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Regularized Green's Function and Group of Reflections in a Cavity with Triangular Cross Section*

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Abstract—For a certain class of triangles (with angles proportional to π/N, $N \geq 3$) we formulate the image method by making use of the group G_N generated by reflections with respect to three lines which form the triangle under consideration. A regularized Green's function (which is employed in Casimir energy calculations) is obtained by classification of subgroups of G_N and corresponding fixed points in the triangle.
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1. INTRODUCTION

We would like to present a part of the program which is aimed at the calculation of Casimir energies for new nontrivial geometries [1, 2]. For most of the cases, one needs the Green's functions for the given boundaries. If the geometry permits, it is convenient to employ the method of images in the construction of the Green's function. The well-known example is the original parallel plate geometry [3]. Thinking that the extension of the image method to other geometries would be of obvious convenience, we present a calculation for a class of triangular geometries.

We first study the group generated by the reflections from the boundaries of the triangle and then construct the Dirichlet Green's function for the massless scalar field. The next important thing to do is the regularization of the Green's function, which is necessary for the renormalization of the vacuum energy. The nice observation we have is that the terms to be subtracted from the Green's function can be classified as the terms belonging to the stability subgroups.

2. GROUP OF REFLECTIONS IN A CLASS OF TRIANGLES

For $N = 3, 4, 5, \ldots$ and $k = 1, 2, \ldots N-2$, consider the triangles $\triangle_k^N$ in the x^1x^2 plane formed by the lines

$$L_1 = \{\mathbf{x} \in R^2 : x^2 = 0\}, \tag{1}$$

$$L_2 = \{\mathbf{x} \in R^2 : x^2 = x^1 \tan \upsilon\}, \tag{2}$$

$$L_3 = \{\mathbf{x} \in R^2 : x^2 = (b - x^1)\tan(k\upsilon)\}, \tag{3}$$

where b is the length of the side lying on the line L_1 and $\upsilon = \pi/N$ is the angle between L_1 and L_2.

The actions of the reflections Q_j with respect to the lines L_j, $j = 1, 2, 3$, on the vector

$$\mathbf{x} = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \tag{4}$$

are given by

$$Q_1\mathbf{x} = p\mathbf{x}, \quad Q_2\mathbf{x} = rp\mathbf{x}, \tag{5}$$
$$Q_3\mathbf{x} = pr^k\mathbf{x} + \mathbf{x}_0,$$

where

$$r = \begin{pmatrix} \cos 2\upsilon & -\sin 2\upsilon \\ \sin 2\upsilon & \cos 2\upsilon \end{pmatrix}, \tag{6}$$
$$p = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{x}_0 = (1 - pr^k)\begin{pmatrix} b \\ 0 \end{pmatrix}.$$

Denote by G_N the group generated by these reflections. G_N is one free group with relations. Relations between the elements Q_1, Q_2, and Q_3 can be obtained from (5) and from the properties of the rotation r and reflection p operators:

$$r^N = 1, \quad p^2 = 1, \quad pr = r^{N-1}p, \tag{7}$$
$$r^k\mathbf{x}_0 = -p\mathbf{x}_0.$$

Some of the obvious relations are

$$Q_j^2 = 1, \quad (Q_1Q_2)^N = 1, \tag{8}$$

from which we conclude that the reflections Q_1 and Q_2 generate the finite subgroup

$$D_N = \{r^s, pr^s, s = 0, 1, \ldots, N-1\} \tag{9}$$

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which is the dihedral group of dimension $2N$. Consider the linear space V_N which consists of the vectors

$$\xi = \sum_{s=0}^{N-1} n_s \mathbf{x}_s, \tag{10}$$

where n_s are integers and

$$\mathbf{x}_s = r^s \mathbf{x}_0. \tag{11}$$

The equalities

$$r\mathbf{x}_s = \mathbf{x}_{s+1}, \quad p\mathbf{x}_s = \mathbf{x}_{N-s+k} \tag{12}$$

imply that D_N is the automorphism group of the linear space V_N. The action of D_N is given in the natural way:

$$\pi(q)\xi = q\xi; \quad q \in D_N. \tag{13}$$

Since V_N is a vector space over integer numbers, unlike the spaces over real numbers, the dimension $|V_N|$ is not necessarily equal to the dimension of the vectors $\mathbf{x}_s$. It may be larger—that is, in our case, it may be greater than two. For example, the dimensions of V_5 and V_8 are four, while the dimensions of V_3, V_6, and V_4 are two. (For a detailed discussion of this problem, see [2].)

The group G_N is the subgroup of the semidirect product group $D_N * V_N$. In fact, for any element $g \in G_N$, one can find the pair of elements $q \in D_N$ and $\xi \in V_N$ as

$$g\mathbf{x} = q\mathbf{x} + \xi \equiv (q, \xi)\mathbf{x}; \quad \mathbf{x} \in R^2. \tag{14}$$

In particular,

$$Q_1 = (p, 0), \quad Q_2 = (rp, 0), \quad Q_3 = (pr^k, \mathbf{x}_0). \tag{15}$$

G_N contains two subgroups: D_N and the one generated by Q_3. Since V_N does not contain invariant subspaces with respect to (13), we conclude that there is no subgroup in the semidirect product group which contains D_N and the group generated by Q_3 simultaneously.

3. CONSTRUCTION OF THE GREEN'S FUNCTION IN THE TRIANGLES WITHOUT OBTUSE ANGLES

Consider the representation of the group G_N in the space of functions on the four-dimensional Minkowski space

$$T(g)f(x) = f(gx). \tag{16}$$

Here, the action of the group G_N is given by the substitution $\mathbf{x} \to x, \xi \to \xi, p \to P, r \to R$, where

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{17}$$

are 4×4 matrices, and

$$\xi = \begin{pmatrix} 0 \\ \xi \\ 0 \end{pmatrix}, \quad x = \begin{pmatrix} x^0 \\ \mathbf{x} \\ x^3 \end{pmatrix} \tag{18}$$

are four-dimensional column vectors.

Using (15), one can verify that the operator

$$\mathbf{O} = \sum_{n \in Z} \sum_{s=0}^{N-1} (T((R^s, \xi)) - T((PR^s, \xi))) \tag{19}$$

satisfies the following property:

$$T(Q_j)\mathbf{O} = -\mathbf{O}. \tag{20}$$

In (19), $n = (n_0, n_1, \ldots, n_{|V_N|-1})$ is multi-index and

$$\xi = \sum_{t=0}^{|V_N|-1} n_t x_t, \tag{21}$$

where

$$x_s = \begin{pmatrix} x^0 \\ \mathbf{x}_s \\ x^3 \end{pmatrix} \tag{22}$$

and $\mathbf{x}_s$ are the base vectors described in the previous section.

It is obvious, that if we define a function $\mathbf{O}f(x)$, it must vanish on the lines L_j of reflections Q_j, a fact that we make use of in the construction of the Green's function inside the triangle $\triangle_k^N$, satisfying the Dirichlet boundary conditions. Since the operator $\mathbf{O}$ commutes with the Klein–Gordon operator (which is invariant under translations, rotations and reflections), the function

$$K(x, x') \equiv \mathbf{O}G(x, x') \tag{23}$$
$$= \sum_{n \in Z} \sum_{s=0}^{N-1} (G(R^s x + \xi, x') - G(PR^s x + \xi, x'))$$

satisfies the equation

$$\eta^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} K(x, x') = \mathbf{O}\delta(x - x') \tag{24}$$

for any $x, x' \in M^2 \times \triangle_k^N$, $M^2 = \{(x_0, x_3)\}$; i.e., the two-dimensional Minkowski space, and the boundary condition

$$K(x, x')|_{x \in M^2 \times \partial\triangle_k^N} = 0, \tag{25}$$

where $\partial\triangle_k^N$ is the boundary of the triangle $\triangle_k^N$. G is the Green's function in the Minkowski space with the metric $\eta = \mathrm{diag}(1, -1, -1, -1)$:

$$G(x, x') = -\frac{1}{4\pi^2} \frac{1}{|x - x'|^2}. \tag{26}$$

The function $K(x,x')$ is the Green's function if the right-hand side of (24) is a delta function

$$\mathbf{O}\delta(x-x') = \delta(x-x') \tag{27}$$

for any $x, x' \in M^2 \times \triangle_k^N$. The above condition implies that for any $(q,\xi) \neq (1,0)$ and for any $\mathbf{x}, \mathbf{x}' \in \triangle_k^N$,

$$\delta(q\mathbf{x} + \xi - \mathbf{x}') = 0 \tag{28}$$

must be satisfied. In other words, any points inside the triangle should be representative of different orbits of the coset space R^2/G_N. The orbits of the coset space R^2/D_N are

$$[\mathbf{x}] = \{r^s\mathbf{x}, pr^s\mathbf{x} : \ s = 0, \ldots, N-1\}. \tag{29}$$

It is clear that we can identify this coset space with region X between two lines L_1 and L_2 including the boundaries. For any orbit in R^2/D_N, there exists a unique representative in X. Since the group G_N is generated by the elements of D_N and Q_3, the problem of constructing the coset space R^2/G_N reduces to finding the subspaces Y of X such that the reflection Q_3 maps Y into X. Consider the area between three lines L_j, which is the triangle under consideration. The previous condition implies that the two angles $k\upsilon$ and $s\upsilon$ of the triangle between the lines L_1, L_3 and L_2, L_3 must be less than or equal to $\pi/2$. The restrictions

$$k\upsilon \leq \frac{\pi}{2}, \quad s\upsilon \equiv \pi - (k+1)\upsilon \leq \frac{\pi}{2} \tag{30}$$

with solutions

$$k = \begin{cases} \dfrac{N}{2}, & \text{for even } N, \\ \dfrac{N-1}{2}, & \text{for odd } N \end{cases} \tag{31}$$

imply that, for triangles without an obtuse angle, the function $K(x,x')$ in (23) is indeed the Green's function. Note that Eqs. (30) have also been solved with $k = (N-2)/2$ for even N. In this case, $s = N/2$. For $k = N/2$, we have $s = (N-2)/2$. Therefore, this solution is congruent to the previous one; that is, $\Delta_{N/2}^N$ goes to $\Delta_{(N-2)/2}^N$ when the length b goes to $b\cos\upsilon$.

4. REGULARIZATION OF THE GREEN'S FUNCTION

In polygonal regions, there are three types of singular terms that have to be subtracted to obtain the regularized Green's function: free space term and surface and vertex terms.

Inspecting (23) we observe that the term

$$T(g)G(x,x') = G(gx,x') \tag{32}$$

leads singularity whenever $gx = x$; that is, the singularities arise at the elements of the group G_N which leave the points fixed. The regularization problem is then reduced to the classification of the points of the region and their stability subgroups:

(i) The identity element (which is the trivial subgroup) leaves all points fixed. The term $T((1,0))G(x,x')$ in (23) therefore gives the volume singularity and is the free Green's function.

(ii) The points on the line L_j are left fixed by the reflection Q_j. The group generated by Q_j is then the stability subgroup for the line L_j. Since the identity element of the two-dimensional reflection group is already employed in the volume regularization, the surface singularity terms in (23) are

$$K_S(x,x') = \sum_{j=1}^{3} T(Q_j)G(x,x'). \tag{33}$$

(iii) To discuss the vertex singularities, let us first consider the vertex at the intersection point of the lines L_1 and L_2. The N-dimensional subgroup generated by the element Q_1Q_2 is the stability subgroup of this vertex. The divergence term at the vertex we consider is

$$K_{L_1L_2}(x,x') = \sum_{j=1}^{N-1} T((Q_1Q_2)^j)G(x,x'). \tag{34}$$

The element Q_1Q_3 generates the stability subgroup of the vertex at the intersection point of the lines L_1 and L_3. Due to restriction (31) and $Q_1Q_3 = (r^k, -r^k\mathbf{x}_0)$, we conclude that the dimension of this group is two for even N and N for odd N. Therefore, we have

$$K_{L_1L_3}(x,x') = \sum_{j=1}^{L-1} T((Q_1Q_3)^j)G(x,x'), \tag{35}$$

where L is the dimension of the stability group, that is, $L = 2$ if N is even and $L = N$ if N is odd.

Finally, let us consider the third vertex which is the intersection point of the lines L_2 and L_3. The stability group of this point is generated by the element Q_2Q_3. One can verify that the dimension of this group is

$$D = \begin{pmatrix} N, & \text{for even } N/2, \\ N/2, & \text{for odd } N/2, \\ N, & \text{for odd } N \end{pmatrix} \tag{36}$$

and the corresponding singular line terms are

$$K_{L_2L_3}(x,x') = \sum_{j=1}^{D-1} T((Q_2Q_3)^j)G(x,x'). \tag{37}$$

Collecting all the above terms, we arrive at

$$K_L(x,x') = \sum_{j=1}^{N-1}(T((Q_1Q_2)^j) + T((Q_1Q_3)^j) + T((Q_2Q_3)^j))G(x,x') \quad (38)$$

for odd N;

$$K_L(x,x') = \left(T(Q_1Q_3) + \sum_{j=1}^{N-1}(T((Q_1Q_2)^j) + T((Q_2Q_3)^j))\right)G(x,x') \quad (39)$$

for even $N/2$; and

$$K_L(x,x') = \left(T(Q_1Q_3) + \sum_{j=1}^{N-1} T((Q_1Q_2)^j) + \sum_{j=1}^{N/2-1} T((Q_2Q_3)^j)\right)G(x,x') \quad (40)$$

for odd $N/2$. Subtracting all divergences from (23), we obtain the regularized Green's function

$$K_r(x,x') = K(x,x') - G(x,x') - K_S(x,x') - K_L(x,x'). \quad (41)$$

At this point, we would like to emphasize that, if the method of images is applicable to a geometry, the stability group classification is a quite reliable approach to the regularization.

5. CONCLUSION

The regularized Green's function obtained in the previous section is employed in the well-known coincidence limit formula in deriving the Casimir energy for the massless scalar field [2].

We hope that the technique we presented can be generalized to other polygonal regions, and then even to smooth boundaries in some suitable limits.

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Algebraic Approach to Thermodynamic Properties of Diatomic Molecules*

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Abstract—A simple model extending Lie algebraic techniques is applied to the analysis of thermodynamic vibrational properties of diatomic molecules. Local anharmonic effects are described by means of a Morse-like potential and the corresponding anharmonic bosons are associated with the $SU(2)$ algebra. The total number of anharmonic bosons, fixed by the potential shape, is determined for a large number of diatomic molecules. A vibrational high-temperature partition function and the related thermodynamic functions are derived and studied in terms of the parameters of the model. The idea of a critical temperature is introduced in relation to the specific heat. A physical interpretation in terms of a quantum deformation associated with the model is given. © *2005 Pleiades Publishing, Inc.*

1. INTRODUCTION

The algebraic approach has been used successfully in molecular physics and has led to new insights into the nature of complex many-body systems [1–3]. In the framework of the algebraic method, the Hamiltonian of a given system is written as an algebraic operator using the generators of the appropriate Lie algebra. All other operations in the model are algebraic operators, unlike the differential operators in the standard wave mechanics. The technical advantage of the algebraic approach is the comparative ease of the algebraic operations. Equally important, however, is the conclusion derived from comparison with experiment that there are generic forms of symmetry-adapted algebraic Hamiltonians and that entire classes of molecules can be described by these Hamiltonians, where the parameters vary in a systematic fashion for different molecules. In its initial stage of development [4–13], the algebraic approach sought to show why and how it provides a framework for the understanding of large-amplitude anharmonic motion. The anharmonicities are introduced by means of dynamical groups that correspond to anharmonic potentials constraining the total number of levels to a finite value. Later on, the $SU(2)$ models [14–20] combined Lie algebraic techniques, describing the interatomic interactions, with discrete symmetry techniques associated with the local symmetry of the molecules. Recently, a clear-cut connection could be established between the Morse–$SU(2)$ approach and the traditional potential energy surface methods [11–13].

The algebraic anharmonic model has been developed to analyze molecular vibrational spectra [2–20]. It provides a systematic procedure for studying vibrational excitations in a simple form by describing the stretching and bending modes in a unified scheme based on $SU(2)$ algebras, which incorporate the anharmonicity at the local level.

The deformation of the harmonic oscillator algebra, associated with the Morse potential, has been derived using a quantum analog for the anharmonic oscillator [21]. We have described the anharmonic vibrations as anharmonic q bosons using a first-order expansion of the quantum deformation. We have thus proposed a physical interpretation of quantum deformation in the framework of the algebraic model.

The aim of this paper is to apply the algebraic approach to the vibrational high-temperature thermodynamics of diatomic molecules and to obtain the basic thermodynamic functions in terms of the parameters of the algebraic model. This paper can be considered as a first step in the direction of incorporating anharmonicity and the finite number of bound vibrational states into the thermodynamic description of molecular systems. The rest of the paper is organized as follows. In Section 2, we review the theory of an algebraic anharmonic model based on the Morse potential and calculate the maximal number of bosons per electronic state for a large number of diatomic molecules. In Section 3, we derive a Morse-like vibrational partition function for high temperature and study its properties. In Section 4, the vibrational

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partition function is used to derive the basic thermodynamic functions, such as the mean vibrational energy, specific heat, and free energy. The idea of critical temperature is introduced in relation to the specific heat. In Section 5, the mean number of anharmonic bosons is obtained. The concept of maximal temperature of the anharmonic vibrations is discussed. The q-bosonic deformation of first order is considered. It is shown that this quantum deformation is related to the shape of the anharmonic potential well and the fixed number of anharmonic bosons. The results are applied to the ground electronic state of the diatomic molecule $^1H^{35}Cl$. These results must be combined with the translational and rotational thermodynamic functions in order to compare with experiment, as discussed in [22].

2. ALGEBRAIC MORSE HAMILTONIAN

The algebraic model [1, 2] exploits the isomorphism between the $SU(2)$ algebra and the one-dimensional Morse oscillator:

$$\mathcal{H} = -\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + D(1-e^{-x/d})^2, \qquad (1)$$

where D is the depth of the potential well, d is its width, x is the displacement from the equilibrium, and μ is the reduced mass of the oscillator. The one-dimensional Morse Hamiltonian can be written in terms of the generators of $SU(2)$:

$$\mathcal{H}_M = \frac{A}{4}\left(\hat{N}^2 - 4\hat{J}_Z^2\right) = \frac{A}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+ - \hat{N}), \qquad (2)$$

where A is a constant dependent on the parameters of the Morse potential. An essential difference between the Morse potential and the harmonic potential is that the eigenstates in the Morse potential well are bound and the total number of bosons, N, is fixed by the potential shape. The eigenstates $|[N], v\rangle$ correspond to the $U(2) \supset SU(2)$ symmetry-adapted basis, where v is the number of quanta in the oscillator, $v = 1, 2, \ldots, [N/2]$. The maximal number of quanta per oscillator is $N_0 = [N/2]$, where $[N/2]$ is the largest integer less than or equal to $(N/2)$.

The anharmonic effects can be described by anharmonic boson operators [2],

$$\hat{b} = \frac{\hat{J}_+}{\sqrt{N}}, \quad \hat{b}^\dagger = \frac{\hat{J}_-}{\sqrt{N}}, \quad \hat{v} = \frac{\hat{N}}{2} - \hat{J}_z, \qquad (3)$$

where $\hat{v}$ is the Morse phonon operator with an eigenvalue v. The operators satisfy the commutation relations

$$\left[\hat{b}, \hat{v}\right] = \hat{b}, \quad \left[\hat{b}^\dagger, \hat{v}\right] = -\hat{b}^\dagger, \quad \left[\hat{b}, \hat{b}^\dagger\right] = 1 - \frac{2\hat{v}}{N}. \qquad (4)$$

The harmonic limit is obtained when $N \to \infty$, in which case $\left[\hat{b}, \hat{b}^\dagger\right] \to 1$, giving the usual boson commutation relations.

The Morse Hamiltonian can be written in terms of the operators $\hat{b}$ and $\hat{b}^\dagger$:

$$\mathcal{H}_M \sim \frac{1}{2}\left(\hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b}\right), \qquad (5)$$

which corresponds to vibrational energies

$$\varepsilon_v = \hbar\omega_0\left(v + \frac{1}{2} - \frac{v^2}{N}\right), \qquad (6)$$
$$v = 1, 2, \ldots, \left[\frac{N}{2}\right],$$

where ω_0 is the harmonic oscillator frequency.

The spectrum of the Morse potential leads to a deformation of the harmonic oscillator algebra. A more detailed relationship between the Morse coordinates and momenta and the $SU(2)$ generators can be derived through a comparison of their matrix elements [11] and through the derivation of raising and lowering operators for the Morse potential [12]. Note that, for an infinite potential depth, $N \to \infty$, the Morse potential cannot be distinguished from the harmonic potential.

The value of N depends on the depth D and the width d of the Morse potential well [1, 2, 14]:

$$N = \left(\frac{8\mu D d^2}{\hbar^2}\right)^{1/2} - 1. \qquad (7)$$

By comparing the eigenvalues (6) with the phenomenological Dunham expansion for the vibrational energy cut to the quadratic term, relations between the parameters A and N and the usual harmonic constant, ω_e, and anharmonic constant, $x_e\omega_e$, used in spectroscopy [22] are obtained [4, 5]:

$$\omega_e = A(N+1) = \hbar\left(\frac{2D}{\mu d^2}\right)^{1/2}, \qquad (8)$$

$$x_e\omega_e = A = \frac{\hbar^2}{2d^2D}. \qquad (9)$$

Thus, the total number of bosons N in the algebraic anharmonic model is obtained in terms of the experimental anharmonic constants:

$$N = \frac{1}{x_e} - 1, \qquad (10)$$

where $x_e = \omega_e x_e/\omega_e$. This result is equivalent to (7). In the model, N is an integer, and in what follows, its values are determined by $[1/x_e - 1]$.

We have determined the values of N and N_0 for oscillators corresponding to the ground electronic

Maximal number of anharmonic bosons for electronic states of diatomic molecules

Molecule	State	$x_e = \omega_e x_e/\omega_e$	$N = [1/x_e - 1]$	$N_0 = [N/2]$
$^{27}Al^{1}H$	$X^1\Sigma^+$	1.73217×10^{-2}	56	28
$^{9}Be^{16}O$	$A^1\Pi$	7.3538×10^{-3}	134	67
	$X^1\Sigma^+$	7.95369×10^{-3}	124	62
$^{209}Bi^{35}Cl$	X	3.11688×10^{-3}	319	159
$^{11}B^{16}O$	$A^2\Pi_i$	8.84985×10^{-3}	111	55
	$X^2\Sigma^+$	6.24204×10^{-3}	159	79
$Ca^{35}Cl$	$A^2\Pi$	3.17895×10^{-3}	313	156
$^{12}C^{16}O$	$X^1\Sigma^+$	6.17701×10^{-3}	160	80
$^{1}H_2$	$d^3\Pi_u$	2.79434×10^{-2}	34	17
	$a^3\Sigma_u^+$	2.68873×10^{-2}	36	18
	$X^1\Sigma_g^+$	2.68452×10^{-2}	36	18
$^{1}H^{3}H$	$d^3\Pi_u$	2.24267×10^{-2}	43.0	21
	$a^3\Sigma_g^+$	2.19751×10^{-2}	44.0	22
	$X^1\Sigma_g^+$	2.42718×10^{-2}	40	20
$^{1}H^{35}Cl$	$X^1\Sigma^+$	1.74095×10^{-2}	56	28
$^{1}H^{19}F$	$X^1\Sigma^+$	2.1764×10^{-2}	44	22
$Hg^{2}H$	$X^2\Sigma$	5.01733×10^{-2}	18	9
$^{39}K_2$	$X^1\Sigma_g^+$	3.82124×10^{-3}	260	130
KBr	$X^1\Sigma^+$	3.0303×10^{-3}	329	164
KCl	$X^1\Sigma^+$	3.21429×10^{-3}	310	155
$K^{127}I$	$X^1\Sigma^+$	3.30189×10^{-3}	301	150
$^{7}Li_2$	$X^1\Sigma_g^+$	7.37558×10^{-3}	134	67
$Li^{127}I$	$X^1\Sigma^+$	3.33333×10^{-3}	299	149
$^{14}N_2$	$X^1\Sigma_g^+$	6.12644×10^{-3}	162	81
$^{23}NaBr$	$X^1\Sigma^+$	3.65079×10^{-3}	272	136
$^{23}Na^{127}I$	$X^1\Sigma^+$	2.62238×10^{-3}	380	190
$^{16}O_2$	$X^3\Sigma_g^-$	7.63939×10^{-3}	129	64
$^{28}Si^{16}O$	$X^1\Sigma^+$	4.86864×10^{-3}	204	102
$Sr^{19}F$	$A^2\Pi$	4.4673×10^{-3}	222	111
	$X^2\Sigma^+$	1.40938×10^{-2}	69	34
$Zn^{1}H$	$X^2\Sigma^+$	0.0343	28	14

state or excited electronic state of a large number of diatomic molecules, using the experimental values of ω_e and $x_e\omega_e$ published in [22]. Some of the results for selected molecules are displayed in the table.

