m\,\mathrm{sn}(x)\mathrm{cn}(x) \times \mathrm{dn}(x)/\psi_0^{(-)}(x)$; hence, the partner potential $V_+(x)$ for the potential $V_-(x)$ given in Eq. (8) is

$$V_+(x) = -V_-(x) + \frac{72m^2\mathrm{sn}^2(x)\mathrm{cn}^2(x)\mathrm{dn}^2(x)}{[1 + m + \delta - 3m\,\mathrm{sn}^2(x)]^2}. \quad (9)$$

Although supersymmetry guarantees that the potentials $V_\pm$ are isospectral, they are not self-isospectral in this example. Therefore, $V_+(x)$ as given by Eq. (9) is a new periodic potential, which is strictly isospectral to the potential (8); hence, it also has two bound bands and a continuum band. Figure 1 shows the potentials $V_\pm(x)$ corresponding to $a = 2$ for $m = 0.8$. Using Eqs. (4) and (5) and the known eigenstates of $V_-(x)$, we can immediately compute all the band-edge Bloch wave functions for $V_+(x)$. Table 1 gives the energy eigenvalues and wave functions for the isospectral partner potentials $V_\pm(x)$. In summary, for integral a, Lamé potentials with $a \geq 2$ are not self-isospectral. They have distinct supersymmetric-partner potentials, even though both potentials have the same $(2a + 1)$ band-edge eigenvalues.

(b) Associated Lamé potentials. It is easily checked from Table 2 that the solution corresponding to $q = a(a - 1)$, as well as one of the $q = (a - 2)(a - 3)$ solutions, is nodeless and corresponds to the ground state. It follows that, for these cases, one can obtain the superpotential and, hence, the partner potential V_+. For example, let us consider the case of $p = a(a + 1)$ and $q = a(a - 1)$. In this case, W is given by $W = am\,\mathrm{sn}(x)\mathrm{cn}(x)/\mathrm{dn}(x)$, so that the corresponding partner potentials are

$$V_\pm = (a \pm 1)am\frac{\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)} + ma(a \mp 1)\mathrm{sn}^2(x) - ma^2.$$

These partner potentials are self-isospectral; therefore, supersymmetry yields nothing new.

Let us now consider the partner potential computed from the ground state for the $p = a(a + 1)$, $q = (a - 2)(a - 3)$ case. Here, $\psi_0(x) = [m(a - 1) - 1 - \delta' + m(2a - 1)\mathrm{sn}^2(x)](\mathrm{dn}(x))^{a-2}$, where $\delta' = \sqrt{1 - m + m^2(a - 1)^2}$. The corresponding superpotential W proves to be

$$W = \frac{m(a - 2)\mathrm{sn}(x)\mathrm{cn}(x)}{\mathrm{dn}(x)} \qquad (10)$$
$$- \frac{2m(2a - 1)\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}(x)}{[m(1 - a) - 1 - \delta' + m(2a - 1)\mathrm{sn}^2(x)]}.$$

Hence, the corresponding partner potentials are

$$V_- = ma(a + 1)\mathrm{sn}^2(x) \qquad (11)$$
$$+ m(a - 3)(a - 2)\frac{\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)} - 2$$
$$- m(a^2 - 2a + 2) + 2\delta', \quad V_+ = -V_- + 2W^2.$$

These potentials are not self-isospectral. Thus, one has discovered a whole class of new elliptic periodic potentials $V_+(x)$ for which three states are analytically known, no matter what a is. In particular, the energy eigenfunctions for V_+ of these three states are easily obtained by taking the corresponding energy eigenstates of V_- as given in Table 2 and using Eqs. (4) and (5).

In conclusion, our analysis yields eigenstates for a large class of associated Lamé potentials and provides a deeper understanding of why Lamé potentials are fully solvable. Further, using the formalism of supersymmetric quantum mechanics, we have discovered many new exactly solvable and quasi exactly solvable periodic potentials involving Jacobi elliptic functions.

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Separation of Variables for Integrable Systems on Poisson Manifolds*

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Abstract—For an integrable system on Poisson manifolds, a construction of separated variables is discussed. We suppose that, for a given integrable system, we know a realization of the corresponding Lagrangian submanifold as the product of plane curves. In this case, we can use properties of the foliation of the initial Poisson manifold on symplectic leaves and values of the Casimir functions in order to construct separated variables. © *2002 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

A systematic way to realize integrable Hamiltonian systems on coadjoint orbits of Lie algebras is provided by the classical inverse scattering method [1], which is a working machine to produce examples of classical integrable systems, together with their solutions. One of the key points of this method is duality between integrals of motion and the Casimir elements in the underlying hidden-symmetry algebra.

The objective of this paper is to show that not only do the same Casimir elements give rise to integrals of motion, but they can generate separated variables for this family of integrals.

Let us consider some $2n$-dimensional symplectic manifold $(\mathcal{M}, \Omega)$ endowed with a symplectic form Ω. The Hamilton–Jacobi equation

$$\mathcal{Q}^{2n-1}: \quad H - E = 0, \quad E \in \mathbb{R},$$

is a hyperplane $\mathcal{Q}^{2n-1}$ in $\mathcal{M}$ [2, 3]. A solution to this equation is an n-dimensional Lagrangian submanifold F lying on the hyperplane $\mathcal{Q}^{2n-1}$. By definition, a Lagrangian submanifold is the one where the symplectic form Ω vanishes when restricted to it; i.e., $\Omega|_{\mathcal{C}} = 0$.

An integrable system on $\mathcal{M}$ is defined by n functionally independent integrals of the motion, I_j. The inverse images $\cap I_j^{-1}(\alpha_j)$ of the corresponding moment map is a Lagrangian submanifold F. Thus, any integrable system is associated with the Lagrangian fibration $\rho_i : \mathcal{M} \to \mathcal{A}^n$ whose fibers F are Lagrangian submanifolds depending at least on n arbitrary parameters α_j, which are identified with the values of integrals of motion, $I_j = \alpha_j$ [2, 3].

Let us consider the Poisson manifold $(\mathcal{M}, \mathcal{P})$ with a degenerate Poisson form $\mathcal{P}$. Any Poisson manifold proves to be foliated in symplectic leaves or minimal Poisson submanifolds S_a for which the induced Poisson structure is nondegenerate. This foliation $\rho_{\mathrm{C}} : \mathcal{M} \to \mathcal{A}^m$ may be described by the Casimir functions whose restrictions to symplectic leaves S_a are constants. Thus, fixing the values of Casimir functions provides a rough classification of symplectic leaves, although it is not true in general that different symplectic leaves are separated by the Casimir functions.

Thus, for an integrable system on Poisson manifolds, we can consider the composition of foliations $\rho = \rho_i \circ \rho_{\mathrm{C}} : \mathcal{M} \to \mathcal{A}^{n+m}$ whose fibers F depend on n values of the integrals α_j of motion and m values a_i of the Casimir functions. Using some additional assumptions on foliation ρ, we can define a canonical symplectic connection ∇ that gives rise to parallel translations T_γ of fibers F under some smooth curve γ in base $\mathcal{A}^{n+m}$.

Let a given integrable system on $\mathcal{M}$ and the corresponding separated variables be invariant with respect to translations T_γ; i.e., separated variables are independent of the values of integrals α_j and the values of the Casimir functions a_i. In this case $\nabla N = 0$ where ∇ is symplectic connection on ρ and N is the Nijenhuis tensor, whose eigenvalues are separated variables [4]. Thus, we suppose that the separated variables may be explicitly defined by using these translations T_γ. In this paper, we prove this proposition for some examples. The general geometric construction will be studied in a forthcoming publication.

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For separable integrable systems on a Poisson manifold, the Lagrangian submanifolds

$$F \simeq \mathcal{C}^{(n)}: \quad \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_n \times \mathcal{A}^m \subset \mathbb{R}^m \quad (1)$$

are the product of n plane curves $\mathcal{C}_j$, which may be defined explicitly as

$$\mathcal{C}_j: \quad \Phi_j(\mu_j, \lambda_j, \alpha_1, \ldots, \alpha_n, a_1, \ldots, a_m) = 0, \quad (2)$$
$$j = 1, \ldots, n.$$

Here, (μ_j, λ_j) are coordinates on the jth plane. Below, we shall drop these indices in the notation used.

In the inverse scattering method, we usually study the Lax form of the equations of motion,

$$\{H, L\} = [L, A],$$

where L and A are matrix-valued functions on $\mathcal{M}$. The coefficients of the characteristic polynomial of the Lax matrix L are integrals of motion and Casimir functions. If a given Lax matrix $L(\lambda)$ depends on the second spectral parameter λ, then the characteristic equation

$$\det(\mu - L(\lambda)) = \Phi(\mu, \lambda, \alpha_1, \ldots, \alpha_n) = 0 \quad (3)$$

defines only one plane curve $\mathcal{C}$. The corresponding Lagrangian submanifold

$$\mathcal{C}^{(n)}: \quad \mathrm{Sym}^n(\mathcal{C}) \times \mathcal{A}^m \subset \mathbb{R}^m$$

is the symmetric product of this spectral curve of the Lax matrix. This allows us to identify Lagrangian foliation with an affine variety of $\mathcal{C}$ and to switch on the powerful algebraic–geometric machinery [5–7]. In this case, the phase space $\mathcal{M}$ may be regarded as a fiber bundle $\mathcal{M} \to \mathcal{A}^{n+m}$. The fibers are (generalized) Jacobians of these curves [6, 7].

2. SEPARATION OF VARIABLES

Let $\{p_i, q_i\}_{i=1}^n$ be some local coordinates on the phase space $\mathcal{M}$. In the method of separation of variables [3, 8], we seek a special canonical transformation $(p, q) \mapsto (P, Q)$ such that separated variables have to satisfy the separated equations

$$\Phi_j\Big(P_j, Q_j, I_1(P, Q), \ldots, I_n(P, Q)\Big) = 0, \quad (4)$$
$$j = 1, \ldots, n.$$

These equations can be obtained from the equations of the plane curves (2) after the substitution of the integrals of motion $I_j(P, Q)$ as functions of the separated variables for their values α_j and the separated variables (P, Q) for the variables (μ, λ).

We have to emphasize that the variables μ and λ are complex variables related by numerical Eq. (2). The separated variables P_i and Q_i are canonical coordinates of $\mathcal{M}$. Constructing the separated Eqs. (4), we have to understand that the numerical parameters α_j and the plane coordinates $\mu = \phi(P_i, Q_i)$ and $\lambda = \psi(P_i, Q_i)$ become some functions on $\mathcal{M}$.

Thus, in the method of separation of variables, two related problems might be distinguished. First, we must determine the decomposition of the Lagrangian submanifold $\mathcal{C}^{(n)}$ (1) on the product of some plane curves. Second, we have to introduce canonical separated variables $\{P_i, Q_i\}_{i=1}^n$ satisfying the corresponding separated Eqs. (4).

The objective of this note is to discuss how to solve the second problem if we know solution of the first problem. Below, we shall consider Lagrangian submanifolds depending on $n + m$ arbitrary parameters. As above, the n constants $\alpha_1, \ldots, \alpha_n$ are identified with the values $I_j = \alpha_j$ of the integrals of the motion [2]. These integrals I_j depend on the remaining constants a_k (charges or coupling constants),

$$I_j(p, q, a) = I_j(p_1, \ldots, p_n, q_1, \ldots, q_n, a_1, \ldots, a_m), \quad a_k \in \mathbb{R}. \quad (5)$$

Let us substitute the functions $I_j(p, q)$ for their values α_j into the known equations of the plane curves (2),

$$\Phi_j(\mu, \lambda, I_1(p, q, a), \ldots, I_n(p, q, a), \quad a_1 \ldots, a_m) = 0, \quad j = 1, \ldots, n. \quad (6)$$

We can consider these equations at different values of parameters, for instance, at $a_k = 0$,

$$\Phi_j(\mu, \lambda, I_1(p, q, a), \quad (7)$$
$$\ldots, I_n(p, q, a), a_1 \ldots, a_m)\Big|_{a=0} = 0,$$

or differentiate these equations with respect to the parameters a_k,

$$\frac{\partial}{\partial a_k}\Phi_j\Big(\mu, \lambda, I_1(p, q, a), \quad (8)$$
$$\ldots, I_n(p, q, a), a_1 \ldots, a_m\Big) = 0, \quad k = 1, \ldots, m.$$

The enlarged system of Eqs. (6)–(8) may be useful in explicitly constructing separated variables.

Proposition 1. If the desired canonical transformation of variables $(p, q) \mapsto (P, Q)$ is independent of the parameters a_k, then the solutions to Eqs. (6)–(8) are the separated variables

$$Q_i = f_i(p, q), \quad P_i = g_i(p, q), \quad i = 1, \ldots, n, \quad (9)$$

as functions of initial variables (p, q).

Here, we suppose that the generating function of the unknown canonical transformation $(p, q) \mapsto (P, Q)$ is independent of the parameters a_k. Clearly, if we can calculate solutions to Eqs. (6)–(8), then we can check that these solutions are independent of parameters and that they are canonically conjugate.

As an example, let us consider the two-dimensional Kepler problem. The corresponding Lagrangian manifold $\mathcal{C}^{(2)} = \mathcal{C}_1 \times \mathcal{C}_2$ depends on the values of two integrals of motion,

$$I_1 = p_x^2 + p_y^2 + \frac{Z}{\sqrt{x^2+y^2}},$$
$$I_2 = 2p_x(p_x\, y - p_y\, x) + \frac{Zy}{\sqrt{x^2+y^2}},$$

and an arbitrary value of the charge Z. We can substitute the functions $I_k(p_x, x, p_y, y)$ for their values α_k into the equations of the corresponding plane curves,

$$\mathcal{C}_{1,2}:\ \Phi_{1,2}(\mu,\lambda) = \Big(\mu^2 + \alpha_1\lambda + Z \pm \alpha_2\Big) = 0,$$
$$I_1 = \alpha_1, \quad I_2 = \alpha_2,$$

and differentiate these equations with respect to Z. Solutions to the resulting system of algebraic equations are usual parabolic coordinates,

$$\frac{\partial}{\partial Z}\Phi_{1,2}(\,\mu,\lambda) = 0 \ \leftrightarrow\ \lambda_{1,2} = y \pm \sqrt{x^2+y^2},$$

which are independent of the charge Z.

In the proposed construction of separated variables, the main problem is associated with the existence of the additional arbitrary parameters a_k. For an integrable Hamiltonian system on coadjoint orbits of Lie algebras, values of the Casimir operators may be considered as these necessary parameters. Recall that the Casimir operators give rise to a stratification of the underlying phase space $\mathcal{M}$ on usually equivalent symplectic leaves. Thus, we can suppose that the desired separated variables are independent of the "labels" of symplectic leaves.

3. NEUMANN SYSTEM

Let us consider the motion of a particle on the $(n-1)$-dimensional sphere $\mathcal{S}^{n-1}$ under the effect of a quadratic potential (see [1] and references within). If x and p are canonical variables on $T^*\mathcal{S}^{n-1}$, then the corresponding Lax representation is given by

$$L(\lambda) = A + J\lambda + X\lambda^2, \qquad (10)$$
$$J = p \wedge x, \quad X = x \otimes x.$$

Here, $A = \mathrm{diag}(a_1, \ldots, a_n)$ is an arbitrary numerical diagonal matrix, and the Hamiltonian has the form

$$H = \frac{1}{2}\sum p_j^2 - \sum a_j x_j^2.$$

At $n = 3$, this system is referred to as a "Neumann system."

The corresponding phase space may be identified with coadjoint orbits of Euclidean Lie algebras [3]. Let two vectors $J \in so(3) \simeq \mathbb{R}^3$ and $x \in \mathbb{R}^3$ be coordinates in the dual space $e^*(3)$ equipped with natural Lie–Poisson brackets:

$$\{J_i, J_j\} = \varepsilon_{ijk}\, J_k, \quad \{J_i, x_j\} = \varepsilon_{ijk}\, x_k, \qquad (11)$$
$$\{x_i, x_j\} = 0, \quad i,j,k = 1,2,3.$$

Here, ε_{ijk} is the standard totally skew-symmetric tensor.

The Poisson structure is degenerate, and the initial phase space $e^*(3)$ decomposes into minimal Poisson submanifolds for which the induced Poisson structure is nondegenerate. The generic coadjoint orbits of $E(3)$ in $e^*(3)$ are four-dimensional symplectic leaves specified by the two second-order Casimir elements

$$I_1 = (x,x) = x_i x_i; \quad I_2 = (J,x) = J_i x_i. \qquad (12)$$

Here, (x,y) means the inner product in $\mathbb{R}^3$. Thus, the dual space $e^*(3)$ decomposes into the coadjoint orbits,

$$\mathcal{O}_{b,c} = \Big\{\{J,x\} \in \mathbb{R}^6\ :\ I_1 = b,\ I_2 = c\Big\}, \qquad (13)$$

which are invariant with respect to the usual Euler–Poisson equations in $e^*(3)$ [1, 3]. The Neumann system is a completely integrable system on the one-parameter subset of orbits $\mathcal{O}_b$ ($I_1 = b,\ I_2 = 0$) in $e^*(3)$.

For the Neumann system, we can consider motion on different spheres $\mathcal{S}_b^2$ of radius $I_1 = b$; i.e., we can consider different orbits labeled with one parameter b. To construct Eqs. (6)–(8), we have to determine a restriction of the integrals of motion to symplectic leaves. Instead of this, we shall use the scaling transformation $x \to \sqrt{b}\,x$ which relates different orbits $\mathcal{O}_b$, changes the integrals of the motion as

$$I_3 = J_1^2 + J_2^2 + J_3^2$$
$$+ b\Big((a_3+a_2)x_1^2 + (a_1+a_3)x_2^2 + (a_1+a_2)x_3^2\Big),$$
$$I_4 = a_1J_1^2 + a_2J_2^2 + a_3J_3^2$$
$$+ b\Big(a_2a_3x_1^2 + a_1a_3x_2^2 + a_1a_2x_3^2\Big),$$

transforms the corresponding Lax representation as

$$L(\lambda) \to L(\lambda, b) = A + J\lambda + b\,X\lambda^2,$$

and modifies the spectral curve as

$$\mathcal{C}\ :\ \Phi(\mu,\lambda) = \frac{(\mu-a_1)(\mu-a_2)(\mu-a_3)}{\lambda^2}$$
$$+ \mu\, I_3 - I_4 - b\Big(\mu^2\, I_1 + \lambda^2\, I_2^2\Big) = 0.$$

Such a modification of the spectral curve allows us to consider this transformation as a scaling transformation of the Casimir functions: $I_1 \to b\, I_1$ and $I_2 \to \sqrt{b}\ I_2$.

Proposition 2. Solutions to the system of Eqs. (6)–(8) in the form

$$\det\Big(L(\lambda,b)-\mu\Big)=0, \qquad (14)$$
$$\frac{\partial\det\Big(L(\lambda,b)-\mu\Big)}{\partial b}=0$$

are separated variables for the Neumann system.

The last equation in (14) has the form

$$\prod_{j=1}^{3}(\mu-a_j)\left(\frac{x_1^2}{\mu-a_1}+\frac{x_2^2}{\mu-a_2}+\frac{x_3^2}{\mu-a_3}\right)=0.$$

It is the well-known definition of elliptic spherical coordinates. These coordinates are separated variables for the Neumann system [5].

Notice that the third additional Eq. (7) is equivalent to

$$\det\Big(L(\lambda,b=0)-\mu\Big)=\det\Big(L(\lambda,b)-\mu\Big) \qquad (15)$$
$$-\,b\,\frac{\partial\det\Big(L(\lambda,b)-\mu\Big)}{\partial b}=0$$

and that the Lax matrix $L(\lambda,b=0)$ may be associated with the Euler top.

4. ON THE STATIONARY FLOW OF THE KdV HIERARCHY

The separation of variables for constrained flows of soliton equations was studied, for instance, in [9, 10]. Below we shall consider only the KdV hierarchy.

Let $\{\mathbf{e},\mathbf{f},\mathbf{h}\}$ be generators of the $sl(2)$ Lie algebra,

$$[\mathbf{h},\mathbf{e}]=\mathbf{e},\quad [\mathbf{h},\mathbf{f}]=-\mathbf{f},\quad [\mathbf{e},\mathbf{f}]=2\mathbf{h}, \qquad (16)$$

and let the element

$$\Delta=\mathbf{h}^2+\frac{1}{2}(\mathbf{e}\cdot\mathbf{f}+\mathbf{f}\cdot\mathbf{e}) \qquad (17)$$

of the universal enveloping algebra be the Laplace operator in $SL(2)$. Let us consider infinite-dimensional irreducible representation $\mathcal{W}_j$ of the $sl(2)$ Lie algebra in the linear space V_j such that

$$\mathcal{W}_j:\ \{\mathbf{e},\mathbf{f},\mathbf{h}\}\to\{e_j,f_j,h_j\}\in\mathrm{End}(V_j).$$

Using these representations and an appropriate completion of the direct sum of $n+1$ copies of the $sl(2,\lambda)$ loop algebra, we can construct the multipole Lax matrix

$$L_0(\lambda)=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \qquad (18)$$
$$+\sum_{j=1}^{n}\frac{1}{2(\lambda-d_j)}\begin{pmatrix}h_j & e_j\\ f_j & -h_j\end{pmatrix}$$

within classical r matrix theory [1].

Below, we shall consider infinite-dimensional representations of $sl(2)$ where the spectra of all Casimir elements are continuous. In this case, the corresponding the $2n$-dimensional Lagrangian foliation is given by

$$\mathcal{C}^{(2n)}:\qquad \overbrace{\mathcal{C}\times\cdots\times\mathcal{C}}^{n\ \text{times}}\times\overbrace{\mathbb{R}\times\cdots\times\mathbb{R}}^{n\ \text{times}},$$

where $\mathcal{C}$ is a spectral curve of the Lax matrix,

$$\mathcal{C}\ :\ \mu^2+h(\lambda)^2+e(\lambda)\,f(\lambda)$$
$$=\mu^2+P(\lambda,I_1,\ldots,I_n)+\sum_{j=1}^{n}\frac{\Delta_k}{(\lambda-d_k)^2}=0.$$

Here, $h(\lambda)$, $f(\lambda)$, and $e(\lambda)$ are the entries of the Lax matrices $L_0(\lambda)$ (18), and the explicit expressions for the function P and the integrals $I_1,\ldots,I_n$ can be found in [10, 11].

This general construction is an ample source of examples of integrable systems. Any specific realization of $sl(2)$ is associated with the family of integrable systems. For instance, the realization

$$h_i=x_ip_i,\quad e_i=x_i^2, \qquad (19)$$
$$f_i=-p_i^2+\frac{a_i}{x_i^2},\quad a_i\in\mathbb{R},$$

of the infinite-dimensional representation of $sl(2)$ may be associated with standard constrained flows and with a geodesic motion on Riemann manifolds with the Stäckel metric [11].

To study eighth-order binary constrained flows, we have to use the so-called m-bosonic realization of the infinite-dimensional representation of $sl(2)$ instead of the one-bosonic realization (19). For example, the two-bosonic realization has the form

$$h_i=\frac{1}{2}(x_ip_{x,i}-y_ip_{y,i}), \qquad (20)$$
$$e_i=x_i\,p_{y,i},\quad f_i=y_i\,p_{x,i}.$$

The first realization of $sl(2)$ (19) is the embedding of the symplectic manifold $\mathbb{R}^{2n}$ into the Poisson manifold $\mathcal{M}=\oplus sl(2)^*$; therefore, the Casimir elements Δ_i (17) describe trivial dynamics. Moreover, all integrals of the motion depend on the values a_i of these Casimir elements Δ_i. Using the parameters a_i, we can construct the system of Eqs. (8) in the form

$$\frac{\partial}{\partial a_k}\Phi\Big(\mu,\lambda,\alpha,a\Big)=0\quad\leftrightarrow\quad\frac{\partial f(\lambda)}{\partial a_k}\,e(\lambda)=0.$$

The zeroes of the great common divisor $e(\lambda)$ of these equations coincide with the poles of the corresponding Baker–Akhiezer function normalized to the vector $\boldsymbol{\alpha} = (1, 0)$. Here, we have explicitly n poles and can introduce $2n$ separated variables.

The two-bosonic realization of $sl(2)$ (20) describes the immersion of the Poisson manifold $\mathcal{M} = \oplus sl(2)^*$ into the symplectic manifold $\mathbb{R}^{4n}$. Hence, the first n integrals of the motion I_j describe a potential motion on the Riemann manifold with the Stäckel metric, whereas the remaining n integrals

$$I_{n+j} = \frac{(x_j p_{x,j} + y_j p_{y,j})^2}{4}, \quad j = 1, \ldots, n,$$

are the Casimir elements Δ_j (17). In contrast to the preceding example, these functions I_{n+j} describe nontrivial dynamics on the $4n$-dimensional symplectic manifold $\mathbb{R}^{4n}$.

The initial Poisson structure on $\mathcal{M}$ is degenerate. A possible way to analyze integrability is to eliminate the Casimir elements of the Poisson tensor by fixing the values of its Casimir functions. On the level surface S_b, the functions $I_{n+j} = b_j$ and the Poisson structure becomes nondegenerate.

Let us consider a restriction of the initial phase space to S_b. On each leaf, the n-dimensional Lagrangian foliation depends on n values of the integrals I_j and n parameters b_j. This allows us to apply the proposed method to construct separated variables.

Proposition 3. Solutions to the system of equations

$$\Phi\left(\mu, \lambda, \widehat{I}_1, \ldots, \widehat{I}_n, b\right) = 0, \qquad (21)$$
$$\frac{\partial}{\partial b_k}\Phi\left(\mu, \lambda, \widehat{I}_1, \ldots, \widehat{I}_n, b\right) = 0$$

are separated variables. Here, $\widehat{I}_j$ are restrictions of the integrals I_j to the generic leaf S_b.

To prove this proposition, we can start with the equations [10]

$$\{I_{n+j}, e(\lambda)\} = \{I_{n+j}, h(\lambda)\} = 0,$$

which are preserved by the reduction procedure. Hence, using the reduction of the Lax matrix $L(\lambda)$ to S_b, we can rewrite Eqs. (8) in the form

$$\frac{\partial f(\lambda)}{\partial b_k}\, e(\lambda) = 0 \quad \leftrightarrow \quad e(\lambda) = 0.$$

As above, the zeroes of the great common divisor $e(\lambda)$ of these equations coincide with the poles of the corresponding Baker–Akhiezer function normalized to the vector $\boldsymbol{\alpha} = (1, 0)$.

Let us return to the initial $4n$-dimensional manifold $\mathbb{R}^{4n}$. By definition of the first $2n$ separated variables (21), the Poisson brackets between the Casimir elements I_{n+j} and these separated variables are equal to zero. To construct a complete family of separated variables on the initial $4n$-dimensional phase space, we can therefore join $2n$ solutions of (21) with n Casimir elements I_{n+j} and n variables conjugate to them. For these additional variables, the separated equations are equivalent to $P_{n+j} = I_{n+j}$.

5. STATIONARY FLOW OF THE BOUSSINESQ HIERARCHY

This example was considered within a general framework for an analysis of stationary flows of n-Gelfand–Dickey hierarchies [4]. According to that study, we introduce a 10-dimensional Poisson manifold $\mathcal{M}$ with coordinates $h_1, k_1, h_2, k_2, \ldots, h_5, k_5$ and with the Poisson tensor

$$P_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
 & & & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & -h_1 & -2h_1 & -k_1 - h_2 \\
 & & & & & 0 & -h_1 & -2k_1 & -k_1 - h_2 & -2k_2 \\
 & & & & & & 0 & 0 & -2h_2 & -k_2 \\
 & & & & & & & 0 & -k_2 & 0 \\
 & & & & & & & & 0 & 0 \\
 & & & & & & & & & 0
\end{pmatrix}.$$

This Poisson structure is degenerate, and two Casimir elements

$$I_1 = 3k_5 - 3k_1k_2 - 3k_2h_2, \qquad (22)$$
$$I_3 = -3h_1k_2 + 3k_4 - 3k_1^2$$

describe trivial dynamics, $\{I_{1,3}, h_j\} = \{I_{1,3}, k_j\} = 0$. On this phase space, we consider the integrable system characterized by the Lax matrices

$$L(\lambda) = \lambda^2 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (23)$$
$$+ \lambda \begin{pmatrix} 0 & 0 & 1 \\ h_2 & h_1 & 0 \\ -h_3 + k_2 & k_1 - h_2 & -h_1 \end{pmatrix}$$

$$+\begin{pmatrix} -k_3 & -k_2 & -k_1 \\ L_0^{21} & -h_2h_1+h_4 & -h_1^2+h_3 \\ L_0^{31} & L_0^{32} & h2h_1-h_4+k_3 \end{pmatrix},$$

where

$$L_0^{21} = -h_1k_2 + k_4 - k_1^2 - h_1h_3 + h_5,$$
$$L_0^{32} = -k_1h_2 + h_2^2 - h_5 + 2k_4 - k_1^2,$$
$$L_0^{31} = h_1k_3 - 2h_1h_4 + 3k_5 + h_2h_1^2 - 2k_1k_2 - 2k_1h_3 + h_1^2k_1 - 2k_2h_2 + h_2h_3.$$

The second matrix of the Lax pair may be found in [4]. This Lax representation may be obtained by using the Lax formulation for the Boussinesq hierarchy or by applying the general r matrix theory to the loop algebra $sl(3,\lambda)$ [4].

The spectral curve of this Lax matrix (23),

$$\Phi(\mu,\lambda) = \mu^3 - \mu\,(I_1\lambda + I_2) - (\lambda^5 + I_3\lambda^3 + I_4\lambda^2 + I_5\lambda + I_6) = 0, \quad (24)$$

depends on two Casimir elements and on four integrals of motion. The explicit expression for the Hamiltonian is

$$I_2 = k_3^2 + 2h_1h_4k_1 - h_1k_3k_1 + h_3k_1^2 - 2h_3k_1h_2 + 3k_1^2k_2 + h_1k_2^2 + h_2h_1k_3 - 2h_2h_1h_4 + h_4^2 - k_4k_2 + h_1h_3k_2 + h_1^2h_5 + 2k_2h_2k_1 - 3k_5k_1 + 2h_3k_4 + h_3h_2^2 - 2h_1^2k_4 - h_4k_3 - h_5k_2 - h_3h_5,$$

whereas other integrals may be found in [4].

In the initial ten-dimensional phase space $\mathcal{M}$, the five-dimensional Lagrangian foliation

$$\mathcal{C}^{(5)}: \quad \mathcal{C}\times\mathcal{C}\times\mathcal{C}\times\mathcal{C}\times\mathbb{R}^2$$

is noncompact. Let us introduce a level surface S_{ab} of the Casimir elements $I_1 = a$ and $I_3 = b$. It is an eight-dimensional symplectic leaf, which may be identified with the coadjoint orbit of the corresponding Lie algebra [4].

On the leaf S_{ab}, the corresponding four-dimensional Lagrangian submanifold depends on four values of integrals of motion, $\alpha_j = \widehat{I}_j$, and on two free parameters a and b.

Proposition 4. The separated variables are solutions to the system of equations

$$\Phi(\mu,\lambda,\widehat{I},a,b) = 0,$$
$$\frac{\partial}{\partial a}\Phi(\mu,\lambda,\widehat{I},a,b) = 0, \quad \frac{\partial}{\partial b}\Phi(\mu,\lambda,\widehat{I},a,b) = 0,$$

where $\Phi(\mu,\lambda)$ determines the spectral curve of the Lax matrix $\mathcal{C}$ (24) and $\widehat{I}_j$ are restrictions of the original integrals I_j to S_{ab}.

To construct the separated variables, we therefore have to compute the integrals $\widehat{I}_j$ on a generic symplectic leaf S_{ab}. This problem was solved in [4]. Substituting the integrals $\widehat{I}_j$ into Eqs. (4), we can directly check that solutions to these equations are independent of the parameters a and b. Of course, these separated variables coincide with the Darboux–Nijenhuis coordinates [4].

In order to avoid calculations of the restrictions of integrals of motion to the symplectic leaf S_{ab}, we can consider the mapping $S_{ab} \to S_{cd}$, which relates different symplectic leaves. To determine such mapping, we can apply canonical transformations of the coordinates (h,k) or the following shift of the Lax matrix:

$$\widetilde{L}(\lambda) = L(\lambda) + \begin{pmatrix} 0 & 0 & 0 \\ b_1\,I_3 + b_2 & 0 & 0 \\ a_1\,I_1 + a_2 & c_1\,I_3 + c_2 & 0 \end{pmatrix}, \quad (25)$$
$$a_k, b_k \in \mathbb{R}.$$

This transformation changes the coefficients of the spectral curve,

$$\Phi(\mu,\lambda) = \mu^3 - \mu\left(\widetilde{I}_1\lambda + \widetilde{I}_2\right) - (\lambda^5 + \widetilde{I}_3\lambda^3 + \widetilde{I}_4\lambda^2 + \widetilde{I}_5\lambda + \widetilde{I}_6) = 0,$$

and yields scaling and shift of the Casimir elements,

$$\widetilde{I}_1 = (a_1+1)\,I_1 + a_2,$$
$$\widetilde{I}_3 = (b_1+c_1+1)\,I_3 + b_2 + c_2.$$

The explicit expressions for other integrals may be simply calculated from the definition of the Lax matrix (25). For instance, the new Hamiltonian is given by

$$\widetilde{I}_2 = I_2 - k_1(a_1I_1 - a_2) - k_2(b_1I_3 - b_2) - (h_1^2 - h_3)(c_1I_3 - c_2).$$

The corresponding Lagrangian foliation depends on six arbitrary parameters. However, the pairs of parameters x_1 and x_2, where $x = a, b$, or c, give rise to the equivalent equations

$$\frac{\partial}{\partial x_1}\Phi(\mu,\lambda) = 0 \;\leftrightarrow\; \frac{\partial}{\partial x_2}\Phi(\mu,\lambda) = 0, \quad x = a,b,c.$$

Differentiating by parameters a, b, or c, one therefore gets three equations in addition to the initial equation of the plane curve.

