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Monodromy Representations of the Braid Group*

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Abstract—Chiral conformal blocks in a rational conformal field theory are a far-going extension of Gauss hypergeometric functions. The associated monodromy representations of Artin's braid group $\mathcal{B}_n$ capture the essence of the modern view on the subject that originates in ideas of Riemann and Schwarz. Physically, such monodromy representations correspond to a new type of braid group statistics which may manifest itself in two-dimensional critical phenomena, e.g., in some exotic quantum Hall states. The associated primary fields satisfy R-matrix exchange relations. The description of the internal symmetry of such fields requires an extension of the concept of a group, thus giving room to quantum groups and their generalizations. We review the appearance of braid group representations in the space of solutions of the Knizhnik–Zamolodchikov equation with an emphasis on the role of a regular basis of solutions which allows us to treat as well the case of indecomposable representations of $\mathcal{B}_n$. The bulk of the paper, starting with the end of Section 2, reflects a joint effort of the authors. © *2001 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

Let us begin with a reminiscence.

O'Raifeartaigh's theorem of 1965 told us that what many of us were trying to do: finding a "relativistic $SU(6)$"—a large group containing Poincaré and $SU(6)$ and reproducing some nice mass formulas—was, in fact, impossible. So our generation managed to learn group theory for a wrong reason. (Most theoretical physicists had missed the previous—better—opportunity when group-representation theory helped understand atomic spectra.) The only noticeable trace from our effort of that time—besides the no-go theorem of Lochlain—was left by the current algebra approach. But that was not a purely group theoretical development: it incorporated ideas of QED and of the $V-A$ theory of weak interactions. Hence, a moral: beware of isolated ideas and methods (for good or bad this lesson kept us later from drowning in the supersymmetry temptation).

Artin's braid group $\mathcal{B}_n$—with its monodromy representations—is a good example of a focal point for important developments in both mathematics and physics.

In mathematics, it appears in the description of topological invariants of algebraic functions [1] and the related study of multiparametric integrals and (generalized) hypergeometric functions [2–4] and in the theory of knot invariants and invariants of three-dimensional manifolds [5–9]. The main physical applications go under the heading of generalized statistics (anticipated by Arnold in [1], see Section 2). The Knizhnik–Zamolodchikov (KZ) equation (Section 3) is a common playground for physicists and mathematicians.

We illustrate high-brow mathematical results of [10, 11] on the relation between the monodromy representations of $\mathcal{B}_n$ in the space of solutions of the KZ equation for a semisimple Lie algebra $\mathcal{G}$ and the universal R matrix for $U_q(\mathcal{G})$ by simple computations for the special case of $\mathcal{G} = su(N)$ step operators and $\mathcal{B}_3$ (Section 4). In fact, we go into our explicit construction beyond these general results by also treating on equal footing the indecomposable representations of $\mathcal{B}_3$ for q an even root of unity ($q^h = -1$).

2. PERMUTATION AND BRAID GROUP STATISTICS

The symmetry of a one-component wave function $\Psi(x_1, \ldots, x_n)$ is described by either of the one-dimensional representations of the group $\mathcal{S}_n$ of permutations giving rise to Bose and/or Fermi statistics. Multicomponent wave functions corresponding to particles with internal quantum numbers may transform under higher dimensional representations of $\mathcal{S}_n$ corresponding to parastatistics. If one allows for multivalued wave functions, then the exchange of two arguments x and y may depend on the (homotopy

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type of the) path along which x and y are exchanged, thus giving rise to a representation of the braid group $\mathcal{B}_n$ of n strands.

$\mathcal{B}_n$ was defined by E. Artin in 1925 as a group with $n-1$ generators, $b_1, \ldots, b_{n-1}$ (b_i "braiding" the strands i and $i+1$), satisfying the following two relations:

$$b_i b_j = b_j b_i, \ |i-j| \geq 2, \qquad (1)$$
$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, \ i = 1, \ldots, n-2.$$

Let $\sigma : \mathcal{B}_n \to \mathcal{S}_n$ be the group homomorphism defined by

$$\sigma(b_i) = \sigma_i, \ {\sigma_i}^2 = 1 (\in \mathcal{S}_n), \ i = 1, \ldots, n-1, \quad (2)$$

where σ_i are the basic transpositions exchanging i and $i+1$ that generate $\mathcal{S}_n$. The kernel $\mathcal{P}_n$ of this homomorphism is called the monodromy (or pure braid) group. Note that the element

$$c^n := (b_1 b_2 \ldots b_{n-1})^n \qquad (3)$$

generates the centre of $\mathcal{B}_n$.

The braid group $\mathcal{B}_n$ and its invariant subgroup $\mathcal{P}_n$ have a topological interpretation. Consider the n-dimensional manifold

$$Y_n = \mathbb{C}^n \setminus \mathrm{Diag} \equiv \{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n; \quad (4)$$
$$i \neq j \Rightarrow z_i \neq z_j\}$$

(Y_n is the analyticity domain of n-point conformal blocks in chiral conformal field theory). The symmetric group $\mathcal{S}_n$ acts on Y_n by permutations of coordinates. The factor space $X_n = Y_n/\mathcal{S}_n$ is the configuration space of n points ("identical particles") in $\mathbb{C}^n$.

Proposition 1 [1]. *The braid group $\mathcal{B}_n$ is isomorphic to the fundamental group $\pi_1(X_n, \mathbf{z}_0)$ of the configuration space (for, say, $\mathbf{z}_0 = (n, \ldots, 1)$); similarly, $\mathcal{P}_n \simeq \pi_1(Y_n, \mathbf{z}_0)$.*

Clearly, had we substituted the complex plane, $\mathbb{C} \simeq \mathbb{R}^2$, by an s-dimensional space $\mathbb{R}^s$ for $s \geq 3$, the fundamental group $\pi_1((\mathbb{R}^s)^{\otimes n} \setminus \mathrm{Diag}, \mathbf{z}_0)$ would have been trivial and no interesting relation with the braid group could be expected. This simple topological fact explains why the possibilities for generalized statistics are richer in physics of low (one and two) dimensions. One may wonder why it took more than half a century after the appearance of Bose and Fermi statistics in quantum mechanics before such a basic observation found its way into the physics literature (first in the work of Leinaas and Myrheim [12] in the framework of quantum mechanics, later in [13] in the context of local current algebras). Moreover, this pioneer work did not attract much attention before it was repeated by others (starting with [14]) when catchwords like "anyons" were coined. For a discussion of the ancestry of the anyon, see [15], where more early references can be found.

A deeper understanding of particle statistics came from the "algebraic" study of superselection sectors in local quantum field theory (see [16, 17], where earlier work of Doplicher, Haag, and Roberts is also cited). We offer here an informal (physicist's oriented) formulation of the main result of this work.

The starting point of the algebraic (Haag–Kastler) approach is the concept of an algebra $\mathcal{A}$ of local observables. Ignoring technicalities, we shall think of $\mathcal{A}$ as generated by conventional (Wightman) Bose fields—such as the stress-energy tensor and conserved $U(1)$ currents (rather than of a net of C^* algebras corresponding to double cones and their causal complements in Minkowski space). Committing another sin against the purist algebraic view, we shall identify from the outset $\mathcal{A}$ with its vacuum representation in a Hilbert space $\mathcal{H}$ that carries a unitary positive energy representation of the Poincaré group with a unique translation-invariant vacuum state. It is important that gauge-dependent charge carrying (and/or multivalued) fields are excluded from $\mathcal{A}$. They reappear—as derived objects—in the role of intertwiners among inequivalent representations of $\mathcal{A}$.

Superselection sectors are defined by irreducible positive energy representations of $\mathcal{A}$ that can be obtained from the vacuum sector by the action of localizable "charged fields"—i.e., of fields that commute at spacelike separations with the observables (but need not be local among themselves). Products of charged fields acting on the vacuum give rise, typically, to a finite sum of superselection sectors defining the fusion rules of the theory. (To make this precise one needs, in fact, more elaborate tools—such as *-endomorphisms of a completion of $\mathcal{A}$ that are localizable in spacelike cones; see [18] for an updated exposition and references; the shortcoming of a simple-minded use of "charged fields" is the nonuniqueness of their choice and, hence, of the multiplicities entering the above naive definition of fusion rules.) A fancy way to express the fact that there is a well-defined composition law for representations of $\mathcal{A}$ (analogous to tensor product of group representations) is to say thatsuperselection sectors give rise to a tensor category. A memorable result of Doplicher and Roberts [17] crowning two decades of imaginative work of Haag's school says that, for a local quantum theory with no massless excitations in space–time dimensions $D \geq 4$, this category is equivalent to the category of irreducible representations (IR) of a compact group G. In more down-to-earth terms, it means that G acts—by automorphisms—on charged fields as a gauge group of the first kind (recall that a gauge group leaves invariant all observables, not only the Hamiltonian). Superselection sectors are labeled by (equivalence classes of) IR $p \in \hat{G}$ (borrowing the terminology of representation theory of semisimple compact Lie groups, we shall call the labels p weights). The state space of the theory can thus be viewed as a direct sum of tensor product spaces:

$$\mathcal{H} = \bigoplus_{p \in \hat{G}} \mathcal{H}_p \otimes \mathcal{F}_p, \quad d(p) := \dim \mathcal{F}_p < \infty, \qquad (5)$$

where $\mathcal{F}_p$ are irreducible G modules. The statistics of a sector p is characterized by a statistics parameter $\lambda_p = \pm 1/d(p)$. If G is Abelian (the common case of commuting superselection sectors labeled by the "spin parity" $e^{2\pi i s_p}$, where s_p is the spin, and by the values of the electric, baryonic, and leptonic charge), then $d(p) = 1$ for all sectors and we are faced with the familiar Bose–Fermi alternative. If G is non-Abelian and $d(p) = 2, 3, \ldots$, then the sector p and its conjugate $\bar{p}$ (or, in physical language, the particles of type p and their antiparticles $\bar{p}$) obey parastatistics. (Unfortunately, one has no such result for quantum electrodynamics. It is, in fact, known that the electric charge cannot be localized in a spacelike cone. Although there is no indication that, say, electrons may obey braid group statistics, presently we are unable to rule it out.)

These results also extend to space–time dimension $D = 3$, provided the superselection charges can be localized in finite regions. In more realistic $(2+1)$-dimensional systems (like a "quantum Hall fluid" in a strong magnetic field perpendicular to the plane of the layer), charges can only be localized in infinite spacelike cones and there is room for braid group statistics. For $D = 2$, braid group statistics may appear even for (superselection) charges localized in finite domains (see [18–20]). The notion of a statistics parameter extends to this case, too, and is related to the Jones index of inclusion of associated factors of operator algebras [20]. It can be written as (see [18], Definition 6.2)

$$\lambda_p = \frac{1}{d(p)} e^{-2\pi i \theta_{p\bar{p}}}, \quad d(p) (= |\lambda_p|^{-1}) > 0, \qquad (6)$$
$$e^{-2\pi i \theta_{p\bar{p}}} = e^{-i\pi(s_p + s_{\bar{p}})},$$

where s_p and $s_{\bar{p}}$ are the (fractional) spins of the conjugate sectors p and $\bar{p}$. For $d(p) = 1$, $\lambda_p \neq \pm 1$, we are dealing with a one-dimensional representation of the braid group, corresponding to anyonic statistics. For noninteger $d(p)$, the "gauge symmetry" of superselection sectors cannot be described by a group. No fully conclusive attempts are made to use instead weak quasi-quantum groups [21], weak Hopf algebras (or quantum groupoids) [22, 23], or a BRS approach with quantum group symmetry in an extended state space [24, 25].

To cite [18], "braid statistics in two-dimensional systems is more than a theoretical curiosity." Indeed, anyons have made their way into the standard interpretation of the fractional Hall effect. Non-Abelian braid group statistics appears to be strongly indicated for Hall plateaux at the second Landau level with filling fractions $\nu = 2 + m/(m+2)$, $m = 2, 3, \ldots$ (cf. [26, 27]).

3. THE KZ EQUATION

Let $\mathcal{G}$ be a compact Lie algebra, V be a finite-dimensional $\mathcal{G}$ module, and C_{ab} be the (polarized) Casimir invariant acting nontrivially on the factors a and b of the n-fold tensor product $V^{\otimes n}$. For $\mathcal{G} = su(N)$ and $n = 3$,

$$C_{12} \, (= C_{21}) = \left(\sum_{i,j=1}^{N} e_{ij} \otimes e_{ji} - \frac{1}{N} \sum_{i=1} e_{ii} \otimes \sum_{j=1} e_{jj} \right) \otimes \mathbb{1}, \qquad (7)$$

where e_{ij} represent the Weyl generators of $U(N)$ in V.

The KZ equation is a system of partial differential equations which can be written compactly as

$$h d\Psi = \sum_{1 \le a < b \le n} C_{ab} \frac{dz_{ab}}{z_{ab}} \Psi, \qquad (8)$$
$$z_{ab} = z_a - z_b, \quad dz_{ab} = dz_a - dz_b, \quad C_{ab} = C_{ba};$$

here, h is a (say, real) parameter, and $\Psi = \Psi(z_1, \ldots, z_n)$ is a (regular) map, $\Psi : Y_n \to V^{\otimes n}$, where Y_n is $\mathbb{C}^n$ minus the diagonal [see (4)]. The system (8) has a nice geometric interpretation: it defines a connection $\nabla = d - \Gamma$ on the trivial bundle $Y_n \times V^{\otimes n}$, where Γ is the connection one-form

$$\Gamma = \frac{1}{h} \sum_{a<b} C_{ab} \frac{dz_{ab}}{z_{ab}}. \qquad (9)$$

Introducing the corresponding covariant derivative

$$\nabla_a = \frac{\partial}{\partial z_a} - \frac{1}{h} \sum_{b \neq a} \frac{C_{ab}}{z_{ab}}, \qquad (10)$$

we can interpret (8) by saying that Ψ is covariantly constant. This requires as a compatibility condition the flatness of the KZ connection.

Proposition 2. *The KZ connection $\nabla = d - \Gamma$ has zero curvature:*

$$\nabla \circ \nabla = \Gamma \wedge \Gamma - d\Gamma = 0 \;\Leftrightarrow\; [\nabla_a, \nabla_b] = 0. \qquad (11)$$

The *proof* (see, e.g., [28]) uses

$$[C_{ab}, C_{cd}] = 0 \text{ for different } a, b, c, d, \qquad (12)$$
$$[C_{ab}, C_{ac} + C_{bc}] = 0 \qquad (13)$$
$$= [C_{ab} + C_{ac}, C_{bc}] \text{ for different } a, b, c,$$

as well as the following *Arnold's lemma: let*

$$u_{ab} = d(\ln z_{ab}) = \frac{dz_{ab}}{z_{ab}} \left(\equiv \frac{dz_a - dz_b}{z_a - z_b} \right), \qquad (14)$$

then

$$u_{ab} \wedge u_{bc} + u_{bc} \wedge u_{ca} + u_{ca} \wedge u_{ab} = 0 \qquad (15)$$
$$\text{for } a \neq b \neq c \neq a.$$

The flatness of the connection ∇ is a necessary and sufficient condition that the holonomy group $\mathcal{P}_n$ at a point $p \in Y_n$ (the transformation group in $V^{\otimes n}$ obtained by parallel transport of vectors along closed paths with beginning and end in p) gives rise to a (monodromy) representation of the fundamental group $\pi_1(Y_n, p)$.

The KZ equation appears in $2D$ CFT in the context of the Wess–Zumino–Novikov–Witten (WZNW) [29] model [30] and in a related study of chiral current algebras [31]. The idea of the latter approach is simple to summarize. A primary field φ of a conformal current algebra is covariant under two infinite Lie algebras of infinitesimal transformations: under local gauge transformations generated by the currents J and under reparametrization generated by the stress-energy tensor T. On the other hand, T is expressed quadratically in terms of J (by the so-called Sugawara formula). The consistency between the two covariances and this quadratic relation yields the operator KZ equation:

$$h\frac{d\varphi}{dz} =: \varphi(z)\mathbf{t}J(z) : . \qquad (16)$$

Here, $C_{ab} = \mathbf{t}_a \otimes \mathbf{t}_b$, the vector $\mathbf{t}$ spanning a basis of the finite-dimensional representation of $\mathcal{G}$ such that $[\mathbf{J}_0, \varphi(z)] = \varphi(z)\mathbf{t}$ for $\mathbf{J}_0 = \oint \mathbf{J}(z)dz/(2\pi i)$, and the "height" h is an integer ($h \geq N$ for $\mathcal{G} = su(N)$). Using also the current-field Ward identity, we end up with Eq. (8) for the "wave function"

$$\Psi(p; z_1, \ldots, z_n) = \langle p|\varphi(z_1) \otimes \cdots \otimes \varphi(z_n)|0\rangle, \quad (17)$$

where p stands for the weight of the $\mathcal{G}$ module that contains the bra $\langle p|$.

The notation of Eq. (17) is, in fact, ambiguous. There are (for fixed n and p) several (linearly independent) solutions of the KZ equation (called conformal blocks). To distinguish among them, one introduces the concept of a chiral vertex operator (CVO) [32] (the counterpart of an intertwiner between different superselection sectors in the algebraic approach to local quantum field theory [17–19]). We shall use instead a field φ belonging to the tensor product $V \otimes \mathcal{V}$ of a $\mathcal{G}$ and a $U_q(\mathcal{G})$ module, $\varphi = (\varphi_\alpha^A)$; it arises naturally in splitting the group valued field g in the WZNW model into left and right movers, $g_B^A(z, \bar{z}) = \varphi_\alpha^A(z)(\bar{\varphi}^{-1})_B^\alpha(\bar{z})$ (see [33, 25] for an early and a recent paper, the latter containing some 50 more references on the subject). By way of example, we consider the case where $\mathcal{G} = su(N)$, $n = 3$ and φ is an $SU(N)$ *step operator* (i.e., $V = \mathbb{C}^N$ carrying the defining representation of $su(N)$). Then, if we take p to be the highest weight of the IR associated with the Young tableau $\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}$ with respect to both $su(N)$ and $U_q(sl_N)$, we can reduce (8) to a system of ordinary differential equations for the invariant amplitude $F(\eta)$ defined by

$$\Psi(p; z_1, z_2, z_3) = z_{13}^{-\frac{3}{4h}}(\eta(1-\eta))^{-\frac{N+1}{Nh}}F(\eta), \quad (18)$$
$$\eta = \frac{z_{23}}{z_{13}};$$

we find (see Appendix A)

$$(h\frac{d}{d\eta} + \frac{\Omega_{12}}{1-\eta} - \frac{\Omega_{23}}{\eta})F(\eta) = 0, \qquad (19)$$

$$\Omega_{12} = C_{12} + \frac{1}{N} + 1 = P_{12} + 1, \qquad (20)$$
$$\Omega_{23} = \frac{N-2}{N} - C_{12} - C_{13}$$

($P_{12}(x \otimes y) = y \otimes x$ and $F(\eta)$ is an invariant $SU(N)$ tensor, $F(\eta) \in \mathrm{Inv}(V_p^* \otimes V^{\otimes 3})$). The subspace of invariant tensors in $V_p^* \otimes V^{\otimes 3}$ is two-dimensional. We shall choose a basis I_0, I_1 in $\mathrm{Inv}(V_p^* \otimes V^{\otimes 3})$ such that

$$\Omega_{23}I_0 = 0, \ I_1 = (P_{12} - 1)I_0 (\Rightarrow \Omega_{12}I_1 = 0). \quad (21)$$

Setting then

$$F(\eta) = (1-\eta)f^0(\eta)I_0 + \eta f^1(\eta)I_1, \qquad (22)$$

we reduce the KZ equation to a system that does not depend on N:

$$h(1-\eta)\frac{df^0}{d\eta} = (h-2)f^0 + f^1, \qquad (23)$$
$$h\eta\frac{df^1}{d\eta} = (2-h)f^1 - f^0;$$

it implies a hypergeometric equation for each f^ℓ:

$$\eta(1-\eta)\frac{d^2f^\ell}{d\eta^2} + \left(1 + \ell - \frac{2}{h} - \left(3 - \frac{4}{h}\right)\eta\right) \quad (24)$$
$$\times \frac{df^\ell}{d\eta} = \left(1 - \frac{1}{h}\right)\left(1 - \frac{3}{h}\right)f^\ell, \quad \ell = 0, 1.$$

4. DYNAMICAL R-MATRIX EXCHANGE RELATIONS AMONG CVO

In order to derive the exchange properties of two $\widehat{su}(N)$ step operators, we shall consider the slightly more general matrix element

$$\Psi(p'', p'; z_1, z_2, z_3) \qquad (25)$$
$$= \langle p''|\varphi(z_1) \otimes \varphi(z_2) \otimes \phi_{p'}(z_3)|0\rangle$$
$$= D_{p''p'}(z_{ab})F(\eta).$$

Here, p' and p'' are the (shifted) weights of $SU(N)$ IR such that the dimension of the space $\mathcal{I}_{p''p'} = \mathrm{Inv}(V^*_{p''} \otimes V^{\otimes 2} \otimes V_{p'})$ $(\ni F(\eta))$ is maximal, $\dim\mathcal{I}_{p''p'} = 2$, and

$$D_{p''p'}(z_{ab}) = z_{13}^{\Delta(p'')-\Delta(p')-2\Delta} \times\eta^{\frac{\Delta(p'')-\Delta(p')}{2}-\frac{\mathfrak{p}}{2h}+\frac{2-N^2}{2Nh}}(1-\eta)^{-\frac{N+1}{Nh}}. \quad (26)$$

In (26), $\mathfrak{p} = {p'}_{ij}$ (≥ 2) for $\varphi(z_a)$, $a = 1, 2$ identified with the CVO $\varphi_i(z_1)$ and $\varphi_j(z_2)$, $i < j$, respectively (for an explanation of the precise meaning of the above notation see Appendix B). We proceed with a summary of relevant results of [34].

The KZ equation for Ψ again reduces to the form (19); only expression (20) and relation (A.9) for Ω_{23} assume a more general form:

$$\Omega_{23} = \frac{2-N}{2} + h\frac{\Delta(p'')-\Delta(p')}{2} - C_{12} - C_{13}, \quad \Omega_{23}^2 = \mathfrak{p}\Omega_{23}. \quad (27)$$

[We recover (20) and (A.9) for ${p'}_{12} = 2 (= \mathfrak{p})$, ${p'}_{ii+1} = 1$ for $2 \leq i \leq N-1$, in which case $\Delta(p'),\ \Delta(p'') = \Delta_\phi$—see (A.2); another simple special case is $N = 2$, in which $p' = p''$.] The relations (21) for the basis $\{I_0, I_1\}$ of $SU(N)$ invariants remain unchanged, while the hypergeometric system (23) assumes the form

$$h(1-\eta)\frac{df^0}{d\eta} = (h-2)f^0 + (\mathfrak{p}-1)f^1, \quad (28)$$
$$h\eta\frac{df^1}{d\eta} = (\mathfrak{p}-h)f^1 - f^0.$$

A standard basis of (two) solutions is obtained by singling out the possible analytic behavior of the invariant amplitude $F(\eta)$ (22) for $\eta \to 0$. This gives the so-called s-channel basis corresponding, in physical terms, to the operator product expansion of $\varphi_j(z_2)\phi_{p'}(z_3)|0\rangle$ (or of $\langle 0|\phi^*_{p''}(z_0)\varphi_i(z_1)$—cf. Appendix A). In the case at hand, these two solutions, $s_0(\eta)$ and $s_1(\eta)$, are characterized by the property that

$$s_0(\eta) \text{ and } \eta^{-\frac{\mathfrak{p}}{h}}s_1(\eta) \quad (29)$$

are (nonzero) analytic at $\eta = 0$ and $s_0(0) = I_0$. They are expressed in terms of hypergeometric functions:

$$s_0(\eta) = K_0\Big((1-\eta)F\Big(1-\frac{1}{h}, 1-\frac{\mathfrak{p}+1}{h}; 1-\frac{\mathfrak{p}}{h};\eta\Big)I_0 - \frac{\eta}{h-\mathfrak{p}}F\Big(1-\frac{1}{h}, 1-\frac{\mathfrak{p}+1}{h}; 2-\frac{\mathfrak{p}}{h};\eta\Big)I_1\Big), \quad (30)$$

$$s_1(\eta) = K_1\eta^{\frac{\mathfrak{p}}{h}}\Big((1-\eta)F\Big(1-\frac{1}{h}, 1+\frac{\mathfrak{p}-1}{h}; 1+\frac{\mathfrak{p}}{h};\eta\Big)I_0 + \frac{\mathfrak{p}}{\mathfrak{p}-1}F\Big(-\frac{1}{h}, \frac{\mathfrak{p}-1}{h}; \frac{\mathfrak{p}}{h};\eta\Big)I_1\Big). \quad (31)$$

We shall now compute the monodromy representation of the braid group generator $B(B_1)$ corresponding to the exchange of two "identical particles" 1 and 2. Note first that φ (25) is single-valued analytic in the neighborhood of the real configuration of points $\{z_1 > z_2 > z_3 > -z_2\}$. We then choose any path in the homotopy class of

$$\overset{\frown}{1\,2}: \quad z_{1,2}(t) = \frac{z_1+z_2}{2} \pm \frac{1}{2}z_{12}e^{-i\pi t}, \quad z_3(t) = z_3, \quad 0 \leq t \leq 1, \quad (32)$$

which thus exchanges z_1 and z_2 in a clockwise direction, and perform an analytic continuation of Ψ along it, followed by a permutation of the $SU(N)$ indices A_1 and A_2. This gives

$$z_{12} \to e^{-i\pi}z_{12}, \quad z_{13} \leftrightarrow z_{13}, \quad (33)$$
$$1-\eta \to e^{-i\pi}\frac{1-\eta}{\eta} \quad \Big(\eta \to \frac{1}{\eta}\Big),$$

$$D_{p''p'}(z_{ab}) \ \to\ \bar{q}^{\frac{N+1}{N}}\eta^{\frac{\mathfrak{p}+1}{h}}D_{p''p'}(z_{ab}) \quad \text{for } q^{\frac{1}{N}} = e^{-\frac{i\pi}{Nh}}, \quad (34)$$

$$(1-\eta)I_0 \ \to -\frac{1-\eta}{\eta}(I_0+I_1), \quad (35)$$
$$\eta I_1 \to -\frac{1}{\eta}I_1.$$

Using known transformation properties (the "Kummer identities") for hypergeometric functions (or rederiving them from their integral representations—see [35]), we end up with the braid relation

$$\overset{\frown}{1\,2}: \quad Ds^\ell_\lambda(\eta) \ \overset{\frown}{\to}\ Ds^\ell_{\lambda'}(\eta)B^{\lambda'}_\lambda, \quad (36)$$
$$B = \bar{q}^{\frac{1}{N}}\begin{pmatrix} \frac{q^{\mathfrak{p}}}{[\mathfrak{p}]} & Kb_{\mathfrak{p}} \\ K^{-1}b_{-\mathfrak{p}} & -\frac{\bar{q}^{\mathfrak{p}}}{[\mathfrak{p}]}\end{pmatrix},$$

where

$$[p] := \frac{q^p-\bar{q}^p}{q-\bar{q}}, \ b_{\mathfrak{p}} = \frac{\Gamma(1+\frac{\mathfrak{p}}{h})\Gamma(\frac{\mathfrak{p}}{h})}{\Gamma(1+\frac{\mathfrak{p}-1}{h})\Gamma(\frac{\mathfrak{p}+1}{h})} \quad (37)$$
$$\Big(\Rightarrow b_{\mathfrak{p}}b_{-\mathfrak{p}} = \frac{[\mathfrak{p}+1][\mathfrak{p}-1]}{[\mathfrak{p}]^2}\Big), \ K = \frac{K_1}{K_0}.$$

It is remarkable that for a family of choices of the normalization constant $K = K(\mathfrak{p})$, namely, for

$$K = \frac{\Gamma(\frac{1+\mathfrak{p}}{h})\Gamma(-\frac{\mathfrak{p}}{h})}{\Gamma(\frac{1-\mathfrak{p}}{h})\Gamma(\frac{\mathfrak{p}}{h})}\rho(\mathfrak{p}), \quad (38)$$
$$\rho(\mathfrak{p})\rho(-\mathfrak{p}) = 1 (= K(\mathfrak{p})K(-\mathfrak{p})),$$

(36) agrees with the (dynamical) R-matrix exchange relations

$$\varphi^B_i(z_2)\overset{\frown}{\varphi}{}^A_j(z_1) = \varphi^A_s(z_1)\varphi^B_t(z_2)\hat{R}(\mathfrak{p})^{st}_{ij}, \quad (39)$$

linked in [25] with the properties of an intertwining quantum matrix algebra generated by an $N \times N$ matrix (a^i_α) with noncommuting entries and by N commuting unitary operators q^{p_i} $(\prod_{i=1}^N q^{p_i} = 1)$, such that

$$q^{p_i} a^j_\alpha = a^j_\alpha q^{p_i + \delta^j_i - \frac{1}{N}}, \tag{40}$$
$$\hat{R}(p) a_1 a_2 = a_1 a_2 \hat{R}, \quad \varphi^A_\alpha(z) = \varphi^A_i(z) a^i_\alpha.$$

Here, $\hat{R} = (\hat{R}^{\alpha_1\alpha_2}_{\beta_1\beta_2})$ and $\hat{R}(p) = (\hat{R}(p)^{i_1 i_2}_{j_1 j_2}$ are the $U_q(sl_N)$ and the dynamical R matrices, respectively, multiplied by a permutation, $\hat{R} = RP$, and we are using Faddeev's concise notation for tensor products. $\hat{R}(p)$ obeys the Gervais–Neveu [36] "dynamical Yang–Baxter equation" whose general solution satisfying "the ice condition" (the condition that $\hat{R}(p)^{ij}_{kl}$ vanishes unless the unordered pairs (i,j) and (k,l) coincide) was found by Isaev [37]. Its 2×2 block

$$\begin{pmatrix} \hat{R}(p)^{ij}_{ij} & \hat{R}(p)^{ij}_{ji} \\ \hat{R}(p)^{ji}_{ij} & \hat{R}(p)^{ji}_{ji} \end{pmatrix} \tag{41}$$
$$= \bar{q}^{\frac{1}{N}} \begin{pmatrix} \frac{q^{\mathfrak{p}}}{[\mathfrak{p}]} & \frac{\mathfrak{p}-1}{[\mathfrak{p}]}\rho(\mathfrak{p}) \\ \frac{\mathfrak{p}+1}{[\mathfrak{p}]}\rho(-\mathfrak{p}) & -\frac{\bar{q}^{\mathfrak{p}}}{[\mathfrak{p}]} \end{pmatrix} \quad (\rho(\mathfrak{p})\rho(-\mathfrak{p}) = 1)$$

indeed coincides with B for K given by (38).

The monodromy representations of the braid group in the space of solutions of the KZ equation were first studied systematically in [32]. The Drinfeld–Kohno theorem [10, 11] (see also [28], Chap. 19) says, essentially, that for generic q this monodromy representation is always given by (a finite-dimensional representation of) Drinfeld's universal R matrix. In the physically interesting case where q is an (even) root of unity ($q^h = -1$) the situation is more complicated. A problem already appears in Eq. (41): for $\mathfrak{p} = h$, $[h] = 0$ and the right-hand side of (41) makes no sense. In fact, the representation of the braid group is not unitarizable for such values of $\mathfrak{p}$. The corresponding "unphysical" solutions of the KZ equation cannot, on the other hand, be thrown away by decree; otherwise, the chiral field algebra will not be closed under multiplication.

It turns out that a monodromy representation of the braid group can, in fact, be defined on the entire space of solutions of the KZ equation. It is, in general, indecomposable. The above s-channel basis, however, does not extend to $\mathfrak{p} = h$ [as is manifested in Eq. (30)].

5. REGULAR BASIS OF SOLUTIONS OF THE KZ EQUATION AND SCHWARZ FINITE MONODROMY PROBLEM

It follows from (39) and (40) that the chiral fields (unlike the CVO φ^A_j) satisfy $\mathfrak{p}$-independent (and, hence, nonsingular) exchange relations:

$$\varphi^B_\alpha(z_2)\overset{\curvearrowright}{\ }\varphi^A_\beta(z_1) = \varphi^A_\rho(z_1)\varphi^B_\sigma(z_2)\hat{R}^{\rho\sigma}_{\alpha\beta}, \tag{42}$$
$$\hat{R} = \bar{q}^{\frac{1}{N}}(q\mathbb{1} - A), \quad A^{\rho\sigma}_{\alpha\beta} = q^{\epsilon_{\sigma\rho}}\delta^\rho_\sigma\delta^\sigma_\beta - \delta^\rho_\beta\delta^\sigma_\alpha,$$

$$\epsilon_{\sigma\rho} = \begin{cases} 1, & \sigma > \rho, \\ 0, & \sigma = \rho, \\ -1, & \sigma < \rho, \end{cases} \tag{43}$$
$$A^2 = [2]A, \quad [2] = q + \bar{q}.$$

The singularity in the conformal block (30) for $\mathfrak{p} = h$ is thus a consequence of the introduction of CVO which pretend to diagonalize the (in general, nondiagonalizable) monodromy matrix M defined by $\varphi(ze^{2\pi i}) = \varphi M$. A regular basis of conformal blocks is linked to a regular basis in $U_q(sl_N)$ invariant tensors (with respect to the indices $\alpha, \beta, \dots$). Such a basis has been introduced for $N = 2$ in [38] and recently generalized to four-point blocks involving a pair of $U_q(sl_N)$ step operators [34]. Its counterpart in the space of conformal blocks of the $SU(N)$ WZNW model was written down in [35] (for $N = 2$) and in [34] for arbitrary N. We shall display here a regular basis of four-point conformal blocks f^ℓ_λ, only mentioning in conclusion some properties of their quantum group counterparts $\mathcal{I}^\lambda$.

Writing the Möbius invariant amplitude (22) in the form

$$F(\eta) = F_0(\eta)\mathcal{I}^0 + F_1(\eta)\mathcal{I}^1, \tag{44}$$
$$F_\lambda(\eta) = (1-\eta) f^0_\lambda(\eta) I_0 + \eta f^1_\lambda(\eta) I_1,$$

we define the regular basis by

$$B\left(\frac{\mathfrak{p}-1}{h}, \frac{2}{h}\right) f^\ell_0(\eta) \tag{45}$$
$$= \int_\eta^1 t^{\frac{\mathfrak{p}-1}{h} - \ell}(1-t)^{\frac{1}{h} - 1 + \ell}(t-\eta)^{\frac{1}{h}-1} dt$$
$$= B\left(\frac{1}{h}, \ell + \frac{1}{h}\right)(1-\eta)^{\frac{2}{h} - 1 + \ell}$$
$$\times F(\ell - \frac{\mathfrak{p}-1}{h}, \ell + \frac{1}{h}; \ell + \frac{2}{h}; 1-\eta),$$

$$B\left(\frac{\mathfrak{p}-1}{h}, \frac{2}{h}\right) f^\ell_1(\eta) \tag{46}$$

$$= \int_0^\eta t^{\frac{\mathfrak{p}-1}{h}-\ell}(1-t)^{\frac{1}{h}-1+\ell}(\eta-t)^{\frac{1}{h}-1}dt$$

$$= B\left(\frac{1}{h}, 1-\ell+\frac{\mathfrak{p}-1}{h}\right)\eta^{\frac{\mathfrak{p}}{h}-\ell}$$

$$\times F\left(1-\ell-\frac{1}{h}, 1-\ell+\frac{\mathfrak{p}-1}{h}; 1-\ell+\frac{\mathfrak{p}}{h}; \eta\right)$$

$(B(\mu,\nu) = \Gamma(\mu)\Gamma(\nu)/(\Gamma(\mu+\nu))$. A direct computation using the integral representations for $f_\lambda^\ell(\eta)$ yields the following form for the braid matrix B_1 exchanging the arguments 1 and 2 [the counterpart of B (36) in the regular basis, B and B_1 having the same eigenvalues]:

$$B_1 = \bar{q}^{\frac{1}{N}}\begin{pmatrix} q & 1 \\ 0 & -\bar{q}\end{pmatrix}, \quad (47)$$

$$\det(q^{\frac{1}{N}}B_1) = -1 = \det(q^{\frac{1}{N}}B),$$

$$\mathrm{tr}(q^{\frac{1}{N}}B_1) = q-\bar{q} = \mathrm{tr}(q^{\frac{1}{N}}B).$$

B_1 and B are thus related by a similarity transformation whenever both make sense:

$$B_1 = SBS^{-1}, \quad S = \begin{pmatrix} 1 & 0 \\ -\frac{[\mathfrak{p}-1]}{[\mathfrak{p}]} & \rho(\mathfrak{p})\frac{[\mathfrak{p}-1]}{[\mathfrak{p}]}\end{pmatrix}, \quad (48)$$

$$S^{-1} = \begin{pmatrix} 1 & 0 \\ \rho(-\mathfrak{p}) & \rho(-\mathfrak{p})\frac{[\mathfrak{p}]}{[\mathfrak{p}-1]}\end{pmatrix}.$$

Similarly, the exchange matrix B_2 (corresponding to the braiding $\overset{\frown}{2\,3}$) is given by

$$B_2 = Sq^{\frac{1-\mathfrak{p}}{N}}\begin{pmatrix} -\bar{q} & 0 \\ 0 & q^{\mathfrak{p}-1}\end{pmatrix}S^{-1} \quad (49)$$

$$= q^{\frac{1-\mathfrak{p}}{N}}\begin{pmatrix} -\bar{q} & 0 \\ \frac{q^{\mathfrak{p}-2}-\bar{q}^{\mathfrak{p}}}{1-\bar{q}^{\mathfrak{p}}} & q^{\mathfrak{p}-1}\end{pmatrix}.$$

We observe that, unlike B, the matrices B_1 and B_2 are defined for $0 < \mathfrak{p} < 2h$. The singularity in S (48), as well as the nonexistence of the s-channel basis for $\mathfrak{p} = h$, is due to the simple fact that the matrix B_2 (49) is nondiagonalizable in this case (while the s-channel basis could be defined as "the basis in which B_2 is diagonal"). Note that for $\mathfrak{p} = 2$ B_2 becomes similar to B_1,

$$B_2 = \bar{q}^{\frac{1}{N}}\begin{pmatrix} -\bar{q} & 0 \\ 1 & q\end{pmatrix} = \sigma_1 B_1 \sigma_1, \quad (50)$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} \quad (\text{for } \mathfrak{p} = 2),$$

and B_1 and B_2 generate a representation of the braid group $\mathcal{B}_3$ with central element $c^3 = (B_1B_2)^3 = \bar{q}^{\frac{6}{N}}\mathbb{1}$.

Whenever the s-channel basis (30), (31), exists, it is related to the regular basis (44)–(46) by

$$F_0(\eta)\mathcal{I}^0 + F_1(\eta)\mathcal{I}^1 = s_0(\eta)\mathcal{S}^0 + s_1(\eta)\mathcal{S}^1. \quad (51)$$

Here, $\mathcal{I}^0$ and $\mathcal{S}^0$ are both proportional (and can be chosen equal) to

$$(\mathcal{S}^0 = \mathcal{I}^0 =)\langle p''|a^i_{\alpha_1}a^j_{\alpha_2}|p'\rangle \quad (52)$$

$$\text{for } i<j \text{ then } \mathcal{S}^1 = \langle p''|a^j_{\alpha_1}a^i_{\alpha_2}|p'\rangle.$$

[If p' is the symmetric tensor representation—see (A.14)—then we choose $i = 1, j = 2$.] The invariant tensor $\mathcal{I}^1$, on the other hand, is related to $\mathcal{I}^0$ by

$$\mathcal{I}^1_{\ldots\alpha_1\alpha_2\ldots} = -\mathcal{I}^0_{\ldots\sigma_1\sigma_2\ldots}A^{\sigma_1\sigma_2}_{\alpha_1\alpha_2}, \quad (53)$$

where A is the quantum antisymmetrizer defined in (43). The exchange relations (40) with the dynamical R matrix (41) then allow us to relate $\mathcal{S}^\lambda$ with $\mathcal{I}^\lambda$ and conversely:

$$\rho(\mathfrak{p})\mathcal{S}^1_{\alpha\beta} = \mathcal{I}^0_{\alpha\beta} + \frac{[\mathfrak{p}]}{[\mathfrak{p}-1]}\mathcal{I}^1_{\alpha\beta}, \quad (54)$$

$$\mathcal{I}^1_{\alpha\beta} = \frac{[\mathfrak{p}-1]}{[\mathfrak{p}]}\left(\rho(\mathfrak{p})\mathcal{S}^1_{\alpha\beta} - \mathcal{S}^0_{\alpha\beta}\right).$$

For $\mathfrak{p}(= {p'}_{ij}) = h$, $\mathcal{S}^0$ and $\mathcal{S}^1$ are proportional, $\mathcal{S}^0 = \rho(h)\mathcal{S}^1$, so that they do not form a basis; $\mathcal{I}^1$, on the other hand, is defined unambiguously by (53) and is linearly independent of $\mathcal{I}^0$.

The above regular basis also has a remarkable number theoretic property: the matrix elements of $q^{\frac{1}{N}}B_1$ (and of $q^{\frac{\mathfrak{p}-1}{N}}B_2$) belong to the cyclotomic field $\mathbb{Q}(q)$ of polynomials in q with rational coefficients for $q^h = -1$. This fact has been used in [39] to classify all cases in which the monodromy representation of the braid group $\mathcal{B}_3$ ($\mathcal{B}_4$, for $N = 2$) is a finite matrix group or, equivalently, the cases in which the KZ equation has an algebraic solution (a classical problem solved for the hypergeometric equation by H.A. Schwarz in the 1870s). The solution uses one of the oldest and most beautiful concepts in group theory, the Galois group, so it deserves to be summarized.

The space of $U_q(sl_N)$ invariants admits a braid-invariant Hermitian form $(,)$. In the regular basis, $Q^{\lambda\mu} \equiv (\mathcal{I}^\lambda, \mathcal{I}^\mu)$ belong to the real subfield $\mathbb{Q}([2]) = \mathbb{Q}(q+\bar{q})$ of $\mathbb{Q}(q)$. The special case of $N = 2$ is worked out in Appendix C. In that case, the resulting Hermitian form Q is positive semidefinite for $q = e^{\pm i\frac{\pi}{h}}$ and has a kernel of dimension $2\mathfrak{p} - h$ for $2\mathfrak{p} > h$. For the case $\mathfrak{p} = 2$ of interest, this kernel is only nontrivial at level 1, for $h = 3$, when it is one-dimensional.

We define a primitive root of the equation $q^h = -1$. Let $P_h(q)$ be an irreducible element of the ring of polynomials with integer coefficients satisfying $P_h(e^{\pm i\frac{\pi}{h}}) = 0$. There is a unique such irreducible polynomial with coefficient to the highest power of q equal to 1. The *Galois group* Gal_h for P_h, the group that permutes its roots, consists of all substitutions of the form

$$\mathrm{Gal}_h = \{q \to q^\ell,\ 0 < \ell < 2h,\ (\ell, 2h) = 1\} \tag{55}$$

(in the last condition in the definition we use the familiar notation (ℓ, m) for the greatest common divisor of ℓ and m).

A Hermitian form with entries in a cyclotomic field $\mathbb{Q}(q)$ is called totally positive if all its Galois transforms are positive. Our analysis is based on the following theorem. The total positivity of a $\mathcal{B}_n$-invariant form Q is sufficient (and, if the invariant form is unique, also necessary) for the monodromy representation of $\mathcal{B}_n$ in q-$\mathrm{Inv}(V^{\otimes n})/\mathrm{Ker}Q$ to be a finite matrix group. For $N = 2$ and $h > 3$, we find [39] that the total positivity of Q is equivalent to the total positivity of the quantum dimension $[3] = (q^3 - \bar{q}^3)/(q - \bar{q}) \equiv q^2 + 1 + \bar{q}^2$ encountered in the tensor product expansion of the tensor square of the two-dimensional representation: $[2]^2 = 1 + [3]$. This amounts to finding the values of $h \geq 4$ such that

$$1 + \cos\frac{2\pi\ell}{h} > 0 \quad \text{for} \quad (\ell, 2h) = 1. \tag{56}$$

The only solutions are $h = 4, 6, 10$. If we add to these the case $h = 3$ in which the commutator subgroup of $\mathcal{B}_4$ is trivial $(B_0 B_1 B_0^{-1} B_1^{-1} = 1 = B_1 B_2 B_1^{-1} B_2^{-1} = \ldots)$, we see that the four cases of "finite monodromy" correspond to the four integral quadratic algebras of dimension $h - 2 = 1, 2, 4, 8$.

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APPENDIX A

Reduction of the KZ Equation for $SU(N)$ Step Operators to an N-Independent System of Hypergeometric Equations

The "wave function" $\Psi(p; z_1, z_2, z_3)$ can be viewed as a $z_0 \to \infty$ limit of a Möbius and $SU(N)$-invariant four-point function

$$w(z_0; z_1, z_2, z_3) \tag{A.1}$$
$$= \langle 0|\phi^*(z_0) \otimes \varphi(z_1) \otimes \varphi(z_2) \otimes \varphi(z_3)|0\rangle,$$
$$\phi(0)|0\rangle \sim \begin{smallmatrix}\square\square\\ \square\ \end{smallmatrix}.$$

The step operator φ and the field ϕ have $su(N)$ weights Λ_1 and $\Lambda_1 + \Lambda_2$ (Λ_j, $j = 1, \ldots, N-1$, being the fundamental $su(N)$ weights). Their conformal dimensions are

$$\Delta = \Delta(\Lambda_1) = \frac{1}{2h}C_2(\Lambda_1) = \frac{N^2-1}{2hN}, \tag{A.2}$$
$$\Delta_\phi = \Delta(\Lambda_1 + \Lambda_2) = \frac{1}{2h}C_2(\Lambda_1 + \Lambda_2)$$
$$= \frac{3}{2hN}(N^2 - 3),$$

where $C_2(\Lambda)$ stands for the eigenvalue of the second-order Casimir invariant (normalized in such a way that for the adjoint representation $C_2(\Lambda_1 + \Lambda_{N-1}) = 2N$). Möbius (i.e., $SL(2)$) invariance implies that we can write w in the form

$$w(z_0; z_1, z_2, z_3) = D_N(z_{ab})F(\eta), \tag{A.3}$$
$$\eta = \frac{z_{01}z_{23}}{z_{02}z_{13}}.$$

The prefactor D_N is a product of powers of the coordinate differences z_{ab} determined from infinitesimal Möbius invariance,

$$\Bigg(z_0^\nu\Big(z_0\frac{\partial}{\partial z_0} + (\nu+1)\Delta_\phi\Big) \tag{A.4}$$
$$+ \sum_{c=1}^{3} z_c^\nu\left(z_c\frac{\partial}{\partial z_c} + (\nu+1)\Delta\right)\Bigg)D_N(z_{ab}) = 0,$$
$$\nu = 0, \pm 1,$$

up to powers of η and $(1-\eta)$ which are fixed by requiring that there exist a solution $F(\eta)$ of the resulting ordinary differential equation that takes finite nonzero values for both $\eta = 0$ and $\eta = 1$:

$$D_N(z_{ab}) = \left(\frac{z_{13}^{2N+4}}{z_{03}^{3N+5}z_{12}}\right)^{\frac{N-2}{2Nh}} \tag{A.5}$$
$$\times \frac{(1-\eta)^{\frac{(N-4)(N+1)}{2Nh}}}{z_{01}^{\frac{N^2+N-3}{Nh}}\, z_{23}^{\frac{N+1}{Nh}}} = \left(\frac{z_{13}^{-3}(\eta(1-\eta))^{-N-1}}{z_{02}^{N^2-1}(z_{01}z_{03})^{N^4-4}}\right)^{\frac{1}{Nh}}.$$

Comparing the last expression with (18), we find the relation

$$\Psi_p(z_1, z_2, z_3) \tag{A.6}$$
$$= \lim_{z_0 \to \infty}\{z_0^{2\Delta_\phi}w(z_0; z_1, z_2, z_3)\}$$
$$= \left(z_{02}^{N^2-1}(z_{01}z_{03})^{N^2-4}\right)^{\frac{1}{Nh}} w(z_0; z_1, z_2, z_3).$$

Applying to the four-point function w (A.3) the covariant derivative $h\nabla_1$ (10),

$$h\nabla_1 = h\frac{\partial}{\partial z_1} + \frac{C_{01}}{z_{01}} - \frac{C_{12}}{z_{12}} - \frac{C_{13}}{z_{13}}, \tag{A.7}$$

and using the $SU(N)$ invariance condition

$$(C_{01}+C_{12}+C_{13}+C_2(\Lambda_1)) \times w(z_0; z_1, z_2, z_3) = 0, \quad (A.8)$$

we end up with (19) for Ω_{ab} given by (20). The operators Ω_{12} and Ω_{23} have an algebraic characterization of Temperley–Lieb type {see [34], Eq. (2.14)}:

$$\Omega_{12}\Omega_{23}\Omega_{12} = \Omega_{12}, \quad (A.9)$$
$$\Omega_{23}\Omega_{12}\Omega_{23} = \Omega_{23}, \quad \Omega_{ab}^2 = 2\Omega_{ab}.$$

In particular, each Ω_{ab} has eigenvalues 0 and 2. If we regard ϕ^* as a mixed tensor of $2N-3$ indices, $\phi^* = \{(\phi^*)^{B_1\dots B_{N-1}C_1\dots C_{N-2}}\}$, then the $SU(N)$ invariant tensors I_0 and I_1 of Eq. (21) can be presented in the form

$$I_0 = \left(\epsilon^{B_1\dots B_{N-1}A_1}\epsilon^{C_1\dots C_{N-2}A_2A_3}\right), \quad (A.10)$$
$$I_1 = (P_{12}-1)I_0,$$

where ϵ is the totally antisymmetric Levi-Civita tensor and P_{12} permutes the indices A_1 and A_2. In this basis, the operators Ω_{12} and Ω_{23} have the following matrix realization:

$$\Omega_{12} = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Omega_{23} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}, \quad (A.11)$$

i.e., $\Omega_{12}I_0 = 2I_0 + I_1$, $\Omega_{23}I_1 = I_0 + 2I_1$, etc. Remarkably, the relations (A.9) and (A.11), are independent of N. Inserting (22) into (19) and using (A.11), we thus end up with the N-independent system (23) of a hypergeometric type.

APPENDIX B

Shifted $SU(N)$ Weights and CVO: Symmetric Tensor Representations

If $\Lambda = \sum_{i=1}^{N-1}\lambda_i\Lambda_i$, $\lambda_i \in \mathbb{Z}_+$, is an $su(N)$ highest weight (λ_i being the number of columns of height i in the associated Young tableau), then the corresponding shifted weight is written in terms of barycentric coordinates $p = (p_1, \dots, p_N)$ as follows:

$$p = \Lambda + \rho = \sum_{i=1}^{N-1} p_{ii+1}\Lambda_i, \quad p_{ij} = p_i - p_j, \quad (A.12)$$
$$p_{ii+1} = \lambda_i + 1, \quad \sum_{i=1}^{N} p_i = 0$$

($\rho = \sum_{i=1}^{N-1}\Lambda_i$ is the half sum of the positive roots). The conformal dimension of a $\widehat{su}(N)$ primary field of weight p is expressed in terms of the second-order Casimir operator $C_2(p)$:

$$2h\Delta(p) = C_2(p) = \frac{1}{N}\sum_{i<j}\left(p_{ij}^2 - (j-i)^2\right) \quad (A.13)$$
$$= \frac{1}{N}\sum_{i<j} p_{ij}^2 - \frac{N(N^2-1)}{12}.$$

The CVO $\varphi_j (=\varphi_j^A(z))$ is related to the $U_q(sl_N)$ covariant field $\varphi_\alpha (= \varphi_\alpha^A(z))$ by (40).

In the example of a symmetric tensor representation p' and its counterpart p'' defined by the requirement $\dim \mathcal{I}_{p'p''} = 2$, we have

$$p'_{12} = p, \quad p'_{ii+1} = 1, \quad 2 \le i \le N-1, \quad (A.14)$$
$$C_2(p') = \frac{N-1}{N}(p-1)(p+N-1),$$
$$p''_{12} = p, \quad p''_{23} = 2, \quad (A.15)$$
$$p''_{ii+1} = 1, \quad 3 \le i \le N-1,$$
$$C_2(p'') = (p+1)\frac{N^2 + (p-2)N - (p+1)}{N}.$$

The dimensions of these representations are expressed in terms of binomial coefficients:

$$d(p') = \binom{p+N-2}{N-1}, \quad (A.16)$$
$$d(p'') = p\binom{p+N-1}{p+1}.$$

In computing the prefactor (26), one needs

$$\Delta(p'') - \Delta(p') - 2\Delta \quad (A.17)$$
$$= \frac{(N-2)(N+p)}{Nh} - \frac{N^2-1}{Nh}$$
$$= \frac{(p-2)(N-2)-3}{Nh},$$
$$\frac{\Delta(p'')-\Delta(p')}{2} - \frac{p}{2h} - \frac{N^2-2}{2Nh} \quad (A.18)$$
$$= -\frac{N+p-1}{Nh}.$$

APPENDIX C

Basis of $U_q(sl_2)$ Invariants in $V^{\otimes 4}$ for $V = \mathbb{C}^2$: Braid-Invariant Hermitian Form

The basic $U_q(sl_N)$ invariant in $V^{\otimes N}$ for $V = \mathbb{C}^N$ is the q-deformed Levi-Civita tensor

$$\mathcal{E}_{\alpha_1\dots\alpha_N} = \bar{q}^{\frac{1}{2}\binom{N}{2}}(-q^2)^{\ell\begin{pmatrix} N & \dots & 1 \\ \alpha_1 & \dots & \alpha_N\end{pmatrix}}, \quad (A.19)$$

where ℓ is the length of the permutation $\begin{pmatrix} N & \dots & 1 \\ \alpha_1 & \dots & \alpha_N\end{pmatrix}$, i.e., the minimal number of transpositions of neighboring indices; in particular, for $N = 2$,

$$(\mathcal{E}_{\alpha_1\alpha_2}) = \begin{pmatrix} 0 & -q^{\frac{1}{2}} \\ \bar{q}^{\frac{1}{2}} & 0 \end{pmatrix}, \quad (A.20)$$

$$\text{i.e., } \mathcal{E}_{21} = \bar{q}^{\frac{1}{2}}, \ \mathcal{E}_{12} = -q^{\frac{1}{2}}.$$

The regular basis of $U_q(sl_2)$ invariants in $V^{\otimes 4} = (\mathbb{C}^2)^{\otimes 4}$ is

$$\mathcal{I}^0{}_{\alpha_1\alpha_2\alpha_3\alpha_4} = \mathcal{E}_{\alpha_1\alpha_2}\mathcal{E}_{\alpha_3\alpha_4}, \quad \text{(A.21)}$$
$$\mathcal{I}^1{}_{\alpha_1\alpha_2\alpha_3\alpha_4} = \mathcal{E}_{\alpha_1\alpha_4}\mathcal{E}_{\alpha_2\alpha_3}.$$

Their inner products are given by traces,

$$(\mathcal{I}^\lambda, \mathcal{I}^\mu) = \sum_{\alpha_1\ldots\alpha_4} \mathcal{I}^\lambda{}_{\alpha_1\alpha_2\alpha_3\alpha_4}\mathcal{I}^\mu{}_{\alpha_1\alpha_2\alpha_3\alpha_4}, \quad \text{(A.22)}$$

$$(\mathcal{I}^\lambda, \mathcal{I}^\lambda) = [2]^2, \ \lambda = 0, 1, \ (\mathcal{I}^0, \mathcal{I}^1) = -[2].$$

To verify braid invariance, note that

$$B_1{}^0_\lambda \mathcal{I}^\lambda = q^{\frac{1}{2}}\mathcal{I}^0 + \bar{q}^{\frac{1}{2}}\mathcal{I}^1, \quad B_1{}^1_\lambda \mathcal{I}^\lambda = -\bar{q}^{\frac{3}{2}}\mathcal{I}^1, \quad \text{(A.23)}$$
$$(B_1{}^0_\lambda \mathcal{I}^\lambda, B_1{}^0_\mu \mathcal{I}^\mu)$$
$$= 2[2]^2 - (q + \bar{q})[2] = [2]^2 = (\mathcal{I}^0, \mathcal{I}^0), \text{ etc.}$$

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SYMPOSIUM ON QUANTUM GROUPS

Twist of Lie Algebras by a Rank-3 Subalgebra*

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Abstract—A new nonstandard deformation of all types of classical Lie algebras is constructed by means of twisting based on a six-dimensional subalgebra. This is an extension of extended twists introduced by Kulish *et al.* It is also shown that the new nonstandard $so(3,2)$ has a close connection with the symmetry of a discrete analog of the Klein–Gordon equation in $(1+2)$ spacetime. © *2001 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

It is known that there exist two types of Hopf algebras: quasitriangular and triangular [1]. The q-deformed algebras by Drinfeld and Jimbo belong to the first type. The algebras of second type are called Jordanian quantum algebras or nonstandard quantum algebras (NQA). The typical example of this type is a deformation of $sl(2)$ introduced by Ohn [2]. In general, triangular quantum algebras are obtained from Lie algebras by twisting [3].

In this article, we discuss a new nonstandard deformation of all types of classical Lie algebras based on a six-dimensional subalgebra. The obtained deformation can be regarded as an extension of the one by Kulish *et al.* [4]. As an application, it is shown that a certain discrete analog of the Klein–Gordon (KG) equation defined on a uniform lattice has a close connection with deformed $so(3,2)$. A part of these results were already presented at another conference [5]. Very recently, Kulish and Lyakhovsky derived the same deformation independently [6].

Their result is more general than the one presented here, since they use an eight-dimensional subalgebra and our six-dimensional one is regarded as a special case of it. In this article, however, more explicit examples are constructed and physical bases are used for $so(3,2)$. The discussion on the discrete KG equation in Section 4 is a new result.

The advantages of NQA are (i) their irreducible representations are known because NQA has undeformed commutation relations, and (ii) an explicit form of universal R matrix is obtained. Combining (i) and (ii), we obtain matrix representations of the universal R matrix; this gives us further advantages: (iii) dual quantum groups are easily obtained, (iv) covariant differential calculus are easily obtained, and so on.

Note that there exist some inequivalent possibilities of twisting for a given Lie algebra. Recalling that NQA have some physical applications and quantum groups are a convenient tool to describe noncommutative spacetime, one can say that it is important to investigate possible twisting for Lie algebras.

In the next section, a brief survey of the recent development of twisting is given. We shall show new NQA in Section 3. In Section 4, symmetries of a discrete analog of the KG equation in $(1+2)$ Minkowskian spacetime is analyzed. Section 5 is devoted to concluding remarks. Throughout this article, we follow the notation and conventions used in [5].

2. BRIEF SURVEY OF TWISTING

Let $\mathbf{g}$ be a Lie algebra. We look for an invertible element $\mathcal{F} \in U(\mathbf{g}) \otimes U(\mathbf{g})$, called twistor, satisfying

$$\mathcal{F}_{12}(\Delta_0 \otimes id)(\mathcal{F}) = \mathcal{F}_{23}(id \otimes \Delta_0)(\mathcal{F}), \qquad (1)$$

$$(\epsilon_0 \otimes id)(\mathcal{F}) = (id \otimes \epsilon_0)(\mathcal{F}) = 1. \qquad (2)$$

It is obvious that a twistor defined on a subalgebra $\mathcal{A} \subset \mathbf{g}$ can be regarded as a twistor for whole $\mathbf{g}$. In the study of twisting, possible subalgebras for a given $\mathbf{g}$ are investigated.

Probably, the most well-known twistors are the so-called Reshetikhin twist, [7] and Jordanian twist [8, 9]. The Cartan subalgebra is chosen as $\mathcal{A}$ in the Reshetikhin twist, so that any Lie algebras of rank ≥ 2 are twisted. On the other hand, the subalgebra for Jordanian twist is the Borel subalgebra:

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$\{H, E \mid [H, E] = 2E\}$. The explicit form of twistor is given by

$$\mathcal{F}_J = \exp\left(-\frac{1}{2}H \otimes \sigma\right), \quad \sigma = -\ln(1 - zE), \quad (3)$$

where z is a deformation parameter. We denote a deformation parameter by z throughout this article, and $z = 0$ corresponds to the undeformed limit. Recently, nontrivial extensions of Jordanian twist have been extensively studied. In [4], the Borel subalgebra is extended to a four-dimensional one that is a semidirect sum of Borel subalgebra and two additional elements A and B. Let $\mathcal{A}_E = \{H, E, A, B\}$ be the subalgebra subject to the relations

$$[H, E] = \delta E, \quad [H, A] = \alpha A, \quad [H, B] = \beta B, \quad (4)$$
$$[A, B] = \gamma E, \quad [E, A] = [E, B] = 0, \quad \alpha + \beta = \delta.$$

Then, one can check that

$$\mathcal{F}_E = \exp(A \otimes B e^{-\beta\sigma/\delta})\mathcal{F}_J \quad (5)$$

is a twistor. This twisting is called extended Jordanian twist (ET). It is verified that all types of classical Lie algebra have the subalgebra $\mathcal{A}_E$. The extension considered in [10] is certain limits of ET and called peripheric extended twists. The ET have nontrivial limits for $\alpha \to 0$ or $\beta \to 0$. One can verify that the peripheric extended twists are applicable to inhomogeneous Lie algebras such as $isu(n)$ and $iso(n)$ [11]. The subalgebra $\mathcal{A}_E$ is enlarged to an eight-dimensional one in [6].

It is shown in [12] that regular injections $\mathcal{A}_p \subset \mathcal{A}_{p-1} \subset \ldots \subset \mathcal{A}_1 \subset \mathcal{A}_0$ of Lie algebras $sl(n)$ and $so(n)$ can provide a carrier space of twisting; that is, a product of ET twistors corresponding to each subset $\mathcal{A}_k$ produces a new twistor. The case of $sp(n)$ is also considered in [12], and further analysis is given in [13]. This construction of "chain" of twisting is developed further in the case of $so(n)$ [14, 15]. Furthermore, the twisting based on quantum Borel subalgebra is also considered and applied to hybrid (standard–nonstandard) quantization for Lie and Kac–Moody algebras [16].

3. NEW TWISTING FOR CLASSICAL LIE ALGEBRAS

Let us consider an algebra $\mathcal{A}$ of six elements H_i, E_i, A, B $(i = 1, 2)$ satisfying

$$[H_i, E_i] = 2E_i, \quad [H_1, H_2] = [E_1, E_2]$$
$$= [H_1, E_2] = [H_2, E_1] = 0, \quad [H_1, A] = -A,$$
$$[H_1, B] = B, \quad [H_2, A] = A, \quad [H_2, B] = B, \quad (6)$$
$$[A, E_1] = 2B, \quad [A, E_2] = 0,$$
$$[E_i, B] = 0, \quad [A, B] = E_2.$$

The four elements $\{H_2, E_2, A, B\}$ form the subalgebra $\mathcal{A}_E$ of ET $(\alpha = \beta = \gamma = 1)$. The additional elements H_1, E_1 form a Borel subalgebra; thus, the algebra $\mathcal{A}$ is a semidirect sum of $\mathcal{A}_E$ and an extra Borel subalgebra. The following invertible element $\mathcal{F}$ satisfies the definition of twistor

$$\mathcal{F} = \exp\left(-\frac{1}{2}H_1 \otimes \sigma_1\right) \quad (7)$$
$$\times \exp(-zA \otimes Be^{\sigma_2/2}) \exp\left(-\frac{1}{2}H_2 \otimes \sigma_2\right).$$

Here,

$$\sigma_1 = -\ln(1 - z(E_1 + zB^2 e^{\sigma_2})), \quad (8)$$
$$\sigma_2 = -\ln(1 - zE_2).$$

The two factors from the right are an ET, and the leftmost factor does not commute with the remaining part of $\mathcal{F}$. Therefore, this is a nontrivial extension of ET. The twistor (7) has similar properties as ET. Namely, the twisted coproducts for σ_i are primitive: $\Delta(\sigma_i) = \sigma_i \otimes 1 + 1 \otimes \sigma_i$ $(i = 1, 2)$, and the $\mathcal{F}$ is factorizable:

$$(\Delta_0 \otimes id)(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{23}, \ (id \otimes \Delta)(\mathcal{F}) = \mathcal{F}_{12}\mathcal{F}_{13}. \quad (9)$$

These relations guarantee that the twistor (7) satisfies the condition (1).

To prove the above statements, we first calculate the twisted coproducts of the elements of $\mathcal{A}$ by the twistor (7). They are given by

$$\Delta(H_1) = H_1 \otimes e^{\sigma_1} + 1 \otimes H_1,$$
$$\Delta(E_1) = E_1 \otimes e^{-\sigma_1} + 1 \otimes E_1$$
$$- 2zB \otimes \rho e^{-(\sigma_1+\sigma_2)/2} + z^2 E_2 \otimes \rho^2 e^{-\sigma_2},$$
$$\Delta(H_2) = H_2 \otimes e^{\sigma_2} + 1 \otimes H_2$$
$$+ 2zA \otimes \rho e^{(\sigma_1+\sigma_2)/2} + z^2 H_1 \otimes \rho^2 e^{\sigma_1}, \quad (10)$$
$$\Delta(E_2) = E_2 \otimes e^{-\sigma_2} + 1 \otimes E_2,$$
$$\Delta(A) = A \otimes e^{(\sigma_1+\sigma_2)/2} + 1 \otimes A$$
$$+ zH_1 \otimes (B + zE_2\rho)e^{\sigma_1},$$
$$\Delta(B) = B \otimes e^{-(\sigma_1+\sigma_2)/2} + e^{-\sigma_2} \otimes B,$$

where $\rho = Be^{\sigma_2}$. With these coproducts, we can verify that the σ_i's are primitive. It follows that the twistor (7) satisfies the factorizable relations (9). Thus, we have proved that the $\mathcal{F}$ satisfies the condition (1). The condition (2) is easily verified by noticing that $\epsilon_0(X) = 0$ for all elements of any Lie algebras.

We next show that all types of classical Lie algebras have the six-dimensional subalgebra $\mathcal{A}$; namely, we obtain new NQA. For $sl(n)$, it is convenient to work with the canonical basis

$$[E_{ab}, E_{cd}] = E_{ad}\delta_{bc} - E_{cb}\delta_{ad}, \quad (11)$$
$$a, b, c, d = 1, \ldots, n.$$

In this case, the six-dimensional subalgebra $\mathcal{A}$ is found for $n \geq 4$

$$\begin{aligned} H_1 &= \sum_{n/2\geq k\geq 2}(E_{kk} - E_{n-k+1,n-k+1}), \\ E_1 &= \sum_{n/2\geq k\geq 2} E_{k,n-k+1}, \\ H_2 &= E_{11} - E_{nn}, \quad E_2 = E_{1n}, \\ A &= 2\sum_{n/2\geq k\geq 2} b^{1,n-k+1}E_{1k} \\ &\quad - 2\sum_{n-1\geq\lambda>n/2} b^{n-\lambda+1,n}E_{\lambda n}, \\ B &= \sum_{n/2\geq k\geq 2} b^{k,n}E_{kn} + \sum_{n-1\geq\lambda>n/2} b^{1,\lambda}E_{1\lambda}, \end{aligned} \tag{12}$$

where the complex coefficients $b^{a,b}$ have to satisfy

$$4\sum_{n/2\geq k\geq 2} b^{1,n-k+1}b^{k,n} = 1. \tag{13}$$

This condition on the coefficients stems from the commutation relation $[A, B] = E_2$. Other commutation relations hold for any values of $b^{a,b}$.

We also use the canonical basis for $so(n)$

$$\begin{aligned} &[Y_{ab}, Y_{cd}] \\ &= i(Y_{ad}\delta_{bc} + Y_{bc}\delta_{ad} - Y_{ac}\delta_{bd} - Y_{bd}\delta_{ac}), \end{aligned} \tag{14}$$

where $a, b, c, d = 1, \ldots, n$ and $Y_{ab} = -Y_{ba}$. In this case, the subalgebra $\mathcal{A}$ is found for $n \geq 5$:

$$\begin{aligned} H_1 &= Y_{1n} + Y_{1\,n-1}, \\ E_1 &= Y_{1\,n-1} - Y_{2n} + iY_{12} + iY_{n-1\,n}, \\ H_2 &= Y_{1n} - Y_{1\,n-1}, \\ E_2 &= \frac{1}{2}(Y_{1\,n-1} + Y_{2n} - iY_{12} + iY_{n-1\,n}), \\ A &= \sum_{k=3}^{n-2} a^k(Y_{n-1\,k} - iY_{2k}), \\ B &= \sum_{k=3}^{n-2} a^k(Y_{kn} - iY_{1k}). \end{aligned} \tag{15}$$

The commutation relation $[A, B] = E_2$ imposes a condition on the coefficients a^k,

$$2\sum_{k=3}^{n-2}(a^k)^2 = 1, \qquad a^k \in \mathbf{C}. \tag{16}$$

For $sp(2n)$, the subalgebra $\mathcal{A}$ is found for $n \geq 2$. In terms of the canonical basis

$$\begin{aligned} [Z_{ab}, Z_{cd}] &= \mathrm{sgn}(bc)(Z_{ad}\delta_{bc} + Z_{-b\,-c}\delta_{ad} \\ &\quad + Z_{a-c}\delta_{-bd} + Z_{-bd}\delta_{c\,-a}), \end{aligned} \tag{17}$$

where $a, b = \pm 1, \ldots, \pm n$ and $Z_{ab} = -\mathrm{sgn}(ab)Z_{-b\,-a}$, the subalgebra $\mathcal{A}$ is given by

$$\begin{aligned} H_1 &= \sum_{k=2}^{n} Z_{kk}, \quad E_1 = \sum_{k=2}^{n} Z_{k\,-k}, \\ H_2 &= Z_{11}, \quad E_2 = Z_{1\,-1}, \\ A &= \sum_{k=2}^{n} a^k Z_{1k}, \quad B = \sum_{k=2}^{n} a^k Z_{k\,-1}. \end{aligned} \tag{18}$$

A condition on the coefficients a^k is obtained in the same way as $sl(n)$ and $so(n)$,

$$\sum_{k=2}^{n}(a^k)^2 = 1, \quad a^k \in \mathbf{C}. \tag{19}$$

We have seen that all types of classical Lie algebras can be twisted by the six-dimensional subalgebra $\mathcal{A}$. Other combinations of elements of Lie algebras could realize the subalgebra $\mathcal{A}$. An appropriate choice could be found when physical applications of twisted algebras are considered.

4. SYMMETRY OF KLEIN–GORDON EQUATION ON LATTICE

In this section, we consider a discrete analog of the KG equation in $(1+2)$-dimensional Minkowskian spacetime. For the massless KG equation

$$(\partial_0^2 - \partial_1^2 - \partial_2^2)\phi = 0, \tag{20}$$

it is known that its symmetry is given by $so(3, 2)$ [17]. By symmetry, we mean a set of transformations of solutions back into other solutions; namely, if ϕ is a solution of (20) and X generates a symmetry, then $X\phi$ is also a solution of (20).

Let us consider the following discrete version of (20):

$$\begin{aligned} \mathcal{L}\phi \equiv \Bigg\{ &\left(\frac{1 - e^{-z(\partial_0 - \partial_1)}}{z}\right) \\ &\times \left(\frac{1 - e^{-z(\partial_0 + \partial_1)}}{z}\right) - e^{-z(\partial_0 - \partial_1)}\partial_2^2 \Bigg\}\phi = 0. \end{aligned} \tag{21}$$

Note that $e^{\pm z\partial_\mu}$ is a shift operator: $e^{\pm z\partial_0}\phi(x_0, x_1, x_2) = \phi(x_0 \pm z, x_1, x_2)$, etc. Thus, $(1 - e^{\pm z\partial_\mu})/z$ is a difference operator and Eq. (21) is a difference-differential equation defined on a uniform lattice. The constant z is a lattice spacing, and Eq. (21) is reduced to (20) in the limit of $z \to 0$. We now consider the difference-differential operators

$$\begin{aligned} J &= \mathcal{X}_1\mathcal{P}_2 - \mathcal{X}_2\mathcal{P}_1, \quad P_\mu = \mathcal{P}_\mu, \\ K_i &= \mathcal{X}_i\mathcal{P}_0 - \mathcal{X}_0\mathcal{P}_i, \qquad D = -\mathcal{X}^\mu\mathcal{P}_\mu - 1/2, \\ C_\mu &= -\mathcal{X}_\mu\mathcal{X}^\nu\mathcal{P}_\nu + \frac{1}{2}\mathcal{X}^2\mathcal{P}_\mu - \frac{1}{2}\mathcal{X}_\mu, \end{aligned} \tag{22}$$

where $\mu, \nu = 0, 1, 2,\ i = 1, 2$, and

$$\mathcal{P}_0 + \mathcal{P}_1 = \frac{1 - e^{-z(\partial_0 + \partial_1)}}{z} - z e^{-z(\partial_0 - \partial_1)} \partial_2^2,$$

$$\mathcal{P}_0 - \mathcal{P}_1 = \frac{1 - e^{-z(\partial_0 - \partial_1)}}{z}, \qquad \mathcal{P}_2 = e^{-z(\partial_0 - \partial_1)} \partial_2,$$

$$\mathcal{X}_0 + \mathcal{X}_1 = (x_0 + x_1) e^{z(\partial_0 - \partial_1)} \tag{23}$$

$$+ z^2 (x_0 - x_1) e^{z(\partial_0 + \partial_1)} \partial_2^2 - 2 z x_2 e^{z(\partial_0 - \partial_1)} \partial_2,$$

$$\mathcal{X}_0 - \mathcal{X}_1 = (x_0 - x_1) e^{z(\partial_0 + \partial_1)},$$

$$\mathcal{X}_2 = x_2 e^{z(\partial_0 - \partial_1)} - z(x_0 - x_1) e^{z(\partial_0 + \partial_1)} \partial_2.$$

It is straightforward to verify the commutation relations

$$[X, \mathcal{L}] = 0, \quad \text{for } X \in \{P_\mu, J, K_i\}, \tag{24}$$

$$[D, \mathcal{L}] = 2\mathcal{L}, \qquad [C_\mu, \mathcal{L}] = 2\mathcal{X}_\mu \mathcal{L}.$$

It follows that the operators (22) generate the symmetry of (21). It is also straightforward to verify the relation

$$[\mathcal{P}_\mu, \mathcal{P}_\nu] = [\mathcal{X}_\mu, \mathcal{X}_\nu] = 0, \quad [\mathcal{P}_\mu, \mathcal{X}_\nu] = g_{\mu\nu}, \tag{25}$$

where $g = \mathrm{diag}(1, -1, -1)$ is the metric. It follows that the operators (22) give a difference-differential realization of $so(3,2)$. Therefore, the Lie algebra $so(3,2)$ is a symmetry algebra of the difference-differential Eq. (21). Note that the operators (22) are reduced to the realization of $so(3,2)$ given in [17] in the limit of $z \to 0$. In this limit, each operator has the following physical meaning: P_μ spacetime translations, J rotation, K Lorentz boosts, D dilatation, C_μ conformal transformations.

On the other hand, $so(3,2)$ has a nonstandard deformation discussed in Section 3. This deformation corresponds to the one in [18] (see [5, 6]). The subalgebra $\mathcal{A}$ is given by

$$H_1 = D + K_1, \quad E_1 = P_0 + P_1, \tag{26}$$

$$H_2 = D - K_1, \quad E_2 = P_0 - P_1,$$

$$A = K_2 + J, \quad B = P_2.$$

Since the deformation is independent of realizations and deformed $so(3,2)$ has the same commutation relations as the undeformed one, we can say that the nonstandard deformation of $so(3,2)$ is also a symmetry algebra of Eq. (21). Other nonstandard deformations of $so(3,2)$ could also give symmetries of Eq. (21). In this sense, the symmetries of the equation are "degenerate." However, one can find a close relation between Eq. (21) and the particular nonstandard $so(3,2)$ given by (26). The primitive elements of the deformed $so(3,2)$ are given by

$$\sigma_1 = -\ln(1 - z(P_0 + P_1 + z P_2 \rho)), \tag{27}$$

$$\sigma_2 = -\ln(1 - z(P_0 - P_1)),$$

and letting $\rho = P_2 e^{\sigma_2}$. In the undeformed limit, they are reduced to

$$\frac{\sigma_1}{z} \to P_0 + P_1, \quad \frac{\sigma_2}{z} \to P_0 - P_1, \quad \rho \to P_2. \tag{28}$$

These elements may be realized by

$$\frac{\sigma_1}{z} = \partial_0 + \partial_1, \quad \frac{\sigma_2}{z} = \partial_0 - \partial_1, \quad \rho = \partial_2, \tag{29}$$

and by solving (29) with respect to P_μ, we obtain the realization of $\mathcal{P}_\mu$ given in (23). It is clear that the deformation parameter is identified with the lattice spacing. It follows that our deformation parameter has a dimension of length. Such deformation of Lie algebras with dimensional deformation parameters has been considered before, for instance, κ-deformation of Poincaré algebra [19] and $D = 3, 4$ conformal algebras [20].

Before closing this section, we should mention previous works. Relations between discrete Schrödinger equations and NQA are discussed in [21] (see also the references therein) from the viewpoint of deformation map. Similar considerations for other wave equations in $(1+1)$ spacetime are made in [22]. Realizations of undeformed Lie algebras as symmetries of various difference equations are considered in [23–26].

5. CONCLUDING REMARKS

In this article, we have shown a new twistor that is an extension of ET applicable to all types of classical Lie algebras. Consequently, new nonstandard deformations of $sl(n), so(n)$, and $sp(2n)$ were obtained. It is natural to ask whether the peripheric extended twists have a similar extension. Since the twists discussed in Section 3 do not contain free parameters (α, β, and γ of the extended twists), we cannot repeat the same discussion as [10]. However, it turns out that the peripheric extended twists have an extension [11] where a five-dimensional subalgebra is used instead of the six-dimensional one. This extension of peripheric extended twists is appropriate to deform inhomogeneous Lie algebras.

Symmetries of a discrete analog of the KG equation on a uniform lattice were also investigated. We saw that the undeformed $so(3,2)$ realized by difference-differential operators were a symmetry algebra and those operators had a close connection with the twisting presented in Section 3. A systematic study of symmetries of difference equations is possible, and it turns out that difference equations, in many cases, have the same symmetry algebras as their continuum limit [27]. This will be published elsewhere.

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SYMPOSIUM ON QUANTUM GROUPS

$3nj$ Coefficients of $u_q(2)$ and Multiple Basic Hypergeometric Series*

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Abstract—Triple-sum formulas for $9j$ coefficients and multiple-sum expressions [with five or four separate sums of the ${}_{p+1}F_p(1)$ or ${}_{p+1}\phi_p$ type, $p = 2, 3, 4$] for the $12j$ coefficients of both kinds (with or without braiding) of the $SU(2)$ group and the quantum algebra $u_q(2)$ are derived, eliminating sums over the j type parameters [q generalizations of the very well poised (Dougall's type) hypergeometric ${}_4F_3(-1)$, ${}_5F_4(1)$, and ${}_6F_5(-1)$ series] from their expansions in terms of q-$6j$ coefficients. The rearrangements of the derived formulas for generic and stretched q-$9j$ coefficients (related to the q versions of some Kampé de Fériet series) are discussed, as well as the different versions of stretched and doubly stretched q-$12j$ coefficients.

1. INTRODUCTION

The $3nj$ coefficients arise as the recoupling coefficients of the several irreducible representations (irreps) of the $SU(2)$ group in the angular momentum theory [1, 2]. Although many expressions for $9j$ coefficients as multiple series exist, the most compact triple-sum formula was derived originally byAlišauskas and Jucys [3], in context of the coupling problem [3] of the $Sp(4)$ [$SO(5)$] group irreps (see also [2]). Recently, Rosengren [4] proposed two new proofs of related formula for $9j$ coefficients of $SU(1,1)$, in the second case rearranging the usual expansion of $9j$ coefficients in terms of $6j$ coefficients, after the appropriate expressions for the Racah coefficients of $SU(1,1)$ in terms of the balanced hypergeometric ${}_4F_3(1)$ series and Dougall's summation formula [5] of the very well poised ${}_4F_3(-1)$ series were used.

For the quantum algebra $u_q(2)$, the expansion of the $9j$ coefficients in terms of $6j$ coefficients was generalized by Nomura [6, 7] and Smirnov *et al.* [8] and extended to q-$3nj$ coefficients (including the q-$12j$ coefficients of the first and second kind [7]). The corresponding summation formula of the twisted q-factorial series (generalizing Dougall's summation formula for q-factorial sums of the very well poised type, depending on three parameters) needed for our purpose was proposed in [9] as a special case of the twisted very well poised q-factorial series, resembling ${}_7\phi_6$ series (depending on five parameters), which also appear in a new approach [10] to the Clebsch–Gordan (CG) coefficients of $u_q(2)$. The summation formula of the q-factorial series depending on four parameters, which correspond to Dougall's summation formula of ${}_5F_4(1)$ or ${}_6\phi_5$ series [11], was also used in the $u_q(3)$ context [9].

In this paper, the derived new expressions with the triple sums for the q-$9j$ (and usual $9j$) coefficients are discussed, as well as the expressions for the q-$12j$ (and $12j$) coefficients of both kinds, with eliminated cumbersome factorial sums weighted with factors $[2j + 1]$ or $(2j + 1)$. We begin from an expression with five sums for $12j$ and q-$12j$ coefficients of the first kind (with braiding [7]) in terms of $6j$ coefficients, whose specifications correspond to the stretched and doubly stretched q-$12j$ coefficients of the first kind, and, particularly, turn to the known expression [3] for $9j$ coefficients of $SU(2)$. Some from six new triple sum formulas for q-$9j$ coefficients and their mutual rearrangement possibilities by means of the Chu–Vandermonde summation formulas are also considered, as well as some double sum formulas for the stretched $9j$ coefficients, enabling us to get new relations and summation formulas for special Kampé de Fériet functions [12] and their q generalizations (cf. [13]). The q-$12j$ coefficients of the second kind [1, 2] (i.e., without braiding [7]) are rearranged into the fourfold (balanced) series, with specific stretched and doubly stretched q-$12j$ coefficients (for exhaustive investigation see [14]).

Appropriate for our purpose, two expressions for the $6j$ (Racah) coefficients of $SU(2)$ were derived originally by Bandzaitis *et al.* [2], when Smirnov *et al.* [15, 16] rederived them for the Racah coefficients of $u_q(2)$. One of them (cf. (29.1b) of [2]) is

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written as follows:

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q = \frac{\nabla[acf]\nabla[dbf]}{\nabla[abe]\nabla[dce]} \tag{1}$$

$$\times \sum_z \frac{(-1)^{a+b+c+d+z}[c+f-a+z]!}{[z]![a+c-f-z]![b+d-f-z]!}$$

$$\times \frac{[b+f-d+z]![a+d+e-f-z]!}{[e+f-a-d+z]![2f+z+1]!},$$

where only asymmetric triangle coefficients

$$\nabla[abc] = \left(\frac{[a+b-c]![a-b+c]![a+b+c+1]!}{[b+c-a]!}\right)^{1/2}$$

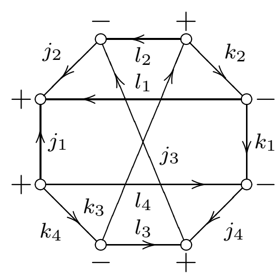
appear. Here and in what follows, $[x] = (q^x - q^{-x})/(q - q^{-1})$ and $[x]! = [x][x-1]...[2][1]$ are, respectively, the q numbers and q factorials ($[1]! = [0]! = 1$), which are invariant under the substitution $q \to q^{-1}$ and turn into usual integers x and factorials $x!$ for $q = 1$.

Each parameter b, c, or e appears only twice in the factorial arguments under the summation sign in (1), as well as parameter f, after some change of summation parameters. Otherwise, in the most symmetric (Racah) and other expressions for q-$6j$ coefficients [2, 15, 16] (including only symmetric triangle coefficients $\Delta[abc]$), all the parameters [and a or d in (1)] appear four times. Note, that only expressions of the type (1) are correlated with the Racah polynomials as introduced by Askey and Wilson (see [11]), which may appear only after some Whipple (Bailey) or Sears transform [11] of the balanced ${}_4F_3(1)$ or ${}_4\phi_3$ series is used.

2. EXPRESSIONS FOR q-$12j$ COEFFICIENTS OF THE FIRST KIND

We begin the rearrangement of expressions for the q-$12j$ coefficients of the first kind [1, 2] (whose graph is not planar), expanded [7] in terms of the factorized four differently transposed q-$6j$ coefficients,

$$\begin{Bmatrix} j_1 & j_2 & j_3 & j_4 & \\ & l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 & \end{Bmatrix}_q = \tag{2a}$$



$$= \sum_x (2x+1)(-1)^{R_4 - x} q^{x(x+1) + Z_{j_1 j_2 j_3 j_4} + Z_{k_1 k_2 k_3 k_4}}$$

$$\times \begin{Bmatrix} j_1 & j_2 & l_1 \\ k_2 & k_1 & x \end{Bmatrix}_q \begin{Bmatrix} k_2 & x & j_2 \\ j_3 & l_2 & k_3 \end{Bmatrix}_q \begin{Bmatrix} j_3 & k_3 & x \\ k_4 & j_4 & l_3 \end{Bmatrix}_q \begin{Bmatrix} k_1 & x & j_1 \\ k_4 & l_4 & j_4 \end{Bmatrix}_q \tag{2b}$$

$$= (-1)^{j_1+j_3+j_4+k_2-k_3-k_4+l_2-l_3+l_4} \frac{\nabla[j_3 j_4 l_3]\nabla[k_4 k_3 l_3]\nabla[k_2 k_3 l_2]\nabla[k_1 j_4 l_4]}{\nabla[j_1 j_2 l_1]\nabla[k_2 k_1 l_1]\nabla[j_3 j_2 l_2]\nabla[k_4 j_1 l_4]}$$

$$\times q^{(j_1+k_2-l_1+1)(l_2+k_2-l_3+k_4+1)-(l_2+k_2-j_3+1)(j_3+k_4-l_3+1)+Z_{j_1 j_2 j_3 j_4}+Z_{k_1 k_2 k_3 k_4}}$$

$$\times \sum_{z_1,z_2,z_3,z_4,u} \frac{(-1)^{z_2+z_3+z_4+u}[j_1+j_2-l_1+z_1]![2l_1-z_1]!}{[z_1]![z_2]![z_3]![k_1-k_2+l_1-z_1]![l_1-j_1+j_2-z_1]!}$$

$$\times \frac{q^{-z_2(j_1-j_3+k_2-k_4-l_1+l_3+z_3)}[j_2+j_3-l_2+z_2]![2l_2-z_2]!}{[l_2+k_2-k_3-z_2]![j_2-j_3+l_2-z_2]![l_2+k_2+k_3-z_2+1]!}$$

$$\times \frac{q^{-z_3(j_1+j_3-l_1-l_2)}[k_3-k_4+l_3+z_3]![j_4-j_3+l_3+z_3]!}{[k_3+k_4-l_3-z_3]![j_3+j_4-l_3-z_3]![2l_3+z_3+1]!}$$

$$\times \frac{[2l_4-z_4]![j_1+k_4-l_4+z_4]![k_2+k_4+l_2-l_3-z_2-z_3]!}{[z_4]![k_1+l_4-j_4-z_4]![j_1-k_1+k_2+k_4-l_1-l_4+z_1+z_4]!}$$

$$\times \frac{q^{-u(k_2+k_4+l_2-l_3-z_2-z_3+1)}[2j_1-k_1+k_2-l_1-u]!}{[k_1+l_4+j_4-z_4+1]![u+z_1]![j_1-k_4+l_4-z_1-z_4-u]!}$$

$$\times \frac{[j_1+k_1+k_2-k_4-l_1+l_4-z_4-u]!}{[j_1+j_3-l_1-l_2+z_2-u]![j_1-j_3+k_2-k_4-l_1+l_3+z_3-u]!}, \qquad (2c)$$

where

$$R_n = \sum_{i=1}^{n}(j_i+k_i+l_i) \quad \text{and}$$

$$Z_{deh} = -d(d+1) - e(e+1) - h(h+1).$$

Inserting (1) with shifted or inverted summation parameters for q-$6j$ coefficients with the parameter x in the right lower or middle upper position, respectively, and straightforwardly (1) in the remaining cases, the x-depending asymmetric triangle coefficients in expansion (2b) cancel, but with exception of the factors $\nabla[j_1k_1x]/\nabla[k_1j_1x] = [j_1-k_1+x]!/[k_1-j_1+x]!$ Then, the sum over x may be rearranged into the ${}_3\phi_2$ or ${}_3F_2[q,x]$ type series using the following formula:

$$\sum_j \frac{q^{j(j+1)}[2j+1][j-p_1-1]![j-p_2-1]![j-p_3-1]![j-p_5-1]!}{[p_1+j+1]![p_2+j+1]![p_3+j+1]![p_4-j]![p_4+j+1]![p_5+j+1]!} \qquad (3)$$

$$= q^{-(p_4+1)(p_5+1)-p_2(p_4+p_5+1)}\frac{[-p_1-p_3-2]![-p_2-p_5-2]!}{[p_1+p_4+1]![p_3+p_4+1]!}$$

$$\times \sum_u \frac{(-1)^u q^{u(p_2+p_5+1)}[p_4-p_3-1-u]![p_4-p_1-1-u]!}{[u]![p_4+p_5+1-u]![p_2+p_4+1-u]![-p_1-p_3-2-u]!},$$

with parameters

$$p_1 = k_1-j_1-1, \quad p_2 = j_3-k_3-z_2-1,$$
$$p_3 = j_1-k_1-z_4-1, \quad p_4 = j_1+k_2-l_1+z_1,$$
$$p_5 = l_3-j_3-k_4+z_3-1.$$

Equation (3) is derived as an analytical continuation from (5.5) of [9], with the r.h.s. replaced using expression [2, 17] with minimal symmetry for the CG coefficients of $SU(2)$ and $u_q(2)$. Particularly, for $p_3 = -p_1-2$, Eq. (3) turns into the q version of Dougall's summation formula, with the r.h.s. including a single term. Note, that formal summation limits of (3) may exceed the interval determined by the triangular conditions in (2b). Although separate q-$6j$ coefficients with spoiled triangular conditions vanish, but for the corresponding pure q-factorial sums of the type (1), only Karlsson's summation formula [11] is helpful. Hence, after applying (3) to (2b), with change $u \to u+z_1$, we obtained expression (2c), including five sums, with four separate sums related to the finite generic or balanced basic hypergeometric series ${}_4\phi_3$ and the fifth sum (over $u+z_1$) of the ${}_3\phi_2$ type.

When the location of the summarized angular momentum in a stretched triangle of the q-$12j$ coefficient of the first kind is along the Hamilton line of the Möbius strip (2a) (e.g., for $k_4 = l_4+j_1$), we obtain from (2c) a triple sum (with fixed $z_4 = u+z_1 = 0$), which does not simplify further for two couples of adjacent diverging stretched triangles (e.g., for $l_4 = k_4-j_1 = |k_1-j_4|$), but for $l_4 = 0,\ k_4 = j_1,\ j_4 = k_1$ it corresponds (cf. (33.20) of [2]) to the general triple-sum expression for the q-$9j$ coefficient,

$$([2j_1+1][2k_1+1])^{-1/2}\begin{Bmatrix} j_1 & l_1 & j_2 \\ k_3 & k_2 & l_2 \\ l_3 & k_1 & j_3 \end{Bmatrix}_q,$$

as a q generalization of the Ališauskas and Jucys [3] formula (see (32.10) of [2]).

Another derived five sum expression [14] turns into a triple sum, when the summarized angular momentum in a stretched triangle corresponds to a crossbar of the Möbius strip (2a), e.g., for $l_4 = k_4+j_1$. There are four possible mutual positions of the couples of the stretched triangles and a great diversity of mutual orientations (22) of the couples of stretched triangles in graph (2a). Corresponding triple-sum expressions do not simplify for two couples of adjacent diverging or merging stretched triangles (e.g., for $l_4 = k_4-j_1 = k_1-j_4$), but for adjacent consecutive stretched triangles (with $k_2 = k_1+l_1 = k_1+j_1+j_2$, with $j_1 = l_1+j_2 = k_4-l_4$, or with $k_4 = l_4+j_1 = l_3-k_3$) the q-$12j$ coefficients may be expressed as the single or double sums, related to the q-$6j$ or the stretched q-$9j$ coefficients. Some doubly stretched q-$12j$ coefficients of the first kind turn into double sums equivalent to compositions of two ${}_{p+1}F_p[...;q,x]$ series, sometimes related to the Kampé de Fériet functions [12], more diverse as in the stretched q-$9j$ case, discussed below.

3. ON TRIPLE SUM EXPRESSIONS FOR q-$9j$ COEFFICIENTS

We present here only one of six alternatives [14] of expression (2c) specified for the q-$9j$ coefficients,

$$\begin{Bmatrix} a & b & e \\ c & d & f \\ h & k & g \end{Bmatrix}_q = q^{(b+e-a)(e-f-h+k)-(a-e+f+1)(a-e+f)+Z_{deh}} \frac{\nabla[abe]\nabla[feg]\nabla[kbd]}{\nabla[ach]\nabla[fcd]\nabla[kgh]}$$
$$\times \sum_{z_1,z_2,z_3} \frac{(-1)^{e-f-h+k+z_1+z_2}[a+c-h+z_1]![g-h+k+z_1]!}{[z_1]![c+h-a-z_1]![g+h-k-z_1]![z_2]!}$$
$$\times \frac{[2h-z_1]![2b-z_2]![b-c+f+k-z_2]!}{[b-d+k-z_2]![b+d+k-z_2+1]![z_3]![a+b-e-z_3]!}$$
$$\times \frac{[b+c+f+k-z_2+1]![b+e-a+z_3]![e-f+g+z_3]!}{[f+g-e-z_3]![2e+z_3+1]![e+k-f-h+z_1+z_3]!}$$
$$\times \frac{q^{z_1(b+e-a-z_2+z_3)+z_3(a+b+f-h+k-z_2+1)-z_2(e+k-f-h)}}{[b+e-a-z_2+z_3]![a+b+f-h+k+z_1-z_2+1]!} \quad (4a)$$
$$= (-1)^{e-f-h+k} \frac{\nabla[abe]\nabla[feg]\nabla[kbd]}{\nabla[ach]\nabla[fcd]\nabla[kgh]} q^{(c+h-a)(b-c+f+k+1)}$$
$$\times q^{(b+e-a)(e-f-h+k)-(f+g-e)(g+h-k)-(a-e+f+1)(a-e+f)+Z_{deh}}$$
$$\times \sum_{z_1,z_2,z_3} \frac{(-1)^{z_1+z_2} q^{-z_1(a-c-g+k+1)-z_2(b+c-f+k)+z_3(a+b+f+g)}[2h-z_1]!}{[z_1]![b-d+k-z_2]![b+d+k-z_2+1]![z_3]![2e+z_3+1]!}$$
$$\times \sum_{s_1,s_2,s_3} \frac{(-1)^{s_1+s_2} q^{-s_1(b+e-a-z_2+z_3+1)-s_2(e-f-h+k+z_1+z_3+1)}[2b-s_1]![2g-s_2]!}{[s_1]![z_2-s_1]![a+b-e-z_3-s_1]![s_2]![g+h-k-z_1-s_2]!}$$
$$\times \frac{q^{-s_3(a+b+f-h+k+z_1-z_2+2)}[2c-s_3]![b-c+f+k-z_2+s_3]!}{[f+g-e-z_3-s_2]![s_3]![c+h-a-z_1-s_3]!}, \quad (4b)$$

as well as the extended version (4b) with six sums, derived after some three blocks (quintuplets) of factorials under the summation sign were identified in (4a) and expanded in (4b), using the Chu–Vandermonde summation formulas (see [2, 11, 16]). All the terms in the last sum of (4a) are of the same sign. The separate sums correspond to the finite basic hypergeometric series ${}_4F_3[...;q,x]$,

$${}_{p+1}F_p\left[\begin{matrix}\alpha_1, & \alpha_2,..., & \alpha_{p+1} \\ & & ;q,x \\ & \beta_1,..., & \beta_p\end{matrix}\right] \quad (5)$$

$$= \sum_k \frac{(\alpha_1|q)_k(\alpha_2|q)_k\cdots(\alpha_{p+1}|q)_k}{(\beta_1|q)_k\cdots(\beta_p|q)_k(1|q)_k} x^k,$$

with $(\alpha|q)_n = \prod_{k=0}^{n-1}[\alpha+k]$, $x = q^{\pm(c+1)}$, $c = \sum_{i=1}^{p+1}\alpha_i - \sum_{j=1}^{p}\beta_j$, as defined by Álvarez-Nodarse and Smirnov [10], instead of the standard basic hypergeometric functions ${}_{p+1}\phi_p$ (see [11]). Parameters $c=-1$ and $x=1$ for the balanced basic hypergeometric series and in expressions for q-$6j$ coefficients [15, 16].

The summations of (4b) over s_1, s_2, s_3 give the original expression (4a), when its summations over z_1, z_2, z_3 give a different expression for q-$9j$ coefficients. The summation intervals for z_i ($i=1,2,3$)

in (4b) and (2c) are mainly restricted by some triangle linear combinations, and with their vanishing we may write 23 different double-sum expressions for the stretched $9j$ coefficients in terms of compositions of ${}_4F_3[...;q,x]$ and ${}_3F_2[...;q,x]$ series. Otherwise, in the case of (4b) (e.g., for $k = g + h$), some couples of parameters z_i and s_j are fixed and summation over z_l and s_l (where i, j, l is some permutation of 1, 2, 3) is possible. Hence, we may derive 13 versions of expressions for the stretched q-$9j$ coefficients as double sums—compositions of both generic ${}_3F_2[...;q,x]$ series. Sometimes, a separate sum corresponds to the CG coefficients of $u_q(2)$ [6, 17, 18] and may be reexpressed by means of another formula, and different expressions for the stretched q-$9j$ coefficients may be derived [14], including a q generalization of standard formula (32.13) of [2].