3. VIBRATIONAL PARTITION FUNCTION

The vibrational partition function of a diatomic anharmonic molecule is

$$Z_N = \sum_{v=0}^{[N/2]} e^{-\beta\varepsilon_v}, \tag{11}$$

where $\beta = 1/(k_B T)$, the vibrational energies ε_v are given by Eq. (6), N is the fixed total number of anharmonic bosons, and we use the notation $N_0 = [N/2]$ for the maximal number of quanta per oscillator as explained in the previous section. Introducing new parameters, $\alpha = \beta\hbar\omega_0/2$ and $l = N_0 - v$, the exact value of the vibrational partition function can be written as

$$Z_N = e^{-\alpha} \sum_{l=0}^{N_0} e^{-\frac{\alpha}{N_0}\left(N_0^2 - l^2\right)}. \qquad (12)$$

When $T \to \infty$, $\alpha \to 0$, the partition function is

$$Z_N(\alpha \to 0) \to N_0 + 1. \qquad (13)$$

At high, but finite, temperatures T, for large N_0 and $\alpha \leq 1$, the sum can be replaced by the integral

$$Z_N = \sqrt{\frac{N_0}{\alpha}} e^{-\alpha(N_0+1)} \int_0^{\sqrt{\alpha N_0}} e^{s^2} ds, \qquad (14)$$

where $s = \sqrt{\alpha l/N_0}$. This integral can be evaluated exactly in terms of the error function, erf $i\left(\sqrt{\alpha N_0}\right)$ (as defined in [23]):

$$Z_N = \frac{1}{2}\sqrt{\frac{N_0\pi}{\alpha}} e^{-\alpha(N_0+1)} \mathrm{erf}\, i\left(\sqrt{\alpha N_0}\right). \qquad (15)$$

Equation (15) represents the high-temperature value of the vibrational partition function in the Morse-like spectrum [24–26]. The partition function is expressed in terms of the parameters of the algebraic model N_0 and α. The dependence on the temperature is given by α,

$$\alpha = \frac{\hbar\omega_0}{2k_B T} = \frac{\Theta}{2T}, \qquad (16)$$

where $\Theta = \hbar\omega_0/k_B$ is the usual characteristic vibrational temperature of the molecule. The contributions of the anharmonic vibrations are essential in the high-temperature region for $T \geq \Theta$, where $T = \Theta$ corresponds to $\alpha = 0.5$.

When $N_0 \to \infty$, the harmonic limit of the model is obtained:

$$Z_\infty \sim \frac{N_0 e^{-\alpha}}{2\alpha N_0 - 1} \sim \frac{e^{-\alpha}}{2\alpha} = \frac{T}{\Theta} e^{-\frac{\Theta}{2T}}, \qquad (17)$$

which coincides with the harmonic vibrational partition function of a diatomic molecule at high temperatures. The expression for the partition function (15) can be generalized for polyatomic molecules by combining the present results with the use of a local-mode model where each interatomic potential is of the Morse form [13].

We take the diatomic molecule $^1H^{35}Cl$ as an example. The total number of anharmonic bosons for the oscillator corresponding to the ground electronic state $X^1\Sigma^+$, described with a Morse potential, is $N = 56$, and the total number of quanta in the oscillator is $N_0 = 28$. The depth of the Morse potential is $D = 5.32$ eV, and the width is $d = 0.57 \times 10^{-10}$ m. The characteristic vibrational temperature of the molecule (as described, for example, in [27]) is $\Theta = 4300$ K.

Substituting the value of $N_0 = 28$ into Eq. (15), we can calculate the partition function, Z_{56}, for the ground electronic state of the molecule $^1H^{35}Cl$ as a function of the parameter α. The graph in Fig. 1 represents the partition function Z_{56} given by Eq. (15) for the values of the parameter α between 0 and 1 (solid curve). The essential contributions of anharmonic vibrations to the partition function are at high temperatures $T \geq \Theta$, which corresponds to $\alpha \leq 0.5$. The exact partition function from Eq. (12) is given for comparison (dash-dotted curve). It is clear that, in the region $0 \leq \alpha \leq 1$, the integral approximation is in excellent agreement with the exact result and does not change the value and appearance of the partition function. The comparison between the function Z_{56} (solid curve) and the harmonic limit Z_∞ (dashed curve) is given in Fig. 2. The finiteness of Z_{56} in the high-T limit is linked, of course, with the finite number of states in the Morse potential. A more realistic description in the high-T region requires the introduction of the continuum states of the Morse potential.

An algebraic approach has been used in [28] to study the thermodynamic properties of molecules. However, the partition function in [28] uses an approximation of the classical density of states, while we have derived an explicit function in terms of the parameters of the algebraic model.

4. THERMODYNAMIC VIBRATIONAL FUNCTIONS

Having the partition function Z_N in terms of the parameters of the algebraic model, we are now in a position to derive the corresponding thermodynamic functions.

4.1. Mean Vibrational Energy

The mean vibrational energy is given by

$$U_N = -\frac{\partial}{\partial\beta}\ln Z_N = -\frac{\hbar\omega_0}{2Z_N}\frac{\partial Z_N}{\partial\alpha}. \qquad (18)$$

Taking into account that

$$\frac{\partial Z_N}{\partial\alpha} = -\frac{Z_N}{2\alpha} - (N_0+1)\, Z_N + \frac{N_0 e^{-\alpha}}{2\alpha}, \qquad (19)$$

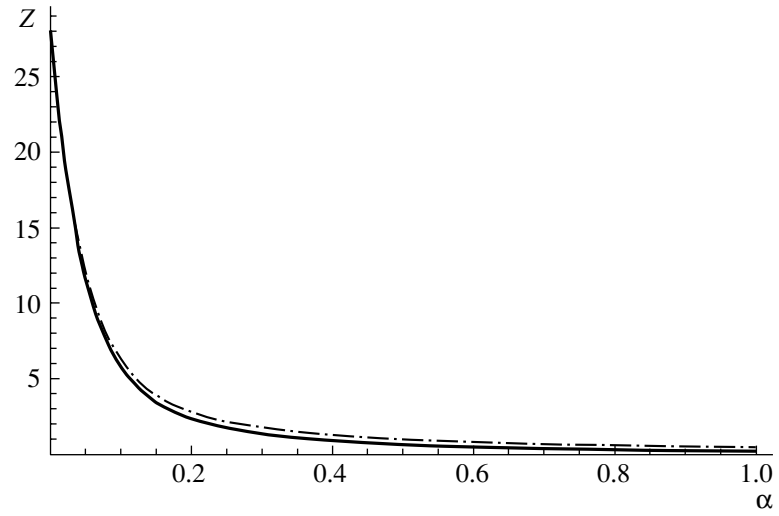

Fig. 1. Vibrational partition function Z_{56} of $^1H^{35}Cl$ as a function of α. The dash-dotted curve represents the exact representation.

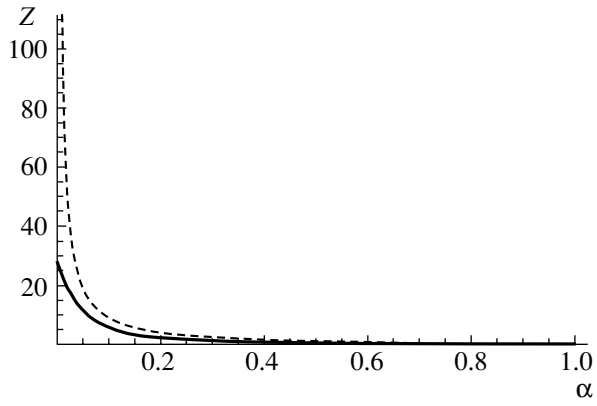

Fig. 2. Vibrational partition function Z_{56} (solid curve) of $^1H^{35}Cl$ and the harmonic limit Z_∞ (dashed curve) as functions of α.

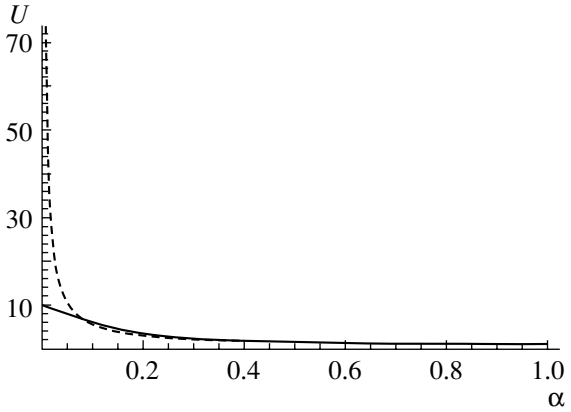

Fig. 3. Mean vibrational energy $U_{56}/\hbar\omega_0$ of $^1H^{35}Cl$ as a function of α. The dashed curve represents the harmonic limit $U_\infty/\hbar\omega_0$.

we obtain the following expression for the mean vibrational energy in terms of the partition function Z_N:

$$U_N = \frac{\hbar\omega_0}{2}\left(1 + N_0 + \frac{1}{2\alpha} - \frac{N_0 e^{-\alpha}}{2\alpha Z_N}\right). \qquad (20)$$

Substituting Z_N (15) gives the following expression for the mean energy U_N in terms of the parameter α:

$$U_N = \frac{\hbar\omega_0}{2} \qquad (21)$$
$$\times \left(1 + N_0 + \frac{1}{2\alpha} - \sqrt{\frac{N_0}{\alpha\pi}}\frac{e^{\alpha N_0}}{\operatorname{erf} i\left(\sqrt{\alpha N_0}\right)}\right).$$

The harmonic limit is obtained from Eq. (20), when $N_0 \to \infty$ and Z_N is given by (17):

$$U_\infty \sim \frac{\hbar\omega_0}{2}\left(1 + \frac{1}{\alpha}\right) = \frac{\hbar\omega_0}{2} + k_B T. \qquad (22)$$

This is the classical mean energy of a diatomic molecule at high temperatures. When $T \to \infty$, $\alpha \to 0$,

$$U_N(\alpha \to 0) \to \hbar\omega_0 \frac{N_0 + 1}{2}. \qquad (23)$$

The graph in Fig. 3 represents the mean vibrational energy, $U_{56}/\hbar\omega_0$, of the ground electronic state of the molecule $^1H^{35}Cl$ for values of α, $0 < \alpha \leq 1$. The high-temperature region corresponds to $\alpha \leq 0.5$. For comparison, the graph of the harmonic limit U_∞ (dashed curve) for $N_0 \to \infty$ is also given. As already mentioned in the previous section, the finiteness of U_N in the high-temperature limit is a result of the finite number of states in the Morse potential.

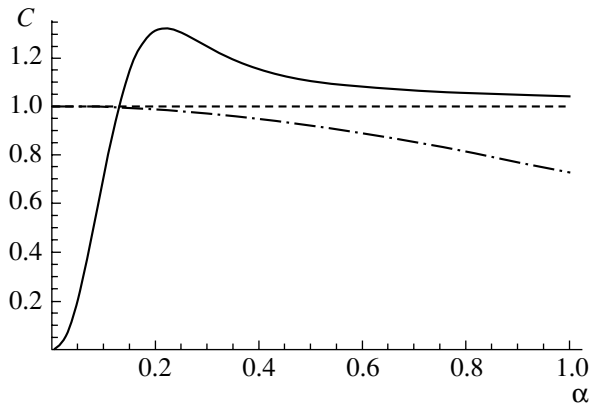

Fig. 4. Vibrational specific heat C_{56}/k_{B} (solid curve) of $^1\mathrm{H}^{35}\mathrm{Cl}$ as a function of α. For comparison, $C_{\mathrm{harm}}/k_{\mathrm{B}}$ (dash-dotted curve) and C_∞/k_{B} (dashed line) are also given.

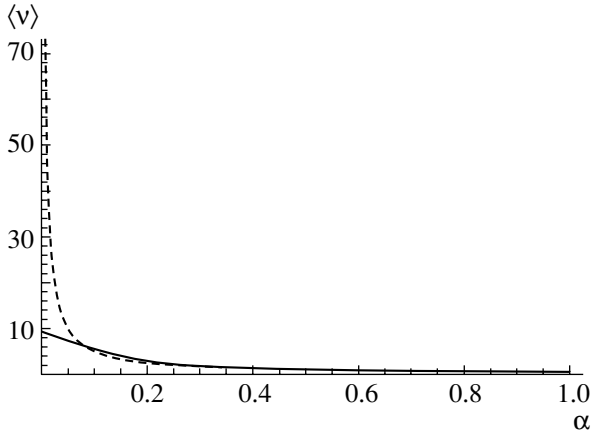

Fig. 6. Mean number of anharmonic bosons $\langle\nu_{56}\rangle$ of $^1\mathrm{H}^{35}\mathrm{Cl}$ as a function of α. The dashed curve represents the harmonic limit $\langle\nu_\infty\rangle$.

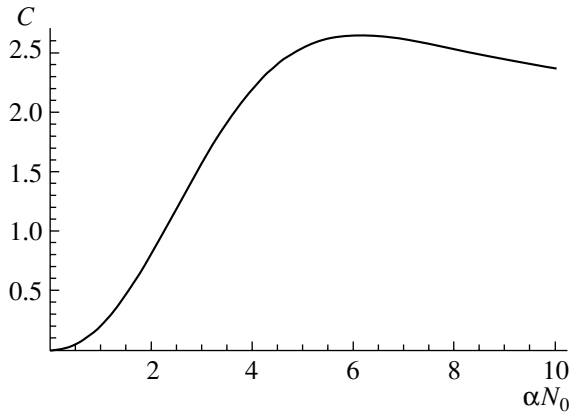

Fig. 5. Vibrational specific heat C_N/k_{B} as a function of αN_0.

4.2. Specific Heat

The vibrational part of the specific heat is

$$C_N = \frac{\partial U_N}{\partial T} = -\frac{\hbar\omega_0}{2k_{\mathrm{B}}T^2}\frac{\partial U_N}{\partial \alpha}. \qquad (24)$$

Substituting U_N (20) and using Eq. (19), we obtain

$$C_N = \frac{k_{\mathrm{B}}}{2} + \frac{k_{\mathrm{B}}N_0 e^{-\alpha}}{2Z_N}\left(\alpha N_0 - \frac{1}{2} - \frac{N_0 e^{-\alpha}}{2Z_N}\right). \qquad (25)$$

This equation represents the vibrational specific heat in the algebraic model in terms of the partition function Z_N. Substituting Z_N (15) into Eq. (25), we obtain the dependence of the specific heat C_N on the parameter α:

$$C_N = \frac{k_{\mathrm{B}}}{2} + k_{\mathrm{B}}\sqrt{\frac{\alpha N_0}{\pi}}\frac{e^{\alpha N_0}}{\mathrm{erf}\, i\left(\sqrt{\alpha N_0}\right)} \qquad (26)$$
$$\times\left(\alpha N_0 - \frac{1}{2} - \sqrt{\frac{\alpha N_0}{\pi}}\frac{e^{\alpha N_0}}{\mathrm{erf}\, i\left(\sqrt{\alpha N_0}\right)}\right).$$

It is clear from relation (26) that all anharmonic contributions to the vibrational part of the specific heat depend on the parameter α and hence on the temperature T. When $N_0 \to \infty$, the harmonic limit of the model gives the vibrational specific heat of a diatomic molecule at high temperatures, $C_\infty \sim k_{\mathrm{B}}$. When $T \to \infty$, $\alpha \to 0$, $C_N(\alpha \to 0) \to 0$.

Figure 4 represents the dependence of the vibrational specific heat, C_{56}/k_{B}, on the parameter α, $0 < \alpha \leq 1$, for the molecule $^1\mathrm{H}^{35}\mathrm{Cl}$ (solid curve). The graph of the harmonic vibrational specific heat of a diatomic molecule, $C_{\mathrm{harm}}/k_{\mathrm{B}}$, is also given (dash-dotted line), as well as the harmonic limit, C_∞/k_{B} (dashed line), where

$$C_{\mathrm{harm}} = 4k_{\mathrm{B}}\alpha^2\frac{e^{2\alpha}}{\left(e^{2\alpha}-1\right)^2}. \qquad (27)$$

The effects of the anharmonicity are strongest for values of $\alpha \leq 0.5$, where $\alpha = 0.5$ corresponds to the characteristic vibrational temperature Θ ($\Theta = 4300$ K for $^1\mathrm{H}^{35}\mathrm{Cl}$).

The graph shows an anomaly in the dependence of the vibrational specific heat on the parameter α (temperature T). The specific heat has a maximum for a value of $\alpha = \alpha_C$, which corresponds to temperature $T = T_C$. We shall call this temperature the critical temperature and the corresponding parameter

α_C the critical parameter. The anomaly of the specific heat is again a result of the finite number of states in the system. The specific heat increases with the temperature as more anharmonic bosons are excited. The maximum is reached when the latter occupy the energy state with $v = N_0 = 28$. (Note that the shape of the curve is similar to the shape associated with the Schottky anomaly of the specific heat of a two-level system [27].)

The maximal vibrational energy is given by Eq. (6) by replacing v with its maximal value N_0,

$$\varepsilon_{\max} = \hbar\omega_0 \left(\frac{N_0}{2} + \frac{1}{2} \right), \tag{28}$$

while the minimum energy is

$$\varepsilon_0 = \frac{1}{2}\hbar\omega_0. \tag{29}$$

Thus,

$$\Delta\varepsilon = \varepsilon_{\max} - \varepsilon_0 = \frac{1}{2}\hbar\omega_0 N_0. \tag{30}$$

This gives $\Delta\varepsilon = 14\hbar\omega_0 = 5.1877$ eV for the molecule ^{1}H^{35}Cl. Comparing $\Delta\varepsilon$ with the dissociation energy of the molecule DE = 4.4703 eV [29], we can conclude that, at the temperature $T = T_C$, $\Delta\varepsilon \geq$ DE, and some of the molecules might have started to dissociate, while others may still be in stable molecular states. Our model, in its present form, does not account for the effects of the dissociation. The shape of the specific-heat curve (Fig. 4) suggests the presence of a second-order phase transition at T_C, which is possibly related to the dissociation process. In addition, this simple version of the model does not yet include the contributions of the translational and rotational degrees of freedom, which at temperatures close to T_C may be substantial. The critical temperature T_C can be considered as a temperature above which the model is no longer valid in its current form and other effects take place, for example, dissociation.

We have studied the behavior of the specific heat with respect to the combined parameter αN_0. The graph of C_N/k_B as a function of αN_0 shows a similar anomaly (Fig. 5).

Solving numerically the equation $\partial C_N/\partial(\alpha N_0) = 0$ with respect to the combined parameter αN_0, we have found a root, $\alpha_C N_0 = 6.133$. Thus, the critical value α_C decreases as the number of fixed anharmonic bosons increases:

$$\alpha_C = \frac{6.133}{N_0}. \tag{31}$$

When $N_0 \to \infty$, $\alpha_C \to 0$ and the anomaly of the specific heat disappears, which is in agreement with the harmonic limit of the model.

For the ground electronic state $X^1\Sigma^+$ of the molecule ^{1}H^{35}Cl, $N_0 = 28$, which gives a value for $\alpha_C = 0.219$. Thus, the value of the critical temperature for this molecule is $T_C = 9815$ K.

4.3. Free Energy

The free vibrational energy in terms of the partition function Z_N is given by

$$F_N = -\frac{1}{\beta}\ln Z_N. \tag{32}$$

Substituting Z_N (15) gives the free vibrational energy in the algebraic model at high temperatures,

$$F_N = \frac{\hbar\omega_0}{2}\left[\frac{1}{\alpha}\ln 2 + \frac{1}{2\alpha}\ln\left(\frac{\alpha}{\pi N_0}\right) + (N_0+1) - \frac{1}{\alpha}\ln\left(\operatorname{erf} i\left(\alpha N_0\right)\right)\right]. \tag{33}$$

When $N_0 \to \infty$, using expression (17) in Eq. (32), we obtain the classical harmonic result for the free vibrational energy at very high temperatures, $F_\infty \sim k_\mathrm{B}T\ln 2$.

5. ANHARMONIC BOSONS

5.1. Mean Number of Anharmonic Bosons

The mean vibrational energy in the anharmonic model can be written in terms of mean number $\langle\nu_N\rangle$ of anharmonic quanta, each with energy $\hbar\omega_0$:

$$U_N = \hbar\omega_0\left(\langle\nu_N\rangle + \frac{1}{2}\right). \tag{34}$$

Substituting U_N (20), we obtain $\langle\nu_N\rangle$ in terms of the partition function Z_N:

$$\langle\nu_N\rangle = \frac{N_0}{2} + \frac{1}{4\alpha} - \frac{N_0 e^{-\alpha}}{4\alpha Z_N}. \tag{35}$$

Using expression (15) in Eq. (35), we obtain the high-temperature value

$$\langle\nu_N\rangle = \frac{N_0}{2} + \frac{1}{4\alpha} - \sqrt{\frac{N_0}{4\pi\alpha}}\frac{e^{\alpha N_0}}{\operatorname{erf} i\left(\sqrt{\alpha N_0}\right)}. \tag{36}$$

The harmonic limit is obtained from Eq. (36) when $N_0 \to \infty$ and Z_N is given by the expression (17),

$$\langle\nu_\infty\rangle \sim \frac{k_\mathrm{B}T}{\hbar\omega_0}. \tag{37}$$

When $T \to \infty$, $\alpha \to 0$, the mean number of anharmonic bosons is

$$\langle\nu_N\rangle(\alpha \to 0) \to \frac{N_0}{2}. \tag{38}$$

The graph of the function $\langle\nu_{56}\rangle$ for the ground electronic state of the molecule ^{1}H^{35}Cl is given in Fig. 6. The dashed curve represents the harmonic limit $\langle\nu_\infty\rangle$. The same reasons apply to the finiteness of $\langle\nu_{56}\rangle$ as those discussed for the partition function in Section 3.

5.2. Maximal Temperature

The maximal vibrational energy (28) is obtained when $v = N_0$. The maximal mean vibrational energy is given by Eq. (34) when $\nu_N = \langle \nu_N \rangle_{\max}$:

$$U_{\max} = \hbar\omega_0 \left(\langle \nu_N \rangle_{\max} + \frac{1}{2} \right). \qquad (39)$$

Comparing Eqs. (28) and (30) gives the maximal mean number of anharmonic bosons

$$\langle \nu_N \rangle_{\max} = \frac{N_0}{2}, \qquad (40)$$

which, as shown above, is for a value $\alpha \to 0$. Substituting $\langle \nu_N \rangle$ (36) and simplifying gives

$$2\sqrt{\frac{\alpha N_0}{\pi}} e^{\alpha N_0} = \operatorname{erf} i \left(\sqrt{\alpha N_0} \right). \qquad (41)$$

The numerical solution of the above equation has a root $\alpha N_0 \to 0$. As N_0 is a fixed number, this solution leads to $\alpha \to 0$, giving $T_{\max} \to \infty$. This result shows that, in practice, the system does not reach a maximal temperature, which shows the need of incorporating the continuum into the partition function (11).

5.3. Quantum Anharmonic Bosons

In [21] we have shown that the anharmonic bosons $b, b^\dagger$ in (4) can be obtained as an approximation of q bosons [30–32]. The q bosons are defined by the following commutation relations:

$$[a, a^\dagger] = q^{\hat{n}}, \quad [\hat{n}, a] = -a, \quad [\hat{n}, a^\dagger] = a^\dagger, \qquad (42)$$

where the deformation parameter q is in general a complex number [31]. As shown in [21], the anharmonic commutation relations (4) can be recovered for real values of the deformation q close to 1, $q < 1$, and a linear expansion of q in terms of a parameter p, $p \equiv 1/(1-q)$,

$$q^{\hat{n}} = 1 - \frac{\hat{n}}{p}. \qquad (43)$$

If we now substitute the approximation for $q^{\hat{n}}$ (43) into the commutation relations (42) and identify the parameter p with $N/2$, $\hat{n}$ with $\hat{v}$, and the creation and annihilation operators a, $a^\dagger$ with b, $b^\dagger$, respectively, we recover the $SU(2)$ anharmonic relations (4).

The form (4) of the $SU(2)$ commutation relations can be considered as a deformation of the usual (harmonic oscillator) commutation relations with a deformation parameter $p = N/2$. This gives a possible physical realization for the quantum deformation obtained in [21]: the quantum deformation parameter p is the fixed number N_0 of the anharmonic bosons in the oscillator. Using the relation between the fixed number of anharmonic bosons N and the characteristics of the Morse potential (7), we arrive at the conclusion that the quantum deformation is also determined by the depth, the width, and in general the shape of the Morse potential well. For the ground electronic state of the molecule $^1H^{35}Cl$, $p = 28$, which gives $q = 27/28$.

Now, substituting $N_0 = p$ into the expressions for the partition function (15), mean energy (21), specific heat (26), free energy (33), and mean number of anharmonic bosons (35), we obtain the thermodynamic properties of diatomic molecules in terms of the deformation parameter p. Equation (31) gives the relation between the quantum deformation parameter and the critical parameter α_C (critical temperature T_C). For large values of p ($q \to 1$), the classic harmonic case is restored.

6. CONCLUSIONS

We have used the vibrational energies obtained in the algebraic Morse model to study the thermodynamic properties of diatomic molecules. Using the experimental data in [22], we have calculated the number of bosons per oscillator (electronic state) for a large number of diatomic molecules. We have derived the vibrational partition function, which incorporates the effects of the anharmonicity and depends on the algebraic parameters. As the anharmonic effects are essential at high temperatures, we have obtained a high-temperature expression for the partition function, which is used to derive the important thermodynamic functions, such as mean vibrational energy, specific heat, and mean number of anharmonic bosons in terms of the parameters of the model. We have analyzed the behavior of the specific heat and introduced a critical temperature related to the limitations of the model.