Proposition 5. Solutions to two different systems of equations

$$\Phi\left(\mu,\lambda,\widetilde{I},a,b,c\right) = 0, \quad (26)$$
$$\frac{\partial}{\partial a_k}\Phi(\mu,\lambda) = 0, \quad \frac{\partial}{\partial b_k}\Phi(\mu,\lambda) = 0$$

and

$$\Phi\left(\mu,\lambda,\widetilde{I},a,b,c\right)=0, \qquad (27)$$
$$\frac{\partial}{\partial a_k}\Phi(\mu,\lambda)=0, \quad \frac{\partial}{\partial c_k}\Phi(\mu,\lambda)=0$$

are separated variables depending on parameters $c_{1,2}$ and $b_{1,2}$, respectively.

These two families of variables are related by the canonical transformation of the initial phase space $\mathcal{M}$ associated with the left and right translations on the group $SL(3)$.

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Quantum Many-Body Problems and Perturbation Theory*

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Abstract—We show that the existence of algebraic forms of exactly solvable $A-B-C-D$, G_2, and F_4 Olshanetsky–Perelomov Hamiltonians allows one to develop algebraic perturbation theory, where corrections are computed by purely algebraic means. A classification of perturbations leading to such a perturbation theory based on the theory of representations of Lie algebras is given. In particular, this scheme admits an explicit study of anharmonic many-body problems. Some examples are presented.

1. INTRODUCTION

Quantum integrable and exactly solvable many-body problems originating from the projection method [1] (see also [2]) and/or the Hamiltonian reduction method [3] have served as a source of inspiration for many years. The goal of this paper is to explore yet another feature of these problems—they can be used as a zero-approximation or unperturbed problem in order to develop a constructive perturbation theory.

We begin from some preliminary knowledge that is necessary to enter the subject. Take an infinite set of linear functional spaces $\mathcal{V}_n$, $n = 0, 1, \ldots$. If they can be ordered

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots \subset \mathcal{V}_n \subset \ldots \subset \mathcal{V},$$

then such a construction is called an infinite flag (filtration) $\mathcal{V}$. A flag is classical if $\dim V_{n+1} = \dim V_n + 1$; otherwise, it is nonclassical. If there is an operator T such that

$$T : \mathcal{V}_n \mapsto \mathcal{V}_n, \quad n = 0, 1, 2, \ldots,$$

then it is implied that T preserves the flag $\mathcal{V}$.

General Definition [4]. An operator T that preserves an infinite flag of finite-dimensional spaces $\{\mathcal{V}_k\}_{k\in\mathbb{N}}$ (namely, each space $\mathcal{V}_k$ is invariant with respect to the action of T) is called an exactly solvable operator with the flag $\{\mathcal{V}_k\}_{k\in\mathbb{N}}$.

Equivalence. Any two functional spaces $\mathcal{V}_n$ are equivalent if they can be transformed one into another by mean of multiplication by a function and/or by a means of change of variables.

Restriction. We study linear spaces (and flags) of polynomials only (and of objects equivalent to polynomials).

Let us consider a linear space of polynomials in $\mathbf{C}^d(\mathbf{R}^d)$,

$$\mathcal{P}_n^{(f)} = \langle\, x_1^{p_1} x_2^{p_2} \ldots x_d^{p_d} \,|\, 0 \le \sum \alpha_i p_i \le n \,\rangle, \quad (1)$$
$$n = 0, 1, 2, \ldots,$$

where α_i are positive integers. We define the vector

$$\mathbf{f} = (\alpha_1, \ldots, \alpha_d), \quad (2)$$

which is called characteristic vector. Now, one can build a flag

$$\mathcal{P}_0^{(f)} \subset \mathcal{P}_1^{(f)} \ldots \subset \mathcal{P}_n^{(f)} \subset \ldots, \quad (3)$$

which is called $\mathcal{P}^{(f)}$. The vector

$$\mathbf{f}_0 = (\underbrace{1, 1, \ldots, 1}_{d}) \quad (4)$$

defines the so-called basic flag $\mathcal{P}^{(f_0)}$ in $\mathbf{C}^d(\mathbf{R}^d)$.

Let us consider the gl_{d+1} algebra realized by

$$\mathcal{J}_i^- = \frac{\partial}{\partial x_i}, \quad i = 1, 2, \ldots, d; \quad (5)$$

$$\mathcal{J}_{ij}^0 = x_i \frac{\partial}{\partial x_j}, \quad i, j = 1, 2, \ldots, d;$$

$$\mathcal{J}^0 = \sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i} - n;$$

$$\mathcal{J}_i^+ = x_i \mathcal{J}^0 = x_i \left(\sum_{j=1}^{d} x_j \frac{\partial}{\partial x_j} - n \right),$$

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$$i = 1, 2, \ldots, d,$$

where $n \in \mathbf{C}$. If n is a nonnegative integer, this algebra has a finite-dimensional representation and its linear space (finite-dimensional representation space) coincides with $\mathcal{P}_n^{(f_0)}$. Therefore, these finite-dimensional representation spaces as functions of n being properly ordered form the flag $\mathcal{P}^{(f_0)}$. It is obvious that the generators $\mathcal{J}_i^{-,0}$ and $\mathcal{J}_{ij}^0$, which span the maximal affine subalgebra $b \subset gl_{d+1}$, and their nonlinear combinations preserve the flag $\mathcal{P}^{(f_0)}$.

Definition. The operator h is called an algebraic one if it preserves a flag of polynomials.

It is rather obvious that an algebraic operator is characterized by polynomial coefficients, $\sum \mathrm{Pol}_n \cdot \partial^n$. The following theorem can be proven:

Theorem. A linear differential operator h preserves the flag $\mathcal{P}^{(f_0)}$ if and only if $h = P(\mathcal{J}(b \subset gl_{d+1}^{(*)}))$, where P is a polynomial in the generators of the maximal affine subalgebra b of the algebra gl_{d+1} taken in realization (5).

In particular, if the second-order differential operator h preserves the flag $\mathcal{P}^{(f_0)}$, it should have the form

$$h = P_2^{(ij)}(x)\partial_i\partial_j + P_1^{(i)}(x)\partial_i,$$

where $P_2^{(ij)}(x)$ and $P_1^{(i)}(x)$ are second- and first-degree polynomials in the coordinates x. This is the well-known hypergeometric operator.

2. ALGEBRAIC FORMS OF OLSHANETSKY–PERELOMOV HAMILTONIANS

In this section, we present the algebraic form of the A_N, BC_N, G_2, and F_4 Olshanetsky–Perelomov Hamiltonians [1, 5]. All of them will be obtained by the same procedure: (i) a gauge rotation of the Hamiltonian with the ground-state eigenfunction and (ii) a change of variables to new variables that code symmetries of the problem. We denote by E_0 the ground-state energy.

In the **Calogero model** (A_{N-1}-rational model) [6], the Hamiltonian has the form

$$\mathcal{H}_{\mathrm{Cal}} = \frac{1}{2}\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial {x_i}^2} + \omega^2 {x_i}^2\right) + g\sum_{i>j}^{N}\frac{1}{(x_i - x_j)^2},$$

and the ground state is

$$\Psi_0^{(\mathrm{Cal})}(x) = \prod_{i<j}|x_i - x_j|^{\nu} e^{-\frac{\omega}{2}\sum x_i^2}, \qquad (6)$$

$$g = \nu(\nu - 1),$$

where

$$h_{\mathrm{Cal}} = 2(\Psi_0^{(\mathrm{Cal})})^{-1}\,(\mathcal{H}_{\mathrm{Cal}} - E_0)\,\Psi_0^{(\mathrm{Cal})}.$$

The new variables are given by

$$Y = \sum x_i,\ y_i = x_i - \frac{1}{N}Y,\ i = 1, \ldots, N, \qquad (7)$$
$$(x_1, x_2, \ldots, x_N) \to \big(Y, \tau_n(x) = \sigma_n(y(x))|\ n = (2 \div N)\big),$$

where

$$\sigma_k(x) = \sum_{i_1<i_2<\ldots<i_k} x_{i_1}x_{i_2}\ldots x_{i_k}$$

are elementary symmetric polynomials.

Finally, the gauge-rotated Calogero Hamiltonian (after the separation of the center-of-mass motion) takes the form

$$h_{\mathrm{Cal}} = \mathcal{A}_{ij}(\tau)\frac{\partial^2}{\partial\tau_i\partial\tau_j} + \mathcal{B}_i(\tau)\frac{\partial}{\partial\tau_i}, \qquad (8)$$

where

$$\mathcal{A}_{ij} = \frac{(N - i + 1)(j - 1)}{N}\,\tau_{i-1}\,\tau_{j-1} + \sum_{l \geq \max(1, j-i)}(j - i - 2l)\,\tau_{i+l-1}\,\tau_{j-l-1},$$
$$\mathcal{B}_i = -\left(\frac{1}{N} + \nu\right)(N - i + 2) \times (N - i + 1)\tau_{i-2} + 2\omega i\tau_i.$$

In the **Sutherland model** (A_{N-1}-trigonometric model) [6], the Hamiltonian has the form

$$\mathcal{H}_{\mathrm{Suth}} = -\frac{1}{2}\sum_{k=1}^{N}\frac{\partial^2}{\partial x_k^2} + \frac{g}{4}\sum_{k<l}\frac{1}{\sin^2((x_k - x_l)/2)},$$

and the ground state is

$$\Psi_0^{(\mathrm{Suth})}(x) = \prod_{i<j}\sin^{\nu}\left(\frac{1}{2}(x_i - x_j)\right), \qquad (9)$$
$$g = \nu(\nu - 1).$$

$$h_{\mathrm{Suth}} = -2(\Psi_0^{(\mathrm{Suth})})^{-1}\,(\mathcal{H}_{\mathrm{Suth}} - E_0)\,\Psi_0^{(\mathrm{Suth})}.$$

The new variables are given by

$$(x_1, x_2, \ldots, x_N) \to \big(e^{iY}, \eta_n(x) = \sigma_n(e^{iy(x)})|\ n = [1, 2, \ldots, (N - 1)]\big), \qquad (10)$$

where the variables y are defined in (7).

Finally, the gauge-rotated Sutherland Hamiltonian (after the separation of the center-of-mass motion) becomes

$$h_{\rm Suth} = \mathcal{A}_{ij}(\eta)\frac{\partial^2}{\partial\eta_i\partial\eta_j} + \mathcal{B}_i(\eta)\frac{\partial}{\partial\eta_i}, \qquad (11)$$

where

$$\mathcal{A}_{ij} = \frac{(N-i)\,j}{N}\,\eta_i\,\eta_j + \sum_{l\geq\max(1,j-i)}(j-i-2l)\,\eta_{i+l}\,\eta_{j-l},$$
$$\mathcal{B}_i = \left(\frac{1}{N}+\nu\right)\,i\,(N-i)\,\eta_i.$$

In the **BC_N-rational model** [7], the Hamiltonian has the form

$$\mathcal{H}^{(r)}_{BC_N} = -\frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial^2}{\partial x_i^2} - \omega^2 x_i^2\right) + g\sum_{i<j}\left[\frac{1}{(x_i-x_j)^2}+\frac{1}{(x_i+x_j)^2}\right] + \frac{g_2}{2}\sum_{i=1}^{N}\frac{1}{x_i^2},$$

and the ground state is

$$\Psi_0 = \left[\prod_{i<j}|x_i-x_j|^{\nu}|x_i+x_j|^{\nu}\prod_{i=1}^{N}|x_i|^{\nu_2}\right] \times e^{-\frac{\omega}{2}\sum_{i=1}^{N}x_i^2}, \quad g=\nu(\nu-1), \quad g_2=\nu_2(\nu_2-1), \qquad (12)$$

where

$$h^{(r)}_{BC_N} = -2(\Psi_0)^{-1}\,(\mathcal{H}^{(r)}_{BC_N} - E_0)\,\Psi_0.$$

The new variables are given by

$$(x_1, x_2, \ldots, x_N) \to \big(\sigma_k(x^2)|\ k=(1,2,\ldots,N)\big). \qquad (13)$$

Finally, the gauge-rotated BC_N-rational Hamiltonian takes the form

$$h^{(r)}_{BC_N} = \mathcal{A}_{ij}(\sigma)\frac{\partial^2}{\partial\sigma_i\partial\sigma_j} + \mathcal{B}_i(\sigma)\frac{\partial}{\partial\sigma_i}, \qquad (14)$$

where

$$\mathcal{A}_{ij} = 4\sum_{l\geq 0}(2l+1+j-i)\,\sigma_{i-l-1}\,\sigma_{j+l},$$
$$\mathcal{B}_i = 2\,[1+\nu_2+2\nu(N-i)] \times [N-i+1]\,\sigma_{i-1} - 4\,\omega\,i\,\sigma_i.$$

In the **BC_N-trigonometric model** [7], the Hamiltonian has the form

$$\mathcal{H}^{(t)}_{BC_N} = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{g}{4}\sum_{i<j}\left[\frac{1}{\sin^2((x_i-x_j)/2)} + \frac{1}{\sin^2((x_i+x_j)/2)}\right] + \frac{g_2}{4}\sum_{i=1}^{N}\frac{1}{\sin^2 x_i} + \frac{g_3}{4}\sum_{i=1}^{N}\frac{1}{\sin^2(x_i/2)},$$

and the ground state is

$$\Psi_0 = \left[\prod_{i<j}\left|\sin\frac{x_i-x_j}{2}\right|^{\nu}\left|\sin\frac{x_i+x_j}{2}\right|^{\nu} \times \prod_{i=1}^{N}|\sin x_i|^{\nu_2}\left|\sin\frac{x_i}{2}\right|^{\nu_3}\right], \qquad (15)$$
$$g=\nu(\nu-1), \quad g_2=\nu_2(\nu_2-1), \quad g_3=\nu_3(\nu_3+2\nu_2-1),$$

where

$$h^{(t)}_{BC_N} = -2(\Psi_0)^{-1}\,(\mathcal{H}^{(t)}_{BC_N} - E_0)\,\Psi_0.$$

The new variables are given by

$$(x_1, x_2, \ldots, x_N) \to \big(\hat\sigma_k(x) = \sigma_k(\cos x)|\ k=(1,2,\ldots,N)\big). \qquad (16)$$

Finally, the gauge-rotated BC_N-trigonometric Hamiltonian becomes

$$h^{(t)}_{BC_N} = \mathcal{A}_{ij}(\hat\sigma)\frac{\partial^2}{\partial\hat\sigma_i\partial\hat\sigma_j} + \mathcal{B}_i(\hat\sigma)\frac{\partial}{\partial\hat\sigma_i}, \qquad (17)$$

where

$$\mathcal{A}_{ij} = N\,\hat\sigma_{i-1}\,\hat\sigma_{j-1} - \sum_{l\geq 0}\Big[(i-l)\,\hat\sigma_{i-l}\,\hat\sigma_{j+l} + (l+j-1)\,\hat\sigma_{i-l-1}\,\hat\sigma_{j+l-1} - (i-2-l)\,\hat\sigma_{i-2-l}\,\hat\sigma_{j+l} - (l+j+1)\,\hat\sigma_{i-l-1}\,\hat\sigma_{j+l+1}\Big],$$
$$\mathcal{B}_i = \frac{\nu_3}{2}(i-N-1)\,\hat\sigma_{j-1} - \Big[\nu_2+\frac{\nu_3}{2}+1+\nu(2N-i-1)\,i\,\hat\sigma_i - \nu(N-i+1)(N-i+2)\hat\sigma_{i-2}\Big].$$

In the **G_2-rational model** [8], the Hamiltonian has the form

$$\mathcal{H}^{(r)}_{G_2} = -\frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial^2}{\partial x_i^2} - \omega^2 x_i^2\right) + g\sum_{i<j}\frac{1}{(x_i-x_j)^2}$$

$$+ g_1 \sum_{i<j} \frac{1}{(x_k + x_l - 2x_m)^2},$$

and the ground state is

$$\Psi_0 = \prod_{i<j}^{3} |x_i - x_j|^{\nu} \prod_{\substack{i<j \\ i,j\neq k}} |x_i + x_j - 2x_k|^{\mu} \, e^{-\frac{1}{2}\omega \sum x_i^2},$$

$$g = \nu(\nu-1) > -\frac{1}{4}, \quad g_1 = 3\mu(\mu-1) > -\frac{3}{4}, \quad (18)$$

where

$$h^{(r)}_{G_2} = -2(\Psi_0)^{-1} \left(\mathcal{H}^{(r)}_{G_2} - E_0\right) \Psi_0.$$

The new variables are given by

$$Y = \sum x_i, \; y_i = x_i - \frac{1}{3} Y, \; i = 1, 2, 3,$$

where

$$(x_1, x_2, x_3) \to \big(Y, \; \lambda_1(y), \; \lambda_2(y)\big), \qquad (19)$$

$$\lambda_1 = -y_1^2 - y_2^2 - y_1 y_2, \; \lambda_2 = [y_1 y_2 (y_1 + y_2)]^2.$$

Finally, the gauge-rotated G_2-rational Hamiltonian (after the separation of the center-of-mass motion) takes the form

$$h^{(r)}_{G_2} = -2\lambda_1 \partial^2_{\lambda_1\lambda_1} - 12\lambda_2 \partial^2_{\lambda_1\lambda_2} \qquad (20)$$
$$+ \frac{8}{3}\lambda_1^2 \lambda_2 \partial^2_{\lambda_2\lambda_2} - \big\{4\omega\lambda_1 + 2[1 + 3(\mu+\nu)]\big\}\partial_{\lambda_1}$$
$$- \big(12\omega\lambda_2 - \frac{4}{3}\lambda_1^2\big)\partial_{\lambda_2}.$$

In the ***G_2*-trigonometric model** [8], the Hamiltonian has the form

$$\mathcal{H}^{(t)}_{G_2} = -\frac{1}{2}\sum_{k=1}^{3} \frac{\partial^2}{\partial x_k^2}$$
$$+ \frac{g\alpha^2}{4} \sum_{k<l}^{3} \frac{1}{\sin^2(\alpha(x_k - x_l)/2)}$$
$$+ \frac{g_1\alpha^2}{4} \sum_{\substack{k<l \\ k,l\neq m}}^{3} \frac{1}{\sin^2(\alpha(x_k + x_l - 2x_m)/2)},$$

and the ground state is

$$\Psi_0 = \prod_{i<j}^{3} \left|\sin\frac{\alpha(x_i - x_j)}{2}\right|^{\nu} \qquad (21)$$
$$\times \prod_{\substack{k<l \\ k,l\neq m}}^{3} \left|\sin\frac{\alpha(x_i + x_j - 2x_k)}{2}\right|^{\mu},$$

$$g = \nu(\nu-1) > -\frac{1}{4}, \quad g_1 = 3\mu(\mu-1) > -\frac{3}{4},$$

where

$$h^{(t)}_{G_2} = -2(\Psi_0)^{-1} \left(\mathcal{H}^{(t)}_{G_2} - E_0\right)\Psi_0.$$

The new variables are

$$Y = \sum x_i, \quad y_1 = x_1 - x_2, \quad y_2 = x_2 - x_3, \quad (22)$$
$$y_3 = x_3 - x_1, \quad (x_1, x_2, x_3) \to \big(Y, \; \tilde\sigma_1, \; \tilde\sigma_2\big),$$

$$\tilde\sigma_1 = \frac{1}{\alpha^2}\Big[\cos(\alpha(y_1 - y_2)) + \cos(\alpha(y_2 - y_3))$$
$$+ \cos(\alpha(y_3 - y_1)) - 3\Big],$$

$$\tilde\sigma_2 = \frac{4}{\alpha^6}\Big[\sin(\alpha(y_1 - y_2)) + \sin(\alpha(y_2 - y_3))$$
$$+ \sin(\alpha(y_3 - y_1))\Big]^2.$$

Finally, the gauge-rotated G_2-trigonometric Hamiltonian (after the separation of the center-of-mass motion) becomes

$$h^{(t)}_{G_2} = -\left(2\tilde\sigma_1 + \frac{\alpha^2}{2}\tilde\sigma_1^2 - \frac{\alpha^4}{24}\tilde\sigma_2\right)\partial^2_{\tilde\sigma_1\tilde\sigma_1} \qquad (23)$$
$$- \left(12 + \frac{8\alpha^2}{3}\tilde\sigma_1\right)\tilde\sigma_2 \partial^2_{\tilde\sigma_1\tilde\sigma_2}$$
$$+ \left(\frac{8}{3}\tilde\sigma_1^2\tilde\sigma_2 - 2\alpha^2\tilde\sigma_2^2\right)\partial^2_{\tilde\sigma_2\tilde\sigma_2}$$
$$- \left\{2[1 + 3(\mu + 2\nu)] + \frac{2}{3}(1 + 3\mu + 4\nu)\alpha^2\tilde\sigma_1\right\}\partial_{\tilde\sigma_1}$$
$$+ \left\{\frac{4}{3}(1 + 4\nu)\tilde\sigma_1^2 - \Big[\frac{7}{3} + 4(\mu+\nu)\Big]\alpha^2\tilde\sigma_2\right\}\partial_{\tilde\sigma_2}.$$

In the ***F_4*-rational model** [9], the Hamiltonian has the form

$$\mathcal{H}^{(r)}_{F_4} = \frac{1}{2}\sum_{i=1}^{4}\left(-\partial^2_{x_i} + 4\omega^2 x_i^2\right)$$
$$+ 2g\sum_{j>i}\left(\frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2}\right) + 2g_1\sum_{i=1}^{4}\frac{1}{x_i^2}$$
$$+ 8g_1 \sum_{\nu,s=0,1} \frac{1}{\left[x_1 + (-1)^{\nu_2}x_2 + (-1)^{\nu_3}x_3 + (-1)^{\nu_4}x_4\right]^2},$$

and the ground state is

$$\Psi^{(r)}_0(x) = (\Delta_-\Delta_+)^{\nu}\,(\Delta_0\Delta)^{\mu} \exp\left(-\omega\sum_{i=1}^{4} {x_i}^2\right),$$

$$g = \nu(\nu-1)/2, \; g_1 = \mu(\mu-1), \qquad (24)$$

where

$$\Delta_\pm = \prod_{j<i}^{4}(x_i \pm x_j),$$

$$\Delta_0 = \prod_{i=1}^{4} x_i,$$

$$\Delta = \prod_{\nu's=0,1} [x_1 + (-1)^{\nu_2}x_2 + (-1)^{\nu_3}x_3 + (-1)^{\nu_4}x_4].$$

The new variables are given by

$$(x_1, x_2, x_3, x_4) \to \big(t_1, t_3, t_4, t_6\big), \qquad (25)$$

where

$$t_1 = \sigma_1, \quad t_3 = \sigma_3 - \frac{1}{6}\sigma_1\sigma_2,$$

$$t_4 = \sigma_4 - \frac{1}{4}\sigma_1\sigma_3 + \frac{1}{12}\sigma_2^2,$$

$$t_6 = \sigma_4\sigma_2 - \frac{1}{36}\sigma_2^3 - \frac{3}{8}\sigma_3^2 + \frac{1}{8}\sigma_1\sigma_2\sigma_3 - \frac{3}{8}\sigma_1^2\sigma_4,$$

and $\sigma_a = \sigma_a(x^2)$, $a = 1, 2, 3, 4$.

Finally, the gauge-rotated F_4-rational Hamiltonian takes the form

$$h_{F_4}^{(r)} = \mathcal{A}_{ab}\frac{\partial^2}{\partial t_a \partial t_b} + (\mathcal{B}_a + \mathcal{C}_a)\frac{\partial}{\partial t_a}, \qquad (26)$$

where

$$\mathcal{A}_{11} = 4t_1, \quad \mathcal{A}_{13} = 12t_3,$$

$$\mathcal{A}_{14} = 16t_4, \quad \mathcal{A}_{16} = 24t_6,$$

$$\mathcal{A}_{33} = -\frac{2}{3}t_1^2 t_3 + \frac{20}{3}t_1 t_4, \quad \mathcal{A}_{34} = -\frac{4}{3}t_1^2 t_4 + 8t_6,$$

$$\mathcal{A}_{36} = 16t_4^2 - 2t_1^2 t_6, \quad \mathcal{A}_{44} = -4t_3 t_4 - 2t_1 t_6,$$

$$\mathcal{A}_{46} = -4t_1 t_4^2 - 6t_3 t_6,$$

$$\mathcal{A}_{66} = -12t_3 t_4^2 - 6t_1 t_4 t_6,$$

$$\mathcal{A}_{ba} = \mathcal{A}_{ab}, \quad \mathcal{B}_1 = 8, \quad \mathcal{B}_3 = -t_1^2,$$

$$\mathcal{B}_4 = -4t_3, \quad \mathcal{B}_6 = -8t_1 t_4,$$

$$\mathcal{C}_1 = 48(\nu+\mu) - 4\omega t_1,$$

$$\mathcal{C}_3 = -2(2\nu+\mu)t_1^2 - 12\omega t_3,$$

$$\mathcal{C}_4 = -12\nu t_3 - 16\omega t_4,$$

$$\mathcal{C}_6 = -12\nu t_1 t_4 - 24\omega t_6.$$

In the ***F_4*-trigonometric model** [9], the Hamiltonian has the form

$$\mathcal{H}_{F_4}^{(t)}(x) = -\frac{1}{2}\sum_{i=1}^{4}\partial_{x_i}^2 + 2gV_1(x,\beta) + \frac{g_1}{2}V_2(x,2\beta), \qquad (27)$$

where $g = \nu(\nu-1)/2$, $g_1 = \mu(\mu-1)$, and

$$V_1(x,\beta) = \beta^2\sum_{j>i}\left(\frac{1}{\sin^2[\beta(x_i - x_j)]} + \frac{1}{\sin^2[\beta(x_i + x_j)]}\right),$$

$$V_2(x,2\beta) = 4\beta^2\sum_{i=1}^{4}\frac{1}{\sin^2(2\beta x_i)} + 4\beta^2\sum_{\nu's=0,1}^{4} 1/[\sin^2\{\beta\,[x_1+(-1)^{\nu_2}x_2+(-1)^{\nu_3}x_3+(-1)^{\nu_4}x_4]\}].$$

The ground state is

$$\Psi_0^{(t)}(x,\beta) = (\Delta_+(x,\beta)\Delta_-(x,\beta))^\nu \times (\Delta_0(x,2\beta)\Delta(x,2\beta))^\mu, \qquad (28)$$

where

$$\Delta_\pm(x,\beta) = \beta^{-6}\prod_{j<i}\sin[\beta(x_i \pm x_j)],$$

$$\Delta_0(x,2\beta) = \beta^{-4}\prod_i \sin(2\beta x_i),$$

$$\Delta(x,2\beta) = \beta^{-8}\prod_{\nu's=0,1}\sin\big\{\beta\,[x_1 + (-1)^{\nu_2}x_2 + (-1)^{\nu_3}x_3 + (-1)^{\nu_4}x_4]\big\}.$$

Here,

$$h_{F_4}^{(t)} = -2\big(\Psi_0^{(t)}(x)\big)^{-1}(\mathcal{H}_{F_4}^{(t)} - E_0)\big(\Psi_0^{(t)}(x)\big).$$

The new variables are given by

$$(x_1, x_2, x_3, x_4) \to \big(\tau_1, \tau_3, \tau_4, \tau_6\big), \qquad (29)$$

where

$$\tau_1 = \sigma_1 - \frac{2\beta^2}{3}\sigma_2, \qquad (30)$$

$$\tau_3 = \sigma_3 - \frac{1}{6}\sigma_1\sigma_2 - 2\beta^2\left(\sigma_4 - \frac{1}{36}\sigma_2^2\right),$$

$$\tau_4 = \sigma_4 - \frac{1}{4}\sigma_1\sigma_3 + \frac{1}{12}\sigma_2^2,$$

$$\tau_6 = \sigma_4\sigma_2 - \frac{1}{36}\sigma_2^3 - \frac{3}{8}\sigma_3^2 + \frac{1}{8}\sigma_1\sigma_2\sigma_3 - \frac{3}{8}\sigma_1^2\sigma_4,$$

$\sigma_a = \sigma_a(y^2)$, and $y_i = \sin(\beta x_i)/\beta$.

Finally, the gauge-rotated F_4-trigonometric Ha-

miltonian takes the form

$$h^{(t)}_{F_4} = A_{ab}\frac{\partial^2}{\partial\tau_a\partial\tau_b} + (B_a + C_a)\frac{\partial}{\partial\tau_a}, \qquad (31)$$
$$a, b = 1, 3, 4, 6,$$

where the coefficient functions are

$$A_{11} = 4\,\tau_1 - 4\beta^2\tau_1^2 - \frac{32}{3}\beta^4\tau_3 - \frac{128}{9}\beta^6\tau_4,$$
$$A_{13} = 12\,\tau_3 - \frac{8}{3}\beta^2(4\tau_1\tau_3 + \tau_4) - \frac{32}{9}\beta^4\tau_1\tau_4,$$
$$A_{14} = 16\,\tau_4 - \frac{40}{3}\beta^2\tau_1\tau_4 - \frac{16}{3}\beta^4\tau_6,$$
$$A_{16} = 24\,\tau_6 - 20\beta^2\tau_1\tau_6 - \frac{32}{3}\beta^4\tau_4^2,$$
$$A_{33} = -\frac{2}{3}\,\tau_1^2\,\tau_3 + \frac{20}{3}\,\tau_1\,\tau_4 - \frac{8}{9}\beta^2\,(18\tau_3^2 + \tau_1^2\,\tau_4 + 12\tau_6),$$
$$A_{34} = -\frac{4}{3}\,\tau_1^2\,\tau_4 + 8\,\tau_6 - \frac{4}{3}\beta^2\,(\tau_1\,\tau_6 + 12\tau_3\,\tau_4),$$
$$A_{36} = 16\,\tau_4^2 - 2\,\tau_1^2\,\tau_6 - \frac{8}{3}\beta^2(9\tau_3\,\tau_6 + \tau_1\,\tau_4^2),$$
$$A_{44} = -4\,\tau_3\,\tau_4 - 2\,\tau_1\,\tau_6 - 24\beta^2\tau_4^2,$$
$$A_{46} = -4\,\tau_1\,\tau_4^2 - 6\,\tau_3\,\tau_6 - 36\beta^2\tau_4\tau_6,$$
$$A_{66} = -12\tau_3\tau_4^2 - 6\tau_1\tau_4\tau_6 - 8\beta^2(6\tau_6^2 + \tau_4^3),$$
$$A_{b\,a} = A_{a\,b},$$
$$B_1 = 8 - 8\beta^2\tau_1, \quad B_3 = -\tau_1^2 - \frac{56}{3}\beta^2\tau_3 - \frac{32}{9}\beta^4\tau_4,$$
$$B_4 = -4\,\tau_3 - \frac{88}{3}\beta^2\tau_4,$$
$$B_6 = -8\tau_1\tau_4 - 56\beta^2\tau_6,$$
$$C_1 = 48(\nu + \mu) - 8\beta^2(5\nu + 6\mu)\tau_1,$$
$$C_3 = -2(2\nu + \mu)\tau_1^2 - 16\beta^2(3\nu + 5\mu)\tau_3,$$
$$C_4 = -12\nu\tau_3 - 24\beta^2(3\nu + 4\mu)\tau_4,$$
$$C_6 = -12\nu\tau_1\tau_4 - 48\beta^2(2\nu + 3\mu)\tau_6.$$

Remarks and Comments

The A_N- and BC_N-rational and trigonometric models possess algebraic forms; their Hamiltonians (8), (11), (14), and (17) preserve the same basic flag of polynomials $\mathcal{P}^{(f_0)}$.

All A_N- and BC_N-rational and trigonometric Hamiltonians taken in the algebraic form can be written as

$$h = P_2(\mathcal{J}(b \subset gl_{N+1})),$$

where P_2 is a polynomial of second degree in the generators $\mathcal{J}$ of the maximal affine subalgebra of the algebra gl_{N+1} in realization (5). One can state that gl_{N+1} is their hidden algebra.

Both the rational and the trigonometric G_2 models possess algebraic forms; their Hamiltonians preserve the same flag of polynomials $\mathcal{P}^{(f_{G_2})}$ with $\mathbf{f}_{G_2} = (1, 2)$; their hidden algebras coincide, forming some infinite-dimensional, finitely generated algebra $g^{(2)} \subset \mathrm{diff}(\mathbb{C}^2)$ (see [8]). This algebra is generated by eight operators: seven operators are of the form of first-order differential operators, while the eighth one is a second-order differential operator.

Both the rational and the trigonometric F_4 models possess algebraic forms; their Hamiltonians preserve the same flag of polynomials $\mathcal{P}^{(f_{F_4})}$ with $\mathbf{f}_{F_4} = (1, 2, 2, 3)$; their hidden algebras coincide, forming some infinite-dimensional, finitely generated algebra $f^{(4)} \subset \mathrm{diff}(\mathbb{C}^4)$ (see [9]). This algebra is generated by 49 operators that are of the form of first-, second-, and third-order differential operators.

The new variables (7), (10), (13), (16), (19), (22), (25), and (29), in which the algebraic forms occur, usually absorb all external symmetries of the model under investigation; they have the meaning of rational and trigonometric invariants in the corresponding root space; to the best of our knowledge, they were used for the first time to find flat space metrics (denoted by $\mathcal{A}$ in A–B–C–D and F_4 examples) in the rational case by Arnold [10]; we will call these metrics $\mathcal{A}$ the Arnold metrics.

Although the question of the existence of algebraic forms for rational and trigonometric $E_{6,7,8}$ models has not yet been constructively studied, there is almost no doubt that they should exist.