The expressions of both classes correspond to q generalizations of the Kampé de Fériet [12] functions $F^{1:2}_{0:2}$ or (after reversing the order of summations) to $F^{0:3}_{1:1}$ (cf. [13]) and $F^{1:2}_{1:1}$, which are defined as follows:

$$
{}^{\pm}F^{A:B}_{C:D}\left[\begin{matrix}(a)\\(c)\end{matrix} : \begin{matrix}(b)\\(d)\end{matrix} ; \begin{matrix}(b')\\(d')\end{matrix} ; x, y; q\right] \tag{6}
$$
$$
= \sum_{s,t}^{\infty} \frac{\prod_{j=1}^{A}(a_j|q)_{s+t}}{\prod_{j=1}^{C}(c_j|q)_{s+t}} \times \frac{\prod_{j=1}^{B}(b_j|q)_s(b'_j|q)_t}{\prod_{j=1}^{D}(d_j|q)_s(d'_j|q)_t} \frac{x^{\pm s}y^{\pm(1-2\delta)t}}{[s]![t]!} q^{\pm(A-C)st},
$$

with special parameters

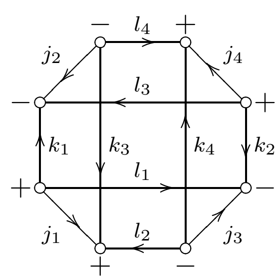
$$
x = q^{p+1}, \quad p = \sum_{j=1}^{A} a_j + \sum_{j=1}^{B} b_j - \sum_{j=1}^{C} c_j - \sum_{j=1}^{D} d_j,
$$
$$
y = q^{p'+1}, \quad p' = \sum_{j=1}^{A} a_j + \sum_{j=1}^{B} b'_j - \sum_{j=1}^{C} c_j - \sum_{j=1}^{D} d'_j
$$

and $\delta = \delta_{AC}$ for $A + B = C + D + 1$ and $|A - C| \le 1$. Series (6) turns into usual Kampé de Fériet function $F^{A:B}_{C:D}[...;1,1]$ for $q = 1$. Upon comparison of different expressions, the rearrangement and summation formulas of the double q-factorial series and related Kampé de Fériet functions of these types were derived [14].

4. EXPRESSIONS FOR q-$12j$ COEFFICIENTS OF THE SECOND KIND

For rearrangement of the q-$12j$ coefficient of the second kind [1, 2] (whose planar graph is cube, i.e., without braiding [7], in contrast with the $3nj$ coefficients of the first kind, whose graphs are possible only on the Möbius strip), we use the expansion

$$
\begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix}_q =
$$



$$
= (-1)^{l_1-l_2-l_3+l_4} \tag{7a}
$$

$$
\times \sum_x [2x+1] \begin{Bmatrix} k_1 & j_1 & l_1 \\ j_3 & k_2 & x \end{Bmatrix}_q \begin{Bmatrix} j_3 & x & j_1 \\ k_3 & l_2 & k_4 \end{Bmatrix}_q \begin{Bmatrix} k_3 & k_4 & x \\ j_4 & j_2 & l_4 \end{Bmatrix}_q \begin{Bmatrix} k_1 & x & k_2 \\ j_4 & l_3 & j_2 \end{Bmatrix}_q \tag{7b}
$$
$$
= (-1)^{j_1-j_3-k_1+k_2-l_1-l_2-l_3+l_4} \frac{\nabla[j_3k_4l_2]\nabla[k_3j_2l_4]\nabla[k_1j_2l_3]\nabla[j_4k_4l_4]}{\nabla[k_1j_1l_1]\nabla[j_3k_2l_1]\nabla[k_3j_1l_2]\nabla[j_4k_2l_3]}
$$
$$
\times \sum_{z_1,z_2,z_3,z_4} \frac{(-1)^{z_2+z_3+z_4}[k_2+j_3-l_1+z_1]![k_1+j_1-l_1+z_1]!}{[z_1]![z_2]![z_3]![z_4]![l_1+k_2-j_3-z_1]![j_1+l_1-k_1-z_1]!}
$$
$$
\times \frac{[2l_1-z_1]![2l_2-z_2]![j_1-l_2+k_3+z_2]!}{[l_2+j_3-k_4-z_2]![j_1+l_2-k_3-z_2]![l_2+j_3+k_4-z_2+1]!}
$$
$$
\times \frac{[j_2-k_3+l_4+z_3]![k_4+l_4-j_4+z_3]!}{[j_4+k_4-l_4-z_3]![j_2+k_3-l_4-z_3]![2l_4+z_3+1]![k_1-j_2+l_3-z_4]!}
$$
$$
\times \frac{[2l_3-z_4]![k_2-l_3+j_4+z_4]!}{[k_2+l_3-j_4-z_4]![k_1+j_2+l_3-z_4+1]![k_1+k_3-l_1-l_2+z_1+z_2]!}
$$

$$\times \frac{[j_3 + j_4 + l_2 - l_4 - z_2 - z_3]![k_1 + k_3 + l_3 - l_4 - z_3 - z_4]!}{[k_1 - k_3 + j_3 - j_4 - l_1 + l_4 + z_1 + z_3]![j_3 + j_4 - l_1 - l_3 + z_1 + z_4]!}$$

$$\times \frac{[j_3 - j_4 + k_1 - k_3 + l_2 + l_3 - z_2 - z_4]!}{[l_1 + l_2 + l_3 - l_4 - z_1 - z_2 - z_3 - z_4]!}. \quad (7c)$$

If we insert expression (1) for q-$6j$ coefficients with the differently changed summation parameters [by analogy with what was done in the (2b) case] into (7b), the asymmetric triangle coefficients, depending on the summation parameter x, cancel. Using the summation formula (3.6) of [9] (cf. (2.4.2) of [11]), we derive (7c), with each separate sum corresponding to the finite balanced basic hypergeometric series ${}_5F_4[q,1]$ (with the fixed sign of all the terms in the first q-factorial sum).

For definite stretched triangles, some summation parameters in (7c) are fixed, and one of three remaining sums turns into balanced series ${}_4F_3[q,1]$, whose rearrangement [11] enables us to transform another ${}_5F_4[q,1]$ into ${}_4F_3[q,1]$-type series, with only the last sum left of the ${}_5F_4[q,1]$ type. Particularly, the doubly stretched q-$12j$ coefficient with $k_1 = j_1 + l_1 = l_3 - j_2$ [i.e., for adjacent consecutive stretched triangles in graph (7a)] is proportional to some q-$6j$ coefficient, when, in the case of $j_1 = k_1 - l_1 = l_2 - k_3$ or $k_1 = j_1 + l_1 = j_2 + l_3$ [with the diverging or merging adjacent stretched triangles in graph (7a)], we obtain the double-sum expressions, corresponding to the q-generalizations of the Kampé de Fériet functions $F^{1:4}_{1:3}$. For $k_1 = j_1 + l_1$ and $j_4 = k_4 + l_4$, as well as for $k_1 = j_1 + l_1$ and $j_4 = k_4 + l_4$ [i.e., for antipode stretched triangles in (7a)], we derive expressions with single sums, related to the balanced basic hypergeometric ${}_6F_5[q,1]$ series. For four different versions of the doubly stretched q-$12j$ coefficients of the second kind with touching angular momenta in remote stretched triangles (e.g., with $k_1 = j_1 + l_1$ and $l_3 = k_2 + j_4$, or with $k_1 = j_1 + l_1$ and $k_3 = j_2 + l_4$) forming the 4-cycles, expressions with the double sums correspond to the Kampé de Fériet functions $F^{1:3}_{1:2}$ (depending on nine parameters, with $b_1 + b'_1 = c_1$). The q-$12j$ coefficients of both kinds turn into single terms in the virtually stretched cases [Eq. (7c) for $l_4 = l_1 + l_2 + l_3$ or, after some effort, Eq. (2c) for $j_3 = j_1 + k_1 + k_3$] with four dependent angular momenta, appearing as disconnected on some Hamilton lines of graphs (7a) or (2a).

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SYMPOSIUM ON QUANTUM GROUPS

On a General Analytic Formula for $U_q(su(3))$ Clebsch–Gordan Coefficients*

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Abstract—We present the projection operator method in combination with the Wigner–Racah calculus of the subalgebra $U_q(su(2))$ for calculation of Clebsch–Gordan coefficients (CGCs) of the quantum algebra $U_q(su(3))$. The key formulas of the method are couplings of the tensor and projection operators and also a tensor form for the projection operator of $U_q(su(3))$. We obtain a very compact general analytic formula for the $U_q(su(3))$ CGCs in terms of the $U_q(su(2))$ Wigner $3nj$ symbols. *© 2001 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

It is well known that the Clebsch–Gordan coefficients (CGCs) of the unitary Lie algebra $u(n)$ ($su(n)$) have numerous applications in various fields of theoretical and mathematical physics. For example, many algebraic models of nuclear theory [interacting boson model (IBM), Elliott $su(3)$ model, $su(4)$ supermultiplet scheme of Wigner, the shell model, and so on] demand the CGCs for $su(6)$, $su(5)$, $su(3)$, $su(4)$, and $su(n)$. Analogously, in quark models of hadrons, we need the CGCs of $su(3)$, $su(4)$, etc. The theory of the $su(n)$ CGCs is connected with the theory of special functions, combinatorial analysis, topology, etc.

There are several methods for the calculation of CGCs of $su(n)$ [$u(n)$] and other Lie algebras: recursion method; method of employment of explicit bases of irreducible representations; method of generating invariants; method of tensor operators, where the Wigner–Eckart theorem is used; projection operator method; coherent state method; combined methods.

It is well known that the method of projection operators for usual (nonquantized) Lie algebras [1, 2] and superalgebras [2] is a powerful and universal method for a solution of many problems in the representation theory. In particular, the method allows one to develop the detailed theory of Clebsch–Gordan coefficients and other elements of Wigner–Racah calculus (including compact analytic formulas of these elements and their symmetry properties) [3] and so on. It is evident that the projection operators of quantum groups [4] play the same role in their representation theory.

In this paper, we present the projection operator method in combination with the Wigner–Racah calculus of the subalgebra $U_q(su(2))$ [5] for calculation of CGCs of the quantum algebra $U_q(su(3))$. The key formulas of the method are couplings of the tensor and projection operators and also a tensor form for the projection operator of $U_q(su(3))$. It should be noted that the first application of this method was for the $su(3)$ case in [3]. Some simple elements of this approach were also used in [6] for the $U_q(su(n))$ case. Also, the coherent state method in combination with the Wigner–Racah calculus was applied in [7] for $u(n)$.

2. GELFAND–TSETLIN BASIS

Let $\Pi := \{\alpha_1, \alpha_2\}$ be a system of simple roots of the Lie algebra $sl(3)$ ($= sl(3,\mathbf{C}) \simeq A_2$), endowed with the following scalar product: $(\alpha_1,\alpha_1) = (\alpha_2,\alpha_2) = 2$, $(\alpha_1,\alpha_2) = (\alpha_2,\alpha_1) = -1$. The root system Δ_+ of $sl(3)$ consists of the roots $\alpha_1, \alpha_1+\alpha_2, \alpha_2$. The quantum Hopf algebra $U_q(sl(3))$ is generated by the Chevalley elements $q^{\pm h_{\alpha_i}}$, $e_{\pm\alpha_i}$ ($i = 1, 2$) with the relations

$$q^{h_{\alpha_i}} q^{-h_{\alpha_i}} = q^{-h_{\alpha_i}} q^{h_{\alpha_i}} = 1, \qquad (1)$$

$$q^{h_{\alpha_i}} q^{h_{\alpha_j}} = q^{h_{\alpha_j}} q^{h_{\alpha_i}}, \quad q^{h_{\alpha_i}} e_{\alpha_j} q^{-h_{\alpha_i}} = q^{(\alpha_i,\alpha_j)} e_{\alpha_j},$$

$$[e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij}[h_{\alpha_i}],$$

$$[[e_{\pm\alpha_i}, e_{\pm\alpha_j}]_q, e_{\pm\alpha_j}]_q = 0 \quad \text{for } |i-j| = 1.$$

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Here and elsewhere, we use the standard notation $[a] := (q^a - q^{-a})/(q - q^{-1})$ and $[e_\alpha, e_\beta]_q := e_\alpha e_\beta - q^{(\alpha,\beta)} e_\beta e_\alpha$. The Hopf structure of $U_q(sl(3))$ is given by

$$\Delta_q(h_{\alpha_i}) = h_{\alpha_i} \otimes 1 + 1 \otimes h_{\alpha_i}, \qquad (2)$$
$$S_q(h_{\alpha_i}) = -h_{\alpha_i},$$
$$\Delta_q(e_{\pm\alpha_i}) = e_{\pm\alpha_i} \otimes q^{\frac{1}{2}h_{\alpha_i}} + q^{-\frac{1}{2}h_{\alpha_i}} \otimes e_{\pm\alpha_i},$$
$$S_q(e_{\pm\alpha_i}) = -q^{\pm 1} e_{\pm\alpha_i}.$$

For construction of the composite root vectors $e_{\pm(\alpha_1+\alpha_2)}$, we fix the normal ordering in Δ_+: $\alpha_1, \alpha_1 + \alpha_2, \alpha_2$. According to this ordering, we set

$$e_{\alpha_1+\alpha_2} := [e_{\alpha_1}, e_{\alpha_2}]_{q^{-1}}, \qquad (3)$$
$$e_{-\alpha_1-\alpha_2} := [e_{-\alpha_2}, e_{-\alpha_1}]_q.$$

Let us introduce another standard notation for the Cartan–Weyl generators:

$$e_{12} := e_{\alpha_1}, \quad e_{21} := e_{-\alpha_1}, \quad e_{11} - e_{22} := h_{\alpha_1}, \qquad (4)$$
$$e_{23} := e_{\alpha_2}, \quad e_{32} := e_{-\alpha_2}, \quad e_{22} - e_{33} := h_{\alpha_2},$$
$$e_{13} := e_{\alpha_1+\alpha_2}, \quad e_{31} := e_{-\alpha_1-\alpha_2},$$
$$e_{11} - e_{33} := h_{\alpha_1} + h_{\alpha_2}.$$

The explicit formula for the extremal projector for the quantum groups [4] specialized to the case of $U_q(sl(3))$ has the form

$$p\big(U_q(sl(3))\big) = p_{12}p_{13}p_{23}, \qquad (5)$$

where the elements p_{ij} $(1 \le i < j \le 3)$ are given by

$$p_{ij} = \sum_{n=0}^{\infty} \frac{(-1)^n}{[n]!} \varphi_{ij,n} e_{ij}^n e_{ji}^n, \qquad (6)$$
$$\varphi_{ij,n} = q^{-(j-i-1)n} \Big\{ \prod_{s=1}^{n} [e_{ii} - e_{jj} + j - i + s] \Big\}^{-1}.$$

The extremal projector $p := p\big(U_q(sl(3))\big)$ satisfies the relations

$$e_{ij}p = pe_{ji} = 0 \quad (i < j), \qquad p^2 = p. \qquad (7)$$

The quantum algebra $U_q(su(3))$ can be considered as the quantum algebra $U_q(sl(3))$ endowed with the additional Cartan involution (*):

$$h_{\alpha_i}^* = h_{\alpha_i}, \qquad e_{\pm\alpha_i}^* = e_{\mp\alpha_i}, \qquad (8)$$
$$q^* = q \text{ or } q^{-1}.$$

Let $(\lambda\mu)$ be a finite-dimensional irreducible representation (IR) of $U_q(su(3))$ with the highest weight $(\lambda\mu)$ (λ and μ are nonnegative integers). The vector of the highest weight, denoted by the symbol $|(\lambda\mu)h\rangle$, satisfies the relations

$$h_{\alpha_1}|(\lambda\mu)h\rangle = \lambda|(\lambda\mu)h\rangle, \qquad (9)$$
$$h_{\alpha_2}|(\lambda\mu)h\rangle = \mu|(\lambda\mu)h\rangle, \quad e_{ij}|(\lambda\mu)h\rangle = 0 \quad (i < j).$$

Labeling of other basis vectors in IR $(\lambda\mu)$ depends upon choice of subalgebras of $U_q(su(3))$ (or, in other words, depends upon which reduction chain from $U_q(su(3))$ to subalgebras is chosen). Here, we use the Gelfand–Tsetlin reduction chain:

$$U_q(su(3)) \supset U_q(u_Y(1)) \otimes U_q(su_T(2)) \qquad (10)$$
$$\supset U_q(u_{T_0}(1)),$$

where the subalgebra $U_q(su_T(2))$ is generated by the elements

$$T_+ := e_{23}, \qquad T_- := e_{32}, \qquad (11)$$
$$T_0 := \tfrac{1}{2}(e_{22} - e_{33}),$$

the subalgebra $U_q(u_{T_0}(1))$ is generated by q^{T_0}, and $U_q(u_Y(1))$ is generated by q^Y [in the classical (nondeformed) case in elementary particle theory, the subalgebra $su_T(2)$ is called the T-spin algebra and the element Y is the hypercharge operator], where

$$Y = -\tfrac{1}{3}\big(2h_{\alpha_1} + h_{\alpha_2}\big). \qquad (12)$$

In the case of the reduction chain (10), the basis vectors of IR $(\lambda\mu)$ are denoted by

$$|(\lambda\mu)jtt_z\rangle. \qquad (13)$$

Here, the set jtt_z characterizes the hypercharge Y and the T spin and its projection:

$$q^{T_0}|(\lambda\mu)jtt_z\rangle = q^{t_z}|(\lambda\mu)jtt_z\rangle, \qquad (14)$$
$$q^Y|(\lambda\mu)jtt_z\rangle = q^y|(\lambda\mu)jtt_z\rangle,$$
$$T_\pm|(\lambda\mu)jtt_z\rangle = \sqrt{[t \mp t_z][t \pm t_z + 1]}\,|(\lambda\mu)jtt_z \pm 1\rangle,$$

where the parameter j is connected with the eigenvalue y of the operator Y as follows: $y = -\frac{1}{3}\big(2\lambda + \mu\big) + 2j$. It is not hard to show that the orthonormalized vectors (13) can be represented in the form

$$|(\lambda\mu)jtt_z\rangle \qquad (15)$$
$$= N_{jt}^{(\lambda\mu)} P_{t_z;t'_z}^t e_{31}^{j+\frac{1}{2}\mu-t} e_{21}^{j-\frac{1}{2}\mu+t} |(\lambda\mu)h\rangle,$$

where $P_{t_z;t'_z}^t$ is the general projection operator of the quantum algebra $U_q(su_T(2))$ [5], and the normalizing factor $N_{jt}^{(\lambda\mu)}$ has the form

$$N_{jt}^{(\lambda\mu)} = \left(\frac{[\lambda + \frac{1}{2}\mu - j + t + 1]![\lambda + \frac{1}{2}\mu - j - t]![\frac{1}{2}\mu + j + t + 1]![\frac{1}{2}\mu - j + t]!}{q^{2j+\mu-2t}[\lambda]![\mu]![\lambda + \mu + 1]![j + \frac{1}{2}\mu - t]![j - \frac{1}{2}\mu + t]![2t + 1]!} \right)^{1/2}. \qquad (16)$$

The quantum numbers jt are all taken to be nonnegative integers and half-integers such that the sum $\frac{1}{2}\mu + j + t$ is an integer and they are subjected to the constraints

$$\begin{cases} \frac{1}{2}\mu + j - t \geq 0, & -\frac{1}{2}\mu + j + t \geq 0, \\ \frac{1}{2}\mu - j + t \geq 0, & \frac{1}{2}\mu + j + t > \lambda + \mu. \end{cases} \quad (17)$$

For every fixed t, the projection t_z runs the values $t_z = -t, -t+1, \ldots, t-1, t$. These results can be obtained from the explicit form of the Gelfand–Tsetlin bases for the case $U_q(su(n))$ [4] specializing to the given case $U_q(su(3))$.

3. COUPLINGS OF TENSOR AND PROJECTION OPERATORS

Let $\{R^{j(q)}_{j_z}\}$ be an irreducible tensor operator (ITO) of the rank j; that is, $(2j+1)$ components $R^{j(q)}_{j_z}$ are transformed with respect to the $U_q(su_T(2))$ adjoint action as the $U_q(su_T(2))$ basis vectors $|jj_z\rangle$ of the spin j:

$$T_i \triangleright R^{j(q)}_{j_z} := (\mathrm{ad}_q T_i) R^{j(q)}_{j_z} \quad (18)$$
$$\equiv \big((\mathrm{id} \otimes S_q)\Delta_q(T_i)\big) \circ R^{j(q)}_{j_z} = \sum_{j'_z} \langle jj'_z | T_i | jj_z \rangle R^{j(q)}_{j'_z},$$

where $(a \otimes b) \circ x = axb$. The tensor operator of the type $\{R^{j(q)}_{j_z}\}$ will be also called the left irreducible tensor operators (LITO) because the generators T_i $(i = \pm,\ 0)$ act on the left side of the components $R^{j(q)}_{j_z}$. (This notation for the ITOs is different from one of the papers [5] by the replacement of q by q^{-1}.) Following [3], we also introduce a right irreducible tensor operator (RITO) denoted by the tilde symbol $\{\tilde{R}^{j(q)}_{j_z}\}$, on which the $U_q(su_T(2))$ generators T_i act on the right side, namely,

$$T_i \triangleleft \tilde{R}^{j(q)}_{j_z} := (\mathrm{ad}^*_q T_i) \tilde{R}^{j(q)}_{j_z} \quad (19)$$
$$\equiv \tilde{R}^{j(q)}_{j_z} \overset{\leftarrow}{\circ} \big((\tilde{S}_q \otimes \mathrm{id})\tilde{\Delta}_q(T^*_i)\big)$$
$$= \sum_{j'_z} \langle jj'_z | T_i | jj_z \rangle \tilde{R}^{j(q)}_{j'_z},$$

where $x \overset{\leftarrow}{\circ} (a \otimes b) = axb$, and $\tilde{\Delta}_q$ is the opposite coproduct ($\tilde{\Delta}_q = \Delta_{q^{-1}}$) and $\tilde{S}_q$ is the corresponding antipode ($\tilde{S}_q = S_{\bar{q}}$). It is not hard to verify that any LITO $\{R^{j(q)}_{j_z}\}$ is the RITO $\{\tilde{R}^{j(q)}_{j_z}\}$: $R^{j(q)}_{j_z} = (-1)^{j_z} q^{j_z} \tilde{R}^{j(q)}_{-j_z}$.

The projection operator set $\{P^t_{t_z;t'_z}\}$ for a fixed IR t and for various t_z and t'_z will be called the $\mathbf{P}^t$ operator. It is not hard to see that the subset of the left components of this operator satisfies the relations for the LITO $\mathbf{R}^{j(q)} := \{R^{j(q)}_{j_z}\}$ if we understand the action "$\triangleright$" of the generator T_i as the usual multiplication of the operators T_i and $P^t_{t_z;t'_z}$ and that the subset of the right components of the $\mathbf{P}^t$ operator satisfies the relations for the RITO $\tilde{\mathbf{R}}^j := \{\tilde{R}^{j(q)}_{j_z}\}$ if we understand the action "$\triangleleft$" as the usual multiplication of the operators $P^t_{t_z;t'_z}$ and T^*_i:

$$T_i \triangleright P^t_{t_z;t'_z} := T_i P^t_{t_z;t'_z} = \sum_{t''_z} \langle tt''_z | T_i | tt_z \rangle P^t_{t''_z;t'_z}, \quad (20)$$

$$T_i \triangleleft P^t_{t_z;t'_z} := P^t_{t_z;t'_z} T^*_i = \sum_{t''_z} \langle tt''_z | T_i | tt'_z \rangle P^t_{t_z;t''_z}. \quad (21)$$

Using the $U_q(su_T(2))$ CGCs, we can couple the LITO $\mathbf{R}^{j(q)}$ with the left components of the $\mathbf{P}^t$ operator and the RITO $\tilde{\mathbf{R}}^{j(q)}$ with the right components of the $\mathbf{P}^t$ operator:

$$^{t'}_{t'_z}\Big\{\mathbf{R}^{j(q)} \dot{\otimes} \mathbf{P}^t_{;t_z}\Big\}_q \quad (22)$$
$$:= \sum_{j_z t''_z} \big(jj_z tt''_z | t't'_z\big)_q R^{j(q)}_{j_z} P^t_{t''_z;t_z},$$

$$\Big\{\mathbf{P}^t_{t_z;} \dot{\otimes} \tilde{\mathbf{R}}^{j(q)}\Big\}_q{}^{t'}_{t'_z} \quad (23)$$
$$:= \sum_{j_z t''_z} \big(jj_z tt''_z | t't'_z\big)_q P^t_{t_z;t''_z} \tilde{R}^{j(q)}_{j_z}.$$

Here, the symbol $\dot{\otimes}$ means that we first take the usual tensor product and then in a resulting expression we replace the tensor product by the usual operator product. It is not hard to show that the couplings (22) and (23) are connected as follows:

$$\mathbb{R}^{j(q)}_{tt_z;t't'_z} := \sqrt{[2t+1]}\ {}^{t}_{t_z}\Big\{\mathbf{R}^{j(q)} \dot{\otimes} \mathbf{P}^{t'}_{;t'_z}\Big\}_q \quad (24)$$
$$= (-1)^{t'-t}\sqrt{[2t'+1]}\Big\{\mathbf{P}^t_{t_z;} \dot{\otimes} \tilde{\mathbf{R}}^{j(q)}\Big\}_q{}^{t'}_{t'_z}.$$

Using (24) and a unitary relation of the $U_q(su(2))$ CGCs [5], one can obtain the following useful permutation relations between the components of the tensors $\mathbf{R}^{j(q)}$, $\tilde{\mathbf{R}}^{j(q)}$ and $\mathbf{P}^t$ operator:

$$R^{j(q)}_{j_z} P^t_{t_z;t'_z} = \sum_{t''t''_z} (-1)^{t-t''} \sqrt{\frac{[2t+1]}{[2t''+1]}} \quad (25)$$
$$\times \big(jj_z tt_z | t''t''_z\big)_q \{\mathbf{P}^{t''}_{t''_z;} \dot{\otimes} \tilde{\mathbf{R}}^{j(q)}\}_q{}^{t}_{t'_z},$$

$$P^{t}_{t_z;t'_z}\tilde{R}^{j(q)}_{j_z}=\sum_{t''t''_z}(-1)^{t-t''}\sqrt{\frac{[2t+1]}{[2t''+1]}} \quad (26)$$
$$\times\big(jj_ztt'_z\big|t''t''_z\big)_q\ {}^{\ \ t}_{t'_z}\{\mathbf{R}^{j(q)}\dot{\otimes}\mathbf{P}^{t''}_{;t''_z}\}_q.$$

We can show that the monomials $e^n_{21}e^m_{31}$ and $e^n_{12}e'^m_{13}$ are components of ITOs with respect to the adjoint action of the subalgebra $U_q(su_T(2))$:

$$R^{j(q)}_{j_z}=\sqrt{\frac{[2j]!}{[j-j_z]![j+j_z]!}} \quad (27)$$
$$\times\, q^{2j^2-j}e^{j+j_z}_{21}e^{j-j_z}_{31}q^{-jh_{\alpha_1}-(j-j_z)T_0},$$
$$R'^{j(q)}_{j_z}=\sqrt{\frac{[2j]!}{[j-j_z]![j+j_z]!}} \quad (28)$$
$$\times\, q^{-2j^2+j}e^{j-j_z}_{12}e'^{j+j_z}_{13}q^{-jh_{\alpha_1}-(j+j_z)T_0},$$

where the generator e'_{13} is defined according to the inverse normal ordering: $\alpha_2,\alpha_1+\alpha_2,\alpha_1$, i.e., $e'_{13}=[e_{23},e_{12}]_{q^{-1}}$. These ITOs have the remarkable properties: *A result of the coupling of two ITOs of the type (27) or (28) is nonzero only for an irreducible component of the maximal rank, e.g.,*

$$\Big\{\mathbf{R}^{j(q)}\dot{\otimes}\mathbf{R}^{j'(q)}\Big\}^{\ j''}_{q\ j''_z}=\delta_{j'',j+j'}R^{j+j'(q)}_{j''_z}. \quad (29)$$

The property is also useful in applications: *For ITOs of the type (29), the relation is valid*

$$R^{j(q)}_{j_z}\mathbb{R}^{j'(q)}_{tt_z;t't'_z}=\sum_{t''t''_z}\sqrt{\frac{[2t+1]}{[2t''+1]}} \quad (30)$$
$$\times\big(jj_ztt_z\big|t''t''_z\big)_qU\big(jj't''t';j+j't\big)_q\mathbb{R}^{j+j'(q)}_{t''t''_z;t't'_z}.$$

Here, $U\big(\ldots;\ldots\big)_q$ is a recoupling coefficient which can be expressed via the stretched q-$6j$ symbols of $U_q(su_T(2))$ [5]:

$$U\big(jj't''t';j+j't\big)_q=(-1)^{j+j'+t'+t''} \quad (31)$$
$$\times\sqrt{[2j+2j'+1][2t+1]}\begin{Bmatrix} j & j' & j+j' \\ t' & t'' & t\end{Bmatrix}_q.$$

Using (26) and (27), we can represent the basis vectors (13) in the form

$$\big|(\lambda\mu)jtt_z\big\rangle=\mathcal{F}^{(\lambda\mu)}_{-}(jtt_z)\big|(\lambda\mu)h\big\rangle \quad (32)$$
$$=\mathcal{N}^{(\lambda\mu)}_{jt}\mathbb{R}^{j(q)}_{tt_z;\frac{1}{2}\mu\frac{1}{2}\mu}\big|(\lambda\mu)h\big\rangle.$$

The normalizing factor $\mathcal{N}^{(\lambda\mu)}_{jt}$ is given by

$$\mathcal{N}^{(\lambda\mu)}_{jt}=(-1)^{2j} \quad (33)$$
$$\times\, q^{(j+\frac{1}{2}\mu-t)(j-\frac{1}{2}\mu+t)+j\lambda+\frac{1}{2}\mu(j+\frac{1}{2}\mu-t)-2j^2+j+t-\frac{1}{2}\mu}$$
$$\times\sqrt{\frac{[j-\frac{1}{2}\mu+t]![j+\frac{1}{2}\mu-t]!}{[2j]![\mu+1]}}$$
$$\times\left(j\,\frac{1}{2}\mu-ttt\Big|\frac{1}{2}\mu\frac{1}{2}\mu\right)_qN^{(\lambda\mu)}_{jt}.$$

With the help of (32), we easily find the action of the ITO (27) on the Gelfand–Tsetlin basis:

$$R^{j'(q)}_{j'_z}\big|(\lambda\mu)jtt_z\big\rangle \quad (34)$$
$$=\sum_{t''t''_z}\big(j'j'_ztt_z\big|t''t''_z\big)_q\big\langle(\lambda\mu)j''t''\big\|R^{j'(q)}\big\|(\lambda\mu)jt\big\rangle_q$$
$$\times\big|(\lambda\mu)j''t''t''_z\big\rangle,$$

where

$$\big\langle(\lambda\mu)j''t''\big\|R^{j'(q)}\big\|(\lambda\mu)jt\big\rangle_q \quad (35)$$
$$=\delta_{j'',j+j'}\sqrt{\frac{[2t+1]}{[2t''+1]}}\frac{\mathcal{N}^{(\lambda\mu)}_{jt}}{\mathcal{N}^{(\lambda\mu)}_{j''t''}}U\big(j'jt''\tfrac{1}{2}\mu;j'+jt\big)_q.$$

4. TENSOR FORM OF THE PROJECTION OPERATOR

It is obvious that the extremal projector (5) can be presented in the form

$$p(U_q(su(3)) \quad (36)$$
$$=p(U_q(su_T(2))\big(p_{12}p_{13}\big)p(U_q(su_T(2)).$$

Now, we present the middle part of (36) in terms of the $U_q(su_T(2))$ tensor operators (27) and (28). To this end, we substitute the explicit expression (6) for the factors p_{12} and p_{13} and combine monomials $e^n_{21}e^m_{31}$ and $e^n_{12}e^m_{13}$. After some summation manipulations, we obtain the following expression for the extremal projection operator $p:=p(U_q(su(3)))$ in terms of the tensor operators (27) and (28):

$$p=p(U_q(su_T(2)) \quad (37)$$
$$\times\Big(\sum_{jj_z}A_{jj_z}\tilde{R}^{j(q)}_{j_z}R'^{j(q)}_{j_z}\Big)p(U_q(su_T(2)).$$

Here,

$$A_{jj_z} \quad (38)$$
$$=\frac{(-1)^{3j}[\varphi_{12}][\varphi_{12}+j+j_z-1]![\varphi_{13}]!}{[2j]![\varphi_{12}+2j]![\varphi_{13}+j+j_z]!}$$
$$\times\, q^{4j^2+j+2jh_{\alpha_1}+2(j+j_z)T_0},$$

where $\varphi_{1i+1}:=e_{11}-e_{i+1i+1}+i$ $(i=1,2)$. Below, we assume that the $U_q(su(3))$ extremal projection operator p acts in a weight space with the weight $(\lambda\mu)$ and in this case the symbol p is supplied with the index $(\lambda\mu)$, $p^{(\lambda\mu)}$, and all the Cartan elements h_{α_i} on the right side of (37) are replaced by the corresponding weight components λ and μ. Now, we multiply the projector $p^{(\lambda\mu)}$ from the left side by the lowering operator $\mathcal{F}_-^{(\lambda\mu)}(jtt_z)$ and from the right side by the rising operator $\left(\mathcal{F}_-^{(\lambda\mu)}(jtt_z)\right)^*$, and by applying a relation of type (30) we finally find the tensor form of the general $U_q(su(3))$ projection operator:

$$P^{(\lambda\mu)}_{jtt_z;j't't'_z} = \sum_{j''t''} B^{(\lambda\mu)}_{j''t''} \mathbb{R}^{j+j''(q)}_{tt_z;t''t''} \mathbb{R}'^{j''+j'(q)}_{t''t'';t't'_z}, \tag{39}$$

where the coefficients $B^{(\lambda\mu)}_{j''t''}$ are given by

$$B^{(\lambda\mu)}_{j''t''} = \frac{(-1)^{2j+j'+j''-t'+t''} q^{\phi}[\lambda+1][\mu+1][\lambda+\mu+2]}{[\lambda+\frac{1}{2}\mu+j''+t''+2]![\lambda+\frac{1}{2}\mu+j''-t''+1]![2j'']!} \begin{Bmatrix} j & j'' & j+j'' \\ t'' & t & \frac{1}{2}\mu \end{Bmatrix}_q \begin{Bmatrix} j' & j'' & j'+j'' \\ t'' & t' & \frac{1}{2}\mu \end{Bmatrix}_q \tag{40}$$
$$\times \left(\frac{[\lambda+\frac{1}{2}\mu-j+t+1]![\lambda+\frac{1}{2}\mu-j-t]![\lambda+\frac{1}{2}\mu-j'+t'+1]![\lambda+\frac{1}{2}\mu-j'-t']![2j+2j''+1][2j'+2j''+1]}{[2j]![2j']![2t+1][2t'+1]} \right)^{1/2},$$

$$\phi = \varphi(\lambda,\mu,j,t) + \varphi(\lambda,\mu,j',t') - 2\varphi(\lambda,\mu,j'',t'') + j''(4\lambda+2\mu-1) + 4t'' - 2\mu - 3j'. \tag{41}$$

Here and elsewhere, we use the notation

$$\varphi(\lambda,\mu,j,t) := \frac{1}{2}\left(\frac{1}{2}\mu + j - t\right) \times \left(\frac{1}{2}\mu + j + t - 3\right) + j(\lambda - 2j + 1).$$

5. GENERAL FORM OF CLEBSCH–GORDAN COEFFICIENTS

For convenience we introduce the short notation $\Lambda := (\lambda\mu)$ and $\gamma := jtt_z$, and therefore the basis vector $|(\lambda\mu)jtt_z\rangle$ will be denoted by $|\Lambda\gamma\rangle$. Let $\{|\Lambda_i\gamma_i\rangle\}$ be bases of two IRs Λ_i $(i = 1, 2)$. Then, let $\{|\Lambda_1\gamma_1\rangle|\Lambda_2\gamma_2\rangle\}$ be a basis in the representation $\Lambda_1 \otimes \Lambda_2$ of $U_q(su(3)) \otimes U_q(su(3))$. In this representation, there is another coupled basis $|\Lambda_1\Lambda_2 : s\Lambda_3\gamma_3\rangle_q$ with respect to $\Delta_q(U_q(su(3)))$, where the index s classifies multiple representations Λ_3. We can expand the coupled basis in terms of the uncoupled basis $\{|\Lambda_1\gamma_1\rangle|\Lambda_2\gamma_2\rangle\}$:

$$|\Lambda_1\Lambda_2 : s\Lambda_3\gamma_3\rangle_q = \sum_{\gamma_1,\gamma_2} \left(\Lambda_1\gamma_1\Lambda_2\gamma_2|s\Lambda_3\gamma_3\right)_q |\Lambda_1\gamma_1\rangle|\Lambda_2\gamma_2\rangle, \tag{42}$$

where the matrix element $\left(\Lambda_1\gamma_1\Lambda_2\gamma_2|s\Lambda_3\gamma_3\right)_q$ is the Clebsch–Gordan coefficient of $U_q(su(3))$. In just the same way as for the nonquantized Lie algebra $su(3)$ (see [3]), we can show that any CGC of $U_q(su(3))$ can be represented in terms of the linear combination of the matrix elements of the projection operator (39)

$$\left(\Lambda_1\gamma_1\Lambda_2\gamma_2|s\Lambda_3\gamma_3\right)_q = \sum_{\gamma'_2} C(\gamma'_2)\langle\Lambda_1\gamma_1|\langle\Lambda_2\gamma_2|\Delta_q(P^{\Lambda_3}_{\gamma_3,h})|\Lambda_1 h\rangle|\Lambda_2\gamma'_2\rangle. \tag{43}$$

Classification of multiple representations Λ_3 in the representation $\Lambda_1 \otimes \Lambda_2$ is a special problem, and we shall not touch it here. For the nondeformed algebra $su(3)$, this problem was considered in detail in [3]. Concerning the matrix elements on the right side of (43), we give here an explicit expression for the more general matrix element:

$$\langle\Lambda_1\gamma_1|\langle\Lambda_2\gamma_2|\Delta_q(P^{\Lambda_3}_{\gamma_3,\gamma'_3})|\Lambda_1\gamma'_1\rangle|\Lambda_2\gamma'_2\rangle. \tag{44}$$

Using (39) and the Wigner–Racah calculus for the subalgebra $U_q(su(2))$ [5] (analogously to the nonquantized Lie algebra $su(3)$ [3]), it is not hard to obtain the following result:

$$\langle\Lambda_1\gamma_1|\langle\Lambda_2\gamma_2|\Delta_q(P^{\Lambda_3}_{\gamma_3,\gamma'_3})|\Lambda_1\gamma'_1\rangle|\Lambda_2\gamma'_2\rangle = \left(t_1t_{1z}t_2t_{2z}|t_3t_{3z}\right)_q \left(t_1t'_{1z}t_2t'_{2z}|t'_3t'_{3z}\right)_q \tag{45}$$
$$\times [\lambda_3+1][\mu_3+1][\lambda_3+\mu_3+2]A \sum_{j''_1j''_2t''_1t''_2t''_3} C_{j''_1j''_2t''_1t''_2t''_3}$$

$$\times \left\{ \begin{matrix} j_1 - j_1'' & j_2 - j_2'' & j_1 + j_2 - j_1'' - j_2'' \\ t_1'' & t_2'' & t_3'' \\ t_1 & t_2 & t_3 \end{matrix} \right\}_q \left\{ \begin{matrix} j_1' - j_1'' & j_2' - j_2'' & j_1' + j_2' - j_1'' - j_2'' \\ t_1'' & t_2'' & t_3'' \\ t_1' & t_2' & t_3' \end{matrix} \right\}_q .$$

Here,

$$A = \Bigg(\frac{[2t_1+1][2t_2+1][2j_1+1]![2j_2+1]![\lambda_3+\frac{1}{2}\mu_3-j_3+t_3+1]![\lambda_3+\frac{1}{2}\mu_3-j_3-t_3]!}{[\lambda_1+\frac{1}{2}\mu_1-j_1+t_1+1]![\lambda_1+\frac{1}{2}\mu_1-j_1-t_1]![\lambda_2+\frac{1}{2}\mu_2-j_2+t_2+1]![\lambda_2+\frac{1}{2}\mu_2-j_2-t_2]![2j_3]!} \tag{46}$$

$$\times \frac{[2t_1'+1][2t_2'+1][2j_1'+1]![2j_2'+1]![\lambda_3+\frac{1}{2}\mu_3-j_3'+t_3'+1]![\lambda_3+\frac{1}{2}\mu_3-j_3'-t_3']!}{[\lambda_1+\frac{1}{2}\mu_1-j_1'+t_1'+1]![\lambda_1+\frac{1}{2}\mu_1-j_1'-t_1']![\lambda_2+\frac{1}{2}\mu_2-j_2'+t_2'+1]![\lambda_2+\frac{1}{2}\mu_2-j_2'-t_2']![2j_3']!} \Bigg)^{1/2},$$

$$C_{j_1''j_2''t_1''t_2''t_3''} = \frac{(-1)^{2(j_1+j_2+j_3'-j_1''-j_2'')} q^{\psi} [2(j_1+j_2-j_1''-j_2'')+1]![2(j_1'+j_2'-j_1''-j_2'')+1]!}{[2j_1'']![2j_2'']![2j_1-2j_1'']![2j_2-2j_2'']![2j_1'-2j_2'']![2j_2'-2j_2'']![2(j_1+j_2-j_3-j_1''-j_2'')]!} \tag{47}$$

$$\times \frac{[\lambda_1+\frac{1}{2}\mu_1-j_1''+t_1''+1]![\lambda_1+\frac{1}{2}\mu_1-j_1''-t_1'']![\lambda_2+\frac{1}{2}\mu_2-j_2''+t_2''+1]![\lambda_2+\frac{1}{2}\mu_2-j_2''-t_2'']![2t_1''+1][2t_2''+1][2t_3''+1]}{[\lambda_3+\frac{1}{2}\mu_3+j_1+j_2-j_3-j_1''-j_2''+t_3''+2]![\lambda_3+\frac{1}{2}\mu_3+j_1+j_2-j_3-j_1''-j_2''-t_3''+1]!}$$

$$\times \left\{ \begin{matrix} j_1 - j_1'' & j_1'' & j_1 \\ \frac{1}{2}\mu_1 & t_1 & t_1'' \end{matrix} \right\}_q \left\{ \begin{matrix} j_2 - j_2'' & j_2'' & j_2 \\ \frac{1}{2}\mu_2 & t_2 & t_2'' \end{matrix} \right\}_q \left\{ \begin{matrix} j_3 & j_1 + j_2 - j_3 - j_1'' - j_2'' & j_1 + j_2 - j_1'' - j_2'' \\ t_3'' & t_3 & \frac{1}{2}\mu_3 \end{matrix} \right\}_q$$

$$\times \left\{ \begin{matrix} j_1' - j_1'' & j_1'' & j_1' \\ \frac{1}{2}\mu_1 & t_1' & t_1'' \end{matrix} \right\}_q \left\{ \begin{matrix} j_2' - j_2'' & j_2'' & j_2' \\ \frac{1}{2}\mu_2 & t_2' & t_2'' \end{matrix} \right\}_q \left\{ \begin{matrix} j_3' & j_1' + j_2' - j_3' - j_1'' - j_2'' & j_1' + j_2' - j_1'' - j_2'' \\ t_3'' & t_3' & \frac{1}{2}\mu_3 \end{matrix} \right\}_q ,$$

where

$$\begin{aligned} \psi = \Big(\sum_{i=1}^{2} 2\varphi(\lambda_i, \mu_i, j_i'', t_i'') - \varphi(\lambda_i, \mu_i, j_i, t_i) \\ - \varphi(\lambda_i, \mu_i, j_i', t_i') - t_i(t_i+1) - t_i'(t_i'+1) \Big) \\ + 4(j_1 - j_1'')(j_2 - j_2'') + 4(j_1' - j_1'')(j_2' - j_2'') \\ - (j_2 + j_2' - 2j_2'')(2\lambda_1 + \mu_1 - 6j_1'') + 2\mu_3 \\ + \varphi(\lambda_3, \mu_3, j_3, t_3) + \varphi(\lambda_3, \mu_3, j_3', t_3') \\ - 2\varphi(\lambda_3, \mu_3, j_3'', t_3'') + j_3''(4\lambda_3 + 2\mu_3 - 1) \\ - 2t_3''(t_3'' - 1) - (j_3 + j_3'')(j_3 + j_3'' + 1) \\ - (j_3' + j_3'')(j_3' + j_3'' + 1) \\ - t_3(t_3 + 1) - t_3'(t_3' - 1), \\ j_3'' := j_1 + j_2 - j_3 - j_1'' - j_2'' \\ = j_1' + j_2' - j_3' - j_1'' - j_2''. \end{aligned}$$

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SYMPOSIUM ON QUANTUM GROUPS

Fourier–Gauss Transforms of Bilinear Generating Functions for the Continuous q-Hermite Polynomials*

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Abstract—The classical Fourier–Gauss transforms of bilinear generating functions for the continuous q-Hermite polynomials of Rogers are studied in detail. Our approach is essentially based on the fact that the q-Hermite functions have simple behavior with respect to the Fourier integral transform with the q-independent exponential kernel. © *2001 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

Bilinear generating functions (or Poisson kernels) are important tools for studying various properties of the corresponding families of orthogonal polynomials. For example, Wiener has used the bilinear generating function for the Hermite polynomials $H_n(x)$ in proving that the Hermite functions $H_n(x)\exp(-x^2/2)$ are complete in the space L_2 over $(-\infty,\infty)$ and a Fourier transform of any function from L_2 belongs to the same space [1]. Also, it turns out that a particular limit value of the Hermite bilinear generating function reproduces the kernel $\exp(ixy)$ of Fourier transformation between two L_2 spaces. This idea was employed for finding an explicit form of the reproducing kernel for the Kravchuk and Charlier functions in [2], whereas the case of the continuous q-Hermite functions was considered in [3].

It is clear that an appropriate q analog of the Fourier transform will be an essential ingredient of a completely developed theory of q special functions. But the point is that the classical Fourier transform with the q-independent kernel turns out to be very useful in revealing close relations between some families of orthogonal q polynomials [4, 5], as well as among various q extensions of the exponential function e^z [6, 7] and of the Bessel function $J_\nu(z)$ [8, 9]. One of the possible explanations of this remarkable circumstance is the simple Fourier–Gauss transformation property

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{irs-s^2/2}x_q(s)ds = q^{1/4}x_{1/q}(r)e^{-r^2/2}, \quad (1)$$

enjoyed by the q-linear $x_q(s)=\exp(i\kappa s)$ and, consequently, by the q-quadratic lattices $x_q(s)=\sin\kappa s$ or $x_q(s)=\cos\kappa s$ as well, where $q=\exp(-2\kappa^2)$. It remains only to remind the reader that there exist a large class of polynomial solutions to the hypergeometric-type difference equation, which are defined in terms of these nonuniform lattices (see [10] for a review). Convinced of the power of classical Fourier transform, we wish to apply it for studying some additional properties of bilinear generating functions for the continuous q-Hermite polynomials of Rogers.

2. LINEAR GENERATING FUNCTIONS

The continuous q-Hermite polynomials $H_n(x|q)$, $|q|<1$, introduced by Rogers [11], are defined by their Fourier expansion

$$H_n(x|q) := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q e^{i(n-2k)\theta}, \quad x=\cos\theta, \quad (2)$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \quad (3)$$

and $(a;q)_0=1$, $(a;q)_n=\prod_{k=0}^{n-1}(1-aq^k)$, $n=1,2,3,\ldots$, is the q-shifted factorial. These polynomials can be generated by the three-term recurrence relation

$$2xH_n(x|q)=H_{n+1}(x|q)+(1-q^n)H_{n-1}(x|q), \quad n\ge 0, \quad (4)$$

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with the initial condition $H_0(x|q) = 1$. They have been found to enjoy many properties analogous to those known for the classical Hermite polynomials [11–15]. In particular, the Rogers generating function [11] for the q-Hermite polynomials has the form

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} H_n(\cos\theta|q) = e_q(te^{i\theta})e_q(te^{-i\theta}), \quad |t| < 1, \tag{5}$$

where the q-exponential function $e_q(z)$ and its reciprocal $E_q(z)$ are defined by

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = (z;q)_\infty^{-1}, \tag{6}$$
$$E_q(z) := \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n}(-z)^n = (z;q)_\infty.$$

It is often more convenient (cf. [4–7]) to make the change of variables $x = \cos\theta \to x_q(s) = \sin\kappa s$ in (5), which is equivalent to the substitution $\theta = \pi/2 - \kappa s$. That is to say, one can represent (5) as

$$g(s;t|q) := \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} H_n(\sin\kappa s|q) \tag{7}$$
$$= e_q(ite^{-i\kappa s})e_q(-ite^{i\kappa s}), \qquad |t| < 1.$$

Observe that it is easy to verify (7) directly, by substituting the explicit form of

$$H_n(\sin\kappa s|q) := i^n \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q e^{i(2k-n)\kappa s} \tag{8}$$

into it and interchanging the order of summations with respect to the indices n and k.

The advantage of such a parametrization $x = \cos\theta = \sin\kappa s$ is that it actually incorporates both cases of the parameter q: $0 < |q| < 1$ and $|q| > 1$. Indeed, to consider the case when $|q| > 1$, one may introduce the continuous q^{-1}-Hermite polynomials $h_n(x|q)$ as [16]

$$h_n(x|q) := i^{-n} H_n(ix|q^{-1}). \tag{9}$$

The corresponding linear generating function for these polynomials [17] is

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} t^n h_n(\sinh\kappa s|q) \tag{10}$$
$$= E_q(te^{-\kappa s})E_q(-te^{\kappa s}),$$

where [cf. (8)]

$$h_n(\sinh\kappa s|q) \tag{11}$$
$$= \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} e^{(n-2k)\kappa s}.$$

From the inversion identity

$$(q^{-1};q^{-1})_n = (-1)^n q^{-n(n+1)/2}(q;q)_n, \quad n = 0, 1, 2, \ldots, \tag{12}$$

and the transformation property [18] of the q-exponential functions (6)

$$e_{1/q}(z) = E_q(qz), \tag{13}$$

it follows that

$$g(s;t|q^{-1}) \tag{14}$$
$$= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n}(-it)^n h_n(\sinh\kappa s|q)$$
$$= E_q(iqte^{\kappa s})E_q(-iqte^{-\kappa s}).$$

The generating function $g(s; iq^{-1}t|q^{-1})$ thus coincides with the left-hand side of (10), so that (14) reproduces (10).

Examination of Eqs. (7) and (14) reveals that the transformation of $q \to 1/q$ actually provides a reciprocal to the (7) function

$$g^{-1}(s;t|q) = g(-is; q^{-1}t|q^{-1}). \tag{15}$$

There is a second linear generating function

$$f(s;t|q) := \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{(q;q)_n} t^n H_n(\sin\kappa s|q) \tag{16}$$
$$= E_{q^2}(qt^2)\mathcal{E}_q(\sin\kappa s;t)$$

for the continuous q-Hermite polynomials [6, 19]. The q-exponential function $\mathcal{E}_q(x;t)$ in (16) is defined [19] by

$$\mathcal{E}_q(\sin\kappa s;t) := e_{q^2}(qt^2)E_{q^2}(t^2) \tag{17}$$
$$\times \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{(q;q)_n}(-it)^n (q^{\frac{1-n}{2}}e^{-i\kappa s}, -q^{\frac{1-n}{2}}e^{i\kappa s}; q)_n,$$

where $(a_1, \ldots, a_k; q)_n = \prod_{j=1}^{k}(a_j;q)_n$ is the conventional contracted notation for the multiple q-shifted factorials [20]. It is also expressible as a sum of two ${}_2\phi_1$ basic hypergeometric series, i.e.,

$$\mathcal{E}_q(\sin\kappa s;t) = e_{q^2}(qt^2)E_{q^2}(t^2) \tag{18}$$
$$\times \Big[{}_2\phi_1(qe^{2i\kappa s}, qe^{-2i\kappa s}; q; q^2, t^2)$$
$$+ \frac{2q^{1/4}t}{1-q}\sin\kappa s\, {}_2\phi_1(q^2e^{2i\kappa s}, q^2e^{-2i\kappa s}; q^3; q^2, t^2)\Big].$$

Introduced in [19] and further explored in [6, 7, 21], this q-analog of the exponential function $\exp(st)$ on the q-quadratic lattice $x_q(s) = \sin\kappa s$ enjoys the property [19]

$$\mathcal{E}_{1/q}(x;t) = \mathcal{E}_q(x; -q^{1/2}t). \tag{19}$$

Therefore, as follows from (9), (12), and (16),

$$f(s;t|q^{-1}) \tag{20}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{(q;q)_n} (-iq^{1/2}t)^n h_n(\sinh \kappa s|q) = e_q(qt^2) f(is; -q^{1/2}t|q).$$

Note that in the limit case when the parameter $q = \exp(-2\kappa^2)$ tends to 1 (and, consequently, $\kappa \to 0$), we have

$$\lim_{q\to 1^-} \kappa^{-n} H_n(\sin \kappa s|q) = \lim_{q\to 1^-} \kappa^{-n} h_n(\sinh \kappa s|q) = H_n(s), \tag{21}$$

where $H_n(s)$ are the classical Hermite polynomials. The generating functions (7) and (16) thus have the same limit value, i.e.,

$$\lim_{q\to 1^-} g(s; 2\kappa t|q) = \lim_{q\to 1^-} f(s; 2\kappa t|q) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(s) = e^{2st-t^2}. \tag{22}$$

To understand the group-theoretical origin of a particular classical generating function, it is useful to know an appropriate differential equation for this function [22]. The continuous q-Hermite polynomials (8) are solutions to the difference equation

$$D_q(s) H_n(\sin \kappa s|q) = q^{n/2} \cos \kappa s H_n(\sin \kappa s|q) \tag{23}$$

with an operator $D_q(s)$ defined by

$$D_q(s) := \frac{1}{2}\left[e^{i\kappa s} e^{-i\kappa \partial_s} + e^{-i\kappa s} e^{i\kappa \partial_s}\right], \quad \partial_s = \frac{d}{ds}. \tag{24}$$

To verify (23), apply the difference operator (24) to both sides of the Rogers generating function (7) and then equate coefficients of the equal powers of the parameter t.

The difference equation (23) coincides in the limit of $q \to 1^-$ with the second-order differential equation

$$(\partial_s^2 - 2s\partial_s + 2n) H_n(s) = 0 \tag{25}$$

for the polynomials $H_n(s)$.

As a consequence of (23), the generating functions (7) and (16) satisfy the same difference equation

$$D_q(s) g(s; t|q) = \cos \kappa s \, g(s; q^{-1/2} t|q). \tag{26}$$

It may be of interest to note that the operator $D_q(s)$ is also well defined in the case of $|q| > 1$. The explicit form of $D_{1/q}(s)$ and its action on the generating functions (14) and (20) is readily obtained from (24) and (23), respectively, by the substitution $q \to q^{-1} (\kappa \to i\kappa)$.

In closing this section, we emphasize that the linear generating functions $g(s; t|q)$ and $f(r; t|q)$ are interrelated by a Fourier transformation with the standard exponential kernel $\exp(isr)$, not involving q. The reason is that the continuous q-Hermite and q^{-1}-Hermite polynomials are related to each other by the Fourier–Gauss transformation [cf. formula (1)]

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_n(\sin \kappa s|q) e^{irs - s^2/2} ds = i^n q^{n^2/4} h_n(\sinh \kappa r|q) e^{-r^2/2}. \tag{27}$$

This integral transform was derived in [4], and it plays an important role in our study of bilinear generating functions for the q-Hermite polynomials. The key point in deriving (27) was the finding that one should use the parametrization $x = \sin \kappa s, q = \exp(-2\kappa^2)$, for the argument of the q-Hermite polynomials $H_n(x|q)$. Once (27) is established, one can readily verify it by employing only the explicit forms (8) and (11) of the q-Hermite and q^{-1}-Hermite polynomials, respectively, and the well-known Fourier integral transform

$$\int_{-\infty}^{\infty} e^{irs - s^2/2} ds = \sqrt{2\pi} e^{-r^2/2}$$

for the Gauss exponential function $\exp(-s^2/2)$.

We return now to a relation between the linear generating functions $g(s; t|q)$ and $f(r; t|q)$. Multiply both sides of (27) by $t^n/(q;q)_n$ and sum over n from zero to infinity. Taking into account (20), this gives

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s; t|q) e^{irs - s^2/2} ds = e_q(t^2) f(ir; t|q) e^{-r^2/2}. \tag{28}$$

3. BILINEAR GENERATING FUNCTIONS

The q-Mehler formula (or the Poisson kernel) for the q-Hermite polynomials

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} H_n(\cos \theta|q) H_n(\cos \varphi|q) = E_q(t^2) e_q(te^{i(\theta-\varphi)}) e_q(te^{i(\varphi-\theta)}) \times e_q(te^{i(\theta+\varphi)}) e_q(te^{-i(\theta+\varphi)}) \tag{29}$$

was originally derived by Rogers [11]; its simple derivation is due to Bressoud [15]. As in the case of linear generating functions, it is more convenient to make the changes $x = \cos \theta \to \sin \kappa s = x_q(s)$ and $y = \cos \varphi \to \sin \kappa r = x_q(r)$ in (29), which are equivalent to substitutions $\theta = \pi/2 - \kappa s$ and $\varphi = \pi/2 - \kappa r$. In other words, one can represent (29) as

$$G(s, r; t|q) := \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} H_n(\sin \kappa s|q) H_n(\sin \kappa r|q)$$

$$= E_q(t^2)e_q(te^{i\kappa(s-r)})e_q(te^{i\kappa(r-s)}) \times e_q(-te^{i\kappa(s+r)})e_q(-te^{-i\kappa(s+r)}) \quad (30a)$$

$$= E_q(t^2)g(s; ite^{-i\kappa r}|q)g(s; -ite^{i\kappa r}|q), \quad (30b)$$

in accordance with the definition (7) of the linear generating function $g(s;t|q)$.

Notice that the bilinear generating functions (29) and (30a) are closely connected with the Rogers linearization formula

$$H_m(x|q)H_n(x|q) = \sum_{k=0}^{m\wedge n} \frac{(q;q)_m(q;q)_n}{(q;q)_{m-k}(q;q)_{n-k}(q;q)_k} H_{m+n-2k}(x|q) \quad (31)$$

and its inverse [11, 15]

$$H_{m+n}(x|q) = (q;q)_m(q;q)_n \sum_{k=0}^{m\wedge n} (-1)^k q^{k(k-1)/2} \times \frac{H_{m-k}(x|q)H_{n-k}(x|q)}{(q;q)_{m-k}(q;q)_{n-k}(q;q)_k}, \quad (32)$$

where $m \wedge n := \min\{m, n\}$. For instance, to verify (30a), one can substitute the explicit form (8) for any one of the two q-Hermite polynomials in (30a) and use (32) for the other one. Then, the sum over the index n in (30a) factorizes into a product of two linear generating functions of the type (7) (see (30b)), which is multiplied by the q-exponential function $E_q(t^2)$.

Later on we shall need an inverse Rogers linearization formula

$$h_{m+n}(x|q) = (q;q)_m(q;q)_n \sum_{k=0}^{m\wedge n} (-1)^k q^{k(k-m-n)} \times \frac{h_{m-k}(x|q)h_{n-k}(x|q)}{(q;q)_{m-k}(q;q)_{n-k}(q;q)_k} \quad (33)$$

for the continuous q^{-1}-Hermite polynomials $h_n(x|q)$, which follows from (32) by replacing x with ix and q with q^{-1} and subsequently employing (9) and (12).

As follows from (9), (12), and (13),

$$G(s,r;t|q^{-1}) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} t^n h_n(\sinh\kappa s|q)h_n(\sinh\kappa r|q)$$
$$= e_q(qt^2)E_q(qte^{\kappa(s-r)})E_q(qte^{\kappa(r-s)}) \times E_q(-qte^{\kappa(s+r)})E_q(-qte^{-\kappa(s+r)}) \quad (34a)$$
$$= e_q(qt^2)g(s; ite^{\kappa r}|q^{-1})g(s; -ite^{-\kappa r}|q^{-1}). \quad (34b)$$

This coincides with the q-Mehler formula [17] for the q^{-1}-Hermite polynomials

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} t^n h_n(\sinh\xi|q)h_n(\sinh\eta|q) \quad (35)$$
$$= e_q(qt^2)E_q(qte^{\xi-\eta})E_q(qte^{\eta-\xi}) \times E_q(-qte^{\xi+\eta})E_q(-qte^{-\xi-\eta}),$$

upon identifying $\kappa s = \xi$ and $\kappa r = \eta$. Similar to the case of the bilinear generating function (30a), one can directly verify (34a), or (35), by using the inverse Rogers linearization formula (33). It is worth noting that Ismail and Masson [17] have employed the Poisson kernel (35) to determine the large-n asymptotics of the q^{-1}-Hermite polynomials $h_n(x|q)$.

Observe also the relation

$$G^{-1}(s,r;qt|q) = (1-qt^2)G(is,ir;t|q^{-1}), \quad (36)$$

which follows from the q-Mehler formulas (31) and (34a).

As a consequence of (9) and (27), the generating functions (30a) and (34a) are related to each other by the Fourier–Gauss transform in the variables s and r, i.e.,

$$G(s,r;t|q^{-1})e^{-(s^2+r^2)/2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(u,v;q^{\frac{1}{2}}t|q)e^{i(su-rv)-(u^2+v^2)/2}dudv. \quad (37)$$

The Fourier–Gauss transform (37) is equivalent to a particular case of Ramanujan's integral

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e_q(ae^{2i\kappa x})e_q(be^{-2i\kappa x})e^{2ixy-x^2}dx \quad (38)$$
$$= e_q(ab)E_q(-aq^{1/2}e^{2\kappa y})E_q(-bq^{1/2}e^{-2\kappa y})e^{-y^2}$$

with a complex parameter [23–25]. Indeed, using (30a) and substituting $s_\pm = (s \pm r)/\sqrt{2}$ and $u_\pm = (v \pm u)/\sqrt{2}$ leads to the separation of variables on the right-hand side of (37) and gives a product of two integrals with respect to u_+ and u_-. Both of these independent integrals are of the type (38) with the equal parameters $a = b = q^{1/2}t$. One thus recovers the left-hand side of (37) with the generating function $G(s,r;t|q^{-1})$, defined in (34a).

Two particular cases of the generating function $G(s,r;t|q)$ are of interest for purposes of its use in the sequel. The first of them is $G(s,0;t|q)$. Since $H_{2k}(0|q) = (-1)^k(q;q^2)_k$, $H_{2k+1}(0|q) = 0$, and $(q;q)_{2k} = (q;q^2)_k(q^2;q^2)_k$, this function represents the sum

$$G(s,0;t|q) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(q^2;q^2)_n} H_{2n}(\sin\kappa s|q). \quad (39)$$

By the q-Mehler formula (31), this sum is equal to

$$G(s,0;t|q) = E_q(t^2)e_{q^2}(t^2e^{2i\kappa s})e_{q^2}(t^2e^{-2i\kappa s}) \quad (40)$$

on account of $e_q(z)e_q(-z) = e_{q^2}(z^2)$.

Similarly, since $h_{2n}(0|q) = (-1)^n q^{-n^2}(q;q^2)_n$ and $h_{2n+1}(0|q) = 0$, from (34a) we have

$$G(s,0;t|q^{-1}) \quad (41)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(q^2;q^2)_n} t^{2n} h_{2n}(\sinh \kappa s|q)$$
$$= e_q(qt^2)E_{q^2}(q^2t^2e^{2\kappa s})E_{q^2}(q^2t^2e^{-2\kappa s}).$$

As a consequence of (23), the generating function $G(s,r;t|q)$ satisfies the following difference equations:

$$D_q(s)G(s,r;t|q) = \cos\kappa s G(s,r;q^{-1/2}t|q), \quad (42a)$$
$$D_q(s)D_q(r)G(s,r;t|q) \quad (42b)$$
$$= \cos\kappa s \cos\kappa r G(s,r;q^{-1}t|q).$$

Observe that, if the independent variables s and r are replaced by their linear combinations $s_\pm = (s \pm r)/\sqrt{2}$, then the product of difference operators $D_q(s)$ and $D_q(r)$ takes the form

$$D_q(s)D_q(r) = \frac{1}{2}\left[D_{q^2}(s_+) + D_{q^2}(s_-)\right]. \quad (43)$$

The integral transform (27), relating the continuous q-Hermite and q^{-1}-Hermite polynomials, also suggests the consideration of the "mixed" generating function

$$F(s,r;t|q) \quad (44)$$
$$= \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{(q;q)_n} t^n H_n(\sin\kappa s|q) h_n(\sinh\kappa r|q),$$

which is different from the known functions (30a) and (34a). Unlike $G(s,r;t|q)$, the generating function $F(s,r;t|q)$ is not symmetric in the variables s and r. Instead, it has the following property

$$F(s,r;t|q^{-1}) = F(r,s;q^{1/2}t|q). \quad (45)$$

Also, a relation between the "mixed" generating function (44) and linear generating function (16) is more complicated than (30b) or (34b). Indeed, substitute the explicit form of the q-Hermite polynomials (8) into (44) and interchange the order of summations with respect to the indices n and k. The subsequent use of the inverse Rogers linearization formula (33) factors out the q-exponential function $e_q(-t^2)$ and gives the following relation:

$$F(s,r;t|q) = e_q(-t^2) \quad (46)$$
$$\times \sum_{n=0}^{\infty} f_n(r, q^{-1/2}te^{i\kappa s}|q^{-1}) f(r, -q^{\frac{n-1}{2}}te^{-i\kappa s}|q^{-1}),$$

where

$$f_n(r;t|q^{-1}) := \frac{q^{n(n+2)/4}}{(q;q)_n}(-it)^n h_n(\sinh\kappa r|q),$$
$$n = 0,1,2,\ldots,$$

are the partial linear generating functions of the second kind (20), i.e,

$$f(r;t|q^{-1}) = \sum_{n=0}^{\infty} f_n(r;t|q^{-1}). \quad (47)$$

From the relation (27) between the q-Hermite and q^{-1}-Hermite polynomials and the definition (44), it follows that the Fourier transform of $F(s,r;t|q) \times \times \exp(-s^2/2)$ is equal to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s,r;t|q)e^{isu-s^2/2}ds \quad (48)$$
$$= G(u,r;iq^{-1/2}t|q^{-1})e^{-u^2/2}.$$

In a like manner, the inverse Fourier transformation with respect to (27) yields

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s,r;t|q)e^{irv-r^2/2}dr \quad (49)$$
$$= G(s,v;it|q)e^{-v^2/2}.$$

Since the explicit form of the bilinear generating function $G(u,v;t|q)$ is given by formulas (31) and (34a) for the values $0<|q|<1$ and $|q|>1$ of the parameter q, respectively, relations (48) and (49) lead to the two integral representations for $F(s,r;t|q)$ of the form

$$F(s,r;t|q)e^{-s^2/2} = \frac{e_q(-t^2)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isu-u^2/2} \quad (50a)$$
$$\times E_q(iq^{1/2}te^{\kappa(u-r)})E_q(iq^{1/2}te^{\kappa(r-u)})$$
$$\times E_q(-iq^{1/2}te^{\kappa(u+r)})E_q(-iq^{1/2}te^{-\kappa(u+r)})du,$$
$$F(s,r;t|q)e^{-r^2/2} = \frac{E_q(-t^2)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-irv-v^2/2}$$
$$\times e_q(ite^{i\kappa(s-v)})e_q(ite^{i\kappa(v-s)}) \quad (50b)$$
$$\times e_q(-ite^{i\kappa(s+v)})e_q(-ite^{-i\kappa(s+v)})dv.$$

Also, combining (48) with (49) shows that the Fourier–Gauss transform of $F(s,r;t|q)$ in both independent variables s and r reproduces this generating function, i.e.,

$$F(v,u;t|q)e^{-(u^2+v^2)/2} \quad (51)$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s,r;t|q)e^{i(su-rv)-(s^2+r^2)/2}dsdr.$$

Similar to $G(s,r;t|q)$, two particular cases of $F(s,r;t|q)$ are easily summed as products of q-exponential functions. The first one is $F(s,0;t|q)$, and it coincides with $G(s,0;t|q)$, given by (39) and (40). The second one is $F(0,r;t|q)=G(r,0;q^{-1/2}t|q^{-1})$, where the function $G(r,0;t|q^{-1})$ is explicitly given in (41).

From (23), it follows at once that the mixed generating function (44) satisfies the difference equations in the variables s and r of the form

$$D_q(s)F(s,r;t|q)=\cos\kappa s F(s,r;q^{-1/2}t|q), \quad (52a)$$

$$D_{1/q}(r)F(s,r;t|q)=\cosh\kappa r F(s,r;q^{1/2}t|q). \quad (52b)$$

In view of the property (45), these two equations are related to each other by the substitution $q\to q^{-1}$. Combining (52a) with (52b) gives the difference equation

$$D_q(s)D_{1/q}(r)F(s,r;t|q) = \cos\kappa s\cosh\kappa r F(s,r;t|q). \quad (53)$$

We note in closing that Ismail and Stanton derived in [26] a closed-form expression for a few generating functions, containing the product of the continuous q-ultraspherical polynomials $C_n(x;\beta|q)$ and $C_n(y;\gamma|q)$ with different parameters β and γ. These generating functions are variations of two typical forms

$$\sum_{n=0}^{\infty}\frac{(q;q)_n}{(\gamma^2;q)_n}C_n(x;\beta|q)C_n(y;\gamma|q)t^n \quad (54)$$

and

$$\sum_{n=0}^{\infty}C_n(x;\beta|q)C_n(y;\gamma|q)t^n. \quad (55)$$

Since the continuous q-ultraspherical polynomials $C_n(x;\beta|q)$ are defined (see, for example, [20]) as

$$C_n(x;\beta|q):=\sum_{k=0}^{n}\frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}}e^{i(n-2k)\theta}, \quad x=\cos\theta, \quad (56)$$

the q-Hermite $H_n(x|q)$ and the q^{-1}-Hermite $H_n(x|q^{-1})$ polynomials can be regarded as the following particular and limiting cases of the continuous q-ultraspherical polynomials (56):

$$H_n(x|q)=(q;q)_nC_n(x;0|q), \quad (57)$$

$$H_n(x|q^{-1}) = (-1)^nq^{-n(n-1)/2}(q;q)_n\lim_{\beta\to\infty}\beta^{-n}C_n(x;\beta|q).$$

Consequently, by considering the particular case of (54) with $\gamma=0$, appropriately rescaling the parameter $t=\tau/\beta$, and subsequently taking the limit as β tends to infinity, one obtains the following mixed generating function for the continuous q-Hermite polynomials:

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q;q)_n}(-\tau)^nH_n(x|q^{-1})H_n(y|q). \quad (58)$$

In a similar manner, from (55) follows another generating function

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q;q)_n^2}(-\tau)^nH_n(x|q)H_n(y|q^{-1}). \quad (59)$$

The mixed generating function (44), studied by us in this paper, is different from both (58) and (59). But it is not hard to show, exactly in the same way as for (44), that (58) and (59) also have a simple transformation property with respect to the classical Fourier integral transform. Indeed, if one introduces a mixed generating function [cf. (58)]

$$F_1^{\rm IS}(s,r;t|q) := \sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q;q)_n}t^nH_n(\sin\kappa s|q)h_n(\sin\kappa r|q), \quad (60)$$

then from [26] it follows that

$$F_1^{\rm IS}(s,r;t|q)=e_q(-e^{2i\kappa s}) \times E_q(ite^{-\kappa(r+is)})E_q(-ite^{\kappa(r-is)}) \times {}_4\phi_3\left(\left.\begin{matrix}0,0,0,0\\ ite^{-\kappa(r+is)},\ -ite^{\kappa(r-is)},\ -qe^{-2i\kappa s}\end{matrix}\right|q,q\right) + (s\to -s, r\to -r). \quad (61)$$

Now using the integral transform (27) with respect to the variable s and its inverse transform with respect to the variable r in (60), leads to [cf. (51)]

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}F_1^{\rm IS}(s,r;t|q)e^{i(sv-ru)-(s^2+r^2)/2}dsdr = F_1^{\rm IS}(u,v;t|q)e^{(u^2+v^2)/2}. \quad (62)$$

Similarly, if one considers a mixed generating function of the form (cf. (59))

$$F_2^{\rm IS}(s,r;t|q) := \sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q;q)_n^2}t^nH_n(\sin\kappa s|q)h_n(\sinh\kappa r|q), \quad (63)$$

then by [26] it can be represented as a sum of two basic hypergeometric series ${}_2\phi_3$, i.e.,

$$F_2^{\rm IS}(s,r;t|q) = e_q(q)e_q(-e^{2i\kappa s})E_q(ite^{-\kappa(r+is)})E_q(-ite^{\kappa(r-is)}) \tag{64}$$
$$\times\ {}_2\phi_3\left(\begin{matrix} 0,0 \\ ite^{-\kappa(r+is)}, -ite^{\kappa(r-is)}, -e^{-2i\kappa s}\end{matrix}\,\middle|\, q, -q^2e^{-2i\kappa s}\right) + (s\to -s, r\to -r).$$

Evidently, the mixed generating function $F_2^{\rm IS}(s,r;t|q)$ has the same transformation property as $F_1^{\rm IS}(s,r;t|q)$ in (62).

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SYMPOSIUM ON QUANTUM GROUPS

The $su_q(2)$ Algebra in the Off-Diagonal Basis and Applications to Quantum Optics*

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Abstract—We consider a new exactly solvable nonlinear quantum model as a Hamiltonian defined in terms of the generators of the $su_q(2)$ algebra. The corresponding matrix elements of finite rotations (the q-deformed Wigner d functions) are introduced. It is shown that the quantum optical model of the three-wave interaction has an approximate $su_q(2)$ dynamical symmetry given by this Hamiltonian. Such q symmetry allows us to investigate the spectral and dynamical properties of the three wave model through new perturbation techniques. © *2001 MAIK "Nauka/Interperiodica".*

1. AN EXACTLY SOLVABLE NONLINEAR QUANTUM MODEL

Let us consider the following operator [1]

$$H_q = q^{J_z/2}(J_+ + J_-)q^{J_z/2}, \tag{1}$$

which is defined on the $su_q(2)$ quantum algebra [2–4]:

$$[J_z, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = [2J_z]_q. \tag{2}$$

The q-number is, as usual, $[x]_q := \dfrac{q^x - q^{-x}}{q - q^{-1}}$ with $q = e^z$. Hereafter, we shall assume that q is not a root of unity. The deformed tensor product representations of $su_q(2)$ are given through the deformed coproduct

$$\Delta(J_\pm) = J_\pm \otimes q^{-J_z} + q^{J_z} \otimes J_\pm, \tag{3}$$

$$\Delta(J_z) = J_z \otimes 1 + 1 \otimes J_z. \tag{4}$$

Note that the $q \to 1$ limit of H_q is just the $2J_x$ operator in $su(2)$.

The representation theory of $su_q(2)$ is a smooth deformation of the $su(2)$ one. Namely, in the "bare" basis $|l, m\rangle$ of eigenvectors of J_z, the $(2l+1)$-dimensional irreducible representations of $su_q(2)$ read

$$2J_z|l, m\rangle = 2m|l, m\rangle, \tag{5}$$

$$J_\pm|l, m\rangle = \sqrt{[l \mp m]_q[l \pm m + 1]_q}\,|l, m \pm 1\rangle. \tag{6}$$

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As a consequence, H_q is a tridiagonal matrix of dimension $(2l+1)$:

$$\begin{pmatrix} 0 & A_l(q) & 0 & \dots & 0 \\ A_l(q) & 0 & A_{l-1}(q) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & A_{-l+2}(q) & 0 & A_{-l+1}(q) \\ 0 & \dots & 0 & A_{-l+1}(q) & 0 \end{pmatrix}, \tag{7}$$

where

$$A_m(q) = q^{m-1/2}\sqrt{[l+m]_q[l-m+1]_q}. \tag{8}$$

It can be proven [1] that the eigenvalues of H_q for a given l are just the q numbers $[2m]$, with $m = -l, \dots, l$ (see Fig. 1). Moreover, the eigenvector $|\underline{l, n}\rangle'$ associated to any eigenvalue $[2n]_q$ can be also deduced, namely,

$$\begin{aligned} \alpha_{mn} &= \langle l, m|\underline{l, n}\rangle' \\ &= \alpha_{ml} q^{(l-n)(l-n-2m)}\sqrt{\frac{[2l]_q!}{[l-n]_q![l+n]_q!}} \\ &\times \sum_{j=0}^{l-n}\Bigg\{(-1)^j q^{-j(l-m-2n)+j(j+1)/2} \\ &\times \frac{[2(l-n)]_q!!}{[j]_q![2(l-n-j)]_q!!} \\ &\times \frac{[l-m]_q[l-m-1]_q\dots[l-m-j+1]_q}{[2l]_q[2l-1]_q\dots[2l-j+1]_q}\Bigg\}, \end{aligned} \tag{9}$$

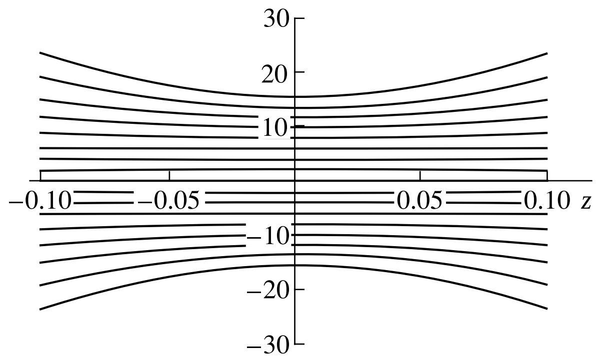

Fig. 1. Spectrum of H_q in the $l = 8$ representation as a function of $z = \ln q$. Note that the spectrum is anharmonic for $z \neq 0$.

where $[2n]_q!! := [2n]_q \cdot [2n-2]_q \dots [2]_q$ and

$$\alpha_{ml} \equiv \langle l, m | \underline{l, l} \rangle' \tag{10}$$
$$= q^{m(l-1/2)} \sqrt{\frac{[2l]_q!}{[l+m]_q![l-m]_q!}}.$$

These eigenvectors can be easily normalized in terms of q numbers (see also [5], where the same eigenvectors are obtained in terms of q-Kravchuk polynomials). In the same manner, formulas for Clebsch–Gordan coefficients in the "dressed basis" $|\underline{l, m}\rangle'$ can be deduced.

2. ON q-DEFORMED WIGNER d FUNCTIONS

A q-deformation of Wigner d functions can be introduced by computing, in a suitable basis, the matrix elements of the exponential of the H_q operator of the quantum algebra $su_q(2)$.

Let us recall that the usual definition of the Wigner d matrix in the $(2l+1)$-dimensional representation is

$$d^{(l)}(\beta) = e^{-i\beta J_y^{(l)}}, \tag{11}$$

where, in general,

$$J_\pm = J_x \pm i J_y. \tag{12}$$

As a consequence,

$$J_+ + J_- = 2J_x \tag{13}$$

and a $\pi/2$ rotation around the z axis is needed in order to define the d functions from the J_x generator:

$$d^{(l)}(\beta) = e^{-i\frac{\pi}{2}J_z^{(l)}} e^{-i\frac{\beta}{2}(J_+^{(l)} + J_-^{(l)})} e^{i\frac{\pi}{2}J_z^{(l)}}. \tag{14}$$

In this context, a natural q deformation of the d matrix is

$$d_q^{(l)}(\beta) = e^{-i\frac{\pi}{2}J_z^{(l)}} e^{-i\frac{\beta}{2}H_q^{(l)}} e^{i\frac{\pi}{2}J_z^{(l)}}, \tag{15}$$

where H_q is the q-deformed $J_+^{(l)} + J_-^{(l)}$ operator (1) (see [6–9] for different constructions of q-Wigner d functions, some of them on noncommutative variables).

Since the eigenvalues of H_q in the dressed basis are the q numbers $[2m]_q$, the exponential of H_q can be easily computed. Afterwards, the eigenvectors of H_q provide the transformation that gives the q-deformed d matrix in an explicit form.

In the $l = 1$ representation, the q-Wigner d functions are the entries of the matrix (14)

$$d_q^{(1)}(\beta)$$
$$= \begin{pmatrix} \frac{q^{-1}+q\cos(\beta[2]_q/2)}{[2]_q} & -\frac{\sqrt{q}\sin(\beta[2]_q/2)}{\sqrt{[2]_q}} & \frac{2\sin^2(\beta[2]_q/4)}{[2]_q} \\ \frac{\sqrt{q}\sin(\beta[2]_q/2)}{\sqrt{[2]_q}} & \cos\frac{\beta[2]_q}{2} & -\frac{\sin(\beta[2]_q/2)}{\sqrt{q}\sqrt{[2]_q}} \\ \frac{2\sin^2(\beta[2]_q/4)}{[2]_q} & \frac{\sin(\beta[2]_q/2)}{\sqrt{q}\sqrt{[2]_q}} & \frac{q+q^{-1}\cos(\beta[2]_q/2)}{[2]_q} \end{pmatrix}.$$

As expected, the $q \to 1$ limit of this matrix is the usual Wigner $d^{(1)}$ matrix. Arbitrary q-$d^{(l)}$ functions will be given in terms of trigonometric functions with q-harmonic arguments $\beta[2l]_q/2$. This fact can be illustrated in the $l = 3/2$ case, for which the matrix $d_q^{(1)}(\beta)$ has the following q-Wigner d functions as matrix elements:

$$\left(d_q^{(1)}\right)_1^1 = \frac{q^{-1}}{[4]_q}\left\{[3]_q \cos\frac{\beta}{2} + q^4 \cos\frac{\beta[3]_q}{2}\right\},$$
$$\left(d_q^{(1)}\right)_1^2 = -\frac{q^{-2}\sqrt{[3]_q}}{[4]_q}\left\{\sin\frac{\beta}{2} + q^4 \sin\frac{\beta[3]_q}{2}\right\},$$
$$\left(d_q^{(1)}\right)_1^3 = \frac{q\sqrt{[3]_q}}{[4]_q}\left\{\cos\frac{\beta}{2} - \cos\frac{\beta[3]_q}{2}\right\},$$
$$\left(d_q^{(1)}\right)_1^4 = \frac{1}{[4]_q}\left\{-[3]_q \sin\frac{\beta}{2} + \sin\frac{\beta[3]_q}{2}\right\},$$
$$\left(d_q^{(1)}\right)_2^2 = \frac{q}{[4]_q}\left\{q^{-4} \cos\frac{\beta}{2} + [3]_q \cos\frac{\beta[3]_q}{2}\right\},$$
$$\left(d_q^{(1)}\right)_2^3 = \frac{[2]_q}{[4]_q}\left\{\sin\frac{\beta}{2} - [3]_q \sin\frac{\beta[3]_q}{2}\right\},$$
$$\left(d_q^{(1)}\right)_2^4 = \frac{q^{-1}\sqrt{[3]_q}}{[4]_q}\left\{\cos\frac{\beta}{2} - \cos\frac{\beta[3]_q}{2}\right\},$$
$$\left(d_q^{(1)}\right)_3^3 = \frac{q^{-1}}{[4]_q}\left\{q^4 \cos\frac{\beta}{2} + [3]_q \cos\frac{\beta[3]_q}{2}\right\},$$
$$\left(d_q^{(1)}\right)_3^4 = -\frac{q^{-2}\sqrt{[3]_q}}{[4]_q}\left\{q^4 \sin\frac{\beta}{2} + \sin\frac{\beta[3]_q}{2}\right\},$$
$$\left(d_q^{(1)}\right)_4^4 = \frac{q}{[4]_q}\left\{[3]_q \cos\frac{\beta}{2} + q^{-1} \cos\frac{\beta[3]_q}{2}\right\}.$$

A complete treatment of these q functions—including symmetry properties and recurrence relations induced from the coproduct—will be given elsewhere [10].

3. CONNECTION WITH THE DICKE MODEL AND SECOND HARMONIC GENERATION

The trilinear boson Hamiltonian

$$\mathcal{H} = \omega_a a^\dagger a + \omega_b b^\dagger b + \omega_c c^\dagger c + \lambda(a^\dagger b^\dagger c + c^\dagger ba) \quad (16)$$

(with a, b, and c being boson operators for three different modes of the radiation field) describes nonlinear quantum optical processes such as frequency conversion and Raman and Brillouin scattering.

Through a Jordan–Schwinger transformation, this model is algebraically equivalent to the resonant interaction of a sample of two-level atoms with a single mode of the quantized radiation field (Dicke model [11]):

$$\mathcal{H} = H_0 + gH_\mathrm{D} = \omega a^\dagger a + \omega_\mathrm{at} S_z + g(a^\dagger S_- + aS_+). \quad (17)$$

Here, $S_i = \sum_{k=1}^N S_i^{(k)}$, where N is the number of atoms in the sample and $S_i^{(k)}$ are the pseudospin operators of the kth atom.

If the initial state of the system is a given eigenstate of $\hat{s} = \hat{n}_a + S_z + N/2$, a block-diagonal form of the Dicke interaction Hamiltonian $H_\mathrm{D} = \left(a^\dagger S_- + aS_+\right)$ is obtained:

$$H_\mathrm{D}^{(s)} = \begin{pmatrix} 0 & A_l & 0 & \dots & 0 \\ A_l & 0 & A_{l-1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & A_{-l+2} & 0 & A_{-l+1} \\ 0 & \dots & 0 & A_{-l+1} & 0 \end{pmatrix}. \quad (18)$$

The dimension of the matrix (18) is $(2l+1)$, and l is related to the number of atoms through $2l = N$. In this basis, the matrix elements of $H_\mathrm{D}^{(s)}$ read

$$A_m = \sqrt{(l+m)(l-m+1)(2s-l-m+1)}, \quad (19)$$
$$m = -l+1, -l+2, \dots, l.$$

Here, $2s$ (with $s \geq l$) is just the (constant) eigenvalue of the excitation number operator in the chosen subspace.

Under certain dynamical conditions (essentially, in either a strong or a weak field regime), the matrix (18) can be written as the $(2l+1)$-dimensional irreducible representation of the $2J_x = J_+ + J_-$ generator of $su(2)$ plus some additional smaller terms. In this way, a perturbative approach to the spectrum and dynamical properties of the Dicke model can be developed (see [12] and references therein).

The second-harmonic-generation (SHG) analog of the Dicke Hamiltonian is obtained when $s = l$.

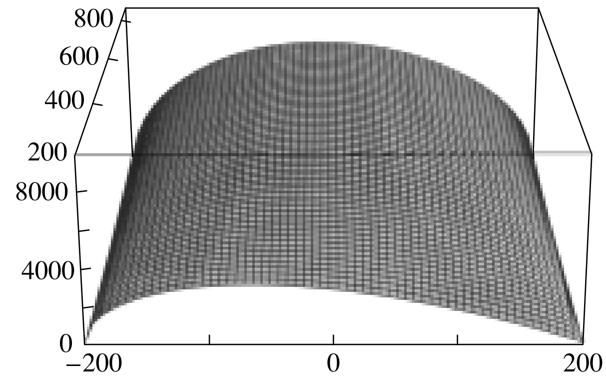

Fig. 2. Matrix elements of the Dicke interaction Hamiltonian $H_\mathrm{D}^{(s)}$ for $200 \leq s \leq 850$ ($l = 200$). Note that, as far as $s/l = 2s/N$ increases, the Hamiltonian approaches pure angular momentum features (symmetric curve). On the other hand, the strongest nonlinearity is obtained in the SHG regime ($s = l$).

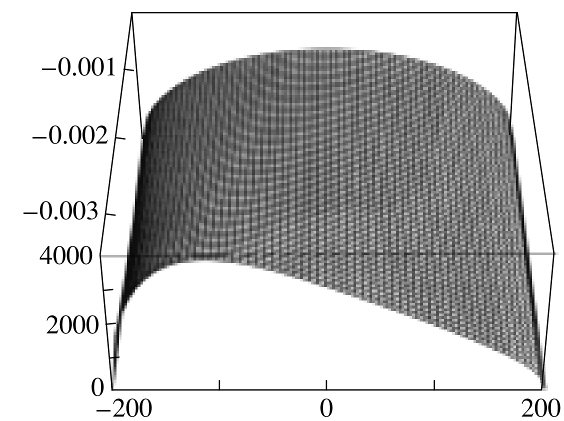

Fig. 3. Matrix elements of the deformed Hamiltonian H_q^Ω for $0 \leq z \leq -0.0035$ ($l = 200$). We stress that the quantum deformation (with parameter $z = \ln q$) introduces a nonlinearity that fits with the one coming from the atom–photon interaction. In the case of SHG with $l = 200$, the best approximation between both models is obtained for $z = -0.0018$.

This is a strongly nonlinear regime of H_D for which the $su(2)$ description is no longer valid, as can be appreciated in Fig. 2, where we have assumed A_m to be a continuous function of m. However, if we compare this plot with the function $A_m(q)$ corresponding to the q-deformed Hamiltonian H, it becomes clear that we could approach the SHG model by fitting an appropriate deformation parameter (see Fig. 3).

Therefore, let us try to approximate analytically the Dicke/SHG operator $H_\mathrm{D}^{(s)}$ through a Hamiltonian of the type

$$T_0 = \Omega H_q, \quad (20)$$

where we shall have two free parameters to be fitted: q and Ω. In order to get the closest T_0 to $H_{\mathrm{D}}^{(s)}$, we can choose both parameters in such a way that the matrix elements A_m of $H_{\mathrm{D}}^{(s)}$ and the matrix elements

$$\tilde{A}_m(q) = \Omega A_m(q) \quad (21)$$
$$= \Omega q^{m-1/2}\sqrt{[l+m]_q[l-m+1]_q}$$
$$= \Omega\langle l,m|2J_x|l,m-1\rangle$$

of the Hamiltonian T_0 coincide in their maxima. This choice gives rise (for $s=l$) to the following relations defining both q and Ω in terms of N:

$$\alpha = N\log q = \frac{3}{2}\log\frac{\sqrt{5}-1}{2} \approx -0.7218, \quad (22)$$

$$\Omega = \frac{4(N+1)^{3/2}}{\sqrt{27}[N+1]_q}. \quad (23)$$

In this way, both the maxima of A_m and $\tilde{A}_m(q)$ (considered as functions of m) occur at the point $m_0 = -(l-1)/3$.

Now, we can find the approximation for the three-wave Hamiltonian in the form

$$A_m \approx \Omega A_m(q)\phi(m), \quad (24)$$
$$\phi(m) = 1 + \phi_1\Delta - \phi_2\Delta^2 + \phi_3\Delta^3, \quad \Delta = m - m_0.$$

We thus restrict the expansion up to the third-order polynomial $\phi(m)$. We can find explicitly the coefficients ϕ_j by equating the corresponding Taylor expansions around the point m_0:

$$\phi(\Delta) = 1 - \left(\frac{27}{8} - \frac{2\alpha^2}{\tanh^2\alpha}\right)\left(\frac{\Delta}{N+1}\right)^2 \quad (25)$$
$$+ \left(\frac{27}{8} + \frac{4\alpha^3}{\tanh^2\alpha}\right)\left(\frac{\Delta}{N+1}\right)^3 + O\left(N^{-4}\right).$$

Now, we may substitute $\Delta = m - m_0 = J_z + (l-1)/3$ and rewrite (24) in the matrix form

$$H_{\mathrm{D}}^{(s)} \approx \Omega\left[J_+\phi(J_z - m_0)\right. \quad (26)$$
$$\left. + \phi(J_z - m_0)J_-\right] = 2\Omega\left\{J_x, f(J_z)\right\}.$$

Here, $J_{\pm,z}$ are generators of $su_q(2)$ and $\{A,B\} = AB + BA$.

The new function $f(J_z)$ is also a polynomial of degree three, whose coefficients can be easily found. Now the ground-state energy is approximately given as

$$\langle\underline{-l,l}|H_{\mathrm{D}}^{(s)}|\underline{-l,l}\rangle \quad (27)$$
$$\approx -\Omega[2l]\sum_{k=0}^{3} f_k\langle\underline{-l,l}|(J_z)^k|\underline{-l,l}\rangle.$$

Therefore, we have reduced the problem to the calculation of averages of the powers of the operators J_z (the moments) in the eigenstates of the operator J_x (see [1]).

ACKNOWLEDGMENTS

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SYMPOSIUM ON QUANTUM GROUPS

Exterior Calculus with $d^3 = 0$ on Two-Dimensional Quantum Plane*

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Abstract—In this paper, we construct the algebra of differential forms with exterior differential satisfying $d^3 = 0$ on the two-dimensional quantum plane assuming that the homomorphism defining first-order differential calculus is linear in variables. Assuming $d^2 \neq 0$, we introduce the second-order differentials d^2x^i. The commutation relations between the generators x^i, dx^i, and d^2x^i of the algebra of differential forms, among dx^i, and among d^2x^i, as well as between noncommutative derivatives with generators, are found. The consistency conditions are described. © *2001 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

The idea to generalize the classical exterior differential calculus with $d^2 = 0$ to the case $d^N = 0$, $N > 2$ has arisen in a recent series of papers [1–5], where the different approaches to this idea have been developed, and these generalizations have been proposed and studied. In the paper [5], such a generalization is provided by the notions of a graded q-differential algebra and a q-differential calculus. According to the definition given in [3], a graded q-differential algebra is an associative unital $\mathbf{N}$-graded algebra endowed with a linear endomorphism d (q differential) of degree 1 satisfying $d^N = 0$ and the graded q-Leibniz rule

$$d(\omega\tau) = d(\omega)\tau + q^{gr(\omega)}\omega d(\tau), \qquad (1)$$

where ω, τ are any elements of the algebra, $gr(\omega)$ is the grading of an element ω, and q is a primitive Nth root of unity.

Following the paper [5], we construct the exterior differential calculus with $d^3 = 0$ on the quantum plane. Assuming that $d^2 \neq 0$, we introduce the second-order differentials d^2x^i. In order to define the exterior differential calculus, we find the consistency conditions and the commutation relations between x^i, dx^i, and d^2x^i, among dx^i, and among d^2x^i, as well as between noncommutative derivatives with first- and second-order differentials. We consider the case where the map ξ defined in Section 2 is linear in variables. Our construction of the exterior differential calculus with $d^3 = 0$ naturally includes the ordinary exterior calculus on the quantum plane with $d^2 = 0$ obtained by Wess and Zumino in [6].

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2. COMMUTATION RELATIONS AND CONSISTENCY CONDITIONS

Let $\mathcal{A}$ be an unital associative $\mathbf{C}$ algebra generated by the variables x^i, $i = 1, \ldots, n$, satisfying the commutative relations

$$x^i x^j = B^{ij}_{kl} x^k x^l \quad \text{or} \quad (\delta^i_k \delta^j_l - B^{ij}_{kl}) x^k x^l = 0. \qquad (2)$$

We are going to construct an exterior calculus on the algebra $\mathcal{A}$ with differential d satisfying the Leibniz rule

$$d(fg) = d(f)g + fd(g), \quad \forall f, g \in \mathcal{A}, \qquad (3)$$

and the property $d^3 = 0$. In this section, we find all commutation relations and all consistency conditions necessary for constructing this exterior calculus.

Following [7], we use the coordinate differential calculi on the algebra $\mathcal{A}$ given by the linear map

$$d : \mathcal{A} \to {}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}, \qquad (4)$$
$$d(f) = dx^i \partial_i(f), \quad \forall f \in \mathcal{A}, \quad i = 1, \ldots, n,$$

where ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ is the bimodule over $\mathcal{A}$ generated by the first-order differentials of generators dx^i, $i = 1, \ldots, n$; the partial derivatives ∂_i are linear maps $\mathcal{A} \to \mathcal{A}$ such that

$$\partial_i(fg) = \partial_i(f)g + \xi^k_i(f)\partial_k(g),$$
$$\forall f, g \in \mathcal{A}, \quad i, k = 1, \ldots, n,$$

where $\xi : \mathcal{A} \to \mathcal{A}_{n\times n}$ is a homomorphism to the algebra of $(n \times n)$ matrices over $\mathcal{A}$ defining the left module structure of the bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ by means of the right module structure.

As follows from the Leibniz rule (3) and the definition (4), the left and right structures of the bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ are consistent by the commutation relations

$$fdx^i = dx^k \xi^i_k(f), \quad \forall f \in \mathcal{A}, \quad i, k = 1, \ldots, n. \qquad (5)$$

Further, we suppose that the homomorphism ξ is linear in the variables x^i, $i = 1, \ldots, n$, that is,

$$\xi^k_i(x^j) = C^{jk}_{il} x^l, \quad i, j, k, l = 1, \ldots, n,$$

are linear functions, and C^{jk}_{il} are numerical coefficients. Then the commutation relation (5) can be rewritten in the form

$$x^i dx^j = C^{ij}_{kl} dx^k x^l, \quad i,j,k,l = 1,\dots,n. \qquad (6)$$

In order to construct a consistent differential calculus with the property $d^3 = 0$, we use the Wess–Zumino first-order differential calculus [6] extending it into a higher order. Since $d^3 = 0$, we have to introduce the second-order differentials d^2x^i of the generators x^i, $i = 1,\dots,n$ [5].

Let $\Omega_C(\mathcal{A})$ be the free right unital associative module over the algebra $\mathcal{A}$ generated by all monomials composed from powers of dx^i,d^2x^i,$i = 1,\dots,n$. If we introduce the grading zero to the elements of $\mathcal{A}$ and the grading 1 and 2, respectively, to the differentials dx^i and d^2x^i, then the module $\Omega_C(\mathcal{A})$ becomes an $\mathbf{N}$-graded module .

Assume that the differential d satisfies the graded q-Leibniz rule (1). Now we mind that q is a third-power primitive root of $1 \in \Omega_C(\mathcal{A})$.

The commutation relations between x^i and dx^j are defined by (6). Let us find four sorts of commutation relations: among the first-order differentials dx^i; between x^i and d^2x^j; between dx^i and d^2x^j; among the second-order differentials d^2x^i.

Differentiating (6), we get commutation relations among the first-order differentials and between x^i and d^2x^j at once. In fact,

$$d(x^i dx^j - C^{ij}_{kl} dx^k x^l)$$
$$= dx^i dx^j + x^i d^2x^j - C^{ij}_{kl} d^2x^k x^l - qC^{ij}_{kl} dx^k dx^l = 0.$$

Here, we assume that the terms $dx^i dx^j$ and $x^i d^2x^j$ must cancel separately. Therefore, we have two kinds of commutation relations

$$dx^i dx^j = qC^{ij}_{kl} dx^k dx^l, \qquad (7)$$

$$x^i d^2x^j = C^{ij}_{kl} d^2x^k x^l. \qquad (8)$$

Differentiating (8), we get the commutation relations between the first- and second-order differentials

$$dx^i d^2x^j = q^2 C^{ij}_{kl} d^2x^k dx^l. \qquad (9)$$

Finally, the commutation relations among second-order differentials can be obtained by differentiating (9)

$$d^2x^i d^2x^j = qC^{ij}_{kl} d^2x^k d^2x^l. \qquad (10)$$

Now, we have five sorts of commutation relations: (6), (7), (8), (9), and (10), where the last four are obtained from relations (6) by differentiating. If we define a multiplication law on the $\Omega_C(\mathcal{A})$ by these commutation relations, then the module $\Omega_C(\mathcal{A})$ becomes an unital associative algebra generated by x^i, dx^i, d^2x^i, $i = 1,\dots, n$. Obviously, the bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ is embedded into the algebra $\Omega_C(\mathcal{A})$.