We have shown that it is possible to interpret these results in terms of a quantum deformation related to the shape of the Morse potential and which is associated with the fixed total number of anharmonic bosons, so that the thermodynamic properties of diatomic molecules depend on the corresponding quantum deformation parameter. We believe that this paper constitutes a first step in the description of thermodynamical properties of diatomic molecules, which in principle can be simply generalized to polyatomic molecules. We are currently studying the introduction of the continuum into the description in order to take into consideration the transition to dissociation.

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Massive Hyper-Kähler Sigma Models and BPS Domain Walls*

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Abstract—With the non-Abelian hyper-Kähler quotient by $U(M)$ and $SU(M)$ gauge groups, we give the massive hyper-Kähler sigma models that are not toric in the $\mathcal{N}=1$ superfield formalism. The $U(M)$ quotient gives $N!/[M!(N-M)!]$ (N is the number of flavors) discrete vacua that may allow various types of domain walls, whereas the $SU(M)$ quotient gives no discrete vacua. We derive a BPS domain-wall solution in the case of $N=2$ and $M=1$ in the $U(M)$ quotient model. © *2005 Pleiades Publishing, Inc.*

1. INTRODUCTION

It is well known that topological solutions are of importance in various areas of particle physics. Recently, there was renewed interest in such solutions because of their crucial role in the brane-world scenario [1, 2]. In this brane-world scenario, our four-dimensional world is to be realized on topological objects like domain walls or brane-junctions. Supersymmetry (SUSY) can be implemented in these models, and it is actually a powerful device for constructing their topological solutions. Viewing the four-dimensional world as a domain wall, we are led to deal with SUSY theories with eight supercharges in five dimensions.

SUSY with eight supercharges is very restrictive. In theories involving only massless scalar multiplets (hypermultiplets), nontrivial interactions can only arise from nonlinearities in the kinetic term, say nonlinear sigma models (NLSMs). Prior to studying the genuine five-dimensional theories with hypermultiplets, it is instructive to start with similar SUSY theories in four dimensions, i.e., $\mathcal{N}=2$, $d=4$ theories. The analysis of the four-dimensional theory could then be of help in studying the brane-world scenario based on SUSY theories in higher dimensions [3].

With regard to rigid $\mathcal{N}=2$ SUSY, the target manifold of the hypermultiplet $d=4$ NLSMs must be Hyper-Kähler (HK) [4]. In these theories, the scalar potential can be obtained only if the hypermultiplets acquire masses by the Scherk–Schwarz mechanism [5] because of the appearance of central charges in the $\mathcal{N}=2$ Poincaré superalgebra [6]. The NLSMs with the scalar potential in $\mathcal{N}=2$ theories are called massive HK NLSMs.

A large class of HK manifold is given by toric HK manifolds that are defined as HK manifolds of real dimension $4n$ admitting mutually commuting n Abelian tri-holomorphic isometries. In the massive HK NLSMs on toric HK manifolds, many interesting BPS solitons were constructed in the component formalism [7–10] as well as off-shell formulation [11–13]. The potential term of the massive $T^*\mathbf{C}P^{N-1}$ model which is toric comes from the mass terms of the hypermultiplets when the NLSM is constructed as the quotient by the $U(1)$ gauge group [11, 12]. We call this formulation of massive HK NLSMs "the massive HK quotient method," since the massless case is just an HK quotient found in [14, 15]. One of the advantages of our massive HK quotient is that the off-shell formulation of the SUSY NLSMs is possible [12]. Off-shell formulation is powerful to extend the models to those with other isometries and/or gauge symmetries and to those coupled with gravity, since (part of) SUSY is manifest. Any *toric* HK manifolds can be constructed using an *Abelian* HK quotient [16, 17]. Therefore, an off-shell formulation of general massive toric HK NLSMs [8] can be obtained using the massive HK quotient with the Abelian gauge theories. On the other hand, a massless HK NLSM other than the toric HK target manifolds has been obtained as a quotient using the non-Abelian gauge group by Lindström and Roček [14] for the massless case only (without potential terms).

In this paper, we discuss massive NLSMs in $\mathcal{N}=2$, $d=4$ theories and their BPS domain-wall solutions. With the HK quotient method, massive NLSMs on cotangent bundles over the Grassmann manifolds, $T^*G_{N,M}$, which are not toric, are obtained along with their generalization. These models are

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constructed in $\mathcal{N}=1$ superfield formalism. The BPS domain-wall solution is given in the simplest case, the Eguchi–Hanson target manifold [18] ($N=2$ and $M=1$). This work is based on our papers [12, 19] in which analysis by a fully off-shell $\mathcal{N}=2$ superspace (the harmonic superspace [20]) formalism is also discussed in detail.

2. MASSIVE HK QUOTIENT BY $U(M)$ GAUGE GROUP

We consider $\mathcal{N}=2$ SUSY QCD with N flavors and a $U(M)$ gauge group. In terms of $\mathcal{N}=1$ superfields, $\mathcal{N}=2$, NM hypermultiplets can be decomposed into $(N\times M)$- and $(M\times N)$-matrix chiral superfields $\Phi(x,\theta,\bar\theta)$ and $\Psi(x,\theta,\bar\theta)$, and $\mathcal{N}=2$ vector multiplets for the $U(M)$ gauge symmetry can be decomposed into $M\times M$ matrices of $\mathcal{N}=1$ vector superfields $V=V^A(x,\theta,\bar\theta)T_A$ and chiral superfields $\Sigma=\Sigma^A(x,\theta,\bar\theta)T_A$, with $M\times M$ matrices T_A ($A=1,\ldots,M$) of the fundamental representation of the generators of the $U(M)$ gauge group. In order that the vector multiplets are treated as Lagrange multipliers, we take the strong coupling limit $g\to\infty$ and drop the kinetic term. The gauge-invariant Lagrangian is given by

$$\mathcal{L}=\int d^4\theta\Big[\mathrm{tr}(\Phi^\dagger\Phi e^V)+\mathrm{tr}(\Psi\Psi^\dagger e^{-V}) - c\,\mathrm{tr}V\Big]+\left[\int d^2\theta\left(\mathrm{tr}\left\{\Sigma(\Psi\Phi-b\mathbf{1}_M)\right\} + \sum_{a=1}^{N-1} m_a\,\mathrm{tr}(\Psi H_a\Phi)\right)+\mathrm{c.c.}\right], \tag{1}$$

where we have absorbed a common mass of hypermultiplets into the field Σ and denoted m_a ($a=1,\ldots,N-1$) as complex mass parameters, and H_a are diagonal traceless matrices, interpreted as the Cartan generators of $SU(N)$ below. The electric and magnetic Fayet–Iliopoulos (FI) parameters are denoted as $c\in\mathbf{R}$ and $b\in\mathbf{C}$, respectively. Note that $U(M)$ gauge symmetry is complexified.

Next, we eliminate the auxiliary superfields V and Σ in the superfield formalism. Their equations of motion read from Eq. (1)

$$\frac{\partial\mathcal{L}}{\partial V}=\Phi^\dagger\Phi e^V-e^{-V}\Psi\Psi^\dagger-c\mathbf{1}_M=0, \tag{2}$$

$$\frac{\partial\mathcal{L}}{\partial\Sigma}=\Psi\Phi-b\mathbf{1}_M=0. \tag{3}$$

From the first equation, V can be solved,

$$e^V=\frac{c}{2}(\Phi^\dagger\Phi)^{-1}\left(\mathbf{1}_M\pm\sqrt{\mathbf{1}_M+\frac{4}{c^2}\Phi^\dagger\Phi\Psi\Psi^\dagger}\right). \tag{4}$$

Substituting this back into (1), we obtain the Kähler potential for the Lindström–Roček metric [14]

$$K=c\,\mathrm{tr}\sqrt{\mathbf{1}_M+\frac{4}{c^2}\Phi^\dagger\Phi\Psi\Psi^\dagger} - c\,\mathrm{tr}\log\left(\mathbf{1}_M+\sqrt{\mathbf{1}_M+\frac{4}{c^2}\Phi^\dagger\Phi\Psi\Psi^\dagger}\right) + c\,\mathrm{tr}\log\Phi^\dagger\Phi. \tag{5}$$

Fixing the complexified $U(M)$ gauge symmetry and solving constraint (3), we obtain the Lagrangian of the NLSM in terms of independent superfields. To this end, we should consider two cases (i) $b=0$ and (ii) $b\neq0$ separately.

Case (i) $b=0$. A gauge can be fixed as

$$\Phi=\begin{pmatrix}\mathbf{1}_M\\ \varphi\end{pmatrix},\quad \Psi=(-\psi\varphi,\psi), \tag{6}$$

with φ and ψ being $[(N-M)\times M]$- and $[M\times(N-M)]$-matrix chiral superfields, respectively. The superpotential becomes

$$W=\sum_a m_a\,\mathrm{tr}\left[(-\psi\varphi,\psi)H_a\begin{pmatrix}\mathbf{1}_M\\ \varphi\end{pmatrix}\right] =\sum_a m_a\,\mathrm{tr}\left[H_a\begin{pmatrix}-\psi\varphi & \psi\\ -\varphi\psi\varphi & \varphi\psi\end{pmatrix}\right]. \tag{7}$$

Case (ii) $b\neq0$. We can take a gauge as [14]

$$\Phi=\begin{pmatrix}\mathbf{1}_M\\ \varphi\end{pmatrix}Q,\quad \Psi=Q(\mathbf{1}_M,\psi), \tag{8}$$

$$Q=\sqrt{b}(\mathbf{1}_M+\psi\varphi)^{-1/2},$$

with φ and ψ being again $[(N-M)\times M]$- and $[M\times(N-M)]$-matrix chiral superfields, respectively. In this case, the superpotential is given by

$$W=b\sum_a m_a\,\mathrm{tr}\left[H_a\begin{pmatrix}\mathbf{1}_M\\ \varphi\end{pmatrix}(\mathbf{1}_M+\psi\varphi)^{-1}(\mathbf{1}_M,\psi)\right]. \tag{9}$$

These two cases are not holomorphically transformed to each other, because they make different complex structures manifest.

We can find the bundle structure of the manifold as follows:

(i) $b=0$. Setting $\psi=0$, the Kähler potential becomes

$$K|_{\psi=0}=c\,\mathrm{tr}\log(1+\varphi^\dagger\varphi), \tag{10}$$

which is the one of the Grassmann manifold. Therefore, φ parametrizes the base Grassmann manifold, whereas ψ parametrizes the cotangent space as the fiber, with the total space being the cotangent bundle over the Grassmann manifold $T^*G_{N,M}$.

(ii) $b \neq 0$. In the case of $T^*\mathbf{C}P^{N-1}$ of $M = 1$, the base manifold is embedded by $\varphi = \psi^\dagger$ [21][4)].

There exists the manifest duality between two theories with $U(M)$ gauge and $U(N-M)$ gauge symmetries and the same flavor $SU(N)$ symmetry. This comes directly from the duality in the base Grassmann manifold $G_{N,M} \simeq G_{N,N-M}$.

For $M = 1$ ($M = N-1$), namely, for the $U(1)$ [$U(N-1)$] gauge symmetry, this model reduces to $T^*\mathbf{C}P^{N-1} \simeq T^*G_{N,1} (\simeq T^*G_{N,N-1})$ [22], which we discussed in detail in [12]. Moreover, if $N = 2$, the manifold $T^*\mathbf{C}P^1$ is the Eguchi–Hanson space. A nontrivial model in the lowest dimensions other than $T^*\mathbf{C}P^{N-1}$ is the case of $N = 4, M = 2$. The manifold is $T^*G_{4,2} = T^*[SU(4)/SU(2) \times SU(2) \times U(1)] = T^*[SO(6)/SO(4) \times U(1)] \equiv T^*Q^4$, in which the base manifold Q^4 is called the Klein quadric space.

3. VACUUM STRUCTURE

3.1. Vacua in the Massive $T^*\mathbf{C}P^{N-1}$ Model

In this subsection, we discuss $T^*\mathbf{C}P^{N-1} = T^*G_{N,1}$ of $M = 1$. Without loss of generality, we consider the case of $b = 0$ and $c \neq 0$. The dynamical matrix fields are column and row vectors like $\varphi^T = (\varphi^1, \ldots, \varphi^{N-1})$ and $\psi = (\psi^1, \ldots, \psi^{N-1})$.

The superpotential given in (7) becomes

$$W = \sum_a m_a \,\mathrm{tr}\left[H_a \begin{pmatrix} -\psi\cdot\varphi & \psi \\ -\varphi(\psi\cdot\varphi) & \varphi\otimes\psi \end{pmatrix}\right]. \quad (11)$$

We take H_a ($a = 1, \ldots, N-1$) as

$$H_a = \frac{1}{\sqrt{a(a+1)}}\mathrm{diag}(1, \ldots, 1, -a, 0, \ldots, 0), \quad (12)$$

where $-a$ is the $(a+1)$th component, with a normalization given by the trace $\mathrm{tr}(H_aH_b) = \delta_{ab}$. Then the superpotential can be calculated as

$$W = -\sum_a M_a\psi^a\varphi^a, \quad (13)$$

$$M_a \equiv \sqrt{\frac{a}{a+1}}m_a + \sum_{b=1}^{a}\frac{m_b}{\sqrt{b(b+1)}}.$$

[4)]This embedding $\varphi = \psi^\dagger$ should hold for a matrix of general M, although we have not proved it yet.

Therefore, the derivatives of W with respect to fields are

$$\partial_{\varphi^a}W = -M_a\psi^a, \quad \partial_{\psi^a}W = -M_a\varphi^a \ \text{(no sum)}. \quad (14)$$

These vanish only at the origin $\varphi = \psi^T = 0$, which is the only vacuum in the regular region of these coordinates because the metric is regular there.

This model, however, contains more vacua, because the whole manifold is covered by several coordinate patches and the vacuum exists at the origin of each coordinate patch. To see this, we concentrate on the base $\mathbf{C}P^{N-1}$ for a while. We consider the fields before the gauge fixing, $\Phi \equiv \phi^A = (\phi^1, \ldots, \phi^N)^T$ ($A = 1, \ldots, N$), called the homogeneous coordinates, in which we need an identification by the gauge transformation $\phi^A \sim e^{i\Lambda}\phi^A$. In the region $\phi^1 \neq 0$, we can take a patch $\varphi^i = \phi^{i+1}/\phi^1$ ($i = 1, \ldots, N-1$), which was used in Eq. (6). Here, let us write these coordinates as $\varphi^i_{(1)} = \phi^{i+1}/\phi^1$. In the same way, in the region of $\phi^A \neq 0$, we can take the Ath patch defined by

$$\varphi^i_{(A)} = \begin{cases} \phi^i/\phi^A & (1 \leq i \leq A-1), \\ \phi^{i+1}/\phi^A & (A \leq i \leq N-1). \end{cases} \quad (15)$$

We thus have N sets of patches $\{\varphi^i_{(A)}\}$ enough to cover the whole base manifold. Corresponding to each patch for the base space, we manifestly have an associated patch for the fiber tangent space $\{\psi^i_{(A)}\}$ from Eq. (6). These sets of coordinates $\{\varphi^i_{(A)}, \psi^i_{(A)}\}$ are enough to cover the whole $T^*\mathbf{C}P^{N-1}$. For each patch, the origin $\varphi^i_{(A)} = \psi^i_{(A)} = 0$ is a vacuum. Therefore, the number of discrete vacua for the massive $T^*\mathbf{C}P^{N-1}$ model is N, which was first found in [9].

To discuss solitons like BPS walls, their junction, and lumps, it may be better to consider the problem in one coordinate patch. The other vacua appear in one patch as the coordinate singularities of the metric in infinities of the coordinates rather than the stationary points of the superpotential [23]. To see this, we consider only the base $\mathbf{C}P^{N-1}$ once again. We discuss how the Ath vacuum ($A \neq 1$) in the origin of the Ath coordinate patch is mapped in the first patch. The Ath vacuum is represented by $\varphi^i_{(A)} = 0$ or $\phi^B/\phi^A = 0$ for all $B(\neq A)$. In the first coordinate patch, this point is mapped to an infinite point represented by $\varphi^i_{(A)} = 0$ or $\phi^B/\phi^A = 0$ for all $B(\neq A)$. In the first coordinate patch, this point is mapped to an infinite point represented by

$$\varphi^{A-1}_{(1)} \to \infty, \quad \varphi^i_{(1)}/\varphi^{A-1}_{(1)} \to 0 \quad (i \neq A-1), \quad (16)$$

which looks like a runaway vacuum in this patch. Hence, the origin and $N-1$ infinities are vacua in each coordinate patch [23]. As a summary, if we include runaway vacua, one patch is enough to describe soliton solutions. However, note that the terminology "runaway" is just a coordinate-dependent concept, because a runaway vacuum in one coordinate patch is a true vacuum in the other coordinate patch.

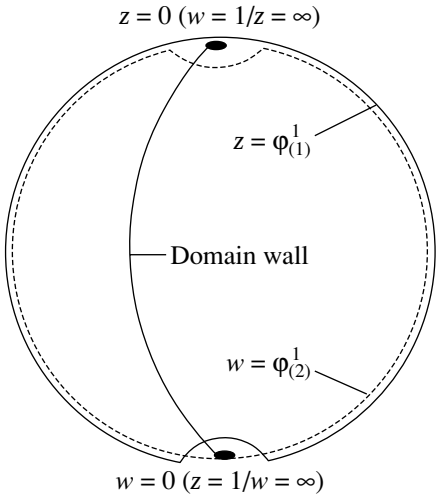
We can also discuss the vacua without referring to the local coordinate patches. We concentrate on the base $\mathbf{C}P^{N-1}$ once again. A point in the $\mathbf{C}P^{N-1}$ corresponds to a complex line through the origin in $\mathbf{C}^N$ with homogeneous coordinates ϕ^A, because of the gauge transformation $\phi^A \sim e^{i\Lambda}\phi^A$ as an equivalence relation. The first vacuum is expressed in region $\phi^1 \neq 0$ by $\varphi^i_{(1)} = \phi^{i+1}/\phi^1 = 0$ $(i = 1, \ldots, N-1)$, namely, $\phi^{i+1} = 0$. Therefore, the first vacuum corresponds to the ϕ^1 axis. In the same way, the Ath vacuum corresponds to the ϕ^A axis. Each vacuum is simply expressed by each orthogonal axis in $\mathbf{C}^N$. Note that each axis is invariant under $U(1)^{N-1}$ transformation of H_a, so that it is a fixed point of this transformation.

If we take N orthogonal normalized basis e_A [with $(e_A)^* \cdot e_B = \delta_{AB}$] whose components are given by

$$(e_A)^B = \delta^B_A, \tag{17}$$

a complex line in $\mathbf{C}^N$ can be spanned by a unit vector $e' = \sum_{A=1}^N a^A e_A = Ue_1$, where a^A is a complex number with $\sum_A |a^A|^2 = 1$ and U is an unitary matrix $U \in U(N)$. Each of the N vacua found above corresponds to each e_A $(A = 1, \ldots, N)$ (with zero value of the cotangent space $\psi = 0$).

Example: the Eguchi–Hanson space [18]. The simplest model is the Eguchi–Hanson space, $T^*\mathbf{C}P^1$ ($N = 2$ and $M = 1$). This model has two discrete vacua and admits a typical domain-wall solution [7, 12]. The vacua are located on the North and South Poles of the base $\mathbf{C}P^1 \simeq S^2$ (see figure). Corresponding to two gauge fixing conditions $\Phi = \begin{pmatrix} 1 \\ z \end{pmatrix}$ and $\Phi = \begin{pmatrix} w \\ 1 \end{pmatrix}$, we have two coordinate patches $z \equiv \varphi^1_{(1)} = \phi^2/\phi^1$ and $w \equiv \varphi^1_{(2)} = \phi^1/\phi^2$, which are related by $z = 1/w$. Two vacua are given by $z = 0$ and $w = 0$. The second (first) vacuum $w = 0$ $(z = 0)$ is mapped to $z = \infty$ $(w = \infty)$ in the first (second) patch, which looks like a runaway vacuum. In homogeneous coordinates, these correspond to $\langle\Phi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv e_1$ and $\langle\Phi\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv e_2$, respectively, with $\langle\Psi\rangle = (0, 0)$. Also, in a coordinate-independent way, these two vacua correspond to the ϕ^1 and ϕ^2 axes spanned by e_1 and e_2, respectively.



The base manifold of $T^*\mathbf{C}P^1$ and vacua. Corresponding to two gauge-fixing conditions, we have two coordinates z and w, covering S^2 except for South (S) and North (N) Poles, respectively. The origins of z and w (N and S, respectively) are both vacua. The domain-wall solution, approaching these two vacua in spatial infinities, is mapped to a trajectory connecting N and S in S^2.

Before closing this subsection, we discuss the case of $b \neq 0$. The superpotential (9) can be calculated to give

$$W = \frac{b}{1+\psi\cdot\varphi}\left(L + \sum_{a=1}^{N-1} N_a \psi^a \varphi^a\right), \tag{18}$$

$$L \equiv \sum_{a=1}^{N-1} \frac{m_a}{\sqrt{a(a+1)}},$$

$$N_a \equiv -\sqrt{\frac{a}{a+1}} m_a + \sum_{b=a+1}^{N-1} \frac{m_b}{\sqrt{b(b+1)}} = L - M_a,$$

with M_a defined in (13). The derivatives of W are

$$\partial_{\varphi^a} W = -\frac{b\psi^a}{(1+\psi\cdot\varphi)^2} \times \left[M_a - \sum_{b=1}^{N-1}(M_b - M_a)\psi^b\varphi^b\right], \tag{19}$$

$$\partial_{\psi^a} W = (\psi^a \leftrightarrow \varphi^a),$$

where an arrow in the second equation represents the exchange of quantities in the first equation. The origin $\varphi^a = \psi^a = 0$ in each patch is a vacuum. There

is no other vacuum than these N vacua. The number of vacua should coincide with the case of $b = 0$ and $c \neq 0$, because they are connected by the R symmetry and the physics does not depend on the difference.

3.2. Vacua in the Massive $T^*G_{N,M}$ Model

To look for vacua of the $T^*G_{N,M}$ model, we consider the case $b = 0$ and $c \neq 0$ again without loss of generality. We label the indices for the matrices as $\varphi = (\varphi_{i\alpha})$ and $\psi = (\psi_{\alpha i})$ in which $i = 1, \ldots, N - M$ and $\alpha = 1, \ldots, M$. The superpotential given in Eq. (7) can be calculated as

$$W = -\sum_{\alpha=1}^{M}\sum_{i=1}^{N-M} M_{\alpha i}\varphi_{i\alpha}\psi_{\alpha i}, \tag{20}$$

$$M_{\alpha i} \equiv \sqrt{\frac{i+M-1}{i+M}}m_{i+M-1} - \sqrt{\frac{\alpha-1}{\alpha}}m_{\alpha-1} + \sum_{a=\alpha}^{i+M-1}\frac{m_a}{\sqrt{a(a+1)}},$$

where we have set $m_0 \equiv 0$. For the case of $M = 1$ ($\alpha = 1$), this reduces to Eq. (13) for $T^*\mathbf{C}P^{N-1}$. From the superpotential (20), its derivatives with respect to the fields are

$$\partial_{\varphi_{i\alpha}}W = -M_{\alpha i}\psi_{\alpha i}, \quad \partial_{\psi_{\alpha i}}W = -M_{\alpha i}\varphi_{i\alpha} \quad \text{(no sum)}. \tag{21}$$

Therefore the origin of these coordinates, $\varphi = \psi^T = 0$, is a vacuum, and this is the only vacuum in the finite region of these coordinates, where the metric is regular. This model contains as many vacua as the coordinate patches, like the $T^*\mathbf{C}P^{N-1}$ case. In the first coordinate patch, we have chosen the first M row vectors in Φ the unit matrix as in Eqs. (6) or (8). The other coordinate patches are given by the other choices of gauge-fixing conditions making the other sets of M row vectors in Φ the unit matrix. The number of such coordinate systems is ${}_NC_M = N!/[M!(N-M)!]$. They are independent and enough to cover the whole manifold, so this model has $N!/[M!(N-M)!]$ vacua. This number is invariant under the duality between $U(M)$ and $U(N-M)$ gauge groups. It also reduces correctly to N for $T^*\mathbf{C}P^{N-1}$ when $M = 1$ or $M = N - 1$.

As in the $T^*\mathbf{C}P^{N-1}$ case, we can understand the vacua of $T^*G_{N,M}$ without local coordinates. A point in the base $G_{N,M}$ corresponds to an M-dimensional complex plane through the origin in $\mathbf{C}^N$. The vacua found above correspond to mutually orthogonal M planes spanned by arbitrary M sets of axes chosen from the N axes. Therefore, the total number of vacua is ${}_NC_M = N!/[M!(N-M)!]$. Since the M planes of vacua are invariant under $U(1)^{N-1}$ generated by H_a, the vacua are fixed points.