3. PERTURBATION THEORY

The existence of algebraic forms leads to the possibility of constructing a special, algebraic perturbation theory—a type of perturbation theory where *finding corrections is an algebraic procedure* and furthermore any correction has a form of a finite-order polynomial in coordinates.

Let us consider the spectral problem

$$(T_0 + \lambda T_1)\phi = E\phi, \qquad (32)$$

where λ is a formal parameter, and let us develop perturbation theory,

$$\phi = \sum \lambda^k \phi_k,\ E = \sum \lambda^k E_k. \qquad (33)$$

Then, the following theorem holds:

Theorem. Let T_0 be an exactly solvable operator with flag $\{\mathcal{V}_k\}_{k\in\mathbb{N}}$. Let the perturbation T_1 be such that T_1 is an element of space $\mathcal{V}_n$ from the flag, and

we seek $\phi \in \mathcal{V}$. Then, the perturbation theory is algebraic: There exists $p(k)$ such that the kth correction ϕ_k belongs to $\mathcal{V}_{p(k)}$; hence, it can be found by algebraic means.

The proof is quite straightforward and is based on the analysis of the equation for the kth correction:

$$(T_0 - E_0)\phi_k = \sum_{i=1}^{k} E_i \phi_{k-i} - T_1 \phi_{k-1}.$$

We can proceed to examples.

Example 1. One-dimensional anharmonic oscillator.

It is characterized by the Hamiltonian

$$\mathcal{H} = \underbrace{-\frac{1}{2}\frac{\partial^2}{\partial y^2} + \omega^2 y^2 + \frac{g}{y^2}}_{A_1\text{-Calogero model}} + \lambda y^4. \qquad (34)$$

The ground state is

$$\psi_0 = y^\nu e^{-\omega y^2/2}, \quad g = \nu(\nu-1), \qquad (35)$$
$$E_0 = \omega(1+2\nu).$$

In terms of the new variable

$$\tau = y^2,$$

the gauge-rotated Hamiltonian has the form

$$h = \frac{1}{\omega}\psi_0^{-1}(\mathcal{H} - E_0)\psi_0$$
$$= -2\tau\partial_\tau^2 + 2(\tau-\mu)\partial_\tau + \lambda\tau^2 \equiv T_0 + \lambda T_1,$$

where $\mu \equiv \nu + 1/2$. It is easy to check that

$$T_0 : \mathcal{P}_n \mapsto \mathcal{P}_n, \quad E_0^{(n)} = 2n, \quad n = 0,1,2,\ldots,$$

$$T_1 = \tau^2 \in \mathcal{P}_{2,3,\ldots},$$

where $\mathcal{P}$ is the basic flag of polynomials in $\mathbf{C}$ [see (1)].

(A) Ground state.

Now the ground state of T_0 is given by $\phi_0^{(0)} = 1$ with $E_0^{(0)} = 0$.

(i) First correction.

The defining equation is

$$-2\tau\partial_\tau^2\phi_1^{(0)} + 2(\tau-\mu)\partial_\tau\phi_1^{(0)} = E_1^{(0)} - \tau^2.$$

Its solution is given by

$$-\phi_1^{(0)} = \frac{1}{4}\tau^2 + \frac{\mu+1}{2}\tau, \qquad (36)$$
$$E_1^{(0)} = \mu(\mu+1). \qquad (37)$$

(ii) Second correction.

$$-2\tau\partial_\tau^2\phi_2^{(0)} + 2(\tau-\mu)\partial_\tau\phi_2^{(0)}$$
$$= E_2^{(0)} + E_1^{(0)}\phi_1^{(0)} - \tau^2\phi_1^{(0)},$$

$$\phi_2^{(0)} = \frac{\tau^4}{32} + \frac{3\mu+4}{24}\tau^3 + \frac{2\mu^2+10\mu+9}{16}\tau^2 + \frac{(\mu+1)(4\mu+5)}{4}\tau,$$

$$E_2^{(0)} = -\frac{\mu(\mu+1)(4\mu+5)}{2}.$$

In general, an arbitrary correction to the ground state has a form

$$\phi_k^{(0)} = a_{2k}\tau^{2k} + a_{2k-1}\tau^{2k-1} + \ldots + a_{2k-m}\tau^{2k-m} + \ldots.$$

The coefficients of leading terms can be found explicitly for any excited state (!)—they are generalized Catalan numbers of the form

$$a_{2k-m} \sim \frac{(2k)!}{k!(k-m/2)!}.$$

In standard Rayleigh–Schrödinger perturbation theory (RSPT), the first correction to energy is $E_1^{(0)} = \langle 0|T_1|0\rangle/\langle 0|0\rangle$; hence,

$$E_1^{(0)} = \frac{\langle 0|y^4|0\rangle}{\langle 0|0\rangle} = \mu(\mu+1).$$

Therefore, we can find the expectation value $\langle 0|y^4|0\rangle$ algebraically {apart from the known normalization factor (see, e.g., [5])}. A comparison of other corrections in the present perturbation theory and in RSPT allows us to find algebraically transition amplitudes between different states (correlation functions).

(B) First excited state.

$$\phi_0^{(1)} = \tau - \mu, \quad E_0^{(1)} = 2.$$

Let us consider the first correction to it.

The defining equation has the form

$$-2\tau\partial_\tau^2\phi_1^{(1)} + 2(\tau-\mu)\partial_\tau\phi_1^{(1)} - 2\phi_1^{(1)}$$
$$= (E_1^{(1)} - \tau^2)(\tau-\mu).$$

The correction itself is given by

$$-\phi_1^{(1)} = \frac{1}{4}\Big[\tau^3 - (\mu-3)\tau^2 + 2(\mu+1)(\mu-3)\tau\Big], \qquad (38)$$

$$E_1^{(1)} = -(\mu+1)(\mu-3). \qquad (39)$$

It is worth noting that, in the present example, the perturbation theory developed here coincides with the so-called Dalgarno–Lewis form of perturbation

theory [11]. In fact, it was precisely this form of perturbation theory that was successfully used by Bender and Wu [12] in their profound study of the problem specified by (34) at $g = 0$.

Example 2. $(N-1)$-dimensional anharmonic oscillator.

Let us consider the following perturbed N-body Calogero model:

$$\mathcal{H} = \mathcal{H}_{\rm Cal} + \lambda\,\tau_4(x), \quad N > 4,$$

$$\tau_4(x) = \sigma_4(y) = \sum_{i_1,i_2,i_3,i_4} y_{i_1} y_{i_2} y_{i_3} y_{i_4},$$

$$h = h_{\rm Cal} + \lambda\,\tau_4 \equiv T_0 + \lambda\, T_1,$$

$$T_0 : \mathcal{P}_n^{(N-1)}(\tau) \mapsto \mathcal{P}_n^{(N-1)}(\tau), \quad n \in \mathbb{N},$$

$$T_1 = \tau_4 \in \mathcal{P}^{(N-1)}_{1,2,3,\ldots}.$$

The ground state is given by

$$\phi_0^{(0)} = 1, \quad E_0^{(1)} = 0.$$

The first correction has the form

$$-\phi_1^{(0)} = \frac{1}{8\omega}\,\tau_4 + \frac{1}{32\omega^2}\left(\frac{1}{N} + \nu\right)(N-2)(N-3)\tau_2,$$

$$E_1^{(0)} = \frac{1}{32\omega^2}\left(\frac{1}{N} + \nu\right)^2 \frac{N!}{(N-4)!}.$$

Again, we can find the expectation value algebraically (apart from the known normalization factor)

$$E_1^{(0)} \;=\; \frac{\langle 0|\tau_4(y)|0\rangle}{\langle 0|0\rangle}.$$

The second correction is of the form

$$\phi_2 = \alpha_1\,\tau_2^2 + \alpha_2\,\tau_3^2 + \alpha_3\,\tau_4^2 + \alpha_4\,\tau_2\tau_4 + \beta_1\,\tau_2 + \beta_2\,\tau_4 + \beta_3\,\tau_6,$$

where the coefficients α and β can easily be computed.

Example 3. Perturbed three-body Sutherland model.

We set

$$\mathcal{H} = \mathcal{H}^{(3)}_{\rm Suth} + \lambda\,\eta_2,$$

where $\mathcal{H}^{(3)}_{\rm Suth}$ is the Hamiltonian of the three-body Sutherland model. Gauging away the ground state (9) and introducing the new variables

$$\eta_2 = \frac{1}{\alpha^2}[\cos(\alpha y_1) + \cos(\alpha y_2) + \cos(\alpha(y_1+y_2)) - 3],$$

$$\eta_3 = \frac{2}{\alpha^3}[\sin(\alpha y_1) + \sin(\alpha y_2) - \sin(\alpha(y_1+y_2))]$$

[compare with (10)], we get the algebraic form

$$h = h_{\rm Suth} + \lambda\,\eta_2 \equiv T_0 + \lambda\, T_1,$$

where

$$h_{\rm Suth} = -\left(2\eta_2 + \frac{\alpha^2}{2}\eta_2^2 - \frac{\alpha^4}{24}\eta_3^2\right)\partial^2_{\eta_2\eta_2} - \left(6 + \frac{4\alpha^2}{3}\eta_2\right)\eta_3\partial^2_{\eta_2\eta_3} + \left(\frac{2}{3}\eta_2^2 - \frac{\alpha^2}{2}\eta_3^2\right)\partial^2_{\eta_3\eta_3} + 2\left(\nu + \frac{1}{3}\right)(3 + \alpha^2\,\eta_2)\partial_{\eta_2} + 2\left(\nu + \frac{1}{3}\right)\alpha^2\eta_3\partial_{\eta_3},$$

$$T_0 : \mathcal{P}_n^{(2)}(\eta) \mapsto \mathcal{P}_n^{(2)}(\eta), \quad n \in \mathbb{N},$$

$$T_1 = \eta_2 \in \mathcal{P}^{(2)}_{1,2,3,\ldots}.$$

For the ground state, we have $\phi_0 = 1$ and $E_0 = 0$.

The first correction is given by

$$-\phi_1 = \frac{3}{2(1+3\nu)\alpha^2}\,\eta_2,$$

$$E_1 = -\frac{3}{\alpha^2}.$$

Since

$$E_1 \;=\; \frac{\langle 0|\eta_2(y)|0\rangle}{\langle 0|0\rangle},$$

we can find the expectation value $\langle 0|\eta_2(y)|0\rangle$ algebraically using the known normalization factor $\langle 0|0\rangle$ [5].

The second correction is given by

$$-\phi_2 = \frac{3}{8\alpha^4(1+3\nu)(1+6\nu)} \times \left[(1+12\nu)\,\eta_2^2 + \frac{1}{4}\,\eta_3^2 + \frac{9(2+13\nu+12\nu^2)}{(1+3\nu)}\,\eta_2\right],$$

$$E_2 = -\frac{27}{4\alpha^4}\,\frac{2+13\nu+12\nu^2}{(1+3\nu)(1+6\nu)}.$$

4. CONCLUSION

Algebraic forms of Calogero–Sutherland models give the opportunity of studying their perturbations by algebraic means through developing a perturbation theory for a single state.

Taking different perturbations and making a comparison of the present perturbation theory with the standard RSPT, we can calculate correlation functions for the Calogero–Sutherland models algebraically.

The algebraic forms of the Calogero–Sutherland models allow one to build their Fock space representation (see [13]) and then develop algebraic perturbation theory in Fock space. This makes it possible to study isospectral discretizations of the Calogero–Sutherland models (on various lattices) and their perturbations [14].

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Lie Symmetries and Superintegrability in Quantum Mechanics*

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Abstract—Starting from the structure of the higher order Lie symmetries of the Schrödinger equation in the Euclidean plane E_2, we establish, in the case of first- and second-order symmetries, the relations between separation of variables and superintegrable systems in quantum mechanics. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

A Hamiltonian system admitting n degrees of freedom in quantum mechanics is called integrable if it possesses n integrals of motion in involution, which are expressed by time-independent linear operators commuting with the Hamiltonian [1]. This is the quantum mechanical formulation of the Liouville–Arnold definition [2] of integrability usually assumed in classical mechanics. The search for such systems, motivated by their good physical properties (e.g., the regularity of orbits in the classical case or the simple time evolution of the representative wave packets in the quantum case), has been actively pursued for a long time in many different ways. A standard approach, pioneered by Bertrand [2], consists in looking for constants of motion that are polynomial in the momenta. This procedure, suitably generalized, led to the discovery [3, 4] of the so-called superintegrable systems in two and three dimensions. By definition, they are systems allowing more than n integrals of motion. More precisely, if the system possesses just $n+1$ integrals, it is called "minimally superintegrable"; when it admits $2n-1$ integrals, it is said to be "maximally superintegrable." In particular, the existence of quadratic integrals of motion turns out to be intimately related to the phenomenon of separation of variables for the Hamilton–Jacobi and the Schrödinger equation (SE). Well-known examples of such systems are those described by the Kepler and the harmonic potential, the nonisotropic oscillator with commensurable frequencies, and the Calogero–Moser system. They all exhibit the property that the classical admissible bounded trajectories are closed and the corresponding quantum eigenstates are multiply degenerate. Many other important features of these systems have been intensively investigated in several configuration spaces, that is, the Euclidean plane E_2, the sphere S_2, and the hyperbolic (Lobachevsky) plane H_2 (see, e.g., [5–9] and references therein). In this work the general problem of integrability of quantum mechanical systems in E_2 is considered in the framework of the Lie group theory. We will establish the correspondence between integrals of motion and generalized Lie symmetries of the SE (see also [10]). In particular, we will present some new results about the general structure of integrals of motion of arbitrary order n allowed by the SE. A complete study of the cases $n=1$ and $n=2$ is also performed.

2. SYMMETRY ANALYSIS

In the Euclidean space E_2, the SE reads

$$H\psi = E\psi, \quad H = -\frac{1}{2}\Delta + V(\mathbf{r}) \tag{1}$$

If we separate the real and the imaginary part of the wave function ψ, the two resulting equations coincide. Therefore, we shall restrict ourselves to the equation

$$-\frac{1}{2}(u_{xx}+u_{yy}) + (V-E)u = 0, \quad \psi = u_1 + iu_2, \tag{2}$$

where $u \in \mathbb{R}$ stands for u_1 or u_2. In complex independent variables, we can simply rewrite Eq. (2) as

$$S = u_{z\overline{z}} - Ru = 0, \quad R = \frac{1}{2}(V(z,\overline{z}) - E) \tag{3}$$

with $z = x+iy$, $\overline{z} = x-iy$. In the so-called evolutionary formalism, an nth order symmetry is expressed by [11] $\widehat{W} =$

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$Q\left(z,\overline{z},u,u_z,u_{\overline{z}},u_{zz},u_{z\overline{z}},u_{\overline{z}\overline{z}},...\right)\partial_u$, where the characteristic function Q depends on derivatives up to order n. According to Lie's theorem [11], the vector field $\widehat{W}$ generates a Lie group of transformations, mapping solutions into solutions, if and only if $pr^{(n)}\widehat{W}S\mid_{S=0}=0$. In the case of Eq. (1), this condition is equivalent to

$$D_{z\overline{z}}Q-RQ\mid_{u_{z\overline{z}}=Ru}=0. \tag{4}$$

In order to find nontrivial symmetries, let us consider Eq. (4). Explicitly, it reads

$$\begin{aligned}(D_{\overline{z}}Q_{u_{nz}})u_{(n+1)z}+Q_{u_{nz}}(R\,u)_{nz} \\ +(D_{\bar{z}}Q_{u_{n\bar{z}}})(R\,u)_{(n-1)\overline{z}}+Q_{u_{n\bar{z}}}(R\,u)_{n\overline{z}} \\ +(D_{\overline{z}}Q_{u_{(n-1)z}})u_{nz}+Q_{u_{(n-1)z}}(R\,u)_{(n-1)z} \\ +(D_{\overline{z}}Q_{u_{(n-1)\overline{z}}})(R\,u)_{(n-2)\overline{z}} \\ +Q_{u_{(n-1)\overline{z}}}(R\,u)_{(n-1)\overline{z}}+...=RQ,\end{aligned} \tag{5}$$

where $u_{nz}=\dfrac{\partial^n u(z,\overline{z})}{\partial z^n}$ and

$$(Ru)_{nz}=\sum_{k=0}^{n}\binom{n}{k}\frac{\partial^k R}{\partial z^k}=Ru_{nz} \tag{6}$$

$$+\binom{n}{1}R_z u_{(n-1)z}+\binom{n}{2}R_{zz}u_{(n-2)z}+\cdots+R_{nz}u.$$

Equating to zero the coefficient of $u_{(n+1)z}$, we find that $D_{\bar{z}}Q_{u_{nz}}=0$, which implies $Q_{u_{nz}}=\alpha_n\left(z\right)$. Iterating the procedure, namely, annulling the coefficients of all independent derivatives, taking into account Eq. (3) with its differential consequences, and the condition $Q=\overline{Q}$ as well, we finally obtain

$$\begin{aligned}Q=\alpha_n(z)u_{nz}+\overline{\alpha_n}(\overline{z})u_{n\overline{z}}+\alpha_{n-1}(z)u_{(n-1)z} \\ +\overline{\alpha_{n-1}}(\overline{z})u_{(n-1)\overline{z}}+\alpha_{n-2}(z,\overline{z})u_{(n-2)z} \\ +\overline{\alpha_{n-2}}(z,\overline{z})u_{(n-2)\overline{z}}+\cdots+\alpha_1(z,\overline{z})u_z \\ +\overline{\alpha_1}(z,\overline{z})u_{\overline{z}}+\eta(z,\overline{z})u+\nu(z,\overline{z}).\end{aligned} \tag{7}$$

Therefore, Q is linear in u and its derivatives. The coefficients of the first and second highest order derivatives depend either on z or on $\overline{z}$, whereas the other ones depend on both z and $\overline{z}$. Here ν is a real solution of the SE, appearing as a consequence of the linear superposition principle. In the following, we will consider in detail the cases $n=1$ and $n=2$. We will also require that symmetries *do not depend on the energy E*, but only on the dynamics, fixed by the Hamiltonian. For these choices of n, the general structure of symmetries can be summarized as follows:

$$\begin{pmatrix}Q_1\\Q_2\end{pmatrix}=M_1\widehat{X}\begin{pmatrix}u_1\\u_2\end{pmatrix} \tag{8}$$

$$+M_2\begin{pmatrix}u_1\\u_2\end{pmatrix}+\begin{pmatrix}h_1(x,y)\\h_2(x,y)\end{pmatrix},$$

where M_1 and M_2 are arbitrary constant matrices, $\widehat{X}$ is a linear time-independent differential operator, and h_1 and h_2 are real solutions of the SE. Moreover, we shall show that $\left[H,\widehat{X}\right]=0$.

3. FIRST-ORDER SYMMETRIES

From the general expression (7), we obtain

$$Q=\alpha(z)u_z+\bar{\alpha}(\bar{z})u_{\bar{z}}+\gamma u+h(z,\bar{z}). \tag{9}$$

Substituting (9) into Eq. (4) and returning to real variables, we get

$$\begin{aligned}(\alpha_{1x}+\alpha_{2y})(V-E)+\alpha_1V_x+\alpha_2V_y=0, \\ \alpha_{1x}-\alpha_{2y}=0,\quad \alpha_{2x}+\alpha_{1y}=0,\end{aligned} \tag{10}$$

where $\alpha_1\left(x,y\right)=\mathrm{Re}\alpha$, $\alpha_2\left(x,y\right)=\mathrm{Im}\alpha$. By imposing the energy independence of symmetries, from Eq. (10) we deduce

$$[\alpha L_3+\beta P_1+\gamma P_2]V(x,y)=0, \tag{11}$$

where $L_3=y\partial_x-x\partial_y$, and $P_1=\partial_x$, $P_2=\partial_y$. The general solution to Eq. (11) is

$$V=V(\xi),\quad \xi=\frac{1}{2}\alpha\left(x^2+y^2\right)-\gamma x+\beta y, \tag{12}$$

For $\alpha\neq 0$, we use a translation to annul β and γ. For $\alpha=0$, we rotate to put $\beta=0$. We have proved the following theorem.

Theorem 1. First-order energy-independent Lie symmetries of the SE exist if and only if the potential allows a geometric symmetry. The corresponding flow is given by formula (8) with

$$\widehat{X}=\alpha L_3+\beta P_1+\gamma P_2,\quad \psi=u_1+iu_2,$$

$\alpha,\beta,\gamma\in\mathbb{R}$, $M_1,M_2\in R^{(2\times 2)}$, and h_1, h_2 being real solutions of the SE. In particular, we reobtain the usual Lie point symmetries if M_1 and M_2 are diagonal.

4. CLASSIFICATION OF THE SECOND-ORDER SYMMETRIES

From formula (8) for $n=2$, we get

$$\begin{aligned}Q=A\left(z\right)u_{zz}+\bar{A}\left(\overline{z}\right)u_{\overline{z}\overline{z}}+B\left(z\right)u_z \\ +\bar{B}\left(\overline{z}\right)u_{\overline{z}}+C\left(z,\overline{z}\right)u+h\left(z,\overline{z}\right),\end{aligned} \tag{13}$$

$$C\left(z,\overline{z}\right)=\overline{C\left(z,\overline{z}\right)},\quad h\left(z,\overline{z}\right)=\overline{h\left(z,\overline{z}\right)}. \tag{14}$$

The corresponding determining equations are

$$RA_z+2AR_z+C_{\overline{z}}=0, \tag{15}$$

$$R\bar{A}_{\bar{z}} + 2\bar{A}R_{\bar{z}} + C_z = 0, \quad h_{z\bar{z}} - Rh = 0,$$

$$C_{z\bar{z}} + \left(B_z + \bar{B}_{\bar{z}}\right) R + BR_z + \bar{B}R_{\bar{z}} \qquad (16)$$
$$+ A_z R_z + \bar{A}_{\bar{z}} R_{\bar{z}} + AR_{zz} + \bar{A}R_{\bar{z}\bar{z}} = 0.$$

Now, we impose the energy independence of symmetries and return to real variables. After some manipulations, we can rewrite Eq. (13) as

$$Q = \{aL_3^2 + b(L_3P_1 + P_1L_3) + c(L_3P_2 + P_2L_3) \qquad (17)$$
$$+ d(P_1^2 - P_2^2) + 2eP_1P_2 + \phi(x,y)\}u + h(x,y)$$

with

$$\phi_x = -2(ay^2 + 2by + d)V_x \qquad (18)$$
$$+ 2(axy + bx - cy - e)V_y,$$
$$\phi_y = 2(axy + bx - cy - e)V_x$$
$$+ 2(-ax^2 + 2cx + d)V_y.$$

The compatibility condition for Eq. (18) is a necessary and sufficient condition for $V(x,y)$ to admit a second order integral of motion. It reads

$$(-axy - bx + cy + e)(V_{xx} - V_{yy}) \qquad (19)$$
$$+ [a(x^2 - y^2) - 2by - 2cx - 2d]V_{xy}$$
$$- 3(ay + b)V_x + 3(ax - c)V_y = 0.$$

The requirement that Eqs. (18) for ϕ and (19) for V be satisfied is equivalent to the condition $[H, \widehat{X}] = 0$. So we have a general theorem.

Theorem 2. A second order Lie symmetry (13) of the SE exists if and only if there exists a second order operator $\widehat{X}$ commuting with the Hamiltonian. This operator has the form

$$\widehat{X} = aL_3^2 + b(L_3P_1 + P_1L_3)$$
$$+ c(L_3P_2 + P_2L_3) + d(P_1^2 - P_2^2)$$
$$+ 2eP_1P_2 + \alpha L_3 + \beta P_1 + \gamma P_2 + \phi(x,y),$$

where Eqs. (18) hold and V satisfies Eqs. (11) and (19).

Let us now assume that $\alpha = \beta = \gamma = 0$; then Eq. (11) is satisfied trivially with no restrictions on the potential $V(x,y)$.

Now, it is useful to classify the operators $\widehat{X}$ under the action of the Euclidean group $E(2)$. We obtain four equivalence classes, each represented by precisely one of the following operators:

(1) $a = 0, \quad b = 0, \quad c = 0, \quad d^2 + e^2 \neq 0,$

$$\widehat{X}_C = -\frac{1}{2}\left(P_1^2 - P_2^2\right) + \phi_C(x,y); \qquad (20)$$

(2) $a \neq 0,$

$$l^2 = \frac{1}{a^3}\left[\left(2a^2d + b^2 - c^2\right)^2 + 4\left(a^2e + bc\right)^2\right]^{1/2} = 0,$$

$$\widehat{X}_R = L_3^2 + \phi_R(x,y); \qquad (21)$$

(3) $a = 0, b^2 + c^2 \neq 0,$,

$$\widehat{X}_P = L_3P_2 + P_2L_3 + \phi_P(x,y); \qquad (22)$$

(4) $a \neq 0, l^2 \neq 0,$

$$\widehat{X}_E = L_3^2 + \frac{l^2}{2}\left(P_1^2 - P_2^2\right) + \phi_E(x,y), \qquad (23)$$

where l^2 is given above. Using Eqs. (17)–(19) as well as Eqs. (20)–(22), we get the following result on the structure of the potentials for the SE compatible with a second-order symmetry.

Theorem 3. A second order operator $\widehat{X}$ exists if and only if the potential $V(x,y)$ admits separation of variables in Cartesian, polar, parabolic, or elliptic coordinates. More precisely, we have this correspondence:

$$\widehat{X}_C \leftrightarrow V(x,y) = f(x) + g(y),$$
$$\widehat{X}_R \leftrightarrow V(r,\vartheta) = f(r) + \frac{1}{r^2}g(\vartheta),$$

$$\widehat{X}_P \leftrightarrow V(\xi,\eta) = \frac{f(\xi) + g(\eta)}{\xi^2 + \eta^2},$$
$$\widehat{X}_E \leftrightarrow V(\sigma,\rho) = \frac{f(\sigma) + g(\rho)}{\cos^2\sigma - \cosh^2\rho},$$

where we introduced the parabolic coordinates

$$x = \frac{1}{2}(\xi^2 - \eta^2), \qquad y = \xi\eta$$

and the elliptic coordinates

$$x = l\cosh\rho\cos\sigma, \quad y = l\sinh\rho\sin\sigma.$$

5. SUPERINTEGRABLE SYSTEMS ALLOWING ONE FIRST-ORDER AND ONE SECOND-ORDER SYMMETRY

From Theorem 1, we know that first-order Lie symmetries exist only for potentials of the form $V = V(r)$ or $V = V(x)$. We will show that these central potentials can also admit a second order invariant.

Case I. $V = V(r)$. One obvious symmetry is given by the angular momentum operator $\widehat{X} = L_3$. An analysis of Eq. (19) for $V = V(r)$ furnishes exactly two different potentials, namely,

$$\text{(case Ia)} \quad V = \frac{\alpha}{r}, \quad \text{(case Ib)} \quad V = \omega^2 r^2.$$

According to Bertrand's theorem [2], they are the only rotationally invariant potentials in which all finite trajectories are closed. Let us see in detail the corresponding dynamical symmetries.

Case Ia. For the Coulomb potential, we get

$$\widehat{X}_1^C = P_1L_3 + L_3P_1 - \frac{2\alpha y}{r}, \qquad (24)$$
$$\widehat{X}_2^C = P_2L_3 + L_3P_2 + \frac{2\alpha x}{r}.$$

Case Ib. For the harmonic oscillator we obtain

$$\widehat{X}_1^h = -\frac{1}{2}P_1^2 + \omega^2x^2 + \frac{1}{2}P_2^2 - \omega^2y^2, \qquad (25)$$
$$\widehat{X}_2^h = -P_1P_2 + 2\omega^2xy.$$

Case II. $V = V(x)$. By virtue of Eq. (19), only two different potentials are allowed:

Case IIa. $V = \alpha/x^2$. We have the following second-order symmetries:

$$\widehat{X}_1 = L_3^2 - 2\alpha\frac{y^2}{x^2}, \quad \widehat{X}_2 = L_3P_1 + P_1L_3 - 4\alpha\frac{y}{x^2}, \qquad (26)$$

$$\widehat{X}_3 = -\frac{1}{2}P_1^2 + \frac{\alpha}{x^2} = H + \frac{1}{2}P_2^2. \qquad (27)$$

Case IIb. $V = \alpha x$. The linear potential admits second-order symmetries given by

$$\widehat{X}_1 = L_3P_2 + P_2L_3 + \alpha y^2, \quad \widehat{X}_2 = P_1P_2 + \alpha y, \qquad (28)$$

$$\widehat{X}_3 = -\frac{1}{2}P_1^2 + \alpha x = H + \frac{1}{2}P_2^2. \qquad (29)$$

The previous results and the structure of the symmetry algebras for the cases shown are summarized as follows:

Theorem 4. Precisely four $E(2)$ classes of potentials exist, allowing one first-order Lie symmetry and at least one second-order one. For $V = \omega^2r^2$, the symmetries $\left\{L_3, \widehat{X}_1, \widehat{X}_2, H\right\}$ form a $u(2)$ algebra. For $V(r) = \alpha/r$, L_3, $\widehat{X}_1$, and $\widehat{X}_2$ form an $o(3)$, $o(2,1)$, or $e(2)$ algebra for a fixed energy $E < 0$, $E > 0$, and $E = 0$, respectively. For $V = \alpha x^{-2}$ and $V = \alpha x$, the second-order Lie symmetries generate infinite-dimensional Lie algebras.

6. SUPERINTEGRABLE SYSTEMS ALLOWING TWO SECOND-ORDER SYMMETRIES

If we solve Eq. (19) to obtain another second order operator, the allowed potentials are separable in more than one coordinate system. The following results hold.

Theorem 5. Four superintegrable systems with two second-order Lie symmetries exist in the Euclidean space E_2. For each of them, the Schrödinger equation allows the separation of variables in at least two coordinate systems.

The four cases are listed below.

I. The SE separates in Cartesian, polar, and also elliptic coordinates.

$$V_{\rm I} = \omega^2(x^2 + y^2) + \frac{\alpha}{x^2} + \frac{\beta}{y^2}, \qquad (30)$$
$$\widehat{X}_1 = P_1^2 - P_2^2 - 2\left[\omega^2(x^2 - y^2) + \frac{\alpha}{x^2} - \frac{\beta}{y^2}\right],$$
$$\widehat{X}_2 = L_3^2 - 2\left(\frac{\alpha}{\cos^2\phi} + \frac{\beta}{\sin^2\phi}\right).$$

II. The SE separates in Cartesian and parabolic coordinates.

$$V_{\rm II} = \omega^2(4x^2 + y^2) + \frac{\alpha}{y^2} + \beta x, \qquad (31)$$
$$\widehat{X}_1 = P_1^2 - P_2^2 - 2\left[\omega^2(4x^2 - y^2) + \beta x - \frac{\alpha}{y^2}\right],$$
$$\widehat{X}_2 = L_3P_2 + P_2L_3 - 4\,\omega^2\,x\,y^2 + \frac{4\,\alpha x}{y^2} - \beta\,y^2.$$

III. The SE separates in polar and parabolic coordinates (and also in appropriately chosen elliptic ones).

$$V_{\rm III} = \frac{\alpha}{r} + \frac{1}{r^2}\left(\frac{\beta + \gamma\cos\vartheta}{\sin^2\vartheta}\right), \qquad (32)$$
$$\widehat{X}_1 = L_3^2 - 2\left(\frac{\beta + \gamma\cos\vartheta}{\sin^2\vartheta}\right),$$
$$\widehat{X}_2 = L_3P_2 + P_2L_3 + 2\alpha\cos\vartheta$$
$$+ 2\left(\frac{\gamma(\cos^2\vartheta + 1) + 2\beta\cos\vartheta}{r\sin^2\vartheta}\right).$$

IV. The SE separates in two different parabolic coordinate systems (and in any parabolic system of coordinates).

$$V_{\rm IV} = \frac{2\alpha + \beta\xi + \gamma\eta}{\xi^2 + \eta^2} \qquad (33)$$
$$= \frac{\alpha}{r} + \frac{1}{\sqrt{2r}}\left(\beta\cos\frac{\vartheta}{2} + \gamma\sin\frac{\vartheta}{2}\right)$$
$$\widehat{X}_1 = L_3P_1 + P_1L_3$$
$$+ \frac{\beta\eta\left(\eta^2 - \xi^2\right) + \gamma\xi\left(\xi^2 - \eta^2\right) - 4\alpha\eta\xi}{(\xi^2 + \eta^2)}$$
$$\widehat{X}_2 = L_3P_2 + P_2L_3$$
$$+ 2\frac{\alpha\left(\xi^2 - \eta^2\right) + \eta\xi\left(\gamma\xi - \beta\eta\right)}{(\xi^2 + \eta^2)}.$$

All details and proofs of theorems are given in [12].

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Solvable $\mathcal{PT}$-Symmetric Hamiltonians*

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Abstract—Within the so-called $\mathcal{PT}$-symmetric version of quantum mechanics, a brief review of exactly solvable models is given. The distinction is made between curved and straight coordinate lines, between their unbounded (aperiodic) and bounded (periodic) choices, and between completely and partially solvable cases. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

$\mathcal{PT}$-symmetric quantum mechanics [1] was independently proposed and used as a methodological laboratory in quantum physics by several groups of authors: by Caliceti *et al.* [2] in perturbation theory, by Bessis *et al.* [3] in field theory, and by Andrianov *et al.* [4] in the supersymmetric context.

This short review will pay attention to (partially or completely) exactly solvable models within this framework, with emphasis on the results obtained by the present author.