Consider the commutation relations (7). If we suppose that relations (9) hold, then the differentiation of (7) gives the following consistency condition for the operator C:

$$d(dx^i dx^j - qC^{ij}_{kl} dx^k dx^l)$$
$$= (\delta^i_k \delta^j_l - qC^{ij}_{kl})(\delta^k_s \delta^l_t + C^{kl}_{st}) d^2x^s dx^t = 0,$$

or in tensor form

$$(E_{12} - qC_{12})(E_{12} + C_{12}) = 0. \qquad (11)$$

According to the paper [6], the differentiation of (2) gives the linear consistency condition for the operators B and C in tensor form

$$(E_{12} - B_{12})(E_{12} + C_{12}) = 0. \qquad (12)$$

If we differentiate relations (2) twice, we obtain the consistency condition

$$(E_{12} - B_{12})(E_{12} + C_{12})(-E_{12} + qC_{12}) = 0. \quad (13)$$

Now, we can see that the conditions (11) and (12) imply the condition (13).

As follows from the paper [6], there exist two other sorts of commutation relations: between the derivatives and the variables

$$\partial_j x^i = \delta^i_j + C^{ik}_{jl} x^l \partial_k \qquad (14)$$

and between the derivatives and the first-order differentials

$$\partial_j dx^i = (C^{-1})^{ik}_{jl} dx^l \partial_k. \qquad (15)$$

The relations (14) follow from the Leibniz rule (3) if we consider both ∂_j and x^i as operators. The relations (15) can be obtained from the assumption $\partial_j dx^i - D^{ik}_{jl} dx^l \partial_k = 0$, where the tensor D is to be determined. Multiplying the last equation by x^r from the right side and using (6) and (15), we see that the equality

$$(\partial_j dx^i - D^{ik}_{jl} dx^i \partial_k) x^r$$
$$= D^{ir}_{st} C^{su}_{jv} x^v (\partial_u dx^t - D^{tm}_{up} dx^p \partial_m) = 0$$

requires $D = C^{-1}$.

For constructing the consistent differential calculus with $d^3 = 0$, we add to the obtained commutation relations the relations between the derivatives and the second-order differentials

$$\partial_j d^2x^i = (C^{-1})^{ik}_{jl} d^2x^l \partial_k.$$

These relations can be obtained if we assume that $\partial_j d^2x^i - K^{ik}_{jl} d^2x^l \partial_k = 0$. Then, we find the tensor K multiplying this equation by x^r from the right side and commuting x^r through to the left by the commutation relations (8) and (14). Then, we have

$$(\partial_j d^2x^i - K^{ik}_{jl} d^2x^l \partial_k) x^r$$
$$= (C^{-1})^{ir}_{jt} d^2x^t + (C^{-1})^{ir}_{st} C^{su}_{jv} x^v \partial_u d^2x^t$$

$$- K^{ir}_{jt} d^2x^t - K^{ir}_{st} C^{su}_{jv} (C^{-1})^{tp}_{ub} x^v d^2x^b \partial_p$$
$$= (C^{-1})^{ir}_{st} C^{su}_{jv} x^v (\partial_u d^2x^t - (C^{-1})^{tp}_{ub} d^2x^b \partial_p) = 0,$$

if $K = C^{-1}$.

Finally, in the paper [6], the authors show that the commutation relations among the derivatives

$$\partial_i \partial_j = F^{lk}_{ji} \partial_k \partial_l$$

lead to the two conditions of consistency

$$(E_{12} + C_{12})(E_{12} - F_{12}) = 0, \quad (16)$$

$$C_{12} C_{23} F_{12} = F_{23} C_{12} C_{23}. \quad (17)$$

Comparing (12) and (16), we can easily see that, if F is equal to B, then (16) holds.

The equation (17) is a Yang–Baxter equation. Another two Yang–Baxter equations appear if we multiply the commutation relations (2) and (7) from the right side by dx^r and d^2x^r, respectively, and, using the corresponding commutation relations, commute dx^r and d^2x^r through to the left

$$(\delta^i_k \delta^j_l - B^{ij}_{kl}) x^k x^l dx^r$$
$$= (\delta^i_k \delta^j_l - B^{ij}_{kl}) C^{lr}_{st} C^{ks}_{uv} dx^u x^v x^t = 0,$$
$$(\delta^i_k \delta^j_l - C^{ij}_{kl}) dx^k dx^l d^2x^r$$
$$= (\delta^i_k \delta^j_l - C^{ij}_{kl}) q^2 C^{lr}_{st} C^{ks}_{uv} d^2x^u dx^v dx^t = 0.$$

We rewrite these two consistency conditions in tensor form

$$(E_{12} - B_{12}) C_{23} C_{12} dx_1 x_2 x_3 = 0, \quad (18)$$

$$(E_{12} - C_{12}) q^2 C_{23} C_{12} d^2x_1 dx_2 dx_3 = 0. \quad (19)$$

If the Yang–Baxter equations

$$B_{12} C_{23} C_{12} = C_{23} C_{12} B_{23}, \quad (20)$$

$$C_{12} C_{23} C_{12} = C_{23} C_{12} C_{23} \quad (21)$$

hold, then the tensors B and C satisfy the conditions (18) and (19).

3. EXTERIOR CALCULUS ON THE QUANTUM PLANE

Now, we consider all commutation relations and conditions obtained above in the case of the quantum plane, which is the free associative unital algebra generated by the variables x^i, $i = 1, \ldots, n$, satisfying the commutation relation $x^i x^j = q\, x^k x^l$, $i < j$.

As follows from the paper [6], these relations can be rewritten by means of the $\widehat{R}$ matrix

$$x^i x^j = \frac{1}{q} \widehat{R}^{ij}_{kl} x^k x^l \quad \text{or} \quad \left(\delta^i_k \delta^j_l - \frac{1}{q} \widehat{R}^{ij}_{kl} \right) x^k x^l = 0,$$

where

$$\widehat{R}^{ij}_{kl} = \delta^i_k \delta^j_l (1 + (q-1)\delta^{ij}) + \left(q - \frac{1}{q} \right) \delta^i_k \delta^j_l \theta(j-i),$$

$$\theta(j-i) = \begin{cases} 1 & \text{if} \quad j > i \\ 0 & \text{if} \quad j \le i; \end{cases}$$

i.e., we have $B = q^{-1} \widehat{R}$. In [6], the authors show that the consistency condition (12) holds if one chooses the values $q\widehat{R}$ or $q^{-1}\widehat{R}^{-1}$ for the tensor C, i.e.,

$$(E - q^{-1}\widehat{R})(E + q\widehat{R}) = 0$$
$$\text{or} \quad (E - q^{-1}\widehat{R})(E + q^{-1}\widehat{R}^{-1}) = 0,$$

respectively, where E is the unit matrix, and q^{-1} and q are the eigenvalues of the $\widehat{R}$ matrix.

We show that the consistency condition (11) is satisfied only for the value $C = q\,\widehat{R}$. Here, we make use of the identities

$$\widehat{R}^2 = E + (q - q^{-1})\widehat{R}$$
$$\text{and} \quad \widehat{R}^{-1} = \widehat{R} + (q^{-1} - q)E.$$

If $C = q\widehat{R}$, we have

$$(E - q^2\widehat{R})(E + q\widehat{R})$$
$$= (1 - q^3)E - (q^2 - q + q^4 - q^2)\widehat{R}.$$

As q is the third-power primitive root of unity, the coefficients are equal to zero.

However, if $C = q^{-1}\,\widehat{R}^{-1}$, then

$$(E - \widehat{R}^{-1})(E + q^{-1}\widehat{R}^{-1}) = (-2q^{-1}E - \widehat{R})$$
$$\times (q^{-2}E + q^{-1}\widehat{R}) = (q-1)(E + q\widehat{R}).$$

Therefore, we can only choose C to be $q\widehat{R}$.

Three Yang–Baxter equations (17), (20), and (21) reduce to the single equation

$$\widehat{R}_{12} \widehat{R}_{23} \widehat{R}_{12} = \widehat{R}_{23} \widehat{R}_{12} \widehat{R}_{23},$$

as was shown in [6].

By means of the $\widehat{R}$ matrix, we rewrite all commutation relations obtained in Section 2. Now, we have

$$x^i dx^j = q\widehat{R}^{ij}_{kl} dx^k x^l, \quad x^i d^2x^j = q\widehat{R}^{ij}_{kl} d^2x^k x^l,$$
$$dx^i dx^j = q^2 \widehat{R}^{ij}_{kl} dx^k dx^l, \quad dx^i d^2x^j = \widehat{R}^{ij}_{kl} d^2x^k dx^l,$$
$$\partial_i \partial_j = \frac{1}{q} \widehat{R}^{lk}_{ji} \partial_k \partial_l, \quad d^2x^i d^2x^j = q^2 \widehat{R}^{ij}_{kl} d^2x^k d^2x^l,$$
$$\partial_j dx^i = \frac{1}{q} (\widehat{R}^{-1})^{ik}_{jl} dx^l \partial_k, \ \partial_j d^2x^i = \frac{1}{q} (\widehat{R}^{-1})^{ik}_{jl} d^2x^l \partial_k.$$

4. ALGEBRA OF DIFFERENTIAL FORMS ON TWO-DIMENSIONAL QUANTUM PLANE

In the case of the two-dimensional quantum plane, we denote $x^1 = x$, $x^2 = y$. Since in the two-dimensional case the $\widehat{R}$ matrix is equal to

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$

we rewrite explicitly all commutation relations of Section 2:

$$xdx = q^2dxx, \quad xd^2x = q^2d^2xx,$$
$$xdy = qdyx + (q^2-1)dxy,$$
$$xd^2y = qd^2yx + (q^2-1)d^2xy,$$
$$ydx = qdxy, \quad yd^2x = qd^2xy,$$
$$ydy = q^2dyy, \quad yd^2y = q^2d^2yy,$$
$$dxdy = qdydx, \quad d^2xd^2y = qd^2yd^2x,$$
$$dxd^2x = qd^2xdx, \quad dyd^2y = qd^2ydy,$$
$$dxd^2y = d^2ydx + (q-q^{-1})d^2xdy,$$
$$dyd^2x = d^2xdy, \quad \partial_x\partial_y = q^{-1}\partial_y\partial_x,$$
$$(dx)^2 = (dy)^2 = (d^2x)^2 = (d^2y)^2 = 0,$$
$$\partial_x dx = q^{-2}dx\partial_x, \quad \partial_x d^2x = q^{-2}d^2xx,$$
$$\partial_x dy = q^{-1}dy\partial_x, \quad \partial_x d^2y = q^{-1}d^2y\partial_x,$$
$$\partial_y dy = q^{-2}dy\partial_y, \quad \partial_y d^2y = q^{-2}d^2y\partial_y,$$
$$\partial_y dx = (q^{-2}-1)dy\partial_x + q^{-1}dx\partial_y,$$
$$\partial_y d^2x = (q^{-2}-1)d^2y\partial_x + q^{-1}d^2x\partial_y.$$

The direct calculation of d^3f shows that the requirements $(dx)^3 = 0$ and $(dy)^3 = 0$ imply $d^3f = 0$. In fact, all the terms except $(dx)^3$ and $(dy)^3$ cancel by use of appropriate commutation relations.

We add these two requirements to the obtained commutation relations defining the multiplication law on the graded algebra $\Omega_C(\mathcal{A})$. This algebra splits into the direct sum $\Omega_C(\mathcal{A}) = \bigoplus_{k=0}^{\infty} \Omega_C^k(\mathcal{A})$ of its subspaces of homogeneous elements of grading k.

Let F_{2k}^{μ} be any monomial of grading $2k$ on second-order differentials

$$F_{2k}^{\mu} = (d^2x)^{m_1}(d^2y)^{m_2},$$

where $k \geq 1$, $m_1 + m_2 = k$, and μ is the multi-index entirely determined by (m_1, m_2). Then, the even form $\omega_e \in \Omega_C^{2k}(\mathcal{A})$ can be written as

$$\omega_e = F_{2k}^{\mu} f_{00} + F_{2(k-1)}^{\nu}(dxdxf_{11} + dxdyf_{12} + dydyf_{22}) + F_{2(k-2)}^{\eta} dxdxdydyh_{22},$$

and the odd form $\omega_o \in \Omega_C^{2k+1}(\mathcal{A})$ as

$$\omega_o = F_{2k}^{\mu}(dxf_{10} + dyf_{01}) + F_{2(k-1)}^{\nu}(dxdxdyh_{21} + dxdydyh_{12}).$$

Differentiating ω_e and ω_o, we get

$$d(\omega_e) = q^{2k}F_{2k}^{\mu}(dx\partial_x + dy\partial_y)f_{00} - q^{2k-1}F_{2(k-1)}^{\nu}((d^2xdx + dxdxdy\partial_y)f_{11} - (qd^2xdy + d^2ydx)f_{12} + (d^2ydy + qdxdydy\partial_x)f_{22}) - q^{2(k-2)}F_{2(k-2)}^{\eta}(q^2d^2xdxdydy + d^2ydxdxdy)h_{22},$$

$$d(\omega_o) = q^{2k}F_{2k}^{\mu}((d^2x + qdxdx\partial_x)f_{10} + dxdy(q\partial_y f_{10} + \partial_x f_{01}) + (d^2y + qdydy)f_{01}) + q^{2(k-1)}F_{2(k-1)}^{\nu}((-d^2xdxdy + qd^2ydxdx)h_{21} + dxdxdydy(q\partial_x h_{12} + \partial_y h_{21}) + (qd^2xdydy - q^2d^2ydxdy)h_{12}),$$

respectively. Hence, one can easily see that the differential d is the linear endomorphism of degree 1.

The direct calculation gives $d^3(\omega) = 0, \forall\omega \in \Omega_C(\mathcal{A})$.

Thus, we have proved the following.

Proposition. *The algebra $\Omega_C(\mathcal{A})$ of differential forms with the requirement $(dx)^3 = (dy)^3 = 0$ is a graded differential algebra with respect to the exterior differential d satisfying the q-Leibniz rule, i.e., $d^3f = 0, \forall f \in \mathcal{A}$, $d^3\omega = 0, \forall\omega \in \Omega_C(\mathcal{A})$.*

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SYMPOSIUM ON QUANTUM GROUPS

Standard Complex for Quantum Lie Algebras*

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Abstract—For a quantum Lie algebra Γ, let $\Gamma^\wedge$ be its exterior extension (the algebra $\Gamma^\wedge$ is canonically defined). We introduce a differential on the exterior extension algebra $\Gamma^\wedge$ which provides the structure of a complex on $\Gamma^\wedge$. In the situation when Γ is a usual Lie algebra, this complex coincides with the "standard complex." The differential is realized as a commutator with a (BRST) operator Q in a larger algebra $\Gamma^\wedge[\Omega]$, with extra generators canonically conjugated to the exterior generators of $\Gamma^\wedge$. A recurrent relation which uniquely defines the operator Q is given. *© 2001 MAIK "Nauka/Interperiodica".*

1. A quantum Lie algebra [1–4] is defined by two tensors C^k_{ij} and σ^{mk}_{ij} (indices belong to some set $\mathcal{N}$, say, $\mathcal{N} = \{1, \ldots, N\}$). By definition, the matrix σ^{mk}_{ij} has an eigenvalue 1; one demands that $(P_{(1)})^{mk}_{ij} C^n_{mk} = 0$, where $P_{(1)}$ is a projector on the eigenspace of σ corresponding to the eigenvalue 1.

By definition, a quantum Lie algebra Γ is generated by elements χ_i, $i = 1, \ldots, N$, subjected to relations

$$\chi_i\chi_j - \sigma^{mk}_{ij}\chi_m\chi_k = C^k_{ij}\chi_k. \quad (1)$$

Here, the structure constants C^k_{ij} obey

$$C^p_{ni}C^l_{pj} = \sigma^{mk}_{ij}C^p_{nm}C^l_{pk} + C^p_{ij}C^l_{np} \quad (2)$$

$$\Leftrightarrow C^{\langle 1|}_{|12\rangle}C^{\langle 4|}_{|13\rangle} = \sigma_{23}C^{\langle 1|}_{|12\rangle}C^{\langle 4|}_{|13\rangle} + C^{\langle 3|}_{|23\rangle}C^{\langle 4|}_{|13\rangle},$$

$$C^k_{ni}\sigma^{pm}_{kq} = \sigma^{sj}_{iq}\sigma^{pk}_{ns}C^m_{kj} \Leftrightarrow C^{\langle 1|}_{|12\rangle}\sigma_{13} = \sigma_{23}\sigma_{12}C^{\langle 3|}_{|23\rangle}, \quad (3)$$

$$(\sigma^{pj}_{im}C^n_{qp} + \delta^n_q C^j_{im})\sigma^{ks}_{nj} = \sigma^{jn}_{qi}(\sigma^{ps}_{nm}C^k_{jp} + \delta^k_j C^s_{nm}) \quad (4)$$

$$\Leftrightarrow (\sigma_{23}C^{\langle 1|}_{|12\rangle} + C^{\langle 3|}_{|23\rangle})\sigma_{13} = \sigma_{12}(\sigma_{23}C^{\langle 1|}_{|12\rangle} + C^{\langle 3|}_{|23\rangle}).$$

The matrix σ^{mk}_{ij} satisfies the Yang–Baxter equation

$$\sigma^{j_1j_2}_{i_1i_2}\sigma^{n_2k_3}_{j_2i_3}\sigma^{k_1k_2}_{j_1n_2} = \sigma^{j_2j_3}_{i_2i_3}\sigma^{k_1n_2}_{i_1j_2}\sigma^{k_2k_3}_{n_2j_3} \quad (5)$$

$$\Leftrightarrow \sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23}.$$

On the right-hand side of (2)–(5), we use Faddeev–Reshetikhin–Takhtajan (FRT) matrix notation [5]; $\{1, 2, 3, \ldots\}$ are the numbers of vector spaces; e.g., $f_1 := f^{i_1}_{j_1}$ is a matrix which acts in the first vector space. Additionally, we use incoming and outcoming indices; e.g., $\Omega^{\langle 1|} := \Omega^{i_1}$ and $\gamma_{|1\rangle} := \gamma_{j_1}$ denote a covector with one outcoming index and a vector with one incoming index, respectively. Thus, in this notation, the matrix f_1 can be written as $f_1 = f^{\langle 1|}_{|1\rangle}$.

Remark. Quantum Lie algebras defined by Eqs. (1)–(5) generalize the usual Lie (super)algebras. Indeed in the nondeformed case, when

$$\sigma^{mk}_{ij} = (-1)^{(m)(k)}\delta^m_j\delta^k_i$$

is a superpermutation matrix [here, $\sigma^2 = 1$ and (5) is fulfilled; $(m) = 0, 1$ is the parity of a generator χ_m], Eqs. (1) and (2) coincide with the defining relations and the Jacobi identities for Lie (super)algebras. Equation (3) is then equivalent to the Z_2-homogeneity condition $C^i_{jk} = 0$ for $(i) \neq (j) + (k)$. Equation (4) follows from (3).

2. The exterior extension $\Gamma^\wedge$ of the quantum algebra Γ (1) is obtained by adding new generators γ_i, $i = 1, \ldots, N$. The generators γ_i form a "generalized" wedge algebra. The definition of the wedge product of the elements γ_i is

$$\gamma_{|1\rangle} \wedge \gamma_{|2\rangle} \ldots \wedge \gamma_{|n\rangle} = A_{1\to n}\gamma_{|1\rangle} \otimes \gamma_{|2\rangle} \cdots \otimes \gamma_{|n\rangle}. \quad (6)$$

Here, the matrix operator $A_{1\to n}$ is an analog of the antisymmetrizer of n-spaces. This operator can be defined inductively (see, e.g., [6]),

$$A_{1\to n} = \left(\mathbf{1} + \sum_{k=1}^{n-1}(-1)^{n-k}\sigma_{k\to n}\right)A_{1\to n-1}, \quad (7)$$

where, for $n > k$,

$$\sigma_{k\to n} := \sigma_{kk+1}\sigma_{k+1k+2}\cdots\sigma_{n-1n}.$$

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Using the Yang–Baxter equation (5), one can rewrite (7) in the following three equivalent forms:

$$A_{1\to n} = A_{1\to n-1}\left(\mathbf{1} + \sum_{k=1}^{n-1}(-1)^k \sigma_{k+1\leftarrow 1}\right)$$
$$= \left(\mathbf{1} + \sum_{k=1}^{n-1}(-1)^k \sigma_{k+1\leftarrow 1}\right) A_{2\to n}$$
$$= A_{2\to n}\left(\mathbf{1} + \sum_{k=1}^{n-1}(-1)^{n-k} \sigma_{k\to n}\right),$$

where

$$\sigma_{n\leftarrow k} := \sigma_{n-1n}\cdots\sigma_{k+1k+2}\sigma_{kk+1}$$

for $n > k$.

If the sequence of operators $A_{1\to n}$ terminates at the step $n = h+1$ ($A_{1\to h} \neq 0$ and $A_{1\to n} = 0$ for $n > h$), then the number h is called the height of the operator σ.

The cross-commutation relations between the generators γ_i and χ_j are

$$\gamma_{|1\rangle}\chi_{|2\rangle} = (\sigma_{12}\chi_{|1\rangle} + C^{\langle 2|}_{|12\rangle})\gamma_{|2\rangle}. \quad (8)$$

The algebra $\Gamma^\wedge$ is graded by the degree in the generators of γ_i.

3. We further introduce a set of generators $\{\Omega^i\}$, $i = 1, \ldots, N$, canonically conjugated to the generators γ_i. The generators Ω^i form a "wedge" algebra as well, with the wedge product defined by

$$\Omega^{\langle r|}\wedge\Omega^{\langle r-1|}\wedge\ldots\wedge\Omega^{\langle 1|} \quad (9)$$
$$= \Omega^{\langle r|}\otimes\Omega^{\langle r-1|}\otimes\cdots\otimes\Omega^{\langle 1|}A_{1\to r}.$$

Here, the operators $A_{1\to n}$ are the same as in (7).

The commutation relations between Ω^i and γ_j are

$$\gamma_j\Omega^i = -\Omega^p(\sigma^{-1})^{si}_{pj}\gamma_s + \delta^i_j \quad (10)$$
$$\Rightarrow \gamma_{|2\rangle}\Omega^{\langle 2|} = -\Omega^{\langle 1|}\sigma^{-1}_{12}\gamma_{|1\rangle} + I_2.$$

Finally, the commutation relations between Ω^i and χ_j are

$$\chi_{|2\rangle}\Omega^{\langle 2|} = \Omega^{\langle 1|}(\sigma_{12}\chi_{|1\rangle} + C^{\langle 2|}_{|12\rangle}). \quad (11)$$

We denote the algebra generated by $\{\chi_i\}$, $\{\gamma_j\}$, and $\{\Omega^k\}$ by $\Gamma^\wedge[\Omega]$. The algebra $\Gamma^\wedge[\Omega]$ is graded by the rule $\deg(\gamma_i) = 1$ and $\deg(\Omega^i) = -1$.

We shall need the following set of consequences of Eq. (10):

$$\gamma_{|1\rangle}\wedge\ldots\wedge\gamma_{|r\rangle}\Omega^{\langle r|} \quad (12)$$
$$= (-1)^r\Omega^{\langle 0|}\sigma^{-1}_{r\leftarrow 0}\gamma_{|0\rangle}\wedge\ldots\wedge\gamma_{|r-1\rangle}$$
$$+ \left(\sum_{k=1}^{r}(-1)^{r-k}\sigma^{-1}_{r\leftarrow k}\right)\gamma_{|1\rangle}\wedge\ldots\wedge\gamma_{|r-1\rangle},$$

where $\sigma^{-1}_{r\leftarrow k} := \sigma^{-1}_{kk+1}\cdots\sigma^{-1}_{r-1r}$ and $\sigma^{-1}_{r\leftarrow r} := \mathbf{1}$.

4. The main result of the present paper is a recursive formula for the BRST operator Q which satisfies $Q^2 = 0$.

Such an operator endows the algebra $\Gamma^\wedge$ with the structure of the differential (chain) complex. To construct the differential (starting with the operator Q), one needs, first, to define the action of the algebra $\Gamma^\wedge[\Omega]$ on the algebra $\Gamma^\wedge$. The elements χ_i and γ_j act on $\Gamma^\wedge$ by left multiplication. To define the action of generators Ω^i on $\Gamma^\wedge$, it suffices [due to relations (10) and (11)] to know $\Omega^i(1)$, where 1 is the unit element of the algebra $\Gamma^\wedge$. We set $\Omega^i(1) = 0$. The definition of the differential d is given by its action on an element ϕ of the algebra $\Gamma^\wedge$,

$$d\phi = [Q, \phi]_\pm(1), \quad (13)$$

where $[,]_\pm$ is the graded commutator.

Now, we are ready to formulate the main Proposition.

Proposition. *The BRST operator Q for the quantum algebra* (1) *has the form*

$$Q = \Omega^i\chi_i + \sum_{r=1}^{h-1} Q_{(r)}, \quad (14)$$

where h is the height of the operator σ_{12}.

Here, the operators $Q_{(r)}$ are given by

$$Q_{(r)} = \Omega^{\langle r+1|}\Omega^{\langle r|}\ldots\Omega^{\langle 1|}X^{\langle\tilde{1}\ldots\tilde{r}|}_{|1\ldots r+1\rangle}\gamma_{|\tilde{1}\rangle}\cdots\gamma_{|\tilde{r}\rangle} \quad (15)$$

(the wedge product is implied); $X^{\langle 1\ldots r|}_{|1\ldots r+1\rangle}$ are tensors which satisfy the recurrent relation

$$A_{1\to r+1}X^{\langle 1\ldots r|}_{|1\ldots r+1\rangle}A_{1\to r} \quad (16)$$
$$= A_{1\to r+1}\left((-1)^r\sigma_{r+1\leftarrow 1} - \mathbf{1}\right)X^{\langle 2\ldots r|}_{|2\ldots r+1\rangle}A_{2\to r}$$

with the initial condition $A_{12}X^{\langle 0|}_{|12\rangle} = -C^{\langle 0|}_{|12\rangle}$.

Proof. We have to verify the identity

$$Q^2 = (\Omega^{\langle 2|}\chi_{|2\rangle})^2 + \left[\Omega^{\langle 2|}\chi_{|2\rangle}, \sum_{r=1}^{h-1}Q_{(r)}\right]_+ \quad (17)$$
$$+ \left(\sum_{r=1}^{h-1}Q_{(r)}\right)^2 = 0.$$

Because of the lack of space, we shall check a part of this identity which includes the terms linear in χ only.

First of all, we find [see (11)]

$$(\Omega^{\langle 2|}\chi_{|2\rangle})^2 = \Omega^{\langle 2|}\left(\Omega^{\langle 1|}(\sigma_{12}\chi_{|1\rangle} + C^{\langle 2|}_{|12\rangle})\right)\chi_{|2\rangle} \quad (18)$$
$$= \Omega^{\langle 2|}\otimes\Omega^{\langle 1|}\sigma_{12}C^{\langle 2|}_{|12\rangle}\chi_{|2\rangle}$$
$$+ \Omega^{\langle 2|}\otimes\Omega^{\langle 1|}(\mathbf{1}-\sigma)_{12}C^{\langle 2|}_{|12\rangle}\chi_{|2\rangle}$$

$$= \Omega^{\langle 2|} \otimes \Omega^{\langle 1|} C^{\langle 2|}_{|12\rangle} \chi_{|2\rangle}.$$

Consider then the anticommutator $[\Omega^{\langle 2|}\chi_{|2\rangle}, Q_{(r)}]_+$ in which we commute all χ_i to the right and extract only the terms which are linear in the generators χ_i:

$$[Q_{(r)}, \Omega^{\langle r|}\chi_{|r\rangle}]_+ = \Omega^{\langle r+1|}\chi_{|r+1\rangle}Q_{(r)} + Q_{(r)}\Omega^{\langle r|}\chi_{|r\rangle} = \Omega^{\langle r+1|} \cdots \Omega^{\langle 0|}(\sigma_{r+1\leftarrow 0} + (-1)^r \mathbf{1})\, X^{\langle 1\ldots r|}_{|1\ldots r+1\rangle}\sigma^{-1}_{r\leftarrow 0}\gamma_{|0\rangle} \cdots \gamma_{|r-1\rangle}\chi_{|r\rangle} + \Omega^{\langle r+1|} \cdots \Omega^{\langle 1|}X^{\langle 1\ldots r|}_{|1\ldots r+1\rangle}\left(\sum_{k=1}^{r}(-1)^{r-k}\sigma^{-1}_{r\leftarrow k}\right)\gamma_{|1\rangle} \cdots \gamma_{|r-1\rangle}\chi_{|r\rangle} + \ldots \quad (19)$$

(ellipses denote the terms independent of χ_i). Here, Eqs. (8), (11), and (12) have been used.

Equations (18) and (19) give the whole contribution to the χ-linear terms in Q^2 since $(\sum_{r=1}^{h-1} Q_{(r)})^2$ is independent of χ_i.

The substitution of (18) and (19) produces the initial data $A_{12}X^{\langle 0|}_{|12\rangle} = -C^{\langle 0|}_{|12\rangle}$ and recurrent relations

$$A_{1\to r+1}X^{\langle 1\ldots r|}_{|1\ldots r+1\rangle}\left(\sum_{k=1}^{r}(-1)^{r-k}\sigma^{-1}_{r\leftarrow k}\right)A_{1\to r-1} = -A_{1\to r+1}\left(\sigma_{r+1\leftarrow 1} + (-1)^{r-1}\mathbf{1}\right) \times X^{\langle 2\ldots r|}_{|2\ldots r+1\rangle}\sigma^{-1}_{r\leftarrow 1}A_{1\to r-1}, \quad (20)$$

where the matrix operator $A_{1\to r}$ is defined in (7). These relations express coefficients $X^{\langle 1\ldots r|}_{|1\ldots r+1\rangle}$ via $X^{\langle 1\ldots r-1|}_{|1\ldots r\rangle}$.

Using an identity $\sigma^{-1}_{r\leftarrow 1}A_{1\to r-1} = A_{2\to r}\sigma^{-1}_{r\leftarrow 1}$ and inductive relations (7) for the projectors $A_{1\to r}$, one can rewrite (20) in the form (16).

5. Comments.

(i) For general σ^{ij}_{kl} and C^i_{jk}, it is rather difficult to solve Eqs. (16) explicitly. However, for the case $\sigma^2 = \mathbf{1}$, the main Eq. (16) becomes simpler and the general solution for Q can be found. Indeed, the relation (16) for $r = 2$ gives

$$A_{1\to 3}X^{\langle 12|}_{|123\rangle}\left(\mathbf{1} - \sigma_{12}\right) = A_{1\to 3}\left(\sigma_{23}\sigma_{12} - \mathbf{1}\right)X^{\langle 2|}_{|23\rangle}.$$

For $\sigma^2 = 1$, we have $A_{1\to 3}\left(\sigma_{23}\sigma_{12} - \mathbf{1}\right) = 0$ and therefore $Q_{(r)} = 0$ for $r \geq 2$. Thus, the BRST operator (14) has the familiar form

$$Q = \Omega^{\langle 1|}\chi_{|1\rangle} - \Omega^{\langle 2|} \otimes \Omega^{\langle 1|}C^{\langle 1|}_{|12\rangle}\gamma_{|1\rangle}.$$

In the case when the matrix σ is the (super)permutation matrix, the algebra $\Gamma^\wedge$ with the differential (13) becomes the standard complex for the Lie (super)algebra Γ (see, e.g., [7]).

In general, for $\sigma^2 \neq 1$, the sum in (14) will be limited only by the height h of the operator σ.

Below, we present an explicit form for Q for the standard quantum deformation $\Gamma = U_q(gl(N))$ of the universal enveloping algebra of the Lie algebra $gl(N)$ ($\sigma^2 \neq 1$ in this case).

(ii) When the algebra (1) is a Hopf algebra, the algebraic structure (6), (8)–(11) is related to the differential calculus on quantum groups (see [2, 8–10]). The BRST operator Q given by (14) generates the differential d (introduced in [2]) on the algebra dual to $\Gamma^\wedge$.

6. Example. The BRST operator Q for the quantum algebra $\Gamma = U_q(gl(N))$.

The quantum algebra $U_q(gl(N))$ is defined (as a Hopf algebra) by the relations [5]

$$\hat{R}L^{\pm}_2L^{\pm}_1 = L^{\pm}_2L^{\pm}_1\hat{R}, \;\; \hat{R}L^{+}_2L^{-}_1 = L^{-}_2L^{+}_1\hat{R}, \quad (21)$$

$$\Delta(L^{\pm}) = L^{\pm} \otimes L^{\pm}, \quad \varepsilon(L^{\pm}) = \mathbf{1}, \quad S(L^{\pm}) = (L^{\pm})^{-1}, \quad (22)$$

where elements of the $N \times N$ matrices $(L^{\pm})^i_j$ are generators of $U_q(gl(N))$; the matrices L^+ and L^- are, respectively, upper and lower triangular; and their diagonal elements are related by $(L^+)^i_i(L^-)^i_i = 1$ for all i. The matrix $\hat{R}$ is defined as $\hat{R} := \hat{R}_{12} = P_{12}R_{12}$ (P_{12} is the permutation matrix); the matrix R_{12} is the standard Drinfeld–Jimbo R matrix for $GL_q(N)$,

$$R_{12} = R^{i_1,\, i_2}_{j_1,\, j_2} = \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}(1 + (q-1)\delta^{i_1 i_2}) + (q - q^{-1})\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\Theta_{i_1 i_2},$$

where

$$\Theta_{ij} = \begin{cases} 1 & \text{if } i > j \\ 0 & \text{if } i \leq j. \end{cases}$$

This R matrix satisfies the Hecke condition $\hat{R}^2 = \lambda\hat{R} + \mathbf{1}$, where $\lambda = (q - q^{-1})$ and q is a parameter of deformation.

The generators of the algebra Γ are defined by the formula [9–11]

$$\chi^l_k = \frac{1}{\lambda}[(D^{-1})^l_k - (D^{-1})^j_i f^{li}_{kj}]. \quad (23)$$

Here, $f^{li}_{kj} = (L^-)^i_k S((L^+)^l_j)$ and the numerical matrix D can be found by means of relations

$$\mathrm{tr}_2\, \hat{R}_{12}\Psi_{23} = P_{13} = \mathrm{tr}_2\, \Psi_{12}\hat{R}_{23},$$

$$D_1 := \mathrm{tr}_2\, \Psi_{12} \Rightarrow \mathrm{tr}_1(D_1^{-1}\hat{R}^{-1}) = \mathbf{1}_2,$$

where tr_1 and tr_2 denote the traces over first and second spaces.

It is convenient to write down the complete set of commutation relations for the exterior algebra $\Gamma^\wedge[\Omega]$ in terms of generators

$$L^i_j = (L^+)^i_k S((L^-)^k_j) = \delta^i_j - \lambda S^{-1}(\chi^i_k) D^k_j,$$
$$J^i_n = -S^{-1}(f^{ik}_{jl})\gamma^l_k D^j_n, \quad \omega^i_j = \Omega^k_m f^{mi}_{kj}.$$

The indices now are pairs of indices; the roles of the elements χ_i, γ_j, and Ω^k are played by the generators χ^i_j, γ^i_j, and Ω^i_j, respectively.

The commutation relations are [10–12]

$$\omega_2 \hat{R}^{-1} \omega_2 \hat{R} = -\hat{R}^{-1} \omega_2 \hat{R}^{-1} \omega_2, \tag{24}$$
$$\omega_2 \hat{R} L_2 \hat{R} = \hat{R} L_2 \hat{R} \omega_2,$$

$$\omega_2 \hat{R} J_2 \hat{R} + \hat{R} J_2 \hat{R} \omega_2 = -\hat{R}, \tag{25}$$
$$L_2 \hat{R} L_2 \hat{R} = \hat{R} L_2 \hat{R} L_2,$$

$$J_2 \hat{R} L_2 \hat{R} = \hat{R} L_2 \hat{R} J_2, \tag{26}$$
$$J_2 \hat{R} J_2 \hat{R} = -\hat{R}^{-1} J_2 \hat{R} J_2.$$

Now, the construction of the BRST operator Q is in order. To begin, we find the first term in the sum (14):

$$\Omega^k_m \chi^m_k = \frac{1}{\lambda} \mathrm{tr}_q \left(\omega(L - \mathbf{1})\right), \tag{27}$$

where we have introduced the quantum trace $\mathrm{tr}_q(X) := \mathrm{tr}(D^{-1}X)$. Then, one can resolve the chain of the recurrent relations (16), where we have to substitute the expressions for the structure constants

$$\sigma^{\langle {jn \atop lq} |}_{| {mi \atop pk} \rangle} = R^{ju}_{sp} (R^{-1})^{sm}_{kr} (D^{-1})^f_o R^{no}_{ut} D^t_l (R^{-1})^{ri}_{qf},$$

$$C^{\langle {q \atop p} |}_{| {im \atop jn} \rangle} = \delta^q_j \delta^i_n \delta^m_p - \sigma^{\langle {qt \atop tp} |}_{| {im \atop jn} \rangle},$$

and find the set of coefficients $X^{\langle 1 \ldots r|}_{|1 \ldots r+1\rangle}$. After straightforward but tiresome calculations, one can obtain the following result:

$$Q = \mathrm{tr}_q \big(\omega(L - \mathbf{1})/\lambda - \omega L(\omega J) \tag{28}$$
$$+ \lambda \omega L(\omega J)^2 - \lambda^2 \omega L(\omega J)^3 + \ldots\big)$$
$$= \mathrm{tr}_q \left(\omega(L - \mathbf{1})/\lambda - \omega L(\omega J)(\mathbf{1} + \lambda \omega J)^{-1}\right)$$
$$= -\frac{1}{\lambda} \mathrm{tr}_q(\omega) + \frac{1}{\lambda} \mathrm{tr}_q(W),$$

where $W = \omega L(\mathbf{1} + \lambda \omega J)^{-1}$ and the sum in the first lines of (28) is limited since monomials of ω's of the order $N^2 + 1$ are equal to zero.

One can check directly that the operator Q given by (28) satisfies

$$Q^2 = 0, \quad [Q, L] = 0, \quad [Q, J]_+ = \frac{1}{\lambda}(\mathbf{1} - L).$$

To obtain these relations, one has to use identities

$$\mathrm{tr}_q(X)\mathbf{1}_2 = \mathrm{tr}_{q1}(\hat{R}^{\pm 1} X_2 \hat{R}^{\mp 1})$$

and relations

$$\hat{R} W_2 \hat{R}^{-1} \omega_2 = -\omega_2 \hat{R}^{-1} W_2 \hat{R},$$
$$\hat{R} W_2 \hat{R}^{-1} W_2 = -W_2 \hat{R}^{-1} W_2 \hat{R}^{-1},$$
$$\hat{R}^{-1} W_2 \hat{R} L_2 = L_2 \hat{R} W_2 \hat{R}^{-1},$$
$$J_2 \hat{R} W_2 \hat{R}^{-1} + \hat{R}^{-1} W_2 \hat{R} J_2$$
$$= -L_2 (\mathbf{1} + \lambda \omega J)^{-1}_2 \hat{R}^{-1} (\mathbf{1} + \lambda \omega J)_2,$$

which follow from (24)–(26).

Remark. The operator Q given by (28) has the correct classical limit for $q \to 1$ ($\lambda \to 0$, $L \to \mathbf{1} + \lambda\tilde{\chi}$, $\omega \to \tilde{\omega}$, $J \to -\tilde{\gamma}$)

$$Q \to Q_{\mathrm{cl}} = \mathrm{tr}(\tilde{\omega}\tilde{\chi} + \tilde{\omega}^2 \tilde{\gamma}) = \mathrm{tr}(\tilde{\omega} X - \tilde{\omega}\tilde{\gamma}\tilde{\omega}),$$

where $X := \tilde{\chi} + \tilde{\omega}\tilde{\gamma} + \tilde{\gamma}\tilde{\omega}$ and the classical algebra is

$$[\tilde{\omega}_2, \tilde{\gamma}_1]_+ = P_{12},\ [\tilde{\omega}_2, \tilde{\omega}_1]_+ = 0 = [\tilde{\gamma}_2, \tilde{\gamma}_1]_+,$$
$$[X_2, X_1] = P_{12}(X_2 - X_1),\ [X_2, \tilde{\omega}_1] = 0 = [X_2, \tilde{\gamma}_1].$$

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SYMPOSIUM ON QUANTUM GROUPS

Induced Representations of Quantum Kinematical Algebras*

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Abstract—We construct the induced representations of the null-plane quantum Poincaré and quantum kappa Galilei algebras in $(1+1)$ dimensions. The induction procedure makes use of the concept of module and is based on the existence of a pair of Hopf algebras with a nondegenerate pairing and dual bases.

1. INTRODUCTION

Quantum kinematical algebras and groups are used for the study of q-deformed symmetries of the q-deformed spacetime, which can be considered as a noncommutative homogeneous space of the quantum kinematical groups [1].

It is well known in nondeformed Lie group theory that, given a Lie group G and a closed Lie subgroup K of it, the space of functions defined in the homogeneous space ($\simeq G/K$) carries a representation of G induced by a representation of K. Hence, symmetries and homogeneous spaces are closely related to induced representations. On the other hand, the physical interest of the induced representations is without doubt [2, 3]. Thus, the study of the induced representations of quantum kinematical groups can be useful for determining the behavior of physical systems endowed with deformed symmetries.

In this work, we present the induced representations of the quantum $\tilde{\kappa}$-Galilei algebra, and the null-plane quantum Poincaré algebra, both in $(1+1)$ dimensions.

The induction procedure used by us has an algebraic character since it makes use of the theory of modules, which is, from our point of view, the appropriate tool to deal with the algebraic structures displayed by quantum groups and algebras [4–7]. A similar method has been developed by Dobrev in [8, 9] and in references therein. Both procedures deal with the dual case, closer to the classical one, constructing representations in the algebra sector. Also, one can find other papers extending the induction technique to the quantum case but constructing corepresentations of quantum groups, i.e., representations of the coalgebra sector, from a mathematical perspective [10, 11] as well as physical one [12–14].

2. INDUCED REPRESENTATIONS OF QUANTUM GROUPS

Let H be a Hopf algebra and V a linear vector space over a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). The triplet $(V, \triangleright, H)$ is said to be a left H module if α is a left action of H on V, i.e., a linear map $\alpha : H \otimes V \to V$ $(\alpha : (h \otimes v) \mapsto \alpha(h \otimes v) \equiv h \triangleright v)$ such that

$$h_1 \triangleright (h_2 \triangleright v) = (h_1 h_2) \triangleright v,$$
$$1_H \triangleright v = v, \quad \forall h_1, h_2 \in H, \ \forall v \in V.$$

Right H modules can be defined in a similar way.

There are two canonical modules associated to any pair of Hopf algebras, H, H' related by a nondegenerate pairing $\langle \cdot, \cdot \rangle$ (under these conditions $(H, H', \langle \cdot, \cdot \rangle)$ will be called a nondegenerate triplet):

(1) The left regular module $(H, \succ, H)$ with action
$$h_1 \succ h_2 = h_1 h_2, \qquad \forall h_1, h_2 \in H.$$

(2) The right coregular module $(H', \prec, H)$ with action defined by
$$\langle h_2, h' \prec h_1 \rangle = \langle h_1 \succ h_2, h' \rangle,$$
$$\forall h_1, h_2 \in H, \quad \forall h' \in H',$$
which using the coproduct in H' $(\Delta(h') = h'_{(1)} \otimes h'_{(2)})$ takes the form $h' \prec h = \langle h, h'_{(1)} \rangle h'_{(2)}$.

The induction and coinduction algorithms of algebra representations are adapted to the Hopf algebras as follows. Let $(H, H', \langle \cdot, \cdot \rangle)$ be a nondegenerate triplet and $(V, \triangleright, K)$ a left K module with K a subalgebra of H. The carrier space, $\mathbb{K}^{\uparrow}$, of the coinduced representation is the subspace of $H' \otimes V$ with elements f such that

$$\langle f, kh \rangle = k \triangleright \langle f, h \rangle, \qquad \forall k \in K, \ \forall h \in H. \qquad (1)$$

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The pairing used in expression (1) is V-valued and is defined by $\langle h'\otimes v, h\rangle = \langle h', h\rangle v$, where $h\in H$, $h'\in H'$, $v\in V$. The action $h\triangleright f$ on the coinduced module is determined by

$$\langle h_1\triangleright f, h_2\rangle = \langle f, h_2h_1\rangle, \qquad \forall h_2\in H.$$

Let $(\mathbb{K},\triangleright,K)$ be a one-dimensional coinducing module. The carrier space of the coinduced representation is the subspace of $H'\otimes\mathbb{K}\simeq H'$ composed by elements φ verifying the equivariance condition $\varphi\prec k = (1\dashv k)\varphi$, $\forall k\in K$. The action of H on $\mathbb{K}^\uparrow$ induced by the action of K on $\mathbb{K}$ is given by

$$\langle h_2\triangleright\varphi, h_1\rangle = \langle\varphi, h_1h_2\rangle, \quad \forall h_1,h_2\in H, \quad \forall\varphi\in\mathbb{K}^\uparrow,$$

or explicitly by $h\triangleright\varphi\equiv h\succ\varphi = \langle h,\varphi_{(2)}\rangle\varphi_{(1)}$.

It is worthy to note that, to describe the induced module the right, $(H',\prec,H)$, and left, $(H',\succ,H)$, coregular modules are both pertinent, the former to determine the carrier space and the last to obtain the induced action.

Let us consider a nondegenerate triplet $(H, H', \langle\cdot,\cdot\rangle)$ with two finite sets of generators, $\{h_1,\ldots,h_n\}$ and $\{\varphi^1,\ldots,\varphi^n\}$, such that the families $\{h_l = h_1^{l_1}\cdots h_n^{l_n}\}_{l\in\mathbb{N}^n}$ and $\{\varphi^m=(\varphi^1)^{m_1}\cdots(\varphi^n)^{m_n}\}_{m\in\mathbb{N}^n}$ $(l=(l_1,\ldots,l_n),\ m=(m_1,\ldots,m_n))$ are bases of H and H', respectively. The action on the coregular module $(H',\succ,H)$ is obtained after computing the action of the generators

$$h_i\succ\varphi^j = \sum_{k\in\mathbb{N}^n}\alpha_{ik}^j\varphi^k, \qquad i,j\in\{1,2,\ldots,n\},$$

and extending it to the ordered polynomial $\varphi^j = (\varphi^1)^{j_1}\cdots(\varphi^n)^{j_n}$ by using of the compatibility relation between the action and the algebra structure in H'

$$h\succ(\varphi\psi) = (h_{(1)}\succ\varphi)(h_{(2)}\succ\psi), \tag{2}$$
$$h\succ 1_{H'} = \epsilon(h)1_{H'}.$$

In order to write explicitly the expression of the action on a general ordered polynomial, we take into account the following:

(1) There is a natural representation ρ, associated to $(H,\prec,H)$, of H

$$[\rho(h_2)](h_1) = h_1\prec h_2.$$

(2) The action on $(H',\succ,H)$ can be expressed in terms of ρ using the adjoint with respect to $\langle\cdot,\cdot\rangle$ ($f^\dagger: H'\to H'$ is the adjoint of $f: H\to H$ if $\langle h, f^\dagger(h')\rangle = \langle f(h), h'\rangle$) defined by

$$h\succ\varphi = [\rho(h)]^\dagger(\varphi). \tag{3}$$

If the bases $\{h_l\}_{l\in\mathbb{N}^n}$ and $\{\varphi^m\}_{m\in\mathbb{N}^n}$ are dual, i.e., $\langle h_l,\varphi^m\rangle = l!\delta_l^m$, $\forall l,m\in\mathbb{N}^n$ (where $l! = \prod_{i=1}^n l_i!$, $\delta_l^m = \prod_{i=1}^n\delta_{l_i}^{m_i}$), we define "multiplication" operators $\bar{h}_i$, $\bar{\varphi}^j$ and formal derivatives $\partial/\partial h_i$, $\partial/\partial\varphi^j$ by

$$\bar{h}_i(h_1^{l_1}\cdots h_i^{l_i}\cdots h_n^{l_n}) = h_1^{l_1}\cdots h_i^{l_i+1}\cdots h_n^{l_n},$$
$$\bar{\varphi}_i\left((\varphi^1)^{m_1}\cdots(\varphi^i)^{m_i}\cdots(\varphi^n)^{m_n}\right) = (\varphi^1)^{m_1}\cdots(\varphi^i)^{m_i+1}\cdots(\varphi^n)^{m_n},$$
$$\frac{\partial}{\partial h_i}(h_1^{l_1}\cdots h_i^{l_i}\cdots h_n^{l_n}) = l_ih_1^{l_1}\cdots h_i^{l_i-1}\cdots h_n^{l_n},$$
$$\frac{\partial}{\partial\varphi^i}\left((\varphi^1)^{m_1}\cdots(\varphi^i)^{m_i}\cdots(\varphi^n)^{m_n}\right) = m_i(\varphi^1)^{m_1}\cdots(\varphi^i)^{m_i-1}\cdots(\varphi^n)^{m_n}.$$

The adjoint operators are given by $\bar{h}_i^\dagger = \partial/\partial\varphi^i$ and $\bar{\varphi}^{i\dagger} = \partial/\partial h_i$.

3. NULL-PLANE QUANTUM POINCARÉ ALGEBRA

The null-plane quantum deformation of the $(1+1)$ Poincaré algebra, $U_z(\mathfrak{p}(1,1))$, is a q-deformed Hopf algebra that in a null-plane basis, $\{P_+,P_-,K\}$, has the form [15]

$$[K,P_+] = \frac{-1}{z}(e^{-2zP_+}-1),$$
$$[K,P_-] = -2P_-, \quad [P_+,P_-]=0;$$
$$\Delta P_+ = P_+\otimes 1 + 1\otimes P_+,$$
$$\Delta X = X\otimes 1 + e^{-2zP_+}\otimes X, \quad X\in\{P_-,K\};$$
$$\epsilon(X) = 0, \quad X\in\{P_\pm,K\};$$
$$S(P_+) = -P_+, \quad S(X) = -e^{2zP_+}X,$$
$$X\in\{P_-,K\}.$$

It has also the structure of bicrossproduct $U_z(\mathfrak{p}(1,1)) = \mathcal{K}\triangleright\!\!\blacktriangleleft\mathcal{L}$, where $\mathcal{K}$ is a commutative and cocommutative algebra generated by K, and $\mathcal{L}$ is the commutative Hopf subalgebra of $U_z(\mathfrak{p}(1,1))$ generated by P_+ and P_-.

The dual Hopf algebra $F_z(P(1,1)) = \mathcal{K}^*\blacktriangleright\!\!\triangleleft\,\mathcal{L}^*$, where $\mathcal{K}^*$ is generated by φ and $\mathcal{L}^*$ by a_+ and a_-, has the following structure:

$$[a_+,a_-] = -2za_-,$$
$$[a_+,\varphi] = 2z(e^{-\varphi}-1), \quad [a_-,\varphi]=0;$$
$$\Delta a_\pm = a_\pm\otimes e^{\mp 2\varphi} + 1\otimes a_\pm, \quad \Delta\varphi = \varphi\otimes 1 + 1\otimes\varphi;$$
$$\epsilon(f) = 0, \quad f\in\{a_\pm,\varphi\};$$
$$S(a_\pm) = -a_\pm e^{\pm\varphi}, \quad S(\varphi) = -\varphi.$$

The duality between $U_z(\mathfrak{p}(1,1))$ and $F_z(P(1,1))$ is explicitly given by the pairing

$$\langle K^mP_-^nP_+^p, \varphi^qa_-^ra_+^s\rangle = m!n!p!\delta_q^m\delta_r^n\delta_s^p.$$

3.1. Coregular Modules

As we mentioned in the previous section, we need to know the left and the right coregular modules, $(F_z(P(1,1)), \succ, U_z(\mathfrak{p}(1,1)))$ and $(F_z(P(1,1)), \prec, U_z(\mathfrak{p}(1,1)))$, respectively, in order to construct the induced representations of $U_z(\mathfrak{p}(1,1))$.

The structure of $(F_z(P(1,1)), \succ, U_z(\mathfrak{p}(1,1)))$ is given by

$$K \succ (\varphi^q a_-^r a_+^s) = q\varphi^{q-1} a_-^r a_+^s + 2r\varphi^q a_-^r a_+^s + \frac{1}{z}\varphi^q a_-^r a_+[(a_+ - 2z)^s - a_+^s], \tag{4}$$
$$P_- \succ (\varphi^q a_-^r a_+^s) = r\varphi^q a_-^{r-1} a_+^s,$$
$$P_+ \succ (\varphi^q a_-^r a_+^s) = s\varphi^q a_-^r a_+^{s-1}.$$

The following equalities are basic in the demonstration of the above result (4)

$$P_-^n K = K P_-^n + 2n P_-^n,$$
$$P_+^n K = K P_+^n - n\frac{1}{z}(1 - e^{-2zP_+})P_+^{n-1},$$
$$\forall n \in N.$$

The structure of $(F_z(P(1,1)), \prec, U_z(\mathfrak{p}(1,1)))$ is given by

$$(\varphi^q a_-^r a_+^s) \prec K = q\varphi^{q-1} a_-^r a_+^s, \tag{5}$$
$$(\varphi^q a_-^r a_+^s) \prec P_- = r e^{2\varphi}\varphi^q a_-^{r-1} a_+^s,$$
$$(\varphi^q a_-^r a_+^s) \prec P_+ = -\frac{1}{2z}\sum_{j=1}^{\infty}\sum_{k=0}^{j}\frac{1}{j}\begin{pmatrix} j \\ k \end{pmatrix}(-1)^k e^{-2\varphi}\varphi^q a_-^r (a_+ + 2kz)^s.$$

The proof of (5) starts characterizing the module $(U_z(\mathfrak{p}(1,1)), \succ, U_z(\mathfrak{p}(1,1)))$. For that we take into account the following

$$P_- K^n = (K+2)^n P_-,$$
$$P_+ K^n = -\frac{1}{2z}\sum_{j=1}^{\infty}\frac{1}{j}(K - 2j)^n (1 - e^{2zP_+})^j,$$
$$\forall n \in N,$$

which allow us to obtain easily the explicit expression of $(U_z(\mathfrak{p}(1,1)), \succ, U_z(\mathfrak{p}(1,1)))$

$$K \succ K^m P_-^n P_+^p = K^{m+1} P_-^n P_+^p,$$
$$P_- \succ K^m P_-^n P_+^p = (K+2)^m P_-^{n+1} P_+^p,$$
$$P_+ \succ K^m P_-^n P_+^p = -\frac{1}{2z}\sum_{j=1}^{\infty}\frac{1}{j}(K - 2j)^m P_-^n (1 - e^{2zP_+})^j P_+^p.$$

The corresponding endomorphisms of $U_z(\mathfrak{p}(1,1))$ are given by

$$\lambda(K) = \bar{K}, \qquad \lambda(P_-) = \bar{P}_- e^{2\frac{\partial}{\partial K}},$$
$$\lambda(P_+) = -\frac{1}{2z}\sum_{j=1}^{\infty}\frac{1}{j}e^{-2j\frac{\partial}{\partial K}}(1 - e^{2z\bar{P}_+})^j.$$

The computation of the adjoints gives

$$\lambda(K)^\dagger = \frac{\partial}{\partial\varphi}, \qquad \lambda(P_-)^\dagger = e^{2\bar{\varphi}}\frac{\partial}{\partial a_-},$$
$$\lambda(P_+)^\dagger = -\frac{1}{2z}\sum_{j=1}^{\infty}\frac{1}{j}(1 - e^{2z\frac{\partial}{\partial a_+}})^j e^{-2j\bar{\varphi}} = \frac{1}{2z}\ln\left[1 - e^{-2\bar{\varphi}}(1 - e^{2z\frac{\partial}{\partial a_+}})\right].$$

Hence, the action on $(F_z(P(1,1)), \prec, U_z(\mathfrak{p}(1,1)))$ is given by

$$f \prec K = \frac{\partial}{\partial\varphi}f, \quad f \prec P_- = e^{2\bar{\varphi}}\frac{\partial}{\partial a_-}f, \tag{6}$$
$$f \prec P_+ = \frac{1}{2z}\ln\left[1 - e^{-2\bar{\varphi}}(1 - e^{2z\frac{\partial}{\partial a_+}})\right] f.$$

The explicit action over the basis elements $\varphi^q a_-^r a_+^s$ (5) is obtained using the series expansions of the above expressions.

3.2. Induced Representations

Let us consider the representation of $\mathcal{L}$

$$1 \dashv (P_-^n P_+^p) = \alpha_-^n \alpha_+^p, \quad n, p \in \mathbb{N}, \ \alpha_-, \alpha_+ \in \mathbb{C}. \tag{7}$$

The carrier space, $\mathbb{C}^\uparrow$, of the representation of $U_z(\mathfrak{p}(1,1))$, induced by the character (7), is constituted by the elements of $F_z(P(1,1))$ having the form

$$\phi(\varphi)e^{\alpha_- a_-}e^{\alpha_+ a_+}.$$

The induced representation can be translated to $\mathbb{C}[[\varphi]]$, where the action of the generators is

$$\phi(\varphi) \dashv K = \phi'(\varphi), \quad \phi(\varphi) \dashv P_- = \phi(\varphi)\alpha_- e^{2\varphi},$$
$$\phi(\varphi) \dashv P_+ = \phi(\varphi)\frac{1}{2z}\ln[1 - e^{-2\varphi}(1 - e^{2z\alpha_+})].$$

A sketch of the construction of the representations induced by the character of $\mathcal{L}$ (7) is as follows [5]. The carrier space of the induced representation is characterized by the equivariance condition which, when is described in terms of the left regular module $(F_z(P(1,1)), \succ, U_z(\mathfrak{p}(1,1)))$, is reduced to the equations

$$\frac{\partial}{\partial a_-}f = \alpha_- f, \qquad \frac{\partial}{\partial a_+}f = \alpha_+ f,$$

which are not really differential equations, except at the limit $z \to 0$. However, their general solution is

$$f = \phi(\varphi)e^{\alpha_- a_-}e^{\alpha_+ a_+},$$

which is the same as that obtained working formally with the derivatives.

The right regular action (6) over f gives the expression of the induced representation

$$[\phi(\varphi)e^{\alpha_- a_-}e^{\alpha_+ a_+}] \prec K = \phi'(\varphi)e^{\alpha_- a_-}e^{\alpha_+ a_+},$$
$$[\phi(\varphi)e^{\alpha_- a_-}e^{\alpha_+ a_+}] \prec P_- = \phi(\varphi)e^{2\varphi}\alpha_- e^{\alpha_- a_-}e^{\alpha_+ a_+},$$
$$[\phi(\varphi)e^{\alpha_- a_-}e^{\alpha_+ a_+}] \prec P_+$$
$$= \phi(\varphi)\frac{1}{2z}\ln\left[1 - e^{-2\varphi}(1 - e^{2z\alpha_+})\right]e^{\alpha_- a_-}e^{\alpha_+ a_+}.$$

Note that in reality we have two kinds of representations labeled by the pairs $(\alpha_+, 0)$ and $(\alpha_+, 1)$, respectively, since we can perform the rescaling $P_- \to P_-/\alpha_-$ and $a_- \to \alpha_- a_-$.

Let us consider now the character of $\mathcal{K}$

$$K^n \vdash 1 = c^n, \qquad n \in \mathbb{N},\ c \in \mathbb{C}. \qquad (8)$$

We can construct a representation of $U_z(\mathfrak{p}(1,1))$ whose carrier space, $\mathbb{C}^\uparrow$, is formed by the elements of $F_z(P(1,1))$

$$e^{c\varphi}\phi(a_-, a_+).$$

The action on $\mathbb{C}^\uparrow$ can be carried to the subalgebra $\mathcal{L}^*$ of $F_z(P(1,1))$, obtaining

$$K \vdash f(a_-, a_+)$$
$$= \left[c + 2\bar{a}_-\frac{\partial}{\partial a_-} + \frac{1}{z}\bar{a}_+(e^{-2z\frac{\partial}{\partial a_+}} - 1)\right]f(a_-, a_+),$$
$$P_\pm \vdash f(a_-, a_+) = \frac{\partial}{\partial a_\pm}f(a_-, a_+).$$

Effectively, the representation induced by the character of $\mathcal{K}$ (8) presents an equivariance condition described in terms of the left regular module by the equation $\partial f/\partial\varphi = cf$, whose general solution is

$$f = e^{c\varphi}\phi(a_-, a_+).$$

The restriction of the right regular action (4) over these elements gives the representation

$$K \succ [e^{c\varphi}\phi(a_-, a_+)]$$
$$= e^{c\varphi}\left[c - 2\bar{a}_-\frac{\partial}{\partial a_-} + \frac{1}{z}\bar{a}_+(e^{-2z\frac{\partial}{\partial a_+}} - 1)\right]\phi(a_-, a_+),$$
$$P_\mp \succ [e^{c\varphi}\phi(a_-, a_+)] = e^{c\varphi}\frac{\partial}{\partial a_\mp}\phi(a_-, a_+).$$

This representation is called "local type" representation because when the deformation parameter goes to zero we recover the called local representations [3]. Note that the coefficient c vanishes after the "gauge transformation" $K \to K - c$.

4. QUANTUM KAPPA-GALILEI ALGEBRA

The quantum algebra $U_{\tilde{\kappa}}(\mathfrak{g}(1,1))$, obtained by contraction of the κ-Poincaré [16], is characterized by the following algebraic structure [17]:

$$[H, K] = -P, \quad [P, K] = \frac{1}{2\tilde{\kappa}}P^2, \quad [H, P] = 0;$$
$$\Delta H = H \otimes 1 + 1 \otimes H,$$
$$\Delta X = X \otimes 1 + e^{-\frac{1}{\tilde{\kappa}}H} \otimes X, \qquad X \in \{P, K\};$$
$$\epsilon(X) = 0, \quad X \in \{H, P, K\};$$
$$S(H) = -H, \quad S(X) = -e^{\frac{1}{\tilde{\kappa}}H}X, \quad X \in \{P, K\},$$

where $\tilde{\kappa} = kc$, κ being the deformation parameter of the above-mentioned κ-Poincaré algebra.

The dual algebra $F_{\tilde{\kappa}}(G(1,1))$ is generated by x, t, and v, and its Hopf structure is

$$[t, x] = -\frac{1}{\tilde{\kappa}}x, \quad [x, v] = \frac{1}{2\tilde{\kappa}}v^2, \quad [t, v] = -\frac{1}{\tilde{\kappa}}v;$$
$$\Delta t = t \otimes 1 + 1 \otimes t, \quad \Delta x = x \otimes 1 + 1 \otimes x - t \otimes v,$$
$$\Delta v = v \otimes 1 + 1 \otimes v;$$
$$\epsilon(f) = 0, \quad f \in \{v, t, x\};$$
$$S(v) = -v, \quad S(x) = -x - tv, \quad S(t) = -t.$$

The pairing between both Hopf algebras is given by

$$\langle K^m P^n H^p, v^q x^r t^s\rangle = m!n!p!\delta_q^m\delta_r^n\delta_s^p.$$

The action of $U_{\tilde{\kappa}}(\mathfrak{g}(1,1))$ on the left coregular module $(F_{\tilde{\kappa}}(G(1,1)), \succ, U_{1/2\tilde{\kappa}}(\mathfrak{g}(1,1)))$ is

$$K \succ f = \left[\frac{\partial}{\partial v} + \frac{1}{2\tilde{\kappa}}\bar{x}\frac{\partial^2}{\partial x^2} - \bar{t}\frac{\partial}{\partial x}\right]f,$$
$$P \succ f = \frac{\partial}{\partial x}f, \quad H \succ f = \frac{\partial}{\partial t}f,$$

where f is an arbitrary element of $F_{\tilde{\kappa}}(G(1,1))$.

The action on the right coregular module $(F_{\tilde{\kappa}}(G(1,1)), \prec, U_{\tilde{\kappa}}(\mathfrak{g}(1,1)))$ is given by

$$f \prec K = \frac{\partial}{\partial v}f, \quad f \prec P = \frac{\partial/\partial x}{1 - \frac{\bar{v}}{2\tilde{\kappa}}\frac{\partial}{\partial x}}f,$$
$$f \prec H = \left[\frac{\partial}{\partial t} - 2\tilde{\kappa}\ln\left(1 - \frac{\bar{v}}{2\tilde{\kappa}}\frac{\partial}{\partial x}\right)\right]f.$$

Now, we can obtain a family of representations of $U_{\tilde{\kappa}}(\mathfrak{g}(1,1))$ coinduced by the character

$$1 \dashv P^n H^p = a^n b^p, \qquad n, p \in \mathbb{N},\ a, b \in \mathbb{C},$$

of the Abelian subalgebra of $U_{\tilde{\kappa}}(\mathfrak{g}(1,1))$ generated by H and P, whose carrier space $\mathbb{C}^\uparrow$ is the set of elements of $F_{\tilde{\kappa}}(G(1,1))$ of the form [4, 6]

$$\phi(v)e^{ax}e^{bt}.$$

The action on $\mathbb{C}^\uparrow$ can be translated to the space of formal power series

$$\phi(v) \dashv K = \phi'(v), \qquad \phi(v) \dashv P = \phi(v)\frac{a}{1-\frac{1}{2\tilde\kappa}av},$$

$$\phi(v) \dashv H = \phi(v)\left[b + 2\tilde\kappa \ln\left(1 - \frac{1}{2\tilde\kappa}av\right)\right].$$

The gauge transformation $H \to H - b$ allows the "gauge equivalence" of the representations labeled by the pair (a, b) and those parameterized by $(a, 0)$.

The "local" representation of $U_{\tilde\kappa}(\mathfrak{g}(1,1))$ coinduced by the character of the Abelian subalgebra of $U_{\tilde\kappa}(\mathfrak{g}(1,1))$ generated by K

$$K^m \vdash 1 = c^m, \qquad m \in \mathbb{N}, \ c \in \mathbb{C},$$

has as support the subspace of $F_{\tilde\kappa}(G(1,1))$ of elements

$$e^{cv}\phi(x,t).$$

The action of $U_{\tilde\kappa}(\mathfrak{g}(1,1))$ carried to the subalgebra of formal power series $\mathbb{C}[[t,x]]$ is

$$K \vdash \phi(x,t) = \left(c - \bar t\frac{\partial}{\partial x} + \frac{1}{2\tilde\kappa}\bar x\frac{\partial^2}{\partial x^2}\right)\phi(x,t),$$

$$P \vdash \phi(x,t) = \frac{\partial}{\partial x}\phi(x,t), \quad H \vdash \phi(x,t) = \frac{\partial}{\partial t}\phi(x,t).$$

Also here, the label c can be reduced to zero.

Note that in the limit when the deformation parameter goes to zero we recover the well-known induced representations of the corresponding nondeformed Lie groups.

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SYMPOSIUM ON QUANTUM GROUPS

Helicity Asymmetry of *q*-Plane Wave Solutions of *q*-Maxwell Equations*

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Abstract—We give solutions of the quantum conformal deformations of the Maxwell and potential equations in terms of deformations of the plane wave. Compatibility of the equations leads to an asymmetry between the q deformations of the fixed helicity constituents $F^{\pm} = E \pm iH$ of the Maxwell field. Namely, only one of $F^{\pm}$ can be written in terms of the q-plane wave, while the other can be expressed only through the components of the q-plane wave. This asymmetry and possible alternatives are discussed. *© 2001 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

One of the purposes of quantum deformations is to provide an alternative of the regularization procedures of quantum field theory. Applied to Minkowski spacetime the quantum deformations approach is also an alternative to Connes' noncommutative geometry [1]. The first problem to tackle in a noncommutative deformed setting is to analyze the behavior of the wave equation analogs. Here, we continue the study of hierarchies of deformed equations derived in [2–4] with the use of quantum conformal symmetry. Actually, we study the q deformation of Maxwell equations and of the potential equations. We give solutions of the q-Maxwell equations using a deformation of the plane wave [5] which is a formal power series in the noncommutative coordinates of q-Minkowski spacetime and four-momenta. (For the latter deformations, we use the one from [2] since, unlike the other known examples [6–8], it is related to a deformation of the conformal group.) Then, we check compatibility of these solutions with the deformation of the potential equations. The restrictions are as in the classical case, except that only one of the fixed helicity constituents of the Maxwell field (the q deformation of the -1 helicity constituent $F_k^- = E_k - iF_k$ in the chosen basis) can be written in terms of the q-plane wave, while the other (the q deformation of the $+1$ helicity constituent $F^+ = E_k + iF_k$ in the chosen basis) can be expressed through the q-plane wave only componentwise. This asymmetry and possible alternatives are discussed at the end of the paper.

2. PRELIMINARIES

First, we introduce new Minkowski variables,

$$x_\pm \equiv x_0 \pm x_3\,, \quad v \equiv x_1 - ix_2, \quad \bar v \equiv x_1 + ix_2, \tag{1}$$

which (unlike the x_μ) have definite group-theoretical interpretation as part of a six-dimensional coset of the conformal group $SU(2,2)$ (as explained in [2]). In terms of these variables, e.g., the d'Alembert equation is

$$\Box\varphi = (\partial_-\partial_+ - \partial_v\partial_{\bar v})\varphi = 0. \tag{2}$$

In the q-deformed case we use the noncommutative q-Minkowski spacetime of [2] which is given by the following commutation relations (with $\lambda \equiv q - q^{-1}$):

$$x_\pm v = q^{\pm1} v x_\pm, \quad x_\pm \bar v = q^{\pm1}\bar v x_\pm, \tag{3}$$
$$x_+x_- - x_-x_+ = \lambda v\bar v, \quad \bar v v = v\bar v,$$

with the deformation parameter being a phase: $|q| = 1$. Relations (3) are preserved by the antilinear anti-involution ω:

$$\omega(x_\pm) = x_\pm, \quad \omega(v) = \bar v, \tag{4}$$
$$\omega(q) = \bar q = q^{-1} \quad (\omega(\lambda) = -\lambda).$$

The solution spaces consist of formal power series in the q-Minkowski coordinates (which we give in two conjugate bases):

$$\varphi = \sum_{j,n,\ell,m\in\mathbb{Z}_+} \mu_{jn\ell m}\,\varphi_{jn\ell m}, \tag{5}$$
$$\varphi_{jn\ell m} = \hat\varphi_{jn\ell m},\ \tilde\varphi_{jn\ell m},$$

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$$\hat{\varphi}_{jn\ell m} = v^j x_-^n x_+^\ell \bar{v}^m, \quad (6)$$

$$\tilde{\varphi}_{jn\ell m} = \bar{v}^m x_+^\ell x_-^n v^j = \omega(\hat{\varphi}_{jn\ell m}). \quad (7)$$

The solution spaces (5) are representation spaces of the quantum algebra $U_q(sl(4))$. For the latter, we use the rational basis of Jimbo [9]. The action of $U_q(sl(4))$ on $\hat{\varphi}_{jn\ell m}$ was given in [10], and on $\tilde{\varphi}_{jn\ell m}$ in [11]. Further, we suppose that q is not a nontrivial root of unity.

In order to write our q-deformed equations in compact form, it is necessary to introduce some additional operators. We first define the operators

$$\hat{M}_\kappa^\pm \varphi = \sum_{j,n,\ell,m \in \mathbb{Z}_+} \mu_{jn\ell m} \hat{M}_\kappa^\pm \varphi_{jn\ell m}, \ \kappa = \pm, v, \bar{v}, \quad (8)$$

$$T_\kappa^\pm \varphi = \sum_{j,n,\ell,m \in \mathbb{Z}_+} \mu_{jn\ell m}\, T_\kappa^\pm \varphi_{jn\ell m}, \ \kappa = \pm, v, \bar{v}, \quad (9)$$

and $\hat{M}_+^\pm$, $\hat{M}_-^\pm$, $\hat{M}_v^\pm$, $\hat{M}_{\bar{v}}^\pm$, respectively, act on $\varphi_{jn\ell m}$ by changing by ± 1 the value of j, n, ℓ, m, respectively, while $T_+^\pm$, $T_-^\pm$, $T_v^\pm$, $T_{\bar{v}}^\pm$, respectively, act on $\varphi_{jn\ell m}$ by multiplication by $q^{\pm j}, q^{\pm n}, q^{\pm \ell}, q^{\pm m}$, respectively. Now, we can define the q-difference operators:

$$\hat{\mathcal{D}}_\kappa \varphi = \frac{1}{\lambda} \hat{M}_\kappa^{-1} \, (T_\kappa - T_\kappa^{-1})\varphi. \quad (10)$$

Note that when $q \to 1$, then $\hat{\mathcal{D}}_\kappa \to \partial_\kappa$. Using (8) and (10), the q-d'Alembert equation may be written as [4, 11], respectively,

$$\left(q\hat{\mathcal{D}}_-\hat{\mathcal{D}}_+ T_v T_{\bar{v}} - \hat{\mathcal{D}}_v \hat{\mathcal{D}}_{\bar{v}}\right) T_v T_- T_+ T_{\bar{v}} \hat{\varphi} = 0, \quad (11)$$

$$\left(\hat{\mathcal{D}}_-\hat{\mathcal{D}}_+ - q\hat{\mathcal{D}}_v \hat{\mathcal{D}}_{\bar{v}} T_v T_{\bar{v}}\right) T_- T_+ \tilde{\varphi} = 0. \quad (12)$$

Note that, when $q \to 1$, both Eqs. (11), (12) go to (2). Note that the operators in (8), (10)–(12) for different variables commute, i.e., we have passed to commuting variables. However, keeping the normal ordering, it is straightforward to pass back to noncommuting variables.

3. SOLUTIONS OF THE q-MAXWELL EQUATIONS

We consider the quantum conformal deformation of Maxwell's equations introduced in [2], as part of Maxwell's hierarchy of equations. The equations of the hierarchy are

$${}_qI_n^+ \, {}_qF_n^+ = {}_qJ^n, \quad {}_qI_n^- \, {}_qF_n^- = {}_qJ^n, \quad (13)$$

where in the basis (6) the operators are [2]

$${}_qI_n^+ = \frac{1}{2}\Bigg(\left(q\hat{\mathcal{D}}_v + \hat{M}_{\bar{z}}\hat{\mathcal{D}}_+ (T_- T_v)^{-1} T_{\bar{v}}\right) \quad (14)$$

$$\times T_-[n+2-N_z]_q - q^{-n-2}\Big(\hat{\mathcal{D}}_- T_- + q^{-1}\hat{M}_{\bar{z}}\hat{\mathcal{D}}_{\bar{v}} - \lambda \hat{M}_v \hat{M}_{\bar{z}} \hat{\mathcal{D}}_- \hat{\mathcal{D}}_+ T_{\bar{v}}\Big) \times T_-^{-1}\hat{\mathcal{D}}_z\Bigg) T_+ T_v T_z T_{\bar{z}}^{-1},$$

$${}_qI_n^- = \frac{1}{2}\Big(\hat{\mathcal{D}}_{\bar{v}} + q\hat{M}_z \hat{\mathcal{D}}_+ \, T_{\bar{v}} T_- T_v^{-1} - q\lambda \hat{M}_v \hat{\mathcal{D}}_- \hat{\mathcal{D}}_+ \, T_{\bar{v}}\Big) T_{\bar{v}}[n+2-N_{\bar{z}}]_q - \frac{1}{2} q^{n+3}\Big(\hat{\mathcal{D}}_- + q\hat{M}_z \hat{\mathcal{D}}_v T_-\Big)\hat{\mathcal{D}}_{\bar{z}} \, T_- T_{\bar{v}}, \quad (15)$$

and where, in the basis (7), the operators are

$${}_qI_n^+ = \frac{1}{2} q\Big(\hat{\mathcal{D}}_v + \hat{M}_{\bar{z}}\hat{\mathcal{D}}_+ T_- \, T_{\bar{v}}^{-1} T_v\Big) \times T_v [n+2-N_z]_q - \frac{1}{2} q^{n+3} \Big(\hat{\mathcal{D}}_- + \hat{M}_{\bar{z}} \hat{\mathcal{D}}_{\bar{v}} T_- + \lambda q^{-1} \hat{M}_v \hat{M}_{\bar{z}} \hat{\mathcal{D}}_- \hat{\mathcal{D}}_+ \, T_{\bar{v}}^{-1} T_-\Big) \hat{\mathcal{D}}_z T_- T_v, \quad (16)$$

$${}_qI_n^- = \frac{1}{2}\Bigg(\Big(\hat{\mathcal{D}}_{\bar{v}} T_{\bar{v}} T_- + \hat{M}_z \hat{\mathcal{D}}_+ T_v + q^{-1}\lambda \hat{M}_v \hat{\mathcal{D}}_- \, \hat{\mathcal{D}}_+ T_-\Big)[n+2-N_{\bar{z}}]_q - q^{-n-2}\Big(\hat{\mathcal{D}}_- + \hat{M}_z \hat{\mathcal{D}}_v T_-^{-1}\Big)\hat{\mathcal{D}}_{\bar{z}} \, T_{\bar{v}}\Bigg) T_+ T_{\bar{z}} T_z^{-1}. \quad (17)$$

Note that, for $q = 1$, (14) and (15) coincide with (16) and (17), respectively. Maxwell's equations $\partial^\mu F_{\mu\nu} = J_\nu$, $\epsilon_{\mu\nu\rho\sigma}\partial^\mu F^{\rho\sigma} = 0$ are obtained from (13) for $n = 0$, $q = 1$, substituting the fixed helicity constituents $F^\pm$ by $F^+ = z^2(F_1^+ + iF_2^+) - 2zF_3^+ - (F_1^+ - iF_2^+)$, $F^- = \bar{z}^2(F_1^- - iF_2^-) - 2\bar{z}F_3^- - (F_1^- + iF_2^-)$, $F_k^\pm = F_{k0} \pm \frac{i}{2}\varepsilon_{k\ell m}F_{\ell m} = E_k \pm iH_k$, $J^0 = \bar{z}z(J_0 + J_3) + z(J_1 + iJ_2) + \bar{z}(J_1 - iJ_2) + (J_0 - J_3)$, and then comparing the coefficients of the resulting first-order polynomials in z and $\bar{z}$.

Further, we consider the free equations, i.e., $J^n = 0$. We shall use the fact that Maxwell's equations also belong to another hierarchy (introduced in [4]) for which we know solutions in terms of deformations of the plane wave. Let us first recall these deformations from [12]. The first deformation is given in the basis (6):

$$\widehat{\exp}_q(k, x) = \sum_{s=0}^{\infty} \frac{1}{[s]_q!} \hat{h}_s, \quad (18)$$

$$[s]_q! \equiv [s]_q [s-1]_q \cdots [1]_q, \quad [0]_q! \equiv 1,$$

$$[n]_q \equiv \frac{q^n - q^{-n}}{q - q^{-1}},$$

$$\hat{h}_s = \beta^s \sum_{a,b,n\in\mathbb{Z}_+} \frac{(-1)^{s-a-b} q^{n(s-2a-2b+2n)+a(s-a-1)+b(-s+a+b+1)}\ q^{P_s(a,b)}}{\Gamma_q(a-n+1)\Gamma_q(b-n+1)\Gamma_q(s-a-b+n+1)[n]_q!} \times k_v^{s-a-b+n} k_-^{b-n} k_+^{a-n} k_{\bar{v}}^n v^n x_-^{a-n} x_+^{b-n} \bar{v}^{s-a-b+n}, \quad (\beta^s)^{-1} = \sum_{p=0}^{s} \frac{q^{(s-p)(p-1)+p}}{[p]_q![s-p]_q!}, \tag{19}$$

where the momentum components $(k_v, k_-, k_+, k_{\bar{v}})$ are supposed to be noncommutative between themselves (obeying the same rules (3) as the q-Minkowski coordinates) and commutative with the coordinates. Further, Γ_q is the q deformation of the Γ function, of which here we use only the properties $\Gamma_q(p) = [p-1]_q!$ for $p \in \mathbb{N}$, $1/\Gamma_q(p) = 0$ for $p \in \mathbb{Z}_-$; $P_s(a,b)$ is a polynomial in a, b. Note that $(\hat{h}_s)|_{q=1} = (k\cdot x)^s$ and thus $(\widehat{\exp}_q(k,x))|_{q=1} = \exp(k\cdot x)$. This q-plane wave has some properties analogous to the classical one but is not an exponent or q exponent, (cf. [13]). This is enabled also by the fact (true also for $q=1$) that solving the equations may be done in terms of the components $\hat{h}_s$. This deformation of the plane wave generalizes the original one from [5] to obtain which one sets $P_s(a,b) = 0$, in which case we shall use the notation f_s for the components from [5] since

$$\left(\hat{h}_s\right)_{P_s(a,b)=0} = f_s. \tag{20}$$

Each $\hat{h}_s$ satisfies the q-d'Alembert equation (11) on the momentum q cone:

$$\mathcal{L}_q^k \equiv k_-k_+ - q^{-1}k_v k_{\bar{v}} = k_+k_- - qk_v k_{\bar{v}} = 0. \tag{21}$$

The second deformation is given in the basis (7):

$$\widetilde{\exp}_q(k,x) = \sum_{s=0}^{\infty} \frac{1}{[s]_q!}\tilde{h}_s, \tag{22}$$

$$\tilde{h}_s = \tilde{\beta}^s \sum_{a,b,n} \frac{(-1)^{s-a-b} q^{n(2a+2b-2n-s)+\ a(a-s-1)+b(s-a-b+1)} q^{Q_s(a,b)}}{\Gamma_q(a-n+1)\Gamma_q(b-n+1)\ \Gamma_q(s-a-b+n+1)[n]_q!} \times k_{\bar{v}}^n k_+^{a-n} k_-^{b-n} k_v^{s-a-b+n}\ \bar{v}^{s-a-b+n} x_+^{b-n} x_-^{a-n} v^n, \quad \left(\tilde{\beta}^s\right)^{-1} = \sum_{p=0}^{s} \frac{q^{(p-s)(p-1)+p}}{[p]_q!\ [s-p]_q!}, \tag{23}$$

where $Q_s(a,b)$ are arbitrary polynomials. If the latter are zero, then $\widetilde{\exp}_q(k,x)$ becomes the q-plane wave deformation found in [11]. The h_s have the same properties as the $\hat{h}_s$, but the conjugated basis is used; in particular, they satisfy the q-d'Alembert equation (12) on the momentum q cone (21).

Further, we shall restrict ourselves to the basis (6).

The solutions of the first equation from (13) are

$$_qF_0^+ = \sum_{s=0}^{\infty} \frac{1}{[s]_q!}\hat{F}_s^+, \tag{24}$$

$$\hat{F}_s^+ = \sum_{m=0}^{2} \hat{\gamma}_m^{s+} \left(\prod_{i=0}^{-m+1} (k_+ - q^{i+B_s+s+4}k_{\bar{v}}z) \right) \times \left(\prod_{j=-m+2}^{1} (k_v - q^{j+B_s+s+4}k_-z) \right) \hat{h}_s^+, \tag{25}$$

where $\hat{h}_s^+$ is $\hat{h}_s$ with

$$P_s(a,b) = P_s^+(a,b) \equiv R_s(a) + B_s b, \tag{26}$$

$\hat{\gamma}_m^{s+}$ and B_s are arbitrary constants, and $R_s(a)$ is an arbitrary polynomial in a. Note that the factors preceding $\hat{h}_s^+$ depend on B_s but not on $P_s(b)$. The check that (24) is a solution is done for commutative Minkowski coordinates and noncommutative momenta on the q cone. In order to be able to write the above solution in terms of the deformed plane wave, we have to suppose that the $\hat{\gamma}_m^{s+}$, $B_s + s$ for different s coincide: $\hat{\gamma}_m^{s+} = \tilde{\gamma}_m^+$, e.g.; we can make the choice $B_s = B' - s - 4$. Then we have

$$_qF_0^+ = \sum_{m=0}^{2} \tilde{\gamma}_m^+ \left(\prod_{i=0}^{-m+1} (k_+ - q^{i+B'}k_{\bar{v}}z) \right) \tag{27}$$

$$\times\left(\prod_{j=-m+2}^{1}(k_v - q^{j+B'}k_- z)\right)\widehat{\exp}_q^+(k,x),$$

where $\widehat{\exp}_q^+(k,x)$ is $\widehat{\exp}_q(k,x)$ with the choice (26).

The solutions of the second equation from (13) are (cf. [12])

$$ {}_qF_0^- = \sum_{s=0}^{\infty}\frac{1}{[s]_q!}\hat{F}_s^-, \tag{28}$$

$$\hat{F}_s^- = \sum_{m=0}^{2}\hat{\gamma}_m^{s-}\left(\prod_{i=-1}^{-m}(k_+ - q^{i-C_s}k_v\bar{z})\right) \tag{29}$$
$$\times\left(\prod_{j=-m+1}^{0}(k_{\bar{v}} - q^{j-C_s}k_-\bar{z})\right)\hat{h}_s^-,$$

where $\hat{h}_s^-$ is $\hat{h}_s$ with

$$P_s(a,b) = P_s^-(a,b) \equiv C_s a + Q_s(b), \tag{30}$$

$\hat{\gamma}_m^{s-}$ and C_s are arbitrary constants, and $Q_s(b)$ is an arbitrary polynomial. For $Q_s(b) = 0 = C_s$, we recover the solutions given in [14] in terms of the q-plane wave from [5]. In order to be able to write this solution in terms of the deformed plane wave, we have to suppose that the $\hat{\gamma}_m^{s-}, C_s$ for different s coincide: $\hat{\gamma}_m^{s-} = \hat{\gamma}_m^-$, $C_s = C$. Then, we have

$$ {}_qF_0^- = \sum_{m=0}^{2}\hat{\gamma}_m^-\left(\prod_{i=-1}^{-m}(k_+ - q^{i-C}k_v\bar{z})\right) \tag{31}$$
$$\times\left(\prod_{j=-m+1}^{0}(k_{\bar{v}} - q^{j-C}k_-\bar{z})\right)\widehat{\exp}_q^-(k,x),$$

where $\widehat{\exp}_q^-(k,x)$ is $\widehat{\exp}_q(k,x)$ with the choice (30).

We shall consider also the potential q-Maxwell hierarchy [3]:

$$ {}_qI_{n-1}^- {}_qA^n = {}_qF_n^+, \quad {}_qI_{n-1}^+ {}_qA^n = {}_qF_n^-. \tag{32}$$

From this for $n = 0$ and $q = 1$, one obtains $\partial_{[\mu}A_{\nu]} = F_{\mu\nu}$ using $A^0 = \bar{z}z(A_0 + A_3) + z(A_1 + iA_2) + \bar{z}(A_1 - iA_2) + (A_0 - A_3)$.

We start with solving the second equation in (32) for ${}_qA^0$ with ${}_qF_0^-$ given by (29). We write

$$ {}_qA^0 = \bar{z}zA_+ + zA_v + \bar{z}A_{\bar{v}} + A_- \tag{33}$$
$$= \sum_{s=0}^{\infty}\frac{1}{[s]_q!}\,{}_qA_s^0\hat{h}_{s+1}^-,$$

$$A_\kappa = A_\kappa(k,x) = \sum_{s=0}^{\infty}\frac{1}{[s]_q!}A_\kappa^s(k)\hat{h}_{s+1}^-, \tag{34}$$
$$\kappa = \pm, v, \bar{v}.$$

Substituting, we take into account that the action of ${}_qI_{-1}^+$ converts $\hat{h}_{s+1}^-$ into $\hat{h}_s^-$, but this requires $P_{s+1}^-(a,b) = P_s^-(a,b) = P^-(a,b) = Ca + Q(b) = Ca + Bb$ (in the last step, we use the fact that $Q(b)$ has to be linear in b and any constant term would be absorbed in the constants $\hat{\gamma}$). Then, comparing the coefficients of 1, $\bar{z}, \bar{z}^2$, we obtain, respectively,

$$(q^{s+2}A_-^s(k)k_{\bar{v}} + q^{-1-B}A_v^s(k)k_+)\hat{h}_s^- \tag{35}$$
$$= -2d_s(\hat{\gamma}_0^{s-}k_+^2 + \hat{\gamma}_1^{s-}k_+k_{\bar{v}} + \hat{\gamma}_2^{s-}k_{\bar{v}}^2)\hat{h}_s^-,$$
$$(q^{s+2+B}A_-^s(k)k_- - q^{s+1+B+C}A_{\bar{v}}^s k_{\bar{v}}$$
$$- q^{C-2}A_+^s(k)k_+ + q^{-1}A_v^s(k)k_v)\hat{h}_s^-$$
$$= -2d_s[2]_q q^{-C}(\hat{\gamma}_0^{s-}k_vk_+ + \hat{\gamma}_1^{s-}k_vk_{\bar{v}} + \hat{\gamma}_2^{s-}k_-k_{\bar{v}})\hat{h}_s^-,$$
$$(q^{s+1+B}A_{\bar{v}}^s(k)k_- + q^{-2}A_+^s(k)k_v)\hat{h}_s^-$$
$$= 2d_s q^{-2C-1}(\hat{\gamma}_0^{s-}k_v^2 + \hat{\gamma}_1^{s-}k_vk_- + \hat{\gamma}_2^{s-}k_-^2)\hat{h}_s^-,$$
$$d_s \equiv \beta^s/\beta^{s+1}.$$

Note, however, that only two of these three equations are independent when they are compatible (see below). Furthermore, we see that $A_\kappa^s(k)$ should be linear in k and in fact should be given as follows:

$$A_+^s(k) = \lambda_+^s k_v + \nu_+^s k_-, A_-^s(k) = \lambda_-^s k_{\bar{v}} + \nu_-^s k_+, \tag{36}$$
$$A_v^s(k) = \lambda_v^s k_+ + \nu_v^s k_{\bar{v}}, A_{\bar{v}}^s(k) = \lambda_{\bar{v}}^s k_- + \nu_{\bar{v}}^s k_v,$$

where for the constants we have

$$\lambda_v^s = -2d_s q^{1+B}\hat{\gamma}_0^{s-}, \quad \lambda_-^s = -2d_s q^{-s-2}\hat{\gamma}_2^{s-}, \tag{37}$$
$$\nu_v^s = -q^{s+4+B}\nu_-^s - 2d_s q^{2+B}\hat{\gamma}_1^{s-},$$
$$\lambda_+^s = 2d_s q^{1-2C}\hat{\gamma}_0^{s-}, \quad \lambda_{\bar{v}}^s = 2d_s q^{-2C-s-2-B}\hat{\gamma}_2^{s-},$$
$$\nu_+^s = -q^{s+4+B}\nu_{\bar{v}}^s + 2d_s q^{2-2C}\hat{\gamma}_1^{s-},$$
$$C = -B,$$

where the last condition arises from compatibility between Eqs. (35).

Now, we substitute this result for ${}_qA^0$ into the first equation in (32). It turns out that we obtain a result compatible with the general solution (25) only when $B = C = 0$. Thus, in fact ${}_qA^0$ is given in terms of the original components f_{s+1} [cf. (20)]. Furthermore, the action of ${}_qI_{-1}^-$ converts f_{s+1} into $\hat{h}_s^+$ with $P_s^+(a,b) = B_s b = -2b$. The result is

$$\hat{F}_s^+ = {}_qI_{-1}^- {}_qA_s^0 = -q^{s+1}\frac{(\nu_{\bar{v}}^s + \nu_-^s)}{2d_s} \tag{38}$$
$$\times(k_+ - q^{s+2}zk_{\bar{v}})(k_v - q^{s+3}zk_-)\hat{h}_s^+,$$

which is a special case of the general solution (25) with $\hat{\gamma}_0^{s+} = \hat{\gamma}_2^{s+} = 0$, $\hat{\gamma}_1^{s+} = -q^{s+1}(\nu_{\bar{v}}^s + \nu_-^s)/2d_s$. Thus, the resulting ${}_qF_0^+$ is not given in terms of the q-plane wave (only componentwise).

Let us now repeat the calculations in the other order; namely, we solve the first equation in (32) for ${}_qA^0$ with ${}_qF_0^+$ given by (24), (25), but, since we want this to be compatible with what we obtained above, we take $P_s^+(a,b) = -2b$. We use again the decomposition (33) but with f_{s+1} instead of $\hat{h}_{s+1}$. Substituting and comparing the coefficients of $1, z, z^2$, we obtain, respectively,

$$\begin{aligned}(q^{s+1}A_-^s(k)k_v + q^{s+2}A_{\bar v}^s(k)k_+)\hat{h}_s^+ \quad (39)\\ = -2d_s(\hat\gamma_0^{s+}k_v^2 + \hat\gamma_1^{s+}k_vk_+ + \hat\gamma_2^{s+}k_+^2)\hat{h}_s^+,\\ (q^sA_-^s(k)k_- + q^{s+1}A_{\bar v}^s(k)k_{\bar v}\\ - qA_+^s(k)k_+ - q^{-1}A_v^s(k)k_v)\hat{h}_s^+\\ = -2d_s[2]_q(\hat\gamma_0^{s+}k_vk_- + \hat\gamma_1^{s+}k_vk_{\bar v} + \hat\gamma_2^{s+}k_+k_{\bar v})\hat{h}_s^+,\\ (q^{-2}A_+^s(k)k_{\bar v} + q^{-3}A_v^s(k)k_-)\hat{h}_s^+\\ = 2d_s(\hat\gamma_0^{s+}k_-^2 + \hat\gamma_1^{s+}k_-k_{\bar v} + \hat\gamma_2^{s+}k_{\bar v}^2)\hat{h}_s^+.\end{aligned}$$

Now, instead of (36), we have

$$A_+^s(k) = \mu_+^sk_{\bar v} + \nu_+^sk_-,\ A_-^s(k) = \mu_-^sk_v + \nu_-^sk_+, \quad (40)$$
$$A_v^s(k) = \mu_v^sk_- + \nu_v^sk_{\bar v},\ A_{\bar v}^s(k) = \mu_{\bar v}^sk_+ + \nu_{\bar v}^sk_v,$$

where from the constants μ^s, ν^s only six can be determined (due to the gauge freedom). Making some choice, we find

$$\begin{aligned}\mu_-^s = -2d_sq^{-s-1}\hat\gamma_0^{s+}, \quad \mu_{\bar v}^s = -2d_sq^{-s-2}\hat\gamma_2^{s+}, \quad (41)\\ \nu_{\bar v}^s = -\nu_-^s - 2d_sq^{-s-2}\hat\gamma_1^{s+},\\ \mu_v^s = 2d_sq^3\hat\gamma_0^{s+}, \quad \mu_+^s = 2d_sq^2\hat\gamma_2^{s+},\\ \nu_+^s = -\nu_v^s + 2d_sq^2\hat\gamma_1^{s+}.\end{aligned}$$

Now, we can substitute this result for ${}_qA^0$ into the second equation in (32). The action of ${}_qI_{-1}^+$ converts f_{s+1} into f_s, and we obtain for the components

$$\hat F_s^- = -\frac{(\nu_v^sq^{-2} + \nu_-^sq^{s+2})}{2d_s} \quad (42)$$
$$\times (k_+ - q^{-1}k_v\bar z)(k_{\bar v} - k_-\bar z)f_s,$$

which is consistent with the solution (29) with $\hat\gamma_0^{s-} = \hat\gamma_2^{s-} = 0$, $\hat\gamma_1^{s-} = -(\nu_v^sq^{-2} + \nu_-^sq^{s+2})/2d_s$. Thus, in general, the resulting ${}_qF_0^-$ is not given in terms of the q-plane wave (only componentwise).

Finally, we impose that we use the same ${}_qA^0$ for $\hat F^+$ and $\hat F^-$. Then, instead of (36) and (40), we have

$$A_+^s(k) = \nu_+^sk_-, \quad A_-^s(k) = \nu_-^sk_+, \quad (43)$$
$$A_v^s(k) = \nu_v^sk_{\bar v}, \quad A_{\bar v}^s(k) = \nu_{\bar v}^sk_v,$$

where from the four constants in (43) only three can be determined since their sum is zero,

$$\nu_+^s + \nu_-^s + \nu_v^s + \nu_{\bar v}^s = 0, \quad (44)$$

and using (37) and (41) we have

$$\begin{aligned}\nu_v^s = -q^{s+4}\nu_-^s - 2d_sq^2\hat\gamma_1^{s-}, \quad (45)\\ \nu_{\bar v}^s = -\nu_-^s - 2d_sq^{-s-2}\hat\gamma_1^{s+},\\ \nu_+^s = q^{s+4}\nu_-^s + 2d_sq^2(\hat\gamma_1^{s+} + \hat\gamma_1^{s-}).\end{aligned}$$

The disappearance of the constants λ^s, μ^s is consistent with $\hat\gamma_0^{s\pm} = \hat\gamma_2^{s\pm} = 0$. Substituting (41) into (38), (42) we obtain, respectively,

$$\hat F_s^+ = \hat\gamma_1^{s+}q^{-1}(k_+ - q^{s+2}zk_{\bar v}) \quad (46)$$
$$\times (k_v - q^{s+3}zk_-)\hat h_s^+,$$

$$\hat F_s^- = \hat\gamma_1^{s-}(k_+ - q^{-1}k_v\bar z)(k_{\bar v} - k_-\bar z)f_s. \quad (47)$$

We stress that for each s there are only three independent constants: $\hat\gamma_1^{s\pm}$, ν_-^s, the latter entering only the expressions for the q potentials and being a manifestation of the gauge freedom. We can eliminate the A_- components by setting $\nu_-^s = 0$ and/or the A_+ components by setting $\hat\gamma_1^{s+} = -\hat\gamma_1^{s-} - q^{s+2}\nu_-^s/2d_s$.

Finally, we note that we can write ${}_qF_0^-$ in terms of $\widehat{\exp}_q(k,x)$ but not ${}_qF_0^+$ because of the s dependence in the prefactors. If we use the basis (7), the roles of ${}_qF_0^-$ and ${}_qF_0^+$ would be exchanged.

If we want ${}_qF_0^\pm$ on an equal footing, one should consider ${}_qF_0^-$ in the basis (6) and ${}_qF_0^+$ in the basis (7). However, then one should use two different q potentials and furthermore should ensure that the two are not mixed because of Eqs. (32); i.e., the q potential obtained from solving from one of the Eqs. (32) should give zero contribution after substitution into the other. This is easy to ensure through the gauge-freedom constants in the q potentials; e.g., setting $\nu_{\bar v}^s + \nu_-^s = 0$, we find that $\hat F_s^+ = 0$ in (38). Thus, the fields ${}_qF_0^+$ and ${}_qF_0^-$ may be seen as living on different copies of q-Minkowski spacetime, similarly to the two four-dimensional sheets in the Connes–Lott model [15]. We shall investigate this possibility elsewhere.