Taking basis (17) in $\mathbf{C}^N$, a point in $G_{N,M}$ expressed by an M plane in $\mathbf{C}^N$ can be spanned by M set of unit vectors

$$(e_i)' = Ue_i, \tag{22}$$

where $i = 1, \ldots, N - M$ and U is an unitary matrix, $U \in U(N)$. The vacua of mutually orthogonal M planes are spanned by arbitrary M sets of basis among orthogonal N basis.

The duality becomes manifest in this framework. We can represent a point in $G_{N,M}$ by an $(N - M)$ plane complement to an M plane.

Example: the cotangent bundle over the Klein quadric. An example is given for the Klein quadric $T^*G_{4,2} = T^*Q^4$ ($N = 4$ and $M = 2$). There exist six coordinate systems $\varphi_{i\alpha}^{(A)}$ ($A = 1, \ldots, 6$) for the base manifold corresponding to six choices of gauge fixing, given by

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \varphi_{11}^{(1)} & \varphi_{12}^{(1)} \\ \varphi_{21}^{(1)} & \varphi_{22}^{(1)} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ \varphi_{11}^{(2)} & \varphi_{12}^{(2)} \\ 0 & 1 \\ \varphi_{21}^{(2)} & \varphi_{22}^{(2)} \end{pmatrix}, \tag{23}$$

$$\begin{pmatrix} 1 & 0 \\ \varphi_{11}^{(3)} & \varphi_{12}^{(3)} \\ \varphi_{21}^{(3)} & \varphi_{22}^{(3)} \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \varphi_{11}^{(4)} & \varphi_{12}^{(4)} \\ 1 & 0 \\ 0 & 1 \\ \varphi_{21}^{(4)} & \varphi_{22}^{(4)} \end{pmatrix},$$

$$\begin{pmatrix} \varphi_{11}^{(5)} & \varphi_{12}^{(5)} \\ 1 & 0 \\ \varphi_{21}^{(5)} & \varphi_{22}^{(5)} \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \varphi_{11}^{(6)} & \varphi_{12}^{(6)} \\ \varphi_{21}^{(6)} & \varphi_{22}^{(6)} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Together with corresponding coordinates $\psi_{\alpha i}^{(A)}$ for the cotangent space in Eq. (6), these six sets of coordinate systems are enough to cover the whole manifold. Therefore, this model has the six vacua given by

$$\langle\Phi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{24}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which are the origins of (23), respectively, with $\langle\Psi\rangle = 0$. A set of two column vectors in each matrix in Eq. (24) is a set of orthogonal basis e_i chosen from the four basis.

In the case of $b \neq 0$, the superpotential (9) is

$$W = b\sum_{a=1}^{N-1}\sum_{n=0}^{\infty}(-1)^n m_a \times \mathrm{tr}\left[H_a \begin{pmatrix} (\psi\varphi)^n & (\psi\varphi)^n\psi \\ \varphi(\psi\varphi)^n & (\varphi\psi)^{n+1} \end{pmatrix}\right] = b\sum_{a=1}^{N-1}\sum_{n=0}^{\infty}(-1)^n m_a \times \mathrm{tr}\left[H_a \begin{pmatrix} (\psi\varphi)^n & 0 \\ 0 & (\varphi\psi)^{n+1} \end{pmatrix}\right], \tag{25}$$

where the last equality holds because H_a are diagonal. Similarly to the $T^*\mathbf{C}P^{N-1}$ case, the origin $\varphi = \psi^T = 0$ of each patch is a vacuum and we cannot have any other vacua.

4. MASSIVE HK QUOTIENT BY $SU(M)$ GAUGE GROUP

In this section, we construct the massive HK NLSM with the $SU(M)$ gauge group. We eliminate the vector multiplets in the superfield formalism and find that this model does not have discrete vacua.

4.1. Massive HK NLSM by SU Gauge Group

In this subsection, we consider $\mathcal{N} = 2$ SUSY QCD with N flavors and the $SU(M)$ gauge group. We take the same matter field contents with $T^*G_{N,M}$ but gauge multiplets take values in the Lie algebra of $SU(M)$: $V = V^A T_A$ and $\Sigma = \Sigma^A T_A$ with T_A generators of $SU(M)$. Then the Lagrangian is given by

$$\mathcal{L} = \int d^4\theta \left[\mathrm{tr}(\Phi^\dagger\Phi e^V) + \mathrm{tr}(\Psi\Psi^\dagger e^{-V})\right] + \left[\int d^2\theta \left(\mathrm{tr}(\Sigma\Psi\Phi) + \sum_{a=1}^{N-1} m_a\,\mathrm{tr}(\Psi H_a \Phi)\right) + \mathrm{c.c.}\right]. \tag{26}$$

We do not have any FI parameters because of the absence of any $U(1)$ gauge symmetry. The $SU(M)$ gauge transformation is given in the same way as in the $U(M)$ case and it is complexified to $SU(M)^{\mathbf{C}} = SL(M,\mathbf{C})$. This model has an additional $U(1)_D$ flavor symmetry,

$$\Phi \to \Phi' = e^{i\lambda}\Phi, \quad \Psi \to \Psi' = e^{-i\lambda}\Psi, \tag{27}$$

which was gauged in the $U(M)$ case.

We eliminate all auxiliary superfields in the superfield formalism. Equations of motion for V, Σ imply

$$\Phi^\dagger\Phi e^V - e^{-V}\Psi\Psi^\dagger = C\mathbf{1}_M, \tag{28}$$

$$\Psi\Phi = B\mathbf{1}_M, \tag{29}$$

respectively, with $C(x,\theta,\bar\theta)$ and $B(x,\theta,\bar\theta)$ being vector and chiral superfields in the $\mathcal{N} = 1$ superfield formalism.

The gauge field V can be solved in terms of the dynamical fields from Eq. (28) as

$$e^V = \frac{1}{2}(\Phi^\dagger\Phi)^{-1}\left(C\mathbf{1}_M \pm \sqrt{C^2\mathbf{1}_M + 4\Phi^\dagger\Phi\Psi\Psi^\dagger}\right). \tag{30}$$

Since the equation $\det e^V = 1$ holds, we get the equation

$$\det\left(C\mathbf{1}_M \pm \sqrt{C^2\mathbf{1}_M + 4\Phi^\dagger\Phi\Psi\Psi^\dagger}\right) = 2^M \det(\Phi^\dagger\Phi), \tag{31}$$

which enables us to express C in terms of dynamical fields implicitly: $C = C(\Phi,\Phi^\dagger;\Psi,\Psi^\dagger)$. On the other hand, Eq. (29) implies

$$B = \frac{1}{M}\mathrm{tr}(\Phi\Psi). \tag{32}$$

Substituting the solution (30) back into the Lagrangian (26), we obtain the Kähler potential

$$K = \pm\mathrm{tr}\sqrt{C^2(\Phi,\Phi^\dagger;\Psi,\Psi^\dagger)\mathbf{1}_M + 4\Phi^\dagger\Phi\Psi\Psi^\dagger}, \tag{33}$$

with C satisfying the constraint (31). We should choose the plus sign for the positivity of the metric.

Let us fix the complex gauge symmetry $SU(M)^{\mathbf{C}} = SL(M,\mathbf{C})$ to express the Lagrangian in terms of independent superfields. We can take the similar gauge as the $b \neq 0$ case in $T^*G_{N,M}$:

$$\Phi = \sigma\begin{pmatrix} \mathbf{1}_M \\ \varphi \end{pmatrix} P, \quad \Psi = P(\mathbf{1}_M, \psi)\rho, \qquad P = (\mathbf{1}_M + \psi\varphi)^{-1/2}, \tag{34}$$

with φ and ψ being $[(N-M)\times M]$- and $[M\times(N-M)]$-matrix chiral superfields, respectively. Here, σ and ρ are chiral superfields satisfying $\sigma\rho = B$ from

Eq. (32). We can consider σ and ρ independent fields among these three fields σ, ρ, and B.

Substituting Eq. (34) into the Kähler potential (33), we obtain the Kähler potential in terms of independent fields φ, ψ, ρ, σ and their conjugates. The superpotential also can be calculated as

$$W = \sum_a m_a \sigma\rho \times \mathrm{tr}\left[H_a \begin{pmatrix} \mathbf{1}_M \\ \varphi \end{pmatrix} (\mathbf{1}_M + \psi\varphi)^{-1} (\mathbf{1}_M, \psi)\right]. \quad (35)$$

This target manifold has the isometry of $U(N) = SU(N) \times U(1)_D$, in which the $SU(N)$ part is the same with $T^*G_{N,M}$. The Kähler potential does not receive the Kähler transformation. As for the symmetry of the Lagrangian, the superpotential is invariant under the $U(1)$ fiber symmetry originated from (27)

$$\sigma \to \sigma' = e^{i\lambda}\sigma, \quad \rho \to \rho' = e^{-i\lambda}\rho, \quad (36)$$

besides the $U(1)^{N-1}$ symmetry of the massive $T^*G_{N,M}$ model. Gauging this $U(1)_D$ symmetry, we obtain the $T^*G_{N,M}$ model. Gauging $U(1)_D$ symmetry implies putting B and C in the constraints (28) and (29) as constants and the constraints then become $T^*G_{N,M}$ ones (2) and (3), respectively. This clarifies the bundle structure: the set of σ and ρ is a fiber of quaternion with the total manifold being the (quaternionic) line bundle over $T^*G_{N,M}$.

4.2. Vacua of SU Gauge Theories

We look for the vacua of the HK NLSM by the SU gauge group. The superpotential (35) of this model can be rewritten as

$$W = \sigma\rho \sum_{a=1}^{N-1} \sum_{n=0}^{\infty} (-1)^n m_a \times \mathrm{tr}\left[H_a \begin{pmatrix} (\psi\varphi)^n & 0 \\ 0 & (\varphi\psi)^{n+1} \end{pmatrix}\right] \equiv \sigma\rho W_U, \quad (37)$$

where W_U (times b) denotes the superpotential (9) or (25) of the $U(M)$ gauge group with $b \neq 0$. The derivatives of the superpotential with respect to fields are given by $\partial_\psi W = \sigma\rho\partial_\psi W_U$, $\partial_\varphi W = \sigma\rho\partial_\varphi W_U$, $\partial_\rho W = \sigma W_U$, and $\partial_\sigma W = \rho W_U$. The vacuum condition is given by $\sigma = \rho = 0$, since $\partial W_U = 0$ holds only at $\varphi = \psi^T = 0$ from the discussion in the last section, but $W_U \neq 0$ there. Therefore, this model has no discrete vacua, and so we cannot expect any wall solutions.

5. BPS EQUATION AND ITS SOLUTION

In this section, we construct the BPS domain wall in the $N = 2$ and $M = 1$ case of $T^*G_{N,M}$, i.e., $T^*\mathbf{C}P^1$. In what follows, we consider the $b \neq 0$ and $c = 0$ case. We assume that there exists a domain-wall solution perpendicular to the $y = x^2$ direction. The BPS domain-wall solution is derived from vanishing of the SUSY transformation for fermions

$$0 = i\sqrt{2}\sigma^\mu\bar{\epsilon}\partial_\mu\Phi^i + \sqrt{2}\epsilon F^i \quad (38)$$

with half SUSY condition $e^{i\alpha}\sigma^2\bar{\epsilon} = i\epsilon$, where $e^{i\alpha}$ is a phase factor, and Φ^i and F^i are scalar and auxiliary fields, respectively. In the case we consider now, the scalar field is given by

$$\Phi^i = \sqrt{\frac{b}{1+\varphi\psi}} \begin{pmatrix} 1 \\ \varphi \end{pmatrix}$$

from Eq. (8). Eliminating the auxiliary fields, the BPS equations are given by

$$\partial_2\varphi^i = -e^{i\alpha} g^{ij^*} \partial_{j^*} W^*, \quad (39)$$

where g^{ij^*} is inverse of the metric $g_{ij^*} = \partial_i\partial_{j^*}K$ and K is given by (5) with (8). Substituting the metric and the superpotential (9), these BPS equations reduce to

$$\partial_2\varphi = e^{i\alpha}\frac{m^*}{4b}K(1+\varphi\psi)^2 \times \left[\frac{|1+\varphi\psi|^2 + (1+|\varphi|^2)(1+|\psi|^2)}{|1+\varphi\psi|^2(1+|\psi|^2)^2}\psi^* + \frac{(\varphi-\psi^*)^2\varphi^*}{|1+\varphi\psi|^2(1+|\varphi|^2)(1+|\psi|^2)}\right], \quad (40)$$

$$\partial_2\psi = e^{i\alpha}\frac{m^*}{4b}K(1+\varphi\psi)^2 \times \left[\frac{|1+\varphi\psi|^2 + (1+|\varphi|^2)(1+|\psi|^2)}{|1+\varphi|^2(1+|\varphi|^2)^2}\varphi^* + \frac{(\psi-\varphi^*)^2\psi^*}{|1+\varphi\psi|^2(1+|\varphi|^2)(1+|\psi|^2)}\right],$$

where m is a mass parameter. Now we must choose the phase $e^{i\alpha}$ to absorb the phase of the parameter [5] m^*/b:

$$e^{i\alpha}\frac{m^*}{b} = \left|\frac{m}{b}\right|. \quad (41)$$

By subtracting the complex conjugate of the second equation from the first one in Eq. (40), we obtain

$$\frac{\partial(\varphi-\psi^*)}{\partial y} = \left|\frac{m}{b}\right|\frac{K}{4}\left[\left\{\left(\frac{1+\varphi\psi}{|1+\varphi\psi|}\right)^2\varphi^* \quad (42)\right.\right.$$

[5] For simplicity, we choose m to be real positive in the following.

$$-\left(\frac{1+\varphi^*\psi^*}{|1+\varphi\psi|}\right)^2\psi\Bigg\}\frac{(\varphi-\psi^*)^2}{(1+|\varphi|^2)(1+|\psi|^2)}$$
$$+\left\{\left(\frac{1+\varphi\psi}{|1+\varphi\psi|}\right)^2\frac{\psi^*}{(1+|\psi|^2)^2}$$
$$-\left(\frac{1+\varphi^*\psi^*}{|1+\varphi\psi|}\right)^2\frac{\varphi}{(1+|\varphi|^2)^2}\Bigg\}\Bigg\{|1+\varphi\psi|^2$$
$$+(1+|\varphi|^2)(1+|\psi|^2)\Bigg\}\Bigg],$$

whose right-hand side vanishes for $\varphi=\psi^*$. The BPS equation (42) dictates that $\varphi=\psi^*$ is valid for arbitrary y if an initial condition $\varphi=\psi^*$ is chosen at some y. Since we can choose the initial condition $\varphi=\psi^*$ at $y=-\infty$, we find the BPS equations (40) simply reduce to

$$\partial_2\varphi=|m|\varphi, \tag{43}$$

which is the BPS equation on the submanifold $\mathbf{C}P^1$ defined by $\varphi=\psi^*$ [12]. Therefore, we obtain a BPS wall configuration connecting two vacua $\varphi=\psi^*=0$ at $y=-\infty$ to $\varphi=\psi^*=\infty$ at $y=\infty$ along $\varphi=\psi^*$ with a constant phase $e^{i\phi_0}$

$$\varphi=\psi^*=e^{|m|(y+y_0)}e^{i\phi_0}, \tag{44}$$

where y_0 is also a constant representing the position of the wall. Thus, we find two collective coordinates (zero modes) corresponding to the spontaneously broken translation (y_0) and $U(1)$ symmetry (ϕ_0).

We can show that BPS solution (44) coincides with that derived in component formalism [8] through the following field redefinition $\varphi\to X,\phi$:

$$\varphi\equiv e^{u+i\phi},\qquad X=|b|\tanh u, \tag{45}$$

where u, ϕ, and X are real scalar fields. After the field redefinition, the theory of the massive $\mathbf{C}P^1$ model is described by X and ϕ, and the wall solution (44) is mapped to

$$X=|b|\tanh|m|(y+y_0),\quad \phi=\phi_0. \tag{46}$$

This solution coincides with that derived in [8].

6. CONCLUSION

We have constructed massive NLSMs on a cotangent bundle over the Grassmann manifold $T^*G_{N,M}$ and its generalization, the line bundle over the $T^*G_{N,M}$ manifold in the $\mathcal{N}=1$ superfield formalism with the quotient method. It was found that the former contains $N!/[M!(N-M)!]$ vacua, while the latter has no discrete vacua.

The BPS wall solution was given in the $N=2$ and $M=1$ case of the $T^*G_{N,M}$ model, which corresponds to the Eguchi–Hanson manifold. A more interesting case is the $N=4$ and $M=2$ case since it is the simplest manifold other than $T^*\mathbf{C}P^{N-1}$. The theory has six discrete vacua and it is expected that the theory has various interesting wall solutions, their junction, and lump.

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The q-Boson–Fermion Realizations of the Quantum Superalgebra $U_q(osp(1/2))$*

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Abstract—We show that our construction of realizations for algebras and quantum algebras can be generalized to quantum superalgebras too. We study an example of quantum superalgebra $U_q(osp(1/2))$ and give the boson–fermion realization with respect to one pair of q-boson operators and one pair of fermions. © *2005 Pleiades Publishing, Inc.*

1. INTRODUCTION

The quantum superalgebra $U_q(osp(1/2))$ is a deformation of the universal enveloping algebra of the Lie superalgebra $osp(1/2)$. This superalgebra can be applied to physical problems such as the trigonometric $osp(1/2)$ Gaudien model [1, 2]. The universal R matrix of the quantum superalgebra $U_q(osp(1/2))$ was developed and studied in [3, 4].

Boson–fermion realizations of a given set of operators via Bose–Fermion creation and annihilation operators are among the main tools for solving various quantum problems. The origin is linked with the Schwinger [5], Dyson [6], and Holstein–Primakoff [7] realizations, which are different boson realizations of the algebra $sl(2)$.

Generalizations of the Dyson realization to the Lie algebra $sl(n)$ were derived in [8]. In our paper [9], we formulated the method starting from the Verma modules for obtaining boson realizations, and in [10], we obtained explicitly a braid class of realizations which generalized the results from [11, 12].

Later the idea was extended to the Lie superalgebra, and the Dyson-type boson–fermion realizations were explicitly given in [13], generalizing the results to $sl(2/1)$ ([14, 15]).

Today these boson–fermion realizations have become a standard technique in quantum many-body physics and we can also find several other applications in all fields of quantum physics.

Quantum groups and quantum supergroups or q-deformed Lie algebras and superalgebras imply some specific deformations of the classical Lie algebras and superalgebras. From a mathematical point of view, those are noncommutative associative Hopf algebras and superalgebras. The structure and representation theory of quantum groups were extensively developed by Jimbo [16] and Drinfeld [17]. The first "quantum" version of Holstein–Primakoff was worked out for $U_q(sl(2))$ [18] and then for $U_q((sl(3))$ [19]. The Schwinger-type realization was written in [20] and [21]. These realizations found immediate applications [22–27].

In our papers [28–30], we studied the Dyson realizations of the series algebras $U_q(sl(2))$, $U_q(gl(n))$, $U_q(B_n)$, $U_q(C_n)$, and $U_q(D_n)$. There is some special case [29] for which the realization of the subalgebra $U_q(gl(n-1))$ in the recurrence is trivial. Such special realizations of the quantum algebra $U_q(sl(n))$ of Dyson type were studied in [31, 32].

The aim of the present paper is to show that there is a possibility of generalizing our method [9] for deriving the boson–fermion realization too. This will be exemplified by the quantum superalgebra $U_q(osp(1/2))$.

2. PRELIMINARIES

In this article, we will use the definition of a quantum superalgebra $U_q(osp(1/2))$ which can be found in [3, 4].

Let q be an independent variable, $\mathcal{A} = C[q, q^{-1}]$, and $\mathcal{C}(q)$ be a division field of $\mathcal{A}$. The superalgebra $U_q(osp(1/2))$ is the associative superalgebra over $\mathcal{C}(q)$ generated by even generators K, K^{-1} and odd generators E, F which satisfy the following relations:

$$KK^{-1} = K^{-1}K = 1, \tag{1}$$
$$KE = qEK, \quad KF = q^{-1}FK,$$

*The text was submitted by the authors in English.

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$$K^{-1}E = q^{-1}EK^{-1}, \quad K^{-1}F = qFK^{-1},$$
$$EF + FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

We do not use the Hopf structure of this algebra for our construction of realizations; therefore, we do not give it explicitly here.

The method of construction used is the same as in the case of the Lie algebras [9] or quantum algebra [30] and is based on using the induced representation. The difference from quantum algebra is that, together with q-deformed boson operators [20, 21], we also use fermion operators.

The algebra $\mathcal{H}$ of the q-deformed boson operators is the associative algebra over the field $\mathcal{C}(q)$ generated by the elements of a^+, $a^- = a$, q^x, and q^{-x}, satisfying the commutation relations

$$q^x q^{-x} = q^{-x}q^x = 1, \quad q^x a^+ q^{-x} = qa^+, \quad (2)$$
$$q^x a q^{-x} = q^{-1}a,$$
$$aa^+ - q^{-1}a^+a = q^x, \quad aa^+ - qa^+a = q^{-x}.$$

The algebra $\mathcal{H}$ has a faithful representation on the vector space with basic elements $\{|n\rangle$, where $n = 0, 1, \dots\}$ of the form

$$q^x|n\rangle = q^n|n\rangle, \quad a^+|n\rangle = |n+1\rangle, \quad (3)$$
$$a|n\rangle = [n]|n-1\rangle,$$

where $[n] = \dfrac{q^n - q^{-n}}{q - q^{-1}}$.

Because of odd generators E and F, we construct the realization by means of the algebra $\mathcal{H}$ for even elements and by fermion elements b^+ and b for odd ones. These fermion elements commute with the elements of $\mathcal{H}$ and together fulfill the relations

$$bb = b^+b^+ = 0, \quad bb^+ + b^+b = 1. \quad (4)$$

The realization of the quantum superalgebra $U_q(osp(1/2))$ is called the homomorphism ρ of the $U_q(osp(1/2))$ to associative superalgebra $\mathcal{W}$ generated by $\mathcal{H}$ and $b^{\pm}$.

3. CONSTRUCTION OF THE REALIZATION OF $U_q(osp(1/2))$

First, for construction of the realization, we find the induced representation of $U_q(osp(1/2))$. As subalgebra $\mathcal{A}_0$ of $U_q(osp(1/2))$, we choose a quantum superalgebra generated by F, K, and K^{-1}. Let φ be a representation of $\mathcal{A}_0$ on vector space V. Let λ be the left regular representation on $U_q(osp(1/2)) \otimes V$; i.e., for $x, y \in U_q(osp(1/2))$ and $v \in V$, the representation λ is defined by

$$\lambda(x)(y \otimes v) = xy \otimes v. \quad (5)$$

Let $\mathcal{I}$ be subspace of $U_q(osp(1/2)) \otimes V$ generated by the relations

$$xy \otimes v = x \otimes \varphi(y)v$$

for all $x \in U_q(osp(1/2))$, $y \in \mathcal{A}_0$, and $v \in V$. It is easy to see that the subspace $\mathcal{I}$ is λ-invariant. Therefore, (5) gives the representation on the factor space $W = [U_q(osp(1/2)) \otimes V]/\mathcal{I}$.

Let $X = E^2$ and $X^N E^M = |N, M\rangle$. Due to the Poincaré–Birkhoff–Witt theorem, the space W of the induced representation is generated by the elements $|N, M\rangle \otimes v$, where $N = 0, 1, 2, \dots$; $M = 0, 1$; and $v \in V$.

To obtain the explicit form of the induced representation, we give some relations. They can be proved by mathematical induction from relations (1).

Lemma 1. *For any $n = 0, 1, 2, \dots$, the following formulas hold:*

$$FX^N = X^N F + \frac{q^N}{q+1}[N]X^{N-1}EK + \frac{q^{-N}}{q^{-1}+1}[N]X^{N-1}EK^{-1},$$
$$FE^M = (-1)^M E^M F + \frac{q^M - (-1)^M}{(q+1)(q-q^{-1})} \times E^{M-1}K - \frac{q^{-M} - (-1)^M}{(q^{-1}+1)(q-q^{-1})}E^{M-1}K^{-1}.$$

We omit the details of the calculations and write the result for the action of the induced representation on the basis elements $|N, M\rangle \otimes v$.

Theorem 1. *The formulas*

$$E|N, M\rangle \otimes v = \frac{1 + (-1)^M}{2}|N, M+1\rangle \otimes v + \frac{1 - (-1)^M}{2}|N+1, M-1\rangle \otimes v,$$
$$K|N, M\rangle \otimes v = q^{2N+M}|N, M\rangle \otimes \varphi(K)v,$$
$$F|N, M\rangle \otimes v = \frac{1 + (-1)^M}{2}[N] \times \left(\frac{q^{N+M}}{q+1}|N-1, M+1\rangle \otimes \varphi(K)v + \frac{q^{-N-M}}{q^{-1}+1}|N-1, M+1\rangle \otimes \varphi(K^{-1})v\right) + \frac{1 - (-1)^M}{2}[N]\left(\frac{q^{N+M}}{q+1}|N, M-1\rangle \otimes \varphi(K)v + \frac{q^{-N-M}}{q^{-1}+1}|N, M-1\rangle \otimes \varphi(K^{-1})v\right) + \frac{1 - (-1)^M}{2}\frac{1}{(q-q^{-1})}$$

$$\times |N, M-1\rangle \otimes \varphi(K-K^{-1})v$$
$$+ (-1)^M |N, M\rangle \otimes \varphi(F)v$$

give the induced representation of the quantum superalgebra $U_q(osp(1/2))$.