2. COMPLETE SOLVABILITY ON CURVED PATHS

One of the first exactly and completely solvable examples of a $\mathcal{PT}$-symmetric system was found by Cannata *et al.* [4] and rediscovered by Bender *et al.* [5] more than one year later. Its modified Schrödinger bound-state problem is defined on certain curved, left–right-symmetric "generalized coordinate" lines in a complex plane. Mathematically, it is defined via an exponential potential and proves to be exactly solvable in terms of Bessel functions. Owing to its relationship to a power-law force in the large-exponent limit, it can be most simply interpreted as a certain smooth and non-Hermitian $\mathcal{PT}$-symmetric analog of the current square well.

Recently, the double-well counterpart of the latter set of models was shown to be exactly solvable, in terms of Laguerre polynomials, in [6]. In a way similar to the above "single-well" example, its paths of integration are the same, curved complex lines again. Their spectra exhibit a puzzling and highly unexpected feature of certain coupling-dependent rearrangements mediated by "unavoided" crossings at critical points. This phenomenon reflects the non-Hermiticity of the Hamiltonian [7].

Via a suitable Liouvillean change of variables in the above double-well-like differential Schrödinger equation, one can immediately obtain another Laguerre-related solvable system with a potential of the Coulombic single-pole form. In this case, the deformation of the integration path plays the beneficial role of a natural regularization prescription. At the same time, it also leads to the need for working with complex charges in a way described in [8]. Also, the related energy spectrum exhibits certain unexpected features: positivity, the coexistence of the growth and decrease with increasing coupling strength, etc.

There is presently no clear mathematical explanation of this behavior; a consistent and/or possible physical interpretation of counterintuitive models of this type has not been developed either. A much better situation emerges in the case of integration curves defined as left–right-symmetric straight lines.

3. STRAIGHT PATHS

By using the language of the so-called Kustaanheimo–Stiefel transformation (see [9] for detailed references), the above two curvilinear examples can be shown to be equivalent to the $\mathcal{PT}$-symmetric harmonic oscillator of [10] with a centrifugal term regularized by a mere downward complex shift of the (full) real axis,

$$\left(-\frac{d^2}{dx^2}+x^2-2icx+\frac{\alpha^2-1/4}{(x-ic)^2}\right)\varphi(x) \quad (1)$$
$$=(E+c^2)\varphi(x), \quad \varphi(x)\in L_2(-\infty,\infty).$$

This oscillator with the Laguerre polynomial normalizable solutions

$$\varphi_{(\pm n)}(x)=\mathcal{N}(x-ic)^{\pm\alpha+1/2}e^{-(x-ic)^2/2}$$

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$$\times L_n^{(\pm\alpha)}\left[(x-ic)^2\right], \quad n=0,1,\ldots,$$

possesses the nonequidistant energy spectrum $E = E_{(\pm n)} = 4n + 2 \pm 2\alpha$ and represents a certain unperturbed limit of the quartic oscillator models formulated by Buslaev and Grecchi [11]. Unfortunately these authors did not notice the existence of the "quasieven," $_{(-n)}$-signed half of the spectrum. This omission can be easily corrected. One just introduces a "two-to-one" isospectrality correspondence between the respective Hermitian and non-Hermitian anharmonic oscillator models of [11].

Interpretation of models living on straight lines becomes significantly facilitated by the easier identification and interpretation of their complex components [12]. Immediate purely analytic constructions recover, e.g., the existence of models that are in a one-to-one correspondence with the so-called shape-invariant real forces in one dimension (compare with their presentation in [13]) and on the half-line (their $\mathcal{PT}$-symmetric counterparts were described and listed in [14]).

The situation is reviewed in [15]. A fully general form of this type of analytic constructions dates back to the introduction of the so-called Natanzon potentials and, in the present context, is thoroughly analyzed and described in [16].

4. MODELS WITH PERIODIC BOUNDARY CONDITIONS

A new and promising development of $\mathcal{PT}$-symmetric considerations has been recently inspired by the study of two- and three-particle models [17]. The $\mathcal{PT}$ symmetrization of the Hamiltonians has been again conjectured to be sufficient for keeping their spectrum real. The related "weakening of the Hermiticity" finds a natural generalization in the new context.

Particular attention was paid to the possible non-Hermitian generalizations of the well-known Calogero model [18]. In this setting, the separability of the underlying partial differential Schrödinger equation in hyperspherical coordinates helps us to reduce the problem to a "hyperangular" ordinary differential equation defined on a finite interval. In this way, one has, in the simplest cases, to solve the complexified ordinary differential equations of the generalized Pöschl–Teller type,

$$\left(-\frac{d^2}{d\phi^2} + \frac{l(l+1)}{\sin^2\phi} + \frac{\lambda(\lambda+1)}{\cos^2\phi}\right)\chi(\phi) = E\chi(\phi), \quad (2)$$

on an interval $\phi \in (-M\pi/2, M\pi/2)$ with a suitable integer M. These equations can be solved exactly in terms of hypergeometric functions [19].

Strongly repulsive singularities at $\phi_j = j\pi/2$ are currently not penetrable [20]. Here, they become regularized in a $\mathcal{PT}$-symmetric manner, which parallels a few older constructions on unbounded intervals [14]. Quasisymmetric and quasiantisymmetric solutions arise from certain *ad hoc* boundary conditions [17, 21].

For the most elementary illustration, let us now set $\lambda = 0$ and $M = 2$. Then, the differential Eq. (2) possesses two independent hypergeometric solutions,

$$\chi^{(\pm)}(\phi) = (\sin\phi)^{1/2\pm\alpha}\,{}_2F_1(u^{(\pm)}, v^{(\pm)}; 1\pm\alpha; \sin^2\phi),$$
$$\alpha = l + 1/2 > 0,$$

where $2u^{(\pm)} = 1/2 - \beta \pm \alpha$ and $2v^{(\pm)} = 1/2 + \beta \pm \alpha$. At the boundary of convergence $\sin^2\phi = 1$, the matching of the logarithmic derivatives is equivalent to the termination of this series, whereupon there arise Gegenbauer polynomials,

$$\chi(\phi) = \chi_{(\pm k)}(\phi) = (\sin\phi)^{1/2\pm\alpha} C_k^{1/2\pm\alpha}(\cos\phi),$$
$$k = 0, 1, \ldots.$$

The construction also quantizes the energies and gives them in the closed form

$$E = E_{(\pm k)} = (k \pm \alpha + 1/2)^2, \quad k = 0, 1, \ldots.$$

This set of eigenvalues is composed of the two subsets in a way that resembles the above-mentioned nonequidistant spectrum of the $\mathcal{PT}$-symmetrized singular harmonic oscillator (1) [10].

5. CONCLUSIONS AND OUTLOOK

Historically, one of the first persuasive manifestations of the merits and power of $\mathcal{PT}$ symmetry was offered by Bender and Boettcher [22], who discovered the quasiexact (i.e., incomplete) solvability of the most common and popular quartic polynomial oscillators.

The plausible reasons for the unexpected delay of such an "obvious" observation are closely related to the above-mentioned "forgotten" energies. One has to keep in mind the "spontaneous" regularity of singular potentials within the new formalism. This observation was made explicit in [23], where the presence of two additional singular terms was shown to be compatible with the quasiexact solvability of quartic potentials.

In the quasiexact context, the changes of variables can play the same role as in the completely solvable models. This was illustrated by the particular constructions of the decadic model [24] and of the harmonic plus Coulomb superposition [25]. Further work in this direction is in progress [26].

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Integrable Models of Black Holes and Their Generalizations in the Theories of Supergravity and Superstrings*

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Abstract—A new class of integrable models of $(0+1)$- and $(1+1)$-dimensional dilaton gravity coupled to any number of scalar fields is introduced and briefly discussed. These models can be reduced to a system of Liouville equations that are coupled through energy and momentum constraints. The constraints can be explicitly solved, thus giving an explicit analytic solution of the theory. In particular, these integrable models describe spherically symmetric black holes and branes of supergravity theories in higher dimensions. © *2002 MAIK "Nauka/Interperiodica".*

The problem of quantizing gravity was not solved in the 20th century. There are deep obstacles to this. In perturbation theory, which could only be applicable at large distances, formally constructed quantum gravity is not renormalizable and cannot be treated as a local field theory. At small distances, gravity becomes strong and it is believed to be incompatible with quantum mechanics at distances less than the Planck length l_{P}. If we try to localize the gravitational field inside a sphere of radius about l_{P}, the uncertainty relation tells us that this requires the Planck energy of about $1/l_{\mathrm{P}}$; therefore, a horizon (black hole) of radius about l_{P} must be produced. Thus, either locality or quantum mechanics (or both?) should be modified at Planck distances.

All attempts at solving the problem of quantum gravity were unsuccessful until the advent of superstring theories. Any version of superstring theory incorporates quantum gravity, which, at small distances, does not coincide with Einstein theory. Superstring theory is inherently nonlocal, but the rules of quantum mechanics are supposed to be valid. At the moment, it seems that the superstring approach to quantum gravity will eventually make it possible to calculate gravitational radiative corrections. However, the way to understanding nonperturbative small-distance quantum gravity has not yet been found, although considerations based on superstring theories allow deeper insights into some rather paradoxical properties of quantum black holes.

One of the paradoxical features of black holes is so-called black-hole thermodynamics. A static (Schwarzschild or Reissner–Nordstrøm) or rotating (Kerr) black hole can be completely characterized by its mass M, electric charge Q (one may also add a magnetic charge), and spin S. One may also introduce the temperature of a black hole (for a Schwarzschild black hole, it is $T_{\mathrm{BH}} = 1/8\pi M$) and its Bekenstein–Hawking entropy $S_{\mathrm{BH}} = A/4G_{\mathrm{N}}$, where A is the area of the horizon and G_{N} is Newton's gravitational constant. It was found that the laws of thermodynamics are valid for black holes and that T is the standard thermodynamic temperature of black-hole radiation (Hawking). However, the statistical meaning of the black-hole entropy is not easy to understand. From statistical mechanics, we know that the entropy must be proportional to $\log N(M, Q, J)$, where N is the number of microscopic states. The problem is the following: What are these states? For example, a Schwarzschild black hole (without matter) is described just by one physical variable M, and all other variables introduced in order to describe it are gauge degrees of freedom that may be removed by gauge transformations.

There are other much discussed problems of the evolution of a black hole produced by collapsing matter and of its quantum radiation (the information paradox and the Hawking radiation). The most convincing attempts at resolving the paradoxes of black holes appeared in the last decade as the by-product of the development of superstring theories. This new understanding of black holes emerged from the development of the theory of p branes in supergravity and of their interpretation in superstring theory. Of greatest importance for black holes are D branes and different sorts of dimensional reduction in supergravity (SUGRA) (compactification, symmetry reduction, duality, AdS/CFT correspondence). The

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simplest and best understood black holes—extreme black holes that are supersymmetric solitons—were found to be related to D branes, which are in fact their higher dimensional analogs. The D-brane description of black holes made it possible to identify and count states contributing to the entropy of black holes.

One cannot say that the paradoxes of quantum black holes are completely solved by string and brane considerations. In fact, there exist alternative approaches that may give different expressions for the black-hole entropy. There is no complete consensus between different approaches even within string–brane theory. Of course, this must be so because we do not know a unique quantum theory of gravity; therefore, quantizing particular sectors of gravity, such as black holes, may depend on the rest of the theory not accessible to us. To dramatize this point, we might recall a recently discussed possibility of the physical existence of extra dimensions accessible to gravity but not to matter fields. It is quite clear that the final theory will strongly depend on these extra dimensions and that this may be crucial for strong coupling.

Although we have no theory of quantum gravity, the quantum theory of its special sectors—black holes—might be consistently constructed. In fact, many physically interesting black holes and branes may be described by low-dimensional dilaton-gravity theories, which are the main subject of this report. Different classes of exactly solvable models are known, and some of them can be consistently quantized. There exist classically integrable dilaton-gravity models in $0+1$, $1+1$, and $2+1$ dimensions. The best studied ones are naturally $(0+1)$- and $(1+1)$-dimensional models. A fairly representative $(1+1)$-dimensional dilaton-gravity theory is the following:

$$\mathcal{L} = \sqrt{-g}\Big[\phi R(g) + V(\phi,\psi) + \sum_{k=1}^{K} Z^{(k)}(\phi;\psi) g^{ij}\psi_i^{(k)}\psi_j^{(k)}\Big]. \quad (1)$$

Here, ϕ and $\psi^{(k)}$ are scalar fields (ϕ is the dilaton), g^{ij} is the two-dimensional metric, R is the scalar curvature, $g = g_{00}g_{11} - g_{01}^2$, $\psi_i^{(k)} \equiv \partial_i\psi^{(k)}$, and V and $Z^{(k)}$ are potentials that define the physical content of the theory.

Well-known $(1+1)$-dimensional dilaton gravity describes d-dimensional spherically symmetric Einstein–Maxwell theory minimally coupled to scalar fields. The effective Lagrangian in $1+1$ dimensions may be written as

$$V = 2[\alpha\phi^{-\nu} + \Lambda\phi^{\nu} - \beta Q^2\phi^{\nu-2}], \quad (2)$$

$$Z^{(k)} = -\gamma_k\phi.$$

Here, $\nu = 1/n$, $n = d-2$, $\alpha = n(n-1)$, Q is the electric charge of the system,[1)] Λ is the cosmological constant, and β and γ_k are numerical constants. Note that, after dimensional reduction, the Weyl transformation $g_{ij} \to \Omega(\phi)g_{ij}$ was used, which allowed the vanishing of the kinetic energy of the dilaton. This is possible because the dilaton is in fact a part of the d-dimensional metric

$$ds^2 = g_{ij}dx^i dx^j + \phi^{2\nu}d\Omega_n^2, \quad (3)$$

where $d\Omega_n^2$ is the metric on an n-dimensional sphere of unit radius. Note that the Lagrangian in (1) also describes spherically symmetric Einstein–Yang–Mills theories and some higher dimensional spherical objects that will be mentioned later.

In superstring theory, the simplest model with $V = g$ and $Z^{(k)} = -\gamma$ was studied in detail at the beginning of the last decade (E. Witten, C.G. Callan *et al.*). In this model (usually called the CGHS model), there exist solutions with a horizon, but their global properties are unrealistic from the higher dimensional point of view, and the geometry of the solutions is too simple because $R(g) \equiv 0$. In the last five years, more complex objects, such as higher dimensional generalizations of black holes, have been studied. These are different sorts of branes in SUGRA, which describe different possible ground states of superstrings (for a review see, e.g., [1–3]). In many interesting cases, a compactification of the original supergravity gives the $(1+1)$-dimensional theory (1), where V depends both on ϕ and on ψ, while the number of the scalar fields K and their coupling to the dilaton gravity is defined by the geometry of compactification (for the majority of the models, $Z^{(k)} \sim \phi$ and V is a sum of linear exponentials in the fields $\psi^{(k)}$).

A typical example of a $(1+1)$-dimensional theory describing such solutions of SUGRA may be obtained by a chain of compactifications and dimensional reductions in the 11-dimensional SUGRA of E. Cremmer, B. Julia, and J. Scherk. The NS–NS bosonic sector in ten-dimensional SUGRA obtained by the Kaluza–Klein (KK) reduction (it is more appropriate to call it the Kaluza–Mandel–Klein–Fock reduction) is described by the action

$$S = \int d^{10}x\sqrt{-G}e^{2\chi}\Big[R(G) + 4(\nabla\chi)^2 - \frac{1}{3}H^2\Big], \quad (4)$$

where χ is the dilaton related to the 11-dimensional metric and H is the three-form gauge field, $H = dB$. A chain of compactifications, which may be found

[1)] In two dimensions, the electromagnetic field does not propagate and its effects may be included in the effective potential.

in [4], finally gives the (1+1)-dimensional theory of the form (1) with the potentials

$$V = 6e^{2\psi_2} - \frac{1}{2}H_0^2 e^{6\psi_2} - 8\phi^{-2}(Q_1^2 e^{\psi_1} + Q_2^2 e^{-\psi_1}), \tag{5}$$

$$Z^{(1)} = -\frac{1}{4}\phi, \quad Z^{(2)} = -3\phi, \quad Z^{(3)} = -\phi. \tag{6}$$

Here, the dilaton ϕ is a linear combination of χ and of three scalar fields $\psi_k \equiv \psi^{(k)}$, which were introduced in compactification and KK and spherical reduction; the constant H_0 is the remnant of the three-form gauge field; the electric charge Q_1 is related to the KK reduction; and Q_2 is related to the dimensionally reduced Abelian field H. We have also used the Weyl transformation and excluded the Abelian gauge fields by solving their equation of motion.

Let us now proceed to discuss the integrability of the $(1+1)$-dimensional models (1) and of the $(0+1)$-dimensional models obtained by their dimensional reduction. The general theory (1) with arbitrary potentials is not integrable. At the beginning of the last decade, the general solutions of the model with $V = g_0 + g_1\phi$ and $Z^{(k)} = -\gamma_k$ were found.[2)]

However, if $Z^{(k)} = 0$ and $V = V(\phi)$, the theory is exactly solvable (and thus integrable) with arbitrary potential $V(\phi)$ (see, e.g., [5, 6]).[3)] Moreover, this two-dimensional field theory actually reduces to $(0+1)$-dimensional theory, which is a Hamiltonian system with a finite number of degrees of freedom and with one constraint. The most interesting feature of this model is that, at arbitrary $V(\phi)$, its solutions have one or more horizons. In this paper, I call a horizon any zero of the metric, $f(u,v) = 0$, in the conformal (light-cone) coordinates, in which $ds^2 = -4f(u,v)dudv$. It is also supposed that the dilaton and matter fields are nonsingular at the horizon and that the solution can be uniquely continued through the horizon (this is true for classical Schwarzschild and Reissner–Nordstrøm black holes). The dimensional reduction means that $\phi(u,v) = \phi(\tau)$ and $f(u,v) = h(\tau)a'(v)b'(u)$, where $\tau \equiv a(u) + b(v)$. The statement that the general solution of the theory with $Z \equiv 0$ can be reduced (is gauge equivalent) to the $(0+1)$-dimensional solution is called the (generalized) Birkhoff theorem.

When $Z^{(k)} \neq 0$, the $(1+1)$-dimensional theories (1) are in general not integrable, but their dimensional reductions to $0+1$ dimensions may be integrable Hamiltonian systems. Their solutions describe static (time-independent) configurations (with or without horizons) or, alternatively, time-dependent homogeneous and isotropic cosmological models (see, e.g., [7]). The spherical reduction of Einstein–Maxwell theory with scalar fields is probably not integrable in $1+1$ dimensions. (But its static reduction is analytically integrable if $\Lambda = 0$.) The static reduction of the model given by Eqs. (5) and (6) is also analytically integrable, as well as many similar models that can be obtained by similar procedures. These models are special cases of a more general class of integrable models that we introduce now.

Let us write the static reduction of the theory (1) (see [6]):

$$\mathcal{L}_{\rm st} = -\frac{1}{l(\tau)}\left(\dot\phi\dot F + \sum_{k=1}^{K} Z^{(k)}\dot\psi_k^2\right) + l(\tau)\exp(F)V(\phi,\psi). \tag{7}$$

Here, the dot denotes differentiation with respect to τ, $l(\tau)$ is a Lagrange multiplier, and $F = \ln|h|$. A class of integrable models (6) with one scalar field ($K = 1$) was found in [6] [it includes the potential (2) with $\Lambda = 0$]. Recently, I have generalized this result and constructed an integrable theory with arbitrary K (even $K = \infty$) [8].

There exist two integrable classes:

$$V = \sum_{m=1}^{M} g_m \phi^{l_m} \exp L_m(\psi), \quad Z^{(k)} = -\gamma_k\phi, \tag{8}$$

$$V = \sum_{m=1}^{M} g_m \exp L_m(\psi,\phi), \quad Z^{(k)} = -\gamma_k, \tag{9}$$

where g_m are numbers, L_m are linear functions, and M must be less than or equal to $K+2$. Using the freedom in defining a Lagrange multiplier, we may reduce the first class to the second one [define $\bar l \equiv l/\phi(\tau)$ and use $\bar l$ instead of l]. Then, we introduce $\psi_1 \equiv (F+\phi)/2$ and $\psi_2 \equiv (F-\phi)/2$ and define $\psi_{k+2} \equiv \sqrt{\gamma_k}\psi^{(k)}$ for $1 \le k \le K$ (recall that γ must be positive). The Lagrangian in (7) can then be written as

$$\mathcal{L}_{\rm st} = -\frac{1}{l(\tau)}\left(-\dot\psi_1^2 + \dot\psi_2^2 + \sum_{n=3}^{N}\dot\psi_n^2\right) + l(\tau)\sum_{n=1}^{N}\bar g_n \exp q_n, \tag{10}$$

[2)] This is the so-called minimal coupling of scalar matter to dilaton gravity.

[3)] One may call it analytically integrable if we regard, as analytic expressions, integrals of arbitrary functions and their inverse functions (this usage was standard in the 19th century).

where $N \equiv K+2$; $\bar{g}_n = g_n$ for $n \le M$ and $\bar{g}_n = 0$ for $n > M$; and

$$q_n = \sum_{m=1}^{N} \psi_m a_{mn}, \quad a_{1n} + a_{2n} = 2. \qquad (11)$$

If the vectors $A_n \equiv (a_{mn})$ are pseudoorthogonal, the Lagrange's equations for q_n can be reduced to the explicitly integrable differential equations

$$\ddot{q}_n = \lambda_n \bar{g}_n \exp q_n, \quad \lambda_n \equiv -a_{1n} + \sum_{m=1}^{N} a_{mn}^2. \quad (12)$$

These equations have N independent integrals

$$\dot{q}_n^2 - 2\lambda_n \bar{g}_n \exp q_n = C_n. \qquad (13)$$

Therefore, they can be explicitly solved in terms of elementary functions. Actually, we have $2N$ integrals: since the equations are independent of τ, the shifts in τ in each solution $\psi_n(\tau)$ give new solutions $\psi_n(\tau + \tau_n)$, and these shifts τ_n are integrals of motion. The constraint equation

$$\sum_{1}^{N} C_n/\lambda_n = 0 \qquad (14)$$

gives one relation between the integrals. The general solution thus depends on $2N-2$ significant integrals (as the differential system has no explicit dependence on τ, only $N-1$ of the τ_n are significant integrals, one of them simply determining a scale).

Here, we will not discuss the properties of these solutions and only mention the main points. The general solutions introduced above may have horizons ($F \to -\infty$ for $\tau \to \pm\infty$, while the other variables are finite) if all C_n are positive and equal, $C_n = C > 0$ (this solution thus depends on N significant independent integrals). In fact, there are two horizons, for $\tau \to \infty$ and $\tau \to -\infty$; thus, we have a Reissner–Nordstrøm-type geometry. The dilaton and the scalar fields nontrivially depend on τ because there is no restriction on the integrals τ_n. The solutions are globally defined and, in general, are restricted by singularities. It is not difficult to construct an extreme black hole for which the two horizons coincide. However, the coordinate τ is somewhat inconvenient, and some other parametrizations are better suited for extreme black holes.

Note that the theories with $V = V(\phi)$ and $Z = Z(\phi)$ have no horizons for τ-dependent ψ (it is sufficient that just one ψ is not constant). This is a local generalization of the no-hair theorem, which was proven in [6]. Recently, it has been shown that, for $\partial_\psi V \neq 0$, there exists at least one nontrivial horizon (with τ-dependent ψ [9]). In the same study, we have found a solution near the horizon as a convergent power-series expansion in h for essentially arbitrary potentials V and Z.

In conclusion, we will briefly summarize the conditions of integrability for $(1+1)$-dimensional field theories. Field theories with the potentials (9) satisfying the $(0+1)$-dimensional integrability conditions are also integrable. Their solutions can be written in terms of a solution of the independent Liouville equation $\Psi_{uv} + 2g \exp \Psi = 0$. The first example of such a theory (reducible to two Liouville equations) was introduced in [6]. Although the Liouville equations are formally independent, their solution must satisfy two constraint equations

$$\phi_{ii} - \phi_i F_i = \sum_{k=1}^{K} Z^{(k)} \psi_{(k),i}^2. \qquad (15)$$

At first glance, this problem looks analytically insolvable. However, it has a conceptually simple analytic solution that, unfortunately, is too lengthy to reproduce here. A general solution of N Liouville equations satisfying the constraints in (15) can be written as an analytic expression depending on $N-1$ arbitrary functions of u and $N-1$ arbitrary functions of v, i.e., in terms of free chiral fields.

Finally, let me briefly mention the problem of quantizing dilaton-gravity models. A great many studies are devoted to quantizing cosmological models and black holes treated as $(0+1)$- or $(1+1)$-dimensional dilaton-gravity models; $(0+1)$-dimensional models were successfully quantized (see, e.g., [10] and references therein). It is much more difficult to quantize $(1+1)$-dimensional models, which are usually plagued by anomalies (see, e.g., [11]). Although some simplest models can be consistently quantized (see, e.g., [12] and references therein), the problem of quantizing dilaton-gravity theories with scalar fields has no unique and convincing solution. The integrable models introduced here are promising from this point of view because they are related to Liouville theory. Recent advances in quantizing the Liouville equation shows that we may hope to have fairly soon proper tools for quantizing integrable dilaton-gravity models describing black holes and branes of supergravity and superstring theories.

The main results were obtained in the winter of 2000–2001 and were first briefly mentioned in the short report [8]. The detailed formulation and solution of a general integrable theory and its analysis will be published elsewhere.

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A Quantum Version of the Inverse Scattering Transformation*

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Abstract—We derive a formula that expresses the local spin and field operators of fundamental graded models in terms of the elements of the monodromy matrix. This formula may be understood as a quantum version of the classical inverse scattering transformation. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

The quantum inverse scattering method was initiated some twenty years ago by Sklyanin and Faddeev [1, 2] and then developed largely by the group from the Steklov Mathematical Institute at Leningrad (see, for instance, [3–6]).

The method acquired its name because it arose as an attempt to develop a quantum version of the (classical) inverse scattering method [7, 8], which was successful in solving nonlinear classical evolution equations, such as the Korteweg–de Vries equation [9], the nonlinear Schrödinger equation [10], or the sine-Gordon equation [11].

The classical inverse scattering method provides a mapping from a set of field variables satisfying nonlinear evolution equations to a set of scattering data of an associated auxiliary problem. While the fields obey nonlinear evolution equations, the scattering data obey linear equations. The solution of the initial-value problem for the original nonlinear evolution equations of the fields is achieved by first mapping the initial data to the scattering data at time $t = 0$, then using the linear time evolution of the scattering data, and finally applying the *inverse transformation* [12, 13] from scattering data to fields at a time $t > 0$.

In this paper, we solve the "inverse scattering problem" for quantum lattice models. The solution is remarkably simple.

Nowadays, the term "quantum inverse scattering method" usually refers to a method formulated for systems of finite length. The relation to the classical case is the following. The elements of the monodromy matrix, which appears in the formulation of the classical problem for systems of finite length, have simple Poisson brackets [4]. In the quantum case, the Poisson brackets are replaced by commutators of quantum operators. These commutators remain simple after quantization. The quantum operators can be grouped into a matrix, which, by analogy to the classical case, is called (quantum) monodromy matrix. The elements of the quantum monodromy matrix obey a set of quadratic relations. They generate the so-called Yang–Baxter algebra. The structure of this algebra is determined by numerical functions of a complex spectral parameter, which again can be arranged in a matrix. This matrix is called the R matrix. It satisfies the famous Yang–Baxter equation (see (23) below). The R matrix and its associated Yang–Baxter algebra are the key concepts of the quantum inverse scattering method. These concepts are *algebraic*.

The Yang–Baxter algebra has two primary applications. First of all, it contains, in general, a rich commutative subalgebra generated by the trace of the monodromy matrix. The elements of this subalgebra have a natural interpretation as a set of commuting operators belonging to a physical system. One of these operators is interpreted as the Hamiltonian. The existence of a large set of commuting operators cannot be directly utilized to diagonalize the Hamiltonian. In many cases, however, the Yang–Baxter algebra can be employed for this task. It can be used to simultaneously diagonalize all of the commuting operators by a procedure called the algebraic Bethe ansatz [3]. This is the most important application of the Yang–Baxter algebra.

In spite of the conceptual differences between classical and quantum inverse scattering method, both methods have an important point in common. They essentially rely on a mapping from local field variables to a set of nonlocal variables, which are the elements of the monodromy matrix. In the quantum case, the inverse transformation, expressing the local

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fields in terms of the elements of the monodromy matrix, was not known until recently. It is this inverse transformation we refer to as the "solution of the quantum inverse problem." It first appeared for the examples of the inhomogeneous XXX and XXZ spin-1/2 Heisenberg chains in the article [14]. In [15], it was noticed that the proof of [14] relies solely on the properties of the shift operator for inhomogeneous models, and, based on this observation, a solution of the quantum inverse problem for general fundamental graded models was constructed.

In this contribution, we summarize the results of [15]. We present an explicit solution of the quantum inverse problem, which is valid (i) in the homogeneous case, (ii) for models with R matrices of arbitrary higher dimension, and (iii) most generally for fundamental graded models [16]. Upon specification to the cases of the inhomogeneous XXX and XXZ spin-1/2 Heisenberg chains, our result reduces to the formula obtained in [14].

Our result in its most general form is given by the formula (45) below. Formula (45) expresses the local operators as *products* of the entries of the monodromy matrix evaluated at the inhomogeneities. The structure of the solution of the quantum inverse problem for periodic lattice models is thus much different from the structure of the solution of the classical inverse scattering problem. In the quantum case, we have an explicit multiplicative formula. In the classical case, the solution is implicit and additive. It reduces to the Gelfand–Levitan–Marchenko integral equations [12, 13].

2. GRADED VECTOR SPACES

In this section, we shall recall the basic concepts of graded vector spaces and graded associative algebras. In the context of the quantum inverse scattering method, these concepts were first used by Kulish and Sklyanin [5, 17]. We shall further recall the notions of "graded local projection operators" and graded permutation operators. Graded local projection operators were introduced in the article [16]. They enable the definition of fundamental graded representations of the Yang–Baxter algebra, which will be given in the following section.

Graded vector spaces are vector spaces equipped with a notion of odd and even that allows us to treat fermions within the formalism of the quantum inverse scattering method. Let us start with a finite-dimensional local space of states V, on which we impose an additional structure, the parity, from the outset. Let $V = V_0 \oplus V_1$, $\dim V_0 = m$, $\dim V_1 = n$. We shall call $v_0 \in V_0$ even and $v_1 \in V_1$ odd. The subspaces V_0 and V_1 are called the homogeneous components of V. The parity p is a function $V_i \to \mathbb{Z}_2$ defined on the homogeneous components of V,

$$p(v_i) = i, \quad i = 0, 1, \quad v_i \in V_i. \tag{1}$$

The vector space V endowed with this structure is called a graded vector space or superspace. Let us fix a basis $\{e_1, \ldots, e_{m+n}\}$ of definite parity and let us define $p(\alpha) := p(e_\alpha)$.

In order to introduce Fermi operators into the formalism of the quantum inverse scattering method, we have to construct an algebra of commuting and anticommuting *operators*. For this purpose, the concept of parity must be extended to operators in $\mathrm{End}(V)$ and to tensor products of these operators. Let $e^\beta_\alpha \in \mathrm{End}(V)$, $e^\beta_\alpha e_\gamma = \delta^\beta_\gamma e_\alpha$. The set $\{e^\beta_\alpha \in \mathrm{End}(V) | \alpha, \beta = 1, \ldots, m+n\}$ is a basis of $\mathrm{End}(V)$. Hence, the definition

$$p(e^\beta_\alpha) = p(\alpha) + p(\beta) \tag{2}$$

induces a grading on $\mathrm{End}(V)$ regarded as a vector space.

It is easy to see that an element $A = A^\alpha_\beta e^\beta_\alpha \in \mathrm{End}(V)$ is homogeneous with parity $p(A)$ if and only if

$$(-1)^{p(\alpha)+p(\beta)} A^\alpha_\beta = (-1)^{p(A)} A^\alpha_\beta. \tag{3}$$

The latter equation implies for two homogeneous elements $A, B \in \mathrm{End}(V)$ that their product AB is homogeneous with parity

$$p(AB) = p(A) + p(B). \tag{4}$$

In other words, multiplication of matrices in $\mathrm{End}(V)$ preserves homogeneity, and, therefore, $\mathrm{End}(V)$ endowed with the grading (2) is a graded associative algebra [5].

Let us consider the L-fold tensorial power $\mathcal{H} = (\mathrm{End}(V))^{\otimes L}$ of $\mathrm{End}(V)$. The definition (2) has a natural extension to $\mathcal{H}$, namely,

$$\begin{aligned} p(e^{\beta_1}_{\alpha_1} \otimes \cdots \otimes e^{\beta_L}_{\alpha_L}) \\ = p(\alpha_1) + p(\beta_1) + \cdots + p(\alpha_L) + p(\beta_L). \end{aligned} \tag{5}$$

From this formula, it can be seen in a similar way as before, that homogeneous elements $A = A^{\alpha_1 \ldots \alpha_L}_{\beta_1 \ldots \beta_L} \times e^{\beta_1}_{\alpha_1} \otimes \cdots \otimes e^{\beta_L}_{\alpha_L}$ of $\mathcal{H}$ with parity $p(A)$ are characterized by the equation

$$\begin{aligned} (-1)^{\sum_{j=1}^L (p(\alpha_j)+p(\beta_j))} A^{\alpha_1 \ldots \alpha_L}_{\beta_1 \ldots \beta_L} \\ = (-1)^{p(A)} A^{\alpha_1 \ldots \alpha_L}_{\beta_1 \ldots \beta_L}, \end{aligned} \tag{6}$$

which generalizes (3). Again the product AB is homogeneous with parity $p(AB) = p(A) + p(B)$ if A and B are homogeneous. Thus, the definition (5)

induces the structure of a graded associative algebra on $\mathcal{H}$.