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Decoupling Braided Tensor Factors*

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Abstract—We briefly report on our result that the braided tensor product algebra of two module algebras $\mathcal{A}_1, \mathcal{A}_2$ of a quasitriangular Hopf algebra H is equal to the ordinary tensor product algebra of $\mathcal{A}_1$ with a subalgebra isomorphic to $\mathcal{A}_2$ and commuting with $\mathcal{A}_1$, provided there exists a realization of H within $\mathcal{A}_1$. As applications of the theorem, we consider the braided tensor product algebras of two or more quantum group covariant quantum spaces or deformed Heisenberg algebras. © *2001 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION AND MAIN THEOREM

As is well known, given two associative unital algebras $\mathcal{A}_1, \mathcal{A}_2$ (over the field $\mathbb{C}$, say), there is an obvious way to build a new algebra $\mathcal{A}$ which is as a vector space the tensor product $\mathcal{A} = \mathcal{A}_1 \otimes_{\mathbb{C}} \mathcal{A}_2$ of the two vector spaces (over the same field) and has a product law such that $\mathcal{A}_1 \otimes \mathbf{1}$ and $\mathbf{1} \otimes \mathcal{A}_2$ are subalgebras isomorphic to $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively: one just completes the product law by postulating the trivial commutation relations

$$(\mathbf{1} \otimes a_2)(a_1 \otimes \mathbf{1}) = (a_1 \otimes \mathbf{1})(\mathbf{1} \otimes a_2) \qquad (1)$$

for any $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$. The resulting algebra is the ordinary tensor product algebra. With a standard abuse of notation, we shall denote in the sequel $a_1 \otimes a_2$ by $a_1 a_2$ for any $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$; consequently, (1) becomes

$$a_2 a_1 = a_1 a_2. \qquad (2)$$

If $\mathcal{A}_1, \mathcal{A}_2$ are module algebras of a Lie algebra $\mathbf{g}$, and we require $\mathcal{A}$ to be too, then (2) has no alternative, because any $g \in \mathbf{g}$ acts as a derivation on the (algebra as well as tensor) product of any two elements, or, in Hopf algebra language, because the coproduct $\Delta(g) = g_{(1)} \otimes g_{(2)}$ (at the rhs we have used Sweedler notation) of the Hopf algebra $H \equiv U\mathbf{g}$ is cocommutative. In this paper, we shall work with right-module algebras (instead of left ones) and denote by $\triangleleft : (a_i, g) \in \mathcal{A}_i \times H \to a_i \triangleleft g \in \mathcal{A}_i$ the right action; the reason is that they are equivalent to left comodule algebras which are used in much of the literature. In [1], we give also the corresponding formulas for the left-module algebras. We recall that a right action $\triangleleft : (a, g) \in \mathcal{A} \times H \to a \triangleleft g \in \mathcal{A}$ by definition fulfills

$$a \triangleleft (gg') = (a \triangleleft g) \triangleleft g', \qquad (3)$$

$$(aa') \triangleleft g = (a \triangleleft g_{(1)})\,(a' \triangleleft g_{(2)}). \qquad (4)$$

If we take as Hopf algebra H a quasitriangular noncocommutative one like the quantum group $U_q\mathbf{g}$, as $\mathcal{A}_i$ some H-module algebras, and we require $\mathcal{A}$ to be a H-module algebra too, then (2) has to be replaced by one of the formulas

$$a_2 a_1 = (a_1 \triangleleft \mathcal{R}^{(1)})\,(a_2 \triangleleft \mathcal{R}^{(2)}), \qquad (5)$$

$$a_2 a_1 = (a_1 \triangleleft \mathcal{R}^{-1(2)})\,(a_2 \triangleleft \mathcal{R}^{-1(1)}). \qquad (6)$$

This yields instead of $\mathcal{A}$ two different braided tensor product algebras [2, 3], which we shall call $\mathcal{A}^+ = \mathcal{A}_1 \underline{\otimes}^+ \mathcal{A}_2$ and $\mathcal{A}^- = \mathcal{A}_1 \underline{\otimes}^- \mathcal{A}_2$, respectively. Here, $\mathcal{R} \equiv \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \in H^+ \otimes H^-$ denotes the so-called universal R matrix of H [4], $\mathcal{R}^{-1}$ its inverse, and $H^{\pm}$ denote the Hopf positive and negative Borel subalgebras of H. If, in particular, H is triangular, then $\mathcal{R}^{-1} = \mathcal{R}_{21}$, $\mathcal{A}^+ = \mathcal{A}^-$, and one has just one braided tensor product algebra. In any case, both $\mathcal{A}^+$ and $\mathcal{A}^-$ go to the ordinary tensor product algebra $\mathcal{A}$ in the limit $q \to 1$, because in this limit $\mathcal{R} \to \mathbf{1} \otimes \mathbf{1}$.

The braided tensor product is a particular example of a more general notion, that of a crossed (or twisted) tensor product [5] of two unital associative algebras.

In view of (5) or (6), studying representations of $\mathcal{A}^{\pm}$ is a more difficult task than just studying the representations of $\mathcal{A}_1, \mathcal{A}_2$ and taking their tensor products. The degrees of freedom of $\mathcal{A}_1, \mathcal{A}_2$ are, so to say, "coupled." One might ask whether one can "decouple" them by a transformation of generators.

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As shown in [1], the answer is positive if there respectively exists an algebra homomorphism φ_1^+ or an algebra homomorphism φ_1^-

$$\varphi_1^\pm : \mathcal{A}_1 \rtimes H^\pm \to \mathcal{A}_1, \tag{7}$$

acting as the identity on $\mathcal{A}_1$, namely, for any $a_1 \in \mathcal{A}_1$

$$\varphi_1^\pm(a_1) = a_1. \tag{8}$$

(Here, $\mathcal{A}_1 \rtimes H^\pm$ denotes the cross product between $\mathcal{A}_1$ and $H^\pm$.) In other words, this amounts to assuming that $\varphi_1^+(H^+)$ [respectively, $\varphi_1^-(H^-)$] provides a realization of H^+ (respectively, H^-) within $\mathcal{A}_1$. In this report, we summarize the main results of [1]. The basic one is the following theorem:

Theorem 1 [1]. *Let $\{H, \mathcal{R}\}$ be a quasitriangular Hopf algebra and H^+, H^- be Hopf subalgebras of H such that $\mathcal{R} \in H^+ \otimes H^-$. Let $\mathcal{A}_1, \mathcal{A}_2$ be respectively an H^+- and an H^--module algebra, so that we can define $\mathcal{A}^+$ as in* (5), *and φ_1^+ be a homomorphism of the type* (7), (8), *so that we can define the map $\chi^+ : \mathcal{A}_2 \to \mathcal{A}^+$ by*

$$\chi^+(a_2) := \varphi_1^+(\mathcal{R}^{(1)})\,(a_2 \triangleleft \mathcal{R}^{(2)}). \tag{9}$$

Alternatively, let $\mathcal{A}_1, \mathcal{A}_2$ be, respectively, an H^-- and an H^+-module algebra, so that we can define $\mathcal{A}^-$ as in (6), *and φ_1^- be a homomorphism of the type* (7), (8), *so that we can define the map χ^-: $\mathcal{A}_2 \to \mathcal{A}^-$ by*

$$\chi^-(a_2) := \varphi_1^-(\mathcal{R}^{-1(2)})\,(a_2 \triangleleft \mathcal{R}^{-1(1)}). \tag{10}$$

In either case, $\chi^\pm$ are then injective algebra homomorphisms and

$$[\chi^\pm(a_2), \mathcal{A}_1] = 0; \tag{11}$$

namely, the subalgebras $\tilde{\mathcal{A}}_2^\pm := \chi^\pm(\mathcal{A}_2) \approx \mathcal{A}_2$ commute with $\mathcal{A}_1$. Moreover, $\mathcal{A}^\pm = \mathcal{A}_1 \otimes \tilde{\mathcal{A}}_2^\pm$.

The last equality means that $\mathcal{A}^\pm$ are, respectively, equal to the ordinary tensor product algebra of $\mathcal{A}_1$ with the subalgebras $\tilde{\mathcal{A}}_2^\pm \subset \mathcal{A}^\pm$, which are isomorphic to $\mathcal{A}_2$!; χ^+ and chi^- will be called "unbraiding" maps.

We recall the content of the hypotheses stated in the theorem. The algebra $\mathcal{A}_1 \rtimes H^\pm$ as a vector space is the tensor product $\mathcal{A}_1 \otimes_{\mathbb{C}} H^\pm$; as an algebra it has subalgebras $\mathcal{A}_1 \otimes \mathbf{1}$, $\mathbf{1} \otimes H$ and has cross commutation relations

$$a_1 g = g_{(1)}\,(a_1 \triangleleft g_{(2)}) \tag{12}$$

for any $a_1 \in \mathcal{A}_1$ and $g \in H^\pm$. $\varphi_1^\pm$ being an algebra homomorphism means that for any $\xi, \xi' \in \mathcal{A}_1 \rtimes H^\pm$ $\varphi_1^\pm(\xi\xi') = \varphi_1^\pm(\xi)\,\varphi_1^\pm(\xi')$. Applying $\varphi_1^\pm$ to both sides of (12) we find $a\varphi^\pm(g) = \varphi^\pm(g_{(1)})(a \triangleleft g_{(2)})$.

Of course, we can use the above theorem iteratively to completely unbraid the braided tensor product algebra of an arbitrary number M of copies of $\mathcal{A}_1$. We end up with the following corollary:

Corollary 1. *If $\mathcal{A}_1$ is a (right-) module algebra of the Hopf algebra H and there exists an algebra homomorphism φ_1^+ of the type* (7), (8), *then there is an algebra isomorphism*

$$\underbrace{\mathcal{A}_1 \underline{\otimes}^+ \cdots \underline{\otimes}^+ \mathcal{A}_1}_{M\ times} \approx \underbrace{\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_1}_{M\ times}. \tag{13}$$

An analogous claim holds for the second braided tensor product if there exists a map φ_1^-.

2. THE UNBRAIDING UNDER THE ∗-STRUCTURES

$\mathcal{A}^+$ (as well as $\mathcal{A}^-$) is a $*$-algebra if H is a Hopf $*$-algebra, $\mathcal{A}_1, \mathcal{A}_2$ are H-module $*$-algebras (we shall use the same symbol $*$ for the $*$-structure on all algebras $H, \mathcal{A}_1$, etc.), and

$$\mathcal{R}^* = \mathcal{R}^{-1} \tag{14}$$

(here, $\mathcal{R}^*$ means $\mathcal{R}^{(1)*} \otimes \mathcal{R}^{(2)*}$). In the quantum group case, (14) requires $|q| = 1$. Under the same assumptions, also $\mathcal{A}_1 \rtimes H$ is a $*$-algebra. If $\varphi_1^\pm$ exist, setting $\varphi_1'^\pm := * \circ \varphi_1^\pm \circ *$, we realize that also $\varphi_1'^\pm$ are algebra homomorphisms of the type (7), (8). If such homomorphisms are uniquely determined, we conclude that $\varphi_1^\pm$ are $*$-homomorphisms. More generally, one may be able to choose $\varphi_1^\pm$ as $*$-homomorphisms. How do the corresponding $\chi^\pm$ behave under $*$?

Proposition 1 [1]. *Assume that the conditions of Theorem 1 for defining χ^+ (respectively, χ^-) are fulfilled. If $\mathcal{R}^* = \mathcal{R}^{-1}$ and φ_1^+ (respectively, φ_1^-) is a $*$-homomorphism, then χ^+ (respectively, χ^-) is too. Consequently, $\mathcal{A}_1$, $\tilde{\mathcal{A}}_2^\pm$ are closed under $*$.*

3. APPLICATIONS

In this section, we illustrate the application of Theorem 1 and Corollary 1 to some algebras $H, \mathcal{A}_i$ for which homomorphisms $\varphi_1^\pm$ are known. H will be the quantum group $U_q sl(N)$ or $U_q so(N)$, and $\mathcal{A}_1$ is the $U_q sl(N)$- or $U_q so(N)$-covariant Heisenberg algebra (Section 3.1), the $U_q so(N)$-covariant quantum space/sphere (Section 3.2). In [1], we have treated also the $U_q so(3)$-covariant q-fuzzy sphere. As generators of H, it will be convenient in either case to use the Faddeev–Reshetikhin–Takhtajan (FRT) generators [6] $\mathcal{L}^{+a}{}_l \in H^+$ and $\mathcal{L}^{-a}{}_l \in H^-$. They are related to $\mathcal{R}$ by

$$\mathcal{L}^{+a}{}_l := \mathcal{R}^{(1)} \rho_l^a(\mathcal{R}^{(2)}), \tag{15}$$

$$\mathcal{L}^{-a}_{\ \ l} := \rho^a_l(\mathcal{R}^{-1(1)})\mathcal{R}^{-1(2)},$$

where $\rho^a_l(g)$ denote the matrix elements of $g \in U_q\mathbf{g}$ in the fundamental N-dimensional representation ρ of $U_q\mathbf{g}$. In fact they provide, together with the square roots of the elements $\mathcal{L}^{\pm i}_{\ \ i}$, a (overcomplete) set of generators of $U_q\mathbf{g}$.

3.1. Unbraiding "Chains" of Braided Heisenberg Algebras

In this subsection, we consider the braided tensor product of $M \geq 2$ copies of the $U_q\mathbf{g}$-covariant deformed Heisenberg algebras $\mathcal{D}_{\epsilon,\mathbf{g}}$, $\mathbf{g} = sl(N), so(N)$. Such algebras have been introduced in [7–9]. They are unital associative algebras generated by x^i, ∂_j fulfilling the relations

$$\mathcal{P}_{a\ hk}^{\ ij}x^hx^k = 0, \quad \mathcal{P}_{a\ hk}^{\ ij}\partial_j\partial_i = 0, \tag{16}$$
$$\partial_i x^j = \delta^i_j + (q\gamma\hat{R})^{\epsilon jk}_{\ \ ih}x^h\partial_k,$$

where $\gamma = q^{1/N}, 1$, respectively, for $\mathbf{g} = sl(N), so(N)$, and the exponent ϵ can take either value $\epsilon = 1, -1$. $\hat{R}$ denotes the braid matrix of $U_q\mathbf{g}$ [given in formulas (A.1)], and the matrix $\mathcal{P}_a$ is the deformed antisymmetric projector appearing in the decomposition (A.2) of the latter. The coordinates x^i transform according to the fundamental N-dimensional representation ρ of $U_q\mathbf{g}$, whereas the "partial derivatives" transform according to the contragredient representation,

$$x^i \triangleleft g = \rho^i_j(g)x^j, \quad \partial_i \triangleleft g = \partial_h\rho^h_i(S^{-1}g). \tag{17}$$

In our conventions, the indices will take the values $i = 1, ..., N$ if $\mathbf{g} = sl(N)$, whereas if $\mathbf{g} = so(N)$ they will take [10] the values $i = -n, \ldots, -1, 0, 1, \ldots n$ for N odd, and $i = -n, \ldots, -1, 1, \ldots n$ for N even; here, $n := [N/2]$ denotes the rank of $so(N)$. We shall enumerate the different copies of $\mathcal{D}_{\epsilon,\mathbf{g}}$ by attaching to them an additional Greek index, e.g., $\alpha = 1, 2, ..., M$. The prescription (6) gives the following "cross" commutation relations between their respective generators ($\alpha < \beta$):

$$x^{\alpha,i}x^{\beta,j} = \hat{R}^{ij}_{hk}x^{\beta,h}x^{\alpha,k}, \ \partial_{\alpha,i}\partial_{\beta,j} = \hat{R}^{kh}_{ji}\partial_{\beta,h}\partial_{\alpha,k}, \tag{18}$$
$$\partial_{\alpha,i}x^{\beta,j} = \hat{R}^{-1jh}_{\ \ \ ik}x^{\beta,k}\partial_{\alpha,h}, \ \partial_{\beta,i}x^{\alpha,j} = \hat{R}^{jh}_{ik}x^{\alpha,k}\partial_{\beta,h}.$$

Algebra homomorphisms $\varphi_1 : \mathcal{A}_1 \rtimes H \to \mathcal{A}_1$ for $H = U_q\mathbf{g}$ and $\mathcal{A}_1$ equal to (a suitable completion of) $\mathcal{D}_{\epsilon,\mathbf{g}}$ have been constructed in [11, 12]. This is the q analog of the well-known fact that the elements of $\mathbf{g}$ can be realized as "vector fields" (first-order differential operators) on the corresponding $\mathbf{g}$-covariant (undeformed) space, e.g., $\varphi_1(E^i_j) = x^i\partial_j - \frac{1}{N}\delta^i_j$ in the $\mathbf{g} = sl(N)$ case. The maps $\varphi_1^\pm$ needed to apply Theorem 1 are simply the restrictions to $\mathcal{A}_1 \rtimes H^\pm$ of φ_1 of [11, 12].

The unbraiding procedure is recursive. We just describe the first step, which consists in using the homomorphism $\varphi_1^\pm$ to unbraid the first copy from the others. According to the main theorem, if we set

$$y^{1,i} \equiv x^{1,i}, \qquad \partial_{y,1,a} \equiv \partial_{1,a}, \tag{19}$$
$$y^{\alpha,i} \equiv \chi^-(x^{\alpha,i}) \tag{20}$$
$$= \varphi_1(\mathcal{R}^{-1(2)})\rho^i_j(\mathcal{R}^{-1(1)})x^{\alpha,j} = \varphi_1(\mathcal{L}^{-i}_{\ \ j})x^{\alpha,j},$$

then

$$\partial_{y,\alpha,a} \equiv \chi^-(\partial_{\alpha,a}) \tag{21}$$
$$= \varphi_1(S\mathcal{R}^{-1(2)})\rho^d_a(\mathcal{R}^{-1(1)})\partial_{\alpha,d} = \varphi_1(S\mathcal{L}^{-d}_{\ \ a})\partial_{\alpha,d}$$

with $\alpha > 1$. By Theorem 1, $y^{1,i} \equiv x^{1,i}$ and $\partial_{y,1,i} \equiv \partial_{1,i}$ will commute with $y^{2,i}, ..., y^{M,i}$ and $\partial_{y,2,i}, ..., \partial_{y,M,i}$. As we see, the FRT generators are special because they appear in the redefinitions (20), (21). The explicit expression of $\varphi_1(\mathcal{L}^{-i}_{\ \ j})$ in terms of $x^{1,i}, \partial_{1,a}$ for $U_qsl(2), U_qso(3)$ has been given in [1]. For different values of N, it can be found from the results of [11, 12] by passing from the generators adopted there to the FRT generators.

By completely analogous arguments, one determines the alternative unbraiding procedure for the braided tensor product stemming from prescription (5).

$\mathcal{A}_1 \rtimes H$ is a $*$-algebra and the map φ_1 is a $*$-homomorphism for both q real and $|q| = 1$. But $\varphi_1^\pm$ are $*$-homomorphisms only for $|q| = 1$. In the latter case, the $*$-structure of $\mathcal{A}_1$ is

$$(x^i)^* = x^i, \quad (\partial_i)^* = -q^{\pm N}g^{kh}g_{ki}\partial_h. \tag{22}$$

Applying Proposition 1 in the latter case, we find that $*$ maps $\mathcal{A}_1$ as well as each of the commuting subalgebras $\tilde{\mathcal{A}}^\pm_i$ into itself.

3.2. Unbraiding "Chains" of Braided Quantum Euclidean Spaces or Spheres

In this subsection, we consider the braided tensor product of $M \geq 2$ copies of the quantum Euclidean space $\mathbb{R}^N_q$ [6] (the $U_qso(N)$-covariant quantum space), i.e., of the unital associative algebra generated by x^i fulfilling the relations $(16)_1$, or of the quotient space of $\mathbb{R}^N_q$ obtained by setting $r^2 := x^ix_i = 1$ [the quantum $(N-1)$-dimensional sphere S^{N-1}_q]. (Thus, these will be subalgebras of the Heisenberg algebras $\mathcal{D}_{+,sl(N)}, \mathcal{D}_{+,so(N)}$ considered in the previous subsection.) Again, the multiplet (x^i) carries the fundamental N-dimensional representation ρ of $U_qso(N)$. As before, we shall enumerate the different copies of the quantum Euclidean space or sphere by attaching an additional Greek index to them, e.g., $\alpha = 1, 2, ..., M$. The prescription (6) gives the cross commutation relations $(18)_1$.

According to [13], to define $\varphi_1^{\pm}$ (for $q \neq 1$), one actually needs a slightly enlarged version of $\mathbb{R}_q^N$ (or S_q^{N-1}). One has to introduce some new generators $\sqrt{r_a}$, with $0 \leq a \leq N/2$, together with their inverses $(\sqrt{r_a})^{-1}$, requiring that

$$r_a^2 = \sum_{h=-a}^{a} x^h x_h = \sum_{h=-a}^{a} g_{hk} x^h x^k \tag{23}$$

(note that, having set $n := [N/2]$, r_n^2 coincides with r^2, whereas for odd N $r_0^2 = (x^0)^2$, so we are adding also $(x^0)^{-1}$ as a new generator). In fact, the commutation relations involving these new generators can be fixed consistently and turn out to be simply q-commutation relations. r plays the role of "deformed Euclidean distance" of the generic "point" (x^i) of $\mathbb{R}_q^N$ from the "origin"; r_a is the "projection" of r on the "subspace" $x^i = 0$, $|i| > a$. In the previous equation, g_{hk} denotes the "metric matrix" of $SO_q(N)$, $g_{ij} = g^{ij} = q^{-\rho_i} \delta_{i,-j}$, which is a $SO_q(N)$-isotropic tensor and a deformation of the ordinary Euclidean metric. Here, $(\rho_i) := (n - 1/2, \ldots, 1/2, 0, -1/2, \ldots, 1/2 - n)$ for N odd and $(\rho_i) := (n-1, \ldots, 0, 0, \ldots, 1-n)$ for N even. g_{ij} is related to the trace projector appearing in (A.2) by $\mathcal{P}_{t\,kl}^{\,ij} = (g^{sm} g_{sm})^{-1} g^{ij} g_{kl}$. The extension of the action of H to these extra generators is uniquely determined by the constraints the latter fulfil. In the case of even N, one needs to include also the FRT generators $\mathcal{L}^{+1}_{\ 1}$, $\mathcal{L}^{-1}_{\ 1}$ (which are generators of H) among the generators of $\mathcal{A}_1$. In the Appendix, we recall the explicit form of $\varphi_1^{\pm}$ in the present case. Note that the maps $\varphi_1^{\pm}$ have no analog in the "undeformed" case ($q = 1$), because $\mathcal{A}_1$ is Abelian, whereas H is not.

The unbraiding procedure is recursive. The first step consists of using the homomorphism $\varphi_1^{\pm}$ found in [13] to unbraid the first copy from the others. Following Theorem 1, we perform the change of generators $(19)_1$, (20) in $\mathcal{A}^-$. In view of formula (A.3), we thus find

$$y^{1,i} := x^{1,i}, \quad y^{\alpha,i} := g^{ih} [\mu_h^1, x^{1,k}]_q g_{kj}\, x^{\alpha,j}, \quad \alpha > 1. \tag{24}$$

The suffix 1 in μ_a^1 means that the special elements μ_a defined in (A.4), must be taken as elements of the first copy. In view of (A.4) we see that $g^{ih}[\mu_h^1, x^{1,k}]_q g_{kj}$ are rather simple polynomials in x^i and r_a^{-1}, homogeneous of total degree 1 in the coordinates x^i and r_a.

Using the results given in the Appendix, we give now the explicit expression of $(24)_2$ for $N = 3$:

$$y^{\alpha,-} = -qh\gamma_1 \frac{r}{x^0} x^{\alpha,-}, \tag{25}$$

$$y^{\alpha,0} = \sqrt{q}(q+1) \frac{1}{x^0} x^+ x^{\alpha,-} + x^{\alpha,0},$$

$$y^{\alpha,+} = \frac{\sqrt{q}(q+1)}{h\gamma_1 r x^0} (x^+)^2 x^{\alpha,-} + \frac{q^{-1}+1}{h\gamma_1 r} x^+ x^{\alpha,0} - \frac{1}{qh\gamma_1 r} x^0 x^{\alpha,+}$$

for any $\alpha = 2, ..., M$. Here, we have set $x^i \equiv x^{1,i}$, $h \equiv \sqrt{q} - 1/\sqrt{q}$; replaced for simplicity the values $-1, 0, 1$ of the indices by the ones $-, 0, +$; and denoted by $\gamma_1 \in \mathbb{C}$ a free parameter. By Theorem 1, $y^{1,i} \equiv x^{1,i}$ commutes with $y^{2,i}, ..., y^{M,i}$.

The alternative unbraiding procedure for the braided tensor product algebra stemming from prescription (5) arises by iterating the change of generators

$$y'^{M,i} := x^{M,i}, \quad y'^{\alpha,i} := \varphi_M(\mathcal{L}^{+i}_{\ \ j}) x^{\alpha,j} \tag{26}$$
$$= g^{ih} [\bar{\mu}_h^M, x^{M,k}]_{q^{-1}} g_{kj}\, x^{\alpha,j}, \quad \alpha < M.$$

The special elements $\bar{\mu}_a$ are defined in (A.4), and the suffix M means that we must take $\bar{\mu}_a$ as an element of the Mth copy of $\mathbb{R}_q^N$ (or S_q^{N-1}). $y^{M,i} \equiv x^{M,i}$ commutes with $y^{1,i}, ..., y^{M-1,i}$.

When $|q| = 1$, by a suitable choice (A.6) of $\gamma_1, \bar{\gamma}_1$, as well as of the other free parameters $\gamma_a, \bar{\gamma}_a$ appearing in the definitions of $\varphi^{\pm}$ for $N > 3$, one can make $\varphi^{\pm}$ into $*$-homomorphisms. Applying Proposition 1 in the latter case, we find that $*$ maps $\mathcal{A}_1$ as well as each of the commuting subalgebras $\tilde{\mathcal{A}}_i^{\pm}$ into itself.

APPENDIX

The braid matrix $\hat{R}$ is related to $\mathcal{R}$ by $\hat{R}^{ij}_{hk} \equiv R^{ji}_{hk} := (\rho_h^j \otimes \rho_k^i)\mathcal{R}$. With the indices' convention described in Sections 3.2 and 3.1, $\hat{R}$ is given by

$$\hat{R} = q^{-1/N} \Big[q \sum_i e_i^i \otimes e_i^i + \sum_{i \neq j} e_i^j \otimes e_j^i + k \sum_{i<j} e_i^i \otimes e_j^j \Big], \tag{A.1}$$

$$\hat{R} = q \sum_{i \neq 0} e_i^i \otimes e_i^i + \sum_{\substack{i \neq j, -j \\ \text{or } i=j=0}} e_i^j \otimes e_j^i + q^{-1} \sum_{i \neq 0} e_i^{-i} \otimes e_{-i}^i + k \sum_{i<j} (e_i^i \otimes e_j^j - q^{-\rho_i + \rho_j} e_i^{-j} \otimes e_{-i}^j)$$

for $\mathbf{g} = sl(N), so(N)$, respectively. Here, e^i_j is the $N \times N$ matrix with all elements equal to zero except for a 1 in the ith column and jth row. The braid matrix of $so(N)$ admits the orthogonal projector decomposition

$$\hat{R} = q\mathcal{P}_s - q^{-1}\mathcal{P}_a + q^{1-N}\mathcal{P}_t. \tag{A.2}$$

$\mathcal{P}_a$, $\mathcal{P}_t$, and $\mathcal{P}_s$ are the q-deformed antisymmetric, trace, and trace-free symmetric projectors. There are just two projectors $\mathcal{P}_a, \mathcal{P}_s$ in decomposition of the braid matrix of $sl(N)$; the latter is obtained from (A.2) just by deleting the third term.

We now recall the explicit form of maps $\varphi^{\pm}$ for the quantum Euclidean spaces or spheres. We introduce the short-hand notation $[A, B]_x = AB - xBA$. In [13], we have found algebra homomorphisms $\varphi^{\pm} : \mathbb{R}^N_q \rtimes U^{\pm}_q so(N) \to \mathbb{R}^N_q$. The images of φ^- (respectively, φ^+) on the negative (respectively, positive) FRT generators read

$$\varphi^-(\mathcal{L}^{-i}{}_j) = g^{ih}[\mu_h, x^k]_q g_{kj}, \tag{A.3}$$
$$\varphi^+(\mathcal{L}^{+i}{}_j) = g^{ih}[\bar{\mu}_h, x^k]_{q^{-1}} g_{kj},$$

where

$$\mu_0 = \gamma_0 (x^0)^{-1}, \quad \bar{\mu}_0 = \bar{\gamma}_0 (x^0)^{-1} \text{ for } N \text{ odd};$$
$$\mu_{\pm 1} = \gamma_{\pm 1}(x^{\pm 1})^{-1}\mathcal{L}^{\mp 1}{}_1, \quad \bar{\mu}_{\pm 1} = \bar{\gamma}_{\pm 1}(x^{\pm 1})^{-1}\mathcal{L}^{\pm 1}{}_1 \quad \text{for } N \text{ even}; \tag{A.4}$$
$$\mu_a = \gamma_a r^{-1}_{|a|} r^{-1}_{|a|-1} x^{-a},$$
$$\bar{\mu}_a = \bar{\gamma}_a r^{-1}_{|a|} r^{-1}_{|a|-1} x^{-a} \quad \text{otherwise},$$

and $\gamma_a, \bar{\gamma}_a \in \mathbb{C}$ are normalization constants fulfilling the conditions

$$\gamma_0 = -q^{-1/2}h^{-1}, \quad \bar{\gamma}_0 = q^{1/2}h^{-1} \quad \text{for } N \text{ odd};$$
$$\gamma_1\gamma_{-1} = \begin{cases} -q^{-1}h^{-2} & \text{for } N \text{ odd} \\ k^{-2} & \text{for } N \text{ even}; \end{cases} \tag{A.5}$$
$$\bar{\gamma}_1\bar{\gamma}_{-1} = \begin{cases} -qh^{-2} & \text{for } N \text{ odd} \\ k^{-2} & \text{for } N \text{ even}; \end{cases}$$
$$\gamma_a\gamma_{-a} = -q^{-1}k^{-2}\omega_a\omega_{a-1}, \;\; \bar{\gamma}_a\bar{\gamma}_{-a} = -qk^{-2}\omega_a\omega_{a-1} \quad \text{for } a > 1.$$

Here, $k := q - 1/q$, $\omega_a := (q^{\rho_a} + q^{-\rho_a})$. Incidentally, for odd N one can choose the free parameters $\gamma_a, \bar{\gamma}_a$ in such a way that φ^+, φ^- can be "glued" into an algebra homomorphism $\varphi : \mathbb{R}^N_q \rtimes U_q so(N) \to \mathbb{R}^N_q$ [13]. When $|q| = 1$, the $*$-structure is given by $(x^i)^* = x^i$ [see (22)]. It turns out that $\varphi^{\pm}$ are $*$-homomorphisms if, in addition,

$$\gamma^*_{\pm 1} = -\gamma_{\pm 1} \quad \text{if } N \text{ even}; \tag{A.6}$$

$$\gamma^*_a = -\gamma_a \begin{cases} 1 & \text{if} \quad a < 0 \\ q^{-2} & \text{if} \quad a > 0 \end{cases} \quad \text{otherwise.}$$

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Possible Contractions of Quantum Orthogonal Groups*

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Abstract—Possible contractions of quantum orthogonal groups which correspond to different choices of primitive elements of Hopf algebra are considered and all allowed contractions in Cayley–Klein scheme are obtained. Quantum deformations of kinematical groups have been investigated and have shown that quantum analogs of (complex) Galilei group $G(1,3)$ do not exist in our scheme. © *2001 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

Contraction of Lie groups (algebras) is a method of obtaining new Lie groups (algebras) from some initial ones with the help of passage to the limit [1]. One may define contraction of algebraic structure $(M,*)$ as the map $\phi_\epsilon : (M,*) \to (N,*')$, where $(N,*)$ is an algebraic structure of the same type, isomorphic to $(M,*)$ for $\epsilon \neq 0$ and nonisomorphic to the initial one for $\epsilon = 0$. Except for Lie group (algebra) contractions, graded contractions [2, 3] are known, which preserve the grading of Lie algebra. Under contractions of bialgebra [4], Lie algebra structure and cocommutator are conserved. Hopf algebra (or quantum group) contractions are introduced in such a way [5, 6] that in the limit $\epsilon \to 0$ new expressions for coproduct, counit, and antipode are consistent with Hopf algebra axioms.

Contractions as a passage to limit correspond with physical intuition. At the same time, it is desirable to investigate contractions of algebraic structures with the help of pure algebraic tools. It is possible for classical and quantum groups and algebras if one takes into consideration Pimenov algebra $\mathbf{D}(\iota)$ with nilpotent commutative generators [7].

In the present paper, contractions of quantum orthogonal groups are studied and the groups under consideration are regarded according to [8] as an algebra of noncommutative functions but with nilpotent generators. From the contraction viewpoint, the Hopf algebra structure of quantum orthogonal group is more rigid as compared with the group one. Possible contractions essentially depend on the choice of primitive elements of Hopf algebra. We have considered all variants of such choice for the quantum orthogonal group $SO_q(N)$ and for each variant have found all admissible contractions in Cayley–Klein scheme.

2. ORTHOGONAL CAYLEY–KLEIN GROUPS

Let us define Pimenov algebra $\mathbf{D}_n(\iota;\mathbb{C})$ as an associative algebra with unit over complex number field and with nilpotent commutative generators ι_k, $\iota_k^2 = 0$, $\iota_k\iota_m = \iota_m\iota_k \neq 0$, $k \neq m$, $k,m = 1,\ldots,n$. The general element of $\mathbf{D}_n(\iota;\mathbb{C})$ is in the form

$$d = d_0 + \sum_{p=1}^{n} \sum_{k_1<\ldots<k_p} d_{k_1\ldots k_p}\iota_{k_1}\ldots\iota_{k_p}, \qquad (1)$$
$$d_0, d_{k_1\ldots k_p} \in \mathbb{C}.$$

It is possible to define the division of nilpotent generator ι_k by itself, namely, $\iota_k/\iota_k = 1$, $k = 1,\ldots,n$. Let us stress that the division of different nilpotent generators ι_k/ι_p, $k \neq p$, as well as the division of complex number by nilpotent generators a/ι_k, $a \in \mathbb{C}$, is not defined.

Let $SO(N;\mathbb{C})$ be an orthogonal matrix group. Its elements are matrices $A = (a_{kp}) \in M_N(\mathbb{C})$, $A^t = A^{-1}$, and under the action $y' = Ay$ on vectors y of complex vector space O_N the quadratic form $y^ty = \sum_{k=1}^N y_k^2$ is preserved, where y_k are Cartesian components of y. Sometimes, it is convenient to consider an orthogonal group in a so-called "symplectic" basis. Transformation from Cartesian to symplectic basis $x = Dy$ is made by matrix D, which is a solution of the equation

$$D^tC_0D = I, \qquad (2)$$

where $C_0 \in M_N$, $(C_0)_{ik} = \delta_{ik'}$, and $k' = N+1-k$. Equation (2) has many solutions; take one of them,

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namely,

$$D = \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 & -i\tilde{C}_0 \\ 0 & \sqrt{2} & 0 \\ \tilde{C}_0 & 0 & iI \end{pmatrix}, \; N = 2n+1, \qquad (3)$$

where $n \times n$ matrix $\tilde{C}_0$ is similar to C_0. For $N = 2n$, the matrix D is given by (3) but without the middle column and row. Matrices B of $SO(N;\mathbb{C})$ in the symplectic basis are obtained from A by similarity transformation $B = DAD^{-1}$ and are subject to orthogonality relations $B^t C_0 B = C_0$. The quadratic form $x^t C_0 x$ is invariant under the action $x' = Bx$.

Complex orthogonal Cayley–Klein group $SO(N;j;\mathbb{C})$ is defined as the group of transformations $\xi'(j) = A(j)\xi(j)$ of complex vector space $O_N(j)$ with Cartesian coordinates $\xi^t(j) = (\xi_1, (1,2)\xi_2, \ldots, (1,N)\xi_N)^t$, which preserve the quadratic form $\mathrm{inv}(j) = \xi^t(j)\xi(j) = \xi_1^2 + \sum_{k=2}^N (1,k)^2 \xi_k^2$, where $j = (j_1, \ldots, j_{N-1})$; each parameter j_k takes *two* values $j_r = 1, \iota_r$, $r = 1, \ldots, N-1$; and

$$(\mu,\nu) = \prod_{l=\min(\mu,\nu)}^{\max(\mu,\nu)-1} j_l, \quad (\mu,\mu) = 1. \qquad (4)$$

Let us stress that Cartesian coordinates of $O_N(j)$ are special elements of Pimenov algebra $\mathbf{D}_{N-1}(j;\mathbb{C})$. Cayley–Klein group $SO(N;j;\mathbb{C})$ in turn may be realized as a matrix group whose elements are taken from algebra $\mathbf{D}_{N-1}(j;\mathbb{C})$ and consist of the $N \times N$ matrices $(A(j))_{kp} = (k,p)a_{kp}$, $a_{kp} \in \mathbb{C}$. Matrices $A(j)$ are subject to the additional j-orthogonality relations $A(j)A^t(j) = A^t(j)A(j) = I$.

The passage to the symplectic description is made by matrices which are solutions of Eq. (2). Let us regard the matrix $D_\sigma = DV_\sigma$, where $V_\sigma \in M_N$, $(V_\sigma)_{ik} = \delta_{\sigma_i,k}$, and $\sigma \in S(N)$ is a permutation of the Nth order. It is easy to verify that D_σ is again a solution to Eq. (2). Then, in the symplectic basis, the orthogonal Cayley–Klein group $SO(N;j;\mathbb{C})$ is described by the matrices $B_\sigma(j) = D_\sigma A(j) D_\sigma^{-1}$ with the additional relations of j orthogonality $B_\sigma(j) C_0 B_\sigma^t(j) = B_\sigma^t(j) C_0 B_\sigma(j) = C_0$.

It should be noted that for orthogonal groups ($j = 1$) the use of different matrices D_σ makes no sense because all Cartesian coordinates of O_N are equivalent up to a choice of its enumerations. A different situation is for Cayley–Klein groups ($j \neq 1$). Cartesian coordinates $(1,k)\xi_k$, $k = 1, \ldots, N$, for nilpotent values of some or all parameters j_k are different elements of the algebra $\mathbf{D}_{N-1}(j;\mathbb{C})$; therefore, the same group $SO(N;j;\mathbb{C})$ may be realized by matrices B_σ with a different disposition of nilpotent generators among their elements. Matrix elements of $B_\sigma(j)$ are as follows:

$$\begin{aligned}
(B_\sigma)_{n+1,n+1} &= b_{n+1,n+1}, \\
(B_\sigma)_{kk} &= b_{kk} + i\tilde{b}_{kk}(\sigma_k, \sigma_{k'}), \\
(B_\sigma)_{k'k'} &= b_{kk} - i\tilde{b}_{kk}(\sigma_k, \sigma_{k'}), \\
(B_\sigma)_{kk'} &= b_{k'k} - i\tilde{b}_{k'k}(\sigma_k, \sigma_{k'}), \\
(B_\sigma)_{k'k} &= b_{k'k} + i\tilde{b}_{k'k}(\sigma_k, \sigma_{k'}), \\
(B_\sigma)_{k,n+1} &= b_{k,n+1}(\sigma_k, \sigma_{n+1}) - i\tilde{b}_{k,n+1}(\sigma_{n+1}, \sigma_{k'}), \\
(B_\sigma)_{k',n+1} &= b_{k,n+1}(\sigma_k, \sigma_{n+1}) + i\tilde{b}_{k,n+1}(\sigma_{n+1}, \sigma_{k'}), \\
(B_\sigma)_{n+1,k} &= b_{n+1,k}(\sigma_k, \sigma_{n+1}) + i\tilde{b}_{n+1,k}(\sigma_{n+1}, \sigma_{k'}), \\
(B_\sigma)_{n+1,k'} &= b_{n+1,k}(\sigma_k, \sigma_{n+1}) \\
&\quad - i\tilde{b}_{n+1,k}(\sigma_{n+1}, \sigma_{k'}), \qquad (5) \\
(B_\sigma)_{kp} &= b_{kp}(\sigma_k, \sigma_p) + b'_{kp}(\sigma_{k'}, \sigma_{p'}) \\
&\quad + i\tilde{b}_{kp}(\sigma_k, \sigma_{p'}) - i\tilde{b}'_{kp}(\sigma_{k'}, \sigma_p), \\
(B_\sigma)_{kp'} &= b_{kp}(\sigma_k, \sigma_p) - b'_{kp}(\sigma_{k'}, \sigma_{p'}) \\
&\quad - i\tilde{b}_{kp}(\sigma_k, \sigma_{p'}) - i\tilde{b}'_{kp}(\sigma_{k'}, \sigma_p), \\
(B_\sigma)_{k'p} &= b_{kp}(\sigma_k, \sigma_p) - b'_{kp}(\sigma_{k'}, \sigma_{p'}) \\
&\quad + i\tilde{b}_{kp}(\sigma_k, \sigma_{p'}) + i\tilde{b}'_{kp}(\sigma_{k'}, \sigma_p), \\
(B_\sigma)_{k'p'} &= b_{kp}(\sigma_k, \sigma_p) + b'_{kp}(\sigma_{k'}, \sigma_{p'}) \\
&\quad - i\tilde{b}_{kp}(\sigma_k, \sigma_{p'}) + i\tilde{b}'_{kp}(\sigma_{k'}, \sigma_p), \; k \neq p.
\end{aligned}$$

Here, $b, b', \tilde{b}, \tilde{b}' \in \mathbb{C}$ may be easily expressed by matrix elements of A.

3. CONTRACTIONS OF QUANTUM ORTHOGONAL GROUPS

3.1. Formal Definition of the Quantum Group $SO_v(N;j;\sigma)$

The starting point of the definition of quantum groups [8] is an algebra $\mathbb{C}\langle T_{ik}\rangle$ of noncommutative polynomials of N^2 variables. We start with an algebra $\mathbf{D}\langle (T_\sigma)_{ik}\rangle$ of noncommutative polynomials of N^2 variables, which are elements of the direct product $\mathbf{D}_{N-1}(j) \otimes \mathbb{C}\langle T_{ik}\rangle$. More precisely, the elements $(T_\sigma)_{ik}$ are obtained from the elements $(B_\sigma(j))_{ik}$ of Eq. (5) by the replacement of commutative variables $b, b', \tilde{b}, \tilde{b}'$ with the noncommutative variables t, t', τ, τ', respectively. One introduces additionally the transformation of the deformation parameters $q = e^z$ as follows: $z = Jv$, where v is a new deformation parameter and J is some product of parameters j, for the present unknown. Let $\tilde{R}_v(j), C(j)$ be matrices which are obtained from the corresponding matrices of [8] by the replacement of deformation parameter z with Jv :

$$R_v(j) = R_q(z \to Jv), \quad C(j) = C(z \to Jv). \quad (6)$$

The commutation relations of the generators $T_\sigma(j)$ are defined by

$$R_v(j)T_1(j)T_2(j) = T_2(j)T_1(j)R_v(j), \qquad (7)$$

where $T_1(j) = T_\sigma(j) \otimes I$, $T_2(j) = I \otimes T_\sigma(j)$ and the additional relations of (v, j) orthogonality

$$T_\sigma(j)C(j)T^t_\sigma(j) = T^t_\sigma(j)C(j)T_\sigma(j) = C(j) \qquad (8)$$

are imposed.

One defines the quantum orthogonal Cayley–Klein group $SO_v(N; j; \sigma)$ as the quotient algebra of $\mathbf{D}\langle(T_\sigma)_{ik}\rangle$ by relations (7), (8). Formally, $SO_v(N; j; \sigma)$ is a Hopf algebra with the following coproduct Δ, counit ϵ, and antipode S :

$$\Delta T_\sigma(j) = T_\sigma(j)\dot{\otimes}T_\sigma(j), \quad \epsilon(T_\sigma(j)) = I, \qquad (9)$$
$$S(T_\sigma(j)) = C(j)T^t_\sigma(j)C^{-1}(j).$$

As far as only second diagonal elements of the matrix C are different from zero and for $q = 1$ this matrix is equal to C_0, we have the symplectic description of $SO_v(N; j; \sigma)$.

3.2. Allowed Contractions of $SO_v(N; j; \sigma)$

The formal definition of $SO_v(N; j; \sigma)$ should be a real definition of quantum group if the proposed construction is a consistent Hopf algebra structure under nilpotent values of some or all parameters j. Counit $\epsilon(t_{n+1,n+1}) = 1$, $\epsilon(t_{kk}) = 1$, $k = 1, \ldots, n$, and $\epsilon(t) = \epsilon(\tau) = 0$ for the remaining generators do not restrict the values of j. Parameters j are arranged in the expressions for coproduct Δ exactly as in matrix product of $B_\sigma(j)$, and as far as the last ones form the group $SO(N; j; \mathbb{C})$ for any values of j, no restrictions follow from the coproduct. A different situation is with the antipode S. Really, for elements $(T_\sigma)_{k'k} = t_{k'k} + i\tau_{k'k}(\sigma_k, \sigma_{k'})$, $k = 1, \ldots, n$, the antipode is obtained as

$$S((T_\sigma)_{k'k}) = (T_\sigma)_{k'k}e^{2J\rho_k v} \qquad (10)$$

and depends both on ρ_k and for the present undetermined factor J. An antipode is an antihomomorphism of Hopf algebra and therefore has to transform $T_\sigma(j)$ to a matrix with the same distribution of the nilpotent parameters j in its elements; i.e., the right and the left parts of Eq. (10) must be identical elements of $\mathbf{D}_{N-1}(j) \otimes \mathbb{C}\langle T_{ik}\rangle$. For $J = 1$, this condition holds for any values of the parameters j. The case $J \neq 1$ requires additional discussion.

The next condition which must be taken into account is the (v, j)-orthogonality relations (8). We require that the number of equations in (8) not be changed as compared with the initial quantum group. It is possible when nilpotent generators appear in Eq. (8) either with powers greater than or equal two (and then the corresponding terms are equal to zero) or as homogeneous multipliers. Taking into account all these arguments and using the explicit expressions for antipode and (v, j) orthogonality, we can find possible contractions of quantum orthogonal groups, which are described by the following theorems.

Theorem 1. *If the deformation parameter is not transformed $J = 1$, then the following maximal n-dimensional contraction of the orthogonal quantum group $SO_v(N; j; \sigma)$, $N = 2n + 1$ is allowed:*

$$j_{2s} = \iota_{2s},\ s = 1, \ldots, m,\ j_{2r+1} = \iota_{2r+1},$$
$$r = m, \ldots, n-1,\ 0 \le m \le n,$$

for example, for permutation σ,

$$\sigma_{n+1} = 2m + 1,\ \sigma_s = 2s - 1,\ \sigma_{s'} = 2s,$$
$$s = 1, \ldots, m,$$
$$\sigma_r = 2r,\ \sigma_{r'} = 2r + 1,\ r = m + 1, \ldots, n.$$

Theorem 2. *If the deformation parameter is not transformed $J = 1$, then the following maximal n-dimensional contraction of the quantum orthogonal group $SO_v(N; j; \sigma)$, $N = 2n$ is allowed:*

$$j_{2s} = \iota_{2s},\ s = 1, \ldots, m-1,\ j_{2p-1} = \iota_{2p-1},$$
$$p = m, \ldots, u,$$
$$j_{2r} = \iota_{2r},\ r = u, \ldots, n-1,\ 1 \le m \le u \le n,$$

for example, for permutation σ,

$$\sigma_n = 2m - 1,\ \sigma_{n'} = 2u,$$
$$\sigma_s = 2s - 1,\ \sigma_{s'} = 2s,\ s = 1, \ldots, m-1,$$
$$\sigma_p = 2p,\ \sigma_{p'} = 2p + 1,\ p = m, \ldots, u-1,$$
$$\sigma_r = 2r + 1,\ \sigma_{r'} = 2r,\ r = u, \ldots, n-1.$$

Remark 1. It should be noted that any permutation with the properties $(\sigma_k, \sigma_{k'}) = 1$, $k = 1, \ldots, n$ (or $n - 1$) may be taken as σ .

Remark 2. Admissible contractions for number of parameters j_k less than n are obtained from Theorems 1 and 2 by setting part of $j_{2s}, j_{2p-1}, j_{2r}, j_{2r+1}$ equal to one.

We return to the antipode (10) for $J \neq 1$. As far as $\rho_{n+1} = 0$ for $N = 2n + 1$, and $\rho_n = \rho_{n'} = 0$ for $N = 2n$, we shall regard these two cases separately.

Theorem 3. *If the deformation parameter is transformed ($J \neq 1$), then the following contractions of the quantum orthogonal group $SO_v(N; j; \sigma)$, $N = 2n + 1$ are allowed:*

1. For $J = j_{n+1}$,

(a) $j_{n+1} = \iota_{n+1}$ if $1 < \sigma_{n+1} < n + 1$;

(b) $j_{n+1} = \iota_{n+1}$ $j_1 = 1, \iota_1$, if $\sigma_{n+1} = 1$.

2. For $J = j_n$,

(a) $j_n = \iota_n$ if $n + 1 < \sigma_{n+1} < 2n + 1$;

(b) $j_n = \iota_n$ $j_{2n} = 1, \iota_{2n}$, if $\sigma_{n+1} = 2n + 1$.

3. For $J = j_n j_{n+1}$*,*

$j_n = 1, \iota_n,\ j_{n+1} = 1, \iota_{n+1}$ *if* $\sigma_{n+1} = n + 1$.

Theorem 4. *If the deformation parameter is transformed* ($J \neq 1$)*, then the following contractions of the quantum orthogonal group* $SO_v(N; j; \sigma)$*,* $N = 2n$ *are allowed:*

1. For $J = j_n$*,*

(a) $j_n = \iota_n$ *if* $\sigma_n > 1, \sigma_{n'} < 2n$;

(b) $j_n = \iota_n, j_1 = 1, \iota_1$ *if* $\sigma_n = 1, \sigma_{n'} < 2n$;

(c) $j_n = \iota_n,\ j_{2n-1} = 1, \iota_{2n-1}$ *if* $\sigma_n > 1,\ \sigma_{n'} = 2n$;

(d) $j_n = \iota_n,\ j_1 = 1, \iota_1,\ j_{2n-1} = 1, \iota_{2n-1}$ *if* $\sigma_n = 1, \sigma_{n'} = 2n$.

2. For $J = j_{n-1}$*,*

(a) $j_{n-1} = \iota_{n-1}$ *if* $\sigma_{n'} < 2n$;

(b) $j_{n-1} = \iota_{n-1},\ j_{2n-1} = 1, \iota_{2n-1}$ *if* $\sigma_n < 2n - 1, \sigma_{n'} = 2n$;

(c) $j_{n-1} = \iota_{n-1},\ j_{2n-2} = 1, \iota_{2n-2},\ j_{2n-1} = 1, \iota_{2n-1}$ *if* $\sigma_n = 2n - 1, \sigma_{n'} = 2n$.

3. For $J = j_{n+1}$*,*

(a) $j_{n+1} = \iota_{n+1}$ *if* $\sigma_n > 1$;

(b) $j_{n+1} = \iota_{n+1}, j_1 = 1, \iota_1$ *if* $\sigma_n = 1, \sigma_{n'} > 2$;

(c) $j_{n+1} = \iota_{n+1},\ j_1 = 1, \iota_1,\ j_2 = 1, \iota_2$ *if* $\sigma_n = 1, \sigma_{n'} = 2$.

4. For $J = j_{n-1} j_n$*,*

(a) $j_{n-1} = \iota_{n-1}, j_n = \iota_n$ *if* $\sigma_{n'} < 2n$;

(b) $j_{n-1} = \iota_{n-1},\ j_n = \iota_n,\ j_{2n-1} = 1, \iota_{2n-1}$ *if* $\sigma_{n'} = 2n$.

5. For $J = j_n j_{n+1}$*,*

(a) $j_n = \iota_n, j_{n+1} = \iota_{n+1}$ *if* $\sigma_n > 1$;

(b) $j_1 = 1, \iota_1, j_n = \iota_n, j_{n+1} = \iota_{n+1}$ *if* $\sigma_n = 1$.

6. For $J = j_{n-1} j_n j_{n+1}$*,*

(a) $j_{n-1} = 1, \iota_{n-1},\ j_n = 1, \iota_n,\ j_{n+1} = 1, \iota_{n+1}$ *if* $\sigma_n = n, \sigma_{n'} = n + 1$.

Hopf algebra $SO_q(N; j; \sigma)$, $N = 2n + 1$ has n primitive elements which correspond to n diagonal 2×2 submatrices: $\mathrm{diag}((B_\sigma)_{kk}, (B_\sigma)_{k'k'}) = \mathrm{diag}(b_{kk} + i\tilde{b}_{kk}(\sigma_k, \sigma_{k'}), b_{kk} - i\tilde{b}_{kk}(\sigma_k, \sigma_{k'}))$, $k = 1, \ldots, n$ [see (5)]. If the deformation parameter z is fixed ($J = 1$) under contractions, then all primitive elements of the contracted quantum orthogonal group correspond to Euclidean rotation $SO(2)$. If the deformation parameter is transformed $z = \iota v$, then all primitive elements correspond to Galilei transformation $SO(2; j = \iota) = G(1, 1)$. The same is true for the contracted quantum groups $SO_q(N; j; \sigma)$, $N = 2n$. Let us note that contractions of quantum orthogonal algebras with different sets of primitive elements have been discussed in [4, 9].

Quantum orthogonal groups have contractions with the same nilpotent parameters j both with a fixed deformation parameter and with a transformed one. For example, the quantum group $SO_q(2n + 1; j; \sigma)$ for even $n = 2p$ at $\sigma_{n+1} = 1$ according to Theorem 1 has contraction $j_n = \iota_n$, $j_{n+1} = \iota_{n+1}$, $J = 1$ and according to 3 of Theorem 3 has the same two-dimensional contraction, but $J = \iota_n \iota_{n+1}$. Let us stress that the cases $J = 1$ and $J \sim \iota$ are realized for different sets of primitive elements in Hopf algebra.

Let permutation σ be identical, i.e., $\sigma_k = k$, $\sigma_{k'} = k'$, $\sigma_{n+1} = n + 1$. It follows from Theorems 1 and 2 that there are no contractions of $SO_q(N; j)$ with fixed deformation parameter ($J = 1$). For $N = 2n + 1$ from Theorem 3, we obtain three possible contractions: $j_n = 1, \iota_n$, $j_{n+1} = 1, \iota_{n+1}$ (both parameters j_n and j_{n+1} independently take nilpotent values); and the deformation parameter is transformed with $J = j_n j_{n+1}$. For $N = 2n$ from Theorem 4, we obtain seven admissible contractions: $j_{n-1} = 1, \iota_{n-1}$, $j_n = 1, \iota_n$, $j_{n+1} = 1, \iota_{n+1}$, where the deformation parameter is multiplied by $J = j_{n-1} j_n j_{n+1}$. Just these allowed contractions should be considered in [10].

From the contraction viewpoint, the Hopf algebra structure of quantum orthogonal group is more rigid as compared with a group one. Cayley–Klein groups are obtained from $SO(N; j)$ for all nilpotent values of parameters j_k, $k = 1, \ldots, N - 1$, whereas their quantum deformations exist only for some of them ($\leq [N/2]$). It should be noted that, among quantum orthogonal groups contracted for equal number of parameters j, there may be isomorphic, as Hopf algebras, quantum groups. Quantum-group isomorphism is not considered in this paper.

4. QUANTUM COMPLEX KINEMATIC GROUPS

Kinematic groups are motion groups of the maximal homogeneous four-dimensional (one time and three space coordinates) spacetime models [11]. All these groups may be obtained from the real group $SO(5; \mathbb{R})$ by contractions and analytic continuations [7]. There are three types of kinematics: nonrelativistic—Galilei $G(1, 3) = SO(5; \iota_1, \iota_2, 1, 1)$ with zero curvature and Newton $N^{\pm}(1, 3) = SO(5; j_1 = 1, i; \iota_2, 1, 1)$ with positive and negative curvature, respectively; relativistic—Poincaré $P(1, 3) = SO(5; \iota_1, i, 1, 1)$ with zero curvature and (anti) de Sitter $S^{\pm}(1, 3) = SO(5; j_1 = 1, i; i, 1, 1)$ with (positive) negative curvature; exotic—Carroll $C^0(1, 3) = SO(5; \iota_1, 1, 1, \iota_4)$ with zero curvature and $C^{\pm}(1, 3) = SO(5; j_1 = 1, i; 1, 1, \iota_4)$ with positive and negative curvature.

The groups $N^{\pm}(1, 3)$ are the real forms of the complex Newton group $N(4)$, the Poincaré group $P(1, 3)$ is the real form of the complex Euclid group

$E(4)$, and the groups $C^{\pm}(1,3)$ are the real forms of the complex Carroll group $C(4)$. In this paper, the quantum deformations of the complex orthogonal groups are considered; therefore, with the help of contractions quantum analogs of the complex kinematic groups may be obtained. Possible contractions of the complex quantum groups $SO_q(5;j;\sigma)$ are described by Theorems 1 and 3 for $N=5$. If deformation parameter remains unchanged ($J=1$), then we have the quantum analogs of Euclidean group $E_q(4)$, Newton group $N_q(4)$, and Carroll group $C_q(4)$. If the deformation parameter is transformed under contraction $z=\iota_2 v$, then we have one more quantum deformation of Newton group $N_v(4)$, which is not isomorphic to the previous one. Two primitive elements of $N_q(4)$ correspond to the elliptic translation along the time axis t and to the rotation in the space plane $\{r_2, r_3\}$ (both are isomorphic to $SO(2)$), while primitive elements of $N_v(4)$ correspond to the flat translation along the spatial axis r_2 and to Galilei boost in the spacetime plane $\{t, r_1\}$ (both are isomorphic to Galilei group $SO(2; j_2=\iota_2)=G(1,1)$). We did not obtain the quantum deformations of the complex Galilei $G(4)$ and Carroll $C^0(4)$ groups.

According to the correspondence principle, a new physical theory must include an old one as a particular case. For spacetime theory, this principle is realized as the chain of limit transitions: general relativity passes to special relativity when spacetime curvature tends to zero, and special relativity passes to classical physics when light velocity tends to infinity. For kinematical groups, this corresponds to the chain of contractions

$$S^{\pm}(1,3) \stackrel{K\to 0}{\longrightarrow} P(1,3) \stackrel{c\to\infty}{\longrightarrow} G(1,3). \qquad (11)$$

As was mentioned above, there is no quantum deformation of the complex Galilei group; therefore, it is not possible to construct the standard quantum analog of the full chain of contractions (11), even at the level of complex groups. This means that (at least standard) quantum deformation of the flat nonrelativistic spacetime does not exist.

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SYMPOSIUM ON QUANTUM GROUPS

On Quantization of r Matrices for Belavin–Drinfeld Triples*

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Abstract—We suggest a formula for quantum universal R matrices corresponding to quasitriangular classical r matrices classified by Belavin and Drinfeld for all simple Lie algebras. The R matrices are obtained by twisting the standard universal R matrix. © *2001 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

Classical quasitriangular r matrices for semisimple Lie algebras are classified by Belavin–Drinfeld triples [1]. The Belavin–Drinfeld triple $(\Gamma_1, \Gamma_2, \tau)$ for a simple Lie algebra $\mathbf{g} = \mathbf{g}^+ \oplus \mathbf{h} \oplus \mathbf{g}^-$ consists of the following data: Γ_1, Γ_2 are subsets of the set Γ of simple roots of the algebra $\mathbf{g}$, and τ is a one-to-one mapping: $\Gamma_1 \to \Gamma_2$ such that $\langle\tau(\alpha), \tau(\beta)\rangle = \langle\alpha, \beta\rangle$ and $\tau^k(\alpha) \neq \alpha$ for any $\alpha, \beta \in \Gamma_1$ and any natural k. The corresponding quantum R matrices should have the form

$$R_{12} = F_{21}\mathcal{R}_{12}F_{12}^{-1}, \tag{1}$$

where $\mathcal{R}$ is the standard universal Drinfeld–Jimbo R matrix for the Lie algebra $\mathbf{g}$. The twisting operator satisfies the cocycle equation

$$F_{12}(\Delta \otimes id)F = F_{23}(id \otimes \Delta)F. \tag{2}$$

Therefore, the problem of quantization is reduced to the problem of finding the twisting operator F_{12} for each Belavin–Drinfeld triple. In the present paper, we suggest a formula for the twisting operator F_{12}. We present the twisting operator in a factorized form

$$F_{12} = F_{12}^{(N)} \cdot F_{12}^{(N-1)} \cdots F_{12}^{(2)} \cdot F_{12}^{(1)} \cdot K, \tag{3}$$

where the factors $F^{(k)}$ are special canonical elements defined by the powers of the one-to-one map τ, and the operator K belongs to $q^{\mathbf{h}\otimes\mathbf{h}}$. A different formula for the operator F_{12} was given in [2]. We shall say several words about the differences at the end of the present paper.

Our approach heavily uses the modified Cartan–Weyl basis for $U_q(\mathbf{g})$, and the plan of our paper is as follows. The definition of the modified simple root generators is contained in Section 2. In Section 3, we give an interpretation of Belavin–Drinfeld triples in terms of the modified basis. In Section 4, a modified Cartan–Weyl basis is introduced. The twisting operator F_{12} is constructed in Section 5. Finally, in Section 6, several examples are presented.

Everywhere below, we assume the deformation parameter q to be generic (not a root of unity).

2. MODIFIED BASIS FOR QUANTUM UNIVERSAL ENVELOPING ALGEBRAS

Consider a quantum universal enveloping algebra $U_q(\mathbf{g})$ with relations (see, e.g., [3])

$$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j,$$

$$[e_i, f_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}, \tag{4}$$

and Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} (e_i)^k e_j (e_i)^{1-a_{ij}-k} = 0, \tag{5}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} (f_i)^k f_j (f_i)^{1-a_{ij}-k} = 0, \tag{6}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}},$$

a_{ij} is the Cartan matrix for $\mathbf{g}$, $K_i = q^{d_i h_i}$, and d_i are the smallest positive integers (from the set $1, 2, 3$) such that $d_i a_{ij} = a_{ij}^{(s)}$ is the symmetric matrix. The

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algebra $U_q(\mathbf{g})$ is a Hopf algebra with the comultiplication

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i,$$
$$\Delta(e_i) = e_i \otimes K_i + 1 \otimes e_i, \quad (7)$$
$$\Delta(f_i) = f_i \otimes 1 + K_i^{-1} \otimes f_i.$$

The antipode and the counit are

$$S(h_i) = -h_i, \quad S(e_i) = -e_i K_i^{-1},$$
$$S(f_i) = -K_i f_i, \quad \epsilon(h_i) = \epsilon(e_i) = \epsilon(f_i) = 0.$$

Any operator $K \in q^{\mathbf{h}\otimes\mathbf{h}}$,

$$K = q^{(\sum_{ij} b_{ij} h_i \otimes h_j)}, \quad (8)$$

for arbitrary numerical matrix b_{ij}, obviously satisfies the cocycle Eq. (2),

$$K_{12}(\Delta \otimes id)K = K_{23}(id \otimes \Delta)K. \quad (9)$$

Therefore, one can twist the comultiplication by K:

$$\widetilde{\Delta}(a) := K\Delta(a)K^{-1}. \quad (10)$$

We change the basis in the algebra $U_q(\mathbf{g})$ by introducing new generators

$$E_i = X_i e_i, \quad F_i = f_i Y_i, \quad (11)$$

where $X_i = \exp(\sum_j x_{ij} h_j)$, $Y_i = \exp(\sum_j y_{ij} h_j)$, and x_{ij}, y_{ij} are some numerical matrices. We require that the comultiplication (10) for the new generators (11) have the following form:

$$\widetilde{\Delta}(E_i) = K\Delta(E_i)K^{-1} = E_i \otimes R_i^+ + 1 \otimes E_i, \quad (12)$$
$$\widetilde{\Delta}(F_i) = K\Delta(F_i)K^{-1} = F_i \otimes 1 + R_i^- \otimes F_i.$$

Equations (12) relate operators X_i, Y_i, and K.

A comparison of (7) and (12) gives

$$X_i = q^{-\sum_{mn} h_m b_{mn} a_{ni}} \equiv q^{-(hba)_i},$$
$$Y_i = q^{\sum_{mn} h_m b_{nm} a_{ni}} \equiv q^{(h\overline{b}a)_i},$$

and

$$R_i^{\pm} = X_i K_i^{\pm 1} Y_i = K_i^{\pm 1} q^{-(h(b-\overline{b})a)_i}, \quad (13)$$
$$R_i^+ = K_i^2 R_i^-,$$

where $\overline{b}_{mn} = b_{nm}$ is the transposed matrix.

The relations (4) and Serre relations (5), (6) for the quantum algebra $U_q(\mathbf{g})$ in terms of the new generators (11) take the form

$$[E_i, F_j] = \delta_{ij} \frac{R_i^+ - R_i^-}{q^{d_i} - q^{-d_i}}, \quad (14)$$

$$R_i^{\pm} E_j = q^{\pm a_{ij}^{(s)} + A_{ij}} E_j R_i^{\pm}, \quad (15)$$
$$R_i^{\pm} F_j = q^{\mp a_{ij}^{(s)} - A_{ij}} F_j R_i^{\pm},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} \quad (16)$$
$$\times\, q^{-kA_{ij}} (E_i)^k E_j (E_i)^{1-a_{ij}-k} = 0,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} \quad (17)$$
$$\times\, q^{kA_{ij}} (F_i)^k F_j (F_i)^{1-a_{ij}-k} = 0,$$

with a skewsymmetric matrix $A_{ij} = (\overline{a}(b-\overline{b})a)_{ij}$.

In the sequel, we shall use q commutators: $[A, B]_\mu := AB - \mu BA$. Relations (16), (17) can be conveniently rewritten in terms of q commutators. For example, for $a_{ij} = 0$, the relations $[e_i, e_j] = 0 = [f_i, f_j]$ are rewritten as $[E_i, E_j]_{q^{A_{ij}}} = 0$, $[F_i, F_j]_{q^{-A_{ij}}} = 0$, while for $a_{ij} = -1$ we have

$$[[E_i, E_j]_\mu, E_i]_\nu = 0 = [E_j, [E_i, E_j]_\mu]_\nu, \quad (18)$$
$$[[F_i, F_j]_\nu, F_i]_\mu = 0 = [F_j, [F_i, F_j]_\nu]_\mu,$$

where $\mu = q^{d_i + A_{ij}}$, $\nu = q^{d_i - A_{ij}}$.

Remark. The modified basis for multiparametric twistings of $U_q(\mathbf{g})$ has been considered by Hodges [4].

3. MODIFIED BASIS AND BELAVIN–DRINFELD TRIPLES

All the data from the Belavin–Drinfeld triple can be conveniently interpreted in terms of the modified basis for a suitable matrix b_{ij}:

Proposition. *Let Γ be the set of simple roots of $\mathbf{g}$, Γ_1, and Γ_2 subsets of Γ and τ a one-to-one mapping: $\Gamma_1 \to \Gamma_2$. Then the following equations for the matrix b_{ij}*

$$R_{\alpha_i}^+ = R_{\tau(\alpha_i)}^-, \ \ \forall \alpha_i \in \Gamma_1, \quad (19)$$

where $R_{\alpha_i}^{\pm} \equiv R_i^{\pm}$, admit a solution if and only if the triple $(\Gamma_1, \Gamma_2, \tau)$ is the Belavin–Drinfeld triple.

Proof. Assume that a solution of Eq. (19) exists. We then need to prove that the mapping τ satisfies the following conditions:

(1) For any $\alpha \in \Gamma_1$ there is a natural k for which $\tau^k(\alpha) \notin \Gamma_1$. (20)

(2) For any $\alpha, \beta \in \Gamma_1$, $\langle \tau(\alpha), \tau(\beta) \rangle = \langle \alpha, \beta \rangle$. (21)

The condition (20) means that τ has no cycles: $\tau^k(\alpha) \neq \alpha$ for all $\alpha \in \Gamma_1$ and $k > 0$.

Indeed, assume that τ has a cycle, $\tau^k(\alpha) = \alpha$ for some $\alpha \in \Gamma_1$ and a natural k. Take a minimal k with this property.

Then $R_{\tau^k(\alpha)}^+ = R_\alpha^+$ and Eqs. (13) imply

$$R_\alpha^+ = R_{\tau(\alpha)}^- = K_{\tau(\alpha)}^{-2} R_{\tau(\alpha)}^+ \quad (22)$$
$$\ldots = \left(\prod_{i=1}^k K_{\tau^i(\alpha)}^{-2} \right) R_{\tau^k(\alpha)}^+ = \left(\prod_{i=1}^k K_{\tau^i(\alpha)}^{-2} \right) R_\alpha^+.$$

Therefore,

$$\left(\prod_{i=1}^{k} K_{\tau^i(\alpha)}^{-2}\right) = 1,$$

which contradicts the independence of generators in the Cartan subalgebra of $U_q(\mathbf{g})$. Thus, $\tau^k(\alpha)$ never equals α, which proves the condition (20).

To prove the condition (21), note that Eq. (19) is equivalent to the following condition on the skew-symmetric matrix $A_{mn} = (\overline{a}(b - \overline{b})a)_{mn}$:

$$A_{im} + A_{m\tau(i)} + a^{(s)}_{im} + a^{(s)}_{\tau(i)m} = 0, \quad (23)$$

where the subscript m runs over all simple roots, while i numerates only roots from Γ_1. Equation (23) is obtained by commuting both sides of Eq. (19) with e_m (or f_m). Here, it is important that q is not a root of unity.

For indices i, m corresponding to roots $\alpha_i, \alpha_m \in \Gamma_1$, Eq. (23) can be rewritten in the following three equivalent forms:

$$A_{i\tau(m)} + A_{\tau(m)\tau(i)} + a^{(s)}_{i\tau(m)} + a^{(s)}_{\tau(i)\tau(m)} = 0, \quad (24)$$

$$A_{mi} + A_{i\tau(m)} + a^{(s)}_{mi} + a^{(s)}_{\tau(m)i} = 0, \quad (25)$$

$$A_{m\tau(i)} + A_{\tau(i)\tau(m)} + a^{(s)}_{m\tau(i)} + a^{(s)}_{\tau(m)\tau(i)} = 0. \quad (26)$$

The combinations (23)+(25) and (24)+(26) of the equations are, respectively,

$$2a^{(s)}_{im} = -a^{(s)}_{\tau(m)i} - a^{(s)}_{\tau(i)m} - A_{m\tau(i)} - A_{i\tau(m)}, \quad (27)$$

$$2a^{(s)}_{\tau(i)\tau(m)} = -a^{(s)}_{i\tau(m)} - a^{(s)}_{m\tau(i)} - A_{m\tau(i)} - A_{i\tau(m)}. \quad (28)$$

Therefore, $a^{(s)}_{im} = a^{(s)}_{\tau(i)\tau(m)}$, which is equivalent to the second condition $\langle\tau(\alpha_i), \tau(\alpha_m)\rangle = \langle\alpha_i, \alpha_m\rangle$ for the Belavin–Drinfeld triple.

Remark 1. The difference of Eqs. (24) and (25) gives the following relation on the matrix A_{ij}:

$$A_{im} - A_{\tau(i)\tau(m)} = a^{(s)}_{mi} + a^{(s)}_{\tau(m)i} - a^{(s)}_{i\tau(m)} - a^{(s)}_{\tau(i)\tau(m)} = 0. \quad (29)$$

This shows that the map τ does not change the modified basis.

Remark 2. Consider two sequences of sets

$$\Gamma_1 = \Gamma_1^{(0)} \supset \Gamma_1^{(1)} \supset \Gamma_1^{(2)} \ldots \supset \Gamma_1^{(N)} \supset \Gamma_1^{(N+1)} = \emptyset, \quad (30)$$

$$\Gamma_2 = \Gamma_2^{(0)} \supset \Gamma_2^{(1)} \supset \Gamma_2^{(2)} \ldots \supset \Gamma_2^{(N)},$$

defined by

$$\Gamma_1^{(k+1)} = \Gamma_1^{(k)} \cap \Gamma_2^{(k)}, \quad \Gamma_1^{(k)} \xrightarrow{\tau} \Gamma_2^{(k)}.$$

We assume that the set $\Gamma_1^{(N)}$ is not empty. The number N is called the degree of the triple $(\Gamma_1, \Gamma_2, \tau)$.

Introduce a set $\widetilde{\Gamma}_1^{(k)} = \tau^{-k-1}(\Gamma_2^{(k)}) \in \Gamma_1$. Then, the mapping τ^k: $\widetilde{\Gamma}_1^{(k-1)} \xrightarrow{\tau^k} \Gamma_2^{(k-1)} \neq \emptyset$ also defines a Belavin–Drinfeld triple

$$(\widetilde{\Gamma}_1^{(k-1)}, \Gamma_2^{(k-1)}, \tau^k). \quad (31)$$

4. MODIFIED CARTAN–WEYL BASIS AND NORMAL ORDER OF ROOTS

Let Δ_+ be the system of all positive roots of $\mathbf{g}$ with respect to Γ. A construction of Cartan–Weyl basis in terms of the modified generators E_i and F_i is analogous to the usual procedure for $U_q(\mathbf{g})$ (see [5]).

Recall the notion of a normal (convex) order in Δ_+: the set Δ_+ is ordered normally if any root γ which is a sum of roots α and β is placed between α and β.

We write $\alpha < \beta$ if the root α is located to the left of the root β. For $\alpha < \beta$, the interval between roots α and β is denoted by $\{\alpha, \beta\}$.

Given a normal order in Δ_+, the modified Cartan–Weyl basis is constructed by the following inductive procedure. The generators for the simple roots are already defined. For a composite root γ, take a minimal interval $\{\alpha, \beta\}$, $\alpha < \beta$, with $\gamma = \alpha + \beta$ ("minimal" means that there is no subinterval $\{\widetilde{\alpha}, \widetilde{\beta}\} \subset \{\alpha, \beta\}$ for which $\gamma = \widetilde{\alpha} + \widetilde{\beta}$). Assume that generators E_α, E_β, F_α, and F_β were defined at previous steps. Then, generators E_γ and F_γ are defined by

$$E_\gamma = [E_\alpha, E_\beta]_\mu = E_\alpha E_\beta - \mu E_\beta E_\alpha, \quad (32)$$
$$F_\gamma = [F_\alpha, F_\beta]_\nu = F_\alpha F_\beta - \nu F_\beta F_\alpha,$$

where

$$\mu = q^{-\langle\alpha,\beta\rangle + \langle\alpha, A\beta\rangle}, \quad \nu = q^{-\langle\alpha,\beta\rangle - \langle\alpha, A\beta\rangle},$$

and A is the operator with the matrix A_{ij}:

$$\langle\alpha_i, A\alpha_j\rangle = A_{ij}.$$

If there are several possible minimal intervals $\{\alpha, \beta\}$ for which $\gamma = \alpha + \beta$, the definitions (32) give proportional results.

Note. For the case $A_{ij} = 0$, the definition (32) of composite roots does not coincide with the definition in [5] since we use the comultiplication (7), which is different from the comultiplication in [5].

5. TWISTING OPERATORS F_{12} FOR BELAVIN–DRINFELD TRIPLES

For a given simple Lie algebra $\mathbf{g}$, fix a normal order in Δ_+.

We need the expression for the inverse of the universal R matrix for the algebra $U_q(\mathbf{g})$:

$$\mathcal{R}^{-1} = \prod_{\beta\in\Delta_+}^{\rightarrow} \exp_{q_\beta}(-\lambda a_\beta (e_\beta \otimes f_\beta)) \cdot K^{(0)}, \quad (33)$$

where $q_\alpha = q^{\langle\alpha,\alpha\rangle}$, $\lambda = q - q^{-1}$, and $K^{(0)} \in q^{\mathbf{h}\otimes\mathbf{h}}$. The product in Eq. (33) is the ordered product corresponding to the chosen normal order of roots. For precise values of the constants a_β (see [5, 6]). The function $\exp_q$ is the standard q exponent,

$$\exp_q(u) = \prod_{n=0}^{\infty}(1 + (q-1)uq^n)^{-1} = \sum_{k=0}^{\infty}\frac{u^k}{k_q!}, \quad (34)$$

$$k_q = \frac{q^k - 1}{q - 1}.$$

Let $(\Gamma_1, \Gamma_2, \tau)$ be a Belavin–Drinfeld triple of degree N. Define elements $F^{(k)}$ by

$$F_{12}^{(k)} = \prod_{\beta\in\Delta_+^{(k)}}^{\rightarrow} \exp_{q_\beta}(-\lambda a_\beta (E_\beta \otimes F_{\tau^k(\beta)})), \quad (35)$$

where in the ordered product we keep terms corresponding to only those roots β for which $\tau^k(\beta)$ is defined [that is, the element e_β belongs to the subalgebra with generators from the subset $\widetilde{\Gamma}_1^{(k)}$ defined in (31)]. This is reflected in the notation $\beta \in \Delta_+^{(k)}$.

The expression (35) can be given the form

$$F_{12}^{(k)} = (1 \otimes T^k)(K\mathcal{R}^{-1}(K^{(0)})^{-1}K^{-1}), \quad (36)$$

where the operator T on the elements F_β is defined by $T(F_\beta) = F_{\tau(\beta)}$ wherever $\tau(\beta)$ is defined; $T(F_\beta) = 0$ otherwise. The operator K corresponds to the solution of Eq. (19) for the given Belavin–Drinfeld triple.

Theorem. *For the quantum algebra $U_q(\mathbf{g})$ and the Belavin–Drinfeld triple $(\Gamma_1, \Gamma_2, \tau)$ of degree N, the universal twisting element F_{12} is*

$$F_{12} = F_{12}^{(N)} \cdot F_{12}^{(N-1)} \cdots F_{12}^{(2)} \cdot F_{12}^{(1)} \cdot K \equiv \widetilde{F}_{12} \cdot K \quad (37)$$

with the factors $F_{12}^{(k)}$ defined in (35).

We sketch the proof shortly. It is based on explicit formulas for the coproduct of elements $F_{12}^{(k)}$:

$$(\widetilde{\Delta} \otimes id)F_{12}^{(k)} = F_{23}^{(k)}(K_{23}^{(k)})^{-1}F_{13}^{(k)}K_{23}^{(k)}, \quad (38)$$
$$(id \otimes \widetilde{\Delta})F_{12}^{(k)} = F_{12}^{(k)}(K_{12}^{(k)})^{-1}F_{13}^{(k)}K_{12}^{(k)}$$

for some elements $K^{(k)} \in q^{\mathbf{h}\otimes\mathbf{h}}$. The comultiplication $\widetilde{\Delta}$ is twisted as in (10).

Next, one can verify the following identities:

$$\widetilde{F}_{23}^{(k)}\widetilde{F}_{13}^{(k+m)}\widetilde{F}_{12}^{(m)} = \widetilde{F}_{12}^{(m)}\widetilde{F}_{13}^{(k+m)}\widetilde{F}_{23}^{(k)}, \quad (39)$$

where $\widetilde{F}^{(k)} = F^{(k)} \cdot (K^{(k)})^{-1}$.

With the help of (38) and (39), it is straightforward to check the cocycle condition (2).

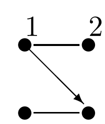
Remark 1. Another expression for the twisting element F was suggested in [2]. The expression in [2] has a factorized form as well. However, the factors $F^{(i)}$ are different; one of the differences is that each factor in [2] contains terms from $q^{\mathbf{h}\otimes\mathbf{h}}$. In our expression (37), all terms from $q^{\mathbf{h}\otimes\mathbf{h}}$ are collected; the price is the appearance of the modified basis.

Remark 2. The element F in (37) satisfies the following analog of the linear ABRR equation [7]:

$$(1 \otimes T)(F_{12}\mathcal{R}^{-1}(K^{(0)})^{-1}K^{-1}) = F_{12}K^{-1}. \quad (40)$$

6. EXAMPLES

(i) $U_q(sl(3))$ case (see [4]). Here, we have only one nontrivial Belavin–Drinfeld triple:



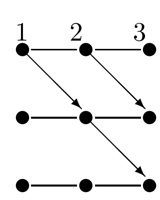
This Cremmer–Gervais-type triple has degree 1 and the basic relations (19) which define this triple are reduced to one equation $R_1^+ = R_2^-$. The antisymmetric matrix A_{ij} is

$$A_{ij} = \delta_{i,j+1} - \delta_{j,i+1}, \quad (41)$$

with $1 \le i, j \le 2$. The corresponding universal twisting element (37) has the form

$$F_{12} = F_{12}^{(1)} \cdot K = \exp_{q^2}(-\lambda E_1 \otimes F_2) \cdot K. \quad (42)$$

(ii) Cremmer–Gervais $U_q(sl(4))$ case. For this case, the triple is given by the following diagram:



It has degree 2. The basic relations (19) which define this triple are $R_1^+ = R_2^-$, $R_2^+ = R_3^-$. The matrix A_{ij} is given by (41), now with $1 \le i, j \le 3$. The corresponding universal twisting element (37) has the form

$$F_{12} = F_{12}^{(2)} \cdot F_{12}^{(1)} \cdot K, \quad (43)$$

where

$$F_{12}^{(2)} = \exp_{q^2}(-\lambda E_1 \otimes F_3), \quad (44)$$

$$F_{12}^{(1)} = \exp_{q^2}(-\lambda E_1 \otimes F_2) \quad (45)$$

$$\times \exp_{q^2}(q^{-1}\lambda [E_{12}] \otimes [F_{23}]_{q^2}) \exp_{q^2}(-\lambda E_2 \otimes F_3).$$

Here, $[E_{12}] = E_1 E_2 - E_2 E_1$ and $[F_{23}]_{q^2} = F_2 F_3 - q^2 F_3 F_2$.

Remark. One can directly check that the universal twisting element in (42) and (43) obeys the cocycle conditions (2). For (42), this check requires only the basic equation for the q exponent, $\exp_q(y)\exp_q(x) = \exp_q(x+y)$ if $xy = qyx$. For (43), one needs two more quantum identities. The first one is the famous pentagon identity (see, e.g., [8] and references therein)

$$\exp_q(u)\exp_q(v) = \exp_q(v)\exp_q([u,v])\exp_q(u), \quad (46)$$

where the operators u and v satisfy the commutation (Serre) relations $u[u,v] = q[u,v]u$, $v[u,v] = q^{-1}[u,v]v$. The second identity is

$$\exp_{q^2}(E)\exp_{q^2}(-R^+)\exp_{q^2}(F) \quad (47)$$
$$= \exp_{q^2}(F)\exp_{q^2}(-R^-)\exp_{q^2}(E),$$

where E, F, and $R^{\pm}$ generate the algebra

$$[E,F] = (R^+ - R^-), \quad [R^+, R^-] = 0,$$
$$R^{\pm}E = q^{\pm 2} E R^{\pm}, \quad R^{\pm}F = q^{\mp 2} F R^{\pm}.$$

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SYMPOSIUM ON QUANTUM GROUPS

On q-Laplace Operator and q-Harmonic Polynomials*

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Abstract—A q-Laplace operator and q-harmonic polynomials on the quantum vector space are studied. A q analog of associated spherical harmonics are constructed. They constitute an orthonormal basis of the space of q-harmonic polynomials. A q analog of separation of variables is given.

1. INTRODUCTION

In the classical analysis, harmonic polynomials are defined by the equation $\Delta p = 0$, where Δ is the Laplace operator and p is a polynomial on the Euclidean space $E_n \sim \mathbb{R}^n$. The space $\mathcal{H}$ of all harmonic polynomials decomposes as a direct sum of the subspaces $\mathcal{H}_m$ of homogeneous harmonic polynomials of degree m: $\mathcal{H} = \bigoplus_{m=0}^{\infty} \mathcal{H}_m$. The Laplace operator Δ on E_n commutes with the natural action of the rotation group $SO(n)$ on this space. This means that the subspaces $\mathcal{H}_m$ are invariant with respect to $SO(n)$. The irreducible representation T_m of the group $SO(n)$ with highest weight $(m, 0, \ldots, 0)$ is realized on $\mathcal{H}_m$. The Laplace operator Δ permits separations of variables on the spaces $\mathcal{H}_m$. They correspond to different coordinate systems (spherical, polyspherical, etc.) on E_n. To different coordinate systems, there correspond different separations of variables and they are determined by chains of subgroups of the group $SO(n)$ (see [1], Chap. 10, for details of this correspondence). The basis of the space $\mathcal{H}_m$ in separated variables consists of products of certain orthogonal polynomials multiplied by r^m, where r is the radius. These polynomials (considered only on the sphere S^{n-1}) are matrix elements of the class-1 irreducible representations of $SO(n)$ belonging to the zero column.

In this paper, we give a q deformation of the classical theory described above. Instead of the Euclidean space, we take the quantum vector space.

The q-Laplace operator Δ_q on the algebra of functions $\mathcal{A}$ on the quantum vector space is defined in terms of q derivatives. In our case, the nonstandard q deformation $U'_q(so_n)$ of the universal enveloping algebra $U(so_n)$, described in [2], plays the role of the rotation group $SO(n)$.

q-Harmonic polynomials on the quantum vector space are defined as elements p of $\mathcal{A}$ for which $\Delta_q p = 0$. By using the algebra $U'_q(so_n)$, we construct for q-harmonic polynomials the theory similar to the theory for classical harmonic polynomials. Our constructions use essentially the results of paper [3], where the operator Δ_q was defined.

Everywhere below, we suppose that q is a positive number.

2. q-HARMONIC POLYNOMIALS ON THE QUANTUM VECTOR SPACE

We denote by $\mathcal{A} \equiv \mathbb{C}[x_1, x_2, \ldots, x_n]$ the associative algebra (with unity) generated by elements $x_1, x_2, \ldots, x_n$ satisfying the defining relations $x_i x_j = q x_j x_i$, $i < j$. It is the algebra of functions on the n-dimensional quantum vector space. We define on $\mathcal{A}$ the q differentiations ∂_i and ∂'_i which are linear operators acting as $\partial_i p = \partial'_i p = 0$ on monomials p not containing x_i and as

$$\partial_i = \check{x}_i^{-1} \frac{\gamma_i - \gamma_i^{-1}}{q - q^{-1}}, \quad \partial'_i = \hat{x}_i^{-1} \frac{\gamma_i - \gamma_i^{-1}}{q - q^{-1}}$$

on monomials containing x_i, where $\hat{x}_i$ and $\check{x}_i$ are the operators of left and right multiplication by x_i, respectively. We also define the operators γ and γ^{-1} acting as

$$\gamma_i^{\pm 1} p(x_1, \ldots, x_n) = p(x_1, \ldots, x_{i-1}, q^{\pm 1} x_i, x_{i+1}, \ldots, x_n).$$

The above operators in general do not commute. We have $\partial_i \hat{x}_j = \hat{x}_j \partial_i$, $i \neq j$, $\partial_i \partial_j = q^{-1} \partial_j \partial_i$, $\partial_i \check{x}_j = q \check{x}_j \partial_i$, $i < j$, $\gamma_i \hat{x}_j = q^{\delta_{ij}} \hat{x}_j \gamma_i$, $\gamma_i \partial_j = q^{-\delta_{ij}} \partial_j \gamma_i$.

We introduce a scalar product on $\mathcal{A}$ defined by the formula

$$\langle p_1, p_2 \rangle = p_1(\partial'_1, \ldots, \partial'_n) p_2^* |_{x=0}, \qquad (1)$$

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where p_2^* is the polynomial p_2 in which numerical coefficients are replaced by complex conjugate ones; $p_1(\partial_1',\dots,\partial_n')$ means the q-differential operator obtained from a polynomial p by replacement of x_i by ∂_i', $i=1,2,\dots,n$; and the symbol $p|_{x=0}$ means a constant term of the polynomial p (see [3]).

The space $\mathcal{A}$ can be decomposed as the orthogonal sum of the subspaces $\mathcal{A}_m$ of homogeneous polynomials of homogeneity degree m: $\mathcal{A}=\bigoplus_{m=0}^{\infty}\mathcal{A}_m$. The formula

$$Q=x_1^2+q^{-1}x_2^2+\ldots+q^{-n+1}x_n^2\in\mathcal{A}_2 \qquad (2)$$

defines the squared q radius on the quantum vector space. We consider on $\mathcal{A}$ the operator

$$\Delta_q\equiv\Delta=q^{n-1}\partial_1^2+q^{n-2}\partial_2^2+\ldots+\partial_n^2, \qquad (3)$$

which is the q-Laplace operator on the quantum vector space.

A polynomial $p\in\mathcal{A}$ is called q-harmonic if $\Delta p=0$. The linear subspace of $\mathcal{A}$ consisting of all q-harmonic polynomials is denoted by $\mathcal{H}$. Let $\mathcal{H}_m=\mathcal{A}_m\cap\mathcal{H}$. Then, $\mathcal{H}=\bigoplus_{m=0}^{\infty}\mathcal{H}_m$. As in the classical case, $\mathcal{A}_m$ can be represented in the form of the direct sum $\mathcal{A}_m=\mathcal{H}_m\oplus Q\mathcal{A}_{m-2}$ (see [3]). This decomposition has the following consequence:

$$\mathcal{A}_m=\bigoplus_{0\le 2j\le m}Q^j\mathcal{H}_{m-2j} \qquad (4)$$

(the summation here is over $j=0,1,2,\dots,\lfloor m/2\rfloor$, where $\lfloor m/2\rfloor$ is the integral part of $m/2$).

The q-harmonic polynomials have the following properties:

(a) If $h_m(\mathbf{x})\in\mathcal{H}_m$, then $\gamma_n^{-1}\partial_n h_m(\mathbf{x})\in\mathcal{H}_{m-1}$ and

$$h_m(\mathbf{x})x_n-\frac{Q\gamma_n^{-1}\partial_n h_m(\mathbf{x})}{[n+2m-2]}\in\mathcal{H}_{m+1}.$$

(b) If $h_m(\mathbf{x})\in\mathcal{H}_m$ and $h_s'(\mathbf{x})\in\mathcal{H}_s$, then

$$\langle Q^k h_m, Q^l h_s'\rangle \qquad (5)$$
$$=\delta_{kl}q^{k(-n+1)}[2l]!!\frac{[2k+n+2s-2]!!}{[n+2s-2]!!}\langle h_m,h_s'\rangle,$$

where $[s]!!=[s][s-2][s-4]\cdots[2]$ (or $[1]$) and $[0]!!=1$.

Remark. In an analogy with the classical case, we may consider the scalar product (1) as an integral of the function $p_1p_2^*$. Then, formula (5) means a fulfillment of "integration" with respect to the q-radial part. As in the classical case, the scalar product $\langle h_m,h_s'\rangle$ can be treated as "integration" over q-spherical coordinates for q-harmonic polynomials.

The decomposition $\mathcal{A}_m=\mathcal{H}_m\oplus Q\mathcal{A}_{m-2}$ is orthogonal with respect to the scalar product (1). We can construct the projector $H_m:\mathcal{A}_m=\mathcal{H}_m\oplus Q\mathcal{A}_{m-2}\to\mathcal{H}_m$. This projector has the form

$$H_m p=\sum_{k=0}^{\lfloor m/2\rfloor}\alpha_k\hat{Q}^k\Delta^k p,\quad \alpha_k\in\mathbb{C},\quad p\in\mathcal{A}_m,$$

where $\lfloor m/2\rfloor$ means the integral part of $m/2$, $\hat{Q}=\hat{x}_1^2+q^{-1}\hat{x}_2^2+\ldots+q^{-n+1}\hat{x}_n^2$ is an operator on $\mathcal{A}$, and

$$\alpha_k=\frac{(-1)^k[n+2m-2k-4]!!}{[2k]!![n+2m-4]!!}. \qquad (6)$$

The coefficients α_k are determined uniquely up to a constant. In (6), we have chosen this constant in such a way that $H_mp=p$ for $p\in\mathcal{H}_m$. This means that $H_m^2=H_m$. The operator H_m is self-adjoint with respect to the scalar product (1).

3. REPRESENTATIONS ON THE QUANTUM VECTOR SPACE

The quantum algebra $U_q(sl_n)$ acts on the space $\mathcal{A}$. This algebra is generated by the elements $k_i\equiv q^{h_i}$, $k_i^{-1}\equiv q^{-h_i}$, e_i, f_i, $i=1,2,\dots,n-1$, satisfying the relations $k_ik_i^{-1}=k_i^{-1}k_i=1$, $k_ik_j=k_jk_i$, $[e_i,e_j]=[f_i,f_j]=0$ and

$$k_ie_jk_i^{-1}=q^{a_{ij}}e_j,\quad k_if_jk_i^{-1}=q^{-a_{ij}}f_j,$$
$$[e_i,f_j]\equiv e_if_j-f_je_i=\delta_{ij}\frac{k_i-k_i^{-1}}{q-q^{-1}},$$
$$e_i^2e_{i\pm1}-(q+q^{-1})e_ie_{i\pm1}e_i+e_{i\pm1}e_i^2=0,$$
$$f_i^2f_{i\pm1}-(q+q^{-1})f_if_{i\pm1}f_i+f_{i\pm1}f_i^2=0,$$

where $a_{ii}=2$, $a_{i,i\pm1}=-1$, and $a_{ij}=0$ for $|i-j|>1$ (see, for example, [4], Chap. 6). The formula $\varphi(I_{i+1,i})=f_i-qq^{-h_i}e_i$ determines the embedding $\varphi:U_q'(so_n)\to U_q(sl_n)$ of the nonstandard q-deformed algebra $U_q'(so_n)$ generated by the elements $I_{i,i-1}$, $i=2,3,\dots,n$, satisfying the defining relations

$$I_{i,i-1}I_{i-1,i-2}^2-(q+q^{-1})I_{i-1,i-2}I_{i,i-1}I_{i-1,i-2}$$
$$+I_{i-1,i-2}^2I_{i,i-1}=-I_{i,i-1},$$
$$I_{i,i-1}^2I_{i-1,i-2}-(q+q^{-1})I_{i,i-1}I_{i-1,i-2}I_{i,i-1}$$
$$+I_{i-1,i-2}I_{i,i-1}^2=-I_{i-1,i-2},$$
$$I_{i,i-1}I_{j,j-1}-I_{j,j-1}I_{i,i-1}=0,\quad |i-j|>1.$$

(For the properties of the algebra $U_q'(so_n)$, see [5, 6].)

The quantum algebra $U_q(sl_n)$ acts on the space $\mathcal{A}$ by the formulas

$$\rho(k_i)=\gamma_i\gamma_{i+1}^{-1},\quad \rho(e_i)=\check{x}_i\gamma_i\partial_{i+1},$$
$$\rho(f_i)=\check{x}_{i+1}\gamma_i^{-1}\partial_i,$$

determining the representation ρ of $U_q(sl_n)$ on $\mathcal{A}$. The subspaces $\mathcal{A}_m$ are invariant with respect to this action of $U_q(sl_n)$. We denote the corresponding subrepresentations of $U_q(sl_n)$ by ρ_m. These subrepresentations are irreducible with highest weights $(m, 0, \ldots, 0)$.

We denote the restriction of the representation ρ of $U_q(sl_n)$ to the subalgebra $U'_q(so_n)$ by T. It is easy to calculate that

$$T(I_{j+1,j}) = \check{x}_{j+1}\gamma_j^{-1}\partial_j - \check{x}_j\gamma_{j+1}\partial_{j+1}, \quad i = 1, 2, \ldots, n-1 \tag{7}$$

(see also [3]). The main property of the representation T is given by the following assertion: *The q-Laplace operator Δ commutes with all operators of the representation T of $U'_q(so_n)$.*

Let $T^{(m)}$ be the restriction of the representation T to the subspace $\mathcal{A}_m$. The representation $T^{(m)}$ is reducible. The q-squared radius (2) is invariant with respect to the representation $T^{(2)}$ (and hence with respect to the representation T), that is, $T^{(2)}(I_{k,k-1})Q = 0$ for $k = 2, 3, \ldots, n$.

The main property of the representation $T^{(m)}$ is given by the following proposition: *The operator H_m commutes with the operators $T^{(m)}(I_{j+1,j})$ of the representation $T^{(m)}$ of $U'_q(so_n)$.* We denote the restriction of the representation $T^{(m)}$ onto the invariant subspace $\mathcal{H}_m$ by T_m. Since Q is invariant with respect to $U'_q(so_n)$, it follows from (4) that $T^{(m)} = \bigoplus_{0\le 2j\le m} T_{m-2j}$.

4. q-ANALOG OF SEPARATION OF VARIABLES

It is well known (see [1], Chaps. 9, 10) that in the space of classical homogeneous harmonic polynomials there exist different orthonormal bases. They correspond to different separations of variables. Each separation of variables corresponds to a certain chain of subgroups of $SO(n)$. We show below that a similar picture takes place for q-harmonic polynomials.

In the classical case, the tree method distinguishes different separations of variables or, equivalently, different chains of subgroups of $SO(n)$. The same tree method can be used for q-harmonic polynomials, but instead of chains of subgroups of $SO(n)$ we have to take the corresponding chains of subalgebras of the algebra $U'_q(so_n)$.

Let us construct an orthonormal basis of the space $\mathcal{H}_m$ corresponding to the chain

$$U'_q(so_n) \supset U'_q(so_{n-1}) \supset \cdots \supset U'_q(so_3) \supset U'_q(so_2), \tag{8}$$

where $U'_q(so_2)$ is the commutative subalgebra generated by the element I_{21}. This basis is a q-analog of the well-known set of associated spherical harmonics which are products of certain Gegenbauer polynomials (see [1], Chap. 9).

We denote by $\hat{t}_s^{n,m}(Q, x_n)$ the expression

$$\hat{t}_s^{n,m}(Q, x_n) = \sum_{k=0}^{\lfloor (m-s)/2 \rfloor} \frac{(-1)^k q^{-2sk}[m-s]![2m+n-2k-4]!!}{[m-s-2k]![2k]!![2m+n-4]!!} Q^k x_n^{m-s-2k}. \tag{9}$$

The following proposition is proved: *Let $h_s(\mathbf{x}')$ be a homogeneous q-harmonic polynomial of degree s in $\mathbf{x}' = (x_1, x_2, \ldots, x_{n-1})$. Then, for $x_n^{m-s}h_s(\mathbf{x}') \in \mathcal{A}_m$, we have*

$$H_m(x_n^{m-s}h_s(\mathbf{x}')) = \hat{t}_s^{n,m}(Q, x_n)h_s(\mathbf{x}'). \tag{10}$$

In order to construct an orthonormal basis of $\mathcal{H}_m$, we have to normalize expression (10). Let τ_s^m denote the expression (10). We calculate that

$$\langle \tau_s^m, \tau_s^m \rangle = \langle x_n^{m-s}h_s(\mathbf{x}'), \tau_s^m \rangle$$

$$= c_m^{(s)} \langle x_n^{m-s}h_s(\mathbf{x}'), x_n^{m-s}h_s(\mathbf{x}') \rangle$$

$$= c_m^{(s)} q^{-s(m-s)} \langle h_s(\mathbf{x}')x_n^{m-s}, x_n^{m-s}h_s(\mathbf{x}') \rangle$$

$$= c_m^{(s)} q^{-s(m-s)} [m-s]! \langle h_s(\mathbf{x}'), h_s(\mathbf{x}') \rangle,$$

where $c_m^{(s)}$ is the coefficient at x_n^{m-s} in the expression (9) for $\hat{t}_s^{n,m}(Q, x_n)$. We find from (9) that

$$c_m^{(s)} = \sum_{k=0}^{\lfloor (m-s)/2 \rfloor} \frac{(q^{-2(m-s)}; q^4)_k (q^{-2(m-s)+2}; q^4)_k}{(q^{-2n-4m+8}; q^4)_k} \frac{q^{k(-2n-4s+6)}}{(q^4; q^4)_k}$$

$$= {}_2\varphi_1(q^{2(s-m)}, q^{2(s-m)+2};\ q^{8-2n-4m}\ ;\ q^4, q^{6-2n-4s}),$$

where ${}_2\varphi_1$ means a basic hypergeometric function (see [7], Chap. 1). This basic hypergeometric series can be summed by means of formula (1.5.2) in [7] and we obtain

$$c_m^{(s)} = \frac{(q^{2(-n-m-s+3+\sigma)}; q^4)_{(m-s-\sigma)/2}}{(q^{2(-n-2m+4)}; q^4)_{(m-s-\sigma)/2}},$$

where $\sigma = 0$ if $m - s$ is even and $\sigma = 1$ if $m - s$ is odd. Thus, along with $\hat{t}_s^{n,m}(Q, x_n)$, we have also the normalized expression

$$t_s^{n,m}(Q, x_n) = \frac{q^{s(m-s)/2}}{\sqrt{c_m^{(s)}[m-s]!}} \hat{t}_s^{n,m}(Q, x_n). \qquad (11)$$

Now, we use the expression $t_s^{n,m}(Q, x_n)$ instead of $\hat{t}_s^{n,m}(Q, x_n)$. In order to construct an orthonormal basis of the space $\mathcal{H}_m$ in an explicit form, we take into account that

$$\langle t_s^{n,m}(Q, x_n) h_s(\mathbf{x}'), t_s^{n,m}(Q, x_n) h_s(\mathbf{x}')\rangle = \langle h_s(\mathbf{x}'), h_s(\mathbf{x}')\rangle.$$

We apply the above reasoning of this section to homogeneous q-harmonic polynomials of $x_1, x_2, \ldots, x_{n-1}$. As a result, we obtain q-harmonic polynomials of the form

$$t_s^{n,m}(Q, x_n) t_r^{n-1,s}(Q_{n-1}, x_{n-1}) h_r(\mathbf{x}''),$$
$$s = 0, 1, 2, \ldots, m; \quad r = 0, 1, 2, \ldots, s,$$

where $Q_{n-1} = x_1^2 + q^{-1}x_2^2 + \ldots + q^{-n+2}x_{n-1}^2$, $\mathbf{x}'' = (x_1, x_2, \ldots, x_{n-2})$, and $h_r(\mathbf{x}'')$ are elements of the space of homogeneous q-harmonic polynomials of degree r in $x_1, x_2, \ldots, x_{n-2}$. Here, $t_r^{n-1,s}(Q_{n-1}, x_{n-1})$ is defined by (9) and (11).