We construct the realization of quantum superalgebra $U_q(osp(1/2))$ from the induced representation given in Theorem 1 as follows:

We choose the representation φ for which $\varphi(F)v = 0$ and $\varphi(K)v = q^\lambda v$, and substitute

$$|N+1\rangle \to a^+, \quad [N]|N-1\rangle \to a, \quad q^N \to q^x,$$
$$\frac{1+(-1)^M}{2}|M+1\rangle \to b^+,$$
$$\frac{1-(-1)^M}{2}|M-1\rangle \to b,$$
$$q^{\pm M} \to bb^+ + q^{\pm 1}b^+b,$$
$$\varphi(F) \to 0, \quad \varphi(K) \to q^\lambda, \quad \varphi(K^{-1}) \to q^{-\lambda}.$$

This substitution leads to the realization of the quantum superalgebra $U_q(osp(1/2))$.

Theorem 2. *The mapping* $\rho : U_q(osp(1/2)) \to \mathcal{W}$ *defined by the formulas*

$$\rho(E) = b^+ + a^+b,$$
$$\rho(K) = q^{2x+\lambda}(bb^+ + qb^+b),$$
$$\rho(F) = \frac{(b^+ + qa^+b)a}{q+1}q^{x+\lambda}$$
$$+ \frac{(b^+ + q^{-1}a^+b)a}{q^{-1}+1}q^{-x-\lambda} + [\lambda]b$$

is the realization of the quantum superalgebra $U_q(osp(1/2))$.

This theorem can be proved by a direct calculation.

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A Modified Schwarzian Korteweg–de Vries Equation in 2 + 1 Dimensions with Lots of Isochronous Solutions*

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Abstract—A modified version of the integrable Schwarzian Korteweg de Vries equation in $2+1$ dimensions is introduced, and it is pointed out that it possesses lots of isochronous solutions.

1. INTRODUCTION AND MAIN RESULTS

Recently, via the symmetry approach, several solutions were exhibited [1] to the following integrable Schwarzian Korteweg–de Vries (SKdV) equation in $2+1$ dimensions [2]:

$$w_\tau + \frac{1}{4}w_{\xi\xi\eta} - \frac{w_\xi w_{\xi\eta}}{2w} - \frac{w_{\xi\xi}w_\eta}{4w} + \frac{w_\xi^2 w_\eta}{2w^2} - \frac{w_\xi \tilde{w}_\eta}{8} = 0, \tag{1a}$$

$$\tilde{w}_\xi = \left(\frac{w_\xi}{w}\right)^2. \tag{1b}$$

Here, ξ, η, and τ are the independent variables; $w \equiv w(\xi,\eta,\tau)$ is the main dependent variable; and $\tilde{w} \equiv \tilde{w}(\xi,\eta,\tau)$ is an auxiliary-dependent variable. Note that, throughout this paper, subscripted variables denote partial derivatives and that we prefer to write this PDE in local form rather than in the nonlocal form mainly used in [1] [where a different local form of this PDE is also mentioned, which is related to the one we prefer to use (see above) by a differential substitution].

In this paper, we introduce the following modified version of this evolution PDE:

$$u_t - i\lambda\omega u - i\mu\omega x u_x + i(2\mu-1)\omega y u_y + \frac{1}{4}u_{xxy} - \frac{u_x u_{xy}}{2u} - \frac{u_{xx}u_y}{4u} + \frac{u_x^2 u_y}{2u^2} - \frac{u_x \tilde{u}_y}{8} = 0, \tag{2a}$$

$$\tilde{u}_x = \left(\frac{u_x}{u}\right)^2. \tag{2b}$$

Here, x, y, and t are the independent variables; $u \equiv u(x,y,t)$ is the main dependent variable; and $\tilde{u} \equiv \tilde{u}(x,y,t)$ is an auxiliary-dependent variable; λ and μ are two arbitrary rational numbers,

$$\lambda = \frac{p_1}{q_1}, \quad \mu = \frac{p_2}{q_2}, \tag{3}$$

with p_1 and q_1 coprime integers and $q_1 > 0$, and likewise p_2 and q_2 coprime integers and $q_2 > 0$; ω is an arbitrary real number (without loss of generality nonnegative, $\omega \geq 0$); and the rest of the notation is, we trust, self-evident. Clearly, for $\omega = 0$, this PDE coincides with the SKdV equation (1), up to trivial notational changes. For $\omega > 0$, this modified Schwarzian Korteweg de Vries (mSKdV) equation (2) features a lot of periodic, indeed isochronous, solutions

$$u(x,y,t+T) = u(x,y,t) \tag{4a}$$

with period

$$T \equiv T(q) = \frac{2\pi q}{\omega} \tag{4b}$$

(or possibly with a period which is a rational multiple of T). Here, q is the minimum common multiple of q_1 and q_2 [see (3)]. Indeed, as we shall detail in Section 3, this mSKdV equation (2) has been obtained from the SKdV equation (1) via a trick—amounting essentially to a change of dependent and independent variables—whose efficacy in yielding evolution equations possessing lots of periodic, indeed isochronous, solutions is obvious (see Section 3), as recently advertised (see [3–5], as well as several other publications where this trick has been used in the context of ODEs rather than PDEs [6–18]).

Note that our claim that the solution to the mSKdV equation (2) be periodic only refers to the main dependent variable $u(x,y,t)$. The corresponding auxiliary dependent variable $\tilde{u}(x,y,t)$ need not always be periodic, but it might instead satisfy a

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shifted periodicity property of the form

$$\tilde{u}(x,y,t+T)=\tilde{u}(x,y,t)+v(x,y). \qquad (4c)$$

We leave to the interested reader a discussion of this issue, including in each case the identification of conditions on the arbitrary functions contained in the solutions reported below sufficient to guarantee that the shift $v(x,y)$ vanishes.

Finally, let us emphasize that, although the mSKdV equation (2) is just as integrable as the SKdV equation (1) since it is related to it via a change of (dependent and independent) variables, in this paper we do not exploit this fact, limiting our treatment merely to the explicit display, and the analysis of the isochronous character, of several solutions, all of which were obtained via an appropriate ansatz based on symmetry considerations, without taking full advantage of the integrable character of these nonlinear PDEs.

2. SOME EXPLICIT SOLUTIONS

We now give some explicit solutions to the mSKdV equation (2) which confirm the claim made in the title of this paper. Clearly, the auxiliary function $\tilde{u}(x,y,t)$ is defined by (2b) up to an additive arbitrary function of the time t, which has no relevance on the time evolution of $u(x,y,t)$ and which itself has to be periodic [with a period that is an appropriate rational multiple of T—see (4b)], in order that $\tilde{u}(x,y,t)$ may be periodic in the variable t.

Hereafter, $f(t)$, $g(z)$, $h(z)$, and $F(z,t)$ denote various a priori arbitrary functions of their arguments, and $F_z(z,t)$ and $F_t(z,t)$ denote the derivatives of $F(z,t)$ with respect to its first and second argument. Of course, we use an appended prime to denote differentiation, so, for instance, $f'(t)\equiv df(t)/dt$. We generally assume $f(t)$ and $F(z,t)$ to be periodic in t with period $T\equiv T(q)$ [see (4b)],

$$f(t+T)=f(t), \quad F(z,t+T)=F(z,t), \qquad (5)$$

although in some cases it will be convenient to assume it is periodic with a period which is an integer multiple or a fraction of T. We assume the functions $g(z)$, $h(z)$, and $F(z,t)$ to be analytic in z; additional restrictions on these functions shall be detailed below, on a case by case basis, as well as on the values of the "space"-independent variables x and y, which are hereafter assumed to be real (although this restriction could be forsaken in most cases). These restrictions shall be introduced to guarantee periodicity (as functions of the real time variable t), indeed isochronicity [see (4)], of the solutions exhibited below to the mSKdV equation (2). We also use hereafter the shorthand notation

$$\rho\equiv\rho(t)=\exp(i\omega t), \qquad (6)$$

and we employ the notation α, β, γ, δ to denote various a priori arbitrary (possibly complex) constants, up to restrictions that shall be specified below on a case by case basis.

The first solution to the mSKdV equation (2) that we exhibit reads

$$u(x,y,t)=\rho^{\lambda}g(y\rho^{1-2\mu})h[x\rho^{\mu}-f(t)], \qquad (7a)$$

$$\tilde{u}(x,y,t)=-8y\rho^{-\mu}f'(t) + \rho^{\mu}\int_{\alpha}^{x\rho^{\mu}-f(t)}\left[\frac{h'(z)}{h(z)}\right]^2 dz. \qquad (7b)$$

This solution is well defined (nonsingular) if the integral in (7b) is.

Sufficient conditions to guarantee that (7a) be periodic as a function of the real independent variable t [for fixed x and y—see (4)] are as follows:

Condition 1. The analytic function $g(z)$ of the complex variable z is meromorphic in the open disc $|z|<|y|$ and has no poles (for fixed y) on the circle $|z|=|y|$; the first of these two requirements is, of course, automatically satisfied if the analytic function $g(z)$ of the complex variable z is meromorphic in the open disc $|z|<Y$, where Y is some positive number, and the independent variable y is then restricted to satisfy the inequality

$$|y|<Y \qquad (8)$$

(of course, if $g(z)$ is meromorphic in the entire complex z plane, one can set $Y=\infty$, entailing that this restriction on the independent variable y disappears); the second of these two requirements is, of course, automatically satisfied if $g(z)$ is holomorphic rather than just meromorphic.

Condition 2. The analytic function $h(z)$ of the complex variable z is meromorphic inside the domain in the complex z plane enclosed by the (closed!) curve $z(t)=x\exp(i\mu\omega t)-f(t)$ and has no poles on this curve; the first of these two requirements is, of course, automatically satisfied if the analytic function $h(z)$ of the complex variable z is meromorphic in the open disc $|z|<X+M$, where M is a positive constant such that $|f(t)|\leq M$ for all real values of t [note that M generally exists, since $f(t)$ is periodic—see (5)] and X is some positive number, and the independent variable x is then restricted to satisfy the inequality

$$|x|<X \qquad (9)$$

(of course, if $h(z)$ is meromorphic in the entire complex z plane, one can set $X = \infty$, entailing that this restriction on the independent variable x disappears); the second of these two requirements is, of course, automatically satisfied if $g(z)$ is holomorphic rather than just meromorphic.

The second solution to the mSKdV equation (2) that we exhibit reads

$$u(x,y,t) = \frac{\rho^\lambda g(y\rho^{1-2\mu})}{[x\rho^\mu + f(t) + h(\gamma y\rho^{1-2\mu} - \rho)]^2}, \tag{10a}$$

$$\tilde{u}(x,y,t) = \frac{8}{\gamma}[\gamma y\rho^{-\mu} f'(t) - i\omega\rho^\mu h(\gamma y\rho^{1-2\mu} - \rho)] - \frac{4\rho^\mu}{x\rho^\mu + f(t) + h(\gamma y\rho^{1-2\mu} - \rho)}. \tag{10b}$$

To write this solution, we assumed $\gamma \neq 0$.

Sufficient conditions to guarantee that solution (10) be nonsingular and periodic as a function of the real independent variable t [for fixed x and y, see (4)] are, in addition to condition 1 (see above), the following:

Condition 3. The analytic function $h(z)$ of the complex variable z is meromorphic inside the domain in the complex z plane enclosed by the (closed!) curve $z(t) = \gamma y \exp[i(1-2\mu)\omega t] - \exp(i\omega t)$ and has no poles on this curve; the first of these two requirements is, of course, automatically satisfied if the analytic function $h(z)$ of the complex variable z is meromorphic in the open disc $|z| < 1 + |\gamma|Y$, where Y is some positive number, and the independent variable y is then restricted to satisfy the inequality $|y| < Y$ (of course, if $h(z)$ is meromorphic in the entire complex z plane, one can set $Y = \infty$, entailing that this restriction on the independent variable y disappears); the second of these two requirements is, of course, automatically satisfied if $g(z)$ is holomorphic rather than just meromorphic.

Condition 4. The following inequality holds for all real values of t:

$$x\rho^\mu + f(t) + h(\gamma y\rho^{1-2\mu} - \rho) \neq 0; \tag{11}$$

since $f(t)$ [see (5)] and the function $h(\gamma y\rho^{1-2\mu} - \rho)$ [thanks to condition 3—see (6)] are periodic functions of the real variable t, it is possible that conditions of type (9) and (8) exist, with an appropriate assignment of the two positive constants X and Y (depending, of course, on the two functions $f(t)$ and $h(z)$), which are sufficient to guarantee validity of inequality (11).

The third solution to the mSKdV equation (2) that we exhibit reads

$$u(x,y,t) = \frac{\rho^\lambda g(y\rho^{1-2\mu})}{\cosh^2[\beta x\rho^\mu + f(t) + h(\gamma y\rho^{1-2\mu} - \rho)]}, \tag{12a}$$

$$\tilde{u}(x,y,t) = \frac{8}{\beta\gamma} \times [\gamma y\rho^{-\mu} f'(t) - i\omega\rho^\mu h(\gamma y\rho^{1-2\mu} - \rho)] + 4\beta\rho^\mu \times (\beta x\rho^\mu - \tanh[\beta x\rho^\mu + f(t) + h(\gamma y\rho^{1-2\mu} - \rho)]). \tag{12b}$$

To write this solution, we assumed $\beta\gamma \neq 0$.

Sufficient conditions to guarantee that solution (12) be nonsingular and periodic as a function of the real independent variable t [for fixed x and y, see (4)] are, in addition to conditions 1 and 3 (see above), the following:

Condition 5. The following inequality holds for all real values of t and integer values of k:

$$\beta x\rho^\mu + f(t) + h(\gamma y\rho^{1-2\mu} - \rho) \neq i(1+2k)\frac{\pi}{2}; \tag{13}$$

since $f(t)$ [see (5)] and the function $h(\gamma y\rho^{1-2\mu} - \rho)$ [thanks to condition 3—see (6)] are periodic functions of the real variable t, it is possible that conditions of type (9) and (8) exist, with an appropriate assignment of the two positive constants X and Y (depending, of course, on the two functions $f(t)$ and $h(z)$), which are sufficient to guarantee validity of inequality (13).

The fourth solution to the mSKdV equation (2) that we exhibit reads

$$u(x,y,t) = \frac{\rho^\lambda g(y\rho^{1-2\mu})}{1 + \sin[\beta x\rho^\mu + f(t) + h(\gamma y\rho^{1-2\mu} - \rho)]}, \tag{14a}$$

$$\tilde{u}(x,y,t) = \frac{8y\rho^{-\mu}}{\beta} f'(t) + \left(\beta - \frac{8i\omega}{\beta\gamma}\right)\rho^\mu h(\gamma y\rho^{1-2\mu} - \rho) + \frac{4\beta\rho^\mu}{1 + \cot\{\frac{1}{2}[\beta x\rho^\mu + f(t) + h(\gamma y\rho^{1-2\mu} - \rho)]\}} - \beta\rho^\mu[\beta x\rho^\mu + f(t) + h(\gamma y\rho^{1-2\mu} - \rho)]. \tag{14b}$$

To write this solution, we again assumed $\beta\gamma \neq 0$.

Sufficient conditions to guarantee that solution (14) be nonsingular and periodic as a function of the real independent variable t [for fixed x and y, see (4)] are, in addition to conditions 1 and 3 (see above), the following:

Condition 6. The following inequalities hold for all real values of t and integer values of k:

$$\beta x\rho^{\mu} + f(t) + h(\gamma y\rho^{1-2\mu} - \rho) \neq (1+2k)\frac{\pi}{2}, \tag{15a}$$

$$\beta x\rho^{\mu} + f(t) + h(\gamma y\rho^{1-2\mu} - \rho) \neq k\pi; \tag{15b}$$

since $f(t)$ [see (5)] and the function $h(\gamma y\rho^{1-2\mu} - \rho)$ [thanks to condition 3—see (6)] are periodic functions of the real variable t, it is possible that conditions of type (9) and (8) exist, with an appropriate assignment of the two positive constants X and Y (depending, of course, on the two functions $f(t)$ and $h(z)$), which are sufficient to guarantee validity of inequalities (15).

The fifth solution to the mSKdV equation (2) that we exhibit reads

$$u(x,y,t) = \frac{\rho^{\lambda+\mu(2\alpha-1)}g(y\rho^{1-2\mu})[x-f(t)]^{2\alpha-1}}{\{y^{\alpha}\rho^{\alpha}[x-f(t)]^{2\alpha}+\beta\}^2}, \tag{16a}$$

$$\tilde{u}(x,y,t) = -8y[i\omega\mu f(t) + f'(t)] - \frac{\beta(2\alpha-1)^2 + (2\alpha+1)^2 y^{\alpha}\rho^{\alpha}[x-f(t)]^{2\alpha}}{\{y^{\alpha}\rho^{\alpha}[x-f(t)]^{2\alpha}+\beta\}[x-f(t)]}. \tag{16b}$$

Sufficient conditions to guarantee that solution (16) be nonsingular and periodic as a function of the real independent variable t [for fixed x and y, see (4)] are, in addition to condition 1 (see above), the following:

Condition 7. α is integer, or perhaps it is a rational number, in which case solution (16) would still be periodic but might feature a period which is an integer multiple of T [see (4b)].

Condition 8. The following inequality holds for all real values of t:

$$x - f(t) \neq 0; \tag{17}$$

this requirement is, of course, satisfied if the independent variable x is restricted to satisfy one of the two inequalities

$$|x| < m \tag{18a}$$

or

$$|x| > M, \tag{18b}$$

where m and M are constants such that $m \leq |f(t)| \leq M$ [note that m and M generally exist, since $f(t)$ is periodic—see (5)].

Condition 9. The following inequality holds for all real values of t:

$$y^{\alpha}\rho^{\alpha}[x-f(t)]^{2\alpha} + \beta \neq 0; \tag{19a}$$

this requirement is, of course, satisfied if

$$|y|^{\alpha}|x-f(t)|^{2\alpha} \neq |\beta|; \tag{19b}$$

and, since there exist two constants m and M such that $m \leq |f(t)| \leq M$ [as $f(t)$ is periodic—see (5)], this condition can be reduced to conditions of type (9) and (8), with an appropriate assignment of the two positive constants X and Y (depending, of course, on the values of the constants α, m, and M).

The sixth solution to the mSKdV equation (2) that we exhibit reads

$$u(x,y,t) = \frac{\rho^{\lambda-\mu}g(y\rho^{1-2\mu})}{[x-f(t)][\ln\{y\rho[x-f(t)]^2\}+\beta]^2}, \tag{20a}$$

$$\tilde{u}(x,y,t) = -8y[i\omega\mu f(t) + f'(t)] - \frac{8+\beta+\ln\{y\rho[x-f(t)]^2\}}{[xf(t)][\ln\{y\rho[x-f(t)]^2\}+\beta]}. \tag{20b}$$

Sufficient conditions to guarantee that solution (20) be nonsingular and periodic as a function of the real independent variable t [for fixed x and y, see (4)] are, in addition to conditions 1 and 8 (see above), the following:

Condition 10. The independent variable x and the arbitrary (periodic) function $f(t)$ are such that the (closed) curve in the complex z plane $z \equiv z(t) = \exp(i\omega t)[x-f(t)]^2$ does not contain the origin $z = 0$. A simple example is $f(t) = X\exp(i\omega t/2)$ (with X an arbitrary positive constant) and $|x| < X$, which entails that $f(t)$ is periodic with period $T(2)$ [see (4b)] and that solution (20) is periodic with period $T(r)$, where r is the minimum common multiple of q and 2 [see (3) and (4)]. Of course, many other examples could be given.

Condition 11. The following inequality holds:

$$y \neq 0. \tag{21}$$

Condition 12. The following inequality holds for all real values of t:

$$\ln\{y\rho[x-f(t)]^2\} + \beta \neq 0; \tag{22a}$$

this requirement is, of course, satisfied if

$$|y||x-f(t)|^2 \neq |\exp(-\beta)| \equiv \exp(-\mathrm{Re}(\beta)); \tag{22b}$$

and, since there exist two constants m and M such that $m \leq |f(t)| \leq M$ [as $f(t)$ is periodic—see (5)], this condition can be reduced to conditions of type (9) and (8), with an appropriate assignment of the two positive constants X and Y (depending, of course, on the values of the constants m and M).

The seventh solution to the mSKdV equation (2) that we exhibit reads

$$u(x,y,t) = \frac{\rho^{\lambda-\mu} g(y\rho^{1-2\mu})[x-f(t)]^{-1}}{-1+\sin[\alpha\ln\{y\rho[x-f(t)]^2\}+\beta]}, \tag{23a}$$

$$\tilde{u}(x,y,t) = -8y[i\omega\mu f(t)+f'(t)] + [x-f(t)]^{-1} \times\left\{4\alpha^2-1+4\alpha\frac{\cos\left[\frac{2\beta-\pi}{4}+\frac{\alpha}{2}\ln\{y\rho[x-f(t)]^2\}\right]}{\cos\left[\frac{2\beta+\pi}{4}+\frac{\alpha}{2}\ln\{y\rho[x-f(t)]^2\}\right]}\right\}. \tag{23b}$$

Sufficient conditions to guarantee that solution (23) be nonsingular and periodic as a function of the real independent variable t [for fixed x and y, see (4)] are, in addition to conditions 1, 8, 10, and 11 (see above), the following:

Condition 13. The following inequality holds for all real values of t and integer values of k:

$$\alpha\ln\{y\rho[x-f(t)]^2\}+\beta \neq k\frac{\pi}{2}; \tag{24a}$$

this requirement is, of course, satisfied if

$$|y|^{\alpha}|x-f(t)|^{2\alpha} \neq \left|\exp\left(k\frac{\pi}{2}-\beta\right)\right| \equiv \exp\left(k\frac{\pi}{2}-\mathrm{Re}(\beta)\right); \tag{24b}$$

and, since there exist two constants m and M such that $m \leq |f(t)| \leq M$ [as $f(t)$ is periodic—see (5)], this condition can be reduced to conditions of type (9) and (8), with an appropriate assignment of the two positive constants X and Y (depending, of course, on the values of the constants α, m, and M).

The eighth solution to the mSKdV equation (2) that we exhibit reads

$$u(x,y,t) = \rho^{\lambda} g(y\rho^{1-2\mu}) \times \mathrm{sn}^{c_1}[\alpha x\rho^{\mu}+F(y,t)|\beta] \times \mathrm{cn}^{c_2}[\alpha x\rho^{\mu}+F(y,t)|\beta]\mathrm{dn}^{c_3}[\alpha x\rho^{\mu}+F(y,t)|\beta], \tag{25a}$$

$$\tilde{u}(x,y,t) = \frac{8\rho^{-\mu}}{\alpha}\int_{\delta}^{y}[F_t(z,t) - i\omega(1-2\mu)yF_z(z,t)]dz + \alpha\big[(\beta+1)c_1^2 + (1-2\beta)c_2^2 + (\beta-2)c_3^2 + 2(2-\beta)c_1c_2 + 2(2\beta-1)c_1c_3 - 2(\beta+1)c_2c_3\big] \times \rho^{\mu}F(y,t) + \alpha\rho^{\mu} \times \int_{\gamma}^{\alpha x\rho^{\mu}+F(y,t)}\left[c_2\frac{\mathrm{sn}(z|\beta)\mathrm{dn}(z|\beta)}{\mathrm{cn}(z|\beta)} - c_1\frac{\mathrm{cn}(z|\beta)\mathrm{dn}(z|\beta)}{\mathrm{sn}(z|\beta)} + c_3\beta\frac{\mathrm{sn}(z|\beta)\mathrm{cn}(z|\beta)}{\mathrm{dn}(z|\beta)}\right]^2 dz. \tag{25b}$$

Here, $\mathrm{sn}(z|\beta)$, $\mathrm{cn}(z|\beta)$, $\mathrm{dn}(z|\beta)$ denote Jacobian elliptic functions and c_1, c_2, c_3 are constants such that each c_i and the sum $c_1+c_2+c_3$ take one of the three values 2, 0, -2.

To write this solution, we assumed $\alpha\gamma \neq 0$.

Sufficient conditions to guarantee that (25a) be nonsingular and periodic as a function of the real independent variable t [for fixed x and y, see (4)] are, in addition to condition 1 (see above), the following:

Condition 14. The arbitrary function $F(y,t)$ is periodic, as a function of the real variable t, with period T [see (4b)]. Of course, $F(y,t)$ could be periodic with a period which is a rational multiple of T, in which case solution (25) would still be periodic, but might feature a period that is an integer multiple of T.