Let us define the superbracket

$$[X,Y]_\pm = XY - (-1)^{p(X)p(Y)}YX \tag{7}$$

for X, Y taken from the homogeneous components of $\mathrm{End}(V)$, and let us extend it linearly to $\mathrm{End}(V)$ in both of its arguments. Then, $\mathrm{End}(V)$ endowed with the superbracket becomes the Lie superalgebra $gl(m|n)$. Note that the above definition of a superbracket makes sense in any graded algebra and is particularly valid in $\mathcal{H}$.

The following definition of "graded local projection operators" [16] will be crucial for our definition of fundamental graded representations of the Yang–Baxter algebra in the next section. Define the matrices

$$\begin{aligned} e_{j_\alpha}^{\ \beta} &= (-1)^{(p(\alpha)+p(\beta))\sum_{k=j+1}^{L} p(\gamma_k)} \\ &\times I_{m+n}^{\otimes(j-1)} \otimes e_\alpha^\beta \otimes e_{\gamma_{j+1}}^{\gamma_{j+1}} \otimes \cdots \otimes e_{\gamma_L}^{\gamma_L}, \end{aligned} \tag{8}$$

where I_{m+n} is the $(m+n)\times(m+n)$ unit matrix, and summation over double *tensor indices* (i.e., over $\gamma_{j+1},\ldots,\gamma_L$) is understood. We shall keep this sum convention throughout the remainder of this article. The index j on the left-hand side of (8) will later refer to the jth site of a physical lattice model and is called the site index. A simple consequence of the definition (8) for $j \neq k$ are the commutation relations

$$e_{j_\alpha}^{\ \beta} e_{k_\gamma}^{\ \delta} = (-1)^{(p(\alpha)+p(\beta))(p(\gamma)+p(\delta))} e_{k_\gamma}^{\ \delta} e_{j_\alpha}^{\ \beta}. \tag{9}$$

It further follows from Eq. (8) that $e_{j_\alpha}^{\ \beta}$ is homogeneous with parity

$$p(e_{j_\alpha}^{\ \beta}) = p(\alpha) + p(\beta). \tag{10}$$

Hence, in agreement with intuition, Eq. (9) says that odd matrices with different site indices mutually anticommute, whereas even matrices commute with each other as well as with the odd matrices. For products of matrices $e_{j_\alpha}^{\ \beta}$ which are acting on the same site (8) implies the projection property

$$e_{j_\alpha}^{\ \beta} e_{j_\gamma}^{\ \delta} = \delta_\gamma^\beta e_{j_\alpha}^{\ \delta}. \tag{11}$$

Equations (9) and (11) justify our terminology. The $e_{j_\alpha}^{\ \beta}$ are graded analogs of local projection operators. We call them graded local projection operators or projection operators, for short. Using the super-bracket (7), Eqs. (9) and (11) can be combined into

$$\begin{aligned} &[e_{j_\alpha}^{\ \beta}, e_{k_\gamma}^{\ \delta}]_\pm \\ &= \delta_{jk}\left(\delta_\gamma^\beta e_{j_\alpha}^{\ \delta} - (-1)^{(p(\alpha)+p(\beta))(p(\gamma)+p(\delta))}\delta_\alpha^\delta e_{j_\gamma}^{\ \beta}\right). \end{aligned} \tag{12}$$

The right-hand side of the latter equation with $j=k$ gives the structure constants of the Lie superalgebra $gl(m|n)$ with respect to the basis $\{e_{j_\alpha}^{\ \beta}\}$.

Since any $(m+n)$-dimensional vector space over the complex numbers is isomorphic to $\mathbb{C}^{m+n}$, we may simply set $V=\mathbb{C}^{m+n}$. We may further assume that our homogeneous basis $\{e_\alpha \in \mathbb{C}^{m+n} | \alpha = 1,\ldots,m+n\}$ is canonical; i.e., we may represent the vector e_α by a column vector having the only nonzero entry $+1$ in row α. Our basic matrices e_α^β are then $(m+n)\times(m+n)$ matrices with a single nonzero entry $+1$ in row α and column β.

Remark. The meaning of (18) becomes more evident by considering a simple example. Let $m=n=1$ and $p(1)=0$, $p(2)=1$. Then, using (12), we obtain

$$[e_{j1}^{\ 2}, e_{k1}^{\ 2}]_\pm = \{e_{j1}^{\ 2}, e_{k1}^{\ 2}\} = 0, \tag{13}$$

$$[e_{j2}^{\ 1}, e_{k2}^{\ 1}]_\pm = \{e_{j2}^{\ 1}, e_{k2}^{\ 1}\} = 0, \tag{14}$$

$$\begin{aligned} [e_{j1}^{\ 2}, e_{k2}^{\ 1}]_\pm &= \{e_{j2}^{\ 1}, e_{k1}^{\ 2}\} \\ &= \delta_{jk}(e_{j1}^{\ 1} + e_{j2}^{\ 2}) = \delta_{jk} \end{aligned} \tag{15}$$

for $j,k=1,\ldots,L$. The curly brackets in (13), (14) denote the anticommutator. The matrices $e_{j1}^{\ 2}$ and $e_{k2}^{\ 1}$ satisfy the canonical anticommutation relations for spinless Fermi operators. We can therefore identify $e_{j1}^{\ 2} \to c_j$ and $e_{k2}^{\ 1} \to c_k^+$. Introducing Pauli matrices $\sigma^+ = e_1^2$, $\sigma^- = e_2^1$, and $\sigma^z = e_1^1 - e_2^2$, we obtain, by carrying out the summation, the following explicit matrix representation from our basic definition (8):

$$c_j = I_2^{\otimes(j-1)} \otimes \sigma^+ \otimes (\sigma^z)^{\otimes(L-j)}, \tag{16}$$

$$c_k^+ = I_2^{\otimes(k-1)} \otimes \sigma^- \otimes (\sigma^z)^{\otimes(L-k)}. \tag{17}$$

This is the well-known Jordan–Wigner transformation [18] expressing Fermi operators for spinless fermions in terms of Pauli matrices. We may thus interpret Eq. (8) as a generalization of the Jordan–Wigner transformation. In general, Eq. (8) provides matrix representations not of Fermi operators but, more generally, of fermionic projection operators. Representations of Fermi operators can be obtained by taking appropriate linear combinations of matrices $e_{j_\alpha}^{\ \beta}$. This issue is explained in [15, 16].

The permutation operator plays an important role in the construction of local integrable lattice models. It enters the expression for the shift operator on homogeneous lattices. In the graded case, the definition of the permutation operator requires the following modifications of signs:

$$P_{jk} = (-1)^{p(\beta)} e_{j_\alpha}^{\ \beta} e_{k_\beta}^{\ \alpha}. \tag{18}$$

As indicated by its name, this operator induces the action of the symmetric group $\mathfrak{S}^L$ on the site indices of the matrices $e_{j_\alpha}^{\ \beta}$. The properties of P_{jk} (for $j \neq k$) are the same as in the nongraded case. They are

easily derived from (9) and (11) and can be found, for instance, in [16].

In the next section, the graded associative algebra $\mathcal{H}$ will be considered as the space of states of a lattice model associated with a solution of the Yang–Baxter equation. We will define a monodromy matrix whose entries are elements of $\mathcal{H}$. The following definitions will prove to be useful.

Consider $(m+n)\times(m+n)$ matrices $A, B, C, \ldots$ with entries in $\mathcal{H}$ such that $p(A^\alpha_\beta)=p(B^\alpha_\beta)=p(C^\alpha_\beta)=\ldots=p(\alpha)+p(\beta)$ for $\alpha,\beta=1,\ldots,m+n$. These matrices form an associative algebra, say $\mathcal{A}$, since $p(A^\alpha_\beta B^\beta_\gamma)=p(\alpha)+p(\gamma)$. For $A,B\in\mathcal{A}$, define a supertensor product (or graded tensor product)

$$(A\otimes_s B)^{\alpha\gamma}_{\beta\delta}=(-1)^{(p(\alpha)+p(\beta))p(\gamma)}A^\alpha_\beta B^\gamma_\delta. \qquad (19)$$

This definition has an interesting consequence. Let $A,B,C,D\in\mathcal{A}$ such that

$$[B^\alpha_\beta, C^\gamma_\delta]_\pm=0. \qquad (20)$$

Then,

$$(A\otimes_s B)(C\otimes_s D)=AC\otimes_s BD. \qquad (21)$$

For matrices $A\in\mathcal{A}$, we further define the supertrace as

$$\mathrm{str}(A)=(-1)^{p(\alpha)}A^\alpha_\alpha. \qquad (22)$$

3. FUNDAMENTAL GRADED MODELS

In this section, we shall recall the notion of *fundamental graded representations* of the Yang–Baxter algebra, which was introduced in [16]. For a given grading, we shall associate a fundamental model with every solution of the Yang–Baxter equation that satisfies a certain compatibility condition [see (24) below].

It is most suitable for our present purpose to interpret the Yang–Baxter equation as a set of functional equations for the matrix elements of an $(m+n)^2\times(m+n)^2$ matrix $R(u,v)$. We may represent the Yang–Baxter equation in graphical form as shown in Fig. 1, where each vertex corresponds to a factor in the equation

$$R^{\alpha\beta}_{\alpha'\beta'}(u,v)R^{\alpha'\gamma}_{\alpha''\gamma'}(u,w)R^{\beta'\gamma'}_{\beta''\gamma''}(v,w) \qquad (23)$$
$$=R^{\beta\gamma}_{\beta'\gamma'}(v,w)R^{\alpha\gamma'}_{\alpha'\gamma''}(u,w)R^{\alpha'\beta'}_{\alpha''\beta''}(u,v).$$

Note that, there is a direction assigned to every line in Fig. 1, which is indicated by the tips of the arrows. Therefore, every vertex has an orientation, and vertices and R matrices can be identified according to Fig. 2, where indices have been supplied to a vertex. Summation is over all inner lines in Fig. 1.

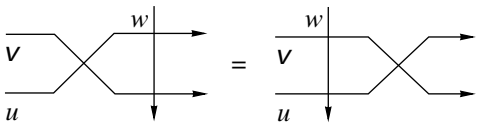

Fig. 1. The Yang–Baxter equation is most easily memorized in graphical form.

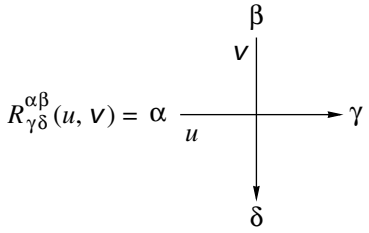

Fig. 2. Identification of the R matrix with a vertex.

Let us assume we are given a solution of (23), which is compatible with the grading in the sense that [5]

$$R^{\alpha\beta}_{\gamma\delta}(u,v)=(-1)^{p(\alpha)+p(\beta)+p(\gamma)+p(\delta)}R^{\alpha\beta}_{\gamma\delta}(u,v). \qquad (24)$$

For such R matrices, we define a graded L matrix at site j,

$$\mathcal{L}_j{}^\alpha_\beta(u,v)=(-1)^{p(\alpha)p(\gamma)}R^{\alpha\gamma}_{\beta\delta}(u,v)e_j{}^\delta_\gamma. \qquad (25)$$

Lemma 1. *Properties of the graded L matrix.*

(i) Homogeneity. The matrix elements of the graded L matrix are homogeneous with parity

$$p(\mathcal{L}_j{}^\alpha_\beta(u,v))=p(\alpha)+p(\beta). \qquad (26)$$

(ii) Commutativity. The entries of the graded L matrix supercommute for different site indices,

$$[\mathcal{L}_j{}^\alpha_\beta(u,v),\mathcal{L}_k{}^\gamma_\delta(w,z)]_\pm=0 \qquad (27)$$

for $j\neq k$.

(iii) Bilinear relation. The entries of the graded L matrix at the same lattice site satisfy the bilinear relation

$$\check{R}(u,v)\big(\mathcal{L}_j(u,w)\otimes_s\mathcal{L}_j(v,w)\big) \qquad (28)$$
$$=\big(\mathcal{L}_j(v,w)\otimes_s\mathcal{L}_j(u,w)\big)\check{R}(u,v),$$

where, as in the nongraded case, the matrix $\check{R}(u,v)$ is defined as $\check{R}^{\alpha\beta}_{\gamma\delta}(u,v)=R^{\beta\alpha}_{\gamma\delta}(u,v)$.

The lemma follows from the Yang–Baxter equation (23) and from Eq. (24). Equation (28) may be interpreted as defining a graded Yang–Baxter algebra with R matrix $\check{R}$. We call $\mathcal{L}_j$ its fundamental graded representation.

Starting from (28), we can construct integrable lattice models as in the nongraded case [19]. Let us

briefly recall the construction with emphasis on the modifications that appear due to the grading. Define a monodromy matrix $\mathcal{T}(u)$ as an L-fold ordered product of fundamental L matrices,

$$\mathcal{T}(u) = \mathcal{T}_{1\dots L}(u) = \mathcal{L}_L(u,\xi_L)\dots\mathcal{L}_1(u,\xi_1). \quad (29)$$

Due to Eq. (4), the matrix elements of $\mathcal{T}(u)$ are homogeneous with parity $p(\mathcal{T}^\alpha_\beta(u)) = p(\alpha)+p(\beta)$. Repeated application of (28) and (21) shows that this monodromy matrix is a representation of the graded Yang–Baxter algebra,

$$\check{R}(u,v)\big(\mathcal{T}(u)\otimes_s\mathcal{T}(v)\big) = \big(\mathcal{T}(v)\otimes_s\mathcal{T}(u)\big)\check{R}(u,v). \quad (30)$$

It follows from (24) and (30) that

$$\big[\operatorname{str}(\mathcal{T}(u)),\operatorname{str}(\mathcal{T}(v))\big] = 0, \quad (31)$$

which is in complete analogy with the nongraded case.

For the sake of completeness, let us recall how to associate a local lattice Hamiltonian with the monodromy matrix $\mathcal{T}(u)$. Assume that $R(u,v)$ is a regular solution of the Yang–Baxter equation, $R^{\alpha\beta}_{\gamma\delta}(\xi,\xi) = \delta^\alpha_\delta\delta^\beta_\gamma$ for some $\xi\in\mathbb{C}$. Then, (25) implies that

$$\mathcal{L}_{j\,\beta}^{\ \alpha}(\xi,\xi) = (-1)^{p(\alpha)p(\beta)}e_{j\,\beta}^{\ \alpha}, \quad (32)$$

and we can easily see [16] that the supertrace of the monodromy matrix evaluated at $u=\xi_1=\dots=\xi_L=\xi$ generates a shift by one site,

$$\operatorname{str}(\mathcal{T}(\xi)) = P_{12}P_{23}\dots P_{L-1L} =: \hat{U}. \quad (33)$$

It follows that $\tau(u) := -i\ln(\operatorname{str}(\mathcal{T}(u)))$ generates a sequence of local operators [20] which, as a consequence of (31), mutually commute,

$$\tau(u) = \hat{\Pi} + (u-\xi)\hat{H} + \mathcal{O}\left((u-\xi)^2\right). \quad (34)$$

$\hat{\Pi}$ in this expansion is the momentum operator. On a lattice, where the minimal possible shift is by one site, and thus $\hat{U}$ rather than $\hat{\Pi}$ is the fundamental geometrical operator, some care is required in the definition of $\hat{\Pi}$. As was shown in [21], a proper definition may be obtained by setting $\hat{\Pi} := -i\ln(\hat{U}) \mod 2\pi$ and expressing the function $f(x) = x \mod 2\pi$ by its Fourier sum. Then, $\hat{\Pi}$ becomes a polynomial in $\hat{U}$.

$$\hat{\Pi} = \phi\sum_{m=1}^{L-1}\left(\frac{1}{2} + \frac{\hat{U}^m}{e^{-i\phi m}-1}\right), \quad (35)$$

where $\phi = 2\pi/L$. The first-order term $\hat{H}$ in the expansion (34) may be interpreted as Hamiltonian. Using (33), it is obtained as

$$\hat{H} = \sum_{j=1}^{L} H_{jj+1}, \quad (36)$$

where $H_{LL+1} = H_{L1}$ and

$$H_{jj+1} = -i\,(-1)^{p(\gamma)(p(\alpha)+p(\gamma))}\,\partial_u \times \check{R}^{\alpha\beta}_{\gamma\delta}(u,v)\Big|_{u=v} e_{j\,\alpha}^{\ \gamma}e_{j+1\,\beta}^{\ \delta}. \quad (37)$$

We would like to emphasize the following points. (i) The R matrix $\check{R}$ in Eq. (28) does *not* undergo a modification due to the grading. (ii) The only necessary compatibility condition which has to be satisfied in order to introduce a fundamental graded representation of the Yang–Baxter algebra associated with a solution of the Yang–Baxter equation is Eq. (24), which was introduced in [5].

The role of the matrix $\check{R}(u,v)$ in the graded Yang–Baxter algebra (28) is to switch the order of the two auxiliary spaces. The definition of an operator that similarly switches the order of quantum spaces in a product of two L matrices requires appropriate use of the grading. Recently, such an operator was introduced for several important models in [22, 23] and was called fermionic R operator. A general definition of the fermionic R operator associated with a solution $R(u,v)$ of the Yang–Baxter equation (23) was introduced in [15]. For a given grading and a solution $R(u,v)$ of the Yang–Baxter equation (23) that is compatible with this grading [see (24)] we define following [15]:

$$\mathcal{R}^f_{jk}(u,v) = (-1)^{p(\gamma)+p(\alpha)(p(\beta)+p(\gamma))} \times R^{\alpha\beta}_{\gamma\delta}(u,v)e_{j\,\alpha}^{\ \gamma}e_{k\,\beta}^{\ \delta}. \quad (38)$$

The fermionic R operator will be an important tool in the proof of our main result (45). Let us summarize its properties in the following lemma.

Lemma 2. *Properties of the fermionic R operator.*

(i) Evenness. The fermionic R operator is even,

$$p(\mathcal{R}^f_{jk}(u,v)) = 0. \quad (39)$$

(ii) Bilinear relation. The fermionic R operator satisfies

$$\mathcal{R}^f_{jk}(\xi_j,\xi_k)\mathcal{L}_k(u,\xi_k)\mathcal{L}_j(u,\xi_j) = \mathcal{L}_j(u,\xi_j)\mathcal{L}_k(u,\xi_k)\mathcal{R}^f_{jk}(\xi_j,\xi_k). \quad (40)$$

(iii) Yang–Baxter equation. The fermionic R operator satisfies the following form of the Yang–Baxter equation:

$$\mathcal{R}^f_{12}(u,v)\mathcal{R}^f_{13}(u,w)\mathcal{R}^f_{23}(v,w) = \mathcal{R}^f_{23}(v,w)\mathcal{R}^f_{13}(u,w)\mathcal{R}^f_{12}(u,v). \quad (41)$$

(iv) Regularity. If $R(u,v)$ is regular, say $R^{\alpha\beta}_{\gamma\delta}(v,v) = \delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma}$, then

$$\mathcal{R}^{f}_{jk}(v,v) = P_{jk}, \tag{42}$$

where P_{jk} is the graded permutation operator (18).

(v) Unitarity. If $R(u,v)$ is unitary, i.e.*, if*

$$R^{\alpha\beta}_{\gamma\delta}(u,v)R^{\delta\gamma}_{\alpha'\beta'}(v,u) = \delta^{\alpha}_{\beta'}\delta^{\beta}_{\alpha'}, \tag{43}$$

then $\mathcal{R}^{f}_{jk}(u,v)$ is unitary in the sense that

$$\mathcal{R}^{f}_{jk}(u,v)\mathcal{R}^{f}_{kj}(v,u) = \mathrm{id}\,. \tag{44}$$

4. SOLUTION OF THE QUANTUM INVERSE PROBLEM

We are now ready to formulate our main result, which is a formula that expresses the local projection matrices $e_{j}{}^{\beta}_{\alpha}$ for fundamental graded models in terms of the elements of the monodromy matrix. We shall assume we are given a solution of the Yang–Baxter equation (23) which is regular and unitary.

Theorem. *Let p be a grading that is compatible with the R matrix in the sense of Eq. (24), and let $\mathcal{T}(u)$ be the corresponding inhomogeneous monodromy matrix (29). Then, we have the following formula:*

$$e_{n}{}^{\beta}_{\alpha} = (-1)^{p(\alpha)p(\beta)}\prod_{j=1}^{n-1}\mathrm{str}(\mathcal{T}(\xi_j)) \tag{45}$$
$$\times\mathcal{T}^{\beta}_{\alpha}(\xi_n)\cdot\prod_{j=n+1}^{L}\mathrm{str}(\mathcal{T}(\xi_j)).$$

Equation (45) is our main result. It constitutes a solution of the quantum inverse problem for fundamental graded models. For $m=2$, $n=0$ ($p(1)=p(2)=0$), Eq. (45) reduces to a result recently obtained in [14]. Note that because of (31) no ordering is required for the products on the right-hand side of (45).

In the following we shall use the fermionic R operator introduced above in order to construct the shift operator for inhomogeneous graded models. We shall explore the properties of the shift operator and shall eventually use these properties to prove our main result (45).

As can be seen from (39) and (40), the fermionic R operator $\mathcal{R}^{f}_{jj+1}(\xi_j,\xi_{j+1})$ interchanges the two neighboring factors $\mathcal{L}_{j+1}(u,\xi_{j+1})$ and $\mathcal{L}_{j}(u,\xi_j)$ in the monodromy matrix. Since the symmetric group $\mathfrak{S}^L$ is generated by the transpositions of nearest neighbors, the L matrices on the right-hand side of (29) can be arbitrarily reordered by application of an appropriate product of fermionic R operators. This means that for every $\tau\in\mathfrak{S}^L$ there exists an operator $\mathcal{R}^{\tau}_{1\ldots L} = \mathcal{R}^{\tau}_{1\ldots L}(\xi_1,\ldots,\xi_L)$, which is a product of fermionic R operators and induces the action of the permutation $\tau\in\mathfrak{S}^L$ on the inhomogeneous monodromy matrix,

$$\mathcal{R}^{\tau}_{1\ldots L}\mathcal{T}_{1\ldots L}(u) = \mathcal{T}_{\tau(1)\ldots\tau(L)}(u)\mathcal{R}^{\tau}_{1\ldots L}. \tag{46}$$

The nongraded analog of this operator was introduced in [24]. An explicit expression for $\mathcal{R}^{\tau}_{1\ldots L}$ in the graded case was constructed in [15]. Here, we only need to consider the special case. where $\tau=\gamma$, the generator of the cyclic subgroup of order L of $\mathfrak{S}^L$, which is defined as

$$\gamma(j) = \begin{cases} j+1, & j=1,\ldots,L-1,\\ 1, & j=L. \end{cases} \tag{47}$$

It is easy to see that

$$\mathcal{R}^{\gamma}_{1\ldots L} = \mathcal{R}^{f}_{1L}\mathcal{R}^{f}_{1L-1}\ldots\mathcal{R}^{f}_{12} \tag{48}$$

induces a shift by one site on the inhomogeneous monodromy matrix,

$$\mathcal{R}^{\gamma}_{1\ldots L}\mathcal{T}_{1\ldots L}(u) = \mathcal{T}_{\gamma(1)\ldots\gamma(L)}(u)\mathcal{R}^{\gamma}_{1\ldots L}. \tag{49}$$

A cyclic shift of all indices by one in the latter equation leads to

$$\mathcal{R}^{\gamma}_{\gamma(1)\ldots\gamma(L)}\mathcal{T}_{\gamma(1)\ldots\gamma(L)}(u) \tag{50}$$
$$= \mathcal{T}_{\gamma^2(1)\ldots\gamma^2(L)}(u)\mathcal{R}^{\gamma}_{\gamma(1)\ldots\gamma(L)}.$$

It follows by multiplication by $\mathcal{R}^{\gamma}_{1\ldots L}$ from the right that

$$\mathcal{R}^{\gamma^2}_{1\ldots L} = \mathcal{R}^{\gamma}_{\gamma(1)\ldots\gamma(L)}\mathcal{R}^{\gamma}_{1\ldots L}. \tag{51}$$

Iterating the above steps, we arrive at the following lemma.

Lemma 3. *The operator*

$$\mathcal{R}^{\gamma^n}_{1\ldots L} = \mathcal{R}^{\gamma}_{\gamma^{n-1}(1)\ldots\gamma^{n-1}(L)} \tag{52}$$
$$\times\,\mathcal{R}^{\gamma}_{\gamma^{n-2}(1)\ldots\gamma^{n-2}(L)}\ldots\mathcal{R}^{\gamma}_{1\ldots L},$$

where

$$\mathcal{R}^{\gamma}_{\gamma^{p-1}(1)\ldots\gamma^{p-1}(L)} \tag{53}$$
$$= \mathcal{R}^{f}_{pp-1}\ldots\mathcal{R}^{f}_{p1}\mathcal{R}^{f}_{pL}\ldots\mathcal{R}^{f}_{pp+1},$$

$p=1,\ldots,n$, generates a shift by n sites on the inhomogeneous lattice, i.e.,

$$\mathcal{R}^{\gamma^n}_{1\ldots L}\,\mathcal{T}_{1\ldots L}(u) = \mathcal{T}_{n+1\ldots L1\ldots n}(u)\,\mathcal{R}^{\gamma^n}_{1\ldots L}. \tag{54}$$

Since $\gamma^L = \mathrm{id}$, we conclude from (54) that

$$\mathcal{R}^{\gamma^L}_{1\ldots L}\,\mathcal{T}_{1\ldots L}(u) = \mathcal{T}_{1\ldots L}(u)\,\mathcal{R}^{\gamma^L}_{1\ldots L}. \tag{55}$$

If $\mathcal{R}^f_{jk}$ is unitary, we have the following stronger result.

Lemma 4. *Let $\mathcal{R}^f_{jk}$ be unitary (cf. Eq. (44)). Then,*

$$\mathcal{R}^{\gamma^L}_{1\ldots L} = \mathrm{id}\,. \tag{56}$$

Our next lemma can be used to establish a connection between the inhomogeneous monodromy matrix (29) and the shift operator (52).

Lemma 5. *Let $X = X^\alpha_\beta e^\beta_\alpha \in \mathrm{End}(\mathbb{C}^{m+n})$, and let $R(u,v)$ be regular, say, $R^{\alpha\beta}_{\gamma\delta}(v,v) = \delta^\alpha_\delta \delta^\beta_\gamma$. Then,*

$$\begin{aligned}&\mathrm{str}(X\mathcal{T}_{n\ldots L1\ldots n-1}(\xi_n)) \qquad (57)\\ &= (-1)^{p(\alpha)+p(\alpha)p(\beta)} X^\alpha_\beta e_{n\alpha}{}^\beta\, \mathcal{R}^{\gamma}_{\gamma^{n-1}1\ldots\gamma^{n-1}L}.\end{aligned}$$

Proofs of Lemma 4 and Lemma 5 are given in [15]. Setting $X^\alpha_\beta = \delta^\alpha_\beta$ in (57) and using the cyclic invariance of the supertrace, we obtain the following corollary to Lemma 4.

Corollary.

$$\mathcal{R}^{\gamma}_{\gamma^{n-1}1\ldots\gamma^{n-1}L} = \mathrm{str}(\mathcal{T}_{1\ldots L}(\xi_n)). \tag{58}$$

Equation (58) is the inhomogeneous analog of Eq. (33).

Lemma 6. *We have the following expression for the shift operator in terms of the elements of the monodromy matrix,*

$$\mathcal{R}^{\gamma^n}_{1\ldots L} = \prod_{j=1}^{n} \mathrm{str}(\mathcal{T}_{1\ldots L}(\xi_j)). \tag{59}$$

If $R(u,v)$ is unitary (cf. Eq. (43)), then $\mathcal{R}^{\gamma^n}_{1\ldots L}$ is invertible with inverse

$$\left(\mathcal{R}^{\gamma^n}_{1\ldots L}\right)^{-1} = \prod_{j=n+1}^{L} \mathrm{str}(\mathcal{T}_{1\ldots L}(\xi_j)). \tag{60}$$

Proof 1. *The lemma follows from Lemma 2, Lemma 3, and Corollary to Lemma 4.*

We are now prepared to prove our main result, equation (45).

Proof 2. (Proof of Eq. (45)) *Using Lemma 2, Lemma 4, the Corollary to Lemma 4, and Lemma 5, we obtain*

$$\begin{aligned}&\mathrm{str}(X\mathcal{T}_{n\ldots L1\ldots n-1}(\xi_n)) \qquad (61)\\ &= \mathcal{R}^{\gamma^{n-1}}_{1\ldots L}\,\mathrm{str}(X\mathcal{T}_{1\ldots L}(\xi_n))\left(\mathcal{R}^{\gamma^{n-1}}_{1\ldots L}\right)^{-1}\\ &= \prod_{j=1}^{n-1}\mathrm{str}(\mathcal{T}_{1\ldots L}(\xi_j))\cdot\mathrm{str}(X\mathcal{T}_{1\ldots L}(\xi_n))\\ &\quad\times\prod_{j=n}^{L}\mathrm{str}(\mathcal{T}_{1\ldots L}(\xi_j))\\ &= (-1)^{p(\alpha)+p(\alpha)p(\beta)} X^\alpha_\beta e_{n\alpha}{}^\beta\,\mathrm{str}(\mathcal{T}_{1\ldots L}(\xi_n)).\end{aligned}$$

It follows that

$$\begin{aligned}&(-1)^{p(\alpha')+p(\alpha')p(\beta')} X^{\alpha'}_{\beta'} e_{n\alpha'}{}^{\beta'} \qquad (62)\\ &= \prod_{j=1}^{n-1}\mathrm{str}(\mathcal{T}_{1\ldots L}(\xi_j))\cdot\mathrm{str}(X\mathcal{T}_{1\ldots L}(\xi_n))\\ &\quad\times\prod_{j=n+1}^{L}\mathrm{str}(\mathcal{T}_{1\ldots L}(\xi_j)).\end{aligned}$$

Finally, by specifying $X^{\alpha'}_{\beta'} = (-1)^{p(\alpha')+p(\alpha')p(\beta')}\times\delta^{\alpha'}_\alpha\delta^\beta_{\beta'}$, we arrive at equation (45).

5. SUMMARY

In this contribution, we presented an explicit solution of the quantum inverse problem for fundamental graded models. Our main result is the general formula (45). This formula expresses the local projection operators $e_{n\alpha}{}^\beta$, which represent local spins and local fields, in terms of the elements of the monodromy matrix. Note that the formula and its proof [15] essentially simplify for translationally invariant models (all inhomogeneities coincide, $\xi_j = \xi$). In the translationally invariant case, the proof is based on the representation of the shift operator as a product of permutation matrices.

Meanwhile, attempts to generalize the solution of the quantum inverse problem into different directions have been published [25, 26]. The article [25] deals with fusion-type models, and the article [26], with models with open boundary conditions. It is our hope that the solution of the quantum inverse problem will prove to be useful for the calculation of correlation functions of integrable models.

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Sturm–Coulomb Problem in a Magnetic Field and Its Implications for Quantum Chaos*

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Abstract—The Sturm–Coulomb problem is an integrable one since its symmetry group is $O(4)$. When we apply to it a magnetic field, this symmetry is broken and reduced to the $O(2)$ group. The problem is then nonintegrable, but we can derive its matrix representation in a basis in which the Sturm–Coulomb problem alone is diagonal. We use this matrix representation to obtain the corresponding eigenvalues and their nearest neighbor spacing distribution. From the histogram of the latter, we discuss the presence or absence of quantum chaos as a function of the intensity H of the magnetic field and the angular momentum m in the direction of this field. © *2002 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

Quantum chaos has been a problem of great interest in physics for many years [1]. Among its main results is the fact that an explicitly integrable quantum-mechanical problem has a histogram associated with the distribution $P(s)$ of the eigenvalues for the value s of the nearest neighbor spacing that follows in general a Poisson distribution, i.e., a function of the type $\exp(-s)$ when normalized to unity.

On the other hand, if the quantum problem is nonintegrable, but can be expressed in terms of a real symmetric numerical matrix, the distribution $P(s)$ as a function of s sometimes follows a Wigner surmise [1], i.e., the histogram of $P(s) = (\pi/2)s\exp[-(\pi/4)s^2]$, as follows from the conjecture of Bohigas *et al.* [1] and as is indicated in Fig. 1.

A single-particle three-dimensional integrable central quantum problem is frequently associated with a symmetry group larger than $O(3)$, and we can make it nonintegrable if we add to it the effect of an external field that breaks the symmetry. An example is the hydrogen atom in a magnetic field, which was extensively studied [2]. The disadvantage of this example is that it has both a discrete but compressed energy spectrum and a continuous one that are associated with different symmetry Lie algebras—i.e., $o(4)$ for the first and $o(3,1)$ for the second.

It seems of interest to have a simpler example that would keep the degeneracies of the Coulomb problem but which would have only a discrete equidistant spectrum and a single symmetry group $O(4)$. This is the well-known Sturm–Coulomb problem [3], and, if we apply to it an external magnetic field, this requires only a minimal substitution in the momentum; that is,

$$\mathbf{p} \to \mathbf{p} - \mathbf{A}, \quad \mathbf{A} = \frac{1}{2}[\mathbf{H} \times \mathbf{r}], \tag{1}$$

where $\mathbf{H}$ is the strength of the magnetic field. In (1) and everywhere below, we shall use atomic units in which

$$\hbar = m = e = 1 \tag{2}$$

with m and e being the mass and the charge of the particle.

In the next sections, we derive the Hamiltonian of the Sturm–Coulomb problem from that of the hydrogen atom and also discuss its classical orbits, quantum eigenvalues, and eigenstates. In Section 5, we introduce a magnetic field in the Sturm–Coulomb problem through the minimal substitution (1) and obtain the matrix representation of the new Hamiltonian, while the statistical properties of its eigenvalues are discussed in Section 6.