Continuing this procedure, we obtain the normalized polynomials of $\mathcal{H}_m$ of the form

$$\Xi_{\mathbf{m}}(\mathbf{x}) \equiv \Xi_{m,m_{n-1},m_{n-2},\ldots,m_2}(\mathbf{x}) = t_{m_{n-1}}^{n,m}(Q, x_n) \qquad (12)$$
$$\times\, t_{m_{n-2}}^{n-1,m_{n-1}}(Q_{n-1}, x_{n-1}) \cdots t_{m_2}^{3,m_3}(Q_3, x_3) \times t^{2,m_2}(x_1, x_2),$$

$$m \ge m_{n-1} \ge m_{n-2} \ge \ldots \ge m_3 \ge |m_2|, \qquad (13)$$

where the polynomials $t^{2,m_2}(x_1, x_2)$ are given by $t^{2,m_2}(x_1, x_2) = (c^{(m_2)})^{-1/2} z^{(m_2)}$, where

$$z^{(0)} \equiv 1,$$
$$z^{(s)} = (\mathrm{i}x_1 + x_2)(\mathrm{i}x_1 + qx_2) \cdots (\mathrm{i}x_1 + q^{s-1}x_2), \quad s > 0,$$
$$z^{(s)} = (\mathrm{i}x_1 - x_2)(\mathrm{i}x_1 - qx_2) \cdots (\mathrm{i}x_1 - q^{-s+1}x_2), \quad s < 0,$$

are linearly independent q-harmonic polynomials in x_1 and x_2 and $c^{(s)} = c^{(-s)} = 2q^{s(s-1)/2}[s][2s-2]!!$ for $s > 0$.

To every set of integers $m_{n-1}, m_{n-2}, \ldots, m_3, m_2$ satisfying the condition (13), there corresponds a polynomial (12). (For fixed m_3, the number m_2 takes the values $-m_3, -m_3+1, \ldots, m_3$.) The number of these polynomials is equal to the dimension of the space $\mathcal{H}_m$ given in Corollary 3.1.4 of [3]. On the other hand, the polynomials (12) are pairwise orthogonal. This means thatthe set of all polynomials (12) constitute an orthonormal basis of the space $\mathcal{H}_m$. This basis corresponds to the chain of subalgebras (8).

Representation of the basis of the space $\mathcal{H}_m$ of solutions of the equation $\Delta p_m = 0$ in the form (12) gives us a q analog of the classical separation of variables, which corresponds to the chain of subalgebras (8). There are q analogs of other types of separations of variables, which are given below.

By direct calculation we can prove the following assertion: *The representation operators* $T_m(I_{k,k-1})$, $k = 2, 3, \ldots, n$, *given by* (7) *act upon the basis elements* $\Xi_{\mathbf{m}} \equiv |\mathbf{m}\rangle$ *from* (12) *as* $T_m(I_{21})|\mathbf{m}\rangle = i[m_2]|\mathbf{m}\rangle$ *and*

$$T_m(I_{k,k-1})|\mathbf{m}\rangle = -([m_k + m_{k-1} + k - 2] \times [m_k - m_{k-1}])^{1/2} A(m_{k-1})|\mathbf{m}_{k-1}^+\rangle$$
$$+ ([m_k + m_{k-1} + k - 3][m_k - m_{k-1} + 1])^{1/2} \times A(m_{k-1} - 1)|\mathbf{m}_{k-1}^-\rangle, \quad k \ne 2,$$

where $m_n \equiv m$, $\mathbf{m}_{k-1}^\pm$ *denote the set of the numbers* $\mathbf{m}$ *with* m_{k-1} *replaced by* $m_{k-1} \pm 1$, *respectively, and*

$$A(m_{k-1}) = \left(\frac{[m_{k-1}+m_{k-2}+k-3][m_{k-1}-m_{k-2}+1]}{[2m_{k-1}+k-3][2m_{k-1}+k-1]}\right)^{1/2}.$$

Let us construct orthonormal bases of the space $\mathcal{H}_m$ corresponding to the reductions

$$U_q'(so_n) \supset U_q'(so_p) \times U_q'(so_{n-p}). \qquad (14)$$

As in the classical case (see [1], Chap. 10), further reductions can be made as in (8) or as in (14). In particular, the usual tree method (see [1], Section 10.2) can be used to describe different chains of subalgebras corresponding to different orthonormal bases of $\mathcal{H}_m$.

We represent the set $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ as $\mathbf{x} = (\mathbf{y}, \mathbf{t})$, where $\mathbf{y} = (x_1, x_2, \ldots, x_p)$ and $\mathbf{t} = (x_{p+1}, x_{p+2}, \ldots, x_n)$. Then the q-Laplace operator Δ can be written as $\Delta = q^{n-p}\Delta_{(\mathbf{y})} + \Delta_{(\mathbf{t})}$, where

$\Delta_{(\mathbf{y})}$ and $\Delta_{(\mathbf{t})}$ are the q-Laplace operators for $\mathbf{y}$ and $\mathbf{t}$, respectively.

In order to find bases of $\mathcal{H}_m$ corresponding to the reduction (14), we take nonnegative numbers s_1 and s_2 such that m and s_1+s_2 are of the same evenness and $m-s_1-s_2 \geq 0$. We denote by $\hat{t}^{n,p;\,m}_{s_1,s_2}(Q_\mathbf{t}, Q_\mathbf{y})$ the expression

$$\hat{t}^{n,p;\,m}_{s_1,s_2}(Q_\mathbf{t}, Q_\mathbf{y}) = \sum_{k=0}^{r} \frac{(-1)^k q^{-2s_2k}[2r]!!}{[n+2m-4]!!} \times \frac{[2r+n-p+2s_1-2]!![n+2m-2k-4]!!}{[2r-2k]!![2r+n-p+2s_1-2k-2]!![2k]!!} \times Q^k Q_\mathbf{t}^{r-k},$$

where $r=(m-s_1-s_2)/2$. It is proved that

$$H_m\left(Q_\mathbf{t}^r h_{s_1}(\mathbf{t}) h_{s_2}(\mathbf{y})\right) = \hat{t}^{n,p;\,m}_{s_1,s_2}(Q_\mathbf{t}, Q_\mathbf{y}) h_{s_1}(\mathbf{t}) h_{s_2}(\mathbf{y}).$$

The normalized expression for $\hat{t}^{n,p;\,m}_{s_1,s_2}(Q_\mathbf{t}, Q_\mathbf{y})$ has the form

$$t^{n,p;\,m}_{s_1,s_2}(Q_\mathbf{t}, Q_\mathbf{y}) = \left(\frac{[n+2s_1-p-2]!!q^{(n-p-1)r+2s_2r+s_1s_2}}{[2r]!![2s_1+n+2r-p-2]!!c_m^{(s_1,s_2)}}\right)^{1/2} \times \hat{t}^{n,p;\,m}_{s_1,s_2}(Q_\mathbf{t}, Q_\mathbf{y}),$$

where

$$c_m^{(s_1,s_2)} = \frac{(q^{4-4s_2-2p-4r}; q^4)_r}{(q^{-2n-4m+8}; q^4)_r}.$$

In order to construct an orthonormal basis of the space $\mathcal{H}_m$ corresponding to the reduction $U'_q(so_n) \supset U'_q(so_p) \times U'_q(so_{n-p})$ in an explicit form, we note that

$$\langle t^{n,p;\,m}_{s_1,s_2}(Q_\mathbf{t}, Q_\mathbf{y}) h_{s_1}(\mathbf{t}) h_{s_2}(\mathbf{y}), t^{n,p;\,m}_{s_1,s_2}(Q_\mathbf{t}, Q_\mathbf{y}) h_{s_1}(\mathbf{t}) h_{s_2}(\mathbf{y})\rangle = \langle h_{s_1}(\mathbf{t}), h_{s_1}(\mathbf{t})\rangle \langle h_{s_2}(\mathbf{y}), h_{s_2}(\mathbf{y})\rangle.$$

Therefore, for construction of such a basis, we have to take orthonormal bases $h^{(i)}_{s_1}(\mathbf{t})$ and $h^{(j)}_{s_2}(\mathbf{y})$ of the spaces $\mathcal{H}^{(\mathbf{t})}_{s_1}$ and $\mathcal{H}^{(\mathbf{y})}_{s_2}$ of homogeneous q-harmonic polynomials in $\mathbf{t}$ and $\mathbf{y}$, respectively, and to construct the products

$$t^{n,p;\,m}_{s_1,s_2}(Q_\mathbf{t}, Q_\mathbf{y}) h^{(i)}_{s_1}(\mathbf{t}) h^{(j)}_{s_2}(\mathbf{y}), \qquad (15)$$

where the indices s_1 and s_2 run over integral values such that

$$s_1+s_2 \equiv m (\mathrm{mod}\ 2), \quad s_1+s_2 \leq m,$$
$$i = 1,2,\ldots,\dim\mathcal{H}^{(\mathbf{t})}_{s_1},$$
$$j = 1,2,\ldots,\dim\mathcal{H}^{(\mathbf{y})}_{s_2}.$$

It is easy to calculate that the number of elements (15) is equal to dim $\mathcal{H}_m$. On the other hand, it is proved that the elements (15) are orthogonal to each other. Therefore, the polynomials (15) constitute an orthonormal basis of the space $\mathcal{H}_m$. In particular, we can take the elements $\Xi_{\mathbf{s}_1}(\mathbf{t})$, $\mathbf{s}_1 = (s_1, s'_1, \ldots)$, and $\Xi_{\mathbf{s}_2}(\mathbf{y})$, $\mathbf{s}_2 = (s_2, s'_2, \ldots)$, of the type (12) as orthonormal bases of the spaces $\mathcal{H}^{(\mathbf{t})}_{s_1}$ and $\mathcal{H}^{(\mathbf{y})}_{s_2}$, respectively. Then, the elements

$$t^{n,p;\,m}_{s_1,s_2}(Q_\mathbf{t}, Q_\mathbf{y}) \Xi_{\mathbf{s}_1}(\mathbf{t}) \Xi_{\mathbf{s}_2}(\mathbf{y})$$

form an orthonormal basis of $\mathcal{H}_m$ corresponding to the chain

$$U'_q(so_n) \supset U'_q(so_p) v \times U'_q(so_{n-p}) \supset U'_q(so_{p-1}) \times U'_q(so_{n-p-1}) \supset \ldots.$$

As is mentioned above, in order to construct different orthonormal bases of $\mathcal{H}_m$, the tree method from Section 10.2 in [1] can be used. To different trees, there correspond different chains of subalgebras of $U'_q(so_n)$ and orthonormal bases corresponding to them. In this way, we obtain q analogs of different separations of variables.

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Star Products on Coadjoint Orbits*

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Abstract—We study properties of a family of algebraic star products defined on coadjoint orbits of semisimple Lie groups. We connect this description with the point of view of differentiable deformations and geometric quantization. © *2001 MAIK "Nauka/Interperiodica"*.

1. FAMILY OF DEFORMATIONS OF THE POLYNOMIALS ON THE ORBIT

Let G be a Lie group of dimension n and $\mathcal{G}$ its Lie algebra. The Kirillov–Poisson structure on the dual space $\mathcal{G}^*$ is given by

$$\{f_1, f_2\}(\lambda) = \langle[(df_1)_\lambda, (df_2)_\lambda], \lambda\rangle, \qquad (1)$$
$$f_1, f_2 \in C^\infty(\mathcal{G}^*), \quad \lambda \in \mathcal{G}^*.$$

The symplectic leaves of this Poisson structure coincide with the orbits of the coadjoint action of G in $\mathcal{G}$,

$$\langle \mathrm{Ad}^*(g)\lambda, Y\rangle = \langle\lambda, \mathrm{Ad}(g^{-1})Y\rangle,$$
$$\forall\, g \in G, \quad \lambda \in \mathcal{G}^*, \quad Y \in \mathcal{G}.$$

Let G be a compact semisimple group of rank n. Then, the coadjoint orbits are algebraic varieties. Let $\{p_i\}_{i=1}^m$ be a set of generators of the algebra of G-invariant polynomials on $\mathcal{G}^*$. The coadjoint orbits are determined by the values of these polynomials, that is, by the equations

$$p_i = c_i, \qquad i = 1, \ldots, m. \qquad (2)$$

The regular orbits are those for which the differentials dp_i are independent [1]. They are algebraic symplectic manifolds of dimension $n - m$. The ideal of polynomials vanishing on a regular orbit Θ is a prime ideal generated by the relations (2), and we will denote it by $\mathcal{I}_0$. The algebra of polynomials on Θ,

$$\mathrm{Pol}(\Theta) = \mathrm{Pol}(\mathcal{G}^*)/\mathcal{I}_0,$$

is a Poisson algebra.

A formal deformation of a Poisson algebra $\mathcal{A}$ over $\mathbb{C}$ is a $\mathbb{C}[[h]]$ module $\mathcal{A}_h$ which is isomorphic as a module to $\mathcal{A}[[h]]$, the isomorphism $\Psi : \mathcal{A}[[h]] \to \mathcal{A}_h$ satisfying the conditions

(a) $\psi^{-1}(F_1F_2) = f_1f_2 \mod(h)$, where $F_i \in \mathcal{A}_{[h]}$ are such that $\psi^{-1}(F_i) = f_i \mod(h)$, $f_i \in \mathcal{A}$;

(b) $\psi^{-1}(F_1F_2 - F_2F_1) = h\{f_1, f_2\} \mod(h^2)$.

In this definition, one can substitute $\mathbb{C}[[h]]$ with $\mathbb{C}[h]$. We will say then that we have a $\mathbb{C}[h]$ deformation. It is clear that a $\mathbb{C}[h]$ deformation can be extended to a formal deformation, while the opposite is not in general true.

Given a deformation $\mathcal{A}_h$, one can make the pullback of the product in $\mathcal{A}_h$ to $\mathcal{A}[[h]]$ by the isomorphism Ψ. The product defined in this way is called a star product and is in general given by a formal series

$$f \star g = \Psi^{-1}(\Psi(f)\Psi(g)) = fg + \sum_{n>1} h^n B_n(f, g),$$

where B_n are bilinear operators. If $\mathcal{A}$ is some space of functions and B_n are bidifferential operators, we say that the star product is differential. It follows that the star product can be extended to the whole space of C^∞ functions, but only as a formal deformation [2]. By choosing another isomorphism Ψ', one could obtain a star product that is not differential. So a star product that is not differential can be isomorphic to a star product that is differential. We will see examples of this situation later.

A formal (and $\mathbb{C}[h]$) deformation of $\mathrm{Pol}(\mathcal{G}^*)$ is given by the enveloping algebra U_h of the Lie algebra with the bracket $h[\cdot,\cdot]$, where $[\cdot,\cdot]$ is the bracket on $\mathcal{G}$. (The tensor algebra needs to be taken over $\mathbb{C}[[h]]$.) One choice for Ψ is the Weyl map or symmetrizer. If $x_1, \ldots, x_n$ are coordinates on $\mathcal{G}^*$ and $X_1, \ldots, X_n$ are the corresponding generators of U_h, the Weyl map is

$$W(x_{i_1} \cdots x_{i_p}) = \frac{1}{p!} \sum_{s \in S_p} X_{i_{s(1)}} \cdots X_{i_{s(p)}}.$$

The star product

$$f \star_S g = W^{-1}(W(f)W(g))$$

can be expressed in terms of bidifferential operators, so it can be extended to the whole $C^\infty(\mathcal{G}^*)$.

$\star_S$ is not tangential to the orbits, so it cannot be restricted to one of them. Nevertheless, the formal deformation $U_{[h]}$ can be used to induce a deformation

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of Pol(Θ). This was developed in [3]. The idea is to find an ideal $\mathcal{I}_h$ such that the diagram

$$\begin{array}{ccc} \mathrm{Pol}(\mathcal{G}^*) & \longrightarrow & U_{[h]} \\ \downarrow & & \downarrow \\ \mathrm{Pol}(\Theta) & \longrightarrow & U_{[h]}/\mathcal{I}_{[h]} \end{array}$$

commutes. The vertical arrows are the natural projections, and the horizontal ones indicate deformations. The ideal $\mathcal{I}_h$ is generated by

$$W(p_i) - c_i(h) = P_i - c_i(h),$$
$$c_i(0) = c_i^0, \quad i = 1, \ldots, n.$$

The ideal is Ad_G-invariant since P_i are Casimirs of $U_{[h]}$, so there is a natural action of G on $U_{[h]}/\mathcal{I}_{[h]}$. The same construction works with $\mathbb{C}[h]$. We will consider only $c^i(h)$ such that its degree in h is not bigger than the degree of p_i. In this context, one can show that $\mathcal{I}_h$ is a prime ideal [4]. Also, the algebras can be specialized to a value of h, say h_0, by quotienting with the proper ideal generated by $h - h_0$. Analyzing the representations of the specialized algebras, one can see that, in general, they are not isomorphic for ideals with $c_i(h) \neq c_i'(h)$.

2. STAR PRODUCTS ON THE POLYNOMIALS ON THE ORBIT

We consider the example of S^2 for clarity, although the argument can be extended to all other compact orbits [3, 4]. $\mathcal{G} = su(2)$ has dimension 3 with basis $\{X, Y, Z\}$,

$$[X, Y] = Z, \quad [Y, Z] = X, \quad [Z, X] = Y.$$

The unique invariant polynomial is $p(x, y, z) = x^2 + y^2 + z^2$, and the Casimir element is $P = X^2 + Y^2 + Z^2$. The regular orbits are given by

$$x^2 + y^2 + z^2 = c$$

for $c > 0$. A basis of $\mathrm{Pol}(S^2)$ is $\{[x^m y^n z^\nu],\ m, n \in \mathbb{N},\ \nu = 0, 1\}$. An isomorphism $\mathrm{Pol}(S^2)[h] \approx U_h/\mathcal{I}_h$ is given by

$$\tilde{\Psi}([x^m y^n z^\nu]) = [X^m Y^n Z^\nu],$$

since $\{[X^m Y^n Z^\nu],\ m, n \in \mathbb{N},\ \nu = 0, 1\}$ is a basis of $U_h/\mathcal{I}_h$. Define the isomorphism $\Psi : \mathrm{Pol}(su(2)^*)[h] \to U_h$

$$\Psi(x^m y^n z^\nu) = X^m Y^n Z^\nu, \qquad m, n \in \mathbb{N},$$
$$\Psi(x^m y^n z^r (p - c^0)) = X^m Y^n Z^r (P - c(h)),$$
$$m, n \in \mathbb{N},$$

which sends the ideal $\mathcal{I}_0$ into the ideal $\mathcal{I}_h$, so it passes to the quotient, where it gives the isomorphism $\tilde{\Psi}$. The corresponding star product on $\mathrm{Pol}(su(2)^*)[h]$ is restricted to $\mathrm{Pol}(S^2)$.

This star product is not differential, as is shown in [4], but it is isomorphic to $\star_S$. In addition, for an orbit in a neighborhood of this one, $p - c^0 - \Delta c^0 = 0$, Ψ does not preserve the ideal.

Another way of giving a basis is using the decomposition

$$\mathrm{Pol}(\mathcal{G}^*) \approx I \otimes H,$$

where I is the algebra of invariant polynomials and H is the space of harmonic polynomials, $H \approx \mathrm{Pol}(\Theta)$. We define the isomorphism $\Phi : \mathrm{Pol}(su(2)^*)[h] \to U_h$

$$\Phi((p - c)^m \otimes \eta_m) = (P - c(h))^m \tilde{\Phi}(\eta_m),$$
$$\eta_m \in H,$$

where $\bar{\Phi}$ is any isomorphism $\tilde{\Phi} : \mathrm{Pol}(S^2)[h] \to U_h/\mathcal{I}_h$. A star product of this kind was first written down in [5], where $\tilde{\Phi}$ was chosen in terms of the Weyl map,

$$\tilde{\Phi}([\eta]) = [W(\eta)],$$

and $c(h) = c$. We will denote this product by $\star_P$. It has the nice properties that it is restricted to all the orbits in a neighborhood of the regular orbit and that it is "covariant,"

$$gf_1 \star gf_2 = g(f_1 \star f_2).$$

Nevertheless, it is not differential, as was shown in [5].

Finally, it was proven in [3] that $U_\hbar/\mathcal{I}_\hbar$, with $c(\hbar) = l(l + \hbar)$, corresponds to the algebra of geometric quantization in the formalism of [6].

3. DIFFERENTIAL AND TANGENTIAL STAR PRODUCTS

In this section, we want to consider differential star products on $\mathcal{G}^*$ and on Θ, and to see the relation with the algebraic approach of the previous section. In [7], the differential deformations of a Poisson manifold X modulo gauge equivalence are shown to be in one-to-one correspondence with the formal Poisson structures

$$\alpha = h\alpha_1 + h^2\alpha_2 + \ldots, \qquad [\alpha, \alpha] = 0$$

(α_i are bivector fields, and $[\cdot, \cdot]$ is the Schouten–Nijenhuis bracket), modulo the action of formal paths in the diffeomorphism group. For each Poisson structure β, one can therefore associate canonically an equivalence class of star products, the one corresponding to $h\beta$. If there are formal structures starting with $h\beta$ which are not equivalent to $h\beta$ through a diffeomorphism path, then one has star products not equivalent to the canonical one such that

$$f \star g - g \star f = h\beta(f, g) \mathrm{mod}(h).$$

In the case of symplectic manifolds, these structures are classified by $H^2(X)[[h]]$. Since the compact coadjoint orbits have a nontrivial second cohomology group, we have more than one equivalence class of differential star products with a term of first order the same Poisson bracket.

In the case of $\mathcal{G}^*$ with the Kirillov–Poisson structure, it depends on the Lie algebra cohomology of $\mathcal{G}$. So, for a semisimple Lie algebra, there is only one equivalence class [4]. $\star_S$ is a representative of this equivalence class. It is not tangential to the orbits, and, in fact, it was shown in [8] that no tangential star product could be extended over 0 for a semisimple Lie algebra.

Nevertheless, a regular orbit has always a neighborhood that is regularly foliated, $\mathcal{N}_\Theta \approx \Theta \times \mathbb{R}^m$. Since the Poisson structure is tangential, the coordinates on $\mathbb{R}^m$ can be considered as parameters, so one has in fact a family of Poisson structures on Θ smoothly varying with the parameters p_i, $\beta_{p_1,\ldots,p_m}$. Kontsevich's construction of the canonical star product gives a star product smoothly varying with the parameters p_i, or, interpreting it in the other way, a tangential star product canonically associated to β. It follows that $\star_S$, when restricted to $\mathcal{N}_\Theta$, is equivalent to a tangential star product. We denote it by $\star_T$.

We have three different products:

$\star_S$. It is differential, not tangential, and defined on $\mathcal{G}^*$.

$\star_P$. It is not differential, tangential, and defined on $\mathcal{G}^*$. (The ideal has been chosen so that $c_i(h) = c_i^0$.)

$\star_T$. It is differential, tangential, and defined only on $\mathcal{N}_\Theta$.

$\star_S$ restricted to the polynomials is isomorphic to $\star_P$. There is then an algebra homomorphism

$$\varphi : (\mathrm{Pol}(\mathcal{G}^*)[[h]], \star_P) \to (C^\infty(\mathcal{G}^*)[[h]], *_S).$$

We have that

$$\varphi(p_i - c_i^0) = p_i - c_i^0,$$

so $\mathcal{I}_0 \subset \mathrm{Pol}(\mathcal{G}^*)[[h]]$ is sent by φ into $\mathcal{I}_0 \subset C^\infty(\mathcal{G}^*)[[h]]$.

Restricting $\star_S$ to $\mathcal{N}_\Theta$, we have an algebra homomorphism

$$\rho : (C^\infty(\mathcal{N}_\Theta)[[h]], \star_S) \to (C^\infty(\mathbb{N})[[h]], *_T).$$

It is not difficult to see that, although in general ρ does not send polynomials into polynomials, the algebra homomorphism structure implies that [4]

$$\rho(p_i - c_i^0) = p_i - c_i^0.$$

By composing $\rho \circ \varphi$, one obtains an homomorphism from a nondifferential star product to a differential one, such that both star products are tangential and the ideal $\mathcal{I}_0$ is mapped into the ideal $\mathcal{I}_0$. The homomorphism passes to the quotient, so the algebraic star product described in Section 2 is shown to be homomorphic to the differentiable star product associated by Kontsevich's map.

We note that we have chosen an algebraic star product with $c_i(h) = c_i$. This star product is not the one obtained from geometric quantization. The differential approach to quantization and geometric quantization, although they have similar features in the case of $\mathbb{R}^{2n}$ [9], seem not to give for compact coadjoint orbits homomorphic algebras.

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SYMPOSIUM ON QUANTUM GROUPS

Noncommutative Parameters of Quantum Symmetries and Star Products*

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Abstract—The star product technique translates the framework of local fields on noncommutative spacetime into nonlocal fields on standard spacetime. We consider the example of fields on κ-deformed Minkowski space, transforming under κ-deformed Poincaré group, with noncommutative parameters. By extending the star product to the tensor product of functions on κ-deformed Minkowski space and κ-deformed Poincaré group we represent the algebra of noncommutative parameters of deformed relativistic symmetries by functions on classical Poincaré group. © *2001 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

It has been recognized recently (see, e.g., [1–3]) that, at very short distances, comparable with Planck length $\lambda \simeq 10^{-33}$ cm, the notion of classical spacetime manifold should be modified. The submicroscopic quantum structure of spacetime implies noncommutativity; i.e., one should replace the classical Minkowski coordinates x_μ by the generators $\widehat{x}_\mu$ of noncommutative algebra. Assuming the formula (see, e.g., [4, 5])[3)]

$$[\widehat{x}_\mu, \widehat{x}_\nu] = \theta_{\mu\nu}(\widehat{x}) = \theta^{(0)}_{\mu\nu} + \theta^{(1)\rho}_{\mu\nu}\widehat{x}_\rho + \theta^{(2)\rho\tau}_{\mu\nu}\widehat{x}_\rho\widehat{x}_\tau + \ldots \quad (1)$$

with $\theta_{\mu\nu}(\widehat{x})$ restricted by Jacobi identities, one arrives at different models of noncommutative spacetime geometry. The simplest case is obtained if $\theta_{\mu\nu}(x)$ is a constant ($\theta_{\mu\nu}(x) \equiv \theta^{(0)}_{\mu\nu}$). Such a deformation, first advocated by Dopplicher, Fredenhagen, and Roberts [1], has been recently extensively studied in string theory as describing world volume coordinates of D branes (see, e.g., [7–9]). In such a deformation, the relativistic symmetries remain classical, which simplifies greatly the formalism of corresponding noncommutative theory. Indeed, if we set $\theta_{\mu\nu}(\widehat{x}) = \theta^{(0)}_{\mu\nu}$, the relations (1) remain invariant under the shifts $\widehat{x}'_\mu = \widehat{x}_\mu + a_\mu$, where a_μ are classical commutative transformations. If the right-hand side of (1) depends on $\widehat{x}_\mu$, the translations preserving the algebraic structure of spacetime become noncommutative. The simplest framework is provided if the rhs of (1) is linear, i.e.,

$$[\widehat{x}_\mu, \widehat{x}_\nu] = \theta^{(1)\rho}_{\mu\nu}\widehat{x}_\rho. \quad (2)$$

In such a case, the translations $\widehat{x}'_\mu = \widehat{x}_\mu + \widehat{v}_\nu$ commute with spacetime algebra

$$[\widehat{x}_\mu, \widehat{v}_\nu] = 0 \quad (3)$$

and form themselves a second copy of the algebra (2)

$$[\widehat{x}_\mu, \widehat{v}_\nu] = \theta^{(1)\rho}_{\mu\nu}\widehat{v}_\rho. \quad (4)$$

In the general case, the translations form another copy of algebra (1), but the invariance under translations implies nontrivial braiding relations between the noncommutative algebra of spacetime coordinates (1) and the translations algebra (we use property $\theta_{\mu\nu} = -\theta_{\nu\mu}$):

$$[\widehat{x}_\mu, \widehat{v}_\nu] = \frac{1}{2}\left\{\theta_{\mu\nu}(\widehat{x}+\widehat{v}) - \theta_{\mu\nu}(\widehat{x}) - \theta_{\mu\nu}(\widehat{v})\right\}. \quad (5)$$

The noncommutativity of spacetime translations implies necessarily the modification of spacetime symmetries. In particular, one can pose the question for which functions $\theta_{\mu\nu}(\widehat{x})$ in (1) the noncommutative translations $\widehat{v}_\mu$ can be extended to the quantum Poincaré group, describing the Hopf algebra of deformed relativistic symmetries. The classification

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3) We shall restrict our considerations to the case when the noncommutative algebra of spacetime is generated only by $\widehat{x}_\mu$. In the general case, as in the first model of noncommutative spacetime by Snyder [6], the operator basis of the algebra is extended by other operators (see, e.g., [7]).

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of noncommutative translations which can be extended to standard (nonbraided) quantum Poincaré group was given by Podleś and Woronowicz [10]. In particular, if we wish to maintain the classical nonrelativistic $O(3)$ symmetries, the choice of the deformation is unique—one obtains the standard form of κ deformation of relativistic symmetries [11–13].

The aim of this talk is to describe the κ-deformed field theory in the commutative framework of classical fields, with the noncommutative parameters of the κ-deformed Poincaré group described by commutative parametrization. For that purpose, the star product on κ-deformed Minkowski space [14], identical to the star product on the subalgebra of noncommutative translations, is extended to ten generators $(\widehat{v}_\mu, \widehat{\Lambda}_\kappa^\nu)$ of κ-deformed Poincaré group.

It appears that, due to the fact that the cross relations between Lorentz generators $\widehat{\Lambda}_{\mu\nu}$ and translations $\widehat{v}_\mu$ are quadratic, our extended star product goes beyond the Baker–Campbell–Hausdorff formula describing star products for Lie algebraic or Lie superalgebraic structures.

The plan of our presentation is the following.

In Section 2, we describe the κ-deformed Poincaré group and recall the star product for the fields defined on κ-deformed Minkowski spacetime. In Section 3, we introduce the star product for functions on κ-deformed Poincaré group. In Section 4, we present final remarks and outlook.

2. κ-DEFORMED POINCARÉ GROUP AND FIELDS ON κ-DEFORMED MINKOWSKI SPACE

The κ-deformed Poincaré group is described by the deformed noncommutative group parameters $(\widehat{v}_\mu, \widehat{\Lambda}_\nu^\mu)$ satisfying the algebraic relations [15, 16]

$$[\widehat{v}_\mu, \widehat{v}_\nu] = \frac{i}{\kappa}\left(\delta_\mu^0 \widehat{v}_\nu - \delta_\nu^0 \widehat{v}_\mu\right), \tag{6a}$$

$$\left[\widehat{\Lambda}_\nu^\mu, \widehat{v}_\rho\right] \tag{6b}$$

$$= -\frac{i}{\kappa}\left\{(\Lambda_0^\mu - \delta_0^\mu)\,\widehat{\Lambda}_{\rho\nu} + \left(\widehat{\Lambda}_{0\nu} - \eta_{0\nu}\right)\delta_\rho^\mu\right\},$$

$$\left[\widehat{\Lambda}_\nu^\mu, \widehat{\Lambda}_\tau^\rho\right] = 0 \tag{6c}$$

with the constraints $\Lambda\Lambda^T = \Lambda^T\Lambda = 1$ or

$$\widehat{\Lambda}_\nu^\mu\,\widehat{\Lambda}_\nu^\tau = \widehat{\Lambda}_\mu^\nu \widehat{\Lambda}_\tau^\nu = \eta^{\mu\tau} = \eta_{\mu\tau}, \tag{7}$$

where $\mathrm{diag}\eta = (1, 1, 1, -1)$.

The relations (6) were first obtained [15] by the quantization of Poisson–Lie bracket for the functions on Poincaré group with the following classical r matrix:

$$r = \frac{1}{\kappa} N_i \,\Lambda\, P_i\,, \tag{8}$$

where P_i are three-momenta and $N_i \equiv M_{i0}$ are Lorentz boost generators. Another way to obtain the relations (6a)–(6c) was to construct the dual Hopf algebra to the κ-deformed Poincaré algebra $\mathcal{U}_\kappa(\mathcal{P}_4)$ written in a bicrossproduct basis [13, 16]. The coproduct for $\widehat{v}_\kappa, \widehat{\Lambda}_\nu^\kappa$ remains undeformed

$$\Delta(\widehat{v}_\mu) = \widehat{v}_\nu \otimes \widehat{\Lambda}_\mu^\nu + \mathbf{1} \otimes \widehat{v}_\mu, \tag{9a}$$

$$\Delta(\widehat{\Lambda}_\nu^\mu) = \widehat{\Lambda}_\rho^\mu \otimes \widehat{\Lambda}_\nu^\rho; \tag{9b}$$

i.e., the composition of two quantum Poincaré group transformations is described by standard classical formulae.

The κ-deformed Minkowski space described in formula (9a) by $\widehat{x} = \widehat{v}_\mu \otimes \mathbf{1}$ satisfies the relations (6a), or, more explicitly,

$$[\widehat{x}_0, \widehat{x}_i] = \frac{i}{\kappa}\widehat{x}_i, \qquad [\widehat{x}_i, \widehat{x}_j] = 0. \tag{10}$$

The κ-deformed field theory is described by the operator functions $\Phi_A(\widehat{x})$. Following the arguments given in [14, 17], we shall use for the fields $\Phi_A(\widehat{x})$ the κ-deformed Fourier transform

$$\Phi_A(\widehat{x}) = \frac{1}{(2\pi)^4}\int d^4p\,\widetilde{\Phi}_\kappa(p) : e^{ip\widehat{x}} :, \tag{11}$$

where

$$: e^{ip\widehat{x}} :\equiv e^{-ip_0\widehat{x}_0} e^{i\mathbf{p}\cdot\widehat{\mathbf{x}}} \tag{12}$$

and

$$\widetilde{\Phi}_\kappa(p) = e^{\frac{3p_0}{\kappa}}\,\widetilde{\Phi}\left(e^{\frac{p_0}{\kappa}}\,\mathbf{p}, p_0\right). \tag{13}$$

We have

$$: e^{ip\widehat{x}} :: e^{ip'\widehat{x}} :=: e^{i\Delta^{(2)}(p,p')\widehat{x}} :, \tag{14}$$

where $\Delta_\mu^{(2)} = (\Delta_0^{(2)} = p_0 + p_0', \Delta_i^{(2)} = p_i e^{\frac{p_0'}{\kappa}} + p_i')$.

The algebraic relation (14) is translated into the star-product framework by the replacement $\widetilde{x}_\mu \to x_\mu$, where x_μ are classical spacetime coordinates and the ordering in Eq. (12) is reflected in explicit choice of the star multiplication:

$$e^{ipx} \star e^{ip'x} = e^{i\Delta^{(2)}(p,p')x}; \tag{15}$$

i.e., after replacement $\Phi(\widehat{x}) \to \phi(x)$, one gets

$$\phi(x) \star \chi(x) \tag{16}$$

$$= \frac{1}{(2\pi)^4}\int d^4p\,d^4p'\,\widetilde{\phi}_\kappa(p)\widetilde{\chi}_\kappa(p') e^{i\Delta^{(2)}(p,p')x}.$$

In the following section, we shall extend the star product (15), (16), valid for the noncommutative translations, to the whole quantum κ-deformed Poincaré group.

3. THE STAR PRODUCT FOR κ-DEFORMED POINCARÉ GROUP

In order to extend the action of Poincaré group on Minkowski space to the noncommutative case, we have to replace the classical Poincaré group by its κ-deformed counterpart

$$(a_\mu, \Lambda^\mu_\nu) \Longrightarrow \left(\widehat{a}_\mu, \widehat{\Lambda}^\mu_\nu\right). \qquad (17)$$

The noncommutativity of the symmetry group parameters raises the question of the physical interpretation of deformed symmetries. In this chapter, we shall show how one can replace the operator algebra of functions on κ-deformed Poincaré group (6a)–(6c) by the functions on classical Poincaré group, with suitably chosen star product multiplication.

We shall consider the algebra of the following ordered exponentials:

$$: e^{i(\alpha_\mu \widehat{v}^\mu + b^\nu_\mu \widehat{\Lambda}^\mu_\nu)} := e^{-i\alpha_0 \widehat{v}_0} e^{i\boldsymbol{\alpha}\mathbf{v}} e^{ib^\mu_\nu \widehat{\Lambda}^\nu_\mu}. \qquad (18)$$

The product of two ordered exponentials (18) is given by the formula

$$: e^{i\alpha_\mu \widehat{v}^\mu + ib^\nu_\mu \widehat{\Lambda}^\mu_\nu} :: e^{i\alpha'_\mu \widehat{v}^\mu + ib'^\nu_\mu \widehat{\Lambda}^\mu_\nu} : \qquad (19)$$
$$=: e^{i\Delta^{(2)}_\mu(\alpha,\alpha')\widehat{v}^\mu} : e^{i(b^\nu_\mu f^\mu_\nu(g^\rho_\sigma(\widehat{\Lambda},\boldsymbol{\alpha}'),\alpha) + b^\nu_\mu \widehat{\Lambda}^\mu_\nu)},$$

where

$$e^{-i\lambda\alpha^0} \widehat{\Lambda}^\kappa_\nu e^{i\lambda\alpha^0} = f^\kappa_\nu\left(\widehat{\Lambda}, \lambda\right), \qquad (20a)$$

$$e^{-i\boldsymbol{\lambda\alpha}} \widehat{\Lambda}^\kappa_\nu e^{i\boldsymbol{\lambda\alpha}} = g^\kappa_\nu\left(\widehat{\Lambda}, \boldsymbol{\lambda}\right). \qquad (20b)$$

The functions f^κ_ν and g^κ_ν can be calculated explicitly. The functions defined by (20a) read

$$f^0_0(\widehat{\Lambda}, \lambda) = \tanh\frac{\lambda}{\kappa}\left(\frac{1 + \coth\frac{\lambda}{\kappa}\widehat{\Lambda}^0_0}{1 + \tanh\frac{\lambda}{\kappa}\widehat{\Lambda}^0_0}\right), \qquad (21)$$

$$f^0_k(\widehat{\Lambda}, \lambda) = \left(\cosh\frac{\lambda}{\kappa}\right)^{-1} \frac{\widehat{\Lambda}^0_k}{1 + \tanh\frac{\lambda}{\kappa}\widehat{\Lambda}^0_0},$$

$$f^k_0(\widehat{\Lambda}, \lambda) = \left(\cosh\frac{\lambda}{\kappa}\right)^{-1} \frac{\Lambda^k_0}{1 + \tanh\frac{\lambda}{\kappa}\widehat{\Lambda}^0_0},$$

$$f^i_k(\widehat{\Lambda}, \lambda) = \frac{\widehat{\Lambda}^i_k + \tanh\frac{\lambda}{\kappa}\left(\widehat{\Lambda}^0_0\widehat{\Lambda}^i_k - \widehat{\Lambda}^i_0\widehat{\Lambda}^0_k\right)}{1 + \tanh\frac{\lambda}{\kappa}\,\widehat{\Lambda}^0_0}.$$

The calculation of (20b) is more complicated, but also possible. They are described by the following set of relations:

$$e^{-i\zeta a^k}\Lambda^0_0 e^{i\zeta a^k} = \frac{\left(\frac{\Lambda^0_0 - 1}{2\kappa^2}\right)\zeta^2 - \frac{\Lambda^k_0}{\kappa}\zeta + \Lambda^0_0}{\frac{\left(\Lambda^0_0 - 1\right)}{2\kappa^2}\zeta^2 - \frac{\Lambda^k_0}{\kappa}\zeta + 1}, \qquad (22a)$$

$$e^{-i\zeta a^k}\Lambda^k_0 e^{i\zeta a^k} = \frac{-\left(\Lambda^0_0 - 1\right)\frac{\zeta}{\kappa} + \Lambda^k_0}{\frac{\left(\Lambda^0_0 - 1\right)}{2\kappa^2}\zeta^2 - \frac{\Lambda^k_0}{\kappa}\zeta + 1}, \qquad (22b)$$

$$e^{-i\zeta a^k}\Lambda^i_0 e^{i\zeta a^k} = \frac{\Lambda^i_0}{\frac{\left(\Lambda^0_0 - 1\right)}{2\kappa^2}\zeta^2 - \frac{\Lambda^k_0}{\kappa}\zeta + 1}, \qquad (22c)$$

$$e^{-i\zeta, a^k}\Lambda^0_k e^{i\zeta a^k} \qquad (22d)$$
$$= \frac{\Lambda^0_k + \frac{\Lambda^k_k\left(\Lambda^0_0 - 1\right) + \Lambda^0_k\Lambda^k_0}{\kappa}\zeta}{\frac{\left(\Lambda^0_0 - 1\right)}{2\kappa^2}\zeta^2 - \frac{\Lambda^k_0}{\kappa}\zeta + 1},$$

$$e^{-i\zeta a^k}\Lambda^k_k e^{i\zeta a^k} \qquad (22e)$$
$$= \frac{-\frac{\left(\Lambda^k_k\left(\Lambda^0_0 - 1\right) + \Lambda^0_k\Lambda^k_0\right)}{2\kappa^2}\zeta^2 - \frac{\Lambda^0_k\zeta}{\kappa} + \Lambda^k_k}{\frac{\left(\Lambda^0_0 - 1\right)}{2\kappa^2}\zeta^2 - \frac{\Lambda^k_0}{\kappa}\zeta + 1},$$

$$e^{-i\zeta a^k}\Lambda^0_i e^{i\zeta a^k} \qquad (22f)$$
$$= \frac{\Lambda^0_i + \frac{\left(\Lambda^0_0 - 1\right) + \Lambda^k_i + \Lambda^k_0\Lambda^0_i}{\kappa}\zeta}{\frac{\left(\Lambda^0_0 - 1\right)}{2\kappa^2}\zeta^2 - \frac{\Lambda^k_0}{\kappa}\zeta + 1},$$

$$e^{-i\zeta a^k}\Lambda^k_i e^{i\zeta a^k} \qquad (22g)$$
$$= \frac{\Lambda^k_i - \frac{\Lambda^0_i}{\kappa}\zeta + \frac{\left(\Lambda^0_0 - 1\right) + \Lambda^k_i + \Lambda^k_0\Lambda^0_i}{2\kappa^2}\zeta^2}{\frac{\left(\Lambda^0_0 - 1\right)}{2\kappa^2}\zeta^2 - \frac{\Lambda^k_0}{\kappa}\zeta + 1},$$

for $i \neq k$

$$e^{-i\zeta a^k}\Lambda_k^i e^{i\zeta a^k} = \Lambda_k^i - \frac{1}{\kappa}\int_0^\zeta d\mu \frac{\Lambda_0^i\left[-\frac{\Lambda_k^k\left(\Lambda_0^0-1\right)+\Lambda_k^0\Lambda_0^k}{2\kappa^2}\mu^2 - \frac{\Lambda_k^0}{\kappa}\mu + \Lambda_k^k\right]}{\left[\frac{\left(\Lambda_0^0-1\right)}{2\kappa^2}\mu^2 - \frac{\Lambda_0^k}{\kappa}\mu + 1\right]^2}, \qquad (22h)$$

for $i \neq k$, $j \neq k$

$$e^{-i\zeta a^k}\Lambda_j^i e^{i\zeta a^k} = \Lambda_j^i - \frac{1}{\kappa}\int_0^\zeta d\mu \frac{\Lambda_0^i\left[-\frac{\left(\Lambda_0^0-1\right)\Lambda_j^k+\Lambda_0^k\Lambda_j^0}{2\kappa^2}\mu^2 - \frac{\Lambda_k^0}{\kappa}\mu + \Lambda_j^k\right]}{\left[\frac{\left(\Lambda_0^0-1\right)}{2\kappa^2}\mu^2 - \frac{\Lambda_0^k}{\kappa}\mu + 1\right]^2}. \qquad (22i)$$

In order to represent the relation (19) in star-product framework, we reproduce the multiplication (19) by a new star product of the basic functions on the classical Poincaré group parameters:

$$e^{i(\alpha_\kappa v^\kappa + b_\mu^\nu \Lambda_\nu^\mu)} \circledast e^{i(\alpha'_\kappa v^\kappa + b_\mu^{\nu\prime}\Lambda_\nu^\mu)} \qquad (23)$$
$$= e^{i(\Delta_\mu(\alpha,\alpha')v^\kappa + b_\mu^\nu f_\nu^\kappa(g_\sigma^\rho(1,\boldsymbol{\alpha}'),\alpha_0) + b_\mu^{\nu\prime}\Lambda_\nu^\kappa)}.$$

As is seen from (22a)–(22i), the functions g_σ^ρ are not linear in $\widehat{\Lambda}_\nu^\kappa$, due to the quadratic commutator (6b). On the other hand, due to the commutativity (6c), the formulae for f_ν^κ and g_ν^κ can be obtained in an explicit form.

4. FINAL REMARKS

We would like to make the following comments.

(i) It should be observed that the nice coproduct formula (15) for noncommutative translations cannot be extended to the Lorentz sector. One can pose the question whether by a suitable choice of noncommutative b_ν^μ such extension can be achieved.

(ii) The relations (6a)–(6c) describe a quadratic algebra, which implies that in the exponential on the rhs of (23) there are arbitrary powers of Λ_ν^κ. It should be observed, however, that the multiplication formula (23) has an explicit form.

(iii) Using the star product (23), one can discuss the covariance of κ-deformed local field theory under the κ-deformed relativistic transformations. At present, it is only clear how to show in the star-product framework the covariance under the subgroup of noncommutative translations (see also [14]).

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SYMPOSIUM ON QUANTUM GROUPS

Poincaré–Lie Algebra and Noncommutative Differential Calculus*

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Abstract—A realization of Poincaré–Lie algebra in terms of noncommutative differential calculus was constructed. Corresponding relativistic quantum mechanics was considered. © *2001 MAIK "Nauka/Interperiodica".*

In this contribution, we apply the noncommutative differential calculus to construct the realizations of the Poincaré group Lie algebra in the relativistic configurational space [1] and related aspects of the relativistic quantum mechanics (RQM) [1–7] in this space. It is sufficient for our purposes to limit ourselves to the simplest case of the commutative algebra of functions. The transfer to the generalized calculus [8–13] can be described by the following vocabulary, in which the first two columns are taken from the book [11] and the third column corresponds to our commutative case.

Classical	Quantum	Present consideration
Complex variable	Operator in $\mathcal{H}$	Function of real variable
Real variable	Self-adjoint operator in $\mathcal{H}$	$\psi^*(z) = \psi(z^*)$
Infinitesimal	Compact operator in $\mathcal{H}$	Compact operator in $\mathcal{H}$
Differential of real or complex variable	$d\psi = [F, \psi] = F\psi - \psi F$	$d\psi = [F, \psi]$

The passage from the classical formula to the operator one is similar to the substitution of the Poisson brackets $\{\psi, \chi\}$ with commutators $[\psi, \chi]$ in the process of quantization. The standard Leibniz rule is valid in the noncommutative differential calculus

$$d(\psi\chi) = [F, \psi\chi] \tag{1}$$
$$= [F, \psi]\chi + \psi[F, \chi] = d(\psi)\chi + \psi d(\chi).$$

This calculus is introduced [8–13] using the theory of differential forms as its deformation.

We refer the reader to [7], where the one-dimensional case was considered, and concentrate here on a case of a particle moving in the space with two spatial dimensions. The mass shell of the particle

$$p^\mu p_\mu = 1 \tag{2}$$

modells the two-dimensional Lobachevsky space.

We are working in the unit system $\hbar = c = m$. The momentum

$$p^\mu = (p^0, \tilde{p}) = (p^0, p^1, p^2) \tag{3}$$

can also be parametrized by hyperpolar coordinates

$$p^0 = \cosh\chi, \quad p^1 = \sinh\chi\cos\phi, \tag{4}$$
$$p^2 = \sinh\chi\sin\phi.$$

The Fourier expansion in the relativistic plane waves

$$\langle\tilde{\rho}|\tilde{p}\rangle = (p^0 - \tilde{p}\tilde{n})^{-i\rho-\frac{1}{2}} \tag{5}$$
$$= (\cosh\chi - \sinh\chi\cos(\phi - \psi))^{-i\rho-\frac{1}{2}}$$

or the Gelfand–Graev transformation brings us into the relativistic configurational ρ space

$$\tilde{\rho} = \rho\tilde{n}, \quad 0 \le \rho < \infty, \tag{6}$$
$$\tilde{n} = (n^1, n^2) = (\cos\psi, \sin\psi).$$

The question arises: How are the three-dimensional Poincaré group $\mathcal{P}(3)$ transformations realized in two-dimensional ρ space? To define the generators, we formulate the evident properties of translation (momentum) operators $\widehat{p}^\mu$ in the ρ space:

1. The correspondence principle with the nonrelativistic two-dimensional QM must be fulfilled. In particular, the spatial components of $\widehat{p}^\mu$ must transfer into the standard first-order differential momentum operators, and the 0-component must transfer into a free nonrelativistic Hamiltonian operator.

2. To realize three components of the relativistic momentum $\widehat{p}^\mu$ in terms of two variables ρ and ψ, the generalized calculus must be unavoidably complex.

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As we shall see below, this can be achieved in the context of noncommutative differential calculus in which the differentials do not commute with independent variables in contradistinction to the usual calculus.

3. As the inclusion $O(2) \subset O(2,1)$ takes place, the angular part of the momentum operators $\widehat{p}^\mu$ contains a standard differentiation $\partial/\partial\psi$ in respect to ψ. This means that only radial differentiations are modified. According to item 1, components $\widehat{p}^{1,2}$ contain the first order of $\partial/\partial\psi$, but $\widehat{p}^0$ contains an angular derivative of the second order.

4. The momentum operators must reproduce the composition law for p^μ in the Lobachevsky space (2),

$$\widetilde{(p(-)k)} = \mathbf{p} - \mathbf{k}\left(\sqrt{1+p^2} - \frac{\mathbf{p}\cdot\mathbf{k}}{1+\sqrt{1+p^2}}\right),$$
$$(p(-)k)_0 = p^\mu k_\mu,$$

through the addition theorem for the plane waves (5), which has the nonlocal form

$$\int\limits_0^{2\pi} \langle\tilde{\rho}|\widetilde{(p(-)k)}\rangle d\psi = \int\limits_0^{2\pi} \langle\tilde{\rho}|\tilde{p}\rangle\langle\tilde{k}|\tilde{\rho}\rangle d\psi. \qquad (7)$$

Trying to derive the calculus which satisfies the requirements listed above, we introduce two triples of differentials, corresponding to $\pm i$ shifts

$$\widehat{d}^\mu_+ = \left(\widehat{d}^0_+, \widehat{d}^0_+ - e^{i\frac{\partial}{\partial\rho}}, \widehat{d}_\psi\right), \qquad (8)$$
$$\widehat{d}^\mu_- = \left(\widehat{d}^0_-, \widehat{d}^0_-, 0\right),$$

where

$$\widehat{d}^0_+ = \frac{1}{2\rho}\left\{\left(\rho + \frac{i}{2}\right) - \frac{1}{\rho + i/2}\frac{\partial^2}{\partial\psi^2}\right\}e^{i\frac{\partial}{\partial\rho}}, \qquad (9)$$
$$\widehat{d}^0_- = \frac{\rho - i/2}{2\rho}e^{-i\frac{\partial}{\partial\rho}}, \quad \widehat{d}_\psi = -\frac{i}{\rho + i/2}e^{i\frac{\partial}{\partial\rho}}\frac{\partial}{\partial\psi}.$$

The algebra of differentials $\widehat{d}^\mu_\pm$ and variables ρ and ψ is not closed in respect to commutators. But if we consider instead the weak commutators, i.e., averages of commutators over the $O(2)$ subgroup, or integrals over $d\psi$, then the algebra becomes closed and we have a consistent noncommutative differential calculus. To define the translations of quantum ρ space, we consider differentials $\widehat{d}^\mu = \widehat{d}^\mu_+ + \widehat{d}^\mu_-$. Then, the generators of translations or momentum operators are defined as the right derivatives corresponding to $\widehat{d}^\mu$. They are expressed as

$$\widehat{p}^0 = \frac{1}{\rho}\left\{\left[\left(\rho + \frac{i}{2}\right) - \frac{1}{\rho + i/2}\frac{\partial^2}{\partial\psi^2}\right]e^{i\frac{\partial}{\partial\rho}}\right\} \qquad (10)$$
$$+ \left(\rho - \frac{i}{2}\right)e^{-i\frac{\partial}{\partial\rho}} = \cosh i\frac{\partial}{\partial\rho} + \frac{i}{2\rho}\sinh i\frac{\partial}{\partial\rho}$$
$$- \frac{1}{2\rho(\rho + i/2)}\frac{\partial^2}{\partial\psi^2}e^{i\frac{\partial}{\partial\rho}},$$
$$\widehat{p}^1 = \cos\psi(p^0 - e^{i\frac{\partial}{\partial\rho}}) + \frac{i\sin\psi}{\rho + i/2}e^{i\frac{\partial}{\partial\rho}}\frac{\partial}{\partial\psi},$$
$$\widehat{p}^2 = \sin\psi(p^0 - e^{i\frac{\partial}{\partial\rho}}) - \frac{i\cos\psi}{\rho + i/2}e^{i\frac{\partial}{\partial\rho}}\frac{\partial}{\partial\psi}.$$

The plane wave (5) is the common eigenfunction for $\widehat{p}^\mu$:

$$\widehat{p}^\mu\langle\tilde{\rho}|\tilde{p}\rangle = p^\mu\langle\tilde{\rho}|\tilde{p}\rangle. \qquad (11)$$

The remaining generators of Poincaré group in ρ space are

$$M^{12} = -i\frac{\partial}{\partial\psi}, \qquad (12)$$
$$M^{01} = -\left(\rho - \frac{i}{2}\right)\cos\psi + i\sin\psi\frac{\partial}{\partial\psi},$$
$$M^{02} = -\left(\rho - \frac{i}{2}\right)\sin\psi - i\cos\psi\frac{\partial}{\partial\psi}.$$

The operators (10) are consistent in a weak sense with the addition theorem (7):

$$\int\limits_0^{2\pi}\left(\widehat{p}^\mu\langle\tilde{\rho}|\widetilde{(p(-)k)}\rangle\right)d\psi \qquad (13)$$
$$= \int\limits_0^{2\pi}\left\{\widehat{p}^\mu\langle\tilde{\rho}|\tilde{p}\rangle\langle\tilde{k}|\tilde{\rho}\rangle\right\}d\psi$$
$$= \int\limits_0^{2\pi}\left\{(\widehat{p}^\mu\langle\tilde{\rho}|\tilde{p}\rangle)\langle\tilde{k}|\tilde{\rho}\rangle + \langle\tilde{\rho}|\tilde{p}\rangle(\widehat{p}^\mu\langle\tilde{k}|\tilde{\rho}\rangle)\right\}d\psi.$$

By the sense of the Gelfand–Graev transformation, the integration is carried along any contour Γ which crosses all the generatrices of the light cone in the three-dimensional Minkowskian p space. In the case considered above, contour Γ emerges as the intersection of the hyperplane $p^0 = \text{const}$; i.e., it is a circle, which is a particular choice. In fact, we can write a more general equation than (13) (we write it for the 0-component) with integration along the contour Γ:

$$\int\limits_\Gamma\left(\widehat{p}^0AB^*\right)d\Gamma \qquad (14)$$
$$= \int\limits_\Gamma\left\{\left(\widehat{p}^0A\right)\left(\widehat{p}^0B\right)^* - \left(\tilde{\widehat{p}}A\right)\left(\tilde{\widehat{p}}B\right)^*\right\}d\Gamma,$$

where $A(\tilde{\rho})$ and $B(\tilde{\rho})$ are arbitrary functions of $\tilde{\rho}$.

In real three-dimensional space (or for higher dimensions in field theory), the integration in similar relations is carried over the closed hypersurfaces in ρ-like configurational space. The important conclusion from this is that field equations appear in the integral form.

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SYMPOSIUM ON QUANTUM GROUPS

Colored Extension of $GL_q(2)$ and Its Dual Algebra*

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Abstract—We address the problem of duality between the colored extension of the quantized algebra of functions on a group and that of its quantized universal enveloping algebra, i.e., its dual. In particular, we derive explicitly the algebra dual to the colored extension of $GL_q(2)$ using the colored RLL relations and exhibit its Hopf structure. This leads to a colored generalization of the R-matrix procedure to construct a bicovariant differential calculus on the colored version of $GL_q(2)$. In addition, we also propose a colored generalization of the geometric approach to quantum group duality given by Sudbery and Dobrev.

1. INTRODUCTION

The quantum group $GL_q(2)$ is known to admit a colored extension by introducing some continuously varying color parameters associated to the generators. In such an extension, the associated algebra and the coalgebra are defined in a way that all Hopf algebraic properties remain preserved. Such extensions have been introduced in [1–3] and studied by various authors [4–7] in recent years. However, some of the basic algebro-geometric structure underlying these colored extensions still need to be established. As such, we shall focus on the colored extension of the most intuitive quantum group $GL_q(2)$. While some aspects of this example have already been studied from both the standard q deformations and the Jordanian (nonstandard) h deformations [6, 7], it has only recently been shown [8] that the contraction procedure could be used to obtain the colored Jordanian quantum groups from their colored q-deformed counterparts. In particular, the colored extension of $GL_q(2)$ was treated in detail in [8] to obtain a new colored extension of Jordanian $GL_h(2)$.

In the present paper, we investigate the algebra dual to the colored extension of $GL_q(2)$ by generalizing two well-known approaches to the problem: the (algebraic) R-matrix approach [9] and the geometric approach [10, 11] due to Sudbery and Dobrev. We first clarify the notion of duality between a colored quantum group and its dual, i.e., the colored quantized universal enveloping algebra. We then generalize the R-matrix approach to establish duality for the colored extension of $GL_q(2)$, and we obtain a new colored quantum algebra corresponding to $gl(2)$ and exhibit its Hopf algebra structure. The colored R-matrix procedure naturally leads us to formulate a constructive differential calculus [12] on the colored extension of $GL_q(2)$.

Furthermore, we propose a colored generalization of the geometric notion of duality for quantum groups, i.e., regarding the dual algebra as the algebra of tangent vectors at the identity of the group. This generalization could also be of significance in establishing the duality for the colored extension of Jordanian quantum groups.

2. COLORED EXTENSION OF $GL_q(2)$

The colored extension of the quantum group $GL_q(2)$ is governed by the colored R matrix [4],

$$R_q^{\lambda,\mu} = \begin{pmatrix} q^{1-(\lambda-\mu)} & 0 & 0 & 0 \\ 0 & q^{\lambda+\mu} & 0 & 0 \\ 0 & q-q^{-1} & q^{-(\lambda+\mu)} & 0 \\ 0 & 0 & 0 & q^{1+(\lambda-\mu)} \end{pmatrix}, \quad (1)$$

which is nonadditive, i.e., $R^{\lambda,\mu} \neq R(\lambda-\mu)$. It satisfies the so-called colored quantum Yang–Baxter equation

$$R_{12}^{\lambda,\mu} R_{13}^{\lambda,\nu} R_{23}^{\mu,\nu} = R_{23}^{\mu,\nu} R_{13}^{\lambda,\nu} R_{12}^{\lambda,\mu}, \quad (2)$$

which is, in general, multicomponent, and λ, μ, ν are considered as "color" variables. The RTT relations are also extended to incorporate the colored extension as

$$R_q^{\lambda,\mu} T_{1\lambda} T_{2\mu} = T_{2\mu} T_{1\lambda} R_q^{\lambda,\mu} \quad (3)$$

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(where $T_{1\lambda} = T_\lambda \otimes \mathbf{1}$ and $T_{2\mu} = \mathbf{1} \otimes T_\mu$) in which the entries of the T matrices carry color dependence, i.e., $T_\lambda = \begin{pmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{pmatrix}$, $T_\mu = \begin{pmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{pmatrix}$. The coproduct and counit for the coalgebra structure are given by $\Delta(T_\lambda) = T_\lambda \dot{\otimes} T_\lambda$, $\varepsilon(T_\lambda) = \mathbf{1}$. The quantum determinant $D_\lambda = a_\lambda d_\lambda - q^{-(1+2\lambda)} c_\lambda b_\lambda$ is grouplike but not central. The antipode is given by

$$S(T_\lambda) = D_\lambda^{-1} \begin{pmatrix} d_\lambda & -q^{1+2\lambda} b_\lambda \\ -q^{-1-2\lambda} c_\lambda & a_\lambda \end{pmatrix} \quad (4)$$

and depends on one color variable at a time. The full Hopf algebraic structure can be constructed resulting in a colored extension of $GL_q(2)$ within the framework of the FRT formalism. Since λ and μ are continuous variables, this implies the colored extension of $GL_q(2)$ has an infinite number of generators. The colorless limit $\lambda = \mu = 0$ gives back the ordinary single-parameter deformed quantum group $GL_q(2)$, and the monochromatic limit $\lambda = \mu \neq 0$ gives rise to the uncolored two-parameter deformed quantum group $GL_{p,q}(2)$.

3. DUALITY (R-MATRIX APPROACH)

In this section, we investigate in detail the dual structure for the colored extension of $GL_q(2)$ employing the R-matrix approach. In doing so, let us denote the generators of the yet unknown dual algebra by $\{A_\lambda, B_\lambda, C_\lambda, D_\lambda\}$ and $\{A_\mu, B_\mu, C_\mu, D_\mu\}$. The following pairings hold:

$$\langle A_{\lambda|\mu}, a_{\lambda|\mu}\rangle = \langle B_{\lambda|\mu}, b_{\lambda|\mu}\rangle = \langle C_{\lambda|\mu}, c_{\lambda|\mu}\rangle = \langle D_{\lambda|\mu}, d_{\lambda|\mu}\rangle = \mathbf{1}. \quad (5)$$

All other pairings give zeros and the notation $\lambda|\mu$ in the subscript in the above relations means either λ or μ. The R^+ and R^- matrices corresponding to the colored extension of $GL_q(2)$ are

$$R^+ = c^+ q^{1/2} \begin{pmatrix} q^{-1/2} q^{1-\lambda+\mu} & 0 & 0 & 0 \\ 0 & q^{-1/2} q^{-(\lambda+\mu)} & q^{-1/2}(q - q^{-1}) & 0 \\ 0 & 0 & q^{-1/2} q^{\lambda+\mu} & 0 \\ 0 & 0 & 0 & q^{-1/2} q^{1+\lambda-\mu} \end{pmatrix}, \quad (6)$$

$$R^- = c^- q^{-1/2} \begin{pmatrix} q^{1/2} q^{-(1-\lambda+\mu)} & 0 & 0 & 0 \\ 0 & q^{1/2} q^{-(\lambda+\mu)} & 0 & 0 \\ 0 & -q^{1/2}(q - q^{-1}) & q^{1/2} q^{\lambda+\mu} & 0 \\ 0 & 0 & 0 & q^{1/2} q^{-(1+\lambda-\mu)} \end{pmatrix}, \quad (7)$$

where $R^+ = c^+ R_{21}$ and $R^- = c^- R_{12}^{-1}$ by definition. The colored $L^\pm$ functionals can be expressed as

$$L^+_{\lambda(\mu)} = c^+ q^{1/2} \begin{pmatrix} q^{H_{\lambda(\mu)}/2} q^{\mu H_{\lambda(\mu)} - \lambda H'_{\lambda(\mu)}} & q^{-1/2}(q - q^{-1}) C_{\lambda(\mu)} \\ 0 & q^{-H_{\lambda(\mu)}/2} q^{\mu H_{\lambda(\mu)} + \lambda H'_{\lambda(\mu)}} \end{pmatrix}, \quad (8)$$

$$L^-_{\lambda(\mu)} = c^- q^{-1/2} \begin{pmatrix} q^{-H_{\lambda(\mu)}/2} q^{\lambda H_{\lambda(\mu)} - \mu H'_{\lambda(\mu)}} & 0 \\ q^{1/2}(q^{-1} - q) B_{\lambda(\mu)} & q^{H_{\lambda(\mu)}/2} q^{\lambda H_{\lambda(\mu)} + \mu H'_{\lambda(\mu)}} \end{pmatrix}, \quad (9)$$

where $H_\lambda = A_\lambda - D_\lambda$, $H'_\lambda = A_\lambda + D_\lambda$ and $H_\mu = A_\mu - D_\mu$, $H'_\mu = A_\mu + D_\mu$. The notation $\lambda(\mu)$ in the subscript means λ (respectively, μ). So, $L^+_{\lambda(\mu)}$ means L^+_λ (respectively, L^+_μ). Each one of $L^\pm_\lambda$ and $L^\pm_\mu$ depends on both λ and μ. The notation $L^\pm_\lambda$ implies that the generators of the dual carry λ dependence, and similarly $L^\pm_\mu$ implies that the generators of the dual carry μ dependence. The duality pairings are then given by the action of the functionals $L^\pm_\lambda$ and $L^\pm_\mu$ on the T matrices T_λ and T_μ

$$(L^+_{\lambda|\mu})^a_b (T_{\lambda|\mu})^c_d = (R^+)^{ac}_{bd}, \quad (10)$$

$$(L^-_{\lambda|\mu})^a_b (T_{\lambda|\mu})^c_d = (R^-)^{ac}_{bd}. \quad (11)$$

Again, according to the notation introduced, $T_{\lambda|\mu}$ implies T_λ or T_μ and $L^{\pm}_{\lambda|\mu}$ implies $L^{\pm}_\lambda$ or $L^{\pm}_\mu$. For vanishing color variables, the colored $L^{\pm}$ functionals reduce to the ordinary $L^{\pm}$ functionals for $GL_q(2)$. The commutation relations of the algebra dual to a colored quantum group can be obtained from the modified or the colored RLL relations

$$R_{12}L^{\pm}_{2\lambda}L^{\pm}_{1\mu} = L^{\pm}_{1\mu}L^{\pm}_{2\lambda}R_{12}, \quad (12)$$

$$R_{12}L^{+}_{2\lambda}L^{-}_{1\mu} = L^{-}_{1\mu}L^{+}_{2\lambda}R_{12}, \quad (13)$$

using the colored $L^{\pm}$ functionals where $L^{\pm}_{1\mu} = L^{\pm}_{\mu} \otimes \mathbf{1}$ and $L^{\pm}_{2\lambda} = \mathbf{1} \otimes L^{\pm}_{\lambda}$. Using the above formulae, we obtain the commutation relations between the generating elements of the algebra dual to the colored extension of $GL_q(2)$

$$[A_\lambda, B_\mu] = B_\mu, \quad [D_\lambda, B_\mu] = -B_\mu, \quad (14)$$
$$[A_\lambda, C_\mu] = -C_\mu, \quad [D_\lambda, C_\mu] = C_\mu,$$
$$[A_\lambda, D_\mu] = 0, \quad [H_\lambda, H_\mu] = 0, \quad \left[H'_\lambda, \bullet\right] = 0,$$

$$q^{-(\lambda+\mu)}C_\lambda B_\mu - q^{\lambda+\mu}B_\mu C_\lambda \quad (15)$$
$$= \frac{q^{\lambda H_\mu + \mu H_\lambda}}{q-q^{-1}}\left[q^{-\frac{1}{2}(H_\lambda+H_\mu)}q^{\lambda H'_\lambda - \mu H'_\mu} - q^{\frac{1}{2}(H_\lambda+H_\mu)}q^{-\lambda H'_\lambda + \mu H'_\mu}\right],$$

$$A_\lambda A_\mu = A_\mu A_\lambda, \quad (16)$$
$$B_\lambda B_\mu = q^{2(\mu-\lambda)}B_\mu B_\lambda,$$
$$C_\lambda C_\mu = q^{2(\lambda-\mu)}C_\mu C_\lambda,$$
$$D_\lambda D_\mu = D_\mu D_\lambda,$$

where H_λ and H'_λ are as before. The relations satisfy the $\lambda \leftrightarrow \mu$ exchange symmetry. The associated coproduct of the elements of the dual algebra is given by

$$\Delta(A_{\lambda(\mu)}) = A_{\lambda(\mu)} \otimes \mathbf{1} + \mathbf{1} \otimes A_{\lambda(\mu)}, \quad (17)$$

$$\Delta(B_{\lambda(\mu)}) = B_{\lambda(\mu)} \otimes q^{A_{\lambda(\mu)}-D_{\lambda(\mu)}} + \mathbf{1} \otimes B_{\lambda(\mu)}, \quad (18)$$

$$\Delta(C_{\lambda(\mu)}) = C_{\lambda(\mu)} \otimes q^{A_{\lambda(\mu)}-D_{\lambda(\mu)}} + \mathbf{1} \otimes C_{\lambda(\mu)}, \quad (19)$$

$$\Delta(D_{\lambda(\mu)}) = D_{\lambda(\mu)} \otimes \mathbf{1} + \mathbf{1} \otimes D_{\lambda(\mu)}. \quad (20)$$

The counit $\varepsilon(Y_{\lambda|\mu}) = 0$, where $Y_{\lambda(\mu)} = \{A_{\lambda(\mu)}, B_{\lambda(\mu)}, C_{\lambda(\mu)}, D_{\lambda(\mu)}\}$ and the antipode is

$$S(A_{\lambda(\mu)}) = -A_{\lambda(\mu)}, \quad (21)$$

$$S(B_{\lambda(\mu)}) = -B_{\lambda(\mu)}q^{-(A_{\lambda(\mu)}-D_{\lambda(\mu)})}, \quad (22)$$

$$S(C_{\lambda(\mu)}) = -C_{\lambda(\mu)}q^{-(A_{\lambda(\mu)}-D_{\lambda(\mu)})}, \quad (23)$$

$$S(D_{\lambda(\mu)}) = -D_{\lambda(\mu)}. \quad (24)$$

Thus, we have defined a new single-parameter colored quantum algebra corresponding to $gl(2)$, which in the monochromatic limit defines the standard uncolored two-parameter quantum algebra for $gl(2)$.

4. CONSTRUCTIVE CALCULUS

We now proceed towards a colored generalization of the constructive differential calculus [12, 13] for the colored extension of $GL_q(2)$. Analogous to the standard uncolored quantum group, a bimodule Γ (space of quantum one-forms ω) is characterized by the commutation relations between ω and $a_{\lambda(\mu)} \in \mathcal{A}$, the colored quantum group corresponding to $GL_q(2)$

$$\omega a_{\lambda(\mu)} = (\mathbf{1} \otimes f_{\lambda,\mu})\Delta(a_{\lambda(\mu)})\omega, \quad (25)$$

and the linear functional $f_{\lambda,\mu}$ is defined in terms of the colored $L^{\pm}$ matrices

$$f_{\lambda,\mu} = S(L^{+}_{\lambda|\mu})L^{-}_{\lambda|\mu}. \quad (26)$$

Thus, we have

$$\omega a_{\lambda(\mu)} = [(\mathbf{1} \otimes S(L^{+}_{\lambda|\mu})L^{-}_{\lambda|\mu})\Delta(a_{\lambda(\mu)})]\omega. \quad (27)$$

In terms of components, this can be written as

$$\omega_{ij}a_{\lambda(\mu)} \quad (28)$$
$$= [(\mathbf{1} \otimes S(l^{+}_{(\lambda|\mu)ki})l^{-}_{(\lambda|\mu)jl})\Delta(a_{\lambda(\mu)})]\omega_{kl}$$

using the expressions $L^{\pm} = l^{\pm}_{ij}$ and $\omega = \omega_{ij}$, where $i, j = 1, 2$. From these relations, one can obtain the commutation relations of all the left-invariant one-forms with the elements of the colored extension of $GL_q(2)$. The left-invariant vector fields χ_{ij} on $\mathcal{A}$ are given by the expression

$$\chi_{ij} = S(l^{+}_{(\lambda|\mu)ik})l^{-}_{(\lambda|\mu)kj} - \delta_{ij}\varepsilon. \quad (29)$$

The vector fields act on the elements $a_{\lambda(\mu)}$ of the colored quantum group as

$$\chi_{ij}a_{\lambda(\mu)} = (S(l^{+}_{(\lambda|\mu)ik})l^{-}_{(\lambda|\mu)kj} - \delta_{ij}\varepsilon)a_{\lambda(\mu)}. \quad (30)$$

Furthermore, using the formula $\mathbf{d}a_{\lambda(\mu)} = \sum_i(\chi_i * *a_{\lambda(\mu)})\omega^i$, we obtain the action of the exterior derivative on the generating elements

$$\mathbf{d}a_{\lambda(\mu)} = (\mathbf{s}q^{-2+2(\lambda-\mu)} - 1)a_{\lambda(\mu)}\omega^1 + \mathbf{s}(q^{-1}-q)q^{\lambda+\mu}b_{\lambda(\mu)}\omega^+ + (\mathbf{s}-1)a_{\lambda(\mu)}\omega^2, \quad (31)$$

$$\mathbf{d}b_{\lambda(\mu)} = (\mathbf{s}(q^{-1}-q)^2 + \mathbf{s} - 1)b_{\lambda(\mu)}\omega^1 + \mathbf{s}(q^{-1}-q)q^{-(\lambda+\mu)}a_{\lambda(\mu)}\omega^- + (\mathbf{s}q^{-2+2(\mu-\lambda)} - 1)b_{\lambda(\mu)}\omega^2, \quad (32)$$

$$\mathbf{d}c_{\lambda(\mu)} = (\mathbf{s}q^{-2+2(\lambda-\mu)} - 1)c_{\lambda(\mu)}\omega^1 + \mathbf{s}(q^{-1}-q)q^{\lambda+\mu}d_{\lambda(\mu)}\omega^+ + (\mathbf{s}-1)c_{\lambda(\mu)}\omega^2, \quad (33)$$

$$\mathbf{d}d_{\lambda(\mu)} = (\mathbf{s}(q^{-1}-q)^2 + \mathbf{s} - 1)d_{\lambda(\mu)}\omega^1 + \mathbf{s}(q^{-1}-q)q^{-(\lambda+\mu)}c_{\lambda(\mu)}\omega^- + (\mathbf{s}q^{-2+2(\mu-\lambda)} - 1)d_{\lambda(\mu)}\omega^2, \quad (34)$$

where $\omega^1 = \omega_{11}, \omega^+ = \omega_{12}, \omega^- = \omega_{21}, \omega^2 = \omega_{22}$, and $\mathbf{s} = (c^+)^{-1}c^-$. $\mathbf{d}\mathcal{A}$ generates Γ as a left $\mathcal{A}$ module. This defines a first-order differential calculus $(\Gamma, \mathbf{d})$ on the colored extension of $GL_q(2)$. Since the color variables λ and μ are continuously varying, the differential calculus obtained is infinite-dimensional. The differential calculus on the uncolored single-parameter quantum group $GL_q(2)$ is easily recovered in the colorless limit, and that of the uncolored two-parameter quantum group $GL_{p,q}(2)$ in the monochromatic limit.

5. DUAL BASIS (GEOMETRIC APPROACH)

It is well known that two bialgebras $\mathcal{U}$ and $\mathcal{A}$ are in duality if there exists a doubly nondegenerate bilinear form

$$\langle , \rangle : \mathcal{U} \otimes \mathcal{A} \to \mathbf{C}; \quad \langle , \rangle : (u, a) \to \langle u, a \rangle; \quad \forall u \in \mathcal{U}, a \in \mathcal{A}, \tag{35}$$

such that, for $u, v \in \mathcal{U}$ and $a, b \in \mathcal{A}$, we have

$$\langle u, ab \rangle = \langle \Delta_{\mathcal{U}}(u), a \otimes b \rangle, \quad \langle uv, a \rangle = \langle u \otimes v, \Delta_{\mathcal{A}}(a) \rangle, \tag{36}$$

$$\langle \mathbf{1}_{\mathcal{U}}, a \rangle = \varepsilon_{\mathcal{A}}(a), \quad \langle u, \mathbf{1}_{\mathcal{A}} \rangle = \varepsilon_{\mathcal{U}}(u). \tag{37}$$

For the two bialgebras to be in duality as Hopf algebras, $\mathcal{U}$ and $\mathcal{A}$ further satisfy

$$\langle S_{\mathcal{U}}(u), a \rangle = \langle u, S_{\mathcal{A}}(a) \rangle. \tag{38}$$

It is enough to define the pairing between the generating elements of the two algebras. Pairing for any other elements of $\mathcal{U}$ and $\mathcal{A}$ follows from these relations and the bilinear form inherited by the tensor product. The geometric approach for duality for quantum groups was motivated by the fact that, at the classical level, an element of the Lie algebra corresponding to a Lie group is a tangent vector at the identity of the Lie group. Let $\mathcal{H}$ be a given Hopf algebra generated by noncommuting elements a, b, c, d. The q analog of the tangent vector at the identity would then be obtained by, first, differentiating the elements of the given Hopf algebra $\mathcal{H}$ (polynomials in a, b, c, d) and then setting $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ later on (i.e., taking the counit operation analogous to the unit element at the group level). The elements thus obtained would belong to the dual Hopf algebra $\mathcal{H}^*$. The approach is due to Sudbery [10] and Dobrev [11] and has proved to be quite a powerful tool in understanding the quantum group duality from a geometric point of view. In what follows in this section, we propose to give a colored generalization of such a geometric picture of duality using the example of $GL_q(2)$. Let $\mathcal{A}_q^{\lambda,\mu}$ denote the colored extension of $GL_q(2)$. Then, as a Hopf algebra $\mathcal{A}_q^{\lambda,\mu}$ is generated by elements $y_\lambda = \{a_\lambda, b_\lambda, c_\lambda, d_\lambda\}$ and $y_\mu = \{a_\mu, b_\mu, c_\mu, d_\mu\}$. The basis is given by all monomials of the form

$$g_\lambda = g_{\lambda;klmn} = a_\lambda^k d_\lambda^l b_\lambda^m c_\lambda^n, \quad g_\mu = g_{\mu;klmn} = a_\mu^k d_\mu^l b_\mu^m c_\mu^n, \tag{39}$$

where $k, l, m, n \in Z_+$, and δ_{0000} is the unit of the algebra $\mathbf{1}_{\mathcal{A}}$. We use a normal ordering as follows: first, put the diagonal elements from the $T_{\lambda(\mu)}$ matrix; then, use the lexicographic order for the others. Let $\mathcal{U}_q^{\lambda,\mu}$ be the algebra generated by tangent vectors at the identity of $\mathcal{A}_q^{\lambda,\mu}$. Then, $\mathcal{U}_q^{\lambda,\mu}$ is dually paired with $\mathcal{A}_q^{\lambda,\mu}$. The pairing is defined through the colored q-tangent vectors as follows:

$$\langle Y_\lambda, g_\lambda \rangle = \left.\frac{\partial g_\lambda}{\partial y_\lambda}\right|_{\left(\begin{smallmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} = \varepsilon\left(\frac{\partial g_\lambda}{\partial y_\lambda}\right), \tag{40}$$

$$\langle Y_\mu, g_\lambda \rangle = \left.\frac{\partial g_\lambda}{\partial y_\lambda}\right|_{\left(\begin{smallmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} = \varepsilon\left(\frac{\partial g_\lambda}{\partial y_\lambda}\right), \tag{41}$$

$$\langle Y_\lambda, g_\mu \rangle = \left.\frac{\partial g_\mu}{\partial y_\mu}\right|_{\left(\begin{smallmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} = \varepsilon\left(\frac{\partial g_\mu}{\partial y_\mu}\right), \tag{42}$$

$$\langle Y_\mu, g_\mu \rangle = \left.\frac{\partial g_\mu}{\partial y_\mu}\right|_{\left(\begin{smallmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} = \varepsilon\left(\frac{\partial g_\mu}{\partial y_\mu}\right), \tag{43}$$

where $Y_\lambda = \{A_\lambda, B_\lambda, C_\lambda, D_\lambda\}$ and $Y_\mu = \{A_\mu, B_\mu, C_\mu, D_\mu\}$ are the sets of generating elements of the dual algebra (which has unit $\mathbf{1}_{\mathcal{U}}$). More compactly, one can write

$$\langle Y_{\lambda|\mu}, g_{\lambda(\mu)} \rangle = \varepsilon\left(\frac{\partial g_{\lambda(\mu)}}{\partial y_{\lambda(\mu)}}\right). \tag{44}$$

Explicitly, we obtain

$$\langle A_{\lambda|\mu}, g_{\lambda(\mu)} \rangle = \varepsilon\left(\frac{\partial g_{\lambda(\mu)}}{\partial a_{\lambda(\mu)}}\right) = k\delta_{m0}\delta_{n0}, \tag{45}$$

$$\langle B_{\lambda|\mu}, g_{\lambda(\mu)} \rangle = \varepsilon\left(\frac{\partial g_{\lambda(\mu)}}{\partial b_{\lambda(\mu)}}\right) = \delta_{m1}\delta_{n0}, \tag{46}$$

$$\langle C_{\lambda|\mu}, g_{\lambda(\mu)} \rangle = \varepsilon\left(\frac{\partial g_{\lambda(\mu)}}{\partial c_{\lambda(\mu)}}\right) = \delta_{m0}\delta_{n1}, \tag{47}$$

$$\langle D_{\lambda|\mu}, g_{\lambda(\mu)} \rangle = \varepsilon\left(\frac{\partial g_{\lambda(\mu)}}{\partial d_{\lambda(\mu)}}\right) = l\delta_{m0}\delta_{n0}, \tag{48}$$

where differentiation is from the right. As a consequence of the above pairings, the following relations hold:

$$\langle A_{\lambda|\mu}, T_{\lambda|\mu} \rangle = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \tag{49}$$

$$\langle B_{\lambda|\mu}, T_{\lambda|\mu} \rangle = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \tag{50}$$

$$\langle C_{\lambda|\mu}, T_{\lambda|\mu} \rangle = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \tag{51}$$

$$\langle D_{\lambda|\mu}, T_{\lambda|\mu} \rangle = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \tag{52}$$

where $T_\lambda = \left(\begin{smallmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{smallmatrix}\right)$ and $T_\mu = \left(\begin{smallmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{smallmatrix}\right)$ as before. Furthermore,

$$\langle Y_{\lambda|\mu}, \mathbf{1}_{\mathcal{A}} \rangle = 0; \tag{53}$$

$$\langle \mathbf{1}_{\mathcal{U}}, g_{\lambda|\mu} \rangle = \varepsilon_{\mathcal{A}}(g_{\lambda|\mu}) = \delta_{m0}\delta_{n0}.$$

The action of the monomials in $\mathcal{U}_q^{\lambda,\mu}$ on g_λ and g_μ then leads to the colored q-commutation relations between the generators of the dual algebra.

6. CONCLUDING REMARKS

We have investigated the structure of the colored extension of the quantum group $GL_q(2)$ and its dual algebra. After establishing the notion of duality, the dual algebra has been derived explicitly using the R-matrix approach. We not only obtain a new colored quantum algebra corresponding to $gl(2)$ but also show that such a colored generalization of the R-matrix approach leads to the formulation of a constructive differential calculus for the colored case. The colorless and the monochromatic limits of both the dual algebra and the differential calculus are in agreement with already known results for $GL_q(2)$ and $GL_{p,q}(2)$.

In the preceding section of the paper, we have proposed a generalization of the geometric picture of duality to incorporate the colored extensions of quantum groups. It would be interesting to investigate in detail this setting in the context of the colored Jordanian quantum groups. The results easily extend to the higher dimensional and multiparametric cases.

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SYMPOSIUM ON QUANTUM GROUPS

Classification of Representations of the Algebra $U_q'(so_3)$ through Examples II*

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Abstract—This paper completes series of articles devoted to classification of the representations of the nonstandard deformation $U_q'(so_3)$ providing examples of such representations in low dimensions. The classification differs substantially when the deformation parameter q is/is not root of unity ($q^n = 1$). When it is a root of unity, the situation differs for odd and even n. The examples presented here cover the first nontrivial case when n is even (namely, $n = 4$), from which the general case follows easily. *© 2001 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

This paper is a continuation of the paper [1], which gave a classification of the representations of the nonstandard deformation $U_q'(so(3,\mathbb{C}))$ (for convenience, we write $U_q'(so_3)$). $U_q'(so_3)$ is defined as the complex associative algebra with unit element generated by the elements I_1, I_2, I_3 satisfying the defining relations ($q \in \mathbb{C} - \{0, \pm 1\}$ is the deformation parameter):

$$[I_1, I_2]_q := q^{1/2} I_1 I_2 - q^{-1/2} I_2 I_1 = I_3,$$
$$[I_2, I_3]_q := q^{1/2} I_2 I_3 - q^{-1/2} I_3 I_2 = I_1,$$
$$[I_3, I_1]_q := q^{1/2} I_3 I_1 - q^{-1/2} I_1 I_3 = I_2.$$

This algebra was studied in [2, 3]. This article supplies examples of representations which were classified in [2]. Therefore, extensive use of the paper [2] is assumed throughout this article.

2. REPRESENTATIONS WHEN q IS NOT ROOT OF UNITY

Finite-dimensional representations when q is not root of unity are studied in [2], Section IV. These are divided into two main groups: representations of classical and nonclassical types.

Representations of classical type admit the limit $q \to 1$. An example in dimension $r = 4$ is given by following formulas:

$$I_3 = -i \begin{pmatrix} \left[-\frac{3}{2}\right]_q & 0 & 0 & 0 \\ 0 & \left[-\frac{1}{2}\right]_q & 0 & 0 \\ 0 & 0 & \left[\frac{1}{2}\right]_q & 0 \\ 0 & 0 & 0 & \left[\frac{3}{2}\right]_q \end{pmatrix},$$

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$$I_1 = \begin{pmatrix} 0 & \frac{-q^{-1/2}}{q^{1/2}+q^{-1/2}} & 0 & 0 \\ -\frac{q^{1/2}[-1]_q[3]_q}{q^{3/2}+q^{-3/2}} & 0 & \frac{-q^{-1/2}}{q^{1/2}+q^{-1/2}} & 0 \\ 0 & -\frac{q^{1/2}[-2]_q[2]_q}{q^{1/2}+q^{-1/2}} & 0 & \frac{-q^{-1/2}}{q^{3/2}+q^{-3/2}} \\ 0 & 0 & -\frac{q^{1/2}[-3]_q[1]_q}{q^{1/2}+q^{-1/2}} & 0 \end{pmatrix},$$

$$C = -q\left[\frac{3}{2}\right]_q\left[\frac{5}{2}\right]_q,$$

where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$ and the second relation determines value of Casimir element $C = q^2I_1^2 + I_2^2 + q^2I_3^2 - (q^{5/2} - q^{1/2})I_1I_2I_3$. There is one representation of classical type per each dimension $r \in \mathbb{N}$.

Representations of nonclassical type do not admit the limit $q \to 1$. An example in dimension $r = 4$ is given by following formulas:

$$I_3 = \varepsilon \begin{pmatrix} \left\{\frac{7}{2}\right\}_q & 0 & 0 & 0 \\ 0 & \left\{\frac{5}{2}\right\}_q & 0 & 0 \\ 0 & 0 & \left\{\frac{3}{2}\right\}_q & 0 \\ 0 & 0 & 0 & \left\{\frac{1}{2}\right\}_q \end{pmatrix},$$

$$I_1 = \varepsilon \begin{pmatrix} 0 & \frac{iq^{1/2}[1]_q[7]_q}{q^{5/2}-q^{-5/2}} & 0 & 0 \\ \frac{iq^{-1/2}}{q^{7/2}-q^{-7/2}} & 0 & \frac{iq^{1/2}[2]_q[6]_q}{q^{3/2}-q^{-3/2}} & 0 \\ 0 & \frac{iq^{-1/2}}{q^{5/2}-q^{-5/2}} & 0 & \frac{iq^{1/2}[3]_q[5]_q}{q^{1/2}-q^{-1/2}} \\ 0 & 0 & \frac{iq^{-1/2}}{q^{3/2}-q^{-3/2}} & \varepsilon'\frac{[4]_q}{q^{1/2}-q^{-1/2}} \end{pmatrix},$$

$$C = q\left\{\frac{7}{2}\right\}_q\left\{\frac{9}{2}\right\}_q,$$

where $\{x\}_q = (q^x + q^{-x})/(q - q^{-1})$ and $\varepsilon, \varepsilon' \in \{-1, 1\}$ (all four combinations). There are precisely four representations of nonclassical type per each dimension $r \in \mathbb{N}$.

The following important classification theorem is proved in [2]. We present a special version suitable for dimension 4:

Theorem 1. *Any four-dimensional irreducible representation of $U'_q(so_3)$, when q is not root of unity, is equivalent to one of five nonequivalent irreducible representations described above.*

3. REPRESENTATIONS WHEN q IS A ROOT OF UNITY

When $q^n = 1$, the situation differs when n is even or odd. Let us consider the first nontrivial example, when n is even, in closer detail, namely, $n = 4$. According to theorems from [2], Section V, all representations of $U'_q(so_3)$ in the case when q is a root of unity can be divided into two groups: nonsingular and singular.

A. Nonsingular representations. A representation is called nonsingular if there exists a vector $x_0 \neq 0$ from representation space and $\nu \in \mathbb{C}$ such that $I_3 x_0 = -i[\nu]_q x_0$ and $q^\nu \notin \{i\varepsilon q^{-k/2} | k = 0, \ldots, n-1; \varepsilon = \pm 1\}$. Otherwise it is called singular.

Nonsingular representations cover in particular one (generic) three-parameter family of representations and some exceptional representations.

At three-parameter family of representations has dimension n (here, $n = 4$) and is given by the following formulas:

$$I_3 = -i \begin{pmatrix} [\nu]_q & 0 & 0 & 0 \\ 0 & [\nu+1]_q & 0 & 0 \\ 0 & 0 & [\nu+2]_q & 0 \\ 0 & 0 & 0 & [\nu+3]_q \end{pmatrix},$$

$$I_1 = \begin{pmatrix} 0 & \frac{(-\alpha\beta + q[1]_q[2\nu]_q)q^{-1/2}}{q^{\nu+1}+q^{-\nu-1}} & 0 & \frac{\alpha q^{-1/2}}{q^{\nu+3}+q^{-\nu-3}} \\ \frac{q^{-1/2}}{q^\nu + q^{-\nu}} & 0 & \frac{(-\alpha\beta + q[2]_q[2\nu+1]_q)q^{-1/2}}{q^{\nu+2}+q^{-\nu-2}} & 0 \\ 0 & \frac{q^{-1/2}}{q^{\nu+1}+q^{-\nu-1}} & 0 & \frac{(-\alpha\beta + q[3]_q[2\nu+2]_q)q^{-1/2}}{q^{\nu+3}+q^{-\nu-3}} \\ -\frac{\beta q^{-1/2}}{q^\nu + q^{-\nu}} & 0 & \frac{q^{-1/2}}{q^{\nu+2}+q^{-\nu-2}} & 0 \end{pmatrix},$$

$$C = -\alpha\beta - q[\nu]_q[\nu-1]_q = \frac{1}{4}(q^{2\nu} - q^{-2\nu}) - \alpha\beta,$$

$$C^{(4)}(I_3) = \frac{1}{64}(q^{2\nu} - q^{-2\nu})^2, \quad C^{(4)}(I_1) = \frac{(\alpha\beta^2 - 1)((q^{2\nu} - q^{-2\nu} - 2\alpha\beta)^2 - 4\alpha)}{16(q^{2\nu} - q^{-2\nu})^2},$$

where $\alpha, \beta, \nu \in \mathbb{C}$, $q^\nu \notin \{i\varepsilon q^{-k/2} | k = 0, 1, 2, 3;\ \varepsilon = \pm 1\}$, and the last two equations determine values of Casimir elements $C^{(4)}(x) = \frac{1}{4}(x^4 + x^2)$.

In dimension $n/2$ (i.e., 2) arises a family of representations which depends on one complex parameter:

$$I_3 = -i \begin{pmatrix} [\nu]_q & 0 \\ 0 & [\nu+1]_q \end{pmatrix},$$

$$I_1 = \begin{pmatrix} 0 & \frac{q^{1/2}[1]_q[2\nu]_q}{q^{\nu+1}+q^{-\nu-1}} \\ \frac{q^{-1/2}}{q^\nu + q^{-\nu}} & 0 \end{pmatrix},$$

$$C = \frac{1}{4}(q^{2\nu} - q^{-2\nu}),$$

$$C^{(4)}(I_1) = C^{(4)}(I_2) = -\frac{1}{16},$$

$$C^{(4)}(I_3) = \frac{1}{64}(q^{2\nu} - q^{-2\nu})^2,$$

where $\nu \in \mathbb{C}$, $q^\nu \notin \{i\varepsilon q^{-k/2} | k = 0, 1, 2, 3; \varepsilon = \pm 1\}$.

Exceptional representations arise in dimensions $<n/2$. For dimension 1, we get a trivial zero representation.

The following classification theorem is proved in [2]. Again, we present special version valid for $q^4 = 1$:

Theorem 2. *Any irreducible nonsingular representation of $U'_q(so_3)$ when $q^4 = 1$ is equivalent to one of the representations described above.*

B. Singular representations. Singular representations form families which depend on less continuous parameters than nonsingular representations, but they are, together with exceptional representations, harder to describe. We say that a singular representation has a weight vector if there exists a vector $x_0 \neq 0$ from representation space and $\nu, \mu \in \mathbb{C}$ such that $I_3 x_0 = -i[\nu]_q x_0$ and $(iI_2 + q^{-\nu+1/2} I_1)x_0 = \mu x_0$.

A family of singular representations which does not have a weight vector arises in dimension n, and an example in dimension $n = 4$ is given by the following formulas:

$$I_3 = \begin{pmatrix} -\varepsilon q & 0 & 0 & 0 \\ 0 & 0 & iq^{3/2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon q \end{pmatrix},$$

$$I_1 = \begin{pmatrix} 0 & \frac{i\varepsilon}{2} q^{1/2} C_1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -i\varepsilon q^{-1/2}(2C_1\gamma - 1) \\ 1 & 0 & 0 & -C_1 \\ 0 & \frac{i\varepsilon}{2} q^{1/2} & \gamma & 0 \end{pmatrix},$$

$$C = C_1,$$

$$C^{(4)}(I_1) = \frac{1}{16}(4C_1^2\gamma^2 - 1),$$

$$C^{(4)}(I_2) = \frac{1}{16}(4C_1^2(1+\gamma)^2 - 1),$$

$$C^{(4)}(I_3) = 0,$$

where $C_1, \gamma \in \mathbb{C}$, $\varepsilon \in \{-1, 1\}$.

Now, we describe singular representations having a weight vector. It is convenient to divide them into two main groups according to eigenvalue $-i[\nu]_q$ of the vector x_0 of the operator I_3. We write $\{i\varepsilon q^{-k/2} | k = 0, \ldots, n-1; \varepsilon = \pm 1\} = M_n \cup M_n'$, where $M_n = \{i\varepsilon q^{-1/2-m} | m = 0, 1, \ldots, n/2 - 1; \varepsilon = \pm 1\}$, $M_n' = \{i\varepsilon q^{-m} | m = 0, 1, ..., n/2 - 1; \varepsilon = \pm 1\}$.

Let us first describe the representations which have eigenvectors from M_n. They arise in dimensions 1, 2, and 4:

$$I_3 = \left(-\frac{\varepsilon q}{\sqrt{2}}\right), I_1 = \left(\frac{\varepsilon\varepsilon' q}{\sqrt{2}}\right),$$

$$C^{(4)}(I_1) = C^{(4)}(I_2) = C^{(4)}(I_3) = -\frac{1}{16}, \quad C = \frac{q}{2},$$

where $\varepsilon, \varepsilon' \in \{-1, 1\}$;

$$I_3 = \begin{pmatrix} \frac{\varepsilon q}{\sqrt{2}} & 0 \\ 0 & -\frac{\varepsilon q}{\sqrt{2}} \end{pmatrix},$$

$$I_1 = \begin{pmatrix} -\frac{\varepsilon\varepsilon'' \sqrt{C_1 + \frac{q}{2}}\, q^{1/2}}{\sqrt{2}} & \frac{i\varepsilon q^{1/2}}{\sqrt{2}} \\ \frac{-i\varepsilon\left(-C_1 + \frac{q}{2}\right) q^{1/2}}{\sqrt{2}} & \frac{\varepsilon\varepsilon' \sqrt{C_1 + \frac{q}{2}}\, q^{1/2}}{\sqrt{2}} \end{pmatrix},$$

$$C^{(4)}(I_1) = \frac{1}{32}(4C^2(1 - \varepsilon'\varepsilon'') - 1 - \varepsilon'\varepsilon''),$$

$$C^{(4)}(I_2) = \frac{1}{32}(4C^2(1 + \varepsilon'\varepsilon'') - 1 + \varepsilon'\varepsilon''),$$

$$C^{(4)}(I_3) = -\frac{1}{16}, \quad C = C_1,$$

where $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$, $C_1 \in \mathbb{C}$;

$$I_3 = \begin{pmatrix} -\frac{\varepsilon q}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{\varepsilon q}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{\varepsilon q}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & -\frac{\varepsilon q}{\sqrt{2}} \end{pmatrix}, \quad I_1 = \begin{pmatrix} \frac{-i\varepsilon q^{1/2}\beta}{\sqrt{2}} & \frac{-i\varepsilon q^{1/2}}{\sqrt{2}} & 0 & 0 \\ \frac{i\varepsilon q^{1/2}\left(q+\beta^2\right)}{\sqrt{2}} & 0 & \frac{i\varepsilon q^{1/2}}{\sqrt{2}} & 0 \\ 0 & \frac{i\varepsilon q^{1/2}\beta^2}{\sqrt{2}} & 0 & \frac{i\varepsilon q^{1/2}}{\sqrt{2}} \\ \frac{-i\varepsilon q^{1/2}\alpha}{\sqrt{2}} & 0 & \frac{-i\varepsilon q^{1/2}\left(q+\beta^2\right)}{\sqrt{2}} & \frac{i\varepsilon q^{1/2}\beta}{\sqrt{2}} \end{pmatrix},$$

$$C^{(4)}(I_1) = \frac{1}{16}q(2\beta^2 + q(1 + \alpha - 2\beta^4)), \quad C^{(4)}(I_2) = -\frac{1}{16}q(-2\beta^2 + q(-1 + \alpha + 2\beta^4)),$$

$$C^{(4)}(I_3) = -\frac{1}{16}, \quad C = -\frac{q}{2} - \beta^2,$$

where $\varepsilon \in \{-1, 1\}$, $\alpha, \beta \in \mathbb{C}$.