Condition 15. The following inequality holds for all real values of t:

$$\alpha x\rho^{\mu}+F(y,t) \neq 0. \tag{26}$$

This condition is imposed to avoid the divergence of the second term in the integral in (25b) at $z=0$. Since $F(y,t)$ is periodic in the real variable t, there generally exist two functions $\bar{m}(y)$ and $\bar{M}(y)$ such that, for all real values of t, $\bar{m}(y) \leq |F(y,t)| \leq \bar{M}(y)$; and, of course, the requirement (26) is satisfied if the variables x and y are restricted to domains such that one of the following two inequalities holds:

$$|\alpha||x| < \bar{m}(y) \tag{27a}$$

or

$$|\alpha||x| > \bar{M}(y). \tag{27b}$$

The ninth solution to the mSKdV equation (2) that we exhibit reads

$$u(x,y,t) = \exp(\alpha x\rho^{\mu})F(y,t), \tag{28a}$$

$$\tilde{u}(x,y,t) = \alpha^2 x\rho^{2\mu} + \frac{8\rho^{-\mu}}{\alpha} \times\left[-i\omega\lambda y + \int_{\gamma}^{y}\frac{F_t(z,t)-i\omega(1-2\mu)zF_z(z,t)}{F(z,t)}dz\right]. \tag{28b}$$

To write this solution, we assumed that $\alpha \neq 0$ and that y is restricted in a domain such that the integral in (28b) is well defined (nonsingular).

A sufficient condition to guarantee that (28a) be periodic as a function of the real independent variable t [for fixed x and y, see (4)] is condition 14 (see above).

The tenth solution to the mSKdV equation (2) that we exhibit reads

$$u(x,y,t) = \exp\left(\pm\frac{2x\rho^{\mu}\sqrt{\alpha+i\omega y\rho^{1-2\mu}}}{\sqrt{\beta-\rho}}\right)F(y,t), \tag{29a}$$

$$\tilde{u}(x,y,t) = \frac{4x(i\omega y\rho+\alpha\rho^{2\mu})}{\beta-\rho} \pm 4\rho^{-\mu}\sqrt{(\beta-\rho)} \times \int_{\gamma}^{y} \frac{F_t(z,t) - i\omega(1-2\mu)zF_z(z,t) - i\omega\lambda F(z,t)}{\sqrt{\alpha+i\omega z\rho^{1-2\mu}}F(z,t)}dz. \tag{29b}$$

To write this solution, we assumed $|\beta| \neq 1$; it is well defined (nonsingular) if the integral in (29b) is.

A sufficient condition to guarantee that (29a) be periodic as a function of the real independent variable t [for fixed x and y, see (4)] is condition 14 (see above).

The eleventh solution to the mSKdV equation (2) that we exhibit reads

$$u(x,y,t) = \exp\Big[\pm x\rho^{\mu}\big\{\alpha\rho+\beta \pm \big[(\alpha\rho+\beta)^2+8i\omega\alpha y\rho^{1-2\mu}\big]^{1/2}\big\}^{1/2}\Big]F(y,t), \tag{30a}$$

$$\tilde{u}(x,y,t) = x\rho^{2\mu}\{\alpha\rho+\beta \pm [(\alpha\rho+\beta)^2+8i\omega\alpha y\rho^{1-2\mu}]^{1/2}\} \pm 8\rho^{-\mu} \times \int_{\gamma}^{y} \frac{F_t(z,t) - i\omega\lambda F(z,t) - i\omega(1-2\mu)zF_z(z,t)}{\left\{\alpha\rho+\beta \pm [(\alpha\rho+\beta)^2+8i\omega\alpha y\rho^{1-2\mu}]^{1/2}\right\}^{1/2}F(z,t)}dz. \tag{30b}$$

This solution is defined if the integral in (30b) is.

A sufficient condition to guarantee that (30a) be periodic as a function of the real independent variable t [for fixed x and y, see (4)] is condition 14 (see above).

The first eight of these solutions, (7), (10), (12), (14), (16), (20), (23), and (25), have been obtained, via the technique described in the following section, from solutions to the SKdV equation (1a) given in [1] and [19] (trivial notational changes have also been introduced: arbitrary functions have been conveniently redefined).

The other three solutions, (28), (29), and (30), have been obtained, via the technique described in the following section, from solutions to the SKdV equation (1) obtained via the following ansatz:

$$w(\xi,\eta,\tau) = \exp[\xi A(\eta,\tau)]B(\eta,\tau). \tag{31}$$

Here, $A(\eta,\tau)$ and $B(\eta,\tau)$ are a priori arbitrary, but analytic, functions. It is indeed easy to verify that ansatz (31) yields a solution to (1) (with a suitable choice of $\tilde{w}(\xi,\eta,\tau)$) provided

$$4A_{\tau}(\eta,\tau) = A^2(\eta,\tau)A_{\eta}(\eta,\tau). \tag{32a}$$

This first-order PDE (32a) can be solved in implicit form: indeed, for any arbitrary (differentiable and with nonvanishing gradient) function $H(z_1,z_2)$, it is equivalent to the nondifferential equation

$$H\left[A(\eta,\tau), \tau+\frac{4\eta}{A^2(\eta,\tau)}\right] = 0. \tag{32b}$$

Using the substitution described in the next section, we thereby get solutions to the mSKdV equation (2): indeed, if $A[\eta,\tau]$ satisfies (32b), we find that

$$u(x,y,t) = \exp[xG(y,t)]F(y,t), \tag{33a}$$

$$\tilde{u}(x,y,t) = xG^2(y,t) - 8\int_{0}^{y} \frac{i\omega\lambda F(z,t) - F_t(z,t) + 8i\omega(1-2\mu)zF_z(z,t)}{F(z,t)G(z,t)}dz \tag{33b}$$

is a solution to the mSKdV equation (2) with

$$G(y,t) = \rho^{\mu} A\left(y\rho^{1-2\mu}, \frac{\rho-1}{i\omega}\right) \tag{34}$$

and $F(y,t) = \rho^{\lambda} B\left(y\rho^{1-2\mu}, \frac{\rho-1}{i\omega}\right)$ being an arbitrary function. On the other hand, it is easy to find some particular solutions to (32), and we got some simple examples by choosing $H(z_1, z_2)$ to be a polynomial (of degree <3) in the complex variables z_1, z_2. After some simplifications, we thus obtained for $G(y,t)$ the following possibilities:

$$G(y,t) = \alpha\rho^{\mu}, \tag{35a}$$

$$G(y,t) = \pm\frac{2\rho^{\mu}\sqrt{\alpha + i\omega y\rho^{1-2\mu}}}{\sqrt{\beta-\rho}}, \tag{35b}$$

$$G(y,t) = \pm\rho^{\mu}\Big\{\alpha\rho + \beta \pm \big[(\alpha\rho+\beta)^2 + 8i\omega\alpha y\rho^{1-2\mu}\big]^{1/2}\Big\}^{1/2}, \tag{35c}$$

which correspond to the last three solutions, (28), (29), and (30), reported above. Of course, many additional examples could be given.

3. DERIVATION OF THE MODIFIED SCHWARZIAN KORTEWEG–DE VRIES EQUATION

In this section, we indicate how the mSKdV equation (2) is related to the SKdV equation (1), and we thereby justify the expectation that this evolution PDE (2) possesses lots of periodic, indeed isochronous, solutions [see (4)]—as already mentioned in the introductory Section 1 and confirmed by the examples reported in Section 2.

Let us introduce [see (6)] the following change of (independent and dependent) variables ("the trick"):

$$u(x,y,t) = \rho^{\lambda} w(\xi,\eta,\tau), \tag{36a}$$

$$\tilde{u}(x,y,t) = \rho^{\mu}\tilde{w}(\xi,\eta,\tau), \tag{36b}$$

$$\xi \equiv \xi(x,t) = x\rho^{\mu}, \tag{36c}$$

$$\eta \equiv \eta(y,t) = y\rho^{\nu}, \tag{36d}$$

$$\tau \equiv \tau(t) = \frac{\rho-1}{i\omega}. \tag{36e}$$

Note that this change of variables entails no change at the "initial" time, $t = 0$:

$$u(x,y,0) = w(x,y,0), \qquad \tilde{u}(x,y,0) = \tilde{w}(x,y,0); \tag{37}$$

and it is plain that this change of variables (36) entails the following relations:

$$u_t(x,y,t) - i\lambda\omega u(x,y,t) - i\mu\omega x u_x(x,y,t) - i\nu\omega y u_y(x,y,t) = \rho^{\lambda+1} w_{\tau}(\xi,\eta,\tau), \tag{38a}$$

$$u_x(x,y,t) = \rho^{\lambda+\mu} w_{\xi}(\xi,\eta,\tau), \tag{38b}$$

$$u_y(x,y,t) = \rho^{\lambda+\nu} w_{\eta}(\xi,\eta,\tau), \tag{38c}$$

$$u_{xx}(x,y,t) = \rho^{\lambda+2\mu} w_{\xi\xi}(\xi,\eta,\tau), \tag{38d}$$

$$u_{xy}(x,y,t) = \rho^{\lambda+\mu+\nu} w_{\xi\eta}(\xi,\eta,\tau), \tag{38e}$$

$$u_{xxy}(x,y,t) = \rho^{\lambda+2\mu+\nu} w_{\xi\xi\eta}(\xi,\eta,\tau), \tag{38f}$$

$$\tilde{u}_x(x,y,t) = \rho^{2\mu}\tilde{w}_{\xi}(\xi,\eta,\tau), \tag{38g}$$

$$\tilde{u}_y(x,y,t) = \rho^{\mu+\nu}\tilde{w}_{\eta}(\xi,\eta,\tau). \tag{38h}$$

Therefore, via this change of dependent and independent variables (36), the SKdV equation (1) gets transformed into the following PDE:

$$\{u_t - i\lambda\omega u - i\mu\omega x u_x - i\nu\omega y u_y\}\rho^{\nu+2\mu-1} + \frac{1}{4}u_{xxy} - \frac{u_x u_{xy}}{2u} - \frac{u_{xx}u_y}{4u} + \frac{u_x^2 u_y}{2u^2} - \frac{u_x\tilde{u}_y}{8} = 0, \tag{39a}$$

$$\tilde{u}_x = \left(\frac{u_x}{u}\right)^2; \tag{39b}$$

and the assignment

$$\nu = 1 - 2\mu \tag{40}$$

immediately yields the autonomous equation (2). Hence, to every solution to the SKdV equation (1) there corresponds, via the change of (independent and dependent) variables (36) with (40), a corresponding solution to the mSKdV equation (2), which often turns out to be periodic in t with period T [see (4)], since the rational character of λ and μ [see (3)], hence of ν as well [see (40) and (3)], clearly entails a periodic dependence on the (real) "time" variable t with period T [see (4b)], of $\xi(x,t)$, $\eta(y,t)$, and $\tau(t)$ [see (36c), (36d), and (36e)], hence as well of the solutions $u(x,y,t)$, $\tilde{u}(x,y,t)$ to the mSKdV equation (2)—at least whenever these solutions $u(x,y,t)$, $\tilde{u}(x,y,t)$ to the mSKdV equation (2) correspond to solutions $w(\xi,\eta,\tau)$, $\tilde{w}(\xi,\eta,\tau)$ to the SKdV equation (1) which are meromorphic in the variables ξ, η, and τ, for τ in the circular disk [see (36e)] $|1 + i\omega\tau| \leq 1$ (with no poles on the boundary of this disk) and for the values of ξ and η yielded by (36c) and (36d) (with t real but otherwise unrestricted, and with x and y possibly restricted to appropriate domains).

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Fractional Supersymmetry as a Superposition of Ordinary Supersymmetry*

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Abstract—It is shown how to derive fractional supersymmetric quantum mechanics of order k as a superposition of $k-1$ copies of ordinary supersymmetric quantum mechanics. © *2005 Pleiades Publishing, Inc.*

1. INTRODUCTION

In recent years, fractional supersymmetry has been the subject of numerous works. Indeed, k-fractional supersymmetry is closely connected to the notion of quantum algebra (deformation theory) and to the concept of intermediate statistics (of anyon [1] and k fermions [2, 3]) interpolating between Bose–Einstein statistics and Fermi–Dirac statistics. Therefore, fractional supersymmetry constitutes a useful tool for dealing with anyonic statistics.

Fractional supersymmetric quantum mechanics of order k can be considered as an extension of ordinary supersymmetric quantum mechanics which corresponds to $k=2$. An ordinary supersymmetric quantum-mechanical system may be generated from a doublet $(H,Q)_2$ of operators satisfying [4, 5]

$$Q^2=0,$$

$$QQ^\dagger+Q^\dagger Q=H.$$

The self-adjoint operator H and the operator Q act on a separable Hilbert space. The operator H is referred to as the Hamiltonian and the operator Q as the supersymmetry operator of the ordinary supersymmetric quantum-mechanical system. The operator Q gives rise to $\mathcal{N}=2$ dependent supercharges $Q_-=Q$ and $Q_+=Q^\dagger$ connected via Hermitian conjugation. They are nilpotent operators of order $k=2$ and commute with the Hamiltonian H.

The ordinary supersymmetric quantum-mechanical system $(H,Q)_2$ can be extended to a fractional supersymmetric quantum-mechanical system $(H,Q)_k$ with $k\in\mathbf{N}\setminus\{0,1,2\}$ as follows. The system $(H,Q)_k$ may be defined by [6, 7]

$$Q_-=Q,\quad Q_+=Q^\dagger\ (\Rightarrow Q_+=Q_-^\dagger),\qquad (1a)$$
$$Q_\pm^k=0,$$

$$Q_-^{k-1}Q_+ + Q_-^{k-2}Q_+Q_- + \ldots \qquad (1b)$$
$$+ Q_+Q_-^{k-1} = Q_-^{k-2}H,$$

$$[H,Q_\pm]=0,\quad H=H^\dagger, \qquad (1c)$$

where the self-adjoint operator H, the Hamiltonian of the system, and the $\mathcal{N}=2$ supercharges Q_- and Q_+ act on a separable Hilbert space. Of course, the case $k=2$ corresponds to an ordinary supersymmetric quantum-mechanical system.

In the present work, we study how it is possible to connect ordinary and k-fractional supersymmetric quantum-mechanical systems.

2. THE ALGEBRA W_k

As an interesting question, we may ask: How to construct a fractional supersymmetric quantum-mechanical system of order k and, thus, fractional supersymmetric quantum mechanics of order k? This question can be answered through the definition of a generalized Weyl–Heisenberg algebra W_k. We now define the generic algebra W_k and shall see in the next section how a fractional supersymmetric quantum-mechanical system of order k may be associated with a given algebra W_k.

For k given, with $k\in\mathbf{N}\setminus\{0,1\}$, the algebra W_k is generated by four linear operators X_-, X_+, N, and K. The operators X_- and $X_+=X_-^\dagger$ are shift operators connected via Hermitian conjugation. The operator N, called number operator, is self-adjoint. Finally,

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the operator K is a Z_k-grading unitary operator. The generators X_-, X_+, N, and K satisfy [8]

$$[X_-, X_+] = \sum_{s=0}^{k-1} f_s(N)\Pi_s,$$

$$[N, X_-] = -X_- \ (+\text{ h.c.}),$$

$$[K, X_-]_q = 0 \ (+\text{ h.c.}),$$

$$[K, N] = 0, \quad K^k = 1.$$

The functions $f_s : N \mapsto f_s(N)$ are such that $f_s(N)^\dagger = f_s(N)$, $[A, B]_q$ stands for $AB - qBA$, and the operators Π_s are defined by

$$\Pi_s = \frac{1}{k}\sum_{t=0}^{k-1} q^{-st}K^t,$$

where

$$q = \exp\left(\frac{2\pi i}{k}\right)$$

is a root of unity. To a given set $\{f_s : s = 0, 1, \ldots, k-1\}$ corresponds one algebra W_k.

The generalized Weyl–Heisenberg algebra W_k covers numerous algebras describing exactly solvable one-dimensional systems. The particular system corresponding to a given set $\{f_s : s = 0, 1, \ldots, k-1\}$ yields, in a Schrödinger picture, a particular dynamical system with a specific potential. Let us mention two interesting cases. The case

$$\forall s \in \{0, 1, \ldots, k-1\} : f_s(N) = f_s \text{ independent of } N$$

corresponds to systems with cyclic shape-invariant potentials (in the sense of [9]), and the case

$$\forall s \in \{0, 1, \ldots, k-1\} : f_s(N) = aN + b, (a, b) \in \mathbf{R}^2$$

corresponds to systems with translational shape-invariant potentials (in the sense of [10]). For instance, the case $(a = 0, b > 0)$ corresponds to the harmonic oscillator potential, the case $(a < 0, b > 0)$ to the Morse potential, and the case $(a > 0, b > 0)$ to the Pöschl–Teller potential. For these various potentials, the part of W_k spanned by X_-, X_+, and N can be identified with the ordinary Weyl–Heisenberg algebra for $(a = 0, b \neq 0)$, with the $su(2)$ Lie algebra for $(a < 0, b > 0)$ and with the $su(1,1)$ Lie algebra for $(a > 0, b > 0)$.

3. A k-FRACTIONAL SYSTEM ASSOCIATED WITH W_k

In order to associate a k-fractional supersymmetric quantum-mechanical system associated with a given generalized Weyl–Heisenberg algebra W_k, we must define a supersymmetry operator Q and a Hamiltonian H. The supersymmetry operator Q is defined by

$$Q \equiv Q_- = X_-(1 - \Pi_1) \Leftrightarrow Q^\dagger \equiv Q_+ = X_+(1 - \Pi_0).$$

Then, the Hamiltonian H associated with W_k can be deduced from Eq. (1b). This yields

$$H = (k-1)X_+X_- - \sum_{s=3}^{k}\sum_{t=2}^{s-1}(t-1)f_t(N-s+t)\Pi_s - \sum_{s=1}^{k-1}\sum_{t=s}^{k-1}(t-k)f_t(N-s+t)\Pi_s.$$

(Note that the summation from $s = k-2$ to $s = k$ appearing in some previous works by the authors [8] should be replaced by a summation from $s = 3$ to $s = k$.) It can be checked that H is self-adjoint and commutes with Q_- and Q_+. As a conclusion, the doublet $(H, Q)_k$ associated to W_k satisfies Eq. (1) and thus defines a k-fractional supersymmetric quantum-mechanical system.

4. CONNECTION BETWEEN FRACTIONAL SUPERSYMMETRY AND ORDINARY SUPERSYMMETRY

In order to establish a connection between fractional supersymmetric quantum mechanics of order k and ordinary supersymmetric quantum mechanics (corresponding to $k = 2$), it is necessary to construct subsystems from the doublet $(H, Q)_k$ that correspond to ordinary supersymmetric quantum-mechanical systems. This may be achieved in the following way [11]. The general Hamiltonian H can be rewritten as

$$H = \sum_{s=1}^{k} H_s\Pi_s,$$

where

$$H_s \equiv H_s(N) = (k-1)X_+X_- - \sum_{t=2}^{k-1}(t-1)f_t(N-s+t) + (k-1)\sum_{t=s}^{k-1} f_t(N-s+t).$$

It can be shown that the operators $H_k \equiv H_0$, $H_{k-1}, \ldots, H_1$ turn out to be isospectral operators. It is possible to factorize H_s as [11]

$$H_s = X(s)_+ X(s)_-.$$

Let us now define (i) the two (supercharge) operators

$$q(s)_- = X(s)_- \Pi_s, \quad q(s)_+ = X(s)_+ \Pi_{s-1}$$

and (ii) the (Hamiltonian) operator

$$h(s) = X(s)_- X(s)_+ \Pi_{s-1} + X(s)_+ X(s)_- \Pi_s.$$

It is then a simple matter of calculation to prove that $h(s)$ is self-adjoint and that

$$q(s)_+ = q(s)_-^\dagger, \quad q(s)_\pm^2 = 0,$$
$$h(s) = \{q(s)_-, q(s)_+\}, \quad [h(s), q(s)_\pm] = 0.$$

Consequently, the doublet $(h(s), q(s))_2$, with $q(s) \equiv q(s)_-$, satisfies Eq. (1) with $k = 2$ and thus defines an ordinary supersymmetric quantum-mechanical system (corresponding to $k = 2$).

The Hamiltonian $h(s)$ is amenable to a form more appropriate for discussing the link between ordinary supersymmetry and fractional supersymmetry. Indeed, we can show that

$$X(s)_- X(s)_+ = H_s(N+1).$$

Then, we can obtain the important relation

$$h(s) = H_{s-1}\Pi_{s-1} + H_s \Pi_s$$

to be compared with the expansion of H in terms of supersymmetric partners H_s.

As a result, the system $(H, Q)_k$, corresponding to k-fractional supersymmetry, can be described in terms of $k-1$ subsystems $(h(s), q(s))_2$, corresponding to ordinary supersymmetry. The Hamiltonian $h(s)$ is given as a sum involving the supersymmetric partners H_{s-1} and H_s. Since the supercharges $q(s)_\pm$ commute with the Hamiltonian $h(s)$, it follows that

$$H_{s-1} X(s)_- = X(s)_- H_s,$$
$$H_s X(s)_+ = X(s)_+ H_{s-1}.$$

As a consequence, the operators $X(s)_+$ and $X(s)_-$ make it possible to pass from the spectrum of H_{s-1} and H_s to the one of H_s and H_{s-1}, respectively. This result is quite familiar for ordinary supersymmetric quantum mechanics (corresponding to $s = 2$).

For $k = 2$, the operator $h(1)$ is nothing but the total Hamiltonian H corresponding to ordinary supersymmetric quantum mechanics. For arbitrary k, the other operators $h(s)$ are simple replicas (except for the ground state of $h(s)$) of $h(1)$. In this sense, fractional supersymmetric quantum mechanics of order k can be considered as a set of $k-1$ replicas of ordinary supersymmetric quantum mechanics corresponding to $k = 2$ and typically described by $(h(s), q(s))_2$. As a further argument, it is to be emphasized that

$$H = q(2)_- q(2)_+ + \sum_{s=2}^{k} q(s)_+ q(s)_-,$$

which can be identified with $h(2)$ for $k = 2$.

5. CONCLUSIONS

Starting from a Z_k-graded algebra W_k characterized by a set $\{f_s : s = 0, 1, \ldots, k-1\}$ of structure functions, it was shown how to associate a k-fractional supersymmetric quantum-mechanical system $(H, Q)_k$ characterized by an Hamiltonian H and a supercharge Q.

The Hamiltonian H for the system $(H, Q)_k$ was developed as a superposition of k isospectral supersymmetric partners $H_k, H_{k-1}, \ldots, H_1$. It was proved that the system $(H, Q)_k$ can be described in terms of $k-1$ subsystems $(h(s), q(s))_2$ which are ordinary supersymmetric quantum-mechanical systems.

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Analytic Proof of the Sutherland Conjecturet*

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Abstract—Using the integral representation of the inverse of the logarithmic derivative of the elliptic theta function, the spectrum of the Lax matrix for the 1D system of particles interacting via an inverse sinh-squared potential is shown to be given by the asymptotic Bethe ansatz in the thermodynamic limit.

The problem of the verification of the asymptotic Bethe ansatz method [1] still remains unsolved more than 30 years after its first presentation. The method consists in using only scattering data for the description of integrable many-body systems in the thermodynamic limit. It is well known, that if the system is integrable in the Yang–Baxter sense, the many-body scattering matrix is expressed via only a two-particle phase shift, but the real structure of the wave functions might be rather complicated if the interaction is nonlocal. In particular, when the two-body potential is of the form $\sin^{-2}(\pi x/L)$, where L is the size of the system, or $1/\sinh^2(x)$, the wave functions differ drastically from the linear combinations of the plane waves inherent for the Bethe ansatz. Despite the fact that the exact results in the thermodynamic limit available for $\sin^{-2}(\pi x/L)$ case are in complete coincidence with the asymptotic Bethe ansatz, the reason for this coincidence is still quite mysterious and it cannot be used as an argument to validate the method for the case of $1/\sinh^2 x$ pair potential, which is more complicated from the mathematical viewpoint.