2. THE STURM–COULOMB PROBLEM

In units specified in (2), the Hamiltonian of the hydrogen atom can be written as

$$\mathcal{H}' = \frac{1}{2}p'^2 - \frac{1}{r'}, \tag{3}$$

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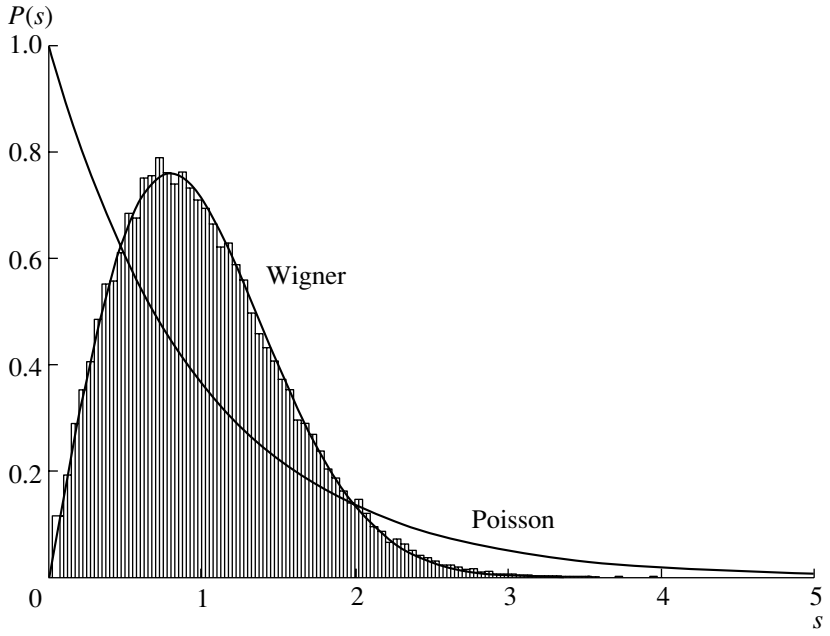

Fig. 1. Example of a distribution $P(s)$ as a function of a nearest neighbor spacing s for the random Gaussian matrix model, approximated by the Wigner surmise $P(s) = (\pi/2)s\exp[-(\pi/4)s^2]$. The Poisson distribution $P(s) = \exp(-s)$ is also drawn. The area under the histograms is normalized to unity.

where all observables here will carry a prime, so as to allow later unprimed letters for the variables we want to use.

The Schrödinger equation for the discrete spectrum of this problem has the form

$$\left(\frac{1}{2}p'^2 - \frac{1}{r'}\right)\psi' = -\frac{1}{2\nu^2}\psi', \quad \nu = 1, 2, \ldots, \quad (4)$$

where ν is the total quantum number taking the integer values indicated in (4).

If we now perform the canonical transformation

$$\mathbf{p} = \frac{\nu}{2}\mathbf{p}', \quad \mathbf{r} = \frac{2\mathbf{r}'}{\nu}, \quad (5)$$

Eq. (4) becomes the Sturm–Coulomb one [3],

$$\left[r\left(p^2 + \frac{1}{4}\right)\right]\psi' = \nu\psi'. \quad (6)$$

While the Sturm–Coulomb and Coulomb problems are related by the simple canonical transformation (5), they are totally different problems, as can be seen from their classical and quantum behavior discussed in the next section.

3. THE CLASSICAL ORBITS FOR THE STURM–COULOMB HAMILTONIAN

Considering now the Hamiltonian

$$\mathcal{H} = r[p^2 + 1/4], \quad (7)$$

we note first of all that it admits the orbital-angular-momentum integral of the motion

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \quad (8)$$

Classically, $\mathbf{L}$ is a vector fixed in space, so that the motion of the particle is restricted to the plane perpendicular to this vector. Considering the polar coordinates r and φ in this plane and the corresponding momenta p_r and p_φ, we can write the Hamiltonian as

$$\mathcal{H} = r\left(p_r^2 + \frac{p_\varphi^2}{r^2} + \frac{1}{4}\right), \quad (9)$$

where the magnitude of the angular momentum p_φ is an integral of the motion.

The Hamilton's equations of motion give the results

$$\dot{r} = \frac{\partial\mathcal{H}}{\partial p_r}, \quad \dot{\varphi} = \frac{\partial\mathcal{H}}{\partial p_\varphi}, \quad (10)$$

where a dot implies a derivative with respect to time. From (9), we then get

$$\dot{r} = 2rp_r, \quad \dot{\varphi} = \frac{2p_\varphi}{r}. \quad (11)$$

From the first equation in (11), we obtain $p_r = (\dot{r}/2r)$ and substitute it into (9). Furthermore, since p_φ is an integral of the motion, we replace it by a constant value denoted by $p_\varphi = m$; $\mathcal{H}$ is also an integral of the motion corresponding to the energy—we will

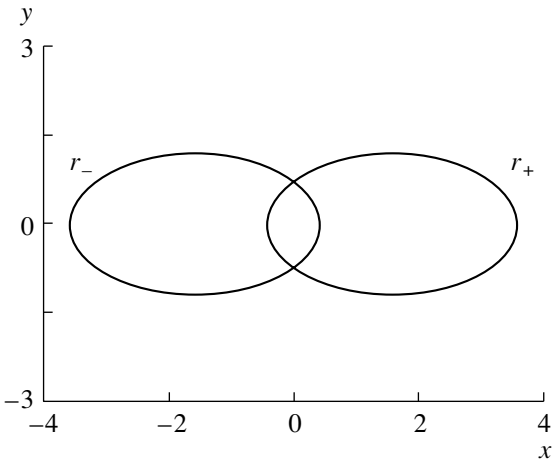

Fig. 2. Classical trajectories for the Sturm–Coulomb problem in the xy plane, where the equation in polar coordinates is $q/r = (1 \pm e\cos\varphi)$ with $q = 2m^2/E$ and $e = [1 - (m^2/E^2)]^{1/2}$; here, $E = 1$ and $m = 0.6$.

denote it by E. We then get from (9) and (11) the equations

$$\frac{\dot r^2}{4r} + \frac{m^2}{r} + \frac{1}{4}r = E, \quad \dot\varphi = \frac{2m}{r}, \tag{12}$$

whence it follows that

$$\frac{dr}{dt} = \sqrt{4r\left(E - \frac{m^2}{r} - \frac{1}{4}r\right)}, \quad \frac{d\varphi}{dt} = \frac{2m}{r}; \tag{13}$$

replacing dt in the first of Eqs. (13) by $dt = (r/2m)d\varphi$, as follows from the second equation in (12), we get

$$\frac{dr}{d\varphi} = \frac{r}{2m}\sqrt{-4m^2 + 4rE - r^2}. \tag{14}$$

Performing integration, we obtain

$$\int_{\varphi_0}^{\varphi} d\varphi = \varphi - \varphi_0 \tag{15}$$

$$= \int \frac{dr}{r\sqrt{-1 + r(E/m^2) - (1/4m^2)r^2}}$$

$$= \arctan \frac{-2 + Er/m^2}{2\sqrt{-1 + r(E/m^2) - (1/4m^2)r^2}},$$

where we used formula (2.266) on page 84 of [4] with $a < 0$.

Taking the tangent of both sides, we have

$$\tan(\varphi - \varphi_0) = \frac{-2 + Er/m^2}{2\sqrt{-1 + r(E/m^2) - (1/4m^2)r^2}}. \tag{16}$$

Upon squaring this expression and rearranging it, we get

$$4\left[1 + \tan^2(\varphi - \varphi_0)\right] \tag{17}$$

$$- \frac{4rE}{m^2}\left[1 + \tan^2(\varphi - \varphi_0)\right]$$

$$+ \left[\frac{1}{m^2}\tan^2(\varphi - \varphi_0) + \frac{E^2}{m^4}\right]r^2 = 0.$$

Multiplying Eq. (17) by $\cos^2(\varphi - \varphi_0)$, using the relation $\sin^2(\varphi - \varphi_0) = 1 - \cos^2(\varphi - \varphi_0)$, and denoting

$$\rho \equiv \frac{1}{r}, \tag{18}$$

we obtain the equation

$$4\rho^2 - \frac{4E}{m^2}\rho \tag{19}$$

$$+ \left[\frac{1}{m^2} + \left(\frac{E^2}{m^4} - \frac{1}{m^2}\right)\cos^2(\varphi - \varphi_0)\right] = 0,$$

whence we get the value of ρ as a function of φ as

$$\rho = \frac{E}{2m^2} \pm \frac{1}{2}\left(\frac{E^2}{m^4} - \frac{1}{m^2}\right)^{1/2}\sin(\varphi - \varphi_0). \tag{20}$$

Selecting $\varphi_0 = -(\pi/2)$ and using (18), we finally obtain

$$\frac{q}{r} = 1 \pm e\cos\varphi, \tag{21}$$

where

$$q = \frac{2m^2}{E}, \quad e = \left(1 - \frac{m^2}{E^2}\right)^{1/2}, \tag{22}$$

which is the equation of an ellipse with q and e being, respectively, its parameter and eccentricity. The major and the minor semiaxis of the ellipse, a and b, are given by

$$a = q/(1 - e^2), \quad b = q/\sqrt{1 - e^2}. \tag{23}$$

We have, in fact, two ellipses corresponding to the $\pm$ signs in (21). This is due to a further degeneracy of the Hamiltonian in (7) under reflection at the origin, i.e., $\mathbf{r} \to -\mathbf{r}$ and $\mathbf{p} \to -\mathbf{p}$. In Fig. 2, we give the two ellipses for $E = 1$ and $m = 0.6$ in the $x = r\cos\varphi$, $y = r\sin\varphi$ plane.

Incidentally for the Coulomb problem [5], the parameter and eccentricity for elliptical orbits, corresponding to $E < 0$, are given by

$$q = m^2, \quad e = (1 - 2Em^2)^{1/2}; \tag{24}$$

for $E = 0$ and $E > 0$, we have, respectively, parabolic and hyperbolic orbits [5], which are not present in the Sturm–Coulomb problem.

4. THE SYMMETRY GROUPS OF THE STURM–COULOMB HAMILTONIAN AND ITS EIGENVALUES AND EIGENSTATES

The Hamiltonian in (7), considered as a quantum-mechanical operator, is obviously invariant under rotations and thus commutes with the angular momentum vector $\mathbf{L}$ of (8).

It is well known that, apart from $\mathbf{L}, \mathcal{H}$ also commutes with the Runge–Lenz vector, which, for the form of (7), is expressed as [6]

$$\mathbf{A} = -\mathbf{r}\left(p^2 - \frac{1}{4}\right) + 2\mathbf{p}(\mathbf{r}\cdot\mathbf{p}). \tag{25}$$

Thus, in the component form, we have

$$[L_i, \mathcal{H}] = 0, \quad [A_i, \mathcal{H}] = 0, \quad i = 1, 2, 3. \tag{26}$$

Since L_i and $A_i, i = 1, 2, 3$, are the generators of a Lie algebra,

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \quad [L_i, A_i] = i\epsilon_{ijk}A_k, \quad [A_i, A_j] = i\epsilon_{ijk}L_k, \tag{27}$$

associated with the $O(4)$ group, we have that the latter is the symmetry group of $\mathcal{H}$.

The eigenstates of the Hamiltonian in (7) can then be characterized by the following chain of groups

$$\begin{array}{ccccc} O(4) & \supset & O(3) & \supset & O(2), \\ \nu & & l & & m \end{array} \tag{28}$$

where, below each one of them, we give the integer quantum numbers associated with their irreducible representations (irreps) subject to the inequalities

$$0 \le l < \nu, \quad -l \le m \le l. \tag{29}$$

If instead of the total quantum number ν, we introduce the radial one n through the relation

$$\nu = n + l + 1; \tag{30}$$

then, Eq. (6) has the well-known solutions [3]

$$\psi' = R_{nl}(r)Y_{lm}(\theta, \varphi), \tag{31}$$

where Y_{lm} is a spherical harmonic,

$$R_{nl}(r) = B_{nl}r^l e^{-r/2}L_n^{2l+1}(r) \tag{32}$$

with

$$B_{nl} = \left[\frac{n!}{(n+2l+1)!}\right]^{1/2}, \tag{33}$$

and $L_n^{2l+1}(r)$ is an associated Laguerre polynomial [4].

The operator in the square brackets on the left-hand side of (6) is obviously non-Hermitian since r and p^2 do not commute, but we can make it Hermitian by introducing the definition

$$\psi' = \sqrt{r}\psi, \tag{34}$$

which converts (6) into

$$\left[\sqrt{r}\left(p^2 + \frac{1}{4}\right)\sqrt{r}\right]\psi = (n + l + 1)\psi. \tag{35}$$

The wave function ψ can then be denoted by the ket$|nlm\rangle$; from (31), we then obtain

$$|nlm\rangle = r^{-1/2}R_{nl}(r)Y_{lm}(\theta, \varphi). \tag{36}$$

The spectrum of this equation is clearly discrete and equidistant but with the same degeneracy as that of the Coulomb problem, as is seen in Fig. 3.

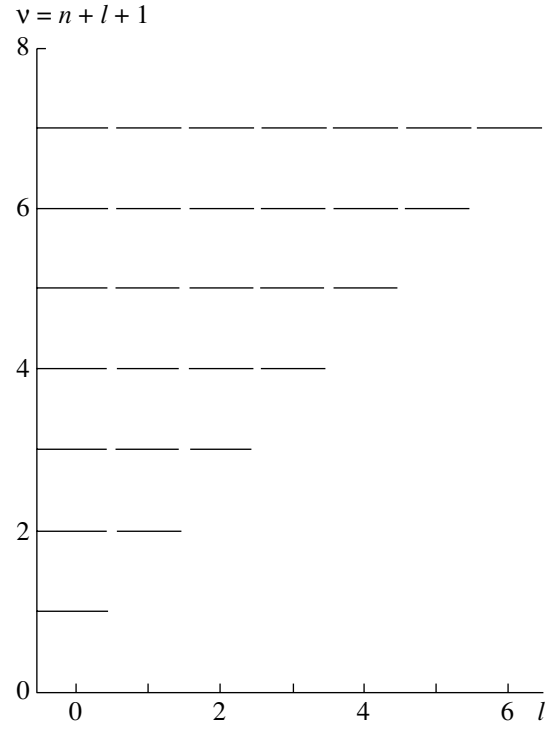

Fig. 3. Energy levels of the quantum Sturm–Coulomb problem. Here, we indicate the orbital angular momentum l along the abscissa and the total quantum number $\nu = n + l + 1$ along the ordinate.

In the next section, we shall consider the change of the operator in the square brackets in (35) when we introduce a constant magnetic field.

5. THE STURM–COULOMB PROBLEM IN A MAGNETIC FIELD

Making the minimal substitution (1) in the operator in the square brackets on the left-hand side of (35), we get a Hamiltonian (to be denoted by $\mathcal{H}$) of the form

$$\mathcal{H} = \sqrt{r}\left\{\left[\mathbf{p} - \frac{1}{2}(\mathbf{H}\times\mathbf{r})\right]^2 + \frac{1}{4}\right\}\sqrt{r} \tag{37}$$

$$= \sqrt{r}\left\{p^2 - \mathbf{H}\cdot\mathbf{L} + \frac{1}{4}[H^2r^2 - (\mathbf{H}\cdot\mathbf{r})^2] + \frac{1}{4}\right\}\sqrt{(}r)$$

with $\mathbf{L}$ being the orbital angular momentum, $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, and $\mathbf{r}$ and $\mathbf{p}$ having the components $\mathbf{r} = (x_1, x_2, x_3)$ and $\mathbf{p} = (p_1, p_2, p_3)$.

Without any loss of generality, we can take x_3 as the direction of the magnetic field and express $\mathcal{H}$ as

$$\mathcal{H} = \sqrt{r}\left\{p^2 - HL_3 + \frac{H^2}{4}(x_1^2 + x_2^2) + \frac{1}{4}\right\}\sqrt{r} \quad (38)$$

$$= \sqrt{r}(p^2 + \frac{1}{4})\sqrt{r} - HrL_0 + (H^2/4)r^3\sin^2\theta,$$

where, in the last line, we have used spherical coordinates in which

$$L_0 = \frac{1}{i}\frac{\partial}{\partial\varphi}. \quad (39)$$

The presence of a magnetic field breaks the $O(4)$ symmetry of the Sturm–Coulomb problem and leaves, in (38), only one integral of motion, L_0, clearly commuting with $\mathcal{H}$. The problem now is nonintegrable, and the only way to discuss its energy levels is to express it in a matrix form with respect to an appropriate basis. We choose the latter as given by the ket $|nlm\rangle$ of (36); thus, we must calculate the matrix elements

$$\langle n'l'm|\mathcal{H}|nlm\rangle = (n + l + 1)\delta_{n'n}\delta_{l'l} \quad (40)$$
$$- Hm\langle n'l'm|r|nlm\rangle$$
$$+ (H^2/4)\langle n'l'm|r^3\sin^2\theta|nlm\rangle.$$

The matrix elements in (40) involve radial integrals of the form

$$b(n'l', nl, k) \equiv \int_0^\infty R_{n'l'}(r)R_{nl}(r)r^k dr, \quad (41)$$

where k is some integer involving the powers of r in (40), as well as the radial volume element r^2dr. These integrals were calculated in [7] by using the generating functions of associated Laguerre polynomials.

For the angular part, we note from page 52 of [8]

$$\sin^2\theta = \frac{2}{3} - \sqrt{\frac{16\pi}{45}}Y_{20}(\theta,\varphi) \quad (42)$$

and, thus, the integral

$$\int_0^\pi\int_0^{2\pi} Y^*_{l'm}(\theta,\varphi)\sin^2\theta Y_{lm}(\theta,\varphi)\sin\theta d\theta d\varphi \quad (43)$$

$$= \frac{2}{3}\delta_{l'l} - \sqrt{\frac{16\pi}{45}}$$
$$\times\int_0^\pi\int_0^{2\pi} Y^*_{l'm}(\theta,\varphi)Y_{20}(\theta,\varphi)Y_{lm}(\theta,\varphi)\sin\theta d\theta d\varphi$$
$$= \frac{2}{3}\left[\delta_{l'l} - \sqrt{\frac{2l+1}{2l'+1}}\langle lm, 20|l'm\rangle\langle l0, 20|l'0\rangle\right],$$

where the last line involves the Clebsch–Gordan coefficients appearing in formula (4.34) on page 62 of [9].

Using the explicit expression for these Clebsch–Gordan coefficients on page 77 of [8], we finally arrive at

$$\int_0^\pi\int_0^{2\pi} Y^*_{l'm}(\theta,\varphi)\sin^2\theta Y_{lm}(\theta,\varphi)\sin\theta d\theta d\varphi \quad (44)$$
$$= -\delta_{l'l+2}u(lm) + \delta_{l'l}v(lm) - \delta_{l'l-2}w(lm),$$

where

$$u(lm) = \frac{1}{2l+3}\sqrt{\frac{(l-m+2)(l-m+1)(l+m+2)(l+m+1)}{(2l+5)(2l+1)}}, \quad (45)$$

$$v(lm) = \frac{2}{3}\left\{1 + \frac{3m^2 - l(l+1)}{(2l-1)(2l+3)}\right\},$$

$$w(lm) = \frac{1}{2l-1}\sqrt{\frac{(l-m)(l-m-1)(l+m)(l+m-1)}{(2l+1)(2l-3)}}.$$

Combining this result with the radial part, we see that the matrix element (40) becomes

$$\langle n'l'm|\mathcal{H}|nlm\rangle = (n + l + 1)\delta_{n'n}\delta_{l'l} - Hm\delta_{l'l}b(n'l, nl, 2) \quad (46)$$
$$+ (H^2/4)[-\delta_{l'l+2}b(n'l+2, nl, 4)u(lm)\delta_{l'l}b(n'l, nl, 4)v(lm) - \delta_{l'l-2}b(n'l-2, nl, 4)w(lm)],$$

where, from the analysis in the Appendix of [7], we have

$$b(n'l', nl, k) = \sqrt{\frac{n'!n!}{(n'+2l'+1)!(n+2l+1)!}}(l - l' + k - 1)!(l' - l + k - 1)!(-1)^{n+n'} \quad (47)$$

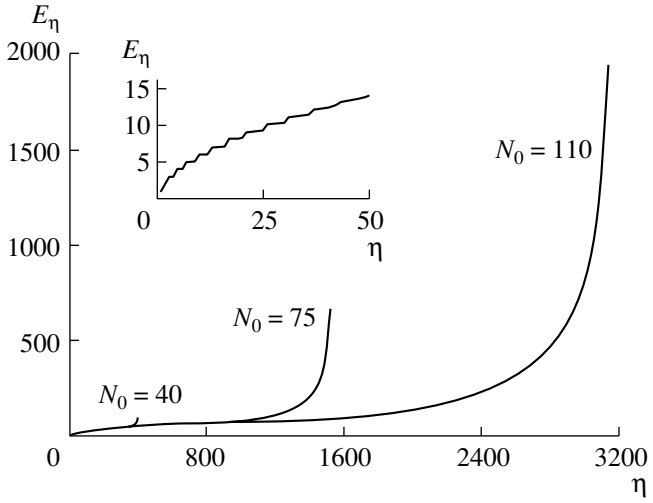

Fig. 4. The curves of the eigenvalues of the matrix whose elements are given by (46). See the text for comments. In the upper left-hand corner, we show a magnification of the curve for low values of η, where the step-function behavior appears owing to the ordering of the levels in Fig. 3, since the contribution of the magnetic field is very small.

$$\times \sum_s \frac{(l+l'+k+s)!}{(l-l'+k-1-n'+s)!(n'-s)!(l'-l+k-1-n+s)!(n-s)!s!}$$

and summation is restricted to a finite number of values s for which the arguments of all the factorials in the denominator of (48) take nonnegative values.

In the next section, we discuss the possible applications of the results obtained.

6. PRESENCE OR ABSENCE OF QUANTUM CHAOS AS A FUNCTION OF H AND m

The quantum Sturm–Coulomb problem in a magnetic field is a nonintegrable problem, and we know that this can lead to spectral statistics from which, on occasions, chaotic behavior can be inferred in the corresponding classical regime [1].

Before we study these spectral statistics, we need first to enumerate appropriately all the states $|nlm\rangle$ of (36). Since m is fixed, we only need to consider n and l. Defining

$$N = n + l, \tag{48}$$

we can enumerate it first in increasing order of N, i.e., $N = 0, 1, 2, \ldots$. At each value of N, we can order (n, l) as follows:

$$\begin{aligned} l = 0, \quad n = N; \quad l = 1, \\ n = N-1; \quad \ldots; \quad l = N, \quad n = 0. \end{aligned} \tag{49}$$

Thus, if we consider states up to a value N_0 of N, we have a maximum of $[(1/2)(N_0+1)(N_0+2)]$ of them and thus at most a $[(1/2)(N_0+1)(N_0+2) \times (1/2)(N_0+1)(N_0+2)]$ matrix whose elements are given by (46). Actually, the matrix will be smaller because there is a selection rule in (46) that limits l to either even or odd values and besides $l \geq m$.

Diagonalizing the matrix mentioned and enumerating the eigenvalues in increasing order, we get a curve in the plane where we indicate the order of the eigenvalue along the abscissa and the corresponding energy along the ordinate. This curve has to suffer the process of "unfolding" [1], and, after selecting an appropriate interval of energy, we obtain the nearest neighbor spacing [1] and construct the histogram of different intervals as a function of their energy. If the histogram can be approximated by a Poisson distribution or by the Wigner surmise, it would imply, respectively, the absence of quantum chaos for the particular strength H of the magnetic field and the value m of the angular momentum in the directions of the field that we assume.

We now proceed to discuss the results of our calculations. First, in Fig. 4, we indicate the curves of the eigenvalues of the matrix whose elements are given by (46). On the abscissa, we denote by $\eta = 1, 2, \ldots, 3136$ the eigenvalues in increasing order of their energies E_η that appear on the ordinate. We take $H = 0.01$ and $m = 0$, while, for N_0, we have the curves corresponding to $N_0 = 40, 75, 110$. Note the very good convergence of the eigenvalues in the appropriate ranges; for example, for $N_0 = 40$ and the values $1 \leq \eta \leq 350$, the curve coincides with those for $N_0 = 75$ or 110.

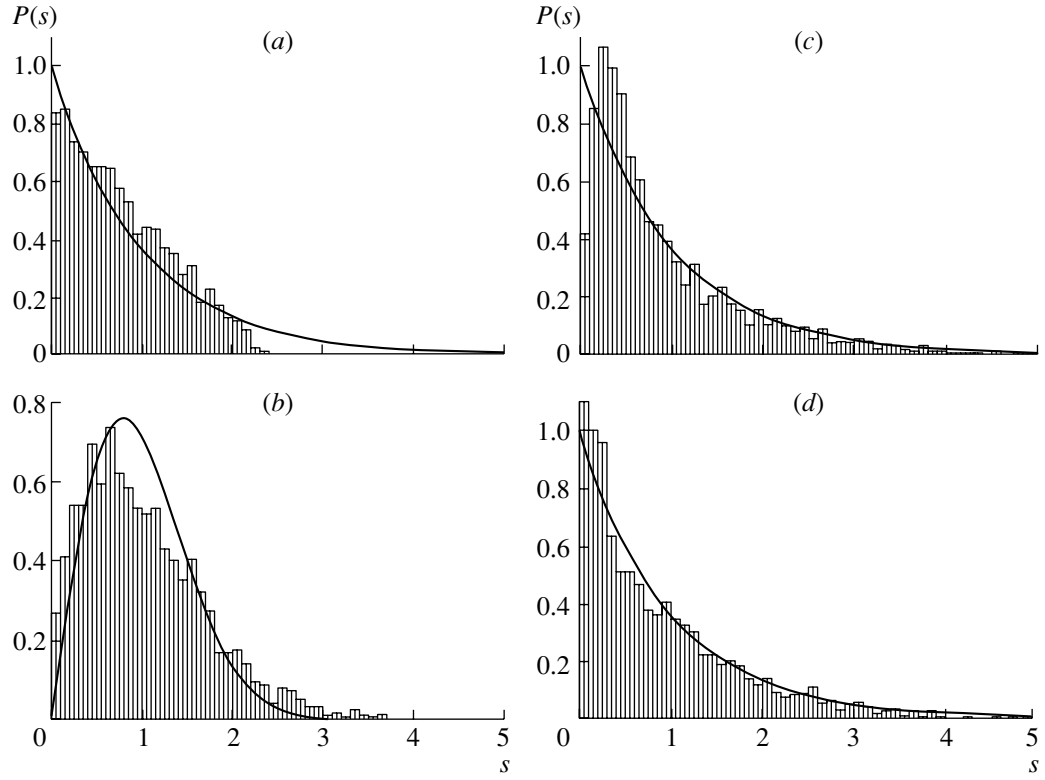

Fig. 5. The distribution $P(s)$ for $m = 0$ and $H = 0.001$ (a), 0.01 (b), 0.1 (c), and 1 (d); $100 \leq \eta \leq 2000$.

The very low magnetic-field value of $H = 0.01$ is due to the fact that the term associated with $(H^2/4)$ in (46) increases very quickly with l and m. In the upper left-hand corner of Fig. 4, we show a magnification of the curve for low values of η, where a step-function behavior appears owing to the ordering of the levels in Fig. 3, since the contribution of the magnetic field can be disregarded.

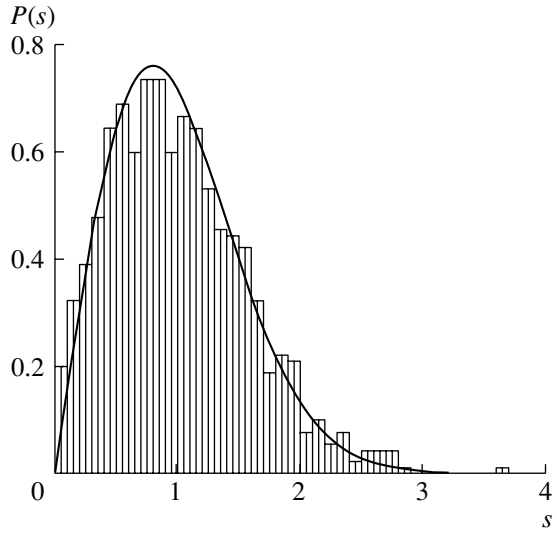

Fig. 6. The distribution $P(s)$ for $m = 5$ and $H = 0.0125$; $100 \leq \eta \leq 1000$.

Figure 5 has four components denoted by the letters a, b, c, and d. The total number of states considered was 3136, which corresponds to $N_0 = 110$, and we selected only those in the range $100 \leq \eta \leq 2000$. The curves were "unfolded" [1], and their nearest neighbor spacing was determined and given by s on the abscissa. With intervals of unfolded energy of width 0.05, the distribution $P(s)$ is given on the ordinate, whereby we obtained the histograms of Fig. 5, in all of which $m = 0$. In Figs. 5a, 5b, 5c, 5d, the values of H are, respectively, 0.001, 0.01, 0.1, and 1, and we see that only $H = 0.01$ gives a histogram resembling the Wigner surmise, while all others are approximated more by a Poisson distribution. The areas under all the histograms are normalized to unity.

We also performed a calculation for $m = 5$ and the same range of values of H, but the histograms look very similar. We just present a particular case where we still have 3136 states but $m = 5$ and $H = 0.0125$, and we take $100 \leq \eta \leq 1000$ (Fig. 6). This particular case is good for showing a Wigner surmise distribution, as is seen in Fig. 6. Compared with results in Fig. 5b, it shows the energy dependence of statistical fluctuations of the spectra.

As a last point, we shall consider what happens when m is large. Since $l \geq m$, it will also be large,

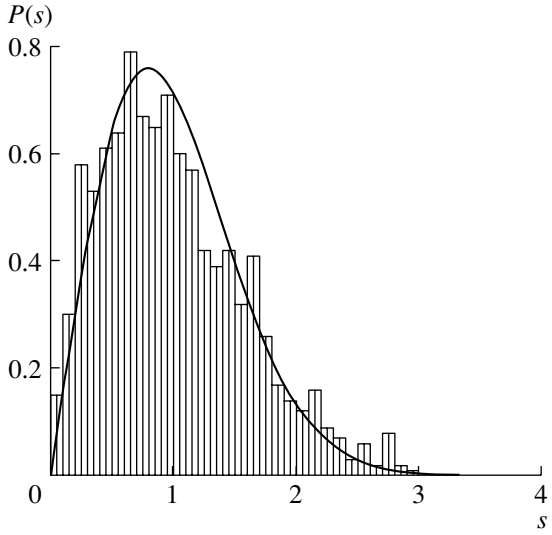

Fig. 7. The distribution $P(s)$ for $m = 350$ and $H = 0.00325$; $N_0 = 460$ and $400 \leq \eta \leq 1400$.

and this increases considerably the terms multiplied by both H and $(H^2/4)$ in (46). To get then a strong repulsion between the levels of the Sturm–Coulomb problem, we need values of H much smaller than those that give the Wigner surmise for $m = 0$. This is illustrated in Fig. 7, where $m = 350$ and $H = 0.00325$.

7. CONCLUSION

The Sturm–Coulomb problem is a very simple example of an integrable one owing to its symmetry group $O(4)$. In the presence of a magnetic field, this symmetry is broken and only the $O(2)$ group remains. What interests us most is the strength of the magnetic field H and the projection m of the angular momentum onto the field direction for which the histogram of $P(s)$ as a function of s behaves close to the Wigner surmise. In this case, a quantum chaotical behavior is expected, and we showed, in the last section, several examples for values of H and m at which it appears.

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Some Exact Results for the Three-Layer Zamolodchikov Model*

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Abstract—In this talk, we present our recent results on the three-layer Zamolodchikov model. We discuss solutions to the Bethe ansatz equations following from functional relations. We consider two regimes I and II that differ by the signs of the spherical sides $(a_1, a_2, a_3) \to (-a_1, -a_2, -a_3)$. Also, we accept the two-line hypothesis for regime I and the one-line hypothesis for regime II. In the thermodynamic limit, we derive integral equations for distribution densities and solve them exactly. Using this solution, we calculate the partition function for the three-layer Zamolodchikov model and check the compatibility of this result with functional relations. We also discuss the reasons for the discrepancy with Baxter's result of 1986.

1. INTRODUCTION

This talk is devoted to our recent results on the three-layer Zamolodchikov model [1–3]. However, we first make a little introduction into the subject of integrable three-dimensional lattice models and formulate our main purposes. We shall consider only the simplest generalization of the two-dimensional lattice models to the three-dimensional case, i.e., the models living on a cubic lattice.

The local Boltzmann weights for the majority of the two-dimensional models depend on one spectral parameter. Geometrically, one can expect that the local Boltzmann weights for a three-dimensional model should depend on three spectral parameters or dihedral angles θ_1, θ_2, and θ_3. Of course, the Boltzmann weights also depend on the spin variables localized at the sites of a lattice.

The partition function is defined as

$$Z = \sum_{\text{all spins}} \prod_{\text{all weights}} W.$$

Usually, we are interested in calculating the free energy or the partition function per site for very large lattices,

$$\kappa = Z^{1/\mathcal{N}},$$
$$\ln(\kappa) = \kappa_0 + (1/\mathcal{N})\kappa_1 + (1/\mathcal{N}^2)\kappa_2 + \ldots,$$

where $\mathcal{N} \gg 1$ is the number of sites in the lattice being considered.

The row-to-row transfer matrix $T(u)$ for a two-dimensional model can be generalized to the three-dimensional case, but it will now depend on three parameters, $T(\theta_1, \theta_2, \theta_3)$.

In terms of the transfer matrix, the partition function has the form

$$Z = \operatorname{tr} T^M,$$

where M is the number of sites of the lattice in the vertical direction.

Now, if we want to have an integrable model, it is necessary that the Boltzmann weights satisfy the local integrability condition. In the two-dimensional case, this condition is the well-known Yang–Baxter equation (YBE) (see, for example, [4]); in three dimensions, there is a natural generalization, namely, a tetrahedron equation (TE) [5, 6].