Representations which have eigenvectors from M'_n arise in dimensions 2, 3, and 4:

$$I_3 = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon q \end{pmatrix},$$

$$I_1 = \begin{pmatrix} 0 & -i\varepsilon q^{3/2} \\ \frac{i}{2}\varepsilon q^{1/2} & 0 \end{pmatrix},$$

$$C^{(4)}(I_1) = C^{(4)}(I_2) = -\frac{1}{16}, \quad C^{(4)}(I_3) = C = 0,$$

where $\varepsilon \in \{-1, 1\}$;

$$I_3 = \begin{pmatrix} \varepsilon q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\varepsilon q \end{pmatrix},$$

$$I_1 = \begin{pmatrix} 0 & \frac{i}{2}\varepsilon q^{1/2} & 0 \\ -i\varepsilon q^{3/2} & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \end{pmatrix},$$

$$C^{(4)}(I_1) = C^{(4)}(I_2) = C^{(4)}(I_3) = C = 0,$$

where $\varepsilon \in \{-1, 1\}$;

$$I_3 = \begin{pmatrix} \varepsilon q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon q \end{pmatrix},$$

$$I_1 = \begin{pmatrix} 0 & \frac{i}{2}\varepsilon q^{1/2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \gamma \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix},$$

$$C^{(4)}(I_1) = C^{(4)}(I_2) = \frac{-i\varepsilon q^{1/2}\gamma}{16},$$

$$C^{(4)}(I_3) = C = 0,$$

where $\varepsilon \in \{-1, 1\}$, $\gamma \in \mathbb{C}$.

The following classification theorem which is proved in [2] is presented here for the case $q^4 = 1$:

Theorem 3. *Any irreducible singular representation of $U'_q(so_3)$ when $q^4 = 1$ is equivalent to one of the representations described above.*

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SYMPOSIUM ON QUANTUM GROUPS

Dynamical Cremmer–Gervais R Matrix*

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Abstract—SL-type zero-graded solutions of the dynamical Yang–Baxter equation in dimension 3 are classified. In addition to the well-known Drinfeld–Jimbo-type dynamical R matrices, the classification of so-called "regular" cases includes a quantization of the classical dynamical r matrix found by O. Schiffmann and a dynamical partner of the constant Cremmer–Gervais R matrix. Nonperturbative effects are exhibited. © *2001 MAIK "Nauka/Interperiodica"*.

The dynamical R matrix is an operator $\hat{R}(p)$ acting on a tensor square of an N-dimensional vector space V and depending on a set of integer parameters $p_i \in \mathbb{Z}$, $i = 1, 2, \ldots, N$. The operator $\hat{R}(p)$ satisfies the dynamical Yang–Baxter equation (DYBE) [1]

$$\hat{R}_{12}X_1\hat{R}_{23}X_1^{-1}\hat{R}_{12} = X_1\hat{R}_{23}X_1^{-1}\hat{R}_{12}X_1\hat{R}_{23}X_1^{-1}. \quad (1)$$

Here, X is a diagonal matrix with operator entries,

$$X := \mathrm{diag}\{x^1, \ldots, x^N\}. \quad (2)$$

The operators x^j act on the dynamical variables p_i by

$$p_i x^j = x^j(p_i + \delta_i^j). \quad (3)$$

In the present paper, we shall address a question of classification of solutions of the DYBE in dimension $N = 3$ which additionally satisfy three more conditions:

$\hat{R}(p)$ has degree zero.

$\hat{R}(p)$ is of the GL type.

$\hat{R}(p)$ admits a dynamical SL reduction.

The "zero-degree" condition means that all components $\hat{R}^{ij}_{kl}$ of the matrix $\hat{R}(p)$ are zeros unless $i + j = k + l$. Therefore, the zero-graded R matrix $\hat{R}^{ij}_{kl}$ takes the block-diagonal form each block corresponds to a fixed value of $\sigma := i + j = k + l$. Let us place the (only possible) nonzero components of such an R matrix with the same number σ into matrices $A^{(\sigma)}$, $\sigma = 2, \ldots, 2N$ ($N = 3$ in the present text). The size of the matrix $A^{(\sigma)}$ is equal to the least of the numbers $(\sigma - 1)$ and $(2N - \sigma + 1)$. The components $\hat{R}^{i(\sigma-i)}_{k(\sigma-k)}$ are arranged in such a way that the values of their first-space indices i and k increase rightwards and downwards in $A^{(\sigma)}$.

The matrix $\hat{R}(p)$ possesses the GL-type property if, by definition, it has two different eigenvalues q and $-q^{-1}$ ($q \in \mathbb{C}^\times$) and, more precisely, for the spectrums of the blocks $A^{(\sigma)}$ one has

$$\begin{cases} A^{(2)} = A^{(6)} = q, \\ \mathrm{Spec}(A^{(3)}) = \mathrm{Spec}(A^{(5)}) = \{q, -q^{-1}\}, \\ \mathrm{Spec}(A^{(4)}) = \{q, q, -q^{-1}\}. \end{cases} \quad (4)$$

This means that the eigenvalues q and $-q^{-1}$ of the $\hat{R}(p)$ are distributed in the same way as the eigenvalues 1 and -1 in the permutation matrix.

Finally, imposing the dynamical SL-reduction condition, one demands the operator $\prod_{i=1}^N x^i$ to commute with the operator $\hat{R}(p)$. This implies that the entries of the $\hat{R}(p)$ depend on the differences $p_{ij} := p_i - p_j$ only.

Our notation for the components of the blocks $A^{(3)}$, $A^{(4)}$, and $A^{(5)}$ is the following:

$$A^{(3)} = \begin{pmatrix} a & b^+ \\ b^- & \lambda - a \end{pmatrix}, \quad A^{(5)} = \begin{pmatrix} \lambda - a' & b'^- \\ b'^+ & a' \end{pmatrix}, \quad (5)$$

$$A^{(4)} = \begin{pmatrix} c & e^+ & f^+ \\ e^- & d & e'^- \\ f^- & e'^+ & c' \end{pmatrix}. \quad (6)$$

Here, $\lambda := q - q^{-1}$ and the elements of the matrices $A^{(*)}$ are supposed to be functions of p_{13} and p_{23}.

Substituting the ansatz (5), (6) into the DYBE, we get a list of equations for the matrix elements.

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The SL property implies that $b^+b^- \neq 0$, $b'^+b'^- \neq 0$. However, the product f^+f^- might vanish. A solution will be called regular if $f^+f^- \neq 0$. We shall first describe regular cases. The independent set of equations contained in the DYBE is (for any function $F := F(p)$, the function shifted in the ith argument is denoted as $F_{(i)} := x^i F(p)(x^i)^{-1}$)

$$a_{(1)} = \frac{q^{-1}a}{q-a}, \quad a'_{(3)} = \frac{q^{-1}a'}{q-a'} \tag{7}$$

and

$$\begin{aligned}(q-d_{(1)})(q-a) &= (q-d)(q^{-1}+a_{(2)}) \\ &= 1-(q-a_{(2)})(q^{-1}+a), \\ (q-c'_{(1)})(q-c) &= (q-a_{(2)})(q^{-1}+a), \\ (q-d_{(3)})(q-a') &= (q-d)(q^{-1}+a'_{(2)}) \\ &= 1-(q-a'_{(2)})(q^{-1}+a'), \\ (q-c_{(3)})(q-c') &= (q-a'_{(2)})(q^{-1}+a'), \\ q-d &= \frac{(q^{-1}+a')(a-a_{(3)})}{q-a_{(3)}} \\ &= \frac{(q^{-1}+a)(a'-a'_{(1)})}{q-a'_{(1)}}, \\ q-d_{(2)} &= \frac{(q-a'_{(1)})(a-a_{(3)})}{q^{-1}+a} \\ &= \frac{(q-a_{(3)})(a'-a'_{(1)})}{q^{-1}+a'}, \\ (q-c)(a_{(3)}-a'_{(1)}) &= (q-c_{(2)})(a-a') \\ &= 1-(q-a)(q^{-1}+a'_{(1)}), \\ (q-c_{(2)}) &= \frac{(q^{-1}+a'_{(1)})(q-a_{(3)})}{(q^{-1}+a')(q-a)}(q-c).\end{aligned} \tag{8}$$

These are equations for the diagonal entries. The evalution of the off-diagonal terms is nontrivial only for the $A^{(4)}$ block. The corresponding equations are

$$\begin{aligned} e'^+_{(1)} &= -\frac{b^-b^-_{(2)}}{(q-a)(q-c)}e^+, \\ e'^-_{(1)} &= -\frac{b^+b^+_{(2)}}{(q-a)(q-c)}e^-, \\ e^+_{(3)} &= -\frac{b'^-b'^-_{(2)}}{(q-a')(q-c')}e'^+, \\ e^-_{(3)} &= -\frac{b'^+b'^+_{(2)}}{(q-a')(q-c')}e'^-, \end{aligned} \tag{9}$$

$$e^+_{(2)} = \frac{b'^-_{(1)}(q-a_{(3)})}{b^+(q^{-1}+a')}e^+, \quad e'^+_{(2)} = \frac{b^-_{(3)}(q-a'_{(1)})}{b'^+(q^{-1}+a)}e'^+,$$

$$e^-_{(2)} = \frac{b'^+_{(1)}(q-a_{(3)})}{b^-(q^{-1}+a')}e^-, \quad e'^-_{(2)} = \frac{b^+_{(3)}(q-a'_{(1)})}{b'^-(q^{-1}+a)}e'^-.$$

The lists (7), (8), and (9) give the full set of equations imposed by the DYBE. The solutions fall into three essentially different cases.

Case 1. $\boldsymbol{e^+ = e^- = e'^+ = e'^- = 0}$. Here,

$$d = q, \quad c' = \lambda - c \tag{10}$$

and

$$c = \frac{a(\lambda - a')}{a-a'}. \tag{11}$$

The dynamics of a and a' is given by

$$\begin{aligned} a_{(2)} &= \frac{qa}{q^{-1}+a}, \quad a_{(3)} = a, \\ a'_{(2)} &= \frac{qa'}{q^{-1}+a'}, \quad a'_{(1)} = a'. \end{aligned} \tag{12}$$

Equations (11) and (12) imply the following dynamics for c:

$$c_{(1)} = \frac{q^{-1}c}{q-c}, \quad c_{(3)} = \frac{qc}{q^{-1}+c}, \quad c_{(2)} = c. \tag{13}$$

The only restrictions on the off-diagonal components $b^\pm$, $b'^\pm$, and $f^\pm$ of the matrices (5), (6) are given by the GL-eigenvalue conditions

$$\begin{aligned} b^+b^- &= (q-a)(q^{-1}+a), \\ b'^+b'^- &= (q-a')(q^{-1}+a'), \\ f^+f^- &= (q-c)(q-c'). \end{aligned} \tag{14}$$

Equations (7) and (12) were solved in [2] in arbitrary dimension N. This solution is a dynamical analog of the multiparametric Drinfeld–Jimbo R matrix.

The general solution for $N = 3$ reads

$$\begin{aligned} a &= a(p_{12}) = \frac{\beta q^{-p_{12}}}{q^{p_{12}} - \beta [p_{12}]_q}, \\ a' &= a'(p_{23}) = \frac{\beta' q^{p_{23}}}{q^{-p_{23}} + \beta' [p_{23}]_q}, \end{aligned} \tag{15}$$

where $[n]_q := (q^n - q^{-n})/\lambda$ is a q number, and β and β' are arbitrary parameters.

It follows that

$$c(p_{13}) = \frac{\beta'' q^{-p_{13}}}{q^{p_{13}} - \beta'' [p_{13}]_q} \tag{16}$$

with $\beta''(\beta - \beta') = \beta(\lambda - \beta')$.

Case 2. $\boldsymbol{e^- = e'^- = 0}$, $\boldsymbol{e^+ \neq 0}$, $\boldsymbol{e'^+ \neq 0}$ **or** $\boldsymbol{e^+ = e'^+ = 0}$, $\boldsymbol{e^- \neq 0}$, $\boldsymbol{e'^- \neq 0}$. These two subcases are related by the transposition symmetry of the R matrix, and we will consider the case where e^+ and e'^+ are different from zero.

The dynamics of a, a', and c is given by the same formulas (15) and (16).

For the off-diagonal elements, one obtains

$$f^+ = -(q^{-1}+c)\frac{e^+}{e'^+}, \quad f^- = -(q-c)\frac{e'^+}{e^+} \qquad (17)$$

and

$$e^+ = a(\lambda - a')c\rho, \quad e'^+ = a(\lambda - a')c\mu, \qquad (18)$$
$$b^- = q^{-1}\nu, \quad b'^- = q\tau,$$

where μ, ν, ρ, τ are given by

$$\mu = \mu(p_{13}) + \overline{\mu}(p_{13})\zeta^{p_{23}} + \overline{\overline{\mu}}(p_{13})\zeta^{-p_{23}}, \qquad (19)$$
$$\nu = \nu(p_{13}) + \overline{\nu}(p_{13})\zeta^{p_{23}} + \overline{\overline{\nu}}(p_{13})\zeta^{-p_{23}},$$

and

$$\rho = -\frac{\mu_{(1)}}{\nu\nu_{(2)}}, \quad \tau = \frac{\mu_{(2)}}{\mu\nu_{(3)}}. \qquad (20)$$

Here, μ, $\overline{\mu}$, $\overline{\overline{\mu}}$, ν, $\overline{\nu}$, $\overline{\overline{\nu}}$ are arbitrary functions, and ζ is a primitive cubic root of unity, $1+\zeta+\zeta^{-1}=0$. The appearance of cubic roots of unity in the SL-reduced dynamical R matrix is a purely "nonperturbative" effect which can not be recognized at the quasiclassical level.

The R matrix appearing in the considered case is a dynamical partner of the (multiparametric) Cremmer–Gervais R matrix [3].

Case 3. $\boldsymbol{e^{\pm}}$, $\boldsymbol{e'^{\pm} \neq 0}$. In this case, the dynamics for a is given by

$$a(p_{13}) = \frac{\beta q^{-p_{13}}}{q^{p_{13}} - \beta [p_{13}]_q}, \qquad (21)$$

where β is an arbitrary constant.

All the other diagonal elements can be expressed in terms of a:

$$q^{-1} + a' = (q^{-1}+a)^{-1}, \qquad (22)$$

$$q - d = \frac{1-(q-a)(q^{-1}+a)}{q^{-1}+a}, \qquad (23)$$

$$q - c = -\frac{1-(q-a)^2}{1-(q^{-1}+a)^2}(q^{-1}+a), \qquad (24)$$

$$q - c' = \frac{(q^{-1}+a)^2 - (q^{-2}+[2]_q a)^2}{\{1-(q^{-1}+a)^2\}(q^{-1}+a)}. \qquad (25)$$

For the off-diagonal elements, one has

$$f^+ = -(q-c')\frac{e^+}{e'^+}, \quad f^- = -(q-c)\frac{e'^+}{e^+}, \qquad (26)$$

$$e^- = \frac{(q-c)(q-d)}{e^+}, \quad e'^- = \frac{(q-c')(q-d)}{e'^+} \qquad (27)$$

and

$$e^+ = \rho, \quad e'^+ = (q-c')(q-a')\mu, \qquad (28)$$
$$b^- = (q-a)(q^{-1}+a)\nu, \quad b'^- = \tau,$$

where ρ, μ, ν, and τ are given by formulas (19) and (20).

The dependence of this R matrix on the second variable p_{23} is discrete. In the absence of nonperturbative effects (when the functions $\overline{\mu}$, $\overline{\overline{\mu}}$, $\overline{\nu}$, and $\overline{\overline{\nu}}$ vanish), the R matrix admits a quasiclassical limit in which it can be identified with the classical r matrix constructed in [4].

The description of regular cases is complete.

Irregular case. $\boldsymbol{f^+f^- = 0}$. In this case, the block $A^{(4)}$ is fixed to be diagonal and constant:

$$A^{(4)} = \mathrm{diag}\{q, -q^{-1}, q\}. \qquad (29)$$

In addition, the diagonal elements a and a' of the blocks $A^{(3)}$ and $A^{(5)}$ are frozen as well, $a = a' = 0$ or λ. The remaining off-diagonal elements $b^{\pm}$ and $b'^{\pm}$ are constrained by the conditions (14) only.

DISCUSSION

From our point of view, the most important issue of this work is the construction of the dynamical analog of the Cremmer–Gervais R matrix (case 2). Explicitly, for this dynamical R matrix, the 2×2 and 3×3 blocks are

$$A^{(3)} = \begin{pmatrix} a & \dfrac{q(q-a)(q^{-1}+a)}{\nu} \\ \dfrac{\nu}{q} & \lambda - a \end{pmatrix}, \quad A^{(5)} = \begin{pmatrix} \lambda - a' & \dfrac{q\mu_{(2)}}{\mu\nu_{(3)}} \\ \dfrac{(q-a')(q^{-1}+a')\mu\nu_{(3)}}{q\mu_{(2)}} & a' \end{pmatrix}, \qquad (30)$$

$$A^{(4)} = \begin{pmatrix} c & -\dfrac{a(\lambda-a')c\mu_{(1)}}{\nu\nu_{(2)}} & \dfrac{(q^{-1}+c)\mu_{(1)}}{\mu\nu\nu_{(2)}} \\ 0 & q & 0 \\ \dfrac{(q-c)\mu\nu\nu_{(2)}}{\mu_{(1)}} & a(\lambda-a')c\mu & \lambda - c \end{pmatrix}.$$

Here, the expressions for a, a', and c are given by Eqs. (15) and (16), and for μ and ν, by Eqs. (19).

The corresponding classical dynamical r matrix is absent in the classification given in [4]. The reason is that the DYBE (1) does not coincide with the quantum analog of equations considered in [4]. The difference is in the range of the dynamical variables p_i. In [4], for the Cremmer–Gervais case, the dynamics is allowed only in the direction p_{13}. There is indeed a particular solution (30) depending exclusively on p_{13} ($\beta = \beta'' = \lambda$, $\beta' = 0$ in Eqs. (15) and (16)). The dynamics of the diagonal elements a, a', and c freezes, and one can show then that the R matrix is gauge-equivalent to the nondynamical one.

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SYMPOSIUM ON QUANTUM GROUPS

Yangians and $\mathcal{W}$ Algebras*

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Abstract—We present a connection between $\mathcal{W}$ algebras and Yangians, in the case of $gl(N)$ algebras, as well as for twisted Yangians and super-Yangians. This connection allows to construct an R matrix for the $\mathcal{W}$ algebras and to classify their finite-dimensional irreducible representations. We illustrate it in the framework of the nonlinear Schrödinger equation in 1 + 1 dimension. © *2001 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

$\mathcal{W}$ algebras have been introduced in the $2d$-conformal models as a tool for the study of these theories. Then, these algebras and their finite-dimensional versions appeared to be relevant in several physical backgrounds. However, a full understanding of their algebraic structure (and of their geometrical interpretation) is lacking. The connection of some of these $\mathcal{W}$ algebras with Yangians appears to be a solution at least for the algebraic structure: it allows the construction of an R matrix for $\mathcal{W}$ algebras, the classification of their irreducible finite-dimensional representations, and the determination of their center.

The paper is structured as follows: In Section 2 (Section 3), we recall some basic definitions for Yangians (for $\mathcal{W}$ algebras). Then, the connection between these two objects is presented in Section 4 for the case of $gl(N)$. Section 5 is devoted to a physical example where the connection explicitly appears, namely, the nonlinear Schrödinger equation in 1 + 1 dimension. The two following sections present various generalizations: the case of $so(M)$ and $sp(2M)$ algebras is studied in Section 6, and the case of superalgebras in Section 7. We conclude in Section 8.

To be reasonably short, we have chosen to detail, as an illustrative case, the study of $Y(N) \equiv Y(gl(N))$ and $\mathcal{W}_p(N) \equiv \mathcal{W}[gl(Np), N.sl(p)]$, while being less precise on the generalizations, sending the interested reader back to the original papers.

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2. YANGIAN $Y(\mathcal{G})$

Yangians $Y(\mathcal{G})$, associated to each simple Lie algebra $\mathcal{G}$, have been introduced by Drinfel'd as deformation of (half) a loop algebra based on $\mathcal{G}$ [1]. They have generators

$$Y(\mathcal{G}) = \mathcal{U}\,(Q^a_n, a = 1, \ldots, \dim(\mathcal{G}); \quad n = 0, 1, \ldots, \infty)\,; \tag{1}$$

n is the loop index and a labels the $\mathcal{G}$-adjoint representation. In other words, we have an infinite set of adjoint representations (labeled by n), the first one being $\mathcal{G}$ itself. This is gathered in the relations

$$[Q^a_0, Q^b_n] = f^{ab}{}_c\, Q^c_n. \tag{2}$$

The deformation appears in the remaining relations

$$[Q^a_m, Q^b_n] = f^{ab}{}_c\, Q^c_{m+n} + P^{ab}_{nm}(Q), \tag{3}$$

where P^{ab}_{nm} is a polynomial in the Q's.

Yangians are Hopf algebras, their coproduct being given by

$$\Delta(Q^a_0) = Q^a_0 \otimes \mathbb{I} + \mathbb{I} \otimes Q^a_0, \tag{4}$$

$$\Delta(Q^a_1) = Q^a_1 \otimes \mathbb{I} + \mathbb{I} \otimes Q^a_1 + \frac{1}{2} f^a{}_{bc}\, Q^b_0 \otimes Q^c_0, \tag{5}$$

which also shows the deformation with respect to the loop algebra coproduct.

There is a consistency relation, which takes the form of a Jacobi-like identity. When $\mathcal{G} \neq sl(2)$, this Jacobi-like identity takes the form

$$f^{bc}{}_d[Q^a_1, Q^d_1] + f^{ca}{}_d[Q^b_1, Q^d_1] + f^{ab}{}_d[Q^c_1, Q^d_1] = f^a{}_{pd} f^b{}_{qx} f^c{}_{ry} f^{xyd}\, s_3(Q^p_0, Q^q_0, Q^r_0), \tag{6}$$

and when $\mathcal{G} = sl(2)$, it takes the form

$$f^{cd}{}_e[[Q^a_1, Q^b_1], Q^e_1] + f^{ab}{}_e[[Q^c_1, Q^d_1], Q^e_1] \tag{7}$$
$$= \Big(f^a{}_{pe} f^b{}_{qx} f^{cd}{}_y f^y{}_{rz} f^{xz}{}_g$$

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$$+ f^{c}{}_{pe} f^{d}{}_{qx} f^{ab}{}_{y} f^{y}{}_{rz} f^{xz}{}_{g} \Big) \eta^{eg}\, s_3(Q_0^p, Q_0^q, Q_1^r),$$

where s_3 is the symmetrized product.

In the case of $\mathcal{G} = gl(N)$, Yangians admit an R-matrix presentation [2, 3]: gathering the generators into an $N \times N$ matrix and using a spectral parameter u, one defines

$$T(u) = \sum_{i,j=1}^{N} \sum_{n=0}^{\infty} u^{-n} T_n^{ij} E_{ij} \quad (8)$$

$$= \sum_{i,j=1}^{N} T^{ij}(u) E_{ij} \text{ with } T_0^{ij} = \delta^{ij}.$$

Then, the defining relations of $Y(gl(N)) = Y(N)$ become

$$R_{12}(u-v) T_1(u) T_2(v) \quad (9)$$
$$= T_2(v) T_1(u) R_{12}(u-v),$$
$$\Delta T(u) = T(u) \otimes T(u), \quad (10)$$
$$S(T(u)) = T(u)^{-1},\ \epsilon(T(u)) = 1,$$

where

$$R_{12}(x) = \mathbb{I}_N \otimes \mathbb{I}_N - \frac{1}{x} P_{12},$$

$$P_{12} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji},$$

$$T_1(u) = T(u) \otimes \mathbb{I}_N = \sum_{i,j=1}^{N} T^{ij}(u)\, E_{ij} \otimes \mathbb{I}_N, \quad (11)$$

$$T_2(u) = \mathbb{I}_N \otimes T(u) = \sum_{i,j=1}^{N} T^{ij}(u)\, \mathbb{I}_N \otimes E_{ij}.$$

R is a rational solution to the Yang–Baxter equation, and P_{12} is the permutation operator of the two auxiliary spaces (spanned by the $N \times N$ matrices).

2.1. Classical Yangians

In the following, we will be interested mainly in a classical version of the Yangians, where the commutators are replaced by Poisson brackets.

For the first presentation, the relations are the same, except for the Poisson bracket, which now replaces the commutator. For instance, relation (6) becomes

$$f^{bc}{}_{d}\{Q_1^a, Q_1^d\} + f^{ca}{}_{d}\{Q_1^b, Q_1^d\} \quad (12)$$
$$+ f^{ab}{}_{d}\{Q_1^c, Q_1^d\} = f^{a}{}_{pd} f^{b}{}_{qx} f^{c}{}_{ry} f^{xyd}\, Q_0^p Q_0^q Q_0^r.$$

In the case of $Y(N)$, the Poisson bracket appears as a classical version of the commutator:

$$R(x) = \mathbb{I} - \hbar\, r(x), \quad (13)$$
$$[.,.] = \hbar\{.,.\}, \quad T(u) = L(u),$$

$$\{L_1(u), L_2(v)\} \quad (14)$$
$$= [r_{12}(u-v), L_1(u) L_2(v)], \quad r(x) = \frac{1}{x} P_{12}.$$

3. $\mathcal{W}(\mathcal{G}, \mathcal{H})$ ALGEBRAS

$\mathcal{W}$ algebras were first introduced in the context of $2d$-conformal theories by Zamolodchikov [4] as a tool for classifying the irreducible unitary representations of these theories. Later, they were shown to be symmetries of Toda field theories [5, 6]. In this context, $\mathcal{W}$ algebras are constructed as a Hamiltonian reduction of affine (Kac–Moody) algebras. Later on, a simpler version of these algebras, called finite $\mathcal{W}$ algebras, was introduced by De Boer and Tjin [7]. They are constructed as a Hamiltonian reduction of finite-dimensional Lie algebras: the resulting algebra is a polynomial algebra with a finite number of generators.

More precisely, starting from a Poisson–Lie algebra $\mathcal{G}$, one constrains some of the generators of $\mathcal{G}$. The constraints are second class, and one considers the Dirac brackets deduced from these constraints: the $\mathcal{W}$ algebra is defined as the set of unconstrained generators provided with the Dirac brackets. The system of constraints is given by a subalgebra $\mathcal{H}$ of $\mathcal{G}$, hence the denomination $\mathcal{W}(\mathcal{G}, \mathcal{H})$ (see [5, 6, 8] for more details).

Here, we will be concerned with a class of finite $\mathcal{W}$ algebras: $\mathcal{W}[gl(Np), N.sl(p)]$ algebras. We will denote these algebras $\mathcal{W}_p(N)$. The generators of $\mathcal{W}_p(N)$ are finite in number:

$$\mathcal{W}_p(N) = \mathcal{U}\left(W_m^a,\ a = 1, 2, \ldots, N^2;\right. \quad (15)$$
$$\left. n = 1, 2, \ldots, p\right).$$

They obey

$$\{W_0^a, W_n^b\} = f^{ab}{}_{c} W_n^c \quad (16)$$
$$\text{and } \{W_m^a, W_n^b\} = f^{ab}{}_{c} W_{m+n}^c + P_{nm}^{ab}(W),$$

where $P_{nm}^{ab}(W)$ are polynomials in the W generators.

Its similarity with the Yangian presentation is quite appealing and has motivated the studies in this direction.

4. $Y(N)$ AND $\mathcal{W}_p(N)$

From the previous presentations, it is natural to seek a relation between $\mathcal{W}_p(N)$ algebras and Yangians $Y(N)$. Indeed, such a relation exists, and it has been proven in [9]:

Theorem 1. *There is an algebra homomorphism between $\mathcal{W}_p(N)$ algebras and Yangians $Y(N)$. More precisely, there is a one-to-one* *connection between the first pN^2 generators of $Y(N)$ and the generators of the $\mathcal{W}_p(N)$ algebra:*

$$Q_n^a \rightarrow \beta_n^a W_n^a + R_n^a(W) \text{ with } \beta_n^a \in \mathbb{R} \setminus \{0\}. \quad (17)$$

$R_n^a(W)$ are polynomials in the W_m^b with $m < n$. The remaining generators of $Y(N)$ are polynomials in the $\mathcal{W}$ generators.

It has been proven that the generators of the $\mathcal{W}$ algebra obey the Jacobi-like relations that define the Yangian.

The R-matrix approach is an easier way to tackle this relation [10]:

Theorem 2. *The $\mathcal{W}_p(N)$ algebra is isomorphic to the truncated Yangian $Y_p(N)$, defined by $Y_p(N) = Y(N)/\mathcal{J}_p$ with $\mathcal{J}_p$ ideal generated by $\mathcal{T}_p = \{T_n^{ij},\ i,j = 1,\ldots,N;\ n > p\}$.*

Thanks to this theorem, one gets an R-matrix formulation of the $\mathcal{W}_p(N)$ algebras:

$$\{W_1(u), W_2(v)\} = [r_{12}(u-v), W_1(u)W_2(v)]$$

with

$$W(u) = \sum_{i,j=1}^{N}\sum_{n=0}^{p} W_n^{ij} u^{-n} E_{ij} \text{ and } r(x) = \frac{1}{x}P_{12}.$$

Remark. The Hopf structure of $Y(N)$ does not survive the coset, so that the algebra isomorphism of Theorem 2 is in this sense a no-go theorem about the existence of a natural Hopf structure for $\mathcal{W}$ algebras.

One can also determine the center of the $\mathcal{W}_p(N)$ algebra:

Theorem 3. *The center of the $\mathcal{W}_p(N)$ has dimension Np and is canonically associated to the center of the underlying $gl(Np)$ algebra.*

Moreover, since the irreducible finite-dimensional representations of the Yangian have been classified, one can prove the following:

Theorem 4. *The finite-dimensional irreducible representations of $\mathcal{W}_p(N)$ are all highest weight representations and are in one-to-one correspondence with the families $\{P_1(u),\ldots,P_{N-1}(u),\rho(u)\}$, where $P_i(u)$ are polynomials of the form*

$$P_i(u) = \prod_{k=1}^{d_i}(u-\gamma_i^k) \quad (18)$$

$$\text{with } \sum_i d_i \le p \text{ and } \gamma_i^k \in \mathbb{C}$$

and $\rho(u) = 1 + \sum_{n=1}^{Np} c_n u^{-n}$ codes the values c_n of the Casimir operators in the representation.

All these representations are highest weight, the highest weight being reconstructed from the polynomials P_i through

$$\frac{\mu^i(u)}{\mu^{i+1}(u)} = \frac{P_i(u+1)}{P_i(u)},\ i = 1,\ldots,N, \quad (19)$$

with the highest weight vector ξ defined by

$$W^{ii}(u)\xi = \mu^i(u)\xi,\ 1 \le i \le N, \quad (20)$$
$$\text{and } W^{ij}(u)\xi = 0,\ 1 \le i < j \le N.$$

Finally, let us remark that a detailed analysis of the decomposition of $\mathcal{W}(\mathcal{G},\mathcal{H})$ algebras with respect to their Lie subalgebras (using the technique developed in [8]) shows that such a connection cannot exist with Yangians $Y(\mathcal{G})$ when $\mathcal{G} = so(N)$ or $sp(2N)$. Indeed, when $\mathcal{G}$ is not a $gl(N)$ algebra, there is no $\mathcal{W}(\mathcal{G},\mathcal{H})$ algebra such that all its generators are in adjoint representations of the Lie subalgebra of $\mathcal{W}(\mathcal{G},\mathcal{H})$. We will see below that the connection applies to objects different from the $Y(\mathcal{G})$ Yangians.

5. NONLINEAR SCHRÖDINGER EQUATION IN TWO DIMENSIONS

The nonlinear Schrödinger equation (NLS) in two dimensions is a nice framework where the connection between $Y(N)$ and $\mathcal{W}_p(N)$ can be visualized.

We start with

$$i\partial_t\Phi = \partial_x^2\Phi + g\,|\Phi|^2\Phi \quad (21)$$
$$\text{with } \Phi^t = (\varphi_1,\ldots,\varphi_N) \text{ and } g < 0.$$

The (quantum) solution to this equation has been known for a long time [11]. It takes the form

$$\Phi = \sum_{n=0}^{\infty} g^n\,\Phi_{(n)};$$

$$\Omega_n = q_0x - q_0^2t + \sum_{i=1}^{n}\Big((q_i-p_i)x - (q_i^2-p_i^2)t\Big),$$

$$\Phi_{(n)} = \int d^{n+1}q\,d^np\,a_1^\dagger(p_1)\ldots a_n^\dagger(p_n)$$
$$\times\, a_n(q_n)\ldots a_0(q_0)\frac{\exp(i\Omega_n)}{\prod_{i=1}^n(p_i-q_{i-1})(p_i-q_i)},$$

where the a's and $a^\dagger$'s obey a Zamolodchikov–Faddeev (ZZF) algebra [12]:

$$a_1(k_1)\,a_2(k_2) = R_{12}(k_2-k_1)\,a_2(k_2)\,a_1(k_1), \quad (22)$$

$$a_1^\dagger(k_1)\,a_2^\dagger(k_2) = a_2^\dagger(k_2)\,a_1^\dagger(k_1)\,R_{12}(k_2-k_1), \quad (23)$$

$$a_1(k_1)\,a_2^\dagger(k_2) = a_2^\dagger(k_2)\,R_{12}(k_1-k_2)\,a_1(k_1) + \delta_{12}(k_1-k_2), \quad (24)$$

where R is the matrix of the Yangian $Y(N)$. We use the notation

$$R_{12}(x) = R_{ij}^{kl}(x)\,E_{ij}\otimes E_{kl}, \quad E_{ij}v_k = \delta_{jk}v_i,$$
$$v_k^\dagger E_{ij} = \delta_{ik}v_j^\dagger,$$
$$a_1(k) = a_i(k)\,v_i\otimes\mathbb{I}, \quad a_2(k) = a_i(k)\,\mathbb{I}\otimes v_i,$$
$$v_iv_j^\dagger = \delta_{ij},$$

$$a_1^\dagger(k) = a_i(k)\, v_i^\dagger \otimes \mathbb{I}, \quad a_2^\dagger(k) = a_i(k)\, \mathbb{I} \otimes v_i^\dagger,$$
$$v_i^\dagger v_j = E_{ij}.$$

The apparition of the Yangian's R matrix is not surprising in this context, since the Yangian is a symmetry of NLS. Indeed, in [13] the generators of this algebra have been expressed in terms of the ZZF algebra. They take the form

$$Q_s^a = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} Q_{s,(n)}^a \text{ with } s = 0, 1, \quad (25)$$

$$Q_{s,(n)}^a = \int d^n k\, a_1^\dagger(k_1) \ldots a_n^\dagger(k_n) \quad (26)$$
$$\times J_{s,(n)}^a\, a_n(k_n) \ldots a_1(k_1),$$

where $J_{s,(n)}^a$ belongs to $M(N, \mathbb{C})^{\otimes n}(k_1, \ldots, k_n)$, $M(N, \mathbb{C})$ being the space of $N \times N$ matrices (see [13] for the exact expression). The Yangian is a symmetry of the whole hierarchy associated with NLS, as can be seen from the expression of the Hamiltonians H_m in terms of the ZZF algebra:

$$H_m = \int dk\, k^m\, a^\dagger(k) a(k) \Rightarrow [H_m, Q_s^a] = 0. \quad (27)$$

In fact, the generators $a^\dagger(k)$ correspond to the asymptotic states of the NLS hierarchy, and it is natural to look at the Fock space $\mathcal{F}$ spanned by the $a^\dagger$'s. This Fock space naturally decomposes into eigenspaces of the particle number H_0: $\mathcal{F} = \oplus_p \mathcal{F}_p$.

Now, on each subspace $\mathcal{F}_p$, the sums (25) truncate at level $n = p$, in the same way one defines $\mathcal{W}_p(N)$ from $Y(N)$.

Thus, on each subspace $\mathcal{F}_p$, the action of $Y(N)$ reduces to the $\mathcal{W}_p(N)$ algebra.

6. ORTHOGONAL AND SYMPLECTIC CASES

6.1. Folding $\mathcal{W}_p(N)$

It has been known for a long time that $so(M)$ and $sp(2M)$ algebras can be obtained from $gl(N)$ ones using their outer automorphism. If τ is such an automorphism, the $so(M)$ and $sp(2M)$ algebras are obtained as $\mathrm{Ker}(\mathbb{I} - \tau)$, i.e., the algebra of τ-invariant generators of $gl(N)$.

It is the same technique that is used for $\mathcal{W}$ algebras. Indeed, it has already been shown that $\mathcal{W}$ algebras based on $so(M)$ and $sp(2M)$ can be constructed from the ones based on $gl(N)$ [14]. The Hamiltonian reduction (i.e., the constraints) must be compatible with the folding (i.e., the automorphism), so that not all the $\mathcal{W}(gl(N), \mathcal{H})$ algebras can be folded. However, it is enough to produce all the $\mathcal{W}$ algebras based on $so(M)$ and $sp(2M)$.

Here, we will consider only the folding of $\mathcal{W}_p(N)$, which can indeed be folded. The automorphism we consider has been defined in [15]. It takes the form

$$\tau_\pm(W_n^{ij}) = (-1)^{n+1}\, \theta^i \theta^j\, W_n^{N+1-j, N+1-i} \quad (28)$$
$$\times \begin{cases} \theta^i = 1 \text{ for } \tau_+ \\ \theta^i = \mathrm{sgn}(\frac{N+1}{2} - i) \text{ for } \tau_- \text{ and } N = 2n. \end{cases}$$

The folded $\mathcal{W}$ algebra is then defined as follows.

Definition 1. *The folded $\mathcal{W}_p(N)^\pm$ algebra is defined by the coset $\mathcal{W}_p(N)/\mathcal{J}$, where $\mathcal{J}$ is the ideal generated by $W_n^{ij} - \tau_\pm(W_n^{ij})$.*

Note that $\mathcal{J}$ is an ideal for the product law, and one can show that the coset can be provided with the bracket of the $\mathcal{W}_p(N)$ algebra (see [15] for more detail). One then proves [14, 15] the following.

Theorem 5. *$\mathcal{W}_p(2n)^+$ (resp. $\mathcal{W}_p(2n)^-$, resp. $\mathcal{W}_p(2n+1)^+$ and $p = 2k+1$) is $\mathcal{W}[so(2np), n.sl(p)]$ (resp. $\mathcal{W}[sp(2np), n.sl(p)]$, resp. $\mathcal{W}[so((2n+1)p), n.sl(p) \oplus so(k)]$).*

6.2. Twisted Yangians

In the same way $\mathcal{U}[gl(Np)]$ and $\mathcal{W}_p(N)$ have been folded into $\mathcal{U}(\mathcal{G})$ and $\mathcal{W}(\mathcal{G}, \mathcal{H})$ with $\mathcal{G} = so(M)$ and $sp(2M)$, one naturally considers the case of $Y(N)$. However, although Yangians based on $so(M)$ and $sp(2M)$ exist, it is not these Hopf algebras that are obtained through this procedure, but another type of algebras, named twisted Yangians [16]. More precisely, the automorphism (28) takes here the form

$$\tau(T(u)) = T^t(-u) \text{ with } T^t(u) \quad (29)$$
$$= \sum_{i,j} T^{ij}(u) E_{ij}^t \text{ and } E_{ij}^t = \theta^i \theta^j E_{N+1-j, N+1-i}.$$

It can be shown that τ is an automorphism of $Y(N)$. From this automorphism, one defines

$$S(u) = T(u) \tau(T(u)). \quad (30)$$

Essentially, two classes of automorphisms appear, labeled by a parameter $\theta_0 = \pm 1$:

$$\text{for } Y^+(N) : \theta^i = 1, \forall i\ (\theta_0 = 1), \quad (31)$$
$$\text{for } Y^-(2n) : \theta^i = \mathrm{sgn}\left(\frac{N+1}{2} - i\right), \forall i\ (\theta_0 = -1).$$

This defines a subalgebra $Y^\pm(N)$ of $Y(N)$, whose commutation relations are coded in

$$R_{12}(u-v)\, S_1(u)\, R'_{12}(u+v)\, S_2(v) \quad (32)$$
$$= S_2(v)\, R'_{12}(u+v)\, S_1(u)\, R_{12}(u-v),$$

where $R(x)$ is given in (11) and

$$R'(x) = (\tau \otimes \mathbb{I})(R(x)) \quad (33)$$

$$= (\mathbb{I} \otimes \tau)(R(x)) = \mathbb{I} - \frac{1}{x} Q_{12}$$

with

$$Q_{12} = \sum_{i,j=1}^{N} \theta^i \theta^j E_{ij} \otimes E_{N+1-i,N+1-j}.$$

The finite-dimensional irreducible representations and the center of $Y^{\pm}(N)$ have been determined in [17].

At the classical level, $S(u)$ generates a Poisson subalgebra $Y(N)$, the Poisson brackets being defined by

$$\{S_1(u), S_2(v)\} = [r_{12}(u-v), S_1(u)S_2(v)] \quad (34)$$
$$+ S_2(v)r'_{12}(u+v)S_1(u) - S_1(u)r'_{12}(u+v)S_2(v),$$

where

$$r'_{12}(x) = (\mathbb{I} \otimes \tau) r_{12}(x) = (\tau \otimes \mathbb{I}) r_{12}(x).$$

The level-one generators of this subalgebra form the Lie algebra $so(M)$ or $sp(2M)$, but the total subalgebra is not the Yangian based on $so(M)$ or $sp(2M)$ (see [18] for more details). However, it is these algebras that are involved in the comparison with $\mathcal{W}$ algebras:

Theorem 6. *The truncated classical Yangians $Y_p^{\pm}(N)$ are $\mathcal{W}$ algebras. More precisely,*

$$Y_p(2n)^- \equiv \mathcal{W}[sp(2np), n.sl(p)], \quad (35)$$
$$Y_p(2n)^+ \equiv \mathcal{W}[so(2np), n.sl(p)],$$
$$Y_p(2n+1)^+ \equiv \mathcal{W}[so((2n+1)p),$$
$$n.sl(p) \oplus so(k)] \quad \textit{with } p = 2k+1,$$

where $\equiv$ denotes algebra isomorphisms. The truncation is defined as in Theorem 2.

Let us remark that, as in the case of $Y_p(N)$, the isomorphism cannot be extended to a Hopf algebra isomorphism, the (untruncated) twisted Yangians $Y(N)$ not even being Hopf algebras (only left coideals in $Y(N)$).

As for the Yangian $Y(N)$, this isomorphism provides a simple way of quantizing the $\mathcal{W}$ algebras. One can also use it to determine the center and the finite-dimensional irreducible representations of these $\mathcal{W}$ algebras (see [15] for more details).

7. GENERALIZATION TO SUPERALGEBRAS

Once again, one can apply the same technique to the case of super-Yangians and $\mathcal{W}$ superalgebras. As for $Y(N)$ and $gl(N)$, the case of $gl(M|N)$ is singled out.

7.1. Super-Yangian $Y(M|N)$

They are based on the superalgebra $gl(M|N)$ in the same way $Y(N)$ is based on $gl(N)$. They have been defined in [19], and their representations are studied in [20]. One defines a $\mathbb{Z}_2$ grading

$$[T^{ij}_{(n)}] = [i] + [j] \quad (36)$$

$$\text{with} \begin{cases} [i] = 0 \text{ for } 1 \le i \le M \\ [i] = 1 \text{ for } M+1 \le i \le M+N, \end{cases}$$

and introduces as usual

$$T(u) = \sum_{i,j=1}^{M+N} \sum_{n \ge 0} u^{-n} T^{ij}_{(n)} E_{ij} \quad (37)$$

$$= \sum_{i,j=1}^{M+N} T^{ij}(u) E_{ij} \text{ and } P_{12} = \sum_{i,j} (-1)^{[i][j]} E_{ij} \otimes E_{ji}.$$

The super-Yangian is then defined by

$$R_{12}(u-v)T_1(u)T_2(v) \quad (38)$$

$$= T_2(v)T_1(u)R_{12}(u-v) \text{ with } R_{12}(u) = \mathbb{I} - \frac{1}{u} P_{12},$$

where we have introduced graded tensor products:

$$T_1(u) = \sum_{i,j,k,l} (-1)^{([i]+[j])[k]} T_{ij}(u) \delta_{kl} \quad (39)$$

$$\times E_{ij} \otimes E_{kl} \text{ and } T_2(u) = \sum_{i,j} T_{ij}(u)\, \mathbb{I} \otimes E_{ij}.$$

It is a graded Hopf algebra, and its R matrix obeys a graded Yang–Baxter algebra. Their classical version is defined as in Section 2.1.We refer to [19–21] for more details.

7.2. $\mathcal{W}(M|N)$ Superalgebras

Starting from the superalgebra $gl(M|N)$ and using $sl(2)$ embeddings, one can construct $\mathcal{W}$ superalgebras. The $sl(2)$ generators being bosonic, they belong to the $gl(M) \oplus gl(N)$ subalgebra, and the procedure is the same as in Section 3. The only difference comes with the fermionic generators which have to be constrained to the Grassmann constant for consistency (see [8] for details).

As for the $gl(N)$ case, one selects a special class of $\mathcal{W}$ superalgebras: the finite $\mathcal{W}$ superalgebras $\mathcal{W}_p(M|N) = \mathcal{W}[gl(pM|pN), (M+N).sl(p)]$, where $(M+N).sl(p)$ denotes the direct sum of $(M+N)$ algebras $sl(p)$, M of them being included in the $gl(M)$ subalgebra of $gl(M|N)$ and the N remaining in its $gl(N)$ subalgebra. Then, one can prove the following.

Theorem 7. *The $\mathcal{W}_p(M|N)$ superalgebras are isomorphic to the truncation at level p of the classical super-Yangian $Y(M|N)$.*

This isomorphism allows us to classify the irreducible finite-dimensional representations of the $\mathcal{W}_p(M|N)$ superalgebras [21].

7.3. Twisted Super-Yangians

Similarly to the twisted Yangians, one can define the twisting of super-Yangians [22]. This leads to subalgebras of $Y(M|N)$ which contain the orthosymplectic superalgebras. Mimicking the case of twisted Yangian, one introduces an automorphism of $Y(M|N)$:

$$\tau(T^{ij}(u)) = (-1)^{[i]([j]+1)}\theta_i\theta_j T^{\bar{j}\bar{i}}(-u) \qquad (40)$$

$$\text{with } \theta_i = \pm 1; \quad (-1)^{[i]}\theta_i\theta_{\bar{i}} = \theta_0 = \pm 1;$$

$$\bar{\imath} = M + 1 - i, \quad \text{for } 1 \le i \le M,$$

$$\bar{\imath} = 2M + N + 1 - i, \quad \text{for } M + 1 \le i \le M + N.$$

However, the presence of fermionic generators forces one to have (up to the Hopf algebra isomorphism $Y(M|N) \leftrightarrow Y(N|M)$ which identifies $M = 2m$ and $\theta_0 = -1$ with $N = 2n$ and $\theta_0 = 1$—see [22]) $N = 2n$ and $\theta_0 = +1$. Thus, one is led to the following definition:

Definition 2. *The twisted Yangian $Y(M|2n)^+$ is the subalgebra of $Y(M|2n)$ generated by $S(u) = T(u)\tau(T(u))$, where τ is defined in* (40) *with*

$$\theta_i = 1 \text{ for } 1 \le i \le M, \qquad (41)$$

$$\theta_i = \mathrm{sgn}\left(\frac{2M+2n+1}{2} - i\right)$$

$$\text{for } M + 1 \le i \le M + 2n.$$

From this definition, one proves that $S(u)$ obeys the rules

$$R_{12}(u-v)S_1(u)R'_{12}(u+v)S_2(v) \qquad (42)$$

$$= S_2(v)\, R'_{12}(u+v)\, S_1(u)\, R_{12}(u-v),$$

$$\tau(S(u)) = S(-u) + \frac{1}{2u}\left(S(u) - S(-u)\right). \qquad (43)$$

The irreducible finite-dimensional representations of $Y(M|2n)^+$ are studied in [22].

As far as $\mathcal{W}$ superalgebras are concerned, their folding has been introduced in [14] and shown to lead to $\mathcal{W}$ superalgebras based on $osp(M|N)$ superalgebras. Considering a special class of folded $\mathcal{W}$ superalgebras, one gets again the following:

Theorem 8. *Let $\mathcal{W}_p(M|2n)^+$ be the $\mathcal{W}[osp(Mp|2np), ([\frac{M}{2}] + n)sl(p) \oplus \epsilon_M\, so(k)]$ superalgebra, where $\epsilon_M \equiv M$ [mod 2] and p is chosen odd $(p = 2k+1)$ when M is odd. $\mathcal{W}_p(M|2n)^+$ is isomorphic to the truncation at level p of the twisted super-Yangian $Y(M|2n)^+$.*

It allows one to classify the finite-dimensional representations of the $\mathcal{W}_p(M|2n)^+$ algebras [22].

8. CONCLUSION

A wide class of $\mathcal{W}$ (super)algebras are shown to be isomorphic to the truncation of (super)(twisted) Yangians. This isomorphism allows one to classify all the irreducible finite-dimensional representations of these $\mathcal{W}$ algebras.

Moreover, since there are many more $\mathcal{W}$ algebras, the connection lets us hope that a generalization of Yangians (as Hopf algebras) is available. The same is valid for affine $\mathcal{W}$ algebras, which should lead to two-parameter generalization of Yangians.

Finally, the application to physical models, such as NLS, has to be studied.

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SYMPOSIUM ON QUANTUM GROUPS

Quantum Algebra $U_q(2,1)$: q Analogs of the Gelfand–Graev Formulas*

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Abstract—The discrete series of unitary irreducible representations of the noncompact quantum algebra $U_q(2,1)$ are studied. For the negative discrete series, two bases of these irreps are considered. One of them corresponds to the reduction $U_q(2,1) \to U_q(2) \times U(1)$. The second basis is connected with the reduction $U_q(2,1) \to U(1) \times U_q(1,1)$. The matrix elements of the $U_q(2,1)$ generators in both bases are calculated. For the intermediate discrete series, only first type of basis is considered and the q analogs of the Gelfand–Graev formulas are obtained. Also, the transformation brackets connecting the two bases are found for the negative discrete series. © *2001 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

The quantum algebras and groups, discovered more than 15 years ago [1], continue to generate a deep interest among both theoreticians and mathematicians. The investigations of this type were realized also by our group at Moscow State University. They were concentrated on the development of the Wigner–Racah formalism for the compact quantum algebras $SU_q(2)$ [2] and $SU_q(3)$ [3–5]. In this talk, we want to extend these studies on the noncompact quantum algebra $U_q(2,1)$. Namely, our aim is to consider discrete series (DS) of unitary irreducible representations (UIR) of this algebra and to obtain the q analog of the formulas given by Gelfand and Graev [6] for the standard $U(n,m)$ algebras many years ago. The work is organized as follows. In Section 2, the DS of UIR with the highest weight are considered in the so-called U basis corresponding to the reduction

$$U_q(2,1) \to SU_q(2) \times U(1). \qquad (1)$$

In Section 3, the same irreps are analyzed in a T basis, corresponding to the reduction

$$U_q(2,1) \to U(1) \times SU_q(1,1). \qquad (2)$$

The transformation brackets between these bases (Weyl coefficients) are described in Section 4. In Section 5, the Gelfand–Graev formulas for the intermediate discrete series are derived. Below, the deformation parameter q is assumed to be real. The standard notation

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [n]! = [n][n-1]\cdots[1]$$

is used for q numbers and q factorials.

2. U BASIS AND GELFAND–GRAEV FORMULAS FOR THE DS OF UIR WITH A HIGHEST WEIGHT

The highest weight vector $|H\rangle$ of the UIR $\{f\} = \{f_1 f_2 f_3\}$ satisfies the relations

$$A_{ik}|H\rangle = 0, \quad i < k,$$
$$A_{ii}|H\rangle = f_i|H\rangle, \qquad (3)$$
$$\langle H|H\rangle = 1.$$

Here, A_{ik} $(i, k = 1, 2, 3)$ are the $U_q(2,1)$ generators for which the same commutation relations are valid as for the $U_q(3)$ generators:

$$[A_{ii}, A_{ik}] = A_{ik}, \quad [A_{kk}, A_{ik}] = -A_{kk},$$
$$[A_{ii}, A_{kk}] = 0, \quad [A_{ik}, A_{ki}] = [A_{ii} - A_{kk}], \qquad (4)$$
$$[A_{12}, A_{32}] = [A_{21}, A_{23}] = 0,$$
$$A_{13} = A_{12}A_{23} - qA_{23}A_{12}, \text{ etc.}$$

(see, for example, [3]). Their properties with respect to an Hermitian conjugation are of the form

$$A_{ii}^+ = A_{ii}, \quad A_{12}^+ = A_{21}, \quad A_{32}^+ = -A_{23}, \qquad (5)$$
$$A_{31}^+ = -\tilde{A}_{13} = -(A_{12}A_{23} - q^{-1}A_{23}A_{12}).$$

The general basis vector $|m_3 U M_U\rangle$, corresponding to the reduction (1), can be written as follows:

$$|m_3 U M_U\rangle \qquad (6)$$
$$= \frac{1}{N(UM_U)N(kl)} A_{21}^{U-M_U} P^U A_{31}^k A_{32}^l |H\rangle,$$

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where $0 \le k \le f_1 - f_2$, $l = 0, 1, 2, \ldots, \infty$, $m_3 = f_3 + k + l, U = (f_1 - f_2 - k + l)/2$, and $-U \le M_U \le U$. The projection operator P^U for the $SU_q(2)$ algebra was obtained in [2]:

$$P^U = \sum_{r=0}^{\infty} \frac{(-1)^r}{[r]!} \times \prod_{k=1}^{r} \frac{1}{[A_{11} - A_{22} + k + 1]} A_{21}^r A_{12}^r. \quad (7)$$

The calculation of the normalization factors $N(UM_U)$ and $N(kl)$ gives the results

$$N^2(UM_U) = \frac{[2U]![U - M_U]!}{[U + M_U]!},$$

$$N^2(k,l) = \frac{[k]![l]![f_3 - f_1 + k - 2]![f_3 - f_2 + l - 1]!}{[f_3 - f_2 - 1]![f_3 - f_1 - 2]![f_1 - f_2 - k]!} \times \frac{[f_1 - f_2]![f_1 - f_2 - k + l + 1]!}{[f_1 - f_2 + l + 1]!}. \quad (8)$$

The requirement of the positiveness of $N^2(kl)$ applies to the following restrictions on the highest weight components:

$$f_3 \ge f_2 + 1, \quad f_3 \ge f_1 + 2, \quad f_1 \ge f_2. \quad (9)$$

Using the explicit form (6)–(9) of the U-basis vectors and the permutation relations for powers of generators found in [3–5], it is possible to find the matrix elements of the $U_q(2,1)$ generators. In such a manner, the following results are obtained:

$$\langle m_3 + 1 U - \frac{1}{2} M_U + \frac{1}{2} | A_{32} | m_3 U M_U \rangle = \left\{ \frac{[U - M_U][k+1][f_1 - f_2 - k][f_3 - f_1 + k - 1]}{[2U+1][2U]} \right\}^{1/2}, \quad (10)$$

$$\langle m_3 + 1 U + \frac{1}{2} M_U + \frac{1}{2} | A_{32} | m_3 U M_U \rangle = \left\{ \frac{[U + M_U + 1][l+1][f_3 - f_2 + l][f_1 - f_2 + l + 2]}{[2U+1][2U+2]} \right\}^{1/2}, \quad (11)$$

$$\langle m_3 + 1 U - \frac{1}{2} M_U - \frac{1}{2} | A_{31} | m_3 U M_U \rangle = q^{U - M_U} \left\{ \frac{[U + M_U][k+1][f_1 - f_2 - k][f_3 - f_1 + k - 1]}{[2U][2U+1]} \right\}^{1/2}, \quad (12)$$

$$\langle m_3 + 1 U + \frac{1}{2} M_U - \frac{1}{2} | A_{31} | m_3 U M_U \rangle = -q^{-(U + M_U + 1)} \left\{ \frac{[U - M_U + 1][l+1][f_1 - f_2 + l + 2][f_3 - f_2 + l]}{[2U+1][2U+2]} \right\}^{1/2}, \quad (13)$$

$$\langle m_3 - 1 U - \frac{1}{2} M_U - \frac{1}{2} | A_{23} | m_3 U M_U \rangle = -\left\{ \frac{[U + M_U][l][f_1 - f_2 + l + 1][f_3 - f_2 + l - 1]}{[2U][2U+1]} \right\}^{1/2}, \quad (14)$$

$$\langle m_3 - 1 U + \frac{1}{2} M_U - \frac{1}{2} | A_{23} | m_3 U M_U \rangle = -\left\{ \frac{[U - M_U + 1][k][f_1 - f_2 - k + 1][f_3 - f_1 + k - 2]}{[2U+1][2U+2]} \right\}^{1/2}, \quad (15)$$

$$\langle m_3 - 1 U - \frac{1}{2} M_U + \frac{1}{2} | A_{13} | m_3 U M \rangle = q^{U + M_U + 1} \left\{ \frac{[U - M_U][l][f_1 - f_2 + l + 1][f_3 - f_2 + l - 1]}{[2U][2U+1]} \right\}^{1/2}, \quad (16)$$

$$\langle m_3 - 1 U + \frac{1}{2} M_U + \frac{1}{2} | A_{13} | m_3 U M_U \rangle = -q^{-U + M_U} \left\{ \frac{[U + M_U + 1][k][f_1 - f_2 - k + 1][f_3 - f_1 + k - 2]}{[2U+1][2U+2]} \right\}^{1/2}. \quad (17)$$

As for the U-spin generators $U_+ = A_{12}$, $U_- = A_{21}$, their matrix elements are given by the standard formulas for the $SU_q(2)$ generators

$$\langle m_3' U' M_U \pm 1 | U_\pm | m_3 U M_U \rangle = \delta_{m_3 m_3'} \delta_{UU'} \{[U \mp M_U][U \pm M_U + 1]\}^{1/2}. \quad (18)$$

It is easy to verify that Eqs. (10)–(17) coincide in a limit $q=1$ with the Gelfand–Graev formulas [6] if we use their notation

$$f_1 = m_{23}-1, \quad f_2 = m_{33}-1, \quad f_3 = m_{13}+2,$$
$$2U = m_{12}-m_{22}, \quad U - M_U = m_{12}-m_{11},$$
$$U + M_U = m_{11}-m_{22}.$$

3. REDUCTION $U_q(2,1) \to U(1)\times SU_q(1,1)$ (T BASIS)

The generators of the T-spin subalgebra $SU_q(1,1)$ are of the form

$$T_+ = A_{23}, \quad T_- = A_{32}, \quad T_0 = (A_{22}-A_{33})/2. \tag{19}$$

The vectors of the T basis can be written in the form

$$|m_1 T M_T\rangle = \frac{1}{N(TM_T)N(sp)} \times A_{32}^{-T-M-1} P^T A_{31}^s A_{31}^p |H\rangle, \tag{20}$$

where

$$m_1 = f_1 - p - s,$$
$$T = \frac{1}{2}(f_3 - f_2 - p + s - 2) = -\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots,$$
$$M_T = -T-1, -T-2, \ldots,$$
$$N^2(TM_T) = [-T-M_T-1]![T-M_T]!/[2T+1]!.$$

In contrast to (7), the projection operator P^T for the noncompact T spin has a form of a finite sum:

$$P^T = \sum_{r=0}^{2T} \frac{[2T-r]!}{[r]![2T]!} A_{32}^r A_{23}^r. \tag{21}$$

The calculation of the normalization factor $N(s,p)$ gives

$$N^2(sp) = q^{-2s}\frac{[s]![p]![f_1-f_2]![f_3-f_2+s-1]![f_3-f_1+s-2]!}{[f_1-f_2-p]![f_3-f_1-2]![f_3-f_2-1]!}\frac{[f_3-f_2-p-2]!}{[f_3-f_2-p+s-2]!}, \tag{22}$$
$$0 \le p \le f_1 - f_2, \quad s = 0,1,2,\ldots,\infty.$$

The matrix elements of the $U_q(2,1)$ generators in the T basis are of the form

$$\left\langle m_1+1\, T+\frac{1}{2} M_T - \frac{1}{2}|A_{12}|m_1 T M_T\right\rangle = \left\{\frac{[p][T-M_T+1][f_3-f_2-p-1][f_1-f_2-p+1]}{[2T+1][2T+2]}\right\}, \tag{23}$$

$$\left\langle m_1+1\, T-\frac{1}{2} M_T - \frac{1}{2}|A_{12}|m_1 T M_T\right\rangle = \left\{\frac{[s][f_3-f_1+s-2][f_3-f_2+s-1][-T-M_T]}{[2T][2T+1]}\right\}^{1/2}, \tag{24}$$

$$\left\langle m_1+1\, T+\frac{1}{2} M_T + \frac{1}{2}|A_{13}|m_1 T M_T\right\rangle \tag{25}$$
$$= q^{T-M_T+1}\left\{\frac{[-T-M_T-1][p+1][f_1-f_2-p+1][f_3-f_2-p-1]}{[2T+1][2T+2]}\right\}^{1/2},$$

$$\left\langle m_1+1\, T-\frac{1}{2} M_T + \frac{1}{2}|A_{13}|m_1 T M_T\right\rangle \tag{26}$$
$$= -q^{-T-M_T}\left\{\frac{[T-M_T][s][f_3-f_2+s-1][f_3-f_1+s-2]}{[2T][2T+1]}\right\}^{1/2},$$

$$\left\langle m_1-1\, T-\frac{1}{2} M_T + \frac{1}{2}|A_{21}|m_1 T M_T\right\rangle = \left\{\frac{[T-M_T][p+1][f_1-f_2-p][f_3-f_2-p-2]}{[2T][2T+1]}\right\}^{1/2}, \tag{27}$$

$$\left\langle m_1-1\, T+\frac{1}{2} M_T + \frac{1}{2}|A_{21}|m_1 T M_T\right\rangle = -\left\{\frac{[-T-M_T-1][s+1][f_3-f_1+s-1][f_3-f_2+s]}{[2T+1][2T+2]}\right\}^{1/2}, \tag{28}$$

$$\left\langle m_1-1\,T-\frac{1}{2}M_T-\frac{1}{2}|A_{31}|m_1TM_T\right\rangle=q^{-T+M_TT-1}\left\{\frac{[-T-M_T][p+1][f_1-f_2-p][f_3-f_2-p-2]}{[2T][2T+1]}\right\}^{1/2}, \tag{29}$$

$$\left\langle m_1-1\,T+\frac{1}{2}M_T-\frac{1}{2}|A_{31}|m_1TM_T\right\rangle \tag{30}$$
$$=q^{T+M}\left\{\frac{[T-M_T+1][s+1][f_3-f_1+s-1][f_3-f_2+s]}{[2T+1][2T+2]}\right\}^{1/2},$$

$$\langle m_1'T'M_T\pm1|T_\pm|m_1TM_T\rangle=\mp\delta_{m_1m_1'}\delta_{TT'}\{[\pm T-M_T][\mp T-M_T\mp1]\}^{1/2}. \tag{31}$$

It should be noted that the reduction (2) was not considered in [6].

4. WEYL COEFFICIENTS FOR THE NEGATIVE DS OF THE $U_q(2,1)$ ALGEBRA

The Weyl coefficient $\langle U|T\rangle$ is a transformation bracket between two bases described in Sections 2 and 3. The direct calculation, similar to one done for the $U_q(3)$ in [7], gives the result

$$\langle U|T\rangle_q\equiv\langle m_1TM_T|m_3UM_U\rangle=(-1)^s\sqrt{[2U+1][2T+1]}\left\{\frac{[k]![U-M_U]![-T-M_T-1]![T-M_T]!}{[l]![s]![p]![U+M_U]!}\right.$$
$$\times\left.\frac{[f_1-f_2-k]![f_1-f_2+l+1]![f_3-f_1+s-2]![f_3-f_2-p-2]!}{[f_3-f_1+k-2]![f_3-f_2+l-1]![f_3-f_2+s-1]![f_1-f_2-p]!}\right\}^{1/2} \tag{32}$$
$$\times\sum_z(-1)^z\frac{[2T+1+p+l-s+z]![U+M_U+z]![l+z]!}{[z]![2U+z+1]![k-z]![l-s+z]![2T+1+l-s+z]!}.$$

It can be expressed in terms of the basic hypergeometric function ${}_4\varphi_3$. Therefore it can be reduced to a definite q-Racah polynomial and coincides with the $SU_q(2)$ Racah coefficient except for a phase factor.

5. INTERMEDIATE DISCRETE SERIES

The UIRs of the intermediate DS are characterized by the existence of the extremal vector $|G\rangle$ of the weight $\{f\}=\{f_1f_2f_3\}$ which satisfies the relations

$$A_{12}|G\rangle=0,\quad Z_{-1}|G\rangle=Z_2|G\rangle=0, \tag{33}$$
$$A_{ii}|G\rangle=f_i|G\rangle,\quad \langle G|G\rangle=1.$$

(The expressions for $Z_{\pm i}$ operators are given below.) The general vector of U basis for this UIR can be denoted as $|m_3UM_U\rangle$, where m_3 is a third component of the weight $(m)=(m_1m_2m_3)$ which is inherent in this vector. It is sufficient to consider only the first-rank vectors with $M_U=U$. The general expression of such a vector is of the form

$$|m_3UU\rangle\equiv|ab)=\frac{1}{N(ab)}Z_1^bZ_{-2}^a|G\rangle,$$
$$a,b=0,1,2,\dots,\infty.$$

Step-up and step-down operators $Z_{\pm i}$ are given by the expressions

$$Z_1=PA_{13}P=A_{13}P, \tag{34}$$
$$Z_{-1}=PA_{31}P=\left(A_{31}+A_{21}A_{32}q^{-(A_{11}-A_{22}+1)}\frac{1}{[A_{11}-A_{22}+1]}\right)P,$$
$$Z_2=PA_{23}P=\left(A_{23}-A_{21}A_{13}\frac{1}{[A_{11}-A_{22}+1]}\right)P,$$
$$Z_{-2}=PA_{32}P=A_{32}P,$$

where P is determined by (7).

The operator $Z_{\pm i}$ transforms the first-rank vector (34) into the neighbor first-rank vector $|m_3\mp1U'U'\rangle$ with $U'=U\mp\frac{1}{2}(-1)^i$.

They form the so-called Mickelsson–Zhelobenko algebra [8, 9] and satisfy the permutation relations (see [10] for the case $q=1$)

$$Z_1Z_{-2}=Z_{-2}Z_1,\quad Z_{-1}Z_2=Z_2Z_{-1},$$
$$Z_1Z_2=Z_2Z_1\frac{[A_{11}-A_{22}+2]}{[A_{11}-A_{22}+1]},$$
$$Z_{-2}Z_{-1}=Z_{-1}Z_{-2}\frac{[A_{11}-A_{22}+2]}{[A_{11}-A_{22}+1]},$$
$$Z_2Z_{-2}=Z_{-2}Z_2\frac{[A_{11}-A_{22}+1]^2}{[A_{11}-A_{22}+2][A_{11}-A_{22}]}$$

$$+ Z_1 Z_{-1}\frac{[A_{11}-A_{22}+1]}{[A_{11}-A_{22}+2][A_{11}-A_{22}]} + \frac{[A_{11}-A_{22}+1][A_{22}-A_{33}-1]}{[A_{11}-A_{22}+2]}P, \tag{35}$$

$$Z_{-1}Z_1 = Z_1 Z_{-1}\frac{[A_{11}-A_{22}+1]^2}{[A_{11}-A_{22}+2][A_{11}-A_{22}]} + Z_{-2}Z_2\frac{[A_{11}-A_{22}+1]}{[A_{11}-A_{22}+2][A_{11}-A_{22}]} + \frac{[A_{11}-A_{22}+1][A_{33}-A_{11}-1]}{[A_{11}-A_{22}+2]}P.$$

In addition,

$$(Z_i)^+ = -Z_{-i}.$$

Using these relations, we can find the normalization factor

$$N^2(a,b) \tag{36}$$

$$= \frac{[a]![b]![f_3-f_2+a]![f_1-f_3+b]![f_1-f_2+b]!}{[f_3-f_2]![f_1-f_3]![f_1-f_2]!} \times \frac{[f_1-f_2+a]!}{[f_1-f_2+a+b+1]!}[f_1-f_2+1].$$

The positiveness of this expression dictates the restrictions

$$f_1 \geq f_3 \geq f_2; \quad 2U = f_1 - f_2 + a + b. \tag{37}$$

Writing the general vector of the U basis as follows,

$$|m_3 U M_U\rangle = \sqrt{\frac{[U+M_U]!}{[2U]![U-M_U]!}}A_{21}^{U-M_U}|m_3 U U\rangle, \tag{38}$$

we can calculate the matrix elements of generators. They are listed below in the Gelfand–Graev notation $m_{13}=f_1-1,\ m_{23}=f_3,\ m_{33}=f_2+1,\ U+M_U = m_{11}-m_{22}$:

$$\left\langle m_3+1 U-\frac{1}{2}M_U+\frac{1}{2}|A_{32}|m_3 U M_U\right\rangle = \left\{\frac{[m_{12}-m_{11}][m_{12}-m_{13}-1][m_{12}-m_{33}+1][m_{12}-m_{23}]}{[m_{12}-m_{22}][m_{12}-m_{22}+1]}\right\}^{1/2}, \tag{39}$$

$$\left\langle m_3+1 U+\frac{1}{2}M_U+\frac{1}{2}|A_{32}|m_3 U M_U\right\rangle = \left\{\frac{[m_{13}-m_{22}+2][m_{23}-m_{22}+1][m_{33}-m_{22}][m_{11}-m_{22}+1]}{[m_{12}-m_{22}+2][m_{12}-m_{22}+1]}\right\}^{1/2}, \tag{40}$$

$$\left\langle m_3+1 U+\frac{1}{2}M_U-\frac{1}{2}|A_{31}|m_3 U M_U\right\rangle = -q^{-(m_{11}-m_{22}+1)}\left\{\frac{[m_{12}-m_{11}+1][m_{33}-m_{22}][m_{13}-m_{22}+2][m_{23}-m_{22}+1]}{[m_{12}-m_{22}+1][m_{12}-m_{22}+2]}\right\}^{1/2}, \tag{41}$$

$$\left\langle m_3+1\, U-\frac{1}{2}M_U-\frac{1}{2}|A_{31}|m_3 U M_U\right\rangle = q^{m_{12}-m_{11}}\left\{\frac{[m_{11}-m_{22}][m_{12}-m_{23}][m_{12}-m_{33}+1][m_{12}-m_{13}-1]}{[m_{12}-m_{22}][m_{12}-m_{22}+1]}\right\}^{1/2}, \tag{42}$$

$$\left\langle m_3-1\, U+\frac{1}{2}\, M_U-\frac{1}{2}|A_{23}|m_3 U M_U\right\rangle = \left\{\frac{[m_{12}-m_{11}+1][m_{12}-m_{13}][m_{12}-m_{23}+2][m_{12}-m_{23}+1]}{[m_{12}-m_{22}+1][m_{12}-m_{22}+2]}\right\}^{1/2}, \tag{43}$$

$$\left\langle m_3-1\, U-\frac{1}{2}\, M_U-\frac{1}{2}|A_{23}|m_3 U M_U\right\rangle = \tag{44}$$

$$= \left\{\frac{[m_{11}-m_{22}][m_{33}-m_{22}-1][m_{13}-m_{22}+1][m_{23}-m_{22}]}{[m_{12}-m_{22}][m_{12}-m_{22}+1]}\right\}^{1/2},$$

$$\left\langle m_3 - 1\, U + \frac{1}{2}\, M_U + \frac{1}{2} | A_{13} | m_3 U M_U \right\rangle \tag{45}$$

$$= q^{-(m_{12}-m_{11})} \left\{ \frac{[m_{11}-m_{22}+1][m_{12}-m_{13}][m_{12}-m_{23}+1][m_{12}-m_{33}+2]}{[m_{12}-m_{22}+1][m_{12}-m_{22}+2]} \right\}^{1/2},$$

$$\left\langle m_3 - 1\, U - \frac{1}{2}\, M_U + \frac{1}{2} | A_{13} | m_3 U M_U \right\rangle \tag{46}$$

$$= -q^{m_{11}-m_{22}+1} \left\{ \frac{[m_{12}-m_{11}][m_{33}-m_{22}-1][m_{13}-m_{22}+1][m_{23}-m_{22}]}{[m_{12}-m_{22}][m_{12}-m_{22}+1]} \right\}^{1/2}.$$

As for the generators $A_{21} = U_-$ and $A_{12} = U_+$, their matrix elements are given by the same formulas (18).

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SYMPOSIUM ON QUANTUM GROUPS

q-Power Function over q-Commuting Variables and Deformed XXX and XXZ Chains*

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Abstract—We find certain functional identities for the Gauss q-power function of a sum of q-commuting variables. Then we use these identities to obtain two-parameter twists of the quantum affine algebra $U_q(\widehat{sl}_2)$ and of the Yangian $Y(sl_2)$. We determine the corresponding deformed trigonometric and rational quantum R matrices, which then are used in the computation of deformed XXX and XXZ Hamiltonians.

1. INTRODUCTION

The most famous R matrices, found by Yang, Baxter, and Zamolodchikov, satisfy the Yang–Baxter (YB) equation due to addition laws for basic rational, trigonometric, and elliptic functions. This note is an attempt to answer the following question: Which elementary functions and which of their properties could be employed to produce other solutions of the YB equation?

There is a general opinion that all the solutions of the YB equation, as well as the corresponding Hopf algebras, can be obtained from the Drinfeld–Jimbo solutions by suitable twists. Recently, all finite-dimensional bialgebras from the Belavin–Drinfeld list [1] were quantized in this way [2]. The first nontrivial infinite-dimensional examples, which cannot be reduced to the finite-dimensional case, are a classical rational and a trigonometric r matrix with values in sl_2, found in [1, 3]. They can be obtained from the classical Yang and Drinfeld–Jimbo r matrices by adding, respectively, a certain (but the same!) polynomial of the first degree in the spectral parameters. We found the corresponding twist for the Yangian $Y(sl_2)$ and extended it to a two-parameter twist of the quantum affine algebra $U_q(\widehat{sl}_2)$.

Surprisingly, it has the simple form of a q-power function, but with q-commuting arguments, its Yangian degeneration becomes the usual power function whose arguments belong to an additive variant of the Manin q plane. In this setting the q-power functions satisfy nontrivial generalizations of their standard properties [see below Eqs. (9)–(11)], which guarantee the cocycle identity for the twists.

We calculate the corresponding deformations of the traditional trigonometric and rational R matrices, putting them into a single family, and compute the related Hamiltonians of the periodic chains. It gives two-parameter integrable deformations of the XXZ and XXX Heisenberg chains. As a particular case, we get the deformed XXX chain treated in [4].

2. q-POWER FUNCTION OVER q-COMMUTING VARIABLES

Denote by $(1-u)_q^{(a)}$ the following q-binomial series [5]:

$$F_a(u) = (1-u)_q^{(a)} = 1 + \sum_{k>0} \frac{(-a)_q(-a+1)_q\cdots(-a+k-1)_q}{(k)_q!}u^k.$$

Here, $(a)_q = (q^a-1)/(q-1)$. This unital formal power series over u satisfies the following additive properties:

$$(1-u)_q^{(a)}(1-q^{-a}u)_q^{(b)} = (1-u)_q^{(a+b)}, \quad (1)$$

$$(1-u)_q^{(a)}(1-v)_q^{(a)} = (1-u-v+q^{-a}uv)_q^{(a)}, \quad (2)$$

$$(1-v)_q^{(a)}(1-u)_q^{(a)} = (1-u-v+uv)_q^{(a)}, \quad (3)$$

where the variables v and u in (2) and in (3) q-commute, $vu = quv$, and each of these is uniquely characterized by the difference equation

$$F_a(u) = \frac{1-q^{-a}u}{1-u}F_a(qu), \quad (4)$$

which follows directly from the definition. The relation (1) can be checked directly on the level of formal

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power series. All the other properties can be deduced from the presentation of the q-power function as a ratio of q-exponential functions and from the corresponding properties of q exponents:

$$(1-u)_q^{(a)} = \frac{\exp_q \frac{u}{1-q}}{\exp_q \frac{uq^{-a}}{1-q}} = \frac{(q^{-a}u;q)_\infty}{(u;q)_\infty}. \quad (5)$$

Here,

$$\exp_q(u) = 1 + \sum_{n>0} \frac{u^n}{(n)_q!},$$
$$(u;q)_\infty = (1-u)(1-qu)\ldots.$$

To prove the relation (5), one can note that both sides satisfy the same difference Eq. (4) under the assumption $|q| < 1$. Clearly, under this assumption, the solution $F_a(u)$ of (4) is unique if $F_a(0) = 1$. Thus both sides of the first equality in (5) coincide as formal power series. Then, relation (1) is a direct corollary of (5), while (2) and (3) follow from the addition law [6] for the q exponents given below and the Faddeev–Volkov [6, 7] identity, where again $vu = quv$:

$$\exp_q(u)\exp_q(v) = \exp_q(u+v), \quad (6)$$

$$\exp_q(v)\exp_q(u) = \exp_q(u+v+(q-1)vu). \quad (7)$$

We refer to (2) and (3) also as to Faddeev–Volkov identities. Below, we will give a different proof of a more general relation and get (2) and (3) as its consequences.

Let us consider now the q-power series as a function of a sum of two q-commuting variables u and v, $vu = quv$:

$$F_a(u+v) = (1-u-v)_q^{(a)}. \quad (8)$$

We claim that this formal power series has, in addition to (1)–(3), the following properties:

$$(1-q^{-b}v-u)_q^{(a)}(1-v-q^{-a}u)_q^{(b)} \quad (9)$$
$$= (1-u-v)_q^{(a+b)},$$

$$(1-w(1-q^{-a}v-q^{-1}u)^{-1})_q^{(a)}(1 \quad (10)$$
$$-u-v)_q^{(a)} = (1-u-v-w)_q^{(a)},$$

$$(1-u-v)_q^{(a)}(1 \quad (11)$$
$$-(1-q^{-1}v-q^{-a}u)^{-1}w)_q^{(a)} = (1-u-v-w)_q^{(a)},$$

where $vu = quv$ everywhere, $vw = qwv$ and $uw = q^{-1}wu$ in (10), (11). Setting $u = 0$ or $v = 0$, we get (1)–(3) as particular cases. The proof of (9)–(11) is based on the following observation:

$$(1-q^av-u)(1-q^bv-q^{-1}u) \quad (12)$$
$$= (1-q^bv-u)(1-q^av-q^{-1}u)$$

for q-commuting variables v and u. Consider first (9). Note that it is sufficient to prove this identity for positive integers a and b only, because in this case both sides are finite power series, and if they are equal for any q-commuting u, v, then their coefficients at ordered monomials are equal. But these coefficients are rational functions of q^a and q^b, so if they are equal for all positive integers a and b, then they are equal identically.

From (1), we know that for any positive integer n

$$(1-u)_q^{(n)} = (1-q^{-1}u)(1-q^{-2}u)\cdots(1-q^{-n}u). \quad (13)$$

Then, we can reorder the factors of the product

$$(1-u-v)_q^{(n)} = (1-q^{-1}u-q^{-1}v)$$
$$\times (1-q^{-2}u-q^{-2}v)\cdots(1-q^{-n}u-q^{-n}v),$$

using (12) and get another presentation:

$$(1-u-v)_q^{(n)} = (1-q^{-n}v-q^{-1}u) \quad (14)$$
$$\times (1-q^{-(n-1)}v-q^{-2}u)\cdots(1-q^{-1}v-q^{-n}u).$$

From this presentation, relation (9) is obvious. Similarly, we prove (10) for an integer positive a. Denote the left-hand side by $F_a(u,v,w)$ and the right-hand side of (10) by $G_a(u,v,w)$. We check first that $F_1(u,v,w) = G_1(u,v,w)$. Next, we see from (9) that the function $F_n(u,v,w)$ satisfies the recurrence relation

$$F_{n+1}(u,v,w) = \Big(1-w(1-q^{-(n+1)}v$$
$$-q^{-1}u)^{-1}\Big)\,F_n(u,q^{-1}v,q^{-1}w)(1-q^{-1}v-q^{-n-1}u).$$

So, it remains to prove the same recurrence relation for $G_n(u,v,w)$. For this, we note that we can, analogously to (14), prove the following identities:

$$(1-q^{-1}v-u-q^{-1}w)_q^{(n)}(1-q^{-1}v-q^{-n-1}u)$$
$$= (1-q^{-n-1}v-q^{-1}u-q^{-2}w)$$
$$\times (1-q^{-n-2}v-q^{-2}u-q^{-3}w)$$
$$\cdots(1-q^{-2}v-q^{-n}u-q^{-n-1}w)$$
$$\times (1-q^{-1}v-q^{-n-1}u),$$

using the identity similar to (12)

$$(1-q^av-q^bu-w)(1-q^{a+1}v-q^{b-1}u)$$
$$= (1-q^av-q^bu)(1-q^{a+1}v-q^{b-1}u-w).$$

Then we get

$$(1-q^{-1}v-u-q^{-1}w)_q^{(n)}(1-q^{-1}v-q^{-n-1}u)$$
$$= (1-q^{-n-1}v-q^{-1}u)$$
$$\times (1-q^{-n-2}v-q^{-2}u-q^{-2}w)$$
$$\cdots(1-q^{-1}v-q^{-n-1}u-q^{-n-1}w).$$

The remaining part is straightforward.

We can get a rational degeneration of the identities above by the following procedure [8]: Set

$$x = u + \frac{\eta}{q^{-1}-1}v, \;\; y = v, \quad z = w. \quad (15)$$

Then, the q-commutativity relation $vu = quv$ transforms into

$$xy - q^{-1}yx = -\eta y^2, \tag{16}$$

and we can rewrite the equalities (9)–(11) in the variables x, y, and z:

$$(1 - x - \eta(c)_{\bar{q}}y)_q^{(a+b)}$$
$$= (1 - x - \eta(c+b)_{\bar{q}}y)_q^{(a)}$$
$$\times (1 - q^{-a}x - q^{-a}\eta(c-a)_{\bar{q}}y)_q^{(b)}$$
$$\times \left(1 - z(1 - \bar{q}x - \eta\bar{q}(c+a-1)_{\bar{q}}y)^{-1}\right)_q^{(a)}$$
$$\times (1 - x - \eta(c)_{\bar{q}}y)_q^{(a)} = (1 - x - \eta(c)_{\bar{q}}y - z)_q^{(a)},$$
$$(1 - x - \eta(c)_{\bar{q}}y)_q^{(a)}(1 - (1 - q^{-a}x$$
$$- \eta q^{-a}(c - a + 1)_{\bar{q}}y)^{-1}z)_q^{(a)}$$
$$= (1 - x - \eta(c)_{\bar{q}}y - z)_q^{(a)}.$$

Here, $\bar{q} = q^{-1}$ and $xy - q^{-1}yx = -\eta y^2$; as before, $yz = qzy$, and $xz - q^{-1}zx = -\eta(2)_{\bar{q}}yz$. All the relations make sense for the Yangian limit $q = 1$. In this case, the q-power series becomes the usual geometric series for the power function $(1 - x)^a$, which is considered now as a function of linear combinations of the Yangian variables x and y, $[x, y] = -\eta y^2$. The basic properties (9)–(11) can be rewritten as

$$(1 - x - \eta cy)^{a+b} \tag{17}$$
$$= (1 - x - \eta(c+b)y)^a(1 - x - \eta(c-a)y)^b,$$
$$(1 - z(1 - x - \eta(c + a - 1)y)^{-1})^a(1 \tag{18}$$
$$- x - \eta cy)^a = (1 - x - \eta cy - z)^a,$$
$$(1 - x - \eta cy)^a(1 - (1 - x - \eta(c \tag{19}$$
$$- a + 1)y)^{-1}z)^a = (1 - x - \eta cy - z)^a,$$

where $[x, y] = -\eta y^2$; as before, $[y, z] = 0$, and $[x, z] = -2\eta\, yz$.

3. TWISTING COCYCLES

Let $e_{\pm\alpha}, e_{\pm(\delta-\alpha)}, q^{h_{\pm\alpha}} = q^{\pm h}$ be the generators of the quantum affine algebra $U_q(\widehat{sl}_2)$ with zero central charge, satisfying the relations

$$q^h e_{\pm\alpha} q^{-h} = q^{\pm 2} e_{\pm\alpha}, \quad q^h e_{\pm(\delta-\alpha)} q^{-h} = q^{\mp 2} e_{\pm(\delta-\alpha)},$$

$$[e_\alpha, e_{-\alpha}] = \frac{q^h - q^{-h}}{q - q^{-1}}, \quad [e_{\delta-\alpha}, e_{-\delta+\alpha}] = \frac{q^{-h} - q^h}{q - q^{-1}},$$
$$[e_{\pm\alpha}, e_{\mp(\delta-\alpha)}] = 0$$

plus q-Serre relations, which we do not use here. The comultiplication is given by the following formulas:

$$\Delta(e_\alpha) = e_\alpha \otimes 1 + q^{-h} \otimes e_\alpha, \tag{20}$$
$$\Delta(e_{\delta-\alpha}) = e_{\delta-\alpha} \otimes 1 + q^h \otimes e_{\delta-\alpha},$$
$$\Delta(e_{-\alpha}) = e_{-\alpha} \otimes q^h + 1 \otimes e_{-\alpha},$$
$$\Delta(e_{-\delta+\alpha}) = e_{-\delta+\alpha} \otimes q^{-h} + 1 \otimes e_{-\delta+\alpha}.$$

We claim that the element

$$\mathcal{F} = \Bigg(1 - (2)_{q^2}\Bigg(a \cdot 1 \otimes e_{\delta-\alpha}$$
$$+ b \cdot q^{-h} \otimes q^{-h}e_{-\alpha}\Bigg)\Bigg)_{q^2}^{(-\frac{h\otimes 1}{2})}$$

satisfies the cocycle identity for any constants a and b.

Let us prove this statement. Set $u = (2)_{q^2}ae_{\delta-\alpha}$, $v = (2)_{q^2}bq^{-h}e_{-\alpha}$. Then, $vu = q^2uv$. We can rewrite the cocycle equation

$$\mathcal{F}_{12}(\Delta \otimes id)\mathcal{F} = \mathcal{F}_{23}(id \otimes \Delta)\mathcal{F},$$

using the tensor notation $a_1 = a \otimes 1 \otimes 1$, $a_2 = 1 \otimes a \otimes 1$, $a_3 = 1 \otimes 1 \otimes a$, as follows:

$$\left(1 - q^{-h_1}v_2 - u_2\right)_{q^2}^{(-h_1/2)} \tag{21}$$
$$\times \left(1 - q^{-h_1-h_2}v_3 - u_3\right)_{q^2}^{(-(h_1+h_2)/2)}$$
$$= \left(1 - q^{-h_2}v_3 - u_3\right)_{q^2}^{(-h_2/2)}$$
$$\times \left(1 - q^{-h_1}v_2 - q^{-h_1-h_2}v_3 - u_2 - q^{h_2}u_3\right)_{q^2}^{(-h_1/2)}.$$

Using (9), we see that we have to prove the following equality:

$$\left(1 - v_3 - q^{h_2}u_3\right)_{q^2}^{(h_2/2)} \left(1 - q^{-h_1}v_2 - u_2\right)_{q^2}^{(-h_1/2)} \tag{22}$$
$$\times \left(1 - q^{-h_1-h_2}v_3 - u_3\right)_{q^2}^{(-(h_1+h_2)/2)}$$
$$= \left(1 - q^{-h_1}v_2 - q^{-h_1-h_2}v_3 - u_2 - q^{h_2}u_3\right)_{q^2}^{(-h_1/2)}.$$

Let us present the second factor of the left-hand side of (22) as a series and permute the first factor with each term of this series. Then, we get on the left-hand side of (22)

$$\sum_{n\geq 0} C_n(q^{-h_1}v_2 + u_2)^n \tag{23}$$
$$\times \left(1 - v_3 - q^{h_2-2n}u_3\right)_{q^2}^{(h_2/2-n)}$$
$$\times \left(1 - q^{-h_1-h_2}v_3 - u_3\right)_{q^2}^{(-(h_1+h_2)/2)},$$

where

$$C_n = \frac{(-h_1/2)_{q^2}(-h_1/2+1)_{q^2}\cdots(-h_1/2+n-1)_{q^2}}{(n)_{q^2}!}.$$

Then, again using (9), we rewrite the left-hand side of (22) as

$$\sum_{n\geq 0} C_n \left(q^{-h_1}v_2+u_2\right)^n \quad (24)$$
$$\times \left(1-q^{-h_2}v_3-q^{h_2-2n}u_3\right)^{(-n)}_{q^2}$$
$$\times \left(1-v_3-q^{h_2}u_3\right)^{(h_2/2)}_{q^2}$$
$$\times \left(1-q^{-h_1-h_2}v_3-u_3\right)^{(-(h_1+h_2)/2)}_{q^2}.$$

Repeating the factorization procedure for negative powers, we can present the product of the first two factors in (24) as a total (usual) power:

$$\left(q^{-h_1}v_2+u_2\right)^n \left(1-q^{-h_2}v_3-q^{h_2-2n}u_3\right)^{(-n)}_{q^2}$$
$$=\left((q^{-h_1}v_2+u_2)(1-q^{-h_2}v_3-q^{h_2-2}u_3)^{-1}\right)^n.$$

Therefore, the left-hand side of (22) is equal to

$$\left(1-(q^{-h_1}v_2+u_2)(1-q^{-h_2}v_3-q^{h_2-2}u_3)^{-1}\right)^{(-h_1/2)}_{q^2}$$
$$\times \left(1-q^{-h_1-h_2}v_3-q^{h_2}u_3\right)^{(-h_1/2)}_{q^2}.$$

One can see that the desired equality (22) is now precisely the generalized Faddeev–Volkov identity (10).

Further, as in the previous section, we can make a change of variables (see [9]):

$$f_1 = e_{\delta-\alpha}+\frac{\eta}{q^{-2}-1}q^{-h}e_{-\alpha}, \quad f_0 = q^{-h}e_{-\alpha}. \quad (25)$$

The elements f_1, f_0, and h generate a Hopf subalgebra of $U_q(\widehat{sl}_2)$, considered now [9] as an algebra over $\mathbb{C}[[\eta]](q)$:

$$[h,f_1]=-2f_1, \quad [h,f_0]=-2f_0, \quad (26)$$
$$f_1f_0-q^{-2}f_0f_1=-\eta f_0^2,$$

$$\Delta(f_0)=f_0\otimes 1+q^{-h}\otimes f_0, \quad (27)$$
$$\Delta(f_1)=f_1\otimes 1+q^h\otimes f_1+\eta q^h\,(h)_{q^{-2}}\otimes f_0.$$

Then, the twisting element $\mathcal{F}$ after a proper normalization of the constants a and b, $a=\xi, b=\xi\eta/(q^{-2}-1)$, has the form

$$\mathcal{F}=\Big(1-(2)_{q^2}\xi(1\otimes f_1 \quad (28)$$
$$+\eta\,(h/2)_{q^{-2}}\otimes f_0)\Big)^{(-\frac{h\otimes 1}{2})}_{q^2}.$$

Again, it makes sense in the Yangian limit $q=1$, where $\mathcal{F}$ has the following form:

$$\mathcal{F}=\left(1-2\xi(1\otimes f_1+\eta\frac{h}{2}\otimes f_0)\right)^{(-\frac{h\otimes 1}{2})}. \quad (29)$$

4. TWISTED R MATRICES AND DEFORMED HAMILTONIANS

Let $\pi_{1/2}(z)$ be the two-dimensional vector representation of the algebra $U_q(\widehat{sl}_2)$. In this representation, the generator $e_{-\alpha}$ acts as a matrix unit e_{21}, $e_{\delta-\alpha}$ as ze_{21}, and h as $e_{11}-e_{22}$. The R matrix in the tensor product $\pi_{1/2}(z_1)\otimes\pi_{1/2}(z_2)$ of $U_q(\widehat{sl}_2)$ is well known. For the comultiplication (20), it is

$$R_0(z_1,z_2)=e_{11}\otimes e_{11}+e_{22}\otimes e_{22}$$
$$+\frac{z_1-z_2}{q^{-1}z_1-qz_2}(e_{11}\otimes e_{22}+e_{22}\otimes e_{11})$$
$$+\frac{q^{-1}-q}{q^{-1}z_1-qz_2}(z_2e_{12}\otimes e_{21}+z_1e_{21}\otimes e_{12}),$$

$$R_0(z_1,z_2)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{z_1-z_2}{q^{-1}z_1-qz_2} & \frac{(q^{-1}-q)z_2}{q^{-1}z_1-qz_2} & 0 \\ 0 & \frac{(q^{-1}-q)z_1}{q^{-1}z_1-qz_2} & \frac{z_1-z_2}{q^{-1}z_1-qz_2} & 0 \\ 0 & 0 & 0 & 1\end{pmatrix}. \quad (30)$$

The image of the element $\mathcal{F}$ has the form

$$F=1+\frac{q^h-1}{q-1}\left(az_2+bq^{-h+1}\right)\otimes e_{21}$$
$$=1+\left((az_2+b)e_{11}-(q^{-1}az_2+qb)e_{22}\right)\otimes e_{21}.$$

Hence, the twisted R matrix $R^F=F^{21}RF^{-1}$ can be written as

$$R^F(z_1,z_2)=R_0(z_1,z_2) \quad (31)$$
$$+\frac{z_1-z_2}{q^{-1}z_1-qz_2}\big((b+az_2)(e_{22}-e_{11})\otimes e_{21}$$
$$+(q^{-1}az_1+qb)e_{21}\otimes(e_{11}-e_{22})$$
$$+(b+az_2)(q^{-1}az_1+qb)e_{21}\otimes e_{21}\big),$$

$$R^F(z_1,z_2)=\frac{z_1-z_2}{q^{-1}z_1-qz_2}\begin{pmatrix} \frac{q^{-1}z_1-qz_2}{z_1-z_2} & 0 & 0 & 0 \\ -(az_2+b) & 1 & \frac{(q^{-1}-q)z_2}{z_1-z_2} & 0 \\ q^{-1}az_1+qb & \frac{(q^{-1}-q)z_1}{z_1-z_2} & 1 & 0 \\ (az_2+b)(q^{-1}az_1+qb) & -(q^{-1}az_1+qb) & az_2+b & \frac{q^{-1}z_1-qz_2}{z_1-z_2}\end{pmatrix}.$$

It satisfies the basic property $R^F(z,z)=P_{12}$, where P_{12} is a permutation of the tensor factors. Let $t(z)=\mathrm{Tr}_0 R^F_{0N}(z,z_2)R^F_{0,N-1}(z,z_2)\cdots R^F_{01}(z,z_2)$ be a family of commuting transfer matrices for the corresponding homogeneous periodic chain, $[t(z'),t(z'')]=0$ (where we treat z_2 as a parameter of the theory and $z=z_1$ is considered as a spectral parameter). Then, the Hamiltonian

$$H_{a,b,z_2}=(q^{-1}-q)z\frac{d}{dz}t(z)|_{z=z_2}t^{-1}(z_2)$$

can be computed by a standard procedure,

$$H_{a,b,z_2}=(q^{-1}-q)\sum_k P_{k,k+1}z\frac{d}{dz}R_{k,k+1}(z,z_2)|_{z=z_2},$$

and is equal to

$$H_{a,b,z_2}=H_{XXZ} \tag{32}$$
$$+\sum_k\left(C\left(\sigma^z_k\sigma^-_{k+1}+\sigma^-_k\sigma^z_{k+1}\right)+D\sigma^-_k\sigma^-_{k+1}\right).$$

Here, $C=((q-1)/2)(b-az_2q^{-1})$, $D=(az_2+b)\times(q^{-1}az_2+qb)$; $\sigma^+=e_{12}$, $\sigma^-=e_{21}$, $\sigma^z=e_{11}-e_{22}$, and

$$H_{XXZ}=\sum_k\biggl(\sigma^+_k\sigma^-_{k+1} \tag{33}$$
$$+\sigma^-_k\sigma^+_{k+1}+\frac{q+q^{-1}}{2}\sigma^z_k\sigma^z_{k+1}\biggr).$$

We see that by a suitable choice of the parameters a,b,z_2 we can add to the XXZ Hamiltonian an arbitrary linear combination of the terms $\sum_k\sigma^z_k\sigma^-_{k+1}+\sigma^-_k\sigma^z_{k+1}$ and $\sum_k\sigma^-_k\sigma^-_{k+1}$ and the model will remain integrable.