Due to the lack of a general approach to the problem, any particular exact results confirming the asymptotic Bethe ansatz are of interest. Some years ago, Sutherland [2] proposed one example of a very good numerical coincidence of the asymptotic results with exact ones. It concerns the densities of the distribution of the eigenvalues of the L matrix from the Lax pair [3] for the systems of particles interacting via $1/\sinh^2(x)$ potential,

$$L_{jk} = p_j\delta_{jk} + (1-\delta_{jk})i\lambda\coth(x_j - x_k),$$

where $p_j = -i\partial/\partial x_j$ obey the canonical commutation relations $[x_j, p_k] = i\delta_{jk}$. The corresponding Hamilton operator reads

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \lambda^2 \sum_{j<k} \frac{1}{\sinh^2(x_j - x_k)}.$$

Asymptotically, if $x_1 \ll x_2 \ll \cdots \ll x_N$, the particles have the momenta $k_1 < k_2 < \cdots < k_N$ and the elements of the Lax matrix become c numbers:

$$(L_{\rm as})_{jk} = k_j\delta_{jk} + i\lambda\,\mathrm{sgn}(j-k). \qquad (1)$$

The asymptotic Bethe ansatz gives the asymptotic momenta as solutions to the equations

$$Lk_j = 2\pi I_j + \sum_{l\neq j}^{N} \tau(k_j - k_l), \qquad (2)$$

where $\tau(k)$ is the two-body phase shift, L is the total size of a system, and $\{I_j\}$ are quantum numbers. In the classical limit $\lambda \to \infty$, for the ground state, (2) becomes an integral equation for large N and L [4]:

$$2a = \int_{-A}^{A} dx'\gamma(x-x')\rho(x'), \qquad (3)$$

where

$$\gamma(x) = \ln(1+x^{-2}),$$

$\rho(x)$ is the density of the momentum distribution in the ground state, and $a = L/N$ is the average nearest-neighbor spacing (or the lattice constant). The kernel of this integral equation is symmetric and positive definite. Thus, it has unique solution at given A [5]. The normalization condition

$$\int_{-A}^{A} \rho(x')dx' = 1 \qquad (4)$$

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defines A as a function of a. The distribution of the eigenvalues of the Lax matrix can be connected with the distribution of the momenta [2]. Indeed, it follows from (1) that the equation for the eigenvalues

$$\det(L_{\rm as} - Iz) = \frac{1}{2}\left[\prod_{j=1}^{N}(k_j - z + i\lambda) + \prod_{j=1}^{N}(k_j - z - i\lambda)\right] = \prod_{s=1}^{N}(\omega_s - z) = 0$$

can be written as

$$(s+1/2)\pi = \frac{1}{2i}\sum_{j=1}^{N}\ln\left[\frac{k_j - \omega_s + i\lambda}{k_j - \omega_s - i\lambda}\right] = \sum_{j=1}^{N}\arctan\left[\frac{\lambda}{k_j - \omega_s}\right].$$

A discontinuous branch of arctan with values in $[0, \pi]$ is used here. In the thermodynamic limit, the eigenvalues $\{\omega\}$ are distributed with the density $\sigma(\omega)$: $N\sigma(\omega)d\omega$ gives the number of $\{\omega\}$ in the interval $(\omega, \omega + d\omega)$. Hence,

$$\frac{ds}{d\omega} = N\sigma(\omega).$$

Differentiating the above relation with respect to ω and taking classical limit gives

$$\sigma(\omega) = \frac{1}{2\pi}\int_{-A}^{A}\frac{dx\rho(x)}{(x-\omega)^2 + 1/4} \tag{5}$$

after recalling of variables [2]. One can see that, in the classical limit, the density $\sigma(\omega)$ can be calculated via the solution of the integral equation of the asymptotic Bethe ansatz method (3).

On the other hand, in this limit, the particles take their equilibrium positions at $x_j = ja$ in the ground state with $k_j = 0$. The form of the Lax matrix becomes simpler,

$$L_{jk} = (1 - \delta_{jk})i\lambda\coth[a(j-k)], \tag{6}$$

and the distribution of its eigenvalues can be calculated directly since (6) is of the Toeplitz form. Its eigenvectors are plane waves, and after imposing a periodic boundary condition (i.e., regularization of the determinant) and taking thermodynamic limit $N \to \infty$, one could introduce the variable $\phi = 2\pi s/N$ defining the continuous distribution

$$\omega(\phi) = \omega_s = \omega_{N\phi/2\pi}.$$

The result can be written upon rescaling $\omega \to 2\lambda\omega$ in the form [2]

$$\omega(\phi) = -\frac{\theta_1'(\phi/2)}{2\theta_1(\phi/2)}, \tag{7}$$

where $\theta_1(x)$ is the standard theta function

$$\theta_1(x) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin(2n+1)x$$

with the nome $q = e^{-a}$. The density of the eigenvalues is given now by the formula

$$\sigma(\omega) = \frac{1}{2\pi}\frac{d\phi}{d\omega}, \tag{8}$$

where the derivative is calculated through the relation

$$\frac{d\omega}{d\phi} = -\frac{1}{4}\left[\frac{\theta_1'(\phi/2)}{\theta_1(\phi/2)}\right]' = -\frac{K^2}{\pi^2}\left[\frac{K-E}{K} - \frac{1}{{\rm sn}^2(K\phi/\pi)}\right]. \tag{9}$$

Thus, one gets two representations for the density of the eigenvalues of the classical Lax matrix, one exact [formulas (7)–(9)] and one obtained by using the asymptotic Bethe ansatz method [formulas (3)–(5)]. If it is true, they should coincide. The main difficulty in verifying this fact is that there is no chance to find an analytic solution to the integral equation (3). In [2], Sutherland found good coincidence of both expressions by solving this equation numerically with high accuracy. However, an analytic solution of the problem has not been found.

In what follows, we propose a construction which uses analytic properties of the elliptic functions and provides the desired proof. Let us introduce the notation $\chi_r = {\rm Re}\chi$, $\chi_i = {\rm Im}\chi$ for any complex χ. Consider at first the problem of explicit construction of the inverse function $\phi(\omega)$ such that

$$\phi(\omega(\lambda)) \equiv \lambda.$$

It is clear that it is no longer holomorphic in the ω plane. Indeed, on the lines $\phi = \phi_r \pm ia$ one finds $\omega(\phi) = \omega_r \pm i/2$ due to the quasiperiodicity property

$$\omega(\phi + 2ia) = \omega(\phi) + i, \quad \omega(\phi + 2\pi) = \omega(\phi).$$

One has also

$$\omega_r(\pm ia) = \omega_r(2\pi \pm ia) = 0.$$

The derivative $d\omega/d\phi$ is double periodic with periods $(2\pi, 2ia)$ and has a double pole in the fundamental domain

$$0 \le {\rm Re}\phi < 2\pi, \quad -a \le {\rm Im}\phi < a.$$

Therefore, it must have just two zeros giving two extremal points of $\omega(\phi)$: one minimum of ω_r located

at $\phi_{\min} + ia$ and one maximum located at $2\pi - \phi_{\min} + ia$. Both of these extrema are considered with respect to line $\mathrm{Im}\phi = a$, and $\phi_{\min} \in (0, 2\pi)$. Let us denote $\Omega_0 = \omega_r(2\pi - \phi_{\min} + ia)$. Then it is evident that the function $\phi(\omega)$ should have two cuts in the ω plane represented by the segments $-\Omega_0 \le \omega_r \le \Omega_0$, $\omega_i = 1/2$ and $-\Omega_0 \le \omega_r \le \Omega_0, \omega_i = -1/2$. Following Haldane [6], let us express ϕ as a Cauchy integral over the contour along the image of the boundary of the fundamental domain and use the symmetry properties of $\omega(\phi)$. We skip these rather long but in fact simple considerations. Only the integral over the finite interval remains, and after integrating by parts, we obtain

$$\phi(\omega) = \int_{-\Omega_0}^{\Omega_0} dx\rho_0(x)\frac{1}{i}\ln\frac{\omega - x - i/2}{\omega - x + i/2}. \qquad (10)$$

The still unknown function $\rho_0(x)$ is normalized due to the properties of the function ϕ, $\omega(\phi + 2\pi) = \omega(\phi)$, and the integral representation (10),

$$\int_{-\Omega_0}^{\Omega_0} \rho_0(x)dx = 1. \qquad (11)$$

On the other hand, we know that $\phi_i(\omega \pm i/2) = \pm ia$ for all real ω in the interval $-\Omega_0 \le \omega \le \Omega_0$. This gives an integral equation for the function $\rho_0(x)$ entering (10) of the form quite similar to (3):

$$\mathrm{Im}\phi(\omega + i/2) = a \qquad (12)$$
$$= \frac{1}{2}\int_{-\Omega_0}^{\Omega_0} dx\rho_0(x)\ln(1 + (\omega - x)^{-2}).$$

Note also that the same equation can be obtained with the use of the quasiperiodicity property $\omega(\phi + 2ia) = \omega(\phi) + i$ and the representation (10).

Equations (3) and (12) become completely identical if one sets $A = \Omega_0$; i.e., the meaning of the parameter A is that it defines the maximal value of $\omega_r(\phi)$ on the segment $0 \le \phi_r \le 2\pi$, $\phi_i = ia$ due to the uniqueness of the solution to (3) mentioned above.

It is straightforward now to verify by differentiating (10) with respect to ω that the derivative $d\omega/d\phi$ has the integral representation

$$\left(\frac{d\omega}{d\phi}\right)^{-1} = \int_{-\Omega_0}^{\Omega_0} \frac{dx\rho_0(x)}{(\omega - x)^2 + 1/4}. \qquad (13)$$

Comparing both sets of formulas (3)–(5) and (11)–(13), one can easily see that the expressions for the spectral density of the Lax matrix in the classical limit coincide after identification $\rho(x) = \rho_0(x)$. This completes the proof.

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Positive Energy Unitary Irreducible Representations of the Superalgebras $osp(1|2n, \mathbb{R})$*

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Abstract—We give the classification of the positive energy (lowest weight) unitary irreducible representations of the superalgebras $osp(1|2n, \mathbb{R})$. © *2005 Pleiades Publishing, Inc.*

1. INTRODUCTION

Recently, superconformal field theories in various dimensions have been attracting more interest, in particular, due to their duality to AdS supergravities (cf. [1–58] and references therein). Until recently, only those for $D \leq 6$ were studied since in these cases the relevant superconformal algebras satisfy [59] the Haag–Lopuszanski–Sohnius theorem [60]. Thus, such a classification was known only for the $D = 4$ superconformal algebras $su(2, 2/N)$ [61] (for $N = 1$), [62–65] (for arbitrary N). More recently, the classification for $D = 3$ (for even N), $D = 5$, and $D = 6$ (for $N = 1, 2$) was given in [66] (some results are conjectural), and then the $D = 6$ case (for arbitrary N) was finalized in [67].

On the other hand, the applications in string theory require the knowledge of the unitary irreducible representations (UIRs) of the conformal superalgebras for $D > 6$. The most prominent role is played by the superalgebras $osp(1|2n)$ (cf. their applications in, e.g., [68–77]). Initially, the superalgebra $osp(1|32)$ was put forward for $D = 10$ [68]. Later, it was realized that $osp(1|2n)$ would fit any dimension, though they are minimal only for $D = 3, 9, 10, 11$ (for $n = 2, 16, 16, 32$, respectively) [74]. In all cases, we need to find first the UIRs of $osp(1|2n, \mathbb{R})$. This can be done for general n. Thus, in this paper, we treat the UIRs of $osp(1|2n, \mathbb{R})$ only, while the implications for conformal supersymmetry for $D = 9, 10, 11$ shall be treated in a follow-up paper.

*The text was submitted by the authors in English.

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2. REPRESENTATIONS OF THE SUPERALGEBRAS $osp(1|2n)$ AND $osp(1|2n, \mathbb{R})$

2.1. The Setting

Our basic references for Lie superalgebras are [78, 79]. The conformal superalgebras in $D = 9, 10, 11$ are $\mathcal{G} = osp(1|2n, \mathbb{R})$, $n = 16, 16, 32$, respectively (cf. [68, 74]). The even subalgebra of $osp(1|2n, \mathbb{R})$ is the algebra $sp(2n, \mathbb{R})$ with maximal compact subalgebra $\mathcal{K} = u(n) \cong su(n) \oplus u(1)$. The algebra $sp(2n, \mathbb{R})$ contains the conformal algebra $\mathcal{C} = so(D, 2)$, while $\mathcal{K}$ contains the maximal compact subalgebra $so(D) \oplus so(2)$ of $\mathcal{C}$, $so(2)$ being identified with the $u(1)$ factor of $\mathcal{K}$.

We label the relevant representations of $\mathcal{G}$ by the signature

$$\chi = [d; a_1, \ldots, a_{n-1}], \tag{1}$$

where d is the conformal weight, and $a_1, \ldots, a_{n-1}$ are nonnegative integers which are Dynkin labels of the finite-dimensional UIRs of the subalgebra $su(n)$ (the simple part of $\mathcal{K}$).

Our aim is to classify the UIRs of $\mathcal{G}$ following the methods used for the $D = 4, 6$ conformal superalgebras (cf. [62–65], [67], respectively). The main tool is an adaptation of the Shapovalov form on the Verma modules V^χ over the complexification $\mathcal{G}^{\mathbb{C}} = osp(1|2n)$ of $\mathcal{G}$.

2.2. Verma Modules

To introduce Verma modules, we use the standard triangular decomposition:

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^-, \tag{2}$$

where $\mathcal{G}^+$ and $\mathcal{G}^-$ are the subalgebras corresponding to the positive and negative roots, and $\mathcal{H}$ denotes the Cartan subalgebra.

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We consider lowest weight Verma modules, so that $V^\Lambda \cong U(\mathcal{G}^+) \otimes v_0$, where $U(\mathcal{G}^+)$ is the universal enveloping algebra of $\mathcal{G}^+$, and v_0 is a lowest weight vector v_0 such that

$$Zv_0 = 0, \quad Z \in \mathcal{G}^-, \qquad (3)$$
$$Hv_0 = \Lambda(H)v_0, \quad H \in \mathcal{H}.$$

Further, for simplicity, we omit the sign $\otimes$; i.e., we write $pv_0 \in V^\Lambda$ with $p \in U(\mathcal{G}^+)$.

The lowest weight Λ is characterized by its values on the simple roots of the superalgebra. In the next subsection, we describe the root system.

2.3. Root Systems

We recall some facts about $\mathcal{G}^{\mathbb{C}} = osp(1|2n)$ (denoted $B(0,n)$ in [78]). Their root systems are given in terms of $\delta_1, \ldots, \delta_n$, $(\delta_i, \delta_j) = \delta_{ij}$, $i, j = 1, \ldots, n$. The even and odd roots systems are [78]

$$\Delta_{\bar{0}} = \{\pm\delta_i \pm \delta_j, 1 \le i < j \le n, \pm 2\delta_i, 1 \le i \le n\}, \qquad (4)$$
$$\Delta_{\bar{1}} = \{\pm\delta_i, 1 \le i \le n\}$$

(we recall that the signs $\pm$ are not correlated). We shall use the following distinguished simple root system [78]:

$$\Pi = \{\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n\}, \qquad (5)$$

or introducing standard notation for the simple roots:

$$\Pi = \{\alpha_1, \ldots, \alpha_n\}, \qquad (6)$$
$$\alpha_j = \delta_j - \delta_{j+1}, \quad j = 1, \ldots, n-1, \quad \alpha_n = \delta_n.$$

The root $\alpha_n = \delta_n$ is odd, the other simple roots are even. The Dynkin diagram is as follows:

$$\overset{\circ}{\underset{1}{}} \text{—} \cdots \text{—} \overset{\circ}{\underset{n-1}{}} \Longrightarrow \overset{\bullet}{\underset{n}{}} \qquad (7)$$

The black dot is used to signify that the simple odd root is not nilpotent, otherwise a gray dot would be used [78]. In fact, the superalgebras $B(0,n) = osp(1|2n)$ have no nilpotent generators, unlike all other types of basic classical Lie superalgebras [78].

The positive root system corresponding to Π is

$$\Delta_{\bar{0}}^+ = \{\delta_i \pm \delta_j, 1 \le i < j \le n, 2\delta_i, 1 \le i \le n\}, \qquad (8)$$
$$\Delta_{\bar{1}}^+ = \{\delta_i, 1 \le i \le n\}.$$

We record how the elementary functionals are expressed through the simple roots:

$$\delta_k = \alpha_k + \ldots + \alpha_n. \qquad (9)$$

The even root system $\Delta_{\bar{0}}$ is the root system of the rank-n complex simple Lie algebra $sp(2n)$, with $\Delta_{\bar{0}}^+$ being its positive roots. The simple roots are

$$\Pi_0 = \{\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, 2\delta_n\} \qquad (10)$$
$$= \{\alpha_1^0, \ldots, \alpha_n^0\},$$
$$\alpha_j^0 = \delta_j - \delta_{j+1}, \quad j = 1, \ldots, n-1, \qquad \alpha_n^0 = 2\delta_n.$$

The Dynkin diagram is as follows:

$$\overset{\circ}{\underset{1}{}} \text{—} \cdots \text{—} \overset{\circ}{\underset{n-1}{}} \Longleftarrow \overset{\circ}{\underset{n}{}} \qquad (11)$$

The superalgebra $\mathcal{G} = osp(1|2n, \mathbb{R})$ is a split real form of $osp(1|2n)$ and has the same root system.

2.4. Lowest Weight through the Signature

Since we use a Dynkin labeling, we have the following relation with the signature χ from (1):

$$(\Lambda, \alpha_k^\vee) = \begin{cases} -a_k, & k < n, \\ \tilde{d}, & k = n, \end{cases} \qquad (12)$$

where $\alpha_k^\vee \equiv 2\alpha_k/(\alpha_k, \alpha_k)$, and $\tilde{d}$ differs from the conformal weight d as explained below. The minus signs in the first row are related to the fact that we work with lowest weight Verma modules (instead of the highest weight modules used in [79]) and to Verma module reducibility with respect to the roots α_k (this is explained in detail in [64]). The value of $\tilde{d}$ is a matter of normalization so as to correspond to some known cases. Thus, our choice is

$$\tilde{d} = 2d + a_1 + \ldots + a_{n-1}. \qquad (13)$$

Having in hand the values of Λ on the basis, we can recover them for any element of $\mathcal{H}^*$. In particular, for the values on the elementary functionals, we have using (9), (12), and (13)

$$(\Lambda, \delta_j) = d + \frac{1}{2}(a_1 + \ldots + a_{j-1} \qquad (14)$$
$$- a_j - \ldots - a_{n-1}).$$

Using (12) and (13), one can easily write $\Lambda = \Lambda(\chi)$ as a linear combination of the simple roots or of the elementary functionals δ_j, but this is not necessary in what follows. We shall need only $(\Lambda, \beta^\vee)$ for all positive roots β and from (14) we have

$$(\Lambda, (\delta_i - \delta_j)^\vee) = (\Lambda, \delta_i - \delta_j) \qquad (15)$$
$$= -a_i - \ldots - a_{j-1},$$
$$(\Lambda, (\delta_i + \delta_j)^\vee) = (\Lambda, \delta_i + \delta_j) = 2d$$
$$+ a_1 + \ldots + a_{i-1} - a_j - \ldots - a_{n-1},$$
$$(\Lambda, \delta_i^\vee) = (\Lambda, 2\delta_i) = 2d$$
$$+ a_1 + \ldots + a_{i-1} - a_i - \ldots - a_{n-1},$$
$$(\Lambda, (2\delta_i)^\vee) = (\Lambda, \delta_i) = d$$
$$+ \frac{1}{2}(a_1 + \ldots + a_{i-1} - a_i - \ldots - a_{n-1}).$$

2.5. Reducibility of Verma Modules

Having established the relation between χ and Λ, we turn our attention to the question of reducibility. A Verma module V^Λ is reducible with respect to the positive root β if the following holds [79]:

$$(\rho - \Lambda, \beta^\vee) = m_\beta, \quad \beta \in \Delta^+, \quad m_\beta \in \mathbb{N}, \tag{16}$$

where $\rho \in \mathcal{H}^*$ is the very important element in representation theory given by the difference of the half-sums $\rho_{\bar 0}$ and $\rho_{\bar 1}$ of the even and odd, respectively, positive roots [cf. (8)]:

$$\rho = \rho_{\bar 0} - \rho_{\bar 1} = \left(n - \frac{1}{2}\right)\delta_1 + \left(n - \frac{3}{2}\right)\delta_2 + \ldots + \frac{3}{2}\delta_{n-1} + \frac{1}{2}\delta_n, \tag{17}$$

$$\rho_{\bar 0} = n\delta_1 + (n-1)\delta_2 + \ldots + 2\delta_{n-1} + \delta_n,$$

$$\rho_{\bar 1} = \frac{1}{2}(\delta_1 + \ldots + \delta_n).$$

To make (16) explicit, we need first the values of ρ on the positive odd roots:

$$(\rho, \delta_i) = n - i + \frac{1}{2}. \tag{18}$$

Then for $(\rho, \beta^\vee)$ we have

$$(\rho, (\delta_i - \delta_j)^\vee) = j - i, \tag{19}$$

$$(\rho, (\delta_i + \delta_j)^\vee) = 2n - i - j + 1,$$

$$(\rho, \delta_i^\vee) = 2n - 2i + 1,$$

$$(\rho, (2\delta_i)^\vee) = n - i + \frac{1}{2}.$$

Naturally, the value of ρ on the simple roots is 1: $(\rho, \alpha_i^\vee) = 1$, $i = 1, \ldots, n$.

Consecutively, we find that the Verma module $V^{\Lambda(\chi)}$ is reducible if one of the following relations holds [following the order of (15) and (19)]:

$$\mathbb{N} \ni m_{ij}^- = j - i + a_i + \ldots + a_{j-1}, \tag{20a}$$

$$\mathbb{N} \ni m_{ij}^+ = 2n - i - j + 1 + a_j + \ldots + a_{n-1} - a_1 - \ldots - a_{i-1} - 2d, \tag{20b}$$

$$\mathbb{N} \ni m_i = 2n - 2i + 1 + a_i + \ldots + a_{n-1} - a_1 + \ldots - a_{i-1} - 2d, \tag{20c}$$

$$\mathbb{N} \ni m_{ii} = n - i + \frac{1}{2}(1 + a_i + \ldots + a_{n-1} - a_1 + \ldots - a_{i-1}) - d. \tag{20d}$$

Note that $m_i = 2m_{ii}$; thus, whenever (20d) is fulfilled, (20c) is also fulfilled.

If a condition from (20) is fulfilled, then V^Λ contains a submodule which is a Verma module $V^{\Lambda'}$ with shifted weight given by the pair m, β: $\Lambda' = \Lambda + m\beta$. The embedding of $V^{\Lambda'}$ in V^Λ is provided by mapping the lowest weight vector v_0' of $V^{\Lambda'}$ to the singular vector $v_s^{m,\beta}$ in V^Λ which is completely determined by the conditions

$$Xv_s^{m,\beta} = 0, \quad X \in \mathcal{G}^-, \tag{21}$$

$$Hv_s^{m,\beta} = \Lambda'(H)v_0, \quad H \in \mathcal{H}, \quad \Lambda' = \Lambda + m\beta.$$

Explicitly, $v_s^{m,\beta}$ is given by a polynomial in the positive root generators:

$$v_s^{m,\beta} = P^{m,\beta}v_0, \quad P^{m,\beta} \in U(\mathcal{G}^+). \tag{22}$$

Thus, the submodule of V^Λ which is isomorphic to $V^{\Lambda'}$ is given by $U(\mathcal{G}^+)P^{m,\beta}v_0$.

Here, we should note that we may eliminate the reducibilities and embeddings related to the roots $2\delta_i$. Indeed, let (20d) hold; then, the corresponding singular vector $v_s^{m_{ii},2\delta_i}$ has the properties prescribed by (21) with $\Lambda' = \Lambda + m_{ii} \times 2\delta_i$. But as we mentioned above, in this situation, (20c) also holds and the corresponding singular vector $v_s^{m_i,\delta_i}$ has the properties prescribed by (21) with $\Lambda'' = \Lambda + m_i\delta_i$. But due to the fact that $m_i = 2m_{ii}$, it is clear that $\Lambda'' = \Lambda'$, which means that the singular vectors $v_s^{m_{ii},2\delta_i}$ and $v_s^{m_i,\delta_i}$ coincide (up to a nonzero multiplicative constant). On the other hand, if (20c) holds with m_i being an odd number, then (20d) does not hold (since $m_{ii} = m_i/2$ is not integer).

Further, we notice that all reducibility conditions in (20a) are fulfilled. In particular, for the simple roots from those conditions, (20a) is fulfilled with $\beta \to \alpha_i = \delta_i - \delta_{i+1}$, $i = 1, \ldots, n-1$, and $m_i^- \equiv m_{i,i+1}^- = 1 + a_i$. The corresponding submodules $I_i^\Lambda = U(\mathcal{G}^+)v_s^i$, where $\Lambda_i = \Lambda + m_i^-\alpha_i$ and $v_s^i = (X_i^+)^{1+a_i}v_0$, where X_i^+ are the root vectors of these simple roots. These submodules generate an invariant submodule which we denote by I_c^Λ. Since these submodules are nontrivial for all our signatures, instead of V^Λ we shall consider the factor-modules:

$$F^\Lambda = V^\Lambda / I_c^\Lambda. \tag{23}$$

We shall denote the lowest weight vector of F^Λ by $\widetilde{|\Lambda\rangle}$ and the singular vectors above become null conditions in F^Λ :

$$(X_i^+)^{1+a_i}\widetilde{|\Lambda\rangle} = 0, \quad i = 1, \ldots, n-1. \tag{24}$$

If the Verma module V^Λ is not reducible with respect to the other roots, i.e., (20b), (20c), and (20d) are not fulfilled, then F^Λ is irreducible and is isomorphic to the irrep L_Λ with this weight.

Other situations shall be discussed below in the context of unitarity.

2.6. Realization of $osp(1|2n)$ and $osp(1|2n,\mathbb{R})$

The superalgebras $osp(m|2n) = osp(m|2n)_{\bar 0} + osp(m|2n)_{\bar 1}$ are defined as follows [78]:

$$osp(m|2n)_s = \{X \in gl(m/2n;\mathbb{C})_s : XW + i^s\, W\, {}^tX = 0\}, \quad s = \bar 0, \bar 1,$$

where W is a matrix of order $m+2n$:

$$W = \begin{pmatrix} iI_m & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{pmatrix}.$$

The even part $osp(m|2n)_{\bar 0}$ consists of matrices X such that

$$X = \begin{pmatrix} S & 0 & 0 \\ 0 & B & C \\ 0 & D & -{}^tB \end{pmatrix}, \qquad (25)$$

$${}^tS = -S, \quad {}^tC = C, \quad {}^tD = D.$$

In our case, $m = 1$ and $S = 0$. The Cartan subalgebra $\mathcal{H}$ consists of diagonal matrices H such that

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & -B \end{pmatrix}.$$

We take the following basis for the Cartan subalgebra:

$$H_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & B_i & 0 \\ 0 & 0 & -B_i \end{pmatrix}, \quad i < n, \qquad (26)$$

$$H_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & -I_n \end{pmatrix},$$

where

$$B_i = \mathrm{diag}(0, \ldots, 0, 1, -1, 0, \ldots, 0)$$

is the first nonzero entry being on the ith place. This basis shall be used also for the real form $osp(1|2n,\mathbb{R})$ and is chosen to be consistent with the fact that the even subalgebra $sp(2n,\mathbb{R})$ of the latter has as maximal noncompact subalgebra the algebra $sl(n,\mathbb{R}) \oplus \mathbb{R}$. Via the Weyl unitary trick, this related to the structure of $sp(2n,\mathbb{R})$ as a Hermitian symmetric space with maximal compact subalgebra $u(n) \cong su(n) \oplus u(1)$.