These conditions ensure the commutativity of the transfer matrices in both two- and three-dimensional cases:

$$[T(u), T(u')] = 0 \to [T(\theta_1, \theta_2, \theta_3), T(\theta_1, \theta_2', \theta_3')] = 0.$$

There are many rather powerful methods developed for investigating two-dimensional models. Our main question we address here is the following: Can we apply the Bethe ansatz method, which is very well known from two dimensions, to the three-dimensional case?

A huge number of solutions to the YBE are known. But what is known about solutions to the TE? Some examples of such solutions are following:

The *two-state Zamolodchikov model (Z)* found by Zamolodchikov in 1980 [7, 8].

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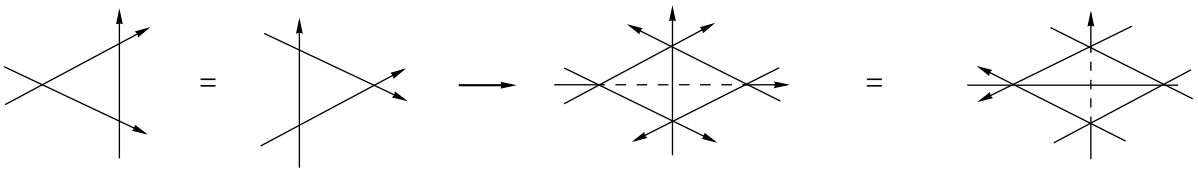


Yang–Baxter equation

$$\sum W(u)W(u+v)W(v) = \sum W(v)W(u+v)W(u)$$

Tetrahedron equation

$$\sum W(\theta_1, \theta_2, \theta_3)\, W(\theta_1', \theta_2, \theta_3')\, W(\theta_1'', \pi-\theta_3, \theta_3')\; W(\theta_1'', \theta_1, \pi-\theta_1') = \sum W(\theta_1'', \theta_1, \pi-\theta_1')W(\theta_1'', \pi-\theta_3, \theta_3')W(\theta_1', \theta_2, \theta_3')W(\theta_1, \theta_2, \theta_3)$$

Fig. 1.

The *N-state Baxter–Bazhanov model (BB),* Baxter and Bazhanov, 1992 [9].

The *elliptic generalizations of Baxter–Bazhanov model*:

the *checkerboard model* due to Mangazeev and Stroganov, 1994 [10].

The *more general elliptic model* due to Boos, Mangazeev, and Sergeev, 1995 [11]; Boos, 1996 [12].

In Baxter's statistical formulation, the Boltzmann weights of the Zamolodchikov model depend on eight spins placed at the cube vertices and three angles θ_1, θ_2, and θ_3 of a spherical triangle or the spherical sides a_1, a_2, and a_3:

$$t_\mu = [\tan(\alpha_\mu/2)]^{1/2}, \quad s_\mu = [\sin(\alpha_\mu/2)]^{1/2},$$
$$c_\mu = [\cos(\alpha_\mu/2)]^{1/2},$$

where $\mu = 0, 1, 2, 3$ and spherical excesses α_μ are defined by

$$\alpha_0 = \theta_1 + \theta_2 + \theta_3 - \pi, \quad \alpha_i = \theta_i - \alpha_0.$$

In fact, any solution to the TE produces an infinite number of solutions to the YBE through the flaky structure of Boltzmann weights. In particular, one has the following equivalence [9]:

$sl(n)$-chiral Potts model [13, 14] at $q^{2N} = 1 \Longleftrightarrow$ n-layer N-state BB model.

Here, we mean "weak" equivalence, i.e., equivalence apart from boundary conditions.

In the further consideration, we need some properties of the Boltzmann weight W:

tetrahedron equation;

invariance under the cube symmetry group Γ,

$$W(\gamma\{a|efg|bcd|h\})(\theta_1^\gamma, \theta_2^\gamma, \theta_3^\gamma) = W(a|efg|bcd|h)(\theta_1, \theta_2, \theta_3),$$

where γ is any element of group Γ.

Let us now briefly discuss Baxter's symmetry method [15] based on the so-called "body-centered-cube" (BCC) structure of the Boltzmann weight, namely,

$$W(a|efg|bcd|h) = \xi \frac{G_{afbg}(\theta_1)}{G_{edhc}(\theta_1)} \frac{G'_{bhdf}(\theta_2)}{G'_{gcea}(\theta_2)} \frac{G''_{edfa}(\theta_3)}{G''_{chbd}(\theta_3)} \times \sum_{\sigma=\pm1} \phi_{\sigma,afch} e^{\sigma(K_1 ag + K_2 bf + K_3 dh + K_4 ce)},$$

where the sum is taken over two values $+1$ and -1 of the additional spin σ situated at a cube center and

$$\phi_{a,b} = -1 \quad \text{if} \quad a = b = -1, \quad \phi_{a,b} = 1 \quad \text{otherwise};$$

$$v_i = \tanh(2K_i),$$

$$v_1 = -zT_1T_2, \quad v_2 = -izT_2/T_1,$$
$$v_3 = -z^{-1}T_1T_2, \quad v_4 = iz^{-1}T_2/T_1$$

with $z = e^{ia_3/2}$, $T_i = [\tan(\theta_i/2)]^{1/2}$, and $\xi = \sqrt{\sin(\theta_3/2)}/(2c_0c_1c_2c_3)$.

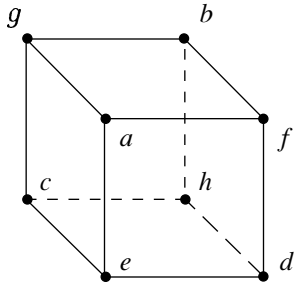
Using the symmetries of the Boltzmann weight, the BCC form, and a modification of boundary conditions, Baxter found the partition function per site for an infinite lattice in all three directions, $\infty \times \infty \times \infty$,

$$\kappa = 2f(v_1)f(-v_2^{-1})f(v_3^{-1})f(-v_4),$$
$$\ln(\kappa/2) = J(\zeta_1) + J(\zeta_2) + J(\zeta_3) + J(\zeta_4),$$

$$J(\zeta) = \ln\left[f(-ie^{-i\zeta})\right] = \frac{1}{2\pi}\int_0^\zeta (\ln(2\cos x) + x\tan x)dx$$

and

$$e^{i\zeta_1} = -i/v_1, \quad e^{i\zeta_2} = iv_2,$$
$$e^{i\zeta_3} = -iv_3, \quad e^{i\zeta_4} = i/v_4.$$



$= W(a\,|\,e, f, g\,|\,b, c, d\,|\,h;\ \theta_1, \theta_2, \theta_3)$
each spin variable = ±1

abeh	*acfh*	*adgh*	$W(a\|efg\|bcd\|h)$
+	+	+	$1 - abcdt_0t_1t_2t_3$
−	+	+	$s_1/(c_0c_2c_3)$
+	−	+	$s_2/(c_0c_1c_3)$
+	+	−	$s_3/(c_0c_1c_2)$
+	−	−	$abt_2t_3 + cdt_0t_1$
−	+	−	$act_1t_3 + bdt_0t_2$
−	−	+	$adt_1t_2 + bct_0t_3$
−	−	−	$s_0/(c_1c_2c_3)$

Fig. 2.

Baxter also obtained a generalization of this result to the case of a lattice that is infinite in the vertical and left-to-right directions and which has n layers in the front-to-back direction. The statement is that the function J must be replaced by the function J_n for which Baxter found the formula

$$J_n(\zeta) = \hat{J}_n(\zeta) + \frac{1}{4}\ln\tan\left(\frac{\zeta}{2} + \frac{\pi}{4}\right),$$

where the function $\hat{J}_n$ is given by

$$\hat{J}_n(\zeta) = \int_{-\infty}^{\infty} \frac{\sinh(2x\zeta)}{4x\sinh(\pi x)}(\mathrm{cosech}(2\pi x) - n^{-1}(\mathrm{cosech}(2\pi x/n)))dx.$$

Baxter also checked that this result was valid for three particular cases: $n = 1$, in which case the model was trivial; $n = 2$, in which case the model became the planar free-fermion model; and $n = \infty$.

2. THE $sl(3)$-CHIRAL POTTS MODEL AT $q^2 = -1$

We intend to examine this result within the Bethe ansatz approach. Unfortunately, we have been able so far to treat only the three-layer case of the Zamolodchikov model. As was mentioned above, the n-layer BB model with modified boundary conditions is equivalent to the $sl(n)$-chiral Potts model at $q^{2N} = 1$. In particular, the modified three-layer Zamolodchikov model ($N = 2$) is equivalent to the $sl(3)$-chiral Potts model at $q^2 = -1$. The model is formulated on a square lattice.[2)] The interaction is defined by two kinds of Boltzmann weights [$\overline{W}_{pq}(\alpha, \beta)$ and $(\overline{W}_{qp}(\alpha, \beta))^{-1}$] depending on the neighboring spin variables and rapidity (spectral) parameters. For the homogeneous case, the Boltzmann weight $\overline{W}_{pq}$ is defined rather simply:

$$\overline{W}_{pq} = \begin{pmatrix} 1 & -w & -w & -w \\ w & 1 & -w & w \\ w & w & 1 & -w \\ w & -w & w & 1 \end{pmatrix}, \quad w = \frac{1 - p/q}{1 + p/q}.$$

The rapidity variables are related to the above parameters by the equations

$$v_1 = -\frac{q}{p'}, \quad v_2 = \frac{q'}{p'}, \quad v_3 = \frac{p}{q'}, \quad v_4 = \frac{p}{q}.$$

3. TRANSFER MATRICES

One can define the transfer matrices as

$$T(p; q, q')^{j_1,\ldots,j_N}_{i_1,\ldots,i_N} = \prod_{k=1}^{N} \frac{\overline{W}_{pq}(i_k, j_k)\overline{W}_{q'p}(j_k, i_{k+1})}{\overline{W}_{q'q}(i_k, i_{k+1})},$$

$$\overline{T}(p; q, q')^{j_1,\ldots,j_N}_{i_1,\ldots,i_N} = \prod_{k=1}^{N} \frac{\overline{W}_{q'q}(j_k, j_{k+1})\overline{W}_{pq'}(j_{k+1}, i_k)}{\overline{W}_{pq}(j_k, i_k)},$$

where we imply the cyclic boundary conditions $i_{N+1} = i_1$ and $j_{N+1} = j_1$.

Let us suppose that the rapidities q and q' are fixed. Below, we shall use the simpler notation $T_p = T(p; q, q')$ and $\overline{T}_p = \overline{T}(p; q, q')$. One can prove that these transfer matrices T_p and $\overline{T}_p$ commute for two arbitrary rapidities p and p':

$$[T_p, T_{p'}] = [\overline{T}_p, \overline{T}_{p'}] = [T_p, \overline{T}_{p'}] = 0.$$

[2)] We use here the definitions from [16].

4. FUNCTIONAL RELATIONS

Since the transfer matrices commute with each other, we can go over to its eigenvalues $t(p;q,q')$ and $\overline{t}(p;q,q')$. An analysis of the eigenvalues $t(p) \equiv t(p;q,q')$ and $\overline{t}(p) \equiv \overline{t}(p;q,q')$ shows that it is convenient to extract some "kinematical" multipliers:

$$t(p) = \frac{2^N}{(p+q)^N(p+q')^N}s(p),$$
$$\overline{t}(p) = \frac{2^N}{(p-q)^N(p+q')^N}\overline{s}(p),$$

where $s(p)$ and $\overline{s}(p)$ are some polynomials of the variable p. Using the fusion technique, we can get the functional relations for $t(p)$ and $\overline{t}(p)$ and, therefore, the functional relations for $s(p)$ and $\overline{s}(p)$ [1, 2] as well:

$$\overline{s}(p)s(p)\overline{s}(-p)s(-\omega p) = \lambda_0^N\overline{s}(p)s(-\omega p) + \lambda_1^N\overline{s}(-p)s(-\omega p) + \lambda_2^N\overline{s}(p)s(\omega p) + \lambda_3^N\overline{s}(-p)s(\omega p),$$
$$\overline{s}(-\omega p)s(-p)\overline{s}(p)s(p) = \lambda_0'^N\overline{s}(-\omega p)s(p) + \lambda_1'^N\overline{s}(-\omega p)s(-p) + \lambda_2'^N\overline{s}(\omega p)s(p) + \lambda_3'^N\overline{s}(\omega p)s(-p),$$
$$\lambda_0 = (p+\omega q)(p+\omega^{-1}q)(p+q')(p-q'),$$
$$\lambda_1 = (p+\omega q')(p+\omega^{-1}q')(p+q)(p-q),$$
$$\lambda_2 = (p-q)(p+\omega^{-1}q)(p-\omega q')(p-q'),$$
$$\lambda_3 = (p-q')(p+\omega^{-1}q')(p-\omega q)(p-q);$$

λ_i' can be obtained from λ_i by the substitution $q \to -q$. Here, ω is a cubic root of unity, $\omega = e^{2\pi i/3}$.

5. BETHE ANSATZ EQUATIONS

Let n be the degree of the polynomials $s(p)$ and $\overline{s}(p)$. In order to construct the Bethe ansatz, we should consider the zeros of the polynomials $s(p)$ and $\overline{s}(p)$:

$$s(p) = a_n(q,q')\prod_{i=1}^{n}(p-p_i),$$
$$\overline{s}(p) = \overline{a}_n(q,q')\prod_{i=1}^{n}(p-\overline{p}_i),$$

where the power n takes only two possible values, $2N$ and $2N-1$. The product of the functions a_n and $\overline{a}_n$ has the very simple form

$$a_{2N}(q,q')\overline{a}_{2N}(q,q') = 4,$$
$$a_{2N-1}(q,q')\overline{a}_{2N-1}(q,q') = N(q'^2 - q^2).$$

Considering the zeros on the left-hand sides of the functional relations, we get Bethe ansatz equations

$$\frac{f(p_i,\omega^{\pm1},-q)^N}{f(p_i,\omega^{\pm1},-q')^N} = (-1)^{n-1}\prod_{j=1}^{n}\frac{p_i+\omega^{\mp1}\overline{p}_j}{p_i-\omega^{\mp1}\overline{p}_j},$$
$$\frac{f(\overline{p}_i,\omega^{\pm1},q)^N}{f(\overline{p}_i,\omega^{\pm1},-q')^N} = (-1)^{n-1}\prod_{j=1}^{n}\frac{\overline{p}_i+\omega^{\mp1}p_j}{\overline{p}_i-\omega^{\mp1}p_j},$$
$$f(p,x,q) = \frac{p-xq}{p+q}.$$

6. TWO REGIMES

Below, we shall consider two regimes, I and II [3]. The Boltzmann weights depend on the spherical angles $(\theta_1,\theta_2,\theta_3)$ and the spherical sides (a_1,a_2,a_3) through the parameters v_1, v_2, v_3, and v_4:

$$v_1 = -zT_1T_2, \quad v_2 = -izT_2/T_1,$$
$$v_3 = -z^{-1}T_1T_2, \quad v_4 = iz^{-1}T_2/T_1.$$

Let us say that this parametrization corresponds to regime I, which can also be called the physical regime. The unphysical regime II corresponds to negating spherical sides, i.e., $(a_1,a_2,a_3) \to (-a_1,-a_2,-a_3)$ with the spherical angles $(\theta_1,\theta_2,\theta_3)$ being unchanged:

$$v_1' = -z^{-1}T_1T_2, \quad v_2' = -iz^{-1}T_2/T_1,$$
$$v_3' = -zT_1T_2, \quad v_4' = izT_2/T_1.$$

7. "POISED" PARAMETRIZATION

If N is a multiple of 3 for regime I and is an arbitrary number for regime II, the largest absolute value of the transfer-matrix eigenvalues is provided if all $2N$ parameters p_j and $\overline{p}_j$ can be divided into two sets with N parameters in each of them, namely, p_j, p_{j+N} and $\overline{p}_j, \overline{p}_{j+N}$, where we already have $j = 1, ..., N$. One can introduce the "poised" parametrization

$$p_j = -e^{x_j+iy_j+\zeta-i\pi/4}, \quad p_{j+N} = -e^{-x_j-iy_j+\zeta-i\pi/4},$$
$$\overline{p}_j = e^{x_j-iy_j+\zeta+i\pi/4}, \quad \overline{p}_{j+N} = e^{-x_j+iy_j+\zeta+i\pi/4},$$

where x_j and y_j are two sets of some real numbers with $j = 1, \ldots, N$ and $e^{-2\zeta} = \tan(\theta_1/2)$ with ζ real. The Bethe ansatz equations can then be rewritten in the form

$$\left[\frac{\cosh(x_i+iy_i\pm i\pi/6)+\cosh(\zeta-i\pi/12)}{\cosh(x_i+iy_i\pm i\pi/6)+\cosh(\zeta-i5\pi/12)}\right]^N = -\prod_{j=1}^{N}\frac{\cosh(x_i+iy_i\pm i\pi/6)-\cosh(x_j-iy_j)}{\cosh(x_i+iy_i\pm i\pi/6)+\cosh(x_j-iy_j)}.$$

8. THE TWO-LINE HYPOTHESIS IN REGIME I

Our numerical data show that, in regime I for $N = 3k$, we deal with the two-line solution to the Bethe ansatz equations, i.e., all solutions (x_j, y_j) can be divided into two sets. The first one (x_j, y_j) with $j = 1, \ldots, 2N/3$ has the imaginary parts y_j near $\pi - \pi/12$, while the second one with $j = 2N/3 + 1, \ldots, N$ has the imaginary parts y_j near $\pi/12$. The precision becomes higher with increasing N.

Conjecture: *In the thermodynamic limit $N \to \infty$, all solutions are distributed on two lines exactly with the imaginary parts $\pi - \pi/12$ and $\pi/12$.*

Suppose that, in the thermodynamic limit, the solutions x_i are distributed with two different densities ρ_- and ρ_+ for the first and the second line, respectively.

9. INTEGRAL EQUATIONS

In the thermodynamic limit, we can show that the densities ρ_+ and ρ_- satisfy the integral equations

$$\ln\left[-\frac{\cosh(k-i\pi/4)-\cosh(\zeta-i\pi/12)}{\cosh(k-i\pi/4)-\cosh(\zeta-i5\pi/12)}\right]$$
$$+i\pi\left(\frac{2}{3}-2\int\limits_{-\infty}^{k}dk'\rho_-(k')\right)-\int\limits_{-\infty}^{\infty}dk'\rho_-(k')$$
$$\times\ln\left[-\frac{\cosh(k-i\pi/4)-\cosh(k'+i\pi/12)}{\cosh(k-i\pi/4)+\cosh(k'+i\pi/12)}\right]$$
$$-\int\limits_{-\infty}^{\infty}dk'\rho_+(k')$$
$$\times\ln\left[-\frac{\cosh(k-i\pi/4)+\cosh(k'-i\pi/12)}{\cosh(k-i\pi/4)-\cosh(k'-i\pi/12)}\right]=0,$$

$$\ln\left[\frac{\cosh(k+i\pi/4)+\cosh(\zeta-i\pi/12)}{\cosh(k+i\pi/4)+\cosh(\zeta-i5\pi/12)}\right]$$
$$-i\pi\left(\frac{1}{3}-2\int\limits_{-\infty}^{k}dk'\rho_+(k')\right)-\int\limits_{-\infty}^{\infty}dk'\rho_-(k')$$
$$\times\ln\left[-\frac{\cosh(k+i\pi/4)+\cosh(k'+i\pi/12)}{\cosh(k+i\pi/4)-\cosh(k'+i\pi/12)}\right]$$
$$-\int\limits_{-\infty}^{\infty}dk'\rho_+(k')$$
$$\times\ln\left[-\frac{\cosh(k+i\pi/4)-\cosh(k'-i\pi/12)}{\cosh(k+i\pi/4)+\cosh(k'-i\pi/12)}\right]=0$$

with the natural normalization conditions

$$\int\limits_{-\infty}^{\infty}dk\rho_+(k)=\frac{1}{3},\quad\int\limits_{-\infty}^{\infty}dk\rho_-(k)=\frac{2}{3}.$$

The solutions to these integral equations have the surprisingly simple form

$$\rho_\pm(k)=\frac{\sqrt{3}/\pi}{2\cosh[2(\zeta-k)]\pm 1}.$$

Substituting these formulas into the initial integral equations, one can check that they are indeed satisfied.

10. ONE-LINE REGIME

Let us now consider the one-line regime corresponding to the substitution $a_i \to -a_i$. For the one-line regime, the imaginary parts y_i are close to $7\pi/12$ for all $i = 1, \ldots, N$. As in the two-line case, the precision of the approximation $y_i \approx 7\pi/12$ becomes higher with increasing N.

Conjecture: *In the thermodynamic limit $N \to \infty$, all solutions are distributed on one line exactly with the imaginary part $7\pi/12$.*

It is not very difficult to repeat the strategy that we used to consider the two-line regime. Now, we have only one distribution density ρ of the real parts x_i in the thermodynamic limit. The corresponding integral equation has the form

$$\ln\left[-\frac{\cosh(k-i\pi/4)-\cosh(\zeta-i\pi/12)}{\cosh(k-i\pi/4)-\cosh(\zeta-i5\pi/12)}\right]$$
$$+i\pi\left(1-2\int\limits_{-\infty}^{k}dk'\rho(k')\right)-\int\limits_{-\infty}^{\infty}dk'\rho(k')$$
$$\times\ln\left[\frac{\cosh(k-i\pi/4)-\cosh(k'+i5\pi/12)}{\cosh(k-i\pi/4)+\cosh(k'+i5\pi/12)}\right]=0$$

with the normalization condition

$$\int\limits_{-\infty}^{\infty}dk\rho(k)=1.$$

The final solution to this equation is given by

$$\rho(k)=\frac{\sqrt{3}/\pi}{2\cosh[2(\zeta+k)]+1}$$
$$+\frac{\sqrt{3}/\pi}{2\cosh[2(\zeta-k)]-1}.$$

11. THE PARTITION FUNCTION IN THERMODYNAMIC LIMIT

Due to the equivalence between the modified three-layer Zamolodchikov model and the $sl(3)$-chiral Potts model, the partition function of the modified Zamolodchikov model has the form

$$\ln(\kappa/2) = K_1 + K_2 + K_3 + K_4$$

$$+ \frac{1}{3N}\ln[t(p;q,q')_0\bar{t}(p';q,q')_0],$$

where $t(p;q,q')_0$ and $\bar{t}(p';q,q')_0$ correspond to the largest absolute value of their product.

The final result is given by

$$\ln(\kappa/2) = \begin{cases} F(v_1) + F(v_2^{-1}) + F(v_3^{-1}) + F(v_4) & \text{in regime I,} \\ F'(v_1'^{-1}) + F'(v_2') + F'(v_3') + F'(v_4'^{-1}) & \text{in regime II} \end{cases}$$

for $0 \le \theta_i \le \pi$, $0 \le a_i \le \pi$, where

$$F(v) = -\frac{i\pi}{27} - \frac{1}{12}\ln(1+v) - \frac{1}{4}\ln(1-v) + I(v),$$

$$F'(v) = \frac{i5\pi}{216} - \frac{1}{12}\ln(1+v) - \frac{1}{4}\ln(1-v) + I(v),$$

$$I(v) = I_1(-ve^{i\pi/6}) + I_2(ve^{i\pi/3}),$$

$$I_1(z) = \frac{1}{\sqrt{3}\pi}\int_{-\infty}^{\infty}\frac{dk}{2\cosh(2k)-1}\ln(1+ze^k),$$

$$I_2(z) = \frac{1}{\sqrt{3}\pi}\int_{-\infty}^{\infty}\frac{dk}{2\cosh(2k)+1}\ln(1+ze^k).$$

We should note that our result for the partition function is different from Baxter's result for three layers. We believe that the main reason is that Baxter chose a wrong analyticity assumption for the function J_n, at least for $n = 3$. Our analysis shows that this function is analytic in a narrower strip. Moreover, a function analogous to J_3 satisfies slightly more complicated inversion relations. However, we can also note that our form somehow resembles Baxter's formula:

$$\ln(\kappa/2) = \Phi(v_1) + \Phi(-v_2^{-1}) + \Phi(v_3^{-1}) + \Phi(-v_4),$$

$$\Phi(-ie^{-ix}) = J_3(x)$$

with the two signs before v_2^{-1} and v_4 being different. Of course, the function Φ is different, but it can also be expressed in terms of the dilogarithmic function.

12. CONCLUSION

It would be very interesting to apply the above calculations to the three-layer Zamolodchikov model without a modification of the boundary conditions. We hope that, proceeding this way, we can get a better understanding of the problem discussed above and study, in thermodynamic limit, the ground state of the Hamiltonian derived by Baxter and Quispel [17].

We also intend to develop the thermodynamic Bethe ansatz technique for studying finite-size corrections and possible conformal properties of the three-layer Zamolodchikov model.

As a further step, we would like to generalize our results obtained for the three-layer case to the generic case of an arbitrary number of layers.

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On the Use of the Lie Group Technique for Differential Equations with a Small Parameter: Approximate Solutions and Integrable Equations*

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Abstract—A new approach to the use of the Lie group technique for partial and ordinary differential equations dependent on a small parameter is developed. In addition to determining approximate solutions to the perturbed equation, the approach allows constructing integrable equations that have solutions with (partially) prescribed features. Examples of application of the approach to partial differential equations are given. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

The symmetry-group method plays an important role in the analysis of differential equations. In the problem of finding particular solutions to a partial differential equation (PDE), the symmetry reduction procedure is mostly used. The classical method for finding similarity reductions of PDEs is the Lie group method of infinitesimal transformations (see, for example, [1, 2]). In this method, the invariance of a PDE (or a set of PDEs) under a Lie group of point transformations is used to construct special solutions that are invariant under some subgroup of the full group admitted by the equation (similarity or invariant solutions). The Lie infinitesimal technique allows one to reduce intractable nonlinear conditions of group invariance of a given equation to linear homogeneous determining equations for infinitesimal generators of the group. If the application of the method is aimed only at constructing invariant solutions, the conditional symmetry approach (nonclassical method [3, 4]) may be applied to enlarge the class of solutions obtainable by the symmetry-reduction method. Some generalizations of the nonclassical method may also be developed (e.g., [5, 6]). The classical Lie group method, based on the invariance of PDEs under point transformations (point symmetries), can be further extended by considering invariance under contact transformations (contact symmetries), Lie–Bäcklund transformations (Lie–Bäcklund symmetries), and nonlocal symmetries [1]. Combinations of these extended symmetries with the conditional-symmetry approach are also possible (e.g., [7, 8]).

The central concept of all those methods is the symmetry of the equation, which is defined as a group of transformations that leaves the equation invariant and, consequently, maps any solution to another solution of the equation. In the present paper, we develop an approach that differs conceptually from the symmetry-group method: it does away with the invariance requirement while using the Lie group machinery. As applied to a differential equation depending on a small parameter ϵ, the approach is aimed at constructing equations that, on one hand, could be reduced by exact transformations to an unperturbed equation and which, on the other hand, would coincide approximately with the original (perturbed) equation. To implement this task, the invariance requirement is replaced by the requirement that the unperturbed equation transform infinitesimally (for small values of the group parameter a) into the perturbed equation. Applying the infinitesimal Lie technique, together with this requirement, yields determining equations for the group generators that differ from those of the symmetry-group method. The corresponding infinitesimal transformations map any solution of the unperturbed equation to an approximate (valid up to first order in ϵ) solution of the perturbed equation. The finite transformations defined on the basis of the group generators, as a solution of the corresponding Cauchy problem, are used to construct an integrable equation depending on the group parameter a, which, for $a \ll 1$, coincides with the perturbed equation and can be converted into the unperturbed equation by an exact transformation. Thus, the method developed allows one to (i) extend any solution of the unperturbed equation to the approximate solution of the perturbed equation and (ii)

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find new integrable equations that have (at least, in some parameter interval) a definite physical meaning.

Some symmetry-based perturbation methods were developed recently. The approach developed in a series of papers by Baikov, Gazizov, and Ibragimov (see, e.g., [9]) and referred to as the approximate symmetry group method represents a perturbation technique embedded into the standard procedure of the classical Lie group method. A comparison of the results it produces for the perturbed nonlinear wave equation with those by our method is presented in Section 3. A natural generalization of the approximate classical symmetry-group method to include conditional symmetries was developed in [10]. The method developed in [11] uses a common perturbation technique at the first stage to replace approximately the original equations by a set of equations for the zero- and first-order parts, whereupon, at the second stage, the usual symmetry-group approach is applied to obtain solutions of this coupled system. Both methods, that of [9] and that of [11], are based on the symmetry of the equations, so that the invariance requirement (the approximate invariance of the original equation in [9] and the exact invariance of the system approximating the original equation in [11]) is a central feature of the methods. Thus, our method, in which the invariance is replaced by another requirement, differs conceptually both from that in [9] and from that in [11].

Discovering related differential equations, one with a definite physical meaning and another of a simpler form, which is another goal of our method, also figures among applications of symmetry methods to differential equations. It is usually implemented by comparing the symmetry groups of a given differential equation and another differential equation (target equation) [1]. Thus, the symmetry of the equations again plays a central role. Our method, on the other hand, deals with the Lie group of transformations that do not leave equations invariant but transform one equation into another.

2. AN EXTENSION OF THE LIE GROUP METHOD AS APPLIED TO DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER

Consider a kth-order scalar differential equation depending on a small parameter ϵ, namely,

$$\Delta(x,u,u_{(1)},u_{(2)},\ldots,u_{(k)};\epsilon) = \Delta_0(x,u,u_{(1)},u_{(2)},\ldots,u_{(k)}) + \epsilon\Delta_1(x,u,u_{(1)},u_{(2)},\ldots,u_{(k)}), \tag{1}$$

where $x=(x^1,x^2,\ldots,x^n)$ denotes n independent variables, u denotes the dependent variable, and $u_{(j)}$ denotes the set of all jth-order partial derivatives of u with respect to x. Consider the one-parameter (a) Lie group of transformations

$$x^{i*} = F^i(x,u;a), \quad u^* = G(x,u;a). \tag{2}$$

Let

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta(x,u)\frac{\partial}{\partial u} \tag{3}$$

be an infinitesimal generator of (2), and let $X^{(k)}$ be the kth extended infinitesimal generator of (3). For the sake of compactness, we will use the notation

$$z=(x,u)=(x^1,x^2,\ldots,x^n,u), \quad \zeta=(\xi,\eta), \quad f=(F,G), \tag{4}$$

so that (2) and (3) become $z^*=f(z;a)$ and $X=\zeta(z)\dfrac{\partial}{\partial z}$, respectively, and Eq. (1) is written as

$$\Delta(z,u_{(1)},\ldots,u_{(k)};\epsilon) = \Delta_0(z,u_{(1)},\ldots,u_{(k)}) + \epsilon\Delta_1(z,u_{(1)},\ldots,u_{(k)}). \tag{5}$$

The main points of the approach are the following:

(i) The one-parameter (a) Lie group of transformation

$$z^* = f(z;a), \tag{6a}$$

$$X = \zeta(z)\frac{\partial}{\partial z} \tag{6b}$$

of the unperturbed equation $\Delta_0=0$ is considered, but the invariance requirement is not imposed, so that the equation

$$\Delta_0(z^*,u^*_{(1)},\ldots,u^*_{(k)}) = 0 \tag{7}$$

is transformed into

$$\tilde{\Delta}_0(z,u_{(1)},\ldots,u_{(k)};a) = 0 \tag{8}$$

or infinitesimally

$$\Delta_0(z^*,u^*_{(1)},\ldots,u^*_{(k)}) = \tilde{\Delta}_0(z,u_{(1)},\ldots,u_{(k)};a) = \Delta_0(z,u_{(1)},\ldots,u_{(k)}) + aP(z,u_{(1)},\ldots,u_{(k)}) + O(a^2) \quad (a\ll 1), \tag{9}$$

where

$$P(z,u_{(1)},\ldots,u_{(k)}) = X^{(k)}\Delta_0(z,u_{(1)},\ldots,u_{(k)})\big|_{\Delta_0=0} \tag{10}$$

and $X^{(k)}$ is the kth extended infinitesimal generator of (6b).

(ii) When $a\ll 1$, the group parameter a is identified with the parameter ϵ of Eq. (5), and the invariance requirement is replaced by the requirement that the unperturbed equation (7) is infinitesimally

transformed into the perturbed equation (5). This requirement may be expressed, in view of (9) and (10), as

$$X^{(k)}\Delta_0(z,u_{(1)},\ldots,u_{(k)})\big|_{\Delta_0=0} = \Delta_1(z,u_{(1)},\ldots,u_{(k)}). \tag{11}$$

(iii) The requirement in (11) yields determining equations for the group generators $\zeta=(\xi,\eta)$. Once the generators have been defined, the finite transformations (6a) can be determined as a solution to the Cauchy problem

$$\frac{df(z;a)}{da}=\zeta(f), \quad f(z;0)=z. \tag{12}$$

Substituting the finite transformations into (7) defines the form of Eq. (8). This equation possesses the following two properties:

(A) When $a \ll 1$, Eq. (8) coincides with the initial perturbed Eq. (5) up to first order in $a=\epsilon$:

$$\tilde{\Delta}_0(z,u_{(1)},\ldots,u_{(k)};a) = \Delta_0(z,u_{(1)},\ldots,u_{(k)}) + a\Delta_1(z,u_{(1)},\ldots,u_{(k)})+O(a^2). \tag{13}$$

(B) There exists the exact transformation $z=f(z^*;-a)$ [inverse to (6a)] that converts Eq. (8) into the unperturbed Eq. (7).

Thus, the new approach allows one (a) to extend any solution of the unperturbed equation to the approximate solution of the perturbed equation and (b) to construct equations that, on one hand, are integrable (if the unperturbed equation is integrable) and which, on the other hand, have solutions with some prescribed (at least, in some parameter interval) features.