In order to get the corresponding XXX degeneration and, moreover, to have a unified description of both models, we use again the realization (25) of $U_q(\widehat{sl}_2)$ and the evaluation homomorphism $\pi_{1/2}(u)(f_1)=\left(u+\eta\,(h/2)_{q^2}\right)q^hf_0$, $\pi_{1/2}(u)(f_0)=q\sigma^-$, which effectively corresponds to a shift of spectral parameter $z=u-\eta/(q^{-2}-1)$. In this notation the nontwisted R matrix $R_0(u_1,u_2)$ has the form

$$R_0(u_1,u_2)=\frac{1}{2}\left(1+\sigma^z\otimes\sigma^z\right)$$
$$+\frac{u_1-u_2}{2(q^{-1}u_1-qu_2-q\eta)}\left(1-\sigma^z\otimes\sigma^z\right)$$
$$+\frac{(q^{-1}-q)u_2-q\eta}{q^{-1}u_1-qu_2-q\eta}\sigma^+\otimes\sigma^-$$
$$+\frac{(q^{-1}-q)u_1-q\eta}{q^{-1}u_1-qu_2-q\eta}\sigma^-\otimes\sigma^+,$$

and the twisted R matrix $R^F(u_1,u_2)$ is equal to

$$R^F(u_1,u_2)=R_0(u_1,u_2) \tag{34}$$
$$+\frac{u_1-u_2}{q^{-1}u_1-qu_2-q\eta}\bigl(-\xi u_2\sigma^z\otimes\sigma^-$$
$$+\xi(q^{-1}u_1-q\eta)\sigma^-\otimes\sigma^z$$
$$+\xi^2u_2(q^{-1}u_1-q\eta)\sigma^-\otimes\sigma^-\bigr),$$

$$R^F(u_1,u_2)=\frac{u_1-u_2}{q^{-1}u_1-qu_2-q\eta}\begin{pmatrix}\frac{q^{-1}u_1-qu_2-q\eta}{u_1-u_2} & 0 & 0 & 0\\ -\xi u_2 & 1 & \frac{(q^{-1}-q)u_2-q\eta}{u_1-u_2} & 0\\ \xi(q^{-1}u_1-q\eta) & \frac{(q^{-1}-q)u_1-q\eta}{u_1-u_2} & 1 & 0\\ \xi^2u_2(q^{-1}u_1-q\eta) & -\xi(q^{-1}u_1-q\eta) & \xi u_2 & \frac{q^{-1}u_1-qu_2-q\eta}{u_1-u_2}\end{pmatrix}.$$

In particular, for $q=1$, we get a deformation of the Yang R matrix:

$$R^F(u_1,u_2)=\frac{u_1-u_2}{u_1-u_2-\eta}\biggl(1-\eta\frac{P_{12}}{u_1-u_2} \tag{35}$$
$$-\xi u_2\sigma^z\otimes\sigma^-+\xi(u_1-\eta)\sigma^-\otimes\sigma^z$$
$$+\xi^2u_2(u_1-\eta)\sigma^z\otimes\sigma^z\biggr).$$

Again, the R matrix $R^F(u_1,u_2)$ satisfies the property $R^F(u,u)=P_{12}$, and the Hamiltonian

$$H_{\eta,\xi,u_2}=((q^{-1}-q)u-q^{-1}\eta)\frac{d}{du}t(u)|_{u=u_2}t^{-1}(u_2)$$

for $t(u)=\mathrm{Tr}_0R^F_{0N}(u,u_2)R^F_{0,N-1}(u,u_2)\cdots R^F_{01}(u,u_2)$ is given by the same formula (32), where $C=\xi((q^{-1}-1)/2)u_2-(q^{-1}\xi\eta)/2$, $D=\xi^2u_2(q^{-1}u_2-q\eta)$, and now also makes sense in the XXX limit $q=1$,

$$H_{\eta,\xi,u_2}=H_{XXX} \tag{36}$$
$$+\sum_k\left(C\left(\sigma^z_k\sigma^-_{k+1}+\sigma^-_k\sigma^z_{k+1}\right)+D\sigma^-_k\sigma^-_{k+1}\right),$$

where $C=-\xi\eta/2$, $D=\xi^2u_2(u_2-\eta)$.

5. DISCUSSIONS

1. One can see that the R matrix (31) is a quantization of the following solution of the classical YB equation:

$$r_{a,b}(z_1,z_2) = r_{\rm DJ}(z_1,z_2) + a(z_1\sigma^-\otimes\sigma^z - z_2\sigma^z\otimes\sigma^-) + b(\sigma^-\otimes\sigma^z - \sigma^z\otimes\sigma^-), \quad (37)$$

where

$$r_{\rm DJ}(z_1,z_2) = \frac{1}{2}\left(\frac{z_1+z_2}{z_1-z_2}t_{12} - \sigma^+\otimes\sigma^- + \sigma^-\otimes\sigma^+\right)$$

is the Drinfeld–Jimbo solution of the classical YB equation. Here, t_{12} is the splitted Casimir operator, $t_{12} = \sigma^-\otimes\sigma^+ + \sigma^+\otimes\sigma^- + \frac{1}{2}\sigma^z\otimes\sigma^z$. The r matrix (37) is gauge-equivalent to

$$\tilde{r}_{a,b}(z_1,z_2) = r_{\rm DJ}(z_1,z_2) + a(z_1\sigma^-\otimes\sigma^z - z_2\sigma^z\otimes\sigma^-) + 4ab(z_1-z_2)(\sigma^-\otimes\sigma^-). \quad (38)$$

The gauge equivalence is given by $Ad(1+2b\sigma^-)\otimes Ad(1+2b\sigma^-)$. It can be shown that for generic a and b the r matrix (38) is gauge-equivalent to the following solution of the YB equation found in [1] (see [10] for the quantum version):

$$r_{\rm BD}(z_1,z_2) = r_{\rm DJ}(z_1,z_2) + (z_1-z_2)(\sigma^-\otimes\sigma^-). \quad (39)$$

Therefore, in the case of sl_2, we have a description of the quantization of all the trigonometric solutions of the YB equation, described in [1], in the universal form. Moreover, the rational degeneration (35) is a quantization of the rational r matrix found in [3],

$$r_{\rm St}(u_1,u_2) = \frac{t_{12}}{u_1-u_2} + \xi(u_1\sigma^-\otimes\sigma^z - u_2\sigma^z\otimes\sigma^-),$$

and thus we answer the similar question of a quantization of the rational sl_2 solutions of the classical YB equation (see also [11]).

2. It will be interesting to study the spectra and the eigenstates of the Hamiltonians (32), (36). The particular case of (36) with $C=0$ was studied in [4]. The study was based on a quantization of a simpler rational r matrix suggested in [11]. It was shown that in this case the spectrum of the Hamiltonian remains unchanged after the deformation. However, the deformed Hamiltonian has Jordanian blocks and thus it is not diagonalizable. Therefore we can expect that at least the deformed XXX chains (36) are not equivalent to the undeformed one.

3. We see that it turned out to be very important to obtain a two-parameter deformation of the algebra $U_q(\widehat{sl}_2)$ and of its fundamental R matrix. Only in such a way did we manage to get the deformation of the Yangian $Y(sl_2)$, the corresponding rational R matrix (35), and the related Hamiltonian (36). On the classical level, the generic r matrices of this family are gauge-equivalent. It is interesting to understand whether these equivalences can be extended to the quantum level and to develop the representation theory of the corresponding deformed two-parameter Hopf algebra.

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SYMPOSIUM ON QUANTUM GROUPS

Super-Drinfeldians and Super-Yangians of Lie Superalgebras of Type $A(n|m)$*

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Abstract—Explicit description of super-Drinfeldians and super-Yangians of Lie superalgebras of the type $A(n|m)(=sl(n+1|m+1))$ is given in terms of the Chevalley basis. Construction of a q analog of Cartan–Weyl generators and their permutation relations for the quantum superalgebra $U_q(sl(n+1|m+1))$, which is a subalgebra of the super-Drinfeldian $D_{q\eta}(sl(n+1|m+1))$, are also presented.

1. INTRODUCTION

It is well known that Yangians [1] and quantum (q-deformed) affine algebras play a very important role in the theory of integrable systems and quantum field theory. One can hope that super-Yangians and quantum affine superalgebras will play the same role.

We recall that the Yangian $Y_\eta(g)$ is a rational deformation of the universal enveloping algebra $U(g[u])$ and the quantum affine algebra $U_q(\hat{g})$ is a trigonometric deformation of $U(g[u,u^{-1}])$, where g is any finite-dimensional simple Lie algebra. In the supercase, we have the same picture: the super-Yangian $Y_\eta(g)$ is a rational deformation of $U(g[u])$, and the quantum affine superalgebra $U_q(g[u,u^{-1}])$ is a trigonometric deformation of $U(g[u,u^{-1}])$, where g is any finite-dimensional contragredient simple Lie superalgebra.

From the point of view of applications, it is useful to know various realizations (in term of various bases) of Yangians, quantum affine algebras, and their superanalogs. In the case of Yangians, there are the following well-known realizations introduced by Drinfeld [1]: in the terms of the Cartesian basis, in the terms of an infinite basis and generating functions (currents), in the terms of L operators (mainly, for the case $Y_\eta(gl(n))$). There are many papers concerned with the Yangian realizations (see references in [2]). In the case of super-Yangians, the situation is very poor. There are only two papers devoted to the super-Yangians, for the superalgebras $sl(n|m)$ [3] and $g(m|n)$ [4].

Recently, it was shown in [5] that the Yangians can be obtained from the quantum affine algebras by means of a singular transformation (at $q=1$) and by subsequent passage to the limit $q\to 1$. It was also shown that this singular transformation results in a new two-parameter deformation called Drinfeldian. Analogous results were obtained for the supercase in paper [2]. Namely, it was shown that, starting from the defining relations of the quantum affine superalgebra $U_q(g[u])$ in terms of the Chevalley basis, where g is any finite-dimensional contragredient simple Lie superalgebra, by means of singular transformation of an affine generator we can obtain a two-parameter deformation called super-Drinfeldian. The super-Drinfeldian $D_{q\eta}(g)$ is a Hopf superalgebra, and moreover, if $q\to 1$, we have

$$D_{q=1\,\eta}(g)\simeq Y_\eta(g), \tag{1}$$

and if $\eta=0$, then

$$D_{q\,\eta=0}(g)\simeq U_q(g[u]). \tag{2}$$

The relations between the super-Drinfeldian $D_{q\eta}(g)$ and the superalgebras $U_q(g[u])$, $Y_\eta(g)$, $U(g[u])$ (and also their subalgebras) are shown in Fig. 1. The general defining relations given in [2] for the super-Drinfeldians and super-Yangians contain an implicit part. They have the singular factor $\eta/(q-q^{-1})$, and, moreover, they depend on the choice of a $U_q(g)$ vector $\tilde{e}_{-\theta}$ of a minimal weight $-\theta$. We can choose the vector $\tilde{e}_{-\theta}$ so as to obtain simpler defining relations.

In this paper, we find the explicit form of the defining relations in terms of the Chevalley basis for super-Drinfeldian and super-Yangian of Lie superalgebras of the type $A(n|m)$. Moreover, we explicitly describe the Cartan–Weyl basis for the quantum superalgebra $U_q(sl(n+1|m+1))$, which is a subalgebra of the super-Drinfeldian $D_{q\eta}(sl(n+1|m+1))$.

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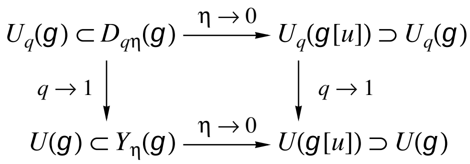

Fig. 1. A diagram of the limit Hopf superalgebras of the super-Drinfeldian $D_{q\eta}(g)$ and their subalgebras. The arrows show passages to the limits.

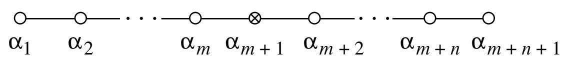

Fig. 2. Dynkin diagram of the Lie superalgebra $A(m|n)$.

2. QUANTUM SUPERALGEBRA $U_q(A(m|n))$

In this section, we construct the Cartan–Weyl generators of the quantum superalgebra $U_q(A(m|n))$ ($A(m|n) = sl(m+1|n+1)$)[1)] and describe their permutation relations which will be used in the next section for calculation of the explicit defining relations of the super-Drinfeldian $D_{q\eta}(sl(m+1|n+1))$ and the super-Yangian $Y_\eta(sl(m+1|n+1))$.

Let $\Pi := \{\alpha_1, \ldots, \alpha_{n+m+1}\}$ be a system of simple roots of $A(m|n)$ endowed with the following scalar product: $(\alpha_i, \alpha_j) = (\alpha_j, \alpha_i)$, $(\alpha_i, \alpha_i) = 2$, $(\alpha_i, \alpha_{i+1}) = -1$ for $i = 1, 2, \ldots, m$, and $(\alpha_{m+1}, \alpha_{m+1}) = 0$, and $(\alpha_{i+1}, \alpha_{i+1}) = -2$, $(\alpha_i, \alpha_{i+1}) = 1$ for $i = m+1, m+2, \ldots, m+n$, and $(\alpha_i, \alpha_j) = 0$ for $|i-j| > 1$ ($i, j = 1, 2, \ldots, m+n+1$). The root α_{m+1} is called an odd gray root.[2)] The corresponding Dynkin diagram of $A(m|n)$ is presented on Fig. 2. Since $U_q(sl(m+1|n+1)) \simeq U_q(sl(n+1|m+1))$ (see [7]), we fix $m \geq n$.

The quantum algebra $U_q(A(m|n))$ is generated by the Chevalley elements $q^{\pm h_{\alpha_i}}$, $e_{\pm\alpha_i}$ ($i = 1, 2, \ldots, m+n+1$) with the defining relations

$$q^{h_{\alpha_i}} q^{-h_{\alpha_i}} = q^{-h_{\alpha_i}} q^{h_{\alpha_i}} = 1, \tag{3}$$

$$q^{h_{\alpha_i}} q^{h_{\alpha_j}} = q^{h_{\alpha_j}} q^{h_{\alpha_i}},$$

$$q^{h_{\alpha_i}} e_{\pm\alpha_j} q^{-h_{\alpha_i}} = q^{\pm(\alpha_i,\alpha_j)} e_{\pm\alpha_j},$$

$$[e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij}\,[h_{\alpha_i}],$$

$$[e_{\pm\alpha_i}, e_{\pm\alpha_j}]_q = 0 \quad (|i-j| \geq 2), \tag{4}$$

$$[[e_{\pm\alpha_i}, e_{\pm\alpha_j}]_q, e_{\pm\alpha_j}]_q = 0 \quad (|i-j| = 1), \tag{5}$$

$$[[e_{\pm\alpha_{m+1}}, e_{\pm\alpha_m}]_q, [e_{\pm\alpha_{m+1}}, e_{\pm\alpha_{m+2}}]_q]_q = 0. \tag{6}$$

Here and elsewhere, we use the standard notation $[a] := (q^a - q^{-a})/(q - q^{-1})$, and the brackets $[\cdot,\cdot]$ and the q-brackets $[\cdot,\cdot]_q$ mean the commutator and the q commutator, respectively:

$$[e_\beta, e_{\beta'}] = e_\beta e_{\beta'} - (-1)^{\deg(e_\beta)\deg(e_{\beta'})} e_{\beta'} e_\beta, \tag{7}$$

$$[e_\beta, e_{\beta'}]_q = e_\beta e_{\beta'} - (-1)^{\deg(e_\beta)\deg(e_{\beta'})} q^{(\beta,\beta')} e_{\beta'} e_\beta. \tag{8}$$

The parity $\deg(\cdot)$ of the Chevalley elements are determined as follows:

$$\deg(h_{\alpha_i}) = 0 \ \ (1 \leq i \leq m+n+1), \tag{9}$$

$$\deg(e_{\pm\alpha_i}) = 0 \ \ (i \neq m+1),$$

$$\deg(e_{\pm\alpha_{m+1}}) = 1.$$

Relations (3)–(6) are invariant with respect to the replacement of q by q^{-1}. Relations (5) for $j = m+1$, $i = m, m+2$ are equivalent to

$$e^2_{\pm\alpha_{m+1}} = 0. \tag{10}$$

The outer q supercommutator in relations (6) is really the usual supercommutator since $(\alpha_{m+1} + \alpha_m, \alpha_{m+1} + \alpha_{m+2}) = 0$. The relations of the type (5) are ordinarily called the Serre relations; therefore, relations (6) may be called the triple Serre relations because they connect tree root vectors e_{α_m}, $e_{\alpha_{m+1}}$, and $e_{\alpha_{m+2}}$.

The Hopf structure on $U_q(A(m|n))$ is given by the following formulas for the comultiplication Δ_q and the antipode S_q:

$$\Delta_q(q^{\pm h_{\alpha_i}}) = q^{\pm h_{\alpha_i}} \otimes q^{\pm h_{\alpha_i}}, \tag{11}$$

$$S_q(q^{\pm h_{\alpha_i}}) = q^{\mp h_{\alpha_i}},$$

$$\Delta_q(e_{\alpha_i}) = e_{\alpha_i} \otimes 1 + q^{-h_{\alpha_i}} \otimes e_{\alpha_i},$$

$$S_q(e_{\alpha_i}) = -q^{h_{\alpha_i}} e_{\alpha_i},$$

$$\Delta_q(e_{-\alpha_i}) = e_{-\alpha_i} \otimes q^{h_{\alpha_i}} + 1 \otimes e_{\alpha_i},$$

$$S_q(e_{-\alpha_i}) = -e_{-\alpha_i} q^{-h_{\alpha_i}}.$$

It is not hard to see that the quantum superalgebra $U_q(A(m|n))$ has the following simple nongraded antilinear anti-involution (or the conjugation) "*":

$$e^*_{\pm\alpha_i} = e_{\mp\alpha_i}, \quad (q^{\pm h_{\alpha_i}})^* = q^{\mp h_{\alpha_i}}, \tag{12}$$

$$(q^{\pm 1})^* = q^{\mp 1}$$

($(xy)^* = y^* x^*$ for $\forall\, x, y \in U_q(A(m|n))$).

Now we introduce another basis in the Cartan subalgebra of $U_q(A(m|n))$, which is natural for $gl(m+1|n+1)$ and $sl(m+1|n+1)$. Let ϑ_i be a

[1)]Generally speaking, in the standard notation, the superalgebra $sl(m+1|m+1)$ differs from $A(m|m)$ by the one-dimensional center $\lambda\mathbf{1}_{2m+2}$: $A(m|m) = sl(m+1|m+1)/\{\lambda\mathbf{1}_{2m+2}\}$, but here we set $A(m|m) := sl(m+1|m+1)$.

[2)]For the sake of simplicity, we choose here such system of simple roots Π, which has only one odd root. The same superalgebra can admit a different number of simple odd roots (e.g., see [6]).

parity function defined on the set $I := 1, 2, \ldots, m + n + 2$ as follows:

$$\vartheta_i = 0 \quad \text{for } 1 \le i \le m+1, \tag{13}$$
$$\vartheta_i = 1 \quad \text{for } m+2 \le i \le N.$$

Here and elsewhere, we also use the short notation $N := m + n + 2$. The natural basis in the Cartan subalgebra of the superalgebra $gl(m+1|n+1)$ ($sl(m+1|n+1)$) consists of the elements e_{ii} ($i \in I$) which are connected with the Cartan elements h_{α_i} by the formulas

$$h_{\alpha_i} = (-1)^{\vartheta_i} e_{ii} - (-1)^{\vartheta_{i+1}} e_{i+1i+1} \tag{14}$$
$$(i = 1, 2, \ldots, N-1).$$

The element

$$c_1 := \sum_{i=1}^{N} e_{ii} \tag{15}$$

is central in $gl(m+1|n+1)$ ($U_q(gl(m+1|n+1))$). In the case $m = n$, the element (15) belongs to $sl(m+1|m+1)$ ($U_q(sl(m+1|m+1))$) and

$$c_1 = \sum_{k=1}^{m} k h_{\alpha_k} + \sum_{k=m+1}^{2m+1} (2m+2-k) h_{\alpha_k}. \tag{16}$$

In the case $n \ne m$, we can find the formulas inverse to (14):

$$e_{ii} = \tfrac{1}{m-n}\Bigg(c_1 + \sum_{k=1}^{m}(m-n-k)h_{\alpha_k} \tag{17}$$
$$- \sum_{k=m+1}^{N-1}(m+n+2-k)h_{\alpha_k}\Bigg) - \sum_{k=1}^{i-1} h_{\alpha_k}.$$

A dual basis to the elements e_{ii} will be denoted by ε_i ($i = 1, 2, \ldots, N$): $\varepsilon_i(e_{jj}) = \delta_{ij}$, $(\varepsilon_i, \varepsilon_j) = \delta_{ij}\varepsilon_i^2 = \delta_{ij}(-1)^{\theta_i}$. In terms of ε_i, the positive root system Δ_+ of $A(m|n)$ is presented as follows:

$$\Delta_+ = \{\varepsilon_i - \varepsilon_j \,|\, 1 \le i < j \le N\}, \tag{18}$$

where $\varepsilon_i - \varepsilon_{i+1}$ are the simple roots: $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ ($i = 1, 2, \ldots, N-1$). The root $\theta := \varepsilon_1 - \varepsilon_N$ is a maximal root: $\theta = \alpha_1 + \alpha_2 + \ldots + \alpha_{N-1}$. For the root vectors $e_{\varepsilon_i - \varepsilon_j}$ ($i \ne j$), the standard notation is also used:

$$e_{ij} := e_{\varepsilon_i - \varepsilon_j}, \quad e_{ji} := e_{\varepsilon_j - \varepsilon_i} \quad (1 \le i < j \le N). \tag{19}$$

In particular, e_{ii+1}, e_{i+1i} are the Chevalley elements: $e_{ii+1} = e_{\alpha_i}$, $e_{i+1i} = e_{-\alpha_i}$.

For construction of the composite root vectors e_{ij} ($j \ne i \pm 1$), we fix the following normal ordering of the positive root system Δ_+ (see [8, 9]):

$$(\varepsilon_1 - \varepsilon_2), (\varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3), \tag{20}$$
$$\ldots, (\varepsilon_1 - \varepsilon_i, \ldots, \varepsilon_{i-1} - \varepsilon_i),$$
$$\ldots, (\varepsilon_1 - \varepsilon_N, \ldots, \varepsilon_{N-1} - \varepsilon_N).$$

According to this ordering, we set

$$e_{ij} := [e_{ik}, e_{kj}]_{q^{-1}}, \tag{21}$$
$$e_{ji} := [e_{jk}, e_{ki}]_q \quad (1 \le i < k < j \le N).$$

It is obvious that $e_{ij}^* = e_{ji}$ and $\deg(e_{ij}) = \vartheta_{ij} \equiv \vartheta_i + \vartheta_j$. Using this explicit construction and the defining relations (3)–(6), it is not hard to calculate all permutation relations of the Cartan–Weyl generators (21). They are

$$q^{e_{kk}} e_{ij} q^{-e_{kk}} = q^{\delta_{ki} - \delta_{kj}} e_{ij} \quad (1 \le i, j, k \le N), \tag{22}$$
$$[e_{ij}, e_{ji}] = [\varepsilon_i^2 e_{ii} - \varepsilon_j^2 e_{jj}] \quad (1 \le i < j \le N), \tag{23}$$
$$[e_{ij}, e_{kl}]_{q^{-1}} = \delta_{jk} e_{il} \; (1 \le i < j \le k < l \le N), \tag{24}$$
$$[e_{ik}, e_{jl}]_{q^{-1}} = -(-1)^{\vartheta_{ij}\vartheta_{jk}} (q^{\varepsilon_j^2} - q^{-\varepsilon_j^2})\, e_{jk} e_{il} \tag{25}$$
$$(1 \le i < j < k < l \le N),$$
$$[e_{jk}, e_{il}]_{q^{-1}} = 0 \quad (1 \le i \le j < k \le l \le N), \tag{26}$$
$$[e_{kl}, e_{ji}] = 0 \quad (1 \le i < j \le k < l \le N), \tag{27}$$
$$[e_{il}, e_{kj}] = 0 \quad (1 \le i < j < k < l \le N), \tag{28}$$
$$[e_{ji}, e_{il}] = e_{jl}\, q^{\varepsilon_j^2 e_{ii} - \varepsilon_i^2 e_{jj}} \; (1 \le i < j < l \le N), \tag{29}$$
$$[e_{kl}, e_{li}] = e_{ki}\, q^{\varepsilon_k^2 e_{kk} - \varepsilon_l^2 e_{ll}} \; (1 \le i < k < l \le N), \tag{30}$$
$$[e_{jl}, e_{ki}] \tag{31}$$
$$= -(-1)^{\vartheta_{jk}\vartheta_{jl}} (q^{\varepsilon_j^2} - q^{-\varepsilon_j^2})\, e_{kl} e_{ji}\, q^{\varepsilon_j^2 e_{jj} - \varepsilon_k^2 e_{kk}},$$

where $1 \le i < j < k < l \le N$ in the last relation (31). All the relations (22)–(31) together with the ones obtained from them by the conjugation "*" describe a complete list of the permutation relations of the Cartan–Weyl basis corresponding to the normal ordering (20). It should be noted that formulas (22)–(31) are valid not only for the diagram of the form in Fig. 2 but for all allowed Dynkin diagrams of the given superalgebra $A(m|n)$ with a different number of odd gray roots [6, 10].

3. SUPER-DRINFELDIAN AND SUPER-YANGIAN OF $sl(m+1|n+1)$

First, we give a general definition of the super-Drinfeldian (see [2]). Let g be a finite-dimensional complex contragredient simple Lie superalgebra of rank r with a symmetric Cartan matrix $A = (a_{ij})_{i,j=1}^r$ and with a system of simple roots $\Pi := \{\alpha_1, \ldots, \alpha_r\}$ ($(\alpha_i, \alpha_j) = a_{ij}$). Moreover, we choose such system of simple roots Π which has only one odd root [10].[3] Let

[3]Such system Π always exists, and, moreover, the same superalgebra can have a different number of odd simple roots (see [6, 10]).

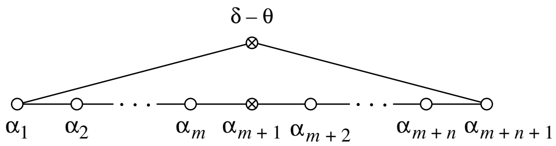

Fig. 3. Dynkin diagram of the affine superalgebra $\widehat{sl}(m+1|n+1)$.

θ be a maximal positive root of g. The root θ can be even or odd. Let $U_q(g)$ be a q analog of $U(g)$ with a Chevalley basis $q^{h_{\alpha_i}}$, $e_{\pm\alpha_i}$ $(i=1,2,\ldots,r)$ and with the standard defining relations which are not written here (relations (3)–(6) for the case $g=A(m|n)$).

The super-Drinfeldian $D_{q\eta}(g)$ is generated by the quantum superalgebra $U_q(g)$ and the elements $\xi_{\delta-\theta}$, $q^{\pm h_\delta}$ with the relations

$$[q^{\pm h_\delta}, \text{everything}] = 0, \tag{32}$$

$$q^{h_{\alpha_i}}\xi_{\delta-\theta}q^{-h_{\alpha_i}} = q^{-(\alpha_i,\theta)}\xi_{\delta-\theta},$$
$$[e_{-\alpha_i},\xi_{\delta-\theta}] = \tfrac{\eta}{q-q^{-1}}\,[e_{-\alpha_i},\tilde{e}_{-\theta}], \tag{33}$$
$$(\mathrm{ad}_q e_{\alpha_i})^{n_{i0}}\xi_{\delta-\theta} = \tfrac{\eta}{q-q^{-1}}\,(\mathrm{ad}_q e_{\alpha_i})^{n_{i0}}\tilde{e}_{-\theta},$$

where $n_{i0}=1$ if $(\alpha_i,\alpha_i)=(\alpha_i,\theta)=0$; $n_{i0}=2$ if $(\alpha_i,\alpha_i)=0$ and $(\alpha_i,\theta)\neq 0$; and $n_{i0}=1+2(\alpha_i,\theta)/(\alpha_i,\alpha_i)$ if $(\alpha_i,\alpha_i)\neq 0$. Moreover,

$$[[e_{\alpha_i},\xi_{\delta-\theta}]_q,\xi_{\delta-\theta}]_q \tag{34}$$
$$= -\tfrac{\eta^2}{(q-q^{-1})^2}\,[[e_{\alpha_i},\tilde{e}_{-\theta}]_q,\tilde{e}_{-\theta}]_q$$
$$+\tfrac{\eta}{q-q^{-1}}\Big([[e_{\alpha_i},\tilde{e}_{-\theta}]_q,\xi_{\delta-\theta}]_q + [[e_{\alpha_i},\xi_{\delta-\theta}]_q,\tilde{e}_{-\theta}]_q\Big)$$

if $(\alpha_i,\theta)\neq 0$, $(\theta,\theta)\neq 0$; and

$$\xi^2_{\delta-\theta} = \tfrac{\eta}{q-q^{-1}}\,[\tilde{e}_{-\theta},\xi_{\delta-\theta}] - \tfrac{\eta^2}{(q-q^{-1})^2}\,\tilde{e}^2_{-\theta} \tag{35}$$
$$\text{if } (\theta,\theta)=0$$

Furthermore, the triple additional Serre relations can be

$$[[\xi_{\delta-\theta},e_{\alpha_i}]_q,[\xi_{\delta-\theta},e_{\alpha_j}]_q]_q \tag{36}$$
$$= -\tfrac{\eta^2}{(q-q^{-1})^2}[[\tilde{e}_{-\theta},e_{\alpha_i}]_q,[\tilde{e}_{-\theta},e_{\alpha_j}]_q]_q$$
$$+\tfrac{\eta}{q-q^{-1}}\Big([[\tilde{e}_{-\theta},e_{\alpha_i}]_q,[\xi_{\delta-\theta},e_{\alpha_j}]_q]_q$$
$$+[[\xi_{\delta-\theta},e_{\alpha_i}]_q,[\tilde{e}_{-\theta},e_{\alpha_j}]_q]_q\Big)$$

if $(\theta,\theta)=(\alpha_i,\alpha_j)=0$, $(\alpha_i,\theta)=-(\alpha_j,\theta)\neq 0$; and

$$[[e_{\alpha_i},\xi_{\delta-\theta}]_q,[e_{\alpha_i},e_{\alpha_j}]_q]_q \tag{37}$$
$$= \tfrac{\eta}{q-q^{-1}}\,[[e_{\alpha_i},\tilde{e}_{-\theta}]_q,[e_{\alpha_i},e_{\alpha_j}]_q]_q$$

if $(\alpha_i,\alpha_i)=(\alpha_j,\theta)=0$, $(\alpha_i,\theta)=-(\alpha_i,\alpha_j)\neq 0$.

The Hopf structure of $D_{q\eta}(g)$ is defined by the formulas $\Delta_{q\eta}(x)=\Delta_q(x)$, $S_{q\eta}(x)=S_q(x)$ for any $x\in U_q(g)$, and $\Delta_{q\eta}(q^{\pm h_\delta})=q^{\pm h_\delta}\otimes q^{\pm h_\delta}$, $S_{q\eta}(q^{\pm h_\delta})=q^{\mp h_\delta}$. The comultiplication and the antipode of the element $\xi_{\delta-\alpha}$ are given by

$$\Delta_{q\eta}(\xi_{\delta-\theta}) = \xi_{\delta-\theta}\otimes 1 + q^{-h_{\delta-\theta}}\otimes\xi_{\delta-\theta} \tag{38}$$
$$+\tfrac{\eta}{q-q^{-1}}\Big(\Delta_q(\tilde{e}_{-\theta}) - \tilde{e}_{-\theta}\otimes 1 - q^{-h_{\delta-\theta}}\otimes\tilde{e}_{-\theta}\Big),$$

$$S_{q\eta}(\xi_{\delta-\theta}) = -q^{h_{\delta-\theta}}\xi_{\delta-\theta} \tag{39}$$
$$+\tfrac{\eta}{q-q^{-1}}\Big(S_q(\tilde{e}_{-\theta}) + q^{h_{\delta-\theta}}\tilde{e}_{-\theta}\Big).$$

Here, in (33)–(39), η is some deformation parameter, $(\mathrm{ad}_q e_\beta)e_\gamma=[e_\beta,e_\gamma]_q$, and the vector $\tilde{e}_{-\theta}$ is any $U_q(g)$ element of the weight $-\theta$, such that $\lim\limits_{q\to 1}\tilde{e}_{-\theta}$ is a nonzero root vector of g with minimal weight $-\theta$. Although the right-hand sides of relations (33)–(39) include the singular factor $\eta/(q-q^{-1})$, we can show that entirely the right-hand sides of these relations are regular at $q=1$.

The super-Drinfeldian $D_{q\eta}(g)$ is a two-parameter quantization of $U(g[u])$ in the direction of a classical r matrix which is a sum of the simplest rational and trigonometric r matrices over the superalgebra g. The Hopf superalgebra $D_{q=1,\eta}(g)$ is isomorphic to the super-Yangian $Y_\eta(g)$. Moreover, $D_{q\eta=0}(g)=U_q(g[u])$.

The right-hand sides of relations (33)–(39) for the super-Drinfeldian $D_{q\eta}(g)$ and the super-Yangian $Y_\eta(g)$ depend on choice of the vector $\tilde{e}_{-\theta}$, but the relations with the different such vectors define the same super-Drinfeldian and the super-Yangian. We can choose the vector $\tilde{e}_{-\theta}$ so as to obtain simpler right-hand sides of the defining relations.

Now, we give an explicit description of the right-hand sides of relations (33)–(39) for the super-Drinfeldian $D_{q\eta}(sl(m+1|n+1))$ and the super-Yangian $Y_\eta(sl(m+1|n+1))$.

Since the defining relations of the super-Drinfeldians and the super-Yangians can be obtained from the defining relations of the quantum nontwisted affine superalgebras (see [2]), therefore the Dynkin diagrams of these affine superalgebras can also be used for classification of the super-Drinfeldians and the super-Yangians. In our case, the Dynkin diagram of the affine superalgebra $\widehat{sl}(m+1|n+1)$ is presented in Fig. 3.

We specialize the general defining relations (32)–(39) of the super-Drinfeldian $D_{q\eta}(g)$ to the case $g=sl(m+1|n+1)$, and in this case we set

$$\tilde{e}_{-\theta} = q^{\varepsilon_1^2 e_{11}+\varepsilon_N^2 e_{NN}}e_{N1}. \tag{40}$$

Using formulas (22)–(31), after some calculations, we obtain the following result.

The super-Drinfeldian $D_{q\eta}(sl(m+1|n+1))$ is generated (as a unital associative algebra over $\mathbb{C}[[\log q,\eta]]$) by the quantum superalgebra $U_q(sl(m+1|n+1))$ and the elements $\xi_{\delta-\theta}$, $q^{\pm h_\delta}$ with the relations

$$[q^{\pm h_\delta}, \text{everything}] = 0, \tag{41}$$

$$q^{e_{11}}\xi_{\delta-\theta} = q^{-1}\xi_{\delta-\theta}\, q^{e_{11}}, \tag{42}$$

$$q^{e_{NN}}\xi_{\delta-\theta} = q\,\xi_{\delta-\theta}\, q^{e_{NN}}, \tag{43}$$

$$q^{e_{ii}}\xi_{\delta-\theta} = \xi_{\delta-\theta}\, q^{e_{ii}} \quad \text{for } i = 2,3,\ldots,N-1, \tag{44}$$

$$[\xi_{\delta-\theta}, e_{i+1i}] = 0 \quad \text{for } i = 1,2,\ldots,N-1, \tag{45}$$

$$[e_{ii+1}, \xi_{\delta-\theta}] = 0 \quad \text{for } i = 2,3,\ldots,N-2, \tag{46}$$

$$[e_{12}, [e_{12}, \xi_{\delta-\theta}]_q]_q = 0, \tag{47}$$

$$[[\xi_{\delta-\theta}, e_{N-1N}]_q, e_{N-1N}]_q = 0, \tag{48}$$

$$\xi^2_{\delta-\theta} = -\eta[\varepsilon^2_N]\, q^{\varepsilon^2_1} q^{\varepsilon^2_1 e_{11}+\varepsilon^2_N e_{NN}}\xi_{\delta-\theta}e_{N1}, \tag{49}$$

$$\begin{aligned}&[[\xi_{\delta-\theta}, e_{12}]_q, [\xi_{\delta-\theta}, e_{N-1N}]_q]_q \\ &= -\eta[\varepsilon^2_N]\Big((-1)^{\vartheta_{1N-1}\vartheta_{2N}} q^{-\varepsilon^2_{N-1}} \\ &\times q^{\varepsilon^2_2 e_{22}+\varepsilon^2_{N-1}e_{N-1N-1}}\xi_{\delta-\theta}e_{N-12} \\ &+ (-1)^{\vartheta_{2N}\vartheta_{N-1N}} q^{\varepsilon^2_1 e_{11}+\varepsilon^2_N e_{NN}} \\ &\times [[\xi_{\delta-\theta}, e_{N-1N}]_q, e_{12}]_q e_{N1} \\ &+ (-1)^{\vartheta_{1N}\vartheta_{2N-1}}\big(1-q^{-2\varepsilon^2_1}\big)q^{-\varepsilon^2_{N-1}} \\ &\times q^{\varepsilon^2_1 e_{11}+\varepsilon^2_{N-1}e_{N-1N-1}} e_{12}\xi_{\delta-\theta}e_{N-11}\Big).\end{aligned} \tag{50}$$

In the case of the super-Drinfeldian $D_{q\eta}(sl(m+1|1))$, we also have the following analog of relation (37):

$$[[e_{m+1m+2}, \xi_{\delta-\theta}]_q, [e_{m+1m+2}, e_{mm+1}]_q]_q = 0. \tag{51}$$

The Hopf structure of $D_{q\eta}(sl(m+1|n+1))$ is defined by the formulas (11) for $U_q(sl(m+1|n+1))$ (i.e., $\Delta_{q\eta}(x) = \Delta_q(x)$, $S_{q\eta}(x) = S_q(x)$ for any $x \in U_q(sl(m+1|n+1))$) and $\Delta_{q\eta}(q^{\pm h_\delta}) = q^{\pm h_\delta}\otimes q^{\pm h_\delta}$, $S_{q\eta}(q^{\pm h_\delta}) = q^{\mp h_\delta}$. The comultiplication and the antipode of $\xi_{\delta-\theta}$ are given by

$$\begin{aligned}\Delta_{q\eta}(\xi_{\delta-\theta}) &= \xi_{\delta-\theta}\otimes 1 + q^{\varepsilon^2_1 e_{11}-\varepsilon^2_N e_{NN}-h_\delta}\otimes\xi_{\delta-\theta} \\ &+ \eta\Big(q^{\varepsilon^2_1 e_{11}}\otimes q^{\varepsilon^2_1 e_{11}}\Big)\Big(q^{\varepsilon^2_N e_{NN}}e_{N1}\otimes[\varepsilon^2_1 e_{11}] \\ &+ \left[\frac{h_\delta}{2}+\varepsilon^2_N e_{NN}\right] q^{-\frac{h_\delta}{2}}\otimes q^{\varepsilon^2_N e_{NN}}e_{N1} \\ &+ \sum_{i=2}^{N-1}[\varepsilon^2_i]q^{\varepsilon^2_N e_{NN}}e_{Ni}\otimes e_{i1}q^{\varepsilon^2_i e_{ii}}\Big),\end{aligned} \tag{52}$$

$$\begin{aligned}S_{q\eta}(\xi_{\delta-\theta}) &= -q^{h_\delta-\varepsilon^2_1 e_{11}+\varepsilon^2_N e_{NN}}\xi_{\delta-\theta} \\ &+ \eta\left[\frac{h_\delta}{2}+\varepsilon^2_1 e_{11}+\varepsilon^2_N e_{NN}+\varepsilon^2_1\right] \\ &\times q^{\frac{h_\delta}{2}-\varepsilon^2_1 e_{11}+\varepsilon^2_N e_{NN}-\varepsilon^2_1}e_{N1} - \frac{\eta}{q-q^{-1}} \\ &\times \sum_{k=1}^{N-1}\sum_{N-1\geq i_k>i_{k-1}>\ldots>i_1\geq 2}\Big(q^{-2\varepsilon^2_{i_k}}-1\Big)\Big(q^{-2\varepsilon^2_{i_{k-1}}}-1\Big) \\ &\cdots\Big(q^{-2\varepsilon^2_{i_1}}-1\Big)e_{n+1i_k}e_{i_k i_{k-1}}\cdots e_{i_1 1}q^{-2\varepsilon^2_1 e_{11}}.\end{aligned} \tag{53}$$

It is not hard to check that the substitution $\xi_{\delta-\theta} = q^{\varepsilon^2_1 e_{11}+\varepsilon^2_N e_{NN}}e_{N1}$, $h_\delta = 0$ satisfies the relations (41)–(51); i.e., there is a simple homomorphism $D_{q\eta}(sl(m+1|n+1)) \to U_q(sl(m+1|n+1))$. Moreover, both sides of relations (45)–(51) are equal to zero independently. Therefore, we can construct the "evaluation representation" $\rho_{\rm ev}$ of $D_{q\eta}(sl(m+1|m+1))$ in $U_q(sl(m+1|n+1))\otimes\mathbb{C}[u]$ as follows:

$$\begin{gathered}\rho_{\rm ev}(q^{h_\delta}) = 1,\;\; \rho_{\rm ev}(\xi_{\delta-\theta}) = uq^{\varepsilon^2_1 e_{11}+\varepsilon^2_N e_{NN}}e_{N1}, \\ \rho_{\rm ev}(q^{\pm e_{ii}}) = q^{\pm e_{ii}}, \quad \rho_{\rm ev}(e_{ij}) = e_{ij} \\ (1\leq i,j\leq N).\end{gathered} \tag{54}$$

By setting $q = 1$ in (41)–(53), we obtain the defining relations of the super-Yangian $Y_\eta(sl(m+1|n+1))$ and its Hopf structure in the Chevalley basis. This result is formulated as follows.

The super-Yangian $Y_\eta(sl(m+1|n+1))$ is generated (as an unital associative algebra over $\mathbb{C}[\eta]$) by the superalgebra $U(sl(m+1|n+1))$ and the elements $\xi_{\delta-\theta}$, h_δ with the relations

$$[h_\delta, \text{everything}] = 0, \tag{55}$$

$$[e_{11},\, \xi_{\delta-\theta}] = -\xi_{\delta-\theta}, \tag{56}$$

$$[e_{NN},\, \xi_{\delta-\theta}] = \xi_{\delta-\theta}, \tag{57}$$

$$[e_{ii},\, \xi_{\delta-\theta}] = 0 \quad \text{for } i = 2,3,\ldots,N-1, \tag{58}$$

$$[\xi_{\delta-\theta},\, e_{i+1i}] = 0 \quad \text{for } i = 1,2,\ldots,N-1, \tag{59}$$

$$[e_{ii+1},\, \xi_{\delta-\theta}] = 0 \quad \text{for } i = 2,3,\ldots,N-2, \tag{60}$$

$$[e_{12},\, [e_{12},\, \xi_{\delta-\theta}]] = 0, \tag{61}$$

$$[[\xi_{\delta-\theta},\, e_{N-1N}], e_{N-1N}] = 0, \tag{62}$$

$$\xi^2_{\delta-\theta} = -\eta\varepsilon^2_N\xi_{\delta-\theta}e_{N1}, \tag{63}$$

$$\begin{aligned}&[[\xi_{\delta-\theta}, e_{12}], [\xi_{\delta-\theta}, e_{N-1N}]] \\ &= -\eta\varepsilon^2_N\Big((-1)^{\vartheta_{1N-1}\vartheta_{2N}}\xi_{\delta-\theta}e_{N-12} \\ &+ (-1)^{\vartheta_{2N}\vartheta_{N-1N}}[[\xi_{\delta-\theta}, e_{N-1N}], e_{12}]e_{N1}\Big).\end{aligned} \tag{64}$$

In the case of the super-Yangian $Y_\eta(sl(m+1|1))$, we also have the following relation:

$$[[e_{m+1m+2}, \xi_{\delta-\theta}], [e_{m+1m+2}, e_{mm+1}]] = 0. \tag{65}$$

The Hopf structure of the super-Yangian $Y_\eta(sl(m+1|n+1))$ is trivial for $U(sl(m+1|m+1))\oplus\mathbb{C}\,h_\delta$ (i.e., $\Delta_\eta(x) = x\otimes 1 + 1\otimes x$, $S_\eta(x) = -x$

for $x \in sl(m+1|n+1) \oplus \mathbb{C}\, h_\delta$), and it is not trivial for the element $\xi_{\delta-\theta}$:

$$\Delta_\eta(\xi_{\delta-\theta}) = \xi_{\delta-\theta} \otimes 1 + 1 \otimes \xi_{\delta-\theta} + \eta\Big(\frac{1}{2}h_\delta \otimes e_{N1} + \sum_{i=1}^{N} \varepsilon_i^2 e_{Ni} \otimes e_{i1}\Big), \tag{66}$$

$$S_\eta(\xi_{\delta-\theta}) = -\xi_{\delta-\theta} + \eta\Big(\frac{1}{2}h_\delta e_{N1} + \sum_{i=1}^{N} \varepsilon_i^2 e_{Ni} e_{i1}\Big). \tag{67}$$

Thus, we obtain the very simple minimal realization of $D_{q\eta}(sl(m+1|n+1))$ and $Y_\eta(sl(m+1|n+1))$, that is, the realization in terms of the Chevalley basis. Analogous results can be also obtained for the super-Drinfeldian and the super-Yangian of the classical superalgebras of series $osp(M|N)$. This work is in progress.

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SYMPOSIUM ON GROUP THEORY AND PATH INTEGRALS

Propagator for the Chebyshev q Object*

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Abstract—We propose an alternative role of the harmonic oscillator algebra. Observing that the q-deformed harmonic oscillator algebra defines the Chebyshev q object, we show that the q-free particle and the pulsed oscillator are special cases of the Chebyshev q object, characterized by a common deformation parameter q and reduced to a usual free particle as q tends to unity. For the deformed free particle, q is a real number, whereas for the pulsed oscillator it belongs to S^1. Then, we derive the propagator for the Chebyshev q object, from which we obtain the propagators for the deformed free particle and the pulsed oscillator. © *2001 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

As is well known, the q-deformed commutator,

$$\hat{a}\hat{a}^\dagger - q\hat{a}^\dagger\hat{a} = q^{-\hat{N}}, \quad (1)$$

proposed by Macfarlane [1] and by Biedenharn [2], has been used to obtain the energy spectrum for the q-deformed harmonic oscillator. For the deformed oscillator, the operators $\hat{a}^\dagger$ and $\hat{a}$ are assumed to take the usual role of raising and lowering energy levels labeled by the eigenvalues of the number operator $\hat{N}$, respectively; and the energy spectrum of the oscillator consists of the eigenvalues of the structure function, $u(\hat{N}) = \hat{a}^\dagger\hat{a}$, of the deformed algebra (1). However, the physical significance of the deformed energy spectrum is not clear.

In this paper, we propose an alternative interpretation by noticing that the eigenvalues of $u(\hat{N})$ obey the Chebyshev recursion relation,

$$u(n+1) - (q+q^{-1})u(n) + u(n-1) = 0 \quad (2)$$
$$(n \in \mathbf{N}_0),$$

which is obeyed by the Chebyshev polynomials of type I and type II (see, e.g., [3]) given, respectively, by

$$T_n[\cos\varphi] = \cos(n\varphi), \quad (3)$$
$$U_n[\cos\varphi] = \frac{\sin[(n+1)\varphi]}{\sin\varphi},$$

when $q = e^{-i\varphi}$ ($\varphi \in \mathbf{R}$). Evidently, $U_{n-1}[\cos\varphi]$ can be the eigenvalues of the structure function $u(\hat{N})$ obeying the algebra (1) [4]. A general process based on the Chebyshev relation (2), which will be referred to as a Chebyshev process, defines a generic q object. We call the q object subjected to the condition, $q + q^{-1} \in \mathbf{R}$, a Chebyshev q object. We show that the time evolution of the q-deformed free particle (q-free particle) and the pulsed oscillator (p oscillator) is a Chebyshev process and that the two systems are both the Chebyshev q objects.

The q-free particle (Section 2) is a q-deformed object whose motion is governed by the q counterpart of Newton's force-free equation,

$$D^2_{s;q}y(s) = 0 \quad (4)$$

or equivalently the q-difference equation,

$$q^{-1}y(q^2s) - (q+q^{-1})y(s) + qy(q^{-2}s) = 0, \quad (5)$$

where s is a time parameter. The p oscillator (Section 3) is a free particle subjected to the periodic pulses of Hooke's force, obeying the difference equation,

$$x(t+T) - (2-\omega^2T^2)\,x(t) + x(t-T) = 0, \quad (6)$$

where T is the period of pulses. As is shown in Section 4, both the q-free particle and the p oscillator can be characterized by a common deformation parameter q. The q-free particle is characterized by $q \in \mathbf{R}$, whereas the p oscillator is characterized by $q \in S^1$.

For the p oscillator, the eigenvalue n of the number operator $\hat{N}$ corresponds to the number of pulses. Now, $\hat{a}^\dagger$ raises the number of pulses, whereas $\hat{a}$ lowers it. In this manner, we interpret the deformed harmonic oscillator algebra (1) as an algebra generating the time progression of a q object rather than an algebra generating the energy spectrum.

In Section 5, to calculate the propagator for the Chebyshev process, we exploit the usual harmonic

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analysis technique applied for the p oscillator with an arbitrary q parameter. We also obtain the propagator for the q-free particle as a special case of the propagator for the q object.

2. THE q-DEFORMED FREE PARTICLE

To describe what we are referring to as the q-deformed free (q-free) particle, let us begin with the q derivative (or the symmetric Jackson derivative) of a function $f(s)$ defined for fixed q by

$$D_{s;q}f(s) = \frac{f(qs) - f(q^{-1}s)}{(q - q^{-1})s}, \tag{7}$$

where s and q may be complex in general. If $f(s)$ is differentiable at s, then $D_{s;q}f(s) \to df(s)/ds$ as $q \to 1$. The q counterpart of Newton's equation is written as

$$D^2_{s;q}y(s) = F(y), \tag{8}$$

where $F(y)$ is a force exerted on the system.

The deformed free particle is the system satisfying the force-free equation ($F(y) = 0$) that can be put into the form of (4). A general solution of (4) is the usual free-particle solution,

$$y(s) = as + b \quad (a, b : \text{const} \in \mathbf{R}). \tag{9}$$

Insofar as the time parameter s changes translationally and uniformly, the q-free particle is nothing more than the ordinary free particle. Yet, the q-free particle equation (5) differs from the difference equation corresponding to the usual force-free Newtonian equation,

$$x(t + T) - 2x(t) + x(t - T) = 0, \tag{10}$$

which is a special case of the difference equation for the p oscillator (6) with $\omega = 0$ (where T is no longer the pulsing period but any finite time interval). The solution of (10), $x(t) = at + b$, is the same as (9) in form, but not in content. The q-free particle equation (4) reduces to the Newtonian form (10) only in the limit $q \to 1$. The difference equation (10) dictates the time evolution of the particle under the time translation $t - T \to t \to t + T$, whereas the deformed difference equation stipulates the progression of the particle under the time scaling $q^{-2}s \to s \to q^2 s$. The time transformation [5]

$$s(t) = s_0 q^{2t/T} \quad (s_0 : \text{const} \in \mathbf{R}) \tag{11}$$

with $q \in \mathbf{C}$ and $q \neq 0$ relates the time translation to the time scaling as

$$s(t + T/2) = q^{2(t+T/2)/T} = qs(t). \tag{12}$$

Therefore the trajectory of a q-free particle evolving with the Newtonian time scale is given by

$$y(s(t)) = as_0 q^{2t/T} + b. \tag{13}$$

If we demand that the time parameter $s(t)$ be real for any value of t, then q must be a positive real number; i.e., $q \in \mathbf{R}^+$. In this case, the trajectory (13) is also real and continuous. However, if the q-free particle is allowed to take a real discrete sequential trajectory $\{y(s(mT))\}$ associated with discrete periodic time translation $t = mT$ ($m \in \mathbf{N}$), then $q \in \mathbf{R}^-$ may be included since $s(mT) = s_0|q|^{2m} \in \mathbf{R}$. Even in the case when $q = |q|e^{i\pi(2k+1)/2}$ with $k \in \mathbf{Z}$, we can have a real sequence $\{y(s(mT))\}$ with $s(mT) = s_0(-|q|^2)^m \in \mathbf{R}$, but the s sequence $\{s(mT)\}$ is acausal (or periodically time-reversed) because of the factor $(-1)^m$. Therefore, taking only account of causal trajectories, whether continuous or discrete, we classify the q-free particle into two: (i) continuous type $q \in \mathbf{R}^+$ and (ii) hopping type $q \in \mathbf{R}^-$.

Next, we focus our attention on the q progression of the q-free particle. Considering m times of the q progression $s(mT) = s_0 q^{2m}$, we substitute $y_m(q) = y(s_0 q^{2m})$ into (4) to obtain the recursion relation,

$$q^{-1}y_{m+1}(q) - (q + q^{-1})y_m(q) + qy_{m-1}(q) = 0. \tag{14}$$

Obviously, $y_m(q) = as_0 q^{2m} + b$ satisfies (14). If we let

$$u_m(q) = y_{2m}(q)/q^m,$$

then the recursion relation (14) is reduced to the Chebyshev form,

$$u_{m+1}(q) - (q + q^{-1})\, u_m(q) + u_{m-1}(q) = 0. \tag{15}$$

Thus, we see that the deformed oscillator algebra (1) may be linked to the q progression of the q-free particle if $q \in \mathbf{R}$ ($q \neq 0$).

3. THE PULSED OSCILLATOR

The pulsed oscillator (p oscillator) is a free particle which undergoes periodic pulses of Hooke's force $F(t) = -M\omega^2 x\delta(t/T - m)$, where $M\omega^2$ is Hooke's constant, T is the period of pulses, and $m \in \mathbf{Z}$. This differs from the so-called kicked oscillator that is a harmonic oscillator subjected to periodic kicks. The Lagrangian is given by

$$L = \frac{1}{2}M\dot{x}^2 - \sum_m \frac{1}{2}M\omega^2 T x^2 \delta(t - t_m), \tag{16}$$

where $t_m = mT$. Hooke's force is exerted not continuously but periodically and instantaneously at $t = t_m$. During the period between two consecutive pulses, the system is a free particle.

The action integral for a time interval $\tau = t'' - t'$ is

$$S(t'', t') \tag{17}$$

$$= \int_{t'}^{t''} \left[\frac{1}{2} M\dot{x}^2 - \frac{1}{2} M\omega^2 T x^2 \sum_m \delta(t - t_m) \right] dt.$$

For a short time interval $\tau_j = t_j - t_{j-1} \ll T = t_m - t_{m-1}$, we choose the action symmetric with respect to x_m and x_{m-1}:

$$S_j = \frac{M}{2\tau_j}(x_j - x_{j-1})^2 \quad (18)$$
$$- \frac{1}{4} M\omega^2 T \{x_m^2 \, \delta(m, j) + x_{m-1}^2 \, \delta(m-1, j)\},$$

where $x_k = x(t_k)$ and $\delta(k, j) = 1$ if $t_{j-1} < kT < t_j$, and $\delta(k, j) = 0$ otherwise. Naturally, the action evaluated along the classical path in the time interval from $t = t_{m-1} + \epsilon$ to $t = t_m - \epsilon$ between two pulses yields that of the free particle:

$$S_0(t_m, t_{m-1}) \quad (19)$$
$$= \lim_{\epsilon \to 0} S(t_m - \epsilon, t_{m-1} + \epsilon) = \frac{M}{2T}(x_m - x_{m-1})^2.$$

The action over one period, involving pulses, resulting from (19) is

$$S(t_m, t_{m-1}) = \frac{M}{2T}(x_m - x_{m-1})^2 \quad (20)$$
$$- \frac{1}{4} M^2 T (x_m^2 + x_{m-1}^2),$$

which is symmetric with respect to x_m and x_{m-1}.

Calculating the canonical momenta from the symmetrized action (20) by

$$p_m = \partial S / \partial x_m$$
$$= (M/T)(x_m - x_{m-1}) - (M\omega^2 T/2) x_m,$$
$$p_{m-1} = -\partial S / \partial x_{m-1}$$
$$= (M/T)(x_m - x_{m-1}) + (M\omega^2 T/2) x_{m-1},$$

we find the area-preserving linear map in phase space:

$$x_m = x_{m-1} \quad (21)$$
$$+ (T/2M)(1 - \frac{1}{4}\omega^2 T^2)^{-1} \, (p_m + p_{m-1}),$$
$$p_m = p_{m-1} - (M\omega^2/2) T (x_m + x_{m-1}).$$

The evolution of the classical trajectory in phase space obeying the linear map (21) is not chaotic and may not be interesting for the chaos study. What is interesting is that both x_m and p_m in (21) obey the Chebyshev recursion relation,

$$u_{m+1}(z) - 2z \, u_m(z) + u_{m-1}(z) = 0, \quad (22)$$

when the following identification is made: $z = 1 - \omega^2 T^2/2$. Apparently (22), if $2z = q + q^{-1}$, is identical in form with (2) and (15). With $z = \cos\varphi$ (or $q = e^{-i\varphi}$), the solutions of the recursion relation (22) are given in terms of the Chebyshev polynomials of (3). If $0 < \omega^2 T^2 < 4$, then $\varphi \in \mathbf{R}$. Hence, the classical discrete solutions for $x(t)$ and $p(t)$ oscillate sinusoidally, which are indeed physical solutions for the proper p oscillator. Thus, we realize that the p oscillator, obeying the Chebyshev process, is indeed a q-deformed object with $q = e^{-i\varphi} \in S^1$, where $\varphi = \cos^{-1}(1 - \omega^2 T^2/2) \in \mathbf{R}$.

If $\omega^2 T^2 < 0$ or $4 < \omega^2 T^2$, then φ has to be complex; so the solutions of (22) are not oscillatory. Nevertheless, we may treat the physically proper solutions and the physically improper solutions together as solutions of the p oscillator in a generalized sense.

4. q OBJECTS

As we have seen above, the time evolution of both the q-free particle and the p oscillator is the Chebyshev process. Furthermore, the two systems approach the free particle in the limit $q \to 1$. Therefore, by using a common deformation parameter q, we should be able to treat both the q-free particle and the p oscillator in a unified manner. In other words, we may consider the two systems as special cases of a generic q object.

The generic q object may be defined with a nonzero complex-valued q. However, we restrict ourselves to the case where $z = \cos\varphi = (q + q^{-1})/2$ is real. Under this condition, $q \in \mathbf{R}$ or $q \in S^1$. In fact, such a q object is equivalent to the generalized p oscillator possessing the proper and improper solutions. Therefore, it is convenient to utilize the oscillator's frequency ω as a parameter even for the q object. We then put solutions of the q object into three classes as follows:

$$\text{(i)} \quad 0 < \omega^2 T^2 < 4;$$
$$\varphi = \cos^{-1}(1 - \omega^2 T^2/2) \in \mathbf{R}; \;\; q \in S^1;$$
$$\text{(ii)} \quad \omega^2 T^2 < 0;$$
$$i\varphi = \cosh^{-1}(1 + |\omega|^2 T^2/2) \in \mathbf{R}; \;\; q \in \mathbf{R}^+;$$
$$\text{(iii)} \quad 4 < \omega^2 T^2;$$
$$i\varphi = i\pi + \cosh^{-1}(\omega^2 T^2/2 - 1) \in \mathbf{R}; \;\; q \in \mathbf{R}^-.$$

Evidently, case (i) corresponds to the proper p oscillator. As has been mentioned earlier, for the real trajectory (13) of the q-free particle, q must be real. However, for a continuous evolution with the Newtonian time t, $y(t)$ can be real only when q is positive. Hence, the (proper) evolution of the q-free particle should belong to case (ii). As each discrete translation of time by T causes the scaling of s by q^2, y_m remains real even if q is a negative real number, provided s_0 is real. For a continuous evolution with a negative q, $y(t)$ takes complex values in general. Thus, case (iii) corresponds to the discrete evolution of the hopping q-free particle. In this manner, the q-free particle may be viewed as a form of the improper p oscillator.

5. THE PROPAGATOR FOR THE CHEBYSHEV PROCESS

In what follows, we calculate the propagator for the p oscillator and see how it depends on the deformation parameter q. Then, we interpret it more generally as the propagator (in the T evolution) for the generic q object obeying the Chebyshev process.

The propagator for the system with the Lagrangian (16) can be calculated from Feynman's path integral,

$$K(x'', x'; \tau) = \lim_{N\to\infty} \int\limits_{x'=x(t_0)}^{x''=x(t_N)} \prod_{j=1}^{N} \exp\left[\frac{i}{\hbar} S_j\right] \times \prod_{j=1}^{N} \left[\frac{M}{2\pi i\hbar\tau_j}\right]^{1/2} \prod_{j=1}^{N-1} dx_j, \quad (23)$$

where $\tau = t_N - t_0$ is a fixed total time interval. This propagator can be easily calculated with the action (18). The propagator evaluated from $t_{m-1}+\epsilon$ to $t_m - \epsilon$, involving no pulse, is in fact the free-particle propagator:

$$K^{\text{free}}(x_m, x_{m-1}; T) = \left[\frac{M}{2\pi i\hbar T}\right]^{1/2} \exp\left[\frac{i}{\hbar} S_0(t_m, t_{m-1})\right], \quad (24)$$

where S_0 is the free action (19). The one-period propagator takes a simple form,

$$K(x_m, x_{m-1}; T) = \left[\frac{M}{2\pi i\hbar T}\right]^{1/2} \exp\left[\frac{i}{\hbar} S(t_m, t_{m-1})\right], \quad (25)$$

where $S(t_m, t_{m-1})$ is the one-period action (20).

For convenience, we rewrite the symmetric one-period (20) as

$$S(x_m, x_{m-1}) = \frac{M}{2T}\left(1 - \frac{1}{2}\omega^2 T^2\right)(x_m^2 + x_{m-1}^2) - \frac{M}{T}(x_m x_{m-1}). \quad (26)$$

If we let $\cos\varphi = 1 - \omega^2 T^2/2$ or $\sin(\varphi/2) = \omega T/2$, and $\xi = \alpha x$ with $\alpha = \sqrt{(M/\hbar T)\sin\varphi}$, the one-period action (26) may further be rewritten as

$$S(x_m, x_{m-1}) = \frac{1}{2}\hbar \cot\varphi\,(\xi_m^2 + \xi_{m-1}^2) - \hbar \csc\varphi\, \xi_m \xi_{m-1}. \quad (27)$$

At this point, we relate φ to the deformation parameter q by $\varphi = i \ln q$ as has been mentioned at the end of the previous section. Then, we may express the one-period propagator for the action (27) as

$$K(x_m, x_{m-1}; T) = \left[\frac{M}{2\pi i\hbar T}\right]^{1/2} \exp\left[-\frac{1}{2}(\xi_m^2 + \xi_{m-1}^2)\right] \times \exp\left[\frac{2\xi_m \xi_{m-1} q - (\xi_m^2 + \xi_{m-1}^2) q^2}{1 - q^2}\right]. \quad (28)$$

The propagator for a double period $2T$ can be found by convolution,

$$K(x_{m+1}, x_{m-1}; 2T) = \int K(x_{m+1}, x_m; T) K(x_m, x_{m-1}; T) dx_m. \quad (29)$$

In finding the two-period propagator via (29), we exploit the idea of harmonic analysis to expand the one-period propagator in a series of orthogonal polynomials and carry out the convolution with the aid of the orthogonality property of the polynomials.

Now, we use Mehler's formula for the Hermite polynomials $H_n(x)$ (see, e.g., [6]),

$$(1 - q^2)^{-1/2} \exp\left[\frac{2xyq - (x^2 + y^2) q^2}{1 - q^2}\right] = \sum_{k=0}^{\infty} \frac{q^k H_k(x) H_k(y)}{2^k k!}, \quad (30)$$

to put the propagator (28) in the series form

$$K(x_m, x_{m-1}; T) = \left[\frac{\alpha}{\pi}\right]^{1/2} \exp\left[-\frac{1}{2}(\xi_m^2 + \xi_{m-1}^2)\right] \times \sum_{k=0}^{\infty} \frac{1}{2^k k!} q^{(k+\frac{1}{2})} H_k(\xi_m) H_k(\xi_{m-1}). \quad (31)$$

Substituting this into the integrand of (29) and performing the integration with the help of the orthogonality relation for the Hermite polynomials,

$$\int\limits_{-\infty}^{\infty} e^{-\xi^2} H_k(\xi) H_{k'}(\xi)\, d\xi = 2^k k! \sqrt{\pi}\,\delta_{k,k'}, \quad (32)$$

we can easily arrive at the double-period propagator. Repeating similar processes, the n-period propagator can be found in the form

$$K(x_n, x_0; nT) = \left[\frac{\alpha}{\pi}\right]^{1/2} \exp\left[-\frac{1}{2}(\xi_n^2 + \xi_0^2)\right] \times \sum_{k=0}^{\infty} \frac{1}{2^k k!} q^{n(k+\frac{1}{2})} H_k(\xi_n) H_k(\xi_0). \quad (33)$$

Now, it is important to note that the n-period propagator is characterized by the nth power of the deformation parameter q.

Again, using the expansion formula (30) and noticing that

$$\alpha = [(M/\hbar T)\sin\varphi]^{1/2} = [M(q - q^{-1})/(2i\hbar T)]^{1/2},$$

we put the series solution (33) back to a closed-form expression:

$$K(x_n, x_0; nT) = \left[\frac{M(q - q^{-1})}{2\pi i\hbar T(q^n - q^{-n})}\right]^{1/2} \quad (34)$$

$$\times \exp\left[\frac{iM(q-q^{-1})}{4\hbar T(q^n-q^{-n})} \times \{(x_n^2+x_0^2)(q^n+q^{-n})-4x_nx_0\}\right].$$

This n-period propagator is the q representation of the propagator for the generalized p oscillator or the Chebyshev q object.

In the following, we consider the propagators for the T evolution of the proper p oscillator and the q-free particle as special cases of the propagator for the q object.

Pulsed harmonic oscillator. The n-period propagator for the proper p oscillator follows immediately from the propagator (34) with $q = e^{-i\varphi}$; namely,

$$K(x_n, x_0; nT) = \left\{\frac{M}{2\pi i\hbar T U_{n-1}[\cos\varphi(T)]}\right\}^{1/2} \tag{35}$$

$$\times \exp\left\{\frac{iM}{2\hbar T U_{n-1}[\cos\varphi(T)]} \times \{(x_n^2+x_0^2)T_n[\cos\varphi(T)]-2x_nx_0\}\right\}.$$

In this case, the angle φ must inevitably be related to the period T by $\varphi(T) = \cos^{-1}(1-\omega^2T^2/2) \in \mathbf{R}$ under the condition $0 < \omega^2T^2 < 4$. The corresponding deformation parameter is $q = e^{-i\varphi} \in S^1$. It is straightforward to show that the propagator (35) reduces to the standard result for the harmonic oscillator in the limit where $T \to 0$ and $n \to \infty$ with a finite time interval $\tau = nT$, that is, in the limit $q^n \to e^{-i\omega\tau} \neq 1$. If $q \to e^{-i\omega T} \to 1$, (35) becomes the usual free-particle propagator. The zeros of $U_{n-1}(\cos\varphi)$ in the prefactor lead the propagator (35) to diverge. The zeros occur only when $n\varphi = k\pi$ $(k = 0, \pm1, \pm2, ...)$, that is, only for the real p oscillator with frequency ω meeting the restriction $0 < \omega^2T^2 < 4$. In other words, $q^n = e^{-ik\pi}$ with $k \in \mathbf{Z}$ corresponding to the caustics of the propagator for the proper p oscillator (35).

***q*-Free particle**. To extract the n-period propagator for the T evolution of the q-free particle from (34), we first remind ourselves that the coordinate variable y_n of the q-free particle is related to the variable x_n satisfying the Chebyshev recursion relation (2) by $x_n = q^{-n}y_n$. Thus, converting x_n into y_n, we obtain

$$K(y_n, y_0; nT) = \left\{\frac{M(q-q^{-1})}{2\pi i\hbar T(q^n-q^{-n})}\right\}^{1/2} \tag{36}$$

$$\times \exp\left\{\frac{iM(q-q^{-1})}{4\hbar T(q^n-q^{-n})} \times \{(q^{-2n}y_n^2+y_0^2)(q^n+q^{-n})-4q^{-n}y_ny_0\}\right\}$$

with $q \in \mathbf{R}$. Here, $y_n = y(q^{2n}s_0)$ with the initial value of the time parameter $s_0 = q^{2t_0/T}$. Since we are dealing the time evolution by $t = nT$ (which corresponds to q progression by q^{2n}), the propagator (36) is valid for the two types of the q-free particle with $q \in \mathbf{R}^+$ and $q \in \mathbf{R}^-$.

6. CONCLUSION

We have suggested that the eigenvalue of the number operator $\hat{N}$ in the q-deformed oscillator algebra (1) may be interpreted as the number of pulses for the pulsed harmonic oscillator (p oscillator). We have then observed that both the q-deformed free particle and the pulsed oscillator obey the Chebyshev recursion relation. By using this interesting nature, we have treated the two systems as special cases of a generic q-deformed system (q object) and evaluated the propagator for the Chebyshev process. From this unified treatment, we have been able to derive the n-period propagator for the T evolution of the q-free particle as well as that of the p oscillator. The boson limit $q = 1$ gives the ordinary free-particle propagator. While $q \in \mathbf{R}$ for the q-free particle, $q \in S^1$ for the p oscillator.

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SYMPOSIUM ON GROUP THEORY AND PATH INTEGRALS

Dual Formulation of Gauge Theories and Confinement*

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Abstract—We review various approaches to the problem of string representation of gauge theories allowing for the analytic description of confinement. The models under study include QCD within the stochastic vacuum model, compact QED, and Abelian-projected $SU(2)$ theory. © *2001 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

The problem of string representation of gauge theories is unambiguously related to the problem of confinement in these theories. Its essence is the quest of a string theory, which is mostly adequate for the description of strings between color objects, which appear in the confining phase of QCD. (Other, Abelian-type gauge theories possessing the confining phase then serve for probing various approaches to the construction of the string representation of QCD.) Quantitatively, the QCD string can be seen by virtue of the Wilson picture of confinement [1]. It states that the criterion of confinement in QCD is the area-law behavior of the Wilson loop

$$\langle W(C)\rangle \equiv \frac{1}{N_c} \quad (1)$$

$$\times \left\langle \mathrm{tr}\mathcal{P}\exp\left(ig\oint\limits_C A_\mu^a T^a dx_\mu\right)\right\rangle \stackrel{|C|\to\infty}{\longrightarrow} e^{-\sigma|\Sigma_{\min}(C)|}.$$

Here, σ is the so-called string tension, i.e., the energy density of the QCD string. The latter one is nothing else but a tube formed by the lines of chromoelectric flux, which appears between two color objects propagating along the contour C. When these objects try to move apart from each other, the QCD string stretches and prevents that, thus ensuring their confinement. According to Eq. (1), during its propagation, such a string sweeps out the surface of the minimal area for a given contour C, $\Sigma_{\min}(C)$. Due to the dimensional reasons,

$$\sigma \propto \Lambda_{\mathrm{QCD}}^2 = \frac{1}{a^2}\exp\left[-\frac{16\pi^2}{\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)g^2\left(a^{-2}\right)}\right] \quad (2)$$

with $a \to 0$ standing for the distance ultraviolet cutoff (e.g., the lattice spacing). Clearly, all the coefficients in the expansion of σ in powers of g^2 vanish, which means that the QCD string is an essentially nonperturbative object.

Owing to this observation, it is nowadays commonly argued that the area law is well saturated by the strong background fields in QCD. Around those, there additionaly exist perturbative fluctuations of the QCD vacuum [2], which excite the string. This means that these fluctuations enable the string to sweep out with a nonvanishing probability not only $\Sigma_{\min}(C)$, but also an arbitrary surface $\Sigma(C)$ bounded by C. Therefore, the final aim in constructing the string representation of QCD is a derivation of the formula $\langle W(C)\rangle = \sum_{\Sigma(C)} e^{-\mathcal{S}[\Sigma(C)]}$. Here, $\sum_{\Sigma(C)}$ and $\mathcal{S}[\Sigma(C)]$ stand for a certain sum over string world sheets and a string effective action, both of which are yet unknown in QCD. Clearly, this formula is just a $2D$ analog of the well-known representation for the propagator of a pointlike particle, which is subject to external forces and/or propagates in external fields. In particular, the role of the classical trajectory of such a particle is played within this analogy by $\Sigma_{\min}(C)$. However, it unfortunately turns out to be difficult to proceed from the $1D$ case, where the sum over paths is universal (i.e., depends only on the dimension of the spacetime), and the world-line action is known for a wide class of potentials and external fields, to the $2D$ case under study. In the next section, we shall discuss various field-theoretical models, where the string effective action and/or the measure in the

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sum over world sheets are either already postulated a priori or can be derived.

2. FIELD-THEORETICAL MODELS AND THEIR STRING REPRESENTATIONS

2.1. String Representation of QCD within the Stochastic Vacuum Model

As a natural origin for the QCD-string effective action serves the stochastic vacuum model (SVM) of QCD [3] (for a review, see [2, 4]). Within the so-called bilocal or Gaussian approximation in SVM, well confirmed by the existing lattice data [5–7], this model is fully described by the irreducible bilocal gauge-invariant field strength correlator (cumulant), $\langle\langle F_{\mu\nu}(x)\Phi(x,x')F_{\lambda\rho}(x')\Phi(x',x)\rangle\rangle$. Here, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ stands for the Yang–Mills field strength tensor,

$$\Phi(x,y) \equiv \frac{1}{N_c}\mathcal{P}\exp\left(ig\int\limits_y^x A_\mu(u)du_\mu\right)$$

is a parallel transporter factor along the straight-line path, and $\langle\langle\mathcal{O}\mathcal{O}'\rangle\rangle \equiv \langle\mathcal{O}\mathcal{O}'\rangle - \langle\mathcal{O}\rangle\langle\mathcal{O}'\rangle$ with the average defined with respect to the Euclidean Yang–Mills action. It is further convenient to parametrize the bilocal cumulant by the following two coefficient functions [3]:

$$\frac{g^2}{2}\langle\langle F_{\mu\nu}(x)\Phi(x,x')F_{\lambda\rho}(x')\Phi(x',x)\rangle\rangle \tag{3}$$
$$= \hat{1}_{N_c\times N_c}\left\{(\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda})D\left(z^2/T_g^2\right) + \frac{1}{2}\left[\partial_\mu(z_\lambda\delta_{\nu\rho} - z_\rho\delta_{\nu\lambda}) + \partial_\nu(z_\rho\delta_{\mu\lambda} - z_\lambda\delta_{\mu\rho})\right]D_1\left(z^2/T_g^2\right)\right\}.$$

Here, $\hat{1}_{N_c\times N_c}$ is the color unity matrix, $z \equiv x - x'$, and T_g is the so-called correlation length of the QCD vacuum, i.e., the distance at which the nonperturbative parts of the functions D and D_1 decrease as $\mathrm{e}^{-|x|/T_g}$. According to the existing lattice data [5], $T_g \simeq 0.13$ fm for the $SU(2)$ case and $T_g \simeq 0.22$ fm for the $SU(3)$ case (see also [6] for related investigations and [7] for reviews). Equation (3) means that the function D plays the role of the propagator of a nonperturbative gluon which propagates between the points x and x' lying on the string world sheet $\Sigma(C)$. Indeed, the respective nonlocal string effective action can be shown to have the form

$$\mathcal{S}[\Sigma(C)] = 2\int\limits_{\Sigma(C)} d\sigma_{\mu\nu}(x)\int\limits_{\Sigma(C)} d\sigma_{\mu\nu}(x')D(z^2/T_g^2). \tag{4}$$

Upon the expansion of this expression in powers of the derivatives with respect to the world-sheet coordinates $\xi = (\xi^1, \xi^2)$, we get [8]

$$\mathcal{S}[\Sigma(C)] = \sigma\int d^2\xi\sqrt{g} + \frac{1}{\alpha}\int d^2\xi\sqrt{g}g^{ab}(\partial_a t_{\mu\nu})(\partial_b t_{\mu\nu}) + O\left(T_g^6\alpha_s\langle F_{\mu\nu}^2(0)\rangle/R^2\right). \tag{5}$$

Here, $R \simeq 1.0$ fm is the size of the contour C at which the area law holds [9], and one can see that the parameter of the expansion is $(T_g/R)^2 \simeq 0.04$, i.e., it is really much less than unity. Next, in Eq. (5), $g^{ab} = (\partial^a x_\mu)(\partial^b x_\mu)$ is the induced metric tensor corresponding to the world sheet $\Sigma(C)$ parametrized by the vector $x_\mu(\xi)$, g is the determinant of this tensor, and $t_{\mu\nu} = \varepsilon^{ab}(\partial_a x_\mu)(\partial_b x_\nu)/\sqrt{g}$ is the extrinsic curvature tensor corresponding to the same world sheet. The first term on the right-hand side of Eq. (5) is the celebrated Nambu–Goto term with the string tension $\sigma = 4T_g^2\int d^2zD\left(z^2\right)$, whereas the second term is the so-called rigidity term [10] with the coupling constant $1/\alpha = -(T_g^4/4)\int d^2zz^2D\left(z^2\right)$. Note that the negative sign of α is important for the stability of string configurations under study [11].

Further developments and applications of the string representation of QCD within the SVM can be found in [12]. However, despite some progress achieved in that direction, an important principal problem cannot be solved by use of the SVM. Namely, this model does not allow one to get the sum over string world sheets, and the string effective action (4) [and, consequently, (5)] is rigorously defined only at $\Sigma_{\min}(C)$. This problem turns out to be soluble in Abelian-type effective theories, which we will consider in the next subsections.

2.2. String Representation of the Wilson Loop in 3D Compact QED

In this model, confinement of an external quark is caused by stochastic magnetic fluxes penetrating through the contour C which are generated by magnetic monopoles. Those form a dilute gas, whose density appears on the right-hand side of the respectively modified Bianchi identity:

$$\frac{1}{2}\varepsilon_{\mu\nu\lambda}\partial_\mu F_{\nu\lambda}^{\mathrm{mon}} = 2\pi\rho_{\mathrm{gas}} \equiv 2\pi\sum_{a=1}^N q_a\delta(\mathbf{x} - \mathbf{z}_a). \tag{6}$$

Here, $F_{\nu\lambda}^{\mathrm{mon}}$ is a certain field strength tensor of the monopole gas, $\mathbf{z}_a$ is the position of the ath monopole, and q_a is its charge (in the units of magnetic coupling constant g). Since the energy of a single monopole can be shown to be the quadratic function of its flux, it is energetically more favorable for the vacuum to support the configuration of two monopoles of a unit magnetic charge than one monopole of the double charge. Thus, surviving monopoles with $|q_a| = 1$ interact with each other by the Coulomb potential. By assigning to each monopole a certain Boltzmann factor $\zeta \propto \exp\left(-\mathrm{const}\, g^2\right)$, one can sum up over the grand canonical ensemble of those as follows [13]:

$$\mathcal{Z}_{\mathrm{mon}} = 1 + \sum_{N=1}^{\infty} \frac{\zeta^N}{N!} \prod_{a=1}^{N} \int d^3 z_a \sum_{q_a=\pm 1} \tag{7}$$
$$\times \exp\left[-\frac{g^2}{8\pi}\int d^3x \int d^3y \rho_{\mathrm{gas}}(\mathbf{x}) \frac{1}{|\mathbf{x}-\mathbf{y}|}\rho_{\mathrm{gas}}(\mathbf{y})\right]$$
$$= \int \mathcal{D}\chi \exp\left\{-\int d^3x \left[\frac{1}{2}(\nabla\chi)^2 - 2\zeta\cos(g\chi)\right]\right\}.$$

We see that the dual (disorder) scalar field χ acquires a nonvanishing (magnetic) mass $m = g\sqrt{2\zeta}$ due to the Debye screening in the Coulomb gas of monopoles. Owing to the Dirac quantization condition $eg = 2\pi$ with e standing for the electric coupling constant, one has $m \propto \exp\left(-\mathrm{const}/e^2\right)$, which is nonanalytic in e. This means that similarly to QCD, the nature of confinement in the model under study is essentially nonperturbative.

In the case of the large enough plane contour C, the area law for the respective Wilson loop describing an external electrically charged particle in the theory (7) has been proved in [13]. However, one can derive the full string representation for an arbitrarily shaped C, which has the form [14]

$$\langle W(C)\rangle \tag{8}$$
$$\equiv \left\langle \exp\left(\frac{i}{2}\int_{\Sigma(C)} d\sigma_{\mu\nu}\left(F_{\mu\nu} + F_{\mu\nu}^{\mathrm{mon}}\right)\right)\right\rangle$$
$$= \exp\left(-\frac{e^2}{8\pi}\oint_C dx_\mu \oint_C dy_\mu \frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\frac{\mathcal{Z}_{\mathrm{mon}}[\eta]}{\mathcal{Z}_{\mathrm{mon}}[0]}.$$

Here, $F_{\mu\nu}$ is the usual field strength tensor of photons, the standard Gaussian average over which was factored out as the first exponential factor. Next, in Eq. (8), $\mathcal{Z}_{\mathrm{mon}}[0]$ coincides with the partition function (7), whereas $\mathcal{Z}_{\mathrm{mon}}[\eta]$ is given by the following integral over monopole densities:

$$\mathcal{Z}_{\mathrm{mon}}[\eta] \tag{9}$$
$$= \int \mathcal{D}\rho \exp\left\{-\left[\frac{g^2}{8\pi}\int d^3x \int d^3y \rho(\mathbf{x})\frac{1}{|\mathbf{x}-\mathbf{y}|}\right.\right.$$
$$\left.\left.\times\ \rho(\mathbf{y}) + V[\rho] - \frac{i}{2}\int d^3x \rho\eta\right]\right\}.$$

Here, $\eta[\mathbf{x}, \Sigma(C)] = \partial_\mu^x \int_{\Sigma(C)} d\sigma_\mu(\mathbf{y})|\mathbf{x}-\mathbf{y}|^{-1}$ is the solid angle under which the surface Σ shows up to an observer located at the point $\mathbf{x}$, and $V[\rho]$ is the following multivalued potential of monopole densities:

$$V[\rho] = \sum_{n=-\infty}^{+\infty}\int d^3x\left\{\rho\left[\ln\left(\frac{\rho}{2\zeta}\right.\right.\right. \tag{10}$$
$$\left.\left.\left. + \sqrt{1+\left(\frac{\rho}{2\zeta}\right)^2}\right) + 2\pi i n\right] - 2\zeta\sqrt{1+\left(\frac{\rho}{2\zeta}\right)^2}\right\}.$$

It is the sum over branches of this potential, which restores the Σ independence of the right-hand side of Eq. (8), which seems to be violated by the last term in Eq. (9). This is the essence of the string representation of the Wilson loop in $3D$ compact QED. Note that upon the change of the integration variables $\rho \to \varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda}/(4\pi)$ [cf. Eq. (6)], $\int \mathcal{D}\rho \to \int \mathcal{D}h_{\mu\nu}$, where $h_{\mu\nu}$ is the antisymmetric tensor field (the so-called Kalb–Ramond field [15]), one can get from Eqs. (8)–(10) the so-called theory of confining strings [16].

2.3. String Representation of the Effective Abelian-Projected SU(2) Theory

Other Abelian-type theories where confinement takes place are the so-called Abelian-projected theories [17]. There, within the so-called Abelian dominance hypothesis [18], one gets as an effective infrared theory corresponding to the $SU(N_c)$ gluodynamics the $[U(1)]^{N_c-1}$ magnetically gauge-invariant dual theory with monopoles. Next, demanding the condensation of monopole Cooper pairs, one arrives at the dual Abelian Higgs-type theory. There, confinement can be described as the dual Meissner effect [19]; i.e., it is due to the formation of the dual Nielsen–Olesen strings [20]. Below, we shall consider these theories in the London limit, i.e., the limit when the mass of the dual Higgs fields is much larger than the mass of the dual vector bosons.[3)]

In our analysis, we shall restrict ourselves to the simplest $SU(2)$ case, referring the reader for the generalization to the case of effective $SU(3)$ Abelian-projected theory [21] to [22, 23]. The partition function of the effective Abelian-projected theory describing $SU(2)$ QCD reads

$$\mathcal{Z} = \int \mathcal{D}B_\mu \mathcal{D}\theta^{\mathrm{sing}}\mathcal{D}\theta^{\mathrm{reg}} \exp\left\{-\int d^4x\right. \tag{11}$$
$$\left.\times\left[\frac{1}{4}\left(F_{\mu\nu} + F_{\mu\nu}^e\right)^2 + \frac{\eta^2}{2}(\partial_\mu\theta - 2g_m B_\mu)^2\right]\right\}.$$

[3)] Note that the size of the core of the string (vortex in $3D$) is equal to the inverse mass of the dual Higgs fields, which means that the London limit corresponds to infinitely thin strings.

Correspondence between various field-theoretical models and approaches to their string representation

Model	QCD within SVM	Abelian-projected theories	$3D$ compact QED
Mechanism of the string representation	No integral over string world sheets. String effective action is rigorously defined only with respect to $\Sigma_{\min}$	Summation over world sheets stems from the integration over the multi-valued part of the phase of the dual Higgs field(s)	Σ independence of $\langle W(C)\rangle$ is realized by the summation over branches of $V[\rho]$
Mechanism of the mass generation	Due to stochastic background fields	Higgs mechanism	Debye screening in the monopole gas
Type of propagator between the elements of the world sheet(s)	Nonperturbative gluon propagator (D function)	Propagators of the Kalb–Ramond fields	Propagator of the Kalb–Ramond field
Parameter of the expansion of the resulting nonlocal interaction between the elements of the world sheet(s)	Correlation length of the QCD vacuum, T_g	Inverse mass of the dual vector bosons	Inverse Debye mass of the dual boson

Here, $\theta = \theta^{\rm sing} + \theta^{\rm reg}$ is the phase of the dual Higgs field describing the condensate of monopole Cooper pairs, η is the vacuum expectation value of this field, and $2g_m$ is its magnetic charge with g_m being the magnetic coupling constant related to the electric one, g, as $g_m g = 4\pi$. Next, in Eq. (11), B_μ stands for the gauge field dual to the diagonal gluonic field A_μ^3, and $F_{\mu\nu}^e$ is a field strength tensor of an external electrically charged particle (quark) obeying the equation $\partial_\mu \tilde{F}^e_{\mu\nu}(x) = g\oint_C dx_\nu(\tau)\delta(x - x(\tau))$, where $\tilde{\mathcal{O}}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}\mathcal{O}_{\lambda\rho}$. The multivalued field $\theta^{\rm sing}$ describes dual closed strings according to the formula

$$\varepsilon_{\mu\nu\lambda\rho}\partial_\lambda\partial_\rho\theta^{\rm sing} \tag{12}$$
$$= 2\pi\Sigma_{\mu\nu}(x) \equiv 2\pi\int_\Sigma d\sigma_{\mu\nu}(x(\xi))\delta(x - x(\xi)),$$

where the vector $x_\mu(\xi)$ parametrizes the world sheet Σ of a closed string. Equation (12), which is actually nothing else but the local form of the Stokes theorem for the gradient of the field $\theta^{\rm sing}$, enables one to pass from the integration over $\theta^{\rm sing}$ to the integration over $x_\mu(\xi)$.[4)] On the other hand, the field $\theta^{\rm reg}$ describes single-valued fluctuations around a given string configuration described by $\theta^{\rm sing}$. Integration over $\theta^{\rm reg}$ can be shown to be reformulated via some field-theoretical constraints as the integration over the Kalb–Ramond field, which is the essence of the so-called path-integral duality transformation [25].

Bringing together the above considerations, we get

$$\mathcal{Z} = \int \mathcal{D}x_\mu(\xi)\mathcal{D}h_{\mu\nu}\exp\left\{-\int d^4x \right.$$
$$\left.\times\left[\frac{1}{12\eta^2}H^2_{\mu\nu\lambda} + g_m^2 h_{\mu\nu}^2 + i\pi h_{\mu\nu}\hat{\Sigma}_{\mu\nu}\right]\right\}.$$

Here, $H_{\mu\nu\lambda} = \partial_\mu h_{\nu\lambda} + \partial_\lambda h_{\mu\nu} + \partial_\nu h_{\lambda\mu}$ is the kinetic term of the Kalb–Ramond field, and $\hat{\Sigma}_{\mu\nu} = 4\Sigma^e_{\mu\nu} - \Sigma_{\mu\nu}$. In the last formula, $\Sigma^e_{\mu\nu}$ is defined in the same way as $\Sigma_{\mu\nu}$ with the replacement $\Sigma \to \Sigma^e$, where Σ^e stands for an arbitrary open surface bounded by the quark trajectory C. Thus, we see that the path-integral duality transformation is an elegant way of getting a coupling of the massive dual vector boson (described by the Kalb–Ramond field) to the string world sheet. Finally, integration over the Kalb–Ramond field yields [26]

$$\mathcal{Z} = \exp\left[-\frac{g^2}{2}\oint_C dx_\mu \oint_C dy_\mu D_m^{(4)}(x-y)\right] \tag{13}$$
$$\times\int \mathcal{D}x_\mu(\xi)\exp\left[-(\pi\eta)^2\int d^4x\right.$$
$$\left.\times\int d^4y\hat{\Sigma}_{\mu\nu}(x)D_m^{(4)}(x-y)\hat{\Sigma}_{\mu\nu}(y)\right].$$

Here, $D_m^{(4)}(x) = mK_1(m|x|)/(4\pi^2|x|)$ is the propagator of the dual vector boson of the mass $m = 2g_m\eta$ with K_1 standing for the modified Bessel function. Clearly, the first exponential factor on the right-hand side of Eq. (13) describes the Yukawa interaction of

4) The Jacobian appearing during this change of variables in the functional integral has been evaluated in [24].

quarks, whereas the integral over world sheets describes the (self-)interaction of both closed and open dual strings. In particular, the derivative expansion of $\Sigma^e_{\mu\nu} \times \Sigma^e_{\mu\nu}$-interaction performed along with the lines of Section 2.1 yields the linear confinement part of the quark—antiquark potential described by the Nambu—Goto term. Finally, the reader is referred to [22, 26] for an extended discussion of the above ideas, as well as their application to the SVM.

3. SUMMARY

As a summary of the present talk serves the table, which summarizes the most important aspects of various approaches to the string representations of gauge theories discussed above.

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SYMPOSIUM ON GROUP THEORY AND PATH INTEGRALS

Geometric Derivation of Nonlinear Sigma Model for the $1D$ Antiferromagnet*

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Abstract—It is argued that a purely geometric derivation of the long-range action for the $1D$ antiferromagnet is available in terms of a Kähler potential. The derivation allows for a natural extension to the $t-J$ model. In particular, it follows that a relevant long-wavelength action of the $t-J$ model exhibits at least at the SUSY ($J=2t$) point the $su(2|1)$ invariance rather than the $so(5)$ one. *© 2001 MAIK "Nauka/Interperiodica".*

Coherent states associated with the $su(2)$ algebra in the spin-s representation are given by

$$|z\rangle = (1+|z|^2)^{-s} e^{zS_-}|s_z=+s\rangle,$$

where operators $\mathbf{S}$ stand for the $su(2)$ generators in the spin-s representation and the coherent states are parametrized by the local coordinates $z \in SU(2)/U(1) = CP^1$. CP^1 appears as a complex symplectic manifold, which amounts to saying that it is a Kähler manifold, the Kähler structure being defined by the potential $F(\bar z, z)$, so that the $SU(2)$-invariant metric and symplectic structure are given by

$$g_{\bar z z} = \frac{\partial^2 F}{\partial \bar z \partial z} \equiv F_{\bar z z}, \quad w^{(2)} = -iF_{\bar z z} dz \wedge d\bar z,$$

respectively. Symplectic one-form A that enters the $su(2)$ path-integral action reads

$$A = A_z dz + A_{\bar z} d\bar z, \quad A_z = \frac{i}{2}\partial_z F \equiv \frac{i}{2} F_z,$$
$$A_{\bar z} = -\frac{i}{2}\partial_{\bar z} F \equiv -\frac{i}{2} F_{\bar z}, \quad A_z = \overline{A_{\bar z}}.$$

An important point (originally due to Berezin and Onofri [1]) is that the potential F can directly be related to the coherent states:

$$F(\bar z_i, z_j) = \log \left.\frac{\langle z_i|z_j\rangle}{\langle z_i|0\rangle\langle 0|z_j\rangle}\right|_{s=1/2} = \log(1+\bar z_i z_j).$$

Based on these preliminary remarks, consider the $1D$ Heisenberg model

$$H = J\sum_{\langle ij\rangle}\left(\mathbf{S}_i\cdot\mathbf{S}_j - s^2\right),$$

where summation over nn sites is implied. Let $H^{\rm cl} := \langle CS|H|CS\rangle$, $|CS\rangle = \prod_i |z_i\rangle$ denote a respective classical Hamiltonian. Let us further make a change $J \to J/2s$ and consider $H^{\rm cl} = 2sH^{\rm cl}_{s=1/2}$. As a consequence, a total classical action turns out to be $\sim 2s$. It is then easily seen that

$$H^{\rm cl}_{s=1/2} = \frac{J}{2}\sum_{\langle ij\rangle}\left(|\langle z_i|z_j\rangle|^2 - 1\right).$$

From now on, only the $s=1/2$ values of $|z\rangle$, F, etc., are considered.

The identity proves helpful for the following:

$$|\langle z_i|z_j\rangle|^2 = \exp\Phi(\bar z_i, z_i|\bar z_j, z_j) \equiv \exp\Phi_{ij},$$
$$\Phi_{ij} = \Phi_{ji},$$
$$\Phi(\bar z_i, z_i|\bar z_j, z_j) = F(\bar z_i, z_j)$$
$$+ F(\bar z_j, z_i) - F(\bar z_i, z_i) - F(\bar z_j, z_j) \le 0.$$

Assuming the ferromagnetic (FM) interaction ($J<0$), one gets

$$H^{\rm cl}_{s=1/2} = \frac{J}{2}\sum_{\langle ij\rangle}(e^{\Phi_{ij}} - 1)$$
$$\approx \frac{J}{2}\sum_{\langle ij\rangle}\Phi_{ij} = J\sum_i \Phi_{i,i+1},$$
$$\Phi_{i,i+1} = F(\bar z_i, z_{i+1})$$
$$+ F(\bar z_{i+1}, z_i) - F(\bar z_i, z_i) - F(\bar z_{i+1}, z_{i+1})$$
$$= -\frac{1}{2}F_{\bar z_i z_i}\frac{\partial \bar z}{\partial x_i}\frac{\partial z}{\partial x_i}a^2 + \mathcal{O}(a^3), \quad a\to 0,$$

where we have denoted $z_i = z(x_i)$, $z_{i+1} = z(x_i + a)$, with a being a lattice spacing. As a result,

$$H^{\rm cl}_{s=1/2} = -\frac{J}{2}\sum_i F_{\bar z_i z_i}\frac{\partial \bar z}{\partial x_i}\frac{\partial z}{\partial x_i}a^2,$$

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which engenders a manifestly $SU(2)$-invariant classical action (note that, under the $SU(2)$ transformations, $A \to A + d\psi$, which results from a transformation law for F: $F(\bar{gz}, gz) = F(\bar{z}, z) + \phi(z) + \overline{\phi(z)}$, $\psi = \frac{i}{2}(\phi - \bar{\phi})$).

We turn now to the most interesting antiferromagnetic (AF) ($J \geq 0$) case. Let us introduce two AF sublattices, A and B, and denote formally $z_k = z_k$, $k \in A$ and $z_k = w_k$ if $k \in B$. In order to obtain the Néel ground state, we should choose w's in such a way that $\mathbf{S}^{\rm cl}(w) = -\mathbf{S}^{\rm cl}(z)$. (Note, that $S^{\rm cl}_z = -\frac{1}{2}\frac{1-|z|^2}{1+|z|^2}$, $S^{\rm cl}_+ = \frac{z}{1+|z|^2}$, $S^{\rm cl}_- = \frac{\bar{z}}{1+|z|^2}$.) It is clear that the choice $w_k = w^{(0)}_k = -1/\bar{z}_k$ does the job. This transformation amounts to a $SU(2)$ rotation followed by a complex conjugation, which preserves the $SU(2)$ path measure up to a sign. It is also very important that, under such a transformation, $A \to -A$. This seems to be the CP^1 analog (in the inhomogeneous, free of constraints, coordinates) of the famous Haldane map [2]. To be more accurate, a full image of the Haldane map in the inhomogeneous CP^1 coordinates can be taken to be

$$z_i \to z_i + \xi_i, \quad i \in A;$$
$$w_i \to w_i(\bar{\xi}) = -1/(\bar{z}_i - \bar{\xi}_i)$$
$$= -1/\bar{z}_i - \bar{\xi}_i/\bar{z}_i^2 + \mathcal{O}(\bar{\xi}_i^2), \quad i \in B,$$

where $\xi_i, \bar{\xi}_i$ stand for a set of auxiliary fields $\sim a$.