In the next section, we use the approach to obtain approximate solutions of the perturbed nonlinear wave equations and to construct some integrable nonlinear wave equations.

3. APPROXIMATE SOLUTIONS OF THE PERTURBED NONLINEAR WAVE EQUATIONS

We will start from the perturbed nonlinear wave equation of the form

$$u_{tt}+\epsilon u_t=(uu_x)_x, \tag{14a}$$

where ϵ is a small parameter, so that the unperturbed equation is

$$u_{tt}=(uu_x)_x. \tag{14b}$$

Equation (14a) arises from one-dimensional gas dynamics [12] and longitudinal-wave propagation along a moving threadline [13].

Following the approach described in Section 2, we consider the one-parameter (a) Lie group of point transformations

$$x^*=f(x,t,u;a), \quad t^*=g(x,t,u;a), \quad u^*=h(x,t,u;a), \tag{15a}$$

$$X=\xi(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\eta(x,t,u)\frac{\partial}{\partial u}, \tag{15b}$$

which convert the unperturbed equation

$$\Delta_0(x^*,t^*,u^*)=u^*_{t^*t^*}-(u^*u^*_{x^*})_{x^*}=0 \tag{16}$$

into another equation $\tilde{\Delta}_0(x,t,u;a)=0$ such that

$$\tilde{\Delta}_0(z,t,u;a)=\Delta_0(x,t,u)+a\Delta_1(x,t,u)+O(a^2) = \{u_{tt}-(uu_x)_x\}+a\{ru_t\}+O(a^2), \tag{17}$$

where the artificial coefficient r before the perturbation term in the equation has been introduced to trace the corresponding terms in the generators of the transformations. The generators of such a group are determined from the requirement in (11) which results in the determining equations for ξ, τ, and η having the solutions

$$\xi=b_4x+b_2, \quad \tau=b_3t+b_1-r\frac{t^2}{10}, \quad \eta=2u(b_4-b_3)+r\frac{2}{5}ut, \tag{18}$$

where b_1, b_2, b_3, and b_4 are arbitrary constants.

Next, we will determine the finite transformations (15a) generated by (15b) with ξ, τ, and η given by (18). It suffices for our purposes to take the simplest form of (18), which includes only the elements additional to the symmetry group of Eq. (16) (which have the coefficient r before them). Solving the corresponding Cauchy problem

$$\frac{dg(t,u;a)}{da}=-\frac{g^2}{10}, \quad \frac{dh(t,u;a)}{da}=\frac{2}{5}gh, \quad g(t,u;0)=t, \quad h(t,u;0)=u, \tag{19}$$

we obtain the transformations in the form

$$t^*=t\left(1+\frac{at}{10}\right)^{-1}, \quad u^*=u\left(1+\frac{at}{10}\right)^{4}. \tag{20}$$

Substituting (20) into (16) yields

$$\tilde{\Delta}_0(x,t,u;a)=u_{tt}-(uu_x)_x+\frac{a}{1+at/10}u_t+\frac{1}{5}\frac{a^2}{(1+at/10)^2}u=0. \tag{21}$$

It is easily seen that the transformed Eq. (21) has the property defined by (17): for $a=\epsilon \ll 1$, it coincides with the original Eq. (14a) up to first order in

a. At the same time, exact solutions of Eq. (21) can be obtained from (exact) solutions of the unperturbed Eq. (16) by the inverse transformation

$$u = u^*(x,t^*)\left(1-\frac{at^*}{10}\right)^4, \qquad (22a)$$
$$t^* = t\left(1+\frac{at}{10}\right)^{-1}.$$

The approximate solutions of Eq. (14a) can be obtained by expanding solutions of Eq. (21) up to the first order in a.

If one is interested only in these approximate solutions, one may directly use the infinitesimal transformations defined by (18) as

$$u \approx u^* - a\frac{2}{5}u^*t^*, \quad t^* \approx t - a\frac{t^2}{10}. \qquad (22b)$$

In accordance with (22a), we have used here only the terms with r before them in (18); more general solutions may be obtained by including other terms (see below a derivation of the approximate solution that stems from the conditional invariant solution).

Equation (21), which has been constructed as the result of the application of our method, contains the independent variable t in the coefficients. By applying the transformation

$$u = \frac{1}{4}\left(1+\frac{at}{10}\right)^{-2}U(x,T), \qquad (23)$$
$$T = \frac{5}{a}\ln\left(1+\frac{at}{10}\right),$$

Equation (21) can be converted into the equation

$$U_{TT} + aU_T + \frac{6}{25}a^2U = (UU_x)_x, \qquad (24)$$

which differs from (14a) only by the source term of order ϵ^2.

Compare the possibilities for constructing approximate solutions of Eq. (14a) provided by the approximate classical symmetry-group method due to Baikov *et al.* [9] and by our approach. (It is worth noting that the applications of Baikov *et al.*'s approach are not restricted in finding approximate solutions of PDEs—they also include calculating approximate conservation laws and approximate symmetry groups of PDEs.) It is useful for the following discussion to list the exact symmetries of the unperturbed Eq. (14b), which are represented by [compare with (18)]

$$X_1^{(0)} = \frac{\partial}{\partial t}, \quad X_2^{(0)} = \frac{\partial}{\partial t}, \qquad (25)$$
$$X_3^{(0)} = t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u},$$
$$X_4^{(0)} = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}.$$

The approximate invariant solutions provided by Baikov *et al.*'s method are based on the approximate symmetries of Eq. (14a). Applying the approximate-symmetry-group method to (14a) yields two approximate symmetries; of these, one coincides with the exact symmetry $X_4^{(0)}$ of this and unperturbed equations, while the other is

$$X_3^{(A)} = \left(t+\epsilon\frac{t^2}{10}\right)\frac{\partial}{\partial t} - 2u\left(1+\epsilon\frac{t}{5}\right)\frac{\partial}{\partial u}, \qquad (26)$$

which is a symmetry inherited from $X_3^{(0)}$. The approximate similarity variables constructed with the generator in (26) will be

$$z = x, \quad w = ut^2\left(1+\epsilon\frac{t}{5}\right), \qquad (27)$$

which gives the approximate invariant solution in the form

$$u = t^{-2}\left(1-\frac{\epsilon}{5}t\right)w(z). \qquad (28)$$

For $\epsilon = 0$, Eq. (28) gives

$$u = t^{-2}w(z), \qquad (29)$$

which represents the invariant solution of the unperturbed Eq. (14b) corresponding to the unperturbed part of the symmetry $X_3^{(A)}$. Note that other possible invariant solutions of the unperturbed equation are of no use in the approximate symmetry-group method and, thus, do not lead to approximate solutions of the perturbed equation.

Let us first show that our approach gives the same approximate solution (28) of Eq. (14a) if the invariant solution (29) of the unperturbed Eq. (14b) is used as a source. Indeed, applying the transformation (22a) to (29) yields the exact solution of Eq. (21) in the form

$$u = \left[\frac{w(x)}{t^{*2}}\right]\left(1-\frac{at^*}{10}\right)^4 \qquad (30)$$
$$= t^{-2}\left(1+\frac{at}{10}\right)^{-2}w(x),$$

which for $a \ll 1$ gives (with a replaced by ϵ) the approximate solution (28) of Eq. (14a). This approximate solution could be obtained by the direct use of the infinitesimal transformation (22b).

While the approximate symmetry-group approach produces the only approximate invariant solution (28), which is based on the only approximate symmetry of the perturbed equation inherited from the symmetries of the unperturbed equation, our approach allows one to use other symmetries of the unperturbed

equation for producing the approximate solutions from the corresponding invariant solutions. Consider, for example, the invariant solution of (14b) based on the symmetry

$$X = b_1\frac{\partial}{\partial t} + b_2\frac{\partial}{\partial x}, \tag{31}$$

which leads to

$$u = \Phi(z), \quad z = x - Ct \quad (C = b_2/b_1), \tag{32a}$$

where $\Phi(z)$ satisfies the equation

$$C^2\Phi'' = (\Phi\Phi')'. \tag{32b}$$

Applying transformation (22a) to solution (32a) (expressed in terms of the variables u^* and t^*) yields the exact solution of Eq. (21), which, for $a = \epsilon \ll 1$, produces the approximate solution of Eq. (14a) in the form

$$u = \left(1 - \frac{2}{5}\epsilon t\right)\Phi(z), \quad z = x - Ct + \frac{\epsilon C t^2}{10}, \tag{33}$$

where $\Phi(z)$ satisfies Eq. (32b).

Not only may the invariant solutions obtained by applying the classical Lie group method to the unperturbed Eq. (14b) be used as a source for constructing approximate solutions of Eq. (14a) by our approach, but one can also construct approximate solutions of Eq. (14a) using any solution of Eq. (14b)—for example, conditional invariant solutions. The conditional symmetries of Eq. (14b) were considered in [14]. We will take, as an example, the conditional symmetry with the generator

$$V_{2,2} = \frac{\partial}{\partial t} + c_1 t\frac{\partial}{\partial x} + 2c_1^2 t\frac{\partial}{\partial u}, \tag{34}$$

where c_1 is a constant. The symmetry in (34) leads to the invariant conditional solution of Eq. (14b) having the form

$$u = c_1^2 t^2 + w(z), \quad z = x - \frac{1}{2}c_1 t^2, \tag{35}$$
$$(ww_z)_z = 2c_1^2 - c_1 w_z,$$

which was discussed in [14]. Our approach allows us to construct an approximate solution of Eq. (14a) by the infinitesimal transformations of the variables u and t that, with ϵ replacing a, take the form [compare with (18)]

$$u \approx u^* - \epsilon\left(-2b_3u^* + \frac{2}{5}u^*t^*\right), \tag{36}$$
$$t^* \approx t + \epsilon\left(b_1 + b_3t - \frac{1}{10}t^2\right),$$

which leads to the approximate solution

$$u = c_1^2t^2 + w(z) + \epsilon\left[2b_1c_1^2t + 4b_3c_1^2t^2 - \frac{3}{5}c_1^2t^3 + \left(2b_3 - \frac{2}{5}t\right)w(z)\right], \tag{37}$$
$$z = x - \frac{1}{2}c_1t^2 - \epsilon\left(b_1c_1t + b_3c_1t^2 - \frac{1}{10}c_1t^3\right),$$

where $w(z)$ satisfies Eq. (35). It is readily verified that the solution in (37) satisfies Eq. (14a) in the first order of precision.

In [10], the approximate conditional symmetry of the perturbed Eq. (14a), inherited from the symmetry in (34), and the corresponding approximate conditional invariant solution were obtained by an extension of the approximate symmetry-group method to conditional symmetries. Unfortunately, it is impossible to compare our results with those in [10] since (evidently due to misprints in the formulas) the approximate solution given by Eqs. (19) and (20) of [10] does not satisfy Eq. (14a) in the first order of precision.

Thus, applying our method to the perturbed nonlinear wave Eq. (14a) enables us, first, to obtain new approximate solutions of this equation and, second, to construct new integrable equations (21) and (24), which do not differ significantly from the initial Eq. (14a) for small values of the equation parameter a.

One may easily apply the method to nonlinear wave equations involving other types of perturbations, for example,

$$u_{tt} + \epsilon r[u_t + k_1uu_x + k_2u_{xt} + k_3(uu_x)_t] = (uu_x)_x. \tag{38}$$

4. CONCLUDING REMARKS

We have presented a new approach to application of the Lie group method to differential equations depending on a small parameter and found new approximate solutions of the perturbed nonlinear wave equations. Below, we will comment on the method developed versus the standard perturbation technique.

As distinct from perturbation methods, our method is designed not for finding solutions of a differential equation but for finding transformations between different equations. At this point, the approach completely differs from standard perturbation methods that involve a straightforward expansion of the dependent variables $u = u_0 + \epsilon u_1 + o(\epsilon^2)$ (sometimes, it is accompanied by a transformation of independent variables as an artificial device), which is inserted into the perturbed differential equation $\Delta = \Delta_0 + \epsilon\Delta_1 = 0$ (ϵ is a small parameter). Our method is aimed at finding transformations from the perturbed equation to the unperturbed equation: which variables are transformed (and in what way) is determined by the requirement that the transformations form a

Lie group. These transformations naturally define an approximate solution of the perturbed equation that has the solution of the unperturbed equation as a zero-order part. Another feature of our method is that applying the corresponding finite transformations, found by solving the Lie equations, results in constructing an equation that is integrable in a sense and which reduces to a given equation for small values of the equation parameter. This should be of interest for the theory of perturbation methods since this method provides an equation that, on one hand, can be transformed into the zero-order equation by an exact transformation and which, on the other hand, has a naturally defined equation for the first-order solution.

It should be indicated once again that finite transformations and the corresponding equations produced by the method are of interest in themselves since it enables one to widen the class of integrable equations having some definite physical meaning. In some cases (we do not consider them here), the method may even provide an opportunity to discover integrability of the initial (perturbed) equation.

The method developed in this paper for the Lie group of point transformations is naturally generalized to include contact and Lie–Bäcklund transformations. Modifications in the spirit of the nonclassical method are also possible. No difficulties arise in applying the same approach to ordinary differential equations.

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Nambu Dynamical System: The Case of Many Triplets*

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Abstract—A modification of the exterior calculus is presented to avoid the vanishing of the square of the canonical 3-form. Once this is done, powers of the canonical 3-form are different from zero up to the volume form. All powers of the canonical 3-form are invariant under the Nambu vector field; this is also valid for the integral of each of these forms. The Darboux theorem is exhibited. A specific realization is constructed.

1. NAMBU DYNAMICAL SYSTEM

The Nambu dynamical system [1] generalizes the Hamiltonian scheme in, at least, that it admits even and odd phase space dimensions N. The phase space is spanned by N variables $\mathbf{x} = (x_1, \ldots, x_N)$. The evolution of a dynamical variable $F(\mathbf{x})$ is determined once a set of $N-1$ functions, $H_1, \ldots, H_{N-1}$, is given. In this case, it is said that there is a single multiplet in phase space. The evolution equation for $F(\mathbf{x})$ is

$$\frac{dF(\mathbf{x})}{dt} = \frac{\partial(F(\mathbf{x}), H_1, \ldots, H_{N-1})}{\partial(x_1, \ldots, x_N)} = \{F(\mathbf{x}), \ldots, H_{N-1}\}, \tag{1}$$

where $\partial(\ldots)/\partial(\ldots)$ is a Jacobian of order N. The Nambu bracket for the one-multiplet case,

$$\{F_1, \ldots, F_N\} = \frac{\partial(F_1, \ldots, F_N)}{\partial(x_1, \ldots, x_N)}, \tag{2}$$

is linear and antisymmetric and is a derivation. The Hamiltonians H_k, $k = 1, \ldots, N-1$, are constants of the motion, $dH_k/dt = 0$, so that the solution to the system of coupled differential equations,

$$\frac{dx_j}{dt} = \{x_j, H_1, \ldots, H_{N-1}\}, \tag{3}$$

lies in the intersection of the hypersurfaces $H_k = C_k$ (C_k are constants). From (3), we have

$$\frac{\partial}{\partial x_k}\frac{dx_k}{dt} = 0, \tag{4}$$

where the summation convention is used.

If $N = KS$, with S being an integer, and $K-1$ functions $H_1, \ldots, H_{K-1}$ are given, then S multiplets of dimension K span the phase space. In this case, the variables are labeled as x_j^α with $j = 1, \ldots, S$ and $\alpha = 1, \ldots, K$. The evolution of $F(\mathbf{x})$ is given by

$$\frac{dF(\mathbf{x})}{dt} = \sum_{\alpha=1}^{S} \frac{\partial(F(\mathbf{x}), H_1, \ldots, H_{K-1})}{\partial(x_1^\alpha, \ldots, x_K^\alpha)} = \{F, H_1, \ldots, H_{K-1}\}, \tag{5}$$

where $\partial(\cdots)/\partial(\cdots)$ is a Jacobian of order K. It is clear from (5) that $\dot{H}_j = 0$, $j = 1, \ldots, K-1$, and that the solution of the set of differential equations

$$\frac{dx_i^\alpha}{dt} = \{x_i^\alpha, H_1, \ldots, H_{K-1}\} = X_i^\alpha \tag{6}$$

is contained in the intersection of the surfaces $H_i = C_i$ (C_i are constants). It follows from (6) that

$$\frac{\partial}{\partial x_i^\alpha}\frac{dx_i^\alpha}{dt} = 0. \tag{7}$$

Equations (4) and (7) represent the Liouville condition for two versions of the Nambu dynamical system. As is well known, the Liouville condition implies volume conservation of a region of phase space during evolution. In [2], it is shown that a dynamical system that satisfies the Liouville condition is not necessarily a Nambu dynamical system.

Remark. The notation $\{\ldots\}$ for the Nambu bracket is used in the case of one or more than one multiplets in phase space. Its general definition is (5), where both cases are considered.

Remark. In [3], the extreme case of a singlet is considered.

The purpose of this paper is to describe changes that are necessary for obtaining a geometric description of the Nambu dynamical system in the particular

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case of S triplets, $N = 3S$. The motivation for this development comes from the following results:

(i) The geometric description is based on a closed nondegenerate 3-form ω [4]:

$$\omega = \sum_{\alpha=1}^{S} dr_1^\alpha \wedge dr_2^\alpha \wedge dr_3^\alpha = \sum_{\alpha=1}^{S} \omega_\alpha. \tag{8}$$

The square of this 3-form vanishes; therefore, no integral invariant of the form

$$\int \omega^m \tag{9}$$

can be defined with $m = 2, \ldots, S$. As a consequence, there is no Liouville condition when $S > 2$.

(ii) A way out of the difficulty mentioned in (i) [5] is to disassemble ω and to consider each ω_α separately. Define a "partial" differential $d_\alpha (d = d_1 + \ldots + d_S)$ acting on a monomial k-form $\theta = f(\mathbf{x}) \wedge_{n=1}^{k} dx_{i_n}^{\alpha_n} = f(\mathbf{x}) dx_I^A$ as [here, $A = (\alpha_1, \ldots, \alpha_k)$ and $I = (i_1, \ldots, i_k)$ are multi-indices]

$$d_\alpha \theta = \sum_{i=1}^{3} \frac{\partial f(\mathbf{x})}{\partial x_i^\alpha} dx_i^\alpha \wedge dx_I^A; \tag{10}$$

no sum over α is implied here.

The Lie derivative acting on ω_α does not vanish; due to $dd_\alpha \neq 0$, the Lie derivative acting on ω does not vanish either. The conclusion is that none of the $\int \Pi_{\alpha=1}^{m} \omega_\alpha$, $m = 1, \ldots, S-1$, is an integral invariant. The volume integral, on the other hand, is an integral invariant; therefore, the Liouville condition is satisfied. This result clearly contradicts that in [4] if $S > 1$.

(iii) In [6], a volume preserving dynamical system is defined, called Liouville dynamics. It is obviously based on the Liouville condition; therefore, the integral of the volume form is invariant. This also contradicts [4].

In a different spirit, integral invariants are introduced in [7].

The inevitable conclusion is that a modification at a very fundamental level is needed in order to avoid the conflicting results contained in [4–6]. Since the origin of the problem is related to the antisymmetry of the exterior product, it is the point where the modification starts [8].

2. MODIFIED EXTERIOR CALCULUS

The phase space of the dynamical system (6) has dimension $3S$. The evolution equations are obtained from the 3-form ω after contraction with the vector field $\mathbf{v} = X_i^\alpha \partial_\alpha^i$. In order to ensure that an odd exterior form has a nonvanishing square, a new exterior product, called an eproduct and denoted by $\overline{\wedge}$, is defined on 1-forms σ_j^α as

$$\sigma_j^\alpha \overline{\wedge} \sigma_k^\beta = (-1)^{\delta_{\alpha\beta}} \sigma_k^\beta \overline{\wedge} \sigma_j^\alpha, \tag{11}$$

so that $\overline{\wedge}$ is a symmetric exterior product if the 1-forms are in different multiplets and is antisymmetric if the 1-forms belong to the same multiplet. With this modified product, the powers ω^m do not vanish for $m \leq S$.

Once the exterior product has been modified, the changes in the definition of the differential denoted the $\hat{d}$ and the partial derivative denoted by $\hat{\partial}$ follow after requiring $\hat{d}(\hat{d}f(\mathbf{x})) = 0$ for an arbitrary function; this leads to

$$\hat{d}f(\mathbf{x}) = \hat{\partial}_\alpha^j f(\mathbf{x}) \hat{d}x_j^\alpha, \tag{12}$$
$$\hat{d}(\hat{d}f(\mathbf{x})) = \hat{\partial}_\beta^k \hat{\partial}_\alpha^j f(\mathbf{x}) \hat{d}x_k^\beta \overline{\wedge} \hat{d}x_j^\alpha,$$

which imposes the condition

$$(\hat{\partial}_\beta^k \hat{\partial}_\alpha^j + (-1)^{\delta_{\alpha\beta}} \hat{\partial}_\alpha^j \hat{\partial}_\beta^k) f(\mathbf{x}) = 0. \tag{13}$$

Let us now define the action of $\hat{d}$ on 1-eforms by $\hat{d}(f(\mathbf{x}) \hat{d}x_i^\alpha) = \hat{d}f(\mathbf{x}) \overline{\wedge} \hat{d}x_i^\alpha = \hat{\partial}_\beta^j f(\mathbf{x}) \hat{d}x_j^\beta \overline{\wedge} \hat{d}x_i^\alpha$; then, if $\theta_1^\alpha = f(\mathbf{x}) \hat{d}x_i^\alpha$ and $\theta_2^\beta = g(\mathbf{x}) \hat{d}x_j^\beta$ are two 1-eforms, it follows that

$$\hat{d}(\theta_1^\alpha \overline{\wedge} \theta_2^\beta) = \hat{d}\theta_1^\alpha \overline{\wedge} \theta_2^\beta + (-1)^{\delta_{\alpha\beta}} \hat{d}\theta_2^\beta \overline{\wedge} \theta_1^\alpha. \tag{14}$$

This result is extended to the action of $\hat{d}$ on the eproduct of a p-eform and a q-eform.

The contraction denoted by $\hat{i}$ is modified so as to satisfy

$$\hat{i}_v f(\mathbf{x}) = 0, \tag{15}$$
$$\hat{i}_v \hat{d}x_j^\alpha = v_j^\alpha,$$
$$\hat{i}_v (\hat{d}x_i^\alpha \overline{\wedge} \hat{d}x_j^\beta) = v_i^\alpha \hat{d}x_j^\beta + (-1)^{\delta_{\alpha\beta}} v_j^\beta \hat{d}x_i^\alpha,$$

so that the contraction is an antiderivation when acting on 2-forms with the same multiplet indices and is a derivation on 2-forms with different multiplet indices. As is clear from (15), $\hat{i}$ is completely determined by its action on 0-, 1-, and 2-eforms.

The combined action of the vector fields $U = X\hat{\partial}_\alpha^i$ and $V = Y\hat{\partial}_\beta^j$ defines a vector field $W = Z\hat{\partial}_\gamma^k$ if the following product is defined in the algebra of vector fields:

$$W = [U, V] = UV + (-1)^{\delta_{\alpha\beta}} VU. \tag{16}$$

This algebra has a Lie subalgebra for $\alpha = \beta$.

From (16), it can be shown by using the vector fields U, V, and W ($[A, B]_\epsilon = AB + \epsilon BA$, $\epsilon = \pm 1$) that

$$[U, [V, W]_-]_- + [V, [W, U]_-]_- \tag{17}$$

$$+ [W,[U,V]_-]_- = 0 \quad (\alpha = \beta = \gamma),$$

$$[U,[V,W]_+]_+ + [V,[W,U]_+]_+ + [W,[U,V]_+]_+ = 0 \quad (\alpha \neq \beta \neq \gamma \neq \alpha), \tag{18}$$

$$[W,[U,V]_-]_+ - [V,[W,U]_+]_- + [U,[V,W]_+]_- = 0 \quad (\alpha = \beta \neq \gamma). \tag{19}$$

The Lie Derivative

The Lie derivative is defined, as usual, to be a derivation on functions, forms, and vectors. Its action on an object gives again an object of the same kind. From (16), it therefore follows that

$$L_U V = [U,V], \tag{20}$$

which generalizes the usual result that the action of the Lie derivative on a vector field is the commutator of both fields.

The action of the Lie derivative on the canonical 3-form ω vanishes when the field is the Nambu one,

$$\mathbf{v} = \sum_\alpha \epsilon_{ijk} \partial^i_\alpha H \partial^j_\alpha G \partial^k_\alpha = v^\alpha_k \partial^k_\alpha, \tag{21}$$

so that this is an invariant form. In fact, we have

$$L_{\mathbf{v}}\omega = \sum_\alpha \left(\sum_i \partial^i_\alpha v^\alpha_i \right) \omega^\alpha = 0 \tag{22}$$

due to the Liouville condition (7).

The result of applying the contraction to ω is

$$i_{\mathbf{v}}\omega = \frac{\hat{d}H \overline{\wedge} \hat{d}G - \hat{d}G \overline{\wedge} \hat{d}H}{2}. \tag{23}$$

3. THE DARBOUX THEOREM

The fundamental identity (FI) [9, 10]—the equivalent of the Jacobi identity—is introduced as part of the defining properties of the Nambu bracket in any of its versions; the case under consideration corresponds to (1) for one triplet or to (5) for a collection of triplets. In this case, the FI reduces to

$$\{\{A,B,C\},D,E\} = \{\{A,D,E\},B,C\} + \{A,\{B,D,E\},C\} + \{A,B,\{C,D,E\}\}. \tag{24}$$

In the particular setting that has been set up in this paper, the fundamental identity is modified after the bracket is written as $\{A,B,C\} = \sum_\alpha \{A,B,C\}_\alpha$; then, the FI takes the form

$$\{\{A,B,C\},D,E\} = \sum (-1)^{\delta_{\alpha\beta}+1} \times \{\{A,B,C\}_\alpha, D, E\}_\beta. \tag{25}$$

Each vector field is specified by two functions; let us then take

$$U_F = \epsilon^{\alpha\beta\gamma}_{ijk} \hat{\partial}^i_\alpha f_1 \hat{\partial}^j_\beta f_2 \hat{\partial}^k_\gamma, \qquad U_G = \epsilon^{\rho\sigma\tau}_{rst} \hat{\partial}^r_\rho g_1 \hat{\partial}^s_\sigma g_2 \hat{\partial}^t_\tau, \tag{26}$$

where the subindex F stands for the pair (f_1, f_2) and, similarly, for G. Consider the case where the two vector fields are labeled with the same triplet index; in this case, it is easy to prove that

$$v_G v_F - v_F v_G = v_{\{f_1 G\} f_2} + v_{f_1 \{f_2 G\}}, \tag{27}$$

where $v_{\{f_1 G\} f_2}$ is the vector field defined by the two functions $\{f_1, g_1, g_2\}$ and f_2 and, similarly, for $v_{f_1\{f_2 G\}}$. To prove the Darboux theorem, this result is sufficient; we follow [11].

The proof of the statement that canonical coordinates exist follows the same pattern as in the simplectic case: straightening of flow lines is indeed possible, and locally a vector field associated with the pair of functions f_2 and f_3 can be reduced to the form ∂^1_α; then, a function Q defined by $dQ/d\lambda = \{Q, f_2, f_3\} = 1$ exists, where λ is the parameter measuring evolution along the flow line defined by ∂^1_α. The coordinates that define this particular triplet are $x^\alpha_1 = Q$, $x^\alpha_2 = f_2$, and $x^\alpha_3 = f_3$. These coordinates define three vector fields taking the pairs of functions (Q, f_2), (Q, f_3), and (f_2, f_3). It is easy to prove that the right-hand side of (27) vanishes for each pair of vector fields; in fact, one vector field vanishes due to $\{Q, f_1, f_2\} = 1$, and the other vanishes because one function appears repeated. Let us now complete the set of coordinates with $3S - 3$ functions x^β_k $(\beta \neq \alpha)$ that have vanishing brackets with the functions in the triplet labeled with α. The set of all brackets form a third-rank tensor whose components in the α sector are defined by the brackets $\{A,B,C\}$, where A, B, and C are any of Q, f_2, and f_3; the remaining part of this third-rank tensor is a function of (x^α_j, x^β_j). This function does not depend on x^α_j. In fact, we call this function $T = \{x^{\beta_1}_{j_1}, x^{\beta_2}_{j_2}, x^{\beta_3}_{j_3}\} (\beta_i \neq \alpha)$ and compute

$$\hat{\partial}^1_\alpha T = \{T, x^\alpha_2, x^\alpha_3\} = 0 \tag{28}$$

after the use of the fundamental identity. The same result is obtained by computing $\hat{\partial}^2_\alpha T$ and $\hat{\partial}^3_\alpha T$. Repeat this argument for T. This proves the Darboux theorem in the case of triplets.

The bracket being completely antisymmetric implies the existence of a constant 3-eform that coincides with the canonical form ω introduced in (8).

The result for two vector fields with different multiplet indices is

$$v^\beta_G v^\alpha_F + v^\alpha_F v^\beta_G = -v^{\beta\alpha}_{\{f_1 G\} f_2} - v^{\beta\alpha}_{f_1 \{f_2 G\}}, \tag{29}$$

where $v^{\beta\alpha}_{\{f_1G\}f_2} = \{.,\{f_1,g_1,g_2\}_\beta,f_2\}_\alpha$.

4. STOKES' THEOREM

To ensure Stokes' theorem, the definition of the integral over the boundary of a region needs some modification; it takes the form

$$\int\limits_{\partial R}\theta = \sum_\alpha\sum_{j=1}^{3}(-1)^j\left[\int\limits_{a_{R_i^\alpha}}\theta - \int\limits_{b_{R_i^\alpha}}\theta\right], \qquad (30)$$

where ∂R is the boundary of an m-dimensional region R of a differentiable manifold; θ is an $(m-1)$- form obtained from an m-form by deleting one of its factors (dx_j^β),

$$\theta = f(\mathbf{x})dx_1^1\overline{\wedge}\ldots\overline{\wedge}\widehat{dx_j^\beta}\overline{\wedge}\ldots\overline{\wedge}dx_3^S; \qquad (31)$$

and $x_j^\beta = C_j^\beta = \text{const}$ is replaced in $f(\mathbf{x})$.

5. A PARTICULAR REALIZATION

It is clear from (13) that the variables involved are not the usual real ones. Let us construct the following set

$$x_i^\alpha = P^\alpha y_i^\alpha, \qquad (32)$$

where the y_i^α are real and the P^α are defined by

$$P^\alpha P^\beta + P^\beta P^\alpha = 2\delta^{\alpha\beta}. \qquad (33)$$

Since $x_i^\alpha x_j^\beta = -(-1)^{\delta_{\alpha\beta}}x_j^\beta x_i^\alpha$, x_i^α are noncommutative. Therefore, in this particular realization, the phase space is spanned by variables that anticommute if they are in different multiplets and commute if they are in the same multiplet. The commutation properties are included in the algebra of P^α, while y_i^α are real. A function $f(\mathbf{x})$ is expanded, after (33), in the form

$$f(\mathbf{x}) = \sum_{A=1}^{S} P^{\tilde{A}}g_{\tilde{A}}(\mathbf{y}), \qquad (34)$$

where $\tilde{A} = (\alpha_1,\ldots,\alpha_A)$ $(\alpha_1<\alpha_2<\ldots<\alpha_A)$ and $P^{\tilde{A}} = \prod_{j=1}^{A}P^{\alpha_j}$; if $A=0$, then $P^{\tilde{0}} = I$. As a result, the function $f(\mathbf{x})$ is defined by the collection of the functions $(g_{\tilde{A}}(\mathbf{y}),\ A=1,\ldots,S)$ that will be denoted as $\mathbf{g}(\mathbf{y})$. It follows from these definitions that

$$\hat{d}x_i^\alpha = P^\alpha dy_i^\alpha, \qquad (35)$$
$$\hat{\partial}_\alpha^i f(\mathbf{x}) = P^\alpha\partial_\alpha^i\mathbf{g}(\mathbf{y}),$$
$$\hat{i}_{\hat{\partial}_\beta^j}(\hat{d}x_i^\alpha) = P^\beta P^\alpha i_{\partial_\beta^j}(dy_i^\alpha)$$
$$= P^\beta P^\alpha\delta^{\alpha\beta}\delta_{ij} = \delta^{\alpha\beta}\delta_{ij}.$$

In (35), $\partial_\alpha^i = \partial/\partial y_i^\alpha$. A vector field has the expression

$$\hat{v} = v_i^\alpha\hat{\partial}_\alpha^i = P^\alpha v_i^\alpha\partial_\alpha^i = P^\alpha P^A v_{iA}^\alpha\partial_\alpha^i. \qquad (36)$$

The action of vector fields $\hat{u}$ and $\hat{v}$ on $f(\mathbf{x})$,

$$\hat{u}[\hat{v}f] = P^\beta P^B P^\alpha P^A u_{jB}^\beta \qquad (37)$$
$$\times(\partial_\beta^j v_{iA}^\alpha\partial_\alpha^i f + v_{iA}^\alpha\partial_\beta^j\partial_\alpha^i f),$$
$$\hat{v}[\hat{u}f] = P^\alpha P^A P^\beta P^B v_{iA}^\alpha$$
$$\times(\partial_\alpha^i u_{jB}^\beta\partial_\beta^j f + u_{jB}^\beta\partial_\alpha^i\partial_\beta^j f),$$

shows that (16) is recovered in the simplest case if $A=B=0$, so that v_i^α in (36) is a function of the real variables y_j^β. Guided by this result, we can easily check that all expressions contained in Section 2 are recovered if the Hamiltonians are taken as functions with $A=0$. This means that, in (35), $\mathbf{g}(\mathbf{y}) = g_0(\mathbf{y}) = g(\mathbf{y})$.

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