To proceed, we consider

$$\sum_{\langle ij\rangle} |\langle z_i|z_j\rangle|^2 = 2\sum_i |\langle z_i|z_{i+1}\rangle|^2$$
$$= 2\sum_{i\in A} |\langle z_i|w_{i+1}\rangle|^2 + 2\sum_{i\in B} |\langle w_i|z_{i+1}\rangle|^2$$
$$= 2\sum_{i\in A} |\langle z_i|w_{i+1}\rangle|^2 + 2\sum_{i\in B} |\langle z_{i+1}|w_i\rangle|^2.$$

Let us evaluate

$$\Phi(\bar{z}_i, z_i|\bar{w}_k, w_k) = F(\bar{z}_i, w_k)$$
$$+ F(\bar{w}_k, z_i) - F(\bar{z}_i, z_i) - F(\bar{w}_k, w_k)$$

at the point $w_k = w^{(0)}_k = -1/\bar{z}_k, \bar{w}_k = \bar{w}^{(0)}_k = -1/z_k$:

$$\Phi(\bar{z}_i, z_i|\bar{w}_k, w_k)|_{w=w^{(0)}} = F(\bar{z}_i, -1/\bar{z}_k)$$
$$+ F(-1/z_k, z_i) - F(\bar{z}_i, z_i) - F(-1/z_k, -1/\bar{z}_k)$$
$$= \log(1 - \bar{z}_i/\bar{z}_k) + \log(1 - z_i/z_k)$$
$$- \log(1+|z_i|^2) - \log(1+|z_k|^2) + \log|z_k|^2$$
$$= \log(\bar{z}_k - \bar{z}_i)(z_k - z_i) - F(\bar{z}_i, z_i) - F(\bar{z}_k, z_k).$$

One may identically rewrite

$$(\bar{z}_k - \bar{z}_i)(z_k - z_i) = (1+|z_i|^2)(1+|z_k|^2)$$
$$\times \left(1 - \frac{(1+\bar{z}_i z_k)(1+\bar{z}_k z_i)}{(1+|z_i|^2)(1+|z_k|^2)}\right);$$

from this, it follows that

$$\log(\bar{z}_k - \bar{z}_i)(z_k - z_i)$$
$$= F(\bar{z}_i, z_i) + F(\bar{z}_k, z_k) + \log(1 - e^{\Phi(z_i|z_k)}).$$

Consequently, one gets

$$|\langle z_i|w_k\rangle|^2|_{w=w^{(0)}}$$
$$= e^{\Phi(z_i|w_k)}|_{w=w^{(0)}} = 1 - e^{\Phi(z_i|z_k)},$$

and therefore on a sublattice A one has

$$|\langle z_i|w_{i+1}\rangle|^2 \approx -\Phi(z_i|z_{i+1}) = \frac{a^2}{2}F_{\bar{z}_i z_i}\frac{\partial\bar{z}}{\partial x_i}\frac{\partial z}{\partial x_i}.$$

The same equation holds for $i \in B$ so that

$$2\sum_{i\in A} |\langle z_i|w_{i+1}\rangle|^2|_{w=w^{(0)}}$$
$$+ 2\sum_{i\in B} |\langle z_{i+1}|w_i\rangle|^2|_{w=w^{(0)}} = \sum_i F_{\bar{z}_i z_i}\frac{\partial\bar{z}}{\partial x_i}\frac{\partial z}{\partial x_i}a^2.$$

In the case when a full mapping that includes $w(\xi)$, $\bar{w}(\xi)$ rather than merely $w^{(0)}$, $\bar{w}^{(0)}$ is involved, one easily obtains

$$2\sum_{i\in A} |\langle z_i|w_{i+1}(\xi)\rangle|^2 + 2\sum_{i\in B} |\langle z_{i+1}|w_i(\xi)\rangle|^2$$
$$= \sum_i F_{\bar{z}_i z_i}\frac{\partial\bar{z}}{\partial x_i}\frac{\partial z}{\partial x_i}a^2 + 4\sum_i F_{\bar{z}_i z_i}\bar{\xi}_i\xi_i + \mathcal{O}(a^3).$$

Finally, for the classical Hamiltonian, we get

$$H^{\rm cl}_{s=1/2} = \frac{J}{2}\sum_i F_{\bar{z}_i z_i}\frac{\partial\bar{z}}{\partial x_i}\frac{\partial z}{\partial x_i}a^2 + 2J\sum_i F_{\bar{z}_i z_i}\bar{\xi}_i\xi_i.$$

We now address the kinetic term that takes the form

$$i\int A = \frac{1}{2}\int(F_{\bar{z}}d\bar{z} - F_z dz).$$

For the A sublattice, one gets

$$F_{\bar{z}}d\bar{z} - F_z dz \to F_{\bar{z}}(z+\xi)(d\bar{z} + d\bar{\xi})$$
$$- F_z(z+\xi)(dz + d\xi)$$
$$= F_{\bar{z}}d\bar{z} - F_z dz + 2\xi F_{\bar{z}z}d\bar{z} - 2\bar{\xi}F_{\bar{z}z}dz + \mathcal{O}(a^2),$$

whereas, on the B sublattice, one has

$$(F_{\bar{w}}d\bar{w} - F_w dw)_{w=-1/\bar{z}-\bar{\xi}/\bar{z}^2;\bar{w}=-1/z-\xi/z^2}$$
$$= -(F_{\bar{z}}d\bar{z} - F_z dz) + 2\xi F_{\bar{z}z}d\bar{z} - 2\bar{\xi}F_{\bar{z}z}dz + \mathcal{O}(a^2).$$

As a result, the total action becomes

$$S_{\rm AF} = \int dt\sum_i F_{\bar{z}_i z_i}$$
$$\times\left\{\xi_i\dot{\bar{z}}_i - \bar{\xi}_i\dot{z}_i - 2J\bar{\xi}_i\xi_i - \frac{J}{2}\frac{\partial\bar{z}}{\partial x_i}\frac{\partial z}{\partial x_i}a^2\right\} + S_{\rm B},$$

where $S_{\rm B}$ is a Berry phase which will turn into the topological term shortly. The auxiliary fields $\bar{\xi}_i$ and ξ_i can be eliminated to yield

$$S_{\rm AF} = \int dt \sum_i F_{\bar{z}_i z_i} \times \left\{ -\frac{J}{2} \frac{\partial \bar{z}}{\partial x_i} \frac{\partial z}{\partial x_i} a^2 - \frac{1}{2J} \dot{\bar{z}}_i \dot{z}_i \right\} + S_{\rm B}.$$

Restoring an explicit s dependence and going over to the continuum ($a \to 0$) limit finally yields

$$S_{\rm AF} = -\frac{1}{g^2} \times \int dx dt\, g_{\bar{z}z} \left(c\partial_x \bar{z} \partial_x z + c^{-1} \dot{\bar{z}} \dot{z} \right) + S_{\rm B},$$

where $c = 2Jsa$ is the spin wave velocity, $g^2 = 1/s$ is the coupling of the sigma model, and use has been made of the fact that $g_{\bar{z}z} = F_{\bar{z}z}$. We are free to choose units so that $c = 1$ and the action becomes Lorentz invariant:

$$S_{\rm AF} = -\frac{1}{g^2} \int dx dt \,(g_{\bar{z}z} \partial_\mu \bar{z} \partial_\mu z) = -\frac{1}{g^2} \int dx dt \frac{\partial_\mu \bar{z} \partial_\mu z}{(1+|z|^2)^2}, \quad \mu = 0, 1.$$

Let us turn now to the Berry phase term, $S_{\rm B}$. First, we rewrite

$$\int A_j = \oint dt\,(A_{z_j} \dot{z}_j + A_{\bar{z}_j} \dot{\bar{z}}_j) = \oint a_0(j) dt,$$

where $a_\mu = A_z \partial_\mu z + A_{\bar{z}} \partial_\mu \bar{z}$ is a pullback of A by z, $\bar{z}$. (Note that $z(x,t), \bar{z}(x,t)$ map $S^2 \to CP^1 \simeq S^2$.) We may also rewrite

$$\exp\left(\sum_j \oint a_0(j) dt \right) = \prod_j \exp\left(\oint_{\Gamma_j} a_\mu dx^\mu \right),$$

where Γ_j denotes a closed path on S^2. As was mentioned above, a one-form a has opposite signs on the A and B sublattices, so that

$$\oint_{\Gamma_j} a_\mu dx^\mu + \oint_{\Gamma_{j+1}} a_\mu dx^\mu = \oint_{\partial\Sigma_j} a_\mu dx^\mu = \int_{\Sigma_j} da,$$

where Σ_j stands for the area of the ribbon enclosed between the adjacent closed loops. The sum of these ribbons is one-half the area of the sphere, which results in

$$S_{\rm B} = \frac{i}{2} \int_{S^2} da.$$

Note that

$$\int_{S^2} da = N \int_{CP^1} dA,$$

where N is an integer, a degree of the map $S^2 \to CP^1 \simeq S^2$. Since

$$\int_{CP^1} dA = 2si \int_{CP^1} \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} = 4\pi s,$$

one obtains

$$S_{\rm B} = i(2\pi s) N \equiv i\theta N$$

and out comes the $\theta = 2\pi s$ sigma model.

It is interesting to note that the only ingredient which essentially enters the above derivation is a purely geometric one, Kähler potential $F(\bar{z}, z)$. This seems to provide a natural framework to generalize this approach to a more general setting. As an instructive example, consider the $t-J$ model. Since the relevant Hamiltonian appears as a bilinear form built out of the $su(2|1)$ generators, it first seems necessary to define coherent states associated with the $su(2|1)$ superalgebra. These are given in the q representation with dim $= 4q + 1$ by

$$|z, \zeta\rangle = (1 + |z|^2 + \bar{\zeta}\zeta)^{-q} e^{-\zeta V_- + z Q_-} |q, q, q\rangle,$$

where V_- and Q_- are the $su(2|1)$ lowering generators with respect to the Cartan–Weyl basis and vector $|q, q, q\rangle$ stands for the highest weight state. Odd- and even-valued Grassmann variables ζ and z parametrize the superprojective space $CP^{1|1} = SU(2|1)/U(1|1)$, the $N = 1$ superextension of the CP^1. Kähler superpotential now becomes

$$F(\bar{z}, \bar{\zeta}; z, \zeta) = \log \frac{\langle z, \zeta | z, \zeta\rangle}{\langle z, \zeta | 0\rangle \langle 0 | z, \zeta\rangle}.$$

Note that $CP^{1|1}$ symplectic and geometric structures can as well be directly related to F.

Consider

$$H_{t-J} = -t \sum_{\langle ij\rangle, \sigma} X_i^{\sigma 0} X_j^{0\sigma} + \text{h.c.} + J \sum_{\langle ij\rangle} \left(\mathbf{Q}_i \cdot \mathbf{Q}_j - \frac{1}{4} n_i n_j \right),$$

where $\mathbf{Q}_i$ is the electron spin operator. A classical counterpart of H_{t-J} takes a form

$$H^{\rm cl}_{t-J} = -t(2q)^2 \times \sum_{\langle ij\rangle} \rho_i \rho_j [\zeta_i \bar{\zeta}_j (1 + \bar{z}_i z_j) + \zeta_j \bar{\zeta}_i (1 + \bar{z}_j z_i)] + 2Jq^2 \sum_{\langle ij\rangle} \rho_i \rho_j [\bar{z}_i z_j + \bar{z}_j z_i - |z_i|^2 - |z_j|^2],$$

where $\rho_i \equiv (1 + |z_i|^2 + \bar{\zeta}_i \zeta_i)^{-1}$. Let us make a change $J \to J/2q$, $t \to t/2q$ and set $H^{\rm cl} = 2q\, H^{\rm cl}_{q=1/2}$. Besides that, we consider the $t-J$ model at the supersymmetric point, $J = 2t$, which results in

$$H^{\rm cl}_{q=1/2;\text{SUSY}} = t \sum_{\langle ij\rangle} \rho_i \rho_j [\bar{z}_i z_j + \bar{z}_j z_i - |z_i|^2$$

$$- |z_j|^2 - \zeta_i\bar{\zeta}_j(1+\bar{z}_i z_j) - \zeta_j\bar{\zeta}_i(1+\bar{z}_j z_i)].$$

After a little algebra, this can be brought into the form

$$H^{\rm cl}_{q=1/2;{\rm SUSY}} = t\sum_{\langle ij\rangle}\left(|\langle z_i,\zeta_i|z_j,\zeta_j\rangle|^2 - 1\right) + {\rm const}\cdot(N_e, N_A),$$

which stands as quite a suggestive result: at least at the SUSY point an effective action of the $t-J$ model can be derived in terms of the manifestly $su(2|1)$ covariant quantity, Kähler superpotential $F(\bar{z},\bar{\zeta};z,\zeta)$. This in turn implies that a relevant long-wavelength action seems to maintain the $SU(2|1)$ invariance rather than the $SO(5)$ one. Unlike the FM case, where explicit calculations are straightforward, the AF case for the $t-J$ model will require due to a presence of the θ term more elaborate consideration.

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SYMPOSIUM ON GROUP THEORY AND PATH INTEGRALS

Reduction in Path Integrals on a Riemannian Manifold with Group Action*

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Abstract—A new method for the factorization of the path-integral measure in path integrals for a particle motion on a compact Riemannian manifold with a free isometric unimodular group action is proposed. It is shown that path-integral measure is not invariant under the factorization. An integral relation between the path integral given on the total space of the principal fiber bundle and the path integral on the base space of this bundle (the orbit space of the group action) is obtained. © *2001 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

In path-integral quantization of gauge theories, one of the main problems is to find a rigorous foundation of a path-integral transformation that relates the path integrals defined over the orbit space of the gauge group action and the path integrals over the whole space of the gauge potentials.

Recently, it has become clear that solving a similar problem for some finite-dimensional system is helpful. For this reason, many investigations have been carried out in this field of path integration [1].

The mechanical system describing the particle motion on a manifold on which the action of a group is given (we will consider the free isometric action of a semisimple unimodular compact Lie group on a smooth compact Riemannian manifold) has many common properties that can be found in gauge theories.

Due to the symmetry, the system under consideration can be reduced to some mechanical system defined on the orbit space of the group action. Path-integral quantization of this system should lead to the relationship between the corresponding path integrals. The path-integral transformation, when the initial space is changed for the reduced one, may be called the path-integral reduction procedure.

We will consider the path-integral reduction procedure for the Wiener-like path integrals in which the integrations are performed over the measures that are generated by stochastic processes. The processes will be defined by the solution of the stochastic differential equations that are also given on the manifold. For these definitions, we will follow the papers by Belopolskaya and Dalecky [2]. It allows us to use mainly a local approach in the investigation of path-integral transformations.

The original manifold in our system, as is well known [3], can be regarded as the total space of the principal fiber bundle over the orbit space.

We will use the Bogolyubov coordinate transformation method [4] in order to introduce the local coordinates that are adapted to a fiber bundle structure. In this method, we suppose that an arbitrary gauge surface, is given. With this gauge surface, it is possible to introduce invariant coordinates—the coordinates on the base of the fiber bundle (on the orbit space) and variable coordinates—the fiber coordinates.

In path integrals, we separate these coordinates by using the solution of the nonlinear filtering equation from the theory of stochastic processes.

The symmetry of our problem helps us to transform this complicated nonlinear equation into a linear matrix equation. Such an approach to the separation of coordinates in path integrals was proposed in an earlier paper [5].

In the second part of this talk, we will consider the path-integral reduction procedure by using the dependent coordinates, i.e., the coordinates that satisfy some constraint equations. These coordinates together with the group coordinates are also used for the principal fiber bundle coordinatization.

In this talk, we present the results of our investigations of the path-integral reduction problem obtained in [5—7].

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2. DEFINITION

Suppose a particle moves on a smooth compact Riemannian manifold $\mathcal{P}$ (without boundaries) on which a smooth isometric action of a semisimple unimodular compact Lie group $\mathcal{G}$ is given.

We assume that the action of a group $\mathcal{G}$ on $\mathcal{P}$ is free, i.e., for every $p \in \mathcal{P}$ the isotropy group $\mathcal{G}_p = \{g \in \mathcal{G} | gp = p\}$ at p consists of only the identity element of $\mathcal{G}$. Therefore, this action is effective; i.e., the homomorphism from $\mathcal{G}$ to the group of the transformation of a manifold $\mathcal{P}$ is an isomorphism.

Our starting equation is the backward Kolmogorov equation

$$\begin{cases} \left(\frac{\partial}{\partial t_a} + \frac{1}{2}\mu^2\kappa\Delta_{\mathcal{P}}(p_a) + \frac{1}{\mu^2\kappa m}V(p_a)\right)\psi_{t_b}(p_a, t_a) = 0, \\ \psi_{t_b}(p_b, t_b) = \phi_0(p_b) \qquad (t_b > t_a) \end{cases} \tag{1}$$

with the potential invariant under the action of the group $\mathcal{G}$: $V(pg) = V(p)$. In this equation $\mu^2 = \hbar/m$, and κ is a real positive parameter. Here, the Laplace–Beltrami operator is given in local coordinates $Q = \varphi(p)$ of the chart (U, φ) as

$$\Delta_{\mathcal{P}}(Q) = G^{-1/2}(Q)\frac{\partial}{\partial Q^A}G^{AB}(Q)G^{1/2}(Q)\frac{\partial}{\partial Q^B},$$

where $G = \det(G_{AB})$ is the determinant of the initial Riemannian metric G_{AB} for the coordinate basis $\left\{\frac{\partial}{\partial Q^A}\right\}$. The indices denoted by the capital letters run from 1 to $N_{\mathcal{P}} = \dim\mathcal{P}$.

According to [2], the solution to Eq. (1) in the case of the proper coefficients and the initial function of Eq. (1) can be presented in the following form:

$$\psi_{t_b}(p_a, t_a) = \mathrm{E}\Big[\phi_0(\eta(t_b)) \tag{2}$$
$$\times \exp\left\{\frac{1}{\mu^2\kappa m}\int\limits_{t_a}^{t_b} V(\eta(u))du\right\}\Big]$$
$$\equiv \int\limits_{\Omega_-} d\mu^\eta(\omega)\phi_0(\eta(t_b))\exp\{\ldots\},$$

where $\eta(t)$ is a stochastic process on a manifold $\mathcal{P}$ and μ^η is generated by this process measure given in the path space $\Omega_- = \{\omega(t) : \omega(t_a) = 0,\ \eta(t) = p_a + \omega(t)\}$.

The local components $\eta^A(t)$ of the stochastic process $\eta(t)$ satisfy the following stochastic differential equation:

$$d\eta^A(t) = \frac{1}{2}\mu^2\kappa G^{-1/2}\frac{\partial}{\partial Q^B}(G^{1/2}G^{AB})dt$$
$$+ \mu\sqrt{\kappa}\mathcal{X}^A_{\bar{M}}(\eta(t))dw^{\bar{M}}(t),$$

where $\mathcal{X}^A_{\bar{M}}$ is defined locally by

$$\sum_{\bar{K}=1}^{n_P}\mathcal{X}^A_{\bar{K}}\mathcal{X}^B_{\bar{K}} = G^{AB}.$$

(The barred indices are the Euclidean indices.) The semigroup determined by the path integral (2) acts in the space of the smooth and bounded functions on $\mathcal{P}$. It is obtained as a result of going to the limit in the superposition of the local semigroups:

$$\psi_{t_b}(p_a, t_a) = U(t_b, t_a)\phi_0(p_a)$$
$$= \lim{}_q\tilde{U}_\eta(t_a, t_1)\cdots\tilde{U}_\eta(t_{n-1}, t_b)\phi_0(p_a),$$

where each local semigroup is defined by the equality

$$\tilde{U}_\eta(s,t)\phi(p) = \mathrm{E}_{s,p}\phi(\eta(t)), \quad s \le t, \eta(s) = p.$$

In the chart $(\mathcal{V}_p, \varphi^{\mathcal{P}})$ with the mapping $\varphi^{\mathcal{P}}$ $(\varphi^{\mathcal{P}}(\eta(t)) = \eta^{\varphi^{\mathcal{P}}}(t) \equiv \{\eta^A(t)\})$, it is given as

$$\tilde{U}_\eta(s,t)\phi(p) = \mathrm{E}_{s,\varphi^{\mathcal{P}}(p)}\phi((\varphi^{\mathcal{P}})^{-1}(\eta^{\varphi^{\mathcal{P}}}(t))),$$
$$\eta^{\varphi^{\mathcal{P}}}(s) = \varphi^{\mathcal{P}}(p).$$

Thus, we see that many properties of the global semigroup can be derived by analyzing the local semigroups $\tilde{U}_\eta$. But the local semigroups are completely determined by the stochastic differential equations, whose solutions—the stochastic processes—generate the corresponding path-integral measures.

Therefore, the study of the transformation of the local stochastic differential equations enables us to get information concerning the transformation of the path integrals and the semigroups acting in the space of functions on a manifold.

3. THE BUNDLE COORDINATES

In our case, the right action of the group $\mathcal{G}$ given in the charts by $\tilde{Q}^A = F^A(Q^B, a^\alpha)$, $\alpha = 1, \ldots, N_{\mathcal{G}}$, leads to the fiber structure of the manifold and the principal bundle $\pi : \mathcal{P} \to \mathcal{P}/\mathcal{G} = \mathcal{M}$, where $\mathcal{M}$ is an orbit space [3].

It implies that the manifold $\mathcal{P}$ can be locally presented as $\pi^{-1}(U_x) \sim U_x \times \mathcal{G}$ $(x = \pi(p)$ belongs to a chart (U_x, φ_x) of the fiber bundle). For an arbitrary point p of the manifold $\mathcal{P}$ with the coordinates Q^A, we should find the local coordinates $(x^i(Q), a^\alpha(Q))$, $i = 1, \ldots, N_{\mathcal{M}} = \dim\mathcal{M}$.

To introduce the bundle coordinates, we use the Bogolyubov coordinate transformation method [4]. In this method, we suppose that, in each sufficiently small neighborhood of an arbitrary point p, there is the set of functions $\{\chi^\alpha(Q), \alpha = 1, \ldots, N_{\mathcal{G}}\}$, which by the equation $\chi^\alpha(Q) = 0$ determines the local submanifold.

From the equation

$$\chi^\alpha(F^A(Q,a^{-1}))=0,$$

one searches for the group element $a^\alpha(Q)$. The group element a^{-1} carries the point p to the submanifold $\{\chi^\alpha(Q)=0\}$ along the orbit $p\mathcal{G}$.

The invariant coordinates $x^i(Q)$ are defined by the following equation:

$$Q^{*A}(x^i)=F^A(Q,a^{-1}).$$

It is supposed that the submanifold $\{\chi^\alpha(Q)=0\}$ has the parametric form representation $Q^A=Q^{*A}(x^i)$ $(\{\chi^\alpha(Q^*(x^i))=0\})$ and x^i are identified with the coordinates of the base of the fiber bundle.

Changing the coordinates Q^A for (x^i,a^α), $Q^A=F^A(Q^*(x^i),a^\alpha)$, makes it possible to rewrite the metric $G_{AB}(Q)$ in the following Kaluza–Klein form:

$$\begin{pmatrix} h_{ij}(x)+A_i^\mu(x)A_j^\nu(x)\bar\gamma_{\mu\nu}(x) & A_i^\mu(x)\bar u_\sigma^\nu(a)\bar\gamma_{\mu\nu}(x)\\ A_i^\mu(x)\bar u_\sigma^\nu(a)\bar\gamma_{\mu\nu}(x) & \bar u_\rho^\mu(a)\bar u_\sigma^\nu(a)\bar\gamma_{\mu\nu}(x)\end{pmatrix},$$

where $\bar u_\beta^\alpha$ is an inverse matrix to the matrix $\bar v_\beta^\alpha(a)=\dfrac{\partial\Phi^\alpha(b,a)}{\partial b^\beta}\Big|_{b=e}$ (Φ is a group function which defines the group multiplication in the space of group parameters).

The projection of the mechanical connection onto the base of the fiber bundle is given by

$$A_i^\nu(x)$$
$$=\gamma^{\nu\sigma}(Q^*(x))G_{EB}(Q^*(x))K_\sigma^B(Q^*(x))\frac{\partial Q^{*E}(x)}{\partial x^i},$$

where K_σ^B are the components of the Killing vector field $K_\sigma^B(Q)\equiv\frac{\partial F^B(Q,a)}{\partial a^\sigma}|_{a=e}$.

The metric h_{ij} on the orbit space can be defined with the help of the projectors $\Pi_B^A=\delta_B^A-K_\alpha^A\gamma^{\alpha\beta}K_{\beta B}$ ($\gamma^{\alpha\beta}$ is inverse to the metric $\gamma_{\alpha\beta}=K_\alpha^A G_{AB}K_\beta^B$ defined along the orbit) and the horizontal metric $G_{AB}^H=\Pi_A^C G_{CD}\Pi_B^D$. It is equal to

$$h_{ij}(x)=Q^{*A}_{\ i}G_{AB}^H Q^{*B}_{\ j},$$

where $Q^{*A}_{\ i}\equiv\frac{\partial Q^{*A}}{\partial x^i}$.

The determinant of the metric G_{AB} is

$$\det G_{AB}=(\det h_{ij}(x))(\det\bar\gamma_{\alpha\beta}(x))(\det\bar u_\rho^\mu(a))^2.$$

4. PATH-INTEGRAL TRANSFORMATION

Changing the coordinates Q^A for (x^i,a^α) leads to the transformation of the components $\eta^A(t)$ of the stochastic process $\eta^{\varphi^{\mathcal P}}(t)$:

$$\eta^A(t)=F^A(Q^*(x^i(t)),a^\alpha(t)).$$

We can regard $(x^i(t),a^\alpha(t))$ as the components $\zeta^A(t)$ of the local stochastic process $\zeta^{\varphi^P}(t)$. This phase-space transformation of the stochastic process $\eta^{\varphi^{\mathcal P}}(t)$ does not change the probabilities and transition probabilities.

With the help of the Itô differentiation formula, it is possible to find the stochastic differential equation for the local components $\zeta^A(t)$ of the stochastic process $\zeta^{\varphi^P}(t)$:

$$dx^i(t)=\frac12\mu^2\kappa\Big[\frac{1}{\sqrt{h\bar\gamma}}\frac{\partial}{\partial x^n}(h^{ni}\sqrt{h\bar\gamma})\Big]dt \qquad (3)$$
$$+\mu\sqrt\kappa X^i_{\bar n}(x(t))dw^{\bar n}(t),$$
$$da^\alpha(t)=\mu^2\kappa\left[-\frac12\frac{1}{\sqrt{h\bar\gamma}}\frac{\partial}{\partial x^k}\left(\sqrt{h\bar\gamma}h^{km}A_m^\nu\right)\right.$$
$$\times\bar v_\nu^\alpha(a(t))+\frac12(\bar\gamma^{\lambda\epsilon}+h^{ij}A_i^\lambda A_j^\epsilon)\bar v_\lambda^\sigma(a(t))$$
$$\left.\times\frac{\partial}{\partial a^\sigma}(\bar v_\epsilon^\alpha(a(t)))\right]dt+\mu\sqrt\kappa\bar v_\lambda^\alpha(a(t))\bar Y_{\bar\epsilon}^\lambda dw^{\bar\epsilon}(t)$$
$$-\mu\sqrt\kappa X^i_{\bar n}A_i^\nu\bar v_\nu^\alpha(a(t))dw^{\bar n}(t).$$

Thus, due to the phase-space transformation of the stochastic process η, the local semigroups $\tilde U_\eta$ are replaced by the semigroups, $\tilde U_{\zeta^{\varphi^P}}$. After going to the limit in the superposition of these new semigroups we get the global semigroup. It can be presented in the following symbolic form:

$$\psi_{t_b}(p_a,t_a)=\mathrm{E}\Bigg[\tilde\phi_0(\xi(t_b),a(t_b)) \qquad (4)$$
$$\times\exp\left\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}\tilde V(\xi(u))du\right\}\Bigg],$$

where $\xi(t_a)=x_a$, $a(t_a)=\theta_a$, and $\varphi^P(p_a)=(x_a,\theta_a)$.

The differential generator of the semigroup associated with the process $\zeta(t)$ in (x^i,a^α) coordinates will be

$$\frac12\mu^2\kappa\Big\{\Delta_M(x)+h^{ij}\frac{1}{\sqrt{\bar\gamma}}\Big(\frac{\partial\sqrt{\bar\gamma}}{\partial x^i}\Big)\frac{\partial}{\partial x^j}$$
$$+h^{ij}A_i^\alpha A_j^\beta\bar L_\alpha\bar L_\beta-2h^{in}A_n^\alpha\bar L_\alpha\frac{\partial}{\partial x^i}-h^{in}\frac{\partial A_n^\alpha}{\partial x^i}\bar L_\alpha$$
$$-\frac{h^{in}}{\sqrt h}\frac{\partial\sqrt h}{\partial x^i}A_n^\alpha\bar L_\alpha-h^{in}\frac{1}{\sqrt{\bar\gamma}}\frac{\partial\sqrt{\bar\gamma}}{\partial x^i}A_n^\alpha\bar L_\alpha$$

$$- \frac{\partial h^{in}}{\partial x^i} A_n^\alpha \bar{L}_\alpha + \bar{\gamma}^{\alpha\beta} \bar{L}_\alpha \bar{L}_\beta \Big\},$$

where Δ_M is the Laplace–Beltrami operator on $\mathcal{M}$ and $\bar{L}_\alpha$ is the right-invariant vector field $\bar{L}_\alpha = \bar{v}^\epsilon_\alpha(a)\frac{\partial}{\partial a^\epsilon}$.

5. FACTORIZATION

We should separate two sorts of variables (the invariant variables and the group variables) in the measure of our path integral (4).

Having in mind our definition of the path integral from [2], it will be done for the local semigroups.

The separation of the variables is based on the method shown in our earlier papers [5, 6]. It was found there that the local stochastic differential equations of the stochastic process given on the principal fiber bundle coincide with the stochastic differential equations that are used in the nonlinear filtering theory.

In the theory, the main problem is to estimate the existing difference between the observation process and the signal process that cannot be directly observed by experiment. The solution of this problem is related to the solution of the nonlinear filtering equation for the conditional expectation of the signal process given by the sub-σ-field associated with the observation process [8, 9]. It is this equation that enables us to perform the path-integral transformation that separates the path-integral variables.

We note that in our case the stochastic process $a^\alpha(t)$ is the signal process and $x^i(t)$ is the observation process. Using the conditional expectation properties ($\zeta(t)$ is the Markov process), we can transform the local semigroup (the local path integral) as follows:

$$\tilde{U}_{\zeta\varphi^P}(s,t)\tilde{\phi}(x_0,\theta_0) = \mathrm{E}\Big[\mathrm{E}\big[\tilde{\phi}(x(t),a(t)) \mid (\mathcal{F}_x)^t_s\big]\Big].$$

For the conditional expectation

$$\hat{\tilde{\phi}}(x(t)) \equiv \mathrm{E}\Big[\tilde{\phi}(x(t),a(t)) \mid (\mathcal{F}_x)^t_s\Big],$$

we will have the following nonlinear filtering equation:

$$d\hat{\tilde{\phi}}(x(t)) = \mu^2\kappa\Big[-\frac{1}{2}\frac{1}{\sqrt{h\bar{\gamma}}}\frac{\partial}{\partial x^k}\left(\sqrt{h\bar{\gamma}}h^{km}A^\mu_m\right)\Big] \quad (5)$$
$$\times \mathrm{E}[\bar{L}_\mu\tilde{\phi}(x(t),a(t)) \mid (\mathcal{F}_x)^t_s]dt$$
$$+ \frac{1}{2}\mu^2\kappa(\bar{\gamma}^{\mu\nu} + h^{ij}A^\mu_i A^\nu_j)$$
$$\times \mathrm{E}[\bar{L}_\mu\bar{L}_\nu\tilde{\phi}(x(t),a(t)) \mid (\mathcal{F}_x)^t_s]dt$$
$$- \mu\sqrt{\kappa}A^\mu_k X^k_{\bar{m}}\mathrm{E}[\bar{L}_\mu\tilde{\phi}(x(t),a(t)) \mid (\mathcal{F}_x)^t_s]dw^{\bar{m}}(t).$$

The function $\tilde{\phi}(x,a)$ can be expanded with the Peter–Weyl theorem in a series over the matrix irreducible representation of a group $\mathcal{G}$, $\tilde{\phi}(x,a) = \sum_{\lambda,p,q} c^\lambda_{pq}(x)D^\lambda_{pq}(a)$. Under the conditional expectation, it will be

$$\mathrm{E}\big[\tilde{\phi}(x(t),a(t)) \mid (\mathcal{F}_x)^t_s\big] = \sum_{\lambda,p,q} c^\lambda_{pq}(x(t))\mathrm{E}\big[D^\lambda_{pq}(a(t)) \mid (\mathcal{F}_x)^t_s\big].$$

Then, from Eq. (5), we find that $\hat{D}^\lambda_{pq}(x(t)) \equiv \mathrm{E}\big[D^\lambda_{pq}(a(t))|(\mathcal{F}_x)^t_s\big]$[1)] is described by the linear matrix equation. Due to [10, 11], its solution is given in terms of the multiplicative stochastic integral:

$$\hat{D}^\lambda_{pq}(x(t)) = (\overleftarrow{\exp})^\lambda_{pn}(x(t),t,s) \times \mathrm{E}\big[D^\lambda_{nq}(a(s)) \mid (\mathcal{F}_x)^t_s\big],$$

where

$$(\overleftarrow{\exp})^\lambda_{pn}(x(t),t,s)$$
$$= \overleftarrow{\exp}\int_s^t\Big\{\mu^2\kappa\Big[\frac{1}{2}\bar{\gamma}^{\mu\nu}(x(u))(J_\mu)^\lambda_{pr}(J_\nu)^\lambda_{rn}$$
$$- \frac{1}{2}\frac{1}{\sqrt{h\bar{\gamma}}}\frac{\partial}{\partial x^k}\left(\sqrt{h\bar{\gamma}}h^{km}A^\mu_m\right)(J_\mu)^\lambda_{pn}\Big]du$$
$$- \mu\sqrt{\kappa}A^\mu_k(x(u))(J_\mu)^\lambda_{pn}X^k_{\bar{m}}(x(u))dw^{\bar{m}}(u)\Big\}.$$

In the above, $(J_\mu)^\lambda_{pn}$ are the generators of the representation $D^\lambda(a)$, and h and $\bar{\gamma}$ are functions of $x(u)$.

Taking into account that initially

$$\mathrm{E}\big[D^\lambda_{nq}(a(s)) \mid (\mathcal{F}_x)^t_s\big] = D^\lambda_{nq}(a(s)) = D^\lambda_{nq}(\theta_0),$$

we obtain

$$\tilde{U}_{\zeta\varphi^P}(s,t)\tilde{\phi}(x_0,\theta_0) = \sum_{\lambda,p,q,q'}\mathrm{E}\big[c^\lambda_{pq}(x(t))(\overleftarrow{\exp})^\lambda_{pq'}(x(t),t,s)\big]D^\lambda_{q'q}(\theta_0).$$

Taking the partition of the time interval, we get the superposition of the local semigroups. And after going to the limit, as in [2], in this superposition, we obtain the global semigroup, which is given in a symbolic form,

$$\psi_{t_b}(p_a,t_a) = \sum_{\lambda,p,q,q'}\mathrm{E}\big[c^\lambda_{pq}(\xi(t_b))(\overleftarrow{\exp})^\lambda_{pq'}(\xi(t),t_b,t_a)\big]D^\lambda_{q'q}(\theta_a), \quad (6)$$

where $\xi(t_a) = \pi\circ p_a$ and the process $\xi(t)$ is a global process on a manifold $\mathcal{M} = \mathcal{P}/\mathcal{G}$. The stochastic equation of the local representatives of the process $\xi(t)$ is defined by the first equation of (3).

[1)] $\hat{D}^\lambda_{pq}(x(t))$ depends as well on $x^i_0 = x^i(s)$ and $\theta^\alpha_0 = a^\alpha(s)$. But, for brevity, this dependence was not explicitly shown in the notation of $\hat{D}^\lambda_{pq}(x(t))$.

It is possible to invert formula (6). Performing this for the corresponding semigroup kernels, we get the integral relation between the path integrals:

$$G^{\lambda}_{mn}(\pi(p_b), t_b; \pi(p_a), t_a) = \int_{\mathcal{G}} G_{\mathcal{P}}(p_b\theta, t_b; p_a, t_a) D^{\lambda}_{nm}(\theta) d\mu(\theta).$$

The path integral for the Green's function $G_{\mathcal{P}}$ is analogous in form to the path integral of Eq. (2), but the paths in its domain of integration have fixed values at the time $t = t_a$ and $t = t_b$.

The path integral for the Green's function G^{λ}_{mn} can be written as follows:

$$G^{\lambda}_{mn}(\pi(p_b), t_b; \pi(p_a), t_a) \quad (7)$$
$$= \tilde{\mathrm{E}}_{\substack{\xi(t_a)=\pi(p_a)\\ \xi(t_b)=\pi(p_b)}} \Bigg[(\overleftarrow{\exp})^{\lambda}_{mn}(\xi(t), t_b, t_a) \times \exp\Bigg\{ \frac{1}{\mu^2\kappa m} \int_{t_a}^{t_b} \tilde{V}(\xi(u)) du \Bigg\} \Bigg]$$
$$= \int_{\substack{\xi(t_a)=\pi(p_a)\\ \xi(t_b)=\pi(p_b)}} d\mu^{\xi} \exp\Bigg\{ \frac{1}{\mu^2\kappa m} \int_{t_a}^{t_b} \tilde{V}(\xi(u)) du \Bigg\}$$
$$\times \overleftarrow{\exp} \int_{t_a}^{t_b} \Bigg\{ \mu^2\kappa \Bigg[\frac{1}{2}\bar{\gamma}^{\mu\nu}(J_\mu)^{\lambda}_{mr}(J_\nu)^{\lambda}_{rn} - \frac{1}{2}\frac{1}{\sqrt{h\bar{\gamma}}} \times \frac{\partial}{\partial x^k}\left(\sqrt{h\bar{\gamma}} h^{kr} A^{\mu}_r\right)(J_\mu)^{\lambda}_{mn} \Bigg] du - \mu\sqrt{\kappa} A^{\mu}_k (J_\mu)^{\lambda}_{mn} X^k_{\bar{m}} dw^{\bar{m}} \Bigg\}.$$

The semigroup with the kernel given by Eq. (7) acts in the space of the sections $\Gamma(\mathcal{M}, V^*)$ of the associated covector bundle $\mathcal{E}^* = \mathcal{P} \times_{\mathcal{G}} V^*_{\lambda}$ (we consider the backward equations), where the scalar product is given by the following form:

$$(\psi_n, \psi_m) = \int_{\mathcal{M}} \langle \psi_n, \psi_m \rangle_{V^*_{\lambda}} \sqrt{\bar{\gamma}(x)} dv_{\mathcal{M}}(x),$$

where $dv_{\mathcal{M}}(x) = \sqrt{h(x)} dx^1 \cdots dx^{N_{\mathcal{M}}}$ in x^i coordinates.

The differential generator of this semigroup is

$$\frac{1}{2}\mu^2\kappa \Bigg\{ \Bigg[\Delta_M + h^{ni}\frac{1}{\sqrt{\bar{\gamma}}}\frac{\partial\sqrt{\bar{\gamma}}}{\partial x^n}\frac{\partial}{\partial x^i} \Bigg] (I^{\lambda})_{pq} - 2h^{ni}A^{\alpha}_n (J_\alpha)^{\lambda}_{pq}\frac{\partial}{\partial x^i} - \frac{1}{\sqrt{h\bar{\gamma}}}\frac{\partial}{\partial x^n}\left(\sqrt{h\bar{\gamma}} h^{nm} A^{\alpha}_m\right)(J_\alpha)^{\lambda}_{pq} + (\bar{\gamma}^{\alpha\nu} + h^{ij}A^{\alpha}_i A^{\nu}_j)(J_\alpha)^{\lambda}_{pq'}(J_\nu)^{\lambda}_{q'q} \Bigg\}.$$

Here, $(I^{\lambda})_{pq}$ is a unit matrix.

Due to the isomorphism between the space of the sections of the associated vector bundle and the space of the equivariant functions given on the total space of the principal fiber bundle, we can say that our semigroups act as well in the space of the functions $\tilde{\psi}_n(p)$ on $\mathcal{P}$ for which the following relation holds: $\tilde{\psi}_n(pg) = \sum_m D^{\lambda}_{mn}(g)\tilde{\psi}_m(p)$. The isomorphism is given locally by $\tilde{\psi}_n(F(Q^*(x), e)) = \psi_n(x)$.

The path-integral reduction for the case $\lambda = 0$ corresponds to the reduction onto the zero-momentum level in the constrained dynamical systems.

In this case, both D^0_{pq} and the multiplicative stochastic integral become unity. Now the resultant semigroup will act in the space of the invariant scalar functions on the total space of the principal fiber bundle.

Then, in order to obtain the diffusion on the base manifold $\mathcal{M}$ with the Laplace–Beltrami operator as the differential generator of the stochastic process, we change the stochastic process ξ for the process $\tilde{\xi}$ and apply the Girsanov transformation to the measure of the path integral for the case of $\lambda = 0$.

As a result, we get the following integral relation:

$$\bar{\gamma}(x_b)^{-1/4}\bar{\gamma}(x_a)^{-1/4} G_{\mathcal{M}}(x_b, t_b; x_a, t_a) = \int_{\mathcal{G}} G_{\mathcal{P}}(p_b\theta, t_b; p_a, t_a) d\mu(\theta),$$

$$G_{\mathcal{M}}(x_b, t_b; x_a, t_a) = \int_{\substack{\tilde{\xi}(t_a)=x_a\\ \tilde{\xi}(t_b)=x_b}} d\mu^{\tilde{\xi}} \times \exp\Bigg\{ \frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b} V(\tilde{\xi}(u))du + \int_{t_a}^{t_b} J(\tilde{\xi}(u))du \Bigg\},$$

where $x = \pi(p)$ and J is the Jacobian depending on the orbit volume $\bar{\gamma}(x)$, given by

$$J(x) = -\frac{\mu^2\kappa}{8}\Big[\Delta_M \ln\bar{\gamma} + \frac{1}{4}h^{ni}\frac{\partial\ln\bar{\gamma}}{\partial x^n}\frac{\partial\ln\bar{\gamma}}{\partial x^i}\Big].$$

This Jacobian can be given in terms of the differential expression [12] which involves the mean curvature of the group orbit at the point x.

The semigroup with the kernel $G_{\mathcal{M}}$ acts in the space of the scalar functions on $\mathcal{M}$ (or in the space of the invariant scalar functions on $\mathcal{P}$) with the scalar product: $(\psi_1, \psi_2) = \int \psi_1(x)\psi_2(x) dv_{\mathcal{M}}(x)$.

Notice that the differential operator of the forward Kolmogorov for the Green's function $G_{\mathcal{M}}$ will be

$$\hat{H}_\kappa = \frac{\hbar\kappa}{2m}\Delta_M - \frac{\hbar\kappa}{8m}[\Delta_M \ln\bar{\gamma} + \frac{1}{4}(\nabla_M \ln\bar{\gamma})^2] + \frac{1}{\hbar\kappa}V.$$

The Hamilton operator $\hat{H}$ of the corresponding Schrödinger equation is obtained by

$$\hat{H} = -\frac{\hbar}{\kappa}\hat{H}_\kappa\big|_{\kappa=i}.$$

6. REDUCTION IN DEPENDENT COORDINATES

Here, we consider the case when the point p of the manifold $\mathcal{P}$ is locally defined by the group coordinates a^α and the dependent coordinates Q^{*A}: $\{\chi^\alpha(Q^{*A}) = 0\}$. Q^{*A} are the coordinates of the local submanifold $\{\chi^\alpha(Q) = 0\}$ in the original manifold $\mathcal{P}$. We assume that these local submanifolds are the parts of some global submanifold Σ. Therefore, in this case, our principal fiber bundle $\mathcal{P}(\mathcal{M},\mathcal{G})$ is trivial.

Moreover, in our case we have a principal fiber bundle $\Sigma \times \mathcal{G} \to \Sigma$, which is isomorphic (locally) to the principal bundle $\mathcal{P}(\mathcal{M},\mathcal{G})$ [13, 14].

As for the coordinate transformations, they have the same form, $Q^A = F^A(Q^{*A}, a^\alpha)$, but now the condition $\{\chi^\alpha(Q^*) = 0\}$ should be fulfilled.

Under this replacement of the coordinates, the vector fields transform as follows:

$$\frac{\partial}{\partial Q^B} = F^C_B(F(Q^*,a),a^{-1})N^A_C(Q^*)\frac{\partial}{\partial Q^{*A}} + F^E_B(F(Q^*,a),a^{-1})\chi^\mu_E(Q^*)(\Phi^{-1})^\beta_\mu(Q^*)\bar{v}^\alpha_\beta(a)\frac{\partial}{\partial a^\alpha}, \tag{8}$$

where $F^C_B(Q,a) \equiv \dfrac{\partial F^C}{\partial Q^B}(Q,a)$, $\chi^\mu_E \equiv \dfrac{\partial \chi^\mu}{\partial Q^E}(Q)$, and $(\Phi^{-1})^\beta_\mu(Q)$ is the inverse of the Faddeev–Popov matrix: $(\Phi)^\beta_\mu(Q) = K^A_\mu(Q)\dfrac{\partial\chi^\beta(Q)}{\partial Q^A}$. In Eq. (8), N^A_C is the projector onto the subspace which is orthogonal to the Killing vector field. This projector is defined by

$$N^A_C(Q) = \delta^A_C - K^A_\alpha(Q)(\Phi^{-1})^\alpha_\mu(Q)\chi^\mu_C(Q)$$

and is given on the submanifold $\{\chi^\alpha = 0\}$ as $N(Q^*) \equiv N(F(Q^*,e))$, so that

$$N^M_D(Q^*) = F^B_D(Q^*,a)N^A_B(F(Q^*,a))F^M_A(F(Q^*,a),a^{-1}).$$

In the new coordinates, the metric G_{AB} takes the form

$$\tilde{G}_{\mathcal{AB}}(Q^*,a) = \begin{pmatrix} G_{CD}(Q^*)(P_\perp)^C_A(P_\perp)^D_B & G_{CD}(Q^*)(P_\perp)^D_A K^C_\mu \bar{u}^\mu_\alpha(a) \\ G_{CD}(Q^*)(P_\perp)^C_A K^D_\nu \bar{u}^\nu_\beta(a) & \gamma_{\mu\nu}(Q^*)\bar{u}^\mu_\alpha(a)\bar{u}^\nu_\beta(a) \end{pmatrix}, \tag{9}$$

where the projectors $P_\perp$ onto the tangent space to the submanifold (given by the gauge) are defined by

$$(P_\perp)^A_B = \delta^A_B - \chi^\alpha_B(\chi\chi^\top)^{-1\beta}_{\ \ \alpha}(\chi^\top)^A_\beta,$$

which depend on Q^*. These projectors have the following properties:

$$(P_\perp)^{\tilde{A}}_B N^C_{\tilde{A}} = (P_\perp)^C_B,\ N^{\tilde{A}}_B(P_\perp)^C_{\tilde{A}} = N^C_B.$$

In Eq. (9), $G_{CD}(Q^*) \equiv G_{CD}(F(Q^*,e))$ and

$$G_{CD}(Q^*) = F^M_C(Q^*,a)F^N_D(Q^*,a)G_{MN}(F(Q^*,a)).$$

In a manner similar to the previous procedure, we can change the stochastic process $\eta(t)$ and find the transformation of the corresponding path-integral measure. As a result, the local stochastic differential equations of a new stochastic process are given by

$$dQ^{*A}(t) = \mu^2\kappa\Big(-\frac{1}{2}G^{EM}N^C_E N^B_M\,{}^H\Gamma^A_{CB} + j^A_{\mathrm{I}} + j^A_{\mathrm{II}}\Big)dt + \mu\sqrt{\kappa}N^A_C\mathcal{X}^C_{\bar{M}}dw^{\bar{M}} \tag{10}$$

and

$$da^\alpha = -\frac{1}{2}\mu^2\kappa\Big[G^{RS}\tilde{\Gamma}^B_{RS}(Q^*)\Lambda^\beta_B\bar{v}^\alpha_\beta + G^{RP}\Lambda^\sigma_R\Lambda^\beta_B K^B_{\sigma P}\bar{v}^\alpha_\beta - G^{CA}N^M_C\frac{\partial}{\partial Q^{*M}}\Big(\Lambda^\beta_A\Big)\bar{v}^\alpha_\beta - G^{MB}\Lambda^\epsilon_M\Lambda^\beta_B\bar{v}^\nu_\epsilon\frac{\partial}{\partial a^\nu}(\bar{v}^\alpha_\beta)\Big]dt + \mu\sqrt{\kappa}\bar{v}^\alpha_\beta\Lambda^\beta_B\mathcal{X}^B_{\bar{M}}dw^{\bar{M}}. \tag{11}$$

In Eq. (10), j^A_{I} is the mean curvature of the orbit space as a submanifold of the Riemannian manifold $(\mathcal{P}, G^H_{AB})$ and the Christoffel coefficients ${}^H\Gamma^A_{CB}$ are obtained from the degenerate metric G^H_{AB}.

The additional drift coefficient j^A_{II} of Eq. (10), the projection of the mean curvature of the orbit onto the gauge submanifold, is equal to

$$j^A_{\rm II}(Q^*)=\frac{1}{2}G^{EU}N^A_E N^D_U\left[\gamma^{\alpha\beta}G_{CD}(\tilde\nabla_{K_\alpha}K_\beta)^C\right],$$

where

$$(\tilde\nabla_{K_\alpha}K_\beta)^C=K^A_\alpha(Q^*)\,\frac{\partial}{\partial Q^A}K^C_\beta(Q)\Big|_{Q=Q^*}+K^A_\alpha(Q^*)K^B_\beta(Q^*)\tilde\Gamma^C_{AB}(Q^*).$$

Also, in Eq. (11), $\Lambda^\alpha_B=(\Phi^{-1})^\alpha_\mu\chi^\mu_B$.

Further transformation of our path-integral measure induces the factorization with the help of the nonlinear filtering equation. This can be done as before. Here, we present only our result for the case of the zero-momentum reduction, i.e., when $\lambda=0$.

After inverting the corresponding semigroup, we have the following integral relation between the two kernels:

$$G_\Sigma(Q^*_b,t_b;Q^*_a,t_a)=\int\limits_{\mathcal G}G_{\mathcal P}(p_b\theta,t_b;p_a,t_a)d\mu(\theta).$$

The path integral for G_Σ is

$$G_\Sigma(Q^*_b,t_b;Q^*_a,t_a)=\int\limits_{\substack{\tilde\xi_\Sigma(t_a)=Q^*_a\\ \tilde\xi_\Sigma(t_b)=Q^*_b}}d\mu^{\tilde\xi_\Sigma}\times\exp\left\{\frac{1}{\mu^2\kappa m}\int\limits_{t_a}^{t_b}V(\tilde\xi_\Sigma(u))du\right\}\times\exp\int\limits_{t_a}^{t_b}\left\{-\frac{1}{8}\mu^2\kappa G^{AB}N^D_A\times N^L_B\left[\gamma^{\alpha\beta}G_{DC}(\tilde\nabla_{K_\alpha}K_\beta)^C\right]\times\left[\gamma^{\mu\nu}G_{LE}(\tilde\nabla_{K_\mu}K_\nu)^E\right]dt+\frac{1}{2}\mu\sqrt{\kappa}N^D_P\times\left[\gamma^{\alpha\beta}G_{CD}(\tilde\nabla_{K_\alpha}K_\beta)^C\right]\mathcal X^P_{\bar M}dw^{\bar M}\right\},$$

where $Q^*=\pi_\Sigma(p)$. In this path integral, the measure $\mu^{\tilde\xi_\Sigma}$ for the stochastic process $\tilde\xi_\Sigma$ has been obtained by eliminating the extra term $j_{\rm II}$ from the stochastic differential equation (10) via the Girsanov transformation.

The semigroup with the kernel G_Σ acts in the space of the functions given on a manifold Σ (or in the space of the invariant functions on $\mathcal P$) where the scalar product is defined by

$$(\psi_1,\psi_2)=\int\limits_\Sigma\psi_1\psi_2\det\Phi^\alpha_\beta(Q^*)\prod_{\alpha=1}^{N_{\mathcal G}}\delta(\chi^\alpha(Q^*))\times\det{}^{1/2}G_{AB}(Q^*)dQ^{*1}\wedge\cdots\wedge dQ^{*N_{\mathcal P}}.$$

In closing, we remark that in [14] it was shown how to use the local measures for the trivial principal bundles in defining the path-integral measure in nontrivial cases.

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Method of Collective Degrees of Freedom in Spin-Coherent-State Path Integral*

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Abstract—We present a detailed field-theoretical description of those collective degrees of freedom (CDF) which are relevant to study macroscopic quantum dynamics of a quasi-one-dimensional ferromagnetic domain wall. We apply the spin-coherent-state path integral in the proper discrete time formalism (a) to extract the relevant CDFs, namely, the center position and the chirality of the domain wall, which originate from the translation and the rotation invariances of the system in question, and (b) to derive an effective action for the CDFs by elimination of environmental zero modes with the help of the Faddeev–Popov technique. The resulting effective action turns out to be such that both the center position and the chirality can be formally described by a boson-coherent-state path integral. However, this is only formal; there is a subtle departure from the latter. © *2001 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

Recent so-called nanostructure technology enables us to study low-dimensional magnetism in mesoscopic magnets from the quantum-mechanical point of view [1–3]. Among others, a magnetic domain wall has attracted much attention, both theoretically and experimentally, because it is expected to exhibit macroscopic quantum phenomena [4–11]. As a theoretical technique to evaluate the quantum dynamics of the domain wall, the spin-coherent-state path integral in the continuous-time formalism [12] is frequently used. However, as noted by some workers [13, 14], it has some fundamental difficulties, which have been recently discussed in detail [15]. Furthermore, it is liable to lead to confusion concerning the interpretation of the collective degrees of freedom, as has been pointed out in [16]. Hence, as yet there is no microscopic theory of the quantum dynamics of the domain wall. In order to pave the way for such a theory, this paper presents a field-theoretical description of collective degrees of freedom by use of the spin-coherent-state path integral in the proper discrete-time formalism.

2. MODEL

We consider a ferromagnet consisting of a spin $\mathbf{S}$ of magnitude S at each site in a quasi-one-dimensional cubic crystal (a linear chain) of lattice constant a. The magnet is assumed to have an easy axis in the z directions. Accordingly, we adopt the Hamiltonian

$$\hat{H} = -\tilde{J}\sum_{\langle i,j\rangle}^{N_L} \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j - \frac{K}{2}\sum_{j}^{N_L} \hat{S}_{j,z}^2, \qquad (1)$$

where the index i or j represents a lattice point, $\langle i,j\rangle$ denotes a nearest neighbor pair, N_L is the total number of lattice points, $\tilde{J}$ is the exchange coupling constant, and K is the longitudinal anisotropy constant; and $\tilde{J}$ and K are all positive.

Since we are interested in those transition amplitudes which are appropriate to describe quantum-mechanical motion of a domain wall, we introduce a spin-coherent state [17] at each site, which is suited for a vector picture of spin. We denote a state of the system as

$$|\xi\rangle \equiv |\xi_1, \xi_2, \ldots, \xi_{N_L}\rangle := \bigotimes_{j}^{N_L} |\xi_j\rangle, \qquad (2)$$

where $|\xi_j\rangle$ is a spin-coherent state at the site j. The transition amplitude between the initial state $|\xi_I\rangle$ and the final state $|\xi_F\rangle$ can be expressed as a spin-coherent-state path integral in the real discrete-time formalism by the standard procedure of the repeated use of the resolution of unity (see, e.g., [15]; on which the present notation is based):

$$\langle \xi_F | e^{-i\hat{H}T/\hbar} | \xi_I \rangle \qquad (3)$$

$$= \lim_{N\to\infty} \int \prod_{n=1}^{N-1} \prod_{j}^{N_L} d\mu(\xi_j(n), \xi_j^*(n)) \exp\left(\frac{i}{\hbar}\mathcal{S}[\xi^*, \xi]\right),$$

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where $N \equiv T/\epsilon$, ϵ is an infinitesimal time interval, n represents discrete time, and the integration measure is

$$d\mu(\xi,\xi^*) := \frac{2S+1}{(1+|\xi|^2)^2}\frac{d\xi d\xi^*}{2\pi i}, \tag{4}$$
$$\frac{d\xi d\xi^*}{2\pi i} \equiv \frac{d\Re\xi d\Im\xi}{\pi}.$$

The action $\mathcal{S}[\xi^*,\xi]$ consists of two parts, $\mathcal{S}^c[\xi^*,\xi]$ and $\mathcal{S}^d[\xi^*,\xi]$, which are to be called the canonical term and the dynamical term, respectively. We shall be interested in those spin configurations whose scale of spatial variation is much larger than the lattice constant a. Accordingly, we take the spatial continuum limit in the action:

$$\mathcal{S}[\xi^*,\xi] := \mathcal{S}^c[\xi^*,\xi] + \mathcal{S}^d[\xi^*,\xi], \tag{5a}$$

$$\frac{i}{\hbar}\mathcal{S}^c[\xi^*,\xi] = S\sum_{n=1}^{N}\int\limits_{-L/2}^{L/2}\frac{dx}{a} \tag{5b}$$
$$\times \ln\frac{(1+\xi^*(x,n)\xi(x,n-1))^2}{(1+|\xi(x,n)|^2)(1+|\xi(x,n-1)|^2)},$$

$$\frac{i}{\hbar}\mathcal{S}^d[\xi^*,\xi] \tag{5c}$$
$$= -\frac{i}{\hbar}\sum_{n=1}^{N}\epsilon\int\limits_{-L/2}^{L/2}\frac{dx}{a}\mathcal{H}(\xi^*(x,n),\xi(x,n-1)),$$

$$\mathcal{H}(\xi^*(x),\eta(x)) \tag{5d}$$
$$:= \frac{S}{(1+\xi^*(x)\eta(x))^2}\Big[2JS\partial_x\xi^*(x)\partial_x\eta(x)$$
$$-\frac{K}{2}\left\{\left(S-\frac{1}{2}\right)(1-\xi^*(x)\eta(x))^2+\frac{1}{2}\right\}\Big],$$

where L is the length of the linear chain and $J \equiv \tilde{J}a^2$.

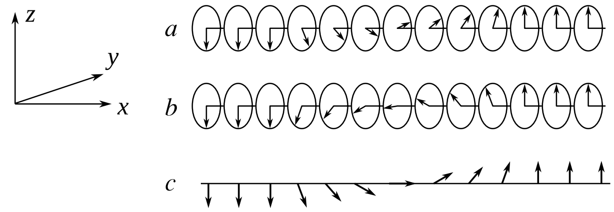
3. METHOD OF COLLECTIVE DEGREES OF FREEDOM

3.1. Kink Configuration

We begin by finding a domain-wall configuration. It is determined by one of the static solutions $\{\xi^s(x),\bar{\xi}^s(x)\}$ of the action $\mathcal{S}[\xi^*,\xi]$. They satisfy the following equations:

$$\lambda^2\left\{\partial_x^2\xi^s(x) - \frac{2\bar{\xi}^s(x)(\partial_x\xi^s(x))^2}{1+\bar{\xi}^s(x)\xi^s(x)}\right\} \tag{6a}$$
$$-\frac{1-\bar{\xi}^s(x)\xi^s(x)}{1+\bar{\xi}^s(x)\xi^s(x)}\xi^s(x) = 0,$$

$$\lambda^2\left\{\partial_x^2\bar{\xi}^s(x) - \frac{2\xi^s(x)(\partial_x\bar{\xi}^s(x))^2}{1+\bar{\xi}^s(x)\xi^s(x)}\right\} \tag{6b}$$
$$-\frac{1-\bar{\xi}^s(x)\xi^s(x)}{1+\bar{\xi}^s(x)\xi^s(x)}\bar{\xi}^s(x) = 0,$$



Domain walls with three chiralities (quoted from [10]): *(a)* right-handed wall ($\phi_0 = \pi/2$), *(b)* left-handed wall ($\phi_0 = -\pi/2$), and *(c)* wall with no chirality ($\phi_0 = 0$). Circles in *(a)* and *(b)* drawn to guide the eye lie in the yz plane, while the spins lie in the zx plane in *(c)*. The quasi-one-dimensional direction of the crystal is here aligned with the spin hard axis for ease of visualization. A different alignment, which may be the case for a real magnet, does not affect the content of the text; for instance, one could rotate all the spins by $\pi/2$ around the y axis if the dominant anisotropy originates from the demagnetizing field.

where $\lambda^2 \equiv JS/K(S-1/2)$. An obvious solution is the "vacuum" solution representing the uniform configuration in which the spins are either all parallel or all antiparallel to the z direction. The other solution is the "kink" solution representing a domain-wall configuration in which the spins at $x \sim +\infty$ are parallel to the z direction, the spins at $x \sim -\infty$ are antiparallel to the z direction, and there is a transition region (i.e., a domain wall) of width λ:

$$\xi^s(x) = \exp\left(-\frac{x-Q}{\lambda} + i\phi_0\right), \tag{7}$$
$$\bar{\xi}^s(x) = \exp\left(-\frac{x-Q}{\lambda} - i\phi_0\right),$$

where Q and ϕ_0 are arbitrary real constants. Q is the center position of the domain wall, and ϕ_0 is a quantitative measure of the chirality of the domain wall with respect to the x axis (the figure); the wall is maximally right-handed if $\phi_0 = \pi/2$ and maximally left-handed if $\phi_0 = -\pi/2$, while it has no chirality if $\phi_0 = 0$. The range of ϕ_0 is chosen as $-\pi \le \phi_0 \le \pi$, with $\phi_0 = \pi$ and $\phi_0 = -\pi$ representing the same situation. $\{\xi^s(-x),\bar{\xi}^s(-x)\}$ is also a solution representing a domain-wall configuration. However, this as well as the vacuum solution belongs to a sector different from that of (7). Since a transition between different sectors is forbidden [18], it is sufficient to consider only sector (7) for the purpose of studying the dynamics of a domain wall.

3.2. Collective Degrees of Freedom and Environment

Study of the domain-wall dynamics is facilitated by introducing relevant collective degrees of freedom.

We note two kinds of invariance possessed by (6). One is the translation invariance in the x direction. The other is the rotation invariance around the z axis. These invariances are embodied by the arbitrariness in the choice of Q and ϕ_0, respectively, in (7). Hence, we elevate them to dynamical variables $Q(n)$ and $\phi_0(n)$ [8, 10, 18]. To deal with these two dynamical variables (collective degrees of freedom), it is convenient to define

$$z(n) := q(n) + i\phi_0(n), \quad z^*(n) := q(n) - i\phi_0(n),$$
$$q(n) \equiv Q(n)/\lambda, \quad n = 1, 2, ..., N-1. \quad (8)$$

By use of these variables, original integration variables $\xi(x,n)$ and $\xi^*(x,n)$ may be decomposed into the domain-wall configuration and the deviation from it:

$$\xi(x,n) = \xi^s(x; z(n)) + \tilde{\eta}(x,n;\{z\}_n^n), \quad (9a)$$
$$\xi^*(x,n) = \bar{\xi}^s(x; z^*(n)) + \tilde{\eta}^*(x,n;\{z^*\}_n^n), \quad (9b)$$

where

$$\xi^s(x; z(n)) := \exp(-x/\lambda + z(n)), \quad (10)$$
$$\bar{\xi}^s(x; z^*(n)) := \exp(-x/\lambda + z^*(n)),$$

and we use the notation $\{z\}_m^n = \{z^*\}_n^m := (z^*(n), z(m))$. At both ends of the discrete time ($n = 0$ or $n = N$), we define

$$z(0) \equiv z_I := q_I + i\phi_I, \quad z(N) \equiv z_F := q_F + i\phi_F, \quad (11)$$

$$\eta(x, 0; \{z\}_0^0) = \eta^*(x, N; \{z\}_N^N) = 0, \quad (12)$$

where z_I represents the center position q_I and the chirality ϕ_I of the domain wall in the initial state, and z_F those in the final state.

The variable $\tilde{\eta}$, which is to be called the environment around the domain wall, could be expanded by a set of some mode function. The expansion is expected to contain the zero modes, which originate from the translation and the rotation invariances and should be eventually eliminated in order to avoid overcounting the degrees of freedom. For this reason, we expand the environment with respect to a set of mode functions $\{\psi_k\}$ as

$$\tilde{\eta}(x,n;\{z\}_n^n) = \eta_0(n)\psi_0(x; z(n)) + \eta(x,n;\{z\}_n^n), \quad (13a)$$

$$\tilde{\eta}^*(x,n;\{z^*\}_n^n) = \eta_0^*(n)\psi_0^*(x; z^*(n)) + \eta^*(x,n;\{z^*\}_n^n), \quad (13b)$$

$$\eta(x,n;\{z\}_n^n) = \sum_k{}' \eta_k(n)\psi_k(x;\{z\}_n^n), \quad (13c)$$

$$\eta^*(x,n;\{z^*\}_n^n) = \sum_k{}' \eta_k^*(n)\psi_k^*(x;\{z^*\}_n^n), \quad (13d)$$

where $\sum_k'$ denotes summation over the modes excluding the zero modes. The zero-modes function is proportional to the first derivative of $\xi^s(x; z(n))$ with respect to x

$$\psi_0(x; z(n)) = -A\lambda \frac{d\xi^s(x; z(n))}{dx}, \quad (14)$$

where A is a real normalization constant. To confirm this relation, one may consider the sum of the domain-wall configuration and the zero-mode part:

$$\xi^s(x; z(n)) + \eta_0(n)\psi_0(x; z(n)) \quad (15)$$
$$= \xi^s(x; z(n)) - A\lambda\eta_0(x)\frac{d\xi^s(x; z(n))}{dx}$$
$$\simeq \xi^s(x - A\lambda\eta_0(n); z(n))$$
$$= \exp\left[-\frac{x}{\lambda} + q(n) + A\Re\eta_0(n) + i(\phi_0(n) + A\Im\eta_0(n))\right].$$

Hence, it is clear that Eq. (14) gives the zero-mode eigenfunction and that the real and the imaginary parts of $\eta_0(n)$ are the zero modes corresponding to the translation and the rotation modes, respectively. We choose $\{\psi_k\}$ so that $\{\psi_0, \psi_k\}$ forms a orthonormal set:

$$\int_{-L/2}^{L/2} \frac{dx}{a} f(x;\{z\}_n^n)\psi_0^*(x; z^*(n))\psi_0(x; z(n)) = 1, \quad (16a)$$

$$\int_{-L/2}^{L/2} \frac{dx}{a} f(x;\{z\}_n^n)\psi_0^*(x; z^*(n))\psi_k(x;\{z\}_n^n) = 0, \quad (16b)$$

$$\int_{-L/2}^{L/2} \frac{dx}{a} f(x;\{z\}_n^n)\psi_k^*(x;\{z^*\}_n^n) \times \psi_{k'}(x;\{z\}_n^n) = \delta_{kk'}, \quad (16c)$$

$$f(x;\{z\}_n^n)\Big\{\psi_0(x; z(n))\psi_0^*(x; z^*(n)) \quad (16d)$$
$$+ \sum_k{}' \psi_k(x;\{z\}_n^n)\psi_k^*(x';\{z^*\}_n^n)\Big\} = \delta\left(\frac{x-x'}{a}\right),$$

where $f(x;\{z\}_n^n)$ is a real weight function to be fixed later. This weight function neglects a nonlinear character of the coherent-state path integral.

3.3. Faddeev–Popov Type Identity

In order to introduce the collective degrees of freedom and eliminate the zero modes, we adopt the Faddeev–Popov method to the spin-coherent-state path integral. To do so, we invoke the following Faddeev–Popov type identity:

$$1 = \int dRg_0^{[\xi,\xi^*]} dIg_0^{[\xi,\xi^*]} \delta(Rg_0^{[\xi,\xi^*]})\delta(Ig_0^{[\xi,\xi^*]}) \quad (17a)$$

$$= \int \frac{dz(n)dz^*(n)}{2i} \Delta^{[\xi,\xi^*]}(\{z\}_n^n)$$
$$\times\, \delta(Rg_0^{[\xi,\xi^*]}(\{z\}_n^n))\delta(Ig_0^{[\xi,\xi^*]}(\{z\}_n^n)),$$

where

$$\frac{dz(n)dz^*(n)}{2i} = dq(n)d\phi_0(n) \tag{17b}$$

and $\Delta^{[\xi,\xi^*]}(\{z\}_n^n)$ is the Faddeev–Popov type determinant

$$\Delta^{[\xi,\xi^*]}(\{z\}_n^n) = \left| \frac{\partial(g_0^{[\xi]}(\{z\}_n^n), g_0^{*[\xi^*]}(\{z^*\}_n^n))}{\partial(z(n), z^*(n))} \right| \tag{18}$$

with $Rg_0^{[\xi,\xi^*]}(\{z\}_n^n)$ and $Ig_0^{[\xi,\xi^*]}(\{z\}_n^n)$ being the real and the imaginary parts of $g_0^{[\xi]}(\{z\}_n^n)$ defined as

$$g_0^{[\xi]}(\{z\}_n^n) := \int_{-L/2}^{L/2} \frac{dx}{a} f(x;\{z\}_n^n)\{\xi(x,n) \tag{19a}$$
$$- \xi^s(x;z(n))\}\psi_0^*(x;z^*(n)),$$
$$g_0^{*[\xi^*]}(\{z^*\}_n^n) := \{g_0^{[\xi]}(\{z\}_n^n)\}^*. \tag{19b}$$

3.4. Transition Amplitude

Hereafter, we consider transition amplitudes between domain-wall states. Namely, we take

$$|\xi_\beta\rangle = |z_\beta\rangle := \bigotimes_j^{N_L} |\xi^s(ja;z_\beta)\rangle, \qquad \beta = I, F, \tag{20}$$

in Eq. (3). Inserting the identity (17a) into the right-hand side of (3) at each discrete time, we get

$$\langle z_F|e^{-i\hat{H}T/\hbar}|z_I\rangle \tag{21}$$
$$= \lim_{N\to\infty} \prod_{n=1}^{N-1} \int \prod_x \frac{d\xi(x,n)d\xi^*(x,n)}{2\pi i}$$
$$\times \frac{2S+1}{(1+\xi^*(x,n)\xi(x,n))^2}$$
$$\times \left\{ \int \prod_{n=1}^{N-1} \frac{dz(n)dz^*(n)}{2i} \Delta^{[\xi,\xi^*]}(\{z\}_n^n) \right.$$
$$\left. \times\, \delta(Rg_0^{[\xi,\xi^*]}(\{z\}_n^n))\delta(Ig_0^{[\xi,\xi^*]}(\{z\}_n^n)) \right\}$$
$$\times \exp\left(\frac{i}{\hbar}\mathcal{S}[\xi^*,\xi]\right).$$

At this stage, we substitute (9) with (16) for ξ and ξ^*. Then, (19) may be replaced by

$$g_0^{[\xi^s+\tilde{\eta}]}(\{z\}_n^n) = \eta_0(n), \quad g_0^{*[\bar{\xi}^s+\tilde{\eta}^*]}(\{z^*\}_n^n) = \eta_0^*(n). \tag{22}$$

Thus,

$$Rg_0^{[\xi^s+\tilde{\eta},\bar{\xi}^s+\tilde{\eta}^*]}(\{z\}_n^n) = \Re\eta_0(n), \tag{23}$$
$$Ig_0^{[\xi^s+\tilde{\eta},\bar{\xi}^s+\tilde{\eta}^*]}(\{z\}_n^n) = \Im\eta_0(n),$$

which are zero-mode degrees of freedom. The integration measure can be rewritten as

$$\prod_x \frac{d\xi(x,n)d\xi^*(x,n)}{2\pi i} \tag{24}$$
$$\times \left.\frac{2S+1}{(1+\xi^*(x,n)\xi(x,n))^2}\right|_{\xi=\xi^s+\tilde{\eta},\xi^*=\bar{\xi}^s+\tilde{\eta}^*}$$
$$= \prod_{k=0} \frac{d\eta_k(n)d\eta_k^*(n)}{2\pi i} 2S\left(1+\frac{1}{2S}\right)\prod_x \Xi[x,n;z,\tilde{\eta}],$$

where

$$\Xi[x,n;z,\tilde{\eta}] := \frac{\{f(x;\{z\}_n^n)\}^{-1}}{\{1+(\bar{\xi}^s(x;z^*(n))+\tilde{\eta}^*(x;\{z^*\}_n^n))(\xi^s(x;z(n))+\tilde{\eta}(x;\{z\}_n^n))\}^2}. \tag{25}$$

Thus, the transition amplitude (21) is reduced to

$$\langle z_F|e^{-i\hat{H}T/\hbar}|z_I\rangle = \lim_{N\to\infty} \int \prod_{n=1}^{N-1} \left[\frac{dz(n)dz^*(n)}{2i} \right. \tag{26}$$
$$\times \prod_{k=0} \frac{d\eta_k(n)d\eta_k^*(n)}{2\pi i} 2S\left(1+\frac{1}{2S}\right)$$
$$\times \prod_x \Xi[x,n;z,\tilde{\eta}] \cdot \Delta^{[\xi^s+\tilde{\eta},\bar{\xi}^s+\tilde{\eta}^*]}(\{z\}_n^n)$$
$$\left. \times\, \delta(\Re\eta_0(n))\delta(\Im\eta_0(n)) \exp\left(\frac{i}{\hbar}\mathcal{S}[\bar{\xi}^s+\tilde{\eta}^*,\xi^s+\tilde{\eta}]\right)\right].$$

With the help of the delta function, we can immediately integrate out the zero modes at each discrete time. Furthermore, in the case of large spin $S \gg 1$, the factors contributing to the integration measure take the simple form [19]

$$\Xi[x,n;z,\eta] = 1 + O(S^{-1/2}), \tag{27a}$$

$$\Delta^{[\xi^s+\eta,\bar{\xi}^s+\eta^*]}(\{z\}_n^n) = \frac{N_{\rm DW}}{2}\left(1+O(S^{-1/2})\right). \tag{27b}$$

In this way, we find

$$\langle z_F|e^{-i\hat{H}T/\hbar}|z_I\rangle \tag{28}$$
$$= \lim_{N\to\infty}\int\prod_{n=1}^{N-1}\left[\prod_k{}' 2S\frac{d\eta_k(n)d\eta_k^*(n)}{2\pi i}\right.$$
$$\left.\times N_{\mathrm{DW}}S\frac{dz(n)dz^*(n)}{2\pi i}\exp\left(\frac{i}{\hbar}\mathcal{S}[\bar{\xi}^s+\eta^*,\xi^s+\eta]\right)\right].$$

This formula determines the quantum dynamics of the domain wall in the path integral.

3.5. Effective Action

The next step is to expand the action $\mathcal{S}[\bar{\xi}^s+\eta^*,\xi^s+\eta]$ with respect to the environment. The expanded action consists of three parts. The first part, $\mathcal{S}_c[z^*,z]$, contains only the collective degrees of freedom. This part was estimated in detail in [16]. The second part, $\mathcal{S}_e[\eta^*,\eta]$, contains only the environment, and the third part, $\mathcal{S}_{c-e}[z^*,z,\eta^*,\eta]$, represents the interaction between the collective degrees of freedom and the environment:

$$\mathcal{S}[\bar{\xi}^s+\eta^*,\xi^s+\eta] := \mathcal{S}_c[z^*,z] \tag{29}$$
$$+\,\mathcal{S}_e[\eta^*,\eta]+\mathcal{S}_{c-e}[z^*,z,\eta^*,\eta].$$

Now, we proceed to find an equation to determine the environmental mode functions $\{\psi_k\}$. Such an equation may be obtained from the dynamical term whose collective part gave the static (6). Accordingly, we examine the dynamical term of $\mathcal{S}_e[\eta^*,\eta]$, which we denote by $\mathcal{S}_e^d[\eta^*,\eta]$:

$$\mathcal{S}_e^d[\eta^*,\eta] := -2KS\left(S-\frac{1}{2}\right) \tag{30a}$$
$$\times\sum_{n=1}^{N}\epsilon\int_{-L/2}^{L/2}\frac{dx}{a}\eta^*(x,n;\{z^*\}_n^n)g^2(x;\{z\}_n^n)$$
$$\times\left[-\lambda^2\frac{\partial^2}{\partial x^2}-4\lambda g(x;\{z\}_n^n)\exp(-2\tilde{x}_{n-1}^n)\frac{\partial}{\partial x}\right.$$
$$\left.-\left\{3\exp(-2\tilde{x}_{n-1}^n)-1\right\}g(x;\{z\}_n^n)\right]$$
$$\times\,\eta(x,n-1;\{z\}_{n-1}^n),$$

where

$$\tilde{x}_m^n = \tilde{x}_n^{*m} \equiv \frac{x}{\lambda}-\frac{z^*(n)+z(m)}{2}, \tag{30b}$$
$$g(x;\{z\}_n^n) \equiv \frac{1}{1+e^{-2\tilde{x}_{n-1}^n}}.$$

Terms proportional to $\eta^*\eta^*$ or $\eta\eta$ vanish due to the static equation (6). Note that this cancellation is not a generally guaranteed theorem. It turns out to be convenient to write $\{\psi_k,\psi_k^*\}$ in (13c) and (13d) as follows:

$$\psi_k(x;\{z\}_n^n) := 2e^{-x/\lambda+z(n)}\cosh\left(\tilde{x}_n^n\right)\varphi_k\left(\tilde{x}_n^n\right), \tag{31a}$$
$$\psi_k^*(x;\{z^*\}_n^n) := \{\psi_k(x;\{z\}_n^n)\}^*. \tag{31b}$$

Putting (13c), (13d), and (31a) into (31b), we find

$$\mathcal{S}_e^d[\eta^*,\eta] = -2\lambda^2KS\left(S-\frac{1}{2}\right) \tag{32}$$
$$\times\sum_{n=1}^{N}\epsilon\sum_{k',k}\eta_{k'}^*(n)\eta_k(n-1)$$
$$\times\int_{-L/2}^{L/2}\frac{dx}{a}\varphi_{k'}^*(x)\left[-\frac{\partial^2}{\partial x^2}+\frac{1}{\lambda^2}\left\{1\right.\right.$$
$$\left.\left.-2\,\mathrm{sech}^2\left(\frac{x}{\lambda}\right)\right\}\right]\varphi_k(x),$$

where we replace $x/\lambda-(z^*(n)+z(n))/2$ by x/λ because $L/\lambda\gg 1$. Thus, we are led to demand $\varphi_k(x)$ to obey the eigenvalue equation

$$\left[-\partial_x^2+\frac{1}{\lambda^2}\left\{1-2\,\mathrm{sech}^2\left(\frac{x}{\lambda}\right)\right\}\right]\varphi_k(x) \tag{33}$$
$$=\omega_k\varphi_k(x),$$

where ω_k is the eigenvalue. This solution to this equation is well known in the literature:

$$\varphi_k(x) = N_k\left(-ik\lambda+\tanh(x/\lambda)\right)e^{ikx}, \tag{34a}$$

where

$$N_k = \left(\frac{L}{a}(k^2\lambda^2+1)\right)^{-1/2},\quad \omega_k = k^2+\lambda^{-2}. \tag{34b}$$

These eigenfunctions diagonalize $\mathcal{S}_e^d[\eta^*,\eta]$ as

$$\mathcal{S}_e^d[\eta^*,\eta] = -2S\sum_{n=1}^{N}\sum_k{}'\hbar\epsilon\Omega_k\eta_k^*(n)\eta_k(n-1), \tag{35a}$$

where

$$\Omega_k \equiv \frac{JS}{\hbar}(k^2+\lambda^{-2}). \tag{35b}$$

The last equation is the dispersion relation of the environmental eigenmode. Note that ω_k is not zero in the limit of $k\to 0$.

What remains is to decide the weight function $f(x;\{z\}_n^n)$. With (31), the condition (16c) takes the form

$$\int_{-L/2}^{L/2}\frac{dx}{a}f(x,\{z\}_n^n)4e^{-2\tilde{x}_n^n}\cosh^2\left(\tilde{x}_n^n\right) \tag{36}$$
$$\times\,\varphi_k^*(x;\{z^*\}_n^n)\varphi_{k'}(x;\{z\}_n^n) = \delta_{k,k'}.$$

This is satisfied if we choose

$$f(x, \{z\}_n^n) = \frac{1}{(1+e^{-2x/\lambda + z^*(n)+z(n)})^2}. \quad (37)$$

It is easy to check that (16b) is also satisfied by this choice. Accordingly, the normalization constant A of the zero-mode function $\psi_0(x; z(n))$ as given by (14) may be determined by use of (37) and (16a):

$$1 = A^2 \int_{-L/2}^{L/2} \frac{dx}{a} \frac{e^{-2x/\lambda + z^*(n)+z(n)}}{(1+e^{-2x/\lambda+z^*(n)+z(n)})^2} \quad (38)$$

$$= \frac{A^2\lambda}{4a} [\tanh(\tilde{x}_n^n)]_{-L/2\to -\infty}^{L/2\to\infty} = \frac{A^2\lambda}{2a}.$$

Thus,

$$A = \sqrt{\frac{2a}{\lambda}}. \quad (39)$$

The closure (16d) can also be shown to hold with these choices.

The remainder of the action can be calculated in the same manner [19], and we obtain the effective action in the following form:

$$\frac{i}{\hbar}\mathcal{S}_s[z^*, z] = N_{\rm DW} S \sum_{n=1}^{N} \Bigg[-\frac{1}{2}\Bigg\{ z^*(n)z(n) \quad (40a)$$

$$+ z^*(n-1)z(n-1)\Bigg\} + z^*(n)z(n-1) - \frac{i}{\hbar}E_{\rm DW}T\Bigg],$$

$$\frac{i}{\hbar}\mathcal{S}_e[\eta^*, \eta] = 2S\sum_{n=1}^{N}\sum_k{}' \Bigg[-\frac{1}{2}\Bigg\{\eta^*(n)\eta(n) \quad (40b)$$

$$+ \eta^*(n-1)\eta(n-1)\Bigg\} + \eta^*(n)\eta(n-1)$$

$$- \frac{i}{\hbar}\epsilon\Omega_k \eta^*(n)\eta(n-1)\Bigg],$$

$$\frac{i}{\hbar}\mathcal{S}_{c-e}[z^*, z, \eta^*, \eta]$$

$$= 2S\sum_{n=1}^{N}\sum_k{}' \Bigg[\Bigg(J_k(\Delta z(n))\eta_k^*(n) \quad (40c)$$

$$- J_k(\Delta z^*(n+1))\eta_k(n)\Bigg) + O(S^{-1})\Bigg],$$

where $\Delta z(n) \equiv z(n) - z(n-1)$, and

$$J_k(z) := \frac{\lambda}{2a}\int_{-L/2\lambda}^{L/2\lambda} dx \frac{\sinh z/2}{\cosh(x+z/2)}\varphi_k^*(x), \quad (40d)$$

which is a function of the collective degrees of freedom, represents a nonlinear interaction with the environment. It is seen that the above form of the effective action is formally the same as that obtained in a boson-coherent-state path integral. Hence, it is concluded that the quantum dynamics of the domain wall is formally represented by that of a "boson" z interacting with environmental bosons $\{\eta\}$. However, it is to be remembered that the imaginary part of the "boson" z is an angular variable. This circumstance can cause a subtle departure from the case of a boson. Details of these subtleties, as well as a concrete evaluation of transition amplitudes, are left for a future work.

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SYMPOSIUM ON GROUP THEORY AND PATH INTEGRALS

Translationally Noninvariant Path Integral for Bose–Einstein Condensate*

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Abstract—The Bogolyubov [Hartree–Fock–Bogolyubov (HFB)] method performs the one-particle (mean-field) approximation in the theory of Bose–Einstein condensation (BEC). Various generalizations of this method are possible. Apart from a nonlinear theory, taking the correlation effects into consideration, the HFB approximation for translationally noninvariant systems describes an instructive phenomenon. This paper is devoted to the treatment of two cases: superfluid ^{4}He in porous media and atomic BEC in traps subjected to the gravitational field. Both these systems show the dependence of a critical BEC temperature T_c on their nonuniform properties in space. © *2001 MAIK "Nauka/Interperiodica".*

1. TRANSLATIONALLY NONINVARIANT BOSE CONDENSATE

The action for bosons with the interaction $G(r - r')$ in an external field U looks like

$$S = \int_0^\beta dt \left[\int dr\, \psi(r,t)\frac{\partial}{\partial t}\psi^*(r,t) - H(t)\right],$$

$$H = H_0 + h,$$

$$H_0 = \int dr \left[\frac{1}{2m}\left(\nabla\psi^*(r,t)\nabla\psi(r,t)\right) + \psi^* U(r)\psi\right], \tag{1}$$

$$h = \frac{1}{2}\int drdr'\, G(r-r')\psi^*(r,t)\psi^*(r',t) \times \psi(r',t)\psi(r,t). \tag{2}$$

The C shift [1] of the Bose operator amplitudes

$$\psi^+(r,t) = B_0^* + B^+(r,t),$$
$$[\psi(r), \psi^+(r')] = \delta(r - r')$$

leads to the Bogolyubov model [2], which is of the square form in the "fast" variables, B and B^+, and the square form in the "slow" C-number variables, B_0 and B_0^*. The components B_0 and B can be considered as the variables of two subsystems in the adiabatic approach [3]

$$\beta E_k \gg 1 \gg \frac{\omega_0}{E_k} \simeq 0,$$
$$\omega_0 \sim dB_0/dt, \quad 0 < E_k < 2\ \mathrm{meV},$$

where E_k is the Bogolyubov spectrum of noncondensate excitation. The trace for a partition function of the Bogolyubov model may be evaluated exactly over the noncondensate variables. In the path-integral method, this average results in the effective action $S_{\mathrm{eff}}(\rho)$ for the condensate density $\rho = |B_0|^2$. The partition function has the form

$$Q = \mathrm{tr}\exp(-\beta H) = \int d^2B_0\, D^2B\, e^{S(0,\beta)},$$

$$S(0,\beta) = \int_0^\beta L d\tau, \quad L = L_B^0 - B^* K_\Omega B + L_{B_0,B},$$

$$K_\Omega = \frac{d}{dt} + \Omega, \quad \Omega = \frac{k^2}{2m}.$$

The average over the variables B, B^* leads to the effective action $S_{\mathrm{eff}}(B_0)$

$$Q = \int d^2\, B_0 e^{S_{\mathrm{eff}}}, \quad e^{S_{\mathrm{eff}}} = \int D^2 B e^{S(0,\beta)}, \tag{3}$$
$$S_{\mathrm{eff}} = S_B - \ln(\mathrm{Det}K_\Omega).$$

With boson-pair correlations, $B^2 \to b_k b_{-k}$, peculiar for Hartree–Fock–Bogolyubov (HFB), we choose a bilinear form

$$S_k = \frac{1}{2}\int_0^\beta (b_k^*, b_{-k})\, M \begin{pmatrix} b_k \\ b_{-k}^* \end{pmatrix} dt,$$

$$M = \begin{pmatrix} K_\Omega & -gB_0 \\ -gB_0^* & -K_{-\Omega} \end{pmatrix},$$

so that the path-integral adiabatic approach gives the effective action (3) for a "slow" condensate variable ρ.

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Here, ρ and μ are found from the equations $\delta S_{\text{eff}} = 0$,

$$R = \rho + r, \quad \frac{\mu}{G_0} - 2r = \rho\left[1 - \frac{D(\rho,\mu)}{g_0}\right],$$

$$D = \frac{c}{2}\int k^2 \coth\left(\frac{\beta E_k}{4}\right)\frac{G_k^2}{E_k}dk > 0,$$

$$r = \frac{c}{2}\int k^2 \Phi\, dk,$$

and

$$\Phi = \frac{\omega_k - \mu}{E_k}\coth\left(\frac{\beta E_k}{4}\right) - 1, \quad N, V \to \infty,$$

$$\beta E_k \gg 1,$$

where R is the total ^{4}He density, r is the noncondensate density, and $\rho = |B_0|^2/V$. The heat capacity

$$C_v = -T\frac{\partial^2 F}{\partial T^2}, \quad F = -\frac{1}{\beta}S_{\text{eff}},$$

has been calculated with the help of the above equations in [4, 5].

In the general case, the C shift of the Bose operator amplitudes may also be written in the form

$$\psi^+(r,t) = \frac{1}{\sqrt{V}}\sum_{n=0} b_n^+(t)u_n(r)$$

$$= \frac{1}{\sqrt{V}}\left(\sum_{n\neq 0} b_n^+(t)u_n(r) + b_0^* u_0(r)\right),$$

$$b_0 = \sqrt{N_0}e^{i\alpha}, \quad n = \{n_x, n_y, n_z\},$$

$$\int dr\, u_n u_{n'} = \delta_{n,n'}.$$

Here, the three-dimensional one-particle eigenfunctions u_n are the solutions of the equation for a boson wave function

$$\left\{\nabla^2 + \frac{2m}{h^2}\left[E_n - U(r)\right]\right\}u_n = 0 \qquad (4)$$

with energy levels E_n.

In terms of Fourier decomposition with a basis $u_n \to \exp(ikr)$, the contributions to the Bogolyubov energy may be classified as follows:

(a) ordinary terms $(k_1 + k_2 = k_3 + k_4)$

$$G(r-r')B_0^* B_0 B_0^* B_0,$$

$$G(r-r')[B^*(r)B(r)B_0^*B_0 + B^*(r')B(r')B_0^*B_0 + \text{h.c.}],$$

(b) broken gauge symmetry terms $(k_1 + k_2 = k_3 + k_4)$

$$G(r-r')[B^*(r)B^*(r')B_0B_0 + B(r)B(r')B_0^*B_0^*],$$

(c) broken gauge and translation symmetries

$$G(r,r')[B^*(r)B_0^*B_0B_0 + B(r')^*B_0^*B_0B_0 + \text{h.c.}]$$

$$\to \begin{cases}\langle k,0|G|0,0\rangle, k_1 + k_2 \neq k_3 + k_4, G = G(r,r'),\\ 0, k_1 + k_2 = k_3 + k_4, G = G(r-r').\end{cases}$$

The δ-interaction approximation for G is valid for a dilute gas:

$$G(r-r') \to G\cdot\delta(r-r').$$

Taking into account the broken translation symmetry terms of type $\langle k,0|G|0,0\rangle$, we get (1) and (2) in the initial action as

$$N, V \to \infty, \quad b_n \to \sqrt{V}b_n, \quad b_n^* \to \sqrt{V}b_n^*,$$

$$H_0 = \rho w_{00} + \sqrt{\rho}\sum_{n\neq 0}(b_n e^{-i\alpha} + b_n^* e^{i\alpha})w_{0n} + \sum_{n,n'\neq 0} b_n^* b_{n'} w_{nn'},$$

$$h = \rho^2\gamma_0 + \rho^{3/2}\sum_{n\neq 0}(b_n e^{-i\alpha} + b_n^* e^{i\alpha})\gamma_{0n} + 2\rho\sum_{n,n'\neq 0}\left[b_n^* b_{n'} + \frac{1}{2}(b_n^* b_{n'}^* e^{2i\alpha} + b_n b_{n'} e^{-2i\alpha})\right]\gamma_{nn'},$$

$$\gamma_0 = \frac{G}{2}\int dr u_0^4, \quad \gamma_{0n} = \frac{G}{2}\int dr u_0^3 u_n,$$

$$\gamma_{nn'} = \frac{G}{2}\int dr\, u_0^2 u_n u_{n'},$$

where ρ is the Bose condensate density. The one-particle energy $w_{nn'}$ in the basis (4)

$$w_{nn'} = \frac{1}{2m}\int \nabla u_n \nabla u_{n'} dr + \int U(r)u_n u_{n'} dr$$

contains both the kinetic energy term $\sim\nabla^2$ and the external field energy term $\sim U(r)$.

In the above, H_0 is the Hamiltonian of the ideal Bose gas, and h is a generalization of the Bogolyubov pair-correlated interaction Hamiltonian in the δ approximation. The coefficients γ_{0n} and w_{0n} are responsible for broken translation symmetry.

2. SUPERFLUID ^{4}He IN POROUS MEDIA

The external stochastic U^*-field theory [6] was suggested as an origin of a translation noninvariant state of bosons, so a field U^* gives a contribution to the one-particle part of the Hamiltonian. Contrary to this approach, we use the aforementioned translation-symmetry-breaking term h' [7] in $H = H_0 + H_B + h'$,

$$h' = \sum_{k\neq 0}\frac{G_{k,0}}{V}(b_k^+ b_0^* b_0^2 + b_k b_0 b_0^{*2}), \qquad (5)$$

$$G_{k,0} = \langle k,0|G|0,0\rangle, \quad h = H_B + h',$$

so that h' gives a contribution to the interaction between atoms. It is clear that the Hamiltonian H

represents broken translation and gauge symmetries of a system. Still, the Hamiltonian (5) conserves the "quasiclassical" number of particles

$$m = \{H, |b_0|^2\} + i\left[H, \sum_{k\neq 0} b_k^+ b_k\right] = 0.$$

Therefore, we can consider the partition function with "quasiclassical" constraint

$$Q = \mathrm{tr}\left(e^{-\beta H}\delta_{N,m}\right) = \int d\rho \prod_{k\neq 0}\int Db_k^* Db_k$$

$$\times \int_{-\pi}^{\pi} dy \exp\left[iy(|b_0|^2 - N) - \beta\frac{g_0}{2V}|b_0|^4 + \Phi\right].$$

The Gaussian integral over noncondensate variables b_k, b_k^* is calculated exactly following the rules as given in [8],

$$\Phi = \sum_{k\neq 0}\Phi_k, \quad \int Db_k^* Db_k \exp(\Phi_k) = \frac{\exp(A_k)}{\mathrm{Det}(-P_k)},$$

$$A_k = \int_0^{\beta} f_k^*(-P_k)^{-1} f_k dt, \quad f_k = \rho\begin{pmatrix} b_0 \\ b_0^* \end{pmatrix}\frac{G_{k,0}}{2},$$

$$\rho = \frac{|b_0|^2}{V}.$$

In these formulas, the "fast" variables b_k, b_k^* and the "slow" variables ρ are separated as in the previous section, so that now the effective action $S_{\rm eff}$ for "slow" condensate bosons in

$$Q = \int d\rho d\mu \, \exp S_{\rm eff}, \quad S_{\rm eff} = S_0 + S_B + S_h', \quad (6)$$

contains $S_{h'}$ as the term breaking translation symmetry,

$$S_{h'} = \beta V\rho^3\frac{c}{2}\int k^2\frac{|G_{k,0}|^2}{E_k}dk, \quad c^{-1} = 2\pi^2\hbar^3.$$

We suppose the nonhomogeneous factor $|G_{0,k}|^2 \ll G_k^2$ to be weak (that means $F_1 \ll F_0$):

$$F_0 = \frac{c}{2}\int k^2 \mathrm{cth}\left(\frac{\beta E_k}{4}\right)\frac{G_k^2}{E_k}dk,$$

$$F_1 = \frac{3c}{2}\int k^2\frac{|G_{k,0}|^2}{E_k}dk, \quad F_1 < 3F_0.$$

After calculations, we get

$$\frac{\rho}{\rho_0} \simeq 1 - \alpha \simeq \frac{T_\lambda}{T_\lambda^0}, \quad \alpha = \frac{2G_0F_1}{F_0^2}.$$

The ratio F_1/F_0 is the measure of the pores' influence on the Bose condensate. Taking into account the experimental value $T_\lambda|_{h_1\neq 0}(T_\lambda|_{h=0})^{-1} = 2.168 \times 2.172^{-1} \simeq 0.998$, we find α as

$$\alpha \simeq 0.002 \simeq \left(\frac{2G_0}{F_0}\right)^2\frac{F_1}{2G_0}, \quad \frac{F_1}{G_0} \sim 0.1, \quad \frac{F_1}{F_0} \sim 0.01.$$

We can consider F_1 as the deformation factor for atomic interaction near the walls of pores. Therefore, the last ratio gives the boson number with the deformed interaction near the walls of pores relative to the number of bosons with the nondeformed interaction in the middle part of pores.

3. ATOMIC TRAP DEFORMED BY THE GRAVITATIONAL FIELD

Another system that displays broken translation symmetry is a trap for atomic gases. Strictly speaking, a trap is a mesoscopic system, the properties of which depend on its size h and potential barrier U_0. The levels of a trap depend on the form of the potential. In the case of the parabolic trap for the alkali atoms, we have the trap frequency ω with the values of parameters [9]

$$h \simeq 2\text{ mm}, \quad U_0 \simeq 10^{-9}\text{ eV}, \quad \omega \simeq 10^{-13}\text{ eV}.$$

Such a trap contains some thousands of atoms and some thousands of levels.

The energy of an atom in the gravitational field mgh is of the same order as the barrier difference $U_+ - U_-$, caused by this field. Therefore, the complete picture of the atomic motion in a trap needs to include the gravitational fields into consideration [10]. The complete potential of a parabolic trap looks like

$$U(z)|_{z>0} = \begin{cases} U_0 + mgz, & \text{outside of the trap,} \\ \frac{1}{2}m\omega^2(r - r_0)^2 + mgz, & \text{inside of the trap,} \end{cases}$$

where m is the mass of an atom and z_0 is the center of the trap in the z-direction. Using a new variable $Z \equiv z - z_0$, we get

$$U(Z) = \begin{cases} U_0 + mgZ, \quad Z < -h/2, \quad Z > h/2, \\ U_g + (m\omega^2/2)\left(Z + \Delta\right)^2, \quad -h/2 < Z < h/2. \end{cases} \quad (7)$$

It means that the shifts of initial $U(Z)$ are $\Delta = g/\omega^2$ (to the left) and $U_g = -mg^2/2\omega^2$ (down). The difference between U_+ (right) and U_- (left) in (7) is

$$U_+ - U_- = mgh, \quad U_\pm = \frac{m\omega^2}{2}\left(\Delta \pm \frac{h}{2}\right)^2 + U_g.$$

The phenomenon of a trap deformation is (a) mesoscopic and (b) nonperturbative:

$$\frac{mg}{\hbar\omega}\left(\frac{\hbar}{m\omega}\right)^{1/2} \sim 100, \; \frac{mg}{GR}\left(\frac{\hbar}{m\omega}\right)^{1/2} \sim 1, \; R = \frac{N}{V}.$$

These relations show that the gravitational field may be considered as a factor more important than the interaction G. Therefore, estimates within the framework of ideal gas theory may have to be done. The radical reconstruction of the system in terms of HFB in the quasiclassical approximation

$$\hbar\omega\beta \ll 1$$

is needed. For a $2D$ finite trap with an ideal gas of N bosons [11],

$$N = N_0 + N_1, \quad N_1 = \sum_{n\neq 0}^{k_{\max}} f_k,$$

$$\varepsilon_k = \begin{cases} \varepsilon_k^0, \ g = 0, \\ \varepsilon_k^{\pm}, \ g \neq 0, \end{cases}$$

with

$$f_k = \frac{1}{\exp[\beta(\varepsilon_k - \mu)] - 1}, \quad k = \{k_x, k_z\},$$

where we make replacements:

energies $\epsilon_k^0 \to U_0$, energies $\epsilon_k^{\pm} \to U_- \sim 10^{-8} U_0 < U_+$, $U_+ - U_- = mgh$ with the corresponding upper cutoff limit $k_{\max}$.

Here, T_c for a finite-size trap is determined by the equations $N_1(T_c) = N$ and $N_0 = 0$. In order to simplify the calculations, we have made the following substitution: *parabolic trap in the gravitational field → rectangular trap with the same potentials U_-, U_+ and the size h.*

The Bose–Einstein condensation (BEC) T_c shift of a trapped ideal gas, induced by gravitation, is determined by a series of equations. At a starting point I,

$$N_0 = 0, \quad N_1 = N = \sum_{k_x, k_z \neq 0}^{10} f[(T_c^{\pm})^*, \epsilon_{k_x k_z}^{\pm}] \quad (8)$$
$$= 4 \times 10^8,$$

$(T_c^{\pm})^* = 2.02 \times 10^{-6}$ K for the initial asymmetrical trap. Then, in position II, the same N atoms in a nondeformed trap show $T_c^0 = 5.05 \times 10^{-7}$ K,

$$N_0 = 0, \quad N = \sum_{k_x, k_z \neq 0}^{k_{\max}} f(T_c^0, \epsilon_{k_x k_z}^0) = 4 \times 10^8. \quad (9)$$

In the end, after including the initial gravitational field once more, $N_- = N - N_1^{\pm}$ atoms become nontrapped. The number of trapped atoms is

$$N_1^{\pm} = \sum_{k_x, k_z \neq 0}^{10} f(T_c^0, \epsilon_{k_x k_z}^0) = 0.96 \times 10^8 < N,$$

which has the temperature $T_c^{\pm} = 5.03 \times 10^{-7}$ K due to equations

$$N_0 = 0, \quad N_1^{\pm} = \sum_{k_x, k_z \neq 0}^{10} f(T_c^{\pm}, \epsilon_{k_x k_z}^0). \quad (10)$$

These transition temperatures

$$(T_c^{\pm})^* = 2.02 \times 10^{-6} \text{ K}$$
$$\to T_c^0 = 5.05 \times 10^{-7} \text{ K} \to T_c^{\pm} = 5.03 \times 10^{-7} \text{ K},$$

determined by Eqs. (8)–(10), may be associated with the assumed motion of the trap in space: *start on the Earth (I) → in space (II) → finish on the Earth (III)*.

For the parabolic trap, the largest shift of T_c is expected to occur just before its destruction $(\omega^2 h - 2g) \to +0$.

4. CONCLUSIONS

1. Translationally noninvariant Bose systems do not obey periodic boundary conditions in space.

2. Extra energy terms via selection rules for momentum appear due to broken translation symmetry.

3. The BEC shift of T_c is an observable effect in porous media and may be expected as a mesoscopic effect in traps for atoms.

Tunneling of atoms out of a trap seems to be a reason for decreasing the number of atoms, if $T > T_c$. Still, if $T < T_c$, a major portion of atoms belongs to the Bose condensate, so they are nearly motionless and have no time to reach the walls of the trap during the experiment (~1 min).

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