The root vectors of the roots $\delta_i - \delta_j$ $(i \neq j)$, $\delta_i + \delta_j$ $(i \leq j)$, $-(\delta_i + \delta_j)$ $(i \leq j)$, respectively, are denoted X_{ij}, X^+_{ij}, X^-_{ij}, respectively. The latter are given by matrices of the type (25) with $S = 0$, given (up to multiplicative normalization) by $B = E_{ij}$, $C = E_{ij} + E_{ji}$, $D = E_{ij} + E_{ji}$, respectively, where E_{ij} is $n \times n$ matrix which has only one nonzero entry equal to 1 on the intersection of the ith row and jth column. Explicitly including some choice of normalization, this is

$$X_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{ij} & 0 \\ 0 & 0 & -E_{ji} \end{pmatrix}, \quad i \neq j, \qquad (27)$$

$$X^+_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -E_{ij} - E_{ji} \\ 0 & 0 & 0 \end{pmatrix}, \quad i < j,$$

$$X^+_{ii} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -E_{ii} \\ 0 & 0 & 0 \end{pmatrix},$$

$$X^-_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E_{ij} + E_{ji} & 0 \end{pmatrix}, \quad i < j,$$

$$X^-_{ii} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E_{ii} & 0 \end{pmatrix}.$$

The odd part $osp(m|2n)_{\bar 1}$ consists of matrices X such that

$$X = \begin{pmatrix} 0 & \xi & -\eta \\ {}^t\eta & 0 & 0 \\ {}^t\xi & 0 & 0 \end{pmatrix}.$$

The root vectors Y^+_i, Y^-_i of the roots $\delta_i, -\delta_i$ correspond to η, ξ, respectively, with only a nonzero ith entry. Explicitly, this is

$$Y^+_i = \begin{pmatrix} 0 & 0 & -E_{1i} \\ E_{i1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (28)$$

$$Y_i^- = \begin{pmatrix} 0 & E_{1i} & 0 \\ 0 & 0 & 0 \\ E_{i1} & 0 & 0 \end{pmatrix}.$$

In the calculations, we need all commutators of the kind $[X_\beta, X_{-\beta}] = H_\beta$, $\beta \in \Delta_0^+$. Explicitly, we have

$$[X_{ij}, X_{ji}] = H_{ij} = H_i + H_{i+1} + \ldots + H_{j-1}, \quad 1 \le i < j \le n, \tag{29a}$$

$$[Y_i^+, Y_i^-]_+ = H'_i \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{ii} & 0 \\ 0 & 0 & -E_{ii} \end{pmatrix}, \quad 1 \le i \le n, \tag{29b}$$

$$[X_{ij}^+, X_{ij}^-] = H'_{ij} = -H'_i - H'_j, \quad 1 \le i < j \le n, \tag{29c}$$

$$[X_{ii}^+, X_{ii}^-] = -H'_i, \quad 1 \le i \le n. \tag{29d}$$

The minus sign in (29d) is consistent with the relations

$$\frac{1}{2}[Y_i^\pm, Y_i^\pm]_+ = (Y_i^\pm)^2 = X_{ii}^\pm. \tag{30}$$

We note also the following relations:

$$[Y_i^+, Y_j^-]_+ = X_{ij}, \quad i \ne j, \tag{31}$$

$$[Y_i^\pm, Y_j^\pm]_+ = X_{ij}^\pm, \quad i \ne j,$$

$$H_n = H'_1 + \ldots + H'_n.$$

We shall also use the abstract defining relations of $osp(1|2n)$ through the Chevalley basis. Let $\hat{H}_i$, $i = 1, \ldots, n$, be the basis of the Cartan subalgebra $\mathcal{H}$ associated with the simple roots, and $X_i^\pm$, $i = 1, \ldots, n$, be the simple root vectors (the Chevalley generators). The connection with the basis above is

$$\hat{H}_i = H_i, \quad i < n, \quad \hat{H}_n = H'_n, \tag{32}$$

$$X_i^+ = X_{i,i+1}^+, \quad i < n, \quad X_n^+ = Y_n^+.$$

Let $A = (a_{ij})$ be the Cartan matrix [78]:

$$A = (a_{ij}) \tag{33}$$

$$= \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 2 & -1 & 0 \\ 0 & 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & 0 & \ldots & 0 & -2 & 2 \end{pmatrix}.$$

We shall also use the decomposition $A = A^d A^s$, where $A^d = \mathrm{diag}\,(1, \ldots, 1, 2)$ and A^s is a symmetric matrix:

$$A^s = \left(a_{ij}^s\right) \tag{34}$$

$$= \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 2 & -1 & 0 \\ 0 & 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & 0 & \ldots & 0 & -1 & 1 \end{pmatrix}.$$

Then the defining relations of $osp(1|2n)$ are

$$[\hat{H}_i, \hat{H}_j] = 0, \quad [\hat{H}_i, X_j^\pm] = \pm a_{ij}^s X_j^\pm, \tag{35}$$

$$[X_i^+, X_j^-] = \delta_{ij} \hat{H}_i,$$

$$(\mathrm{Ad}X_j^\pm)^{n_{j,k}}(X_k^\pm) = 0, \quad j \ne k, \quad n_{j,k} = 1 - a_{jk},$$

where in (35) one uses the supercommutator

$$(\mathrm{Ad}X_j^\pm)(X_k^\pm) = [X_j^\pm, X_k^\pm] \tag{36}$$

$$\equiv X_j^\pm X_k^\pm - (-1)^{\deg X_j^\pm \deg X_k^\pm} X_k^\pm X_j^\pm.$$

2.7. Shapovalov Form and Unitarity

The Shapovalov form is a bilinear $\mathbb{C}$–valued form on $U(\mathcal{G}^+)$ [80], which we extend in the obvious way to Verma modules (cf., e.g., [65]). We also need the involutive antiautomorphism ω of $U(\mathcal{G})$ which will provide the real form we are interested in. Since this is the split real form $osp(1|2n, \mathbb{R})$, we use

$$\omega(X_\beta) = X_{-\beta}, \qquad \omega(H) = H, \tag{37}$$

where X_β is the root vector corresponding to the root β, $H \in \mathcal{H}$.

Thus, an adaptation of the Shapovalov form suitable for our purposes is defined as follows:

$$(u, u') = (pv_0, p'v_0) \tag{38}$$

$$\equiv (v_0, \omega(p)p'v_0) = (\omega(p')pv_0, v_0),$$

$$u = pv_0, \quad u' = p'v_0, \quad p, p' \in U(\mathcal{G}^+), u, u' \in V^\Lambda,$$

supplemented by the normalization condition $(v_0, v_0) = 1$. The norms squared of the states will be denoted by

$$||u||^2 \equiv (u, u). \tag{39}$$

Now, we need to introduce a PBW basis of $U(\mathcal{G}^+)$. We use the so-called normal ordering; namely, if we have the relation

$$\beta = \beta' + \beta'', \quad \beta, \beta', \beta'' \in \Delta^+,$$

then the corresponding root vectors are ordered in the PBW basis as follows:

$$\ldots (X^+_{\beta'})^{k'} \ldots (X^+_{\beta})^{k} \ldots (X^+_{\beta''})^{k''} \ldots, \qquad (40)$$
$$k, k', k'' \in \mathbb{Z}_+.$$

We also have to take into account relation (30) between the root vectors corresponding to the roots δ_i and $2\delta_i$. Because of this relation and consistently with (40), the generators X^+_{ii}, $i = 1, \ldots, n$, are not present in the PBW basis. On the other hand, the PBW basis of the even subalgebra of $U(\mathcal{G}^+)$ will differ from the above only in the fact that the powers of X^+_i, $i = 1, \ldots, n$, are only even representing powers of the even generators X^+_{ii}, $i = 1, \ldots, n$.

3. UNITARITY

3.1. Calculation of Some Norms

In this subsection, we show how to use the form (38) to calculate the norms of the states. We shall use the isomorphism between the Cartan subalgebra $\mathcal{H}$ and its dual $\mathcal{H}^*$. This is given by the following correspondence: to every element $\beta \in \mathcal{H}^*$, there is unique element $H_\beta \in \mathcal{H}$ such that

$$\mu(H_\beta) = (\mu, \beta^\vee) \qquad (41)$$

for every $\mu \in \mathcal{H}^*$, $\mu \neq 0$. Applying this to the positive roots, we have the following: to $\beta = \delta_i - \delta_j, \delta_i, \delta_i + \delta_j$, respectively, correspond $H_\beta = H_{ij}, H'_i, H'_i + H'_j$.

We give now explicitly the norms of the one-particle states, introducing also notation for future use:

$$x^-_{ij} \equiv ||X_{ij} v_0||^2 = (X_{ij} v_0, X_{ij} v_0) \qquad (42)$$
$$= (v_0, X_{ji} X_{ij} v_0) = (v_0, (X_{ij} X_{ji} - H_{ij}) v_0)$$
$$= -\Lambda(H_{ij}) = -(\Lambda, (\delta_i - \delta_j)^\vee) = a_i + \ldots + a_{j-1},$$
$$i < j,$$
$$x^+_{ij} \equiv ||X^+_{ij} v_0||^2 = (X^+_{ij} v_0, X^+_{ij} v_0)$$
$$= (v_0, X^-_{ij} X^+_{ij} v_0) = (v_0, (X^+_{ij} X^-_{ji} - H'_{ij}) v_0)$$
$$= -\Lambda(H'_{ij}) = \Lambda(H'_i + H'_j) = (\Lambda, (\delta_i + \delta_j)^\vee)$$
$$= 2d + a_1 + \ldots + a_{i-1} - a_j - \ldots - a_{n-1},$$
$$x_i \equiv ||X^+_i v_0||^2 = (X^+_i v_0, X^+_i v_0)$$
$$= (v_0, X^-_i X^+_i v_0) = (v_0, (-X^+_i X^-_i + H'_i) v_0)$$
$$= \Lambda(H'_i) = (\Lambda, \delta^\vee_i) = 2d + a_1 + \ldots + a_{i-1}$$
$$- a_i - \ldots - a_{n-1}.$$

Positivity of all these norms gives the following necessary conditions for unitarity:

$$a_i \geq 0, \quad i = 1, \ldots, n-1, \qquad (43)$$
$$d \geq \frac{1}{2}(a_1 + \ldots + a_{n-1}).$$

In fact, the boundary values are possible due to factoring out of the corresponding null states when passing from the Verma module to the unitary irreducible factor module.

Further, we shall discuss only norms which involve the conformal weight since the others are related to unitarity of the irrep restricted to the maximal simple compact subalgebra $su(n)$. The norms that we are going to consider can be written in terms of factors $(d - \ldots)$, and the leading term in d has a positive coefficient. Thus, for d large enough, all norms will be positive. When d is decreasing, there is a critical point at which one (or more) norm(s) will become zero. This critical point (called the "first reduction point" in [61]) can be read off from the reducibility conditions, since at that point the Verma module is reducible (and it is the corresponding submodule that has zero norm states).

The maximal d coming from the different possibilities in (20b) are obtained for $m^+_{ij} = 1$ and they also denote the corresponding root:

$$d_{i,j} \equiv n + \frac{1}{2}(a_j + \ldots + a_{n-1} - a_1 - \ldots \qquad (44)$$
$$- a_{i-1} - i - j),$$

the corresponding root being $\delta_i + \delta_j$. The maximal d coming from the different possibilities in (20c) and (20d), respectively, are obtained for $m_i = 1$ and $m_{ii} = 1$, respectively, and they are

$$d_i \equiv n - i + \frac{1}{2}(a_i + \ldots + a_{n-1} \qquad (45)$$
$$- a_1 - \ldots - a_{i-1}),$$
$$d_{i,i} = d_i - \frac{1}{2},$$

the corresponding roots being δ_i and $2\delta_j$, respectively. These are some orderings between these maximal reduction points:

$$d_1 > d_2 > \ldots > d_n, \qquad (46)$$
$$d_{i,i+1} > d_{i,i+2} > \ldots > d_{i,n},$$
$$d_{1,j} > d_{2,j} > \ldots > d_{j-1,j},$$
$$d_i > d_{j,k} > d_\ell, \quad i \leq j < k \leq \ell.$$

Obviously, the first reduction point is

$$d_1 = n - 1 + \frac{1}{2}(a_1 + \ldots + a_{n-1}). \qquad (47)$$

3.2. Main Result

Theorem. *All positive energy unitary irreducible representations of the superalgebras $osp(1|2n,\mathbb{R})$ characterized by the signature χ in (1) are obtained for real d and are given in the following list*:

$$d \geq d_1 = n - 1 + \frac{1}{2}(a_1 + \ldots + a_{n-1}), \qquad (48)$$

$$\text{no restrictions on } a_j;$$

$$d = d_{1,2} = n - 2 + \frac{1}{2}(a_2 + \ldots + a_{n-1} + 1),$$

$$a_1 = 0;$$

$$\ldots$$

$$d = d_{j-1,j} = n - j + \frac{1}{2}(a_j + \ldots + a_{n-1} + 1),$$

$$a_1 = \ldots = a_{j-1} = 0;$$

$$\ldots$$

$$d = d_{n-1,n} = \frac{1}{2}, \quad a_1 = \ldots = a_{n-1} = 0.$$

Proof.

Necessity.

We give examples of states with negative norm in the excluded intervals $d < d_1$ (cf. [81]).

Sufficiency.

The statement of the theorem for $d > d_1$ is clear from the general considerations above. For $d = d_1$, we have the first zero-norm state which is naturally given by the corresponding singular vector $v^{1,\delta_1} = \mathcal{P}^{1,\delta_1} v_0$. In fact, all states of the embedded submodule $V^{\Lambda+\delta_1}$ built on v^{1,δ_1} have zero norms. Due to the above singular vector, we have the following additional null condition in F^Λ:

$$\mathcal{P}^{1,\delta_1} |\widetilde{\Lambda}\rangle = 0. \qquad (49)$$

The above conditions factorize the submodule built on v^{1,δ_1}. There are no other vectors with zero norm at $d = d_1$ since, by a general result [79], the elementary embeddings between Verma modules are one-dimensional. Thus, F^Λ is the UIR $L_\Lambda = F^\Lambda$.

Further, we consider the remaining discrete points of unitarity for $d < d_1$, i.e., $d = d_{i,i+1}$, $i = 1, \ldots, n-1$. The corresponding roots are $\delta_i + \delta_{i+1} = \alpha_i + 2\alpha_{i+1} + \ldots + 2\alpha_n$. The corresponding singular vectors $v^{1,\delta_i+\delta_{i+1}} = \mathcal{P}^{1,\delta_i+\delta_{i+1}} v_0$.

Now, fix i, where $i \in \{1, \ldots, n-1\}$. All states of the embedded submodule $V^{\Lambda+\delta_i+\delta_{i+1}}$ built on $v^{1,\delta_i+\delta_{i+1}}$ have zero norms for $d = d_{i,i+1}$. Due to the above singular vector, we have the following additional null condition in F^Λ:

$$\mathcal{P}^{1,\delta_i+\delta_{i+1}} |\widetilde{\Lambda}\rangle = 0, \qquad d = d_{i,i+1}. \qquad (50)$$

At this point, the states built on the vector v^{1,δ_1} and on the vectors $v^{1,\delta_k+\delta_{k+1}}$ for $k < i$ (all of these are not singular vectors at $d = d_{i,i+1}$) have a negative norm except when $a_1 = \ldots = a_i = 0$. For this statement, we may use the explicit form of these vectors. This explicit form is the same as the singular vectors of the same weight for the Lie algebra $B_n = so(2n+1)$. For v^{1,δ_1}, this can be read off from [82] (in fact, there it is for the more general situation of the quantum group $U_q(B_n)$):

$$v^{1,\delta_1} = \sum_{k_1=0}^{1} \ldots \sum_{k_{n-1}=0}^{1} b_{k_1 \ldots k_{n-1}} (X_1^+)^{1-k_1} \ldots (X_{n-1}^+)^{1-k_{n-1}} X_n^+ (X_{n-1}^+)^{k_{n-1}} \ldots (X_1^+)^{k_1} v_0 \equiv \mathcal{P}^{1,\delta_1} v_0, \quad (51a)$$

$$b_{k_1 \ldots k_{n-1}} = (-1)^{k_1+\ldots+k_{n-1}} b_0 \frac{(\rho-\Lambda)(H^1)}{(\rho-\Lambda)(H^1) - k_1} \ldots \frac{(\rho-\Lambda)(H^{n-1})}{(\rho-\Lambda)(H^{n-1}) - k_{n-1}} \qquad (51b)$$

$$= (-1)^{k_1+\ldots+k_{n-1}} b_0 \frac{1 + a_1}{1 + a_1 - k_1} \ldots \frac{n - 1 + a_1 + \ldots + a_{n-1}}{n - 1 + a_1 + \ldots + a_{n-1} - k_{n-1}} \qquad (51c)$$

$$= (-1)^{k_1+\ldots+k_{n-1}} (a_1 + k) \frac{2 + a_1 + a_2}{2 + a_1 + a_2 - k_2} \ldots \frac{n - 1 + a_1 + \ldots + a_{n-1}}{n - 1 + a_1 + \ldots + a_{n-1} - k_{n-1}}, \qquad (51d)$$

where $H^s = \hat{H}_1 + \hat{H}_2 + \ldots + \hat{H}_s$ {cf. (13) from [82] with $q = 1$, $t = n-1$, $m = 1$, $l \to -\Lambda$; the last change due to the fact that, in [82], the highest weight modules are considered}; in (51c), we have inserted our signatures

$$(\rho - \Lambda)(H^s) = (\rho - \Lambda, (\alpha_1 + \ldots + \alpha_s)^\vee)$$

$$= (\rho - \Lambda, \delta_1 - \delta_{s+1}) = m^-_{1,s+1} = s + a_1 + \ldots + a_s$$

and in (51d), we have made the choice of constant

$b_0 = a_1$ in order to make the expression valid also for $a_1 = 0$. It is easy to see that, for $a_1 = 0$, the vector v^{1,δ_1} is not independent, but is a descendant of the singular vector $v_s^1 = X_1^+ v_0$:

$$v^{1,\delta_1} = \sum_{k_2=0}^{1} \dots \sum_{k_{n-1}=0}^{1} b_{1,k_2\dots k_{n-1}} \times (X_2^+)^{1-k_2} \dots (X_{n-1}^+)^{1-k_{n-1}} \times X_n^+ (X_{n-1}^+)^{k_{n-1}} \dots (X_2^+)^{k_2} X_1^+ v_0. \quad (52)$$

Thus, v^{1,δ_1} is not present in F^Λ for any d and $a_1 = 0$ since the null condition (49) follows from case $i = 1$ of the null conditions (24). Analogously, if $i > 1$ and fixing now $k < i$, the vector $v^{1,\delta_k+\delta_{k+1}}$ has a negative norm at $d = d_{i,i+1}$ except if $a_{k+1} = 0$, when it is not independent, but is a descendant of the singular vector $v_s^{k+1} = X_{k+1}^+ v_0$, and, hence, is not present in F^Λ (this will be given more explicitly in [81]). Thus, for $d = d_{i,i+1}$ together with $a_1 = \dots = a_i = 0$, the condition (50) factorizing the submodule built on $v^{1,\delta_i+\delta_{i+1}}$ is the only condition—in addition to (24)—needed to obtain the UIR $L_\Lambda = F^\Lambda$ at $d = d_{i,i+1}$, $i = 1, \dots, n-1$.

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A Problem of Classification of Quantum Groups*

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Abstract—An attempt to formulate a precise program of classification of a large family of quantum groups is presented. This family includes the familiar quantum groups and quantum supergroups, but much more, all unified in a very simple structure. The emphasis is on the logic of the classification scheme. Recent results are reported without much explanation and proofs are described only in a general way.

INTRODUCTION

Among famous achievements in classification must be included the classification of the simple Lie algebras by Killing and Cartan, the simple superalgebras by Kac, and the simple Lie bialgebras by Belavin and Drinfel'd. Each case embraces a natural category, not too small and not too large, including large, natural families of algebras but not so large as to defy classification. If one desires to classify the quantum groups, then one had better be prepared to reign in too large an ambition; half the battle is to discover a family of quantum groups that is at once natural (easily defined) and amenable to analysis.

This paper deals with a family dubbed q-algebras. It is indeed easy to define. What is more, it contains a subfamily that stands in a very direct relationship to quantized Kac-Moody algebras, so that the classification of these latter algebras is a corollary of the classification of q-algebras. But let us go back a bit.

Serre Presentation of $su(3)$

The familiar basis for the real Lie algebra $su(3)$ includes three "step-up operators" e_1, e_2, e_3 (a basis for the space of upper triangular 3×3 matrices), their conjugates f_1, f_2, f_3, and Cartan generators H_1, H_2, H_3 (diagonal matrices with $H_1 + H_2 + H_3 = 0$). The relations include $[H_i, e_j] \propto e_j, [H_i, f_j] \propto f_j$, and the ones that concern us the most right now, namely,

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = 0 = [e_2, e_3],$$

and similar commutation relations among f_i.

The nine generators (actually only eight are linearly independent) are "generators" in the linear sense; $su(3)$ consists of all real linear combinations. But in another sense, the elements e_1, e_2 generate e_3; we may take the first relation as the definition of e_3. This requires that we express the second and the third relation in terms of e_1 and e_2:

$$[e_1, [e_1, e_2]] = 0 = [e_2, [e_2, e_1]].$$

This suggestion was made by Chevalley; later, the formulation was completed, for all simple Lie algebras, by Serre. The generators e_1, e_2 are called Serre generators and the double commutation relations are called Serre relations.

To summarize, the relations of $su(3)$ take the form

$$[H_1, e_1] = 2e_1, \quad [H_1, e_2] = -e_2, \quad [H_2, e_1] = -e_1, \quad [H_2, e_2] = 2e_2,$$

$$[e_1, f_1] = H_1, \quad [e_1, f_2] = 0 = [e_2, f_1], \quad [e_2, f_2] = H_2,$$

$$[e_1, [e_1, e_2]] = 0 = [e_2, [e_2 e_1]].$$

Drinfel'd defined his quantum groups in this idiom, and it seems quite essential to do so, especially if one aims at a discussion of simple Lie algebras and the associated quantum groups in general. Of the three sets of relations, the first remains unchanged, while the others are replaced by

$$[e_1, f_1] = \frac{\sin hH_1}{h}, \quad [e_1, f_2] = 0 = [e_2, f_1], \quad [e_2, f_2] = \frac{\sin hH_2}{h},$$

$$[e_1, [e_1, e_2]_h]_h = 0 = [e_1, [e_1, e_2]_h]_h.$$

The Lie algebra $su(3)$ is recovered in the limit, as the parameter h tends to zero.

Let us pass to the general case of a simple Lie algebra and to Drinfel'd's quantization. Then we shall once again specialize to the simplest case.

*The text was submitted by the author in English.

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Serre Presentation of a Simple Lie Algebra

In Serre's presentation, a simple Lie algebra is "generated" by three sets of elements, in direct generalization of the case of $su(3)$, denoted e_i, f_i, and H_i, $i = 1, \ldots, n$. The relations are

$$[H_i, H_j] = 0,$$
$$[H_i, e_j] = H_i(j)e_j, \quad [H_i, f_j] = -H_i(j)f_j,$$
$$[e_i, f_j] = \delta_{ij}H_i^\vee.$$

The elements $H^\vee$ are certain linear combinations of H_i. In addition, there are Serre relations of the type

$$[e_1, [e_1, [\ldots [e_1, e_2] \ldots] = 0.$$

Details are omitted, for what interests us is Drinfel'd's quantization of these algebras, and for this we prefer a better notation.

Drinfel'd's quantum group is defined by the same generators and slightly modified relations. Here is a complete definition.

Definition. *Let $\mathcal{M}$, $\mathcal{N}$ be two countable sets, and ϕ, ψ two maps,*

$$\phi : \mathcal{M} \times \mathcal{M} \to \mathbb{C}, \quad a, b \mapsto \phi^{ab},$$
$$\psi : \mathcal{M} \times \mathcal{N} \to \mathbb{C}, \quad a, i \mapsto H_a(i).$$

Let

$$\phi(i, \cdot) = \sum_{a,b \in \mathcal{M}} \phi^{ab} H_a(i) H_b, \quad \phi(\cdot, i)$$
$$= \sum_{a,b \in \mathcal{M}} \phi^{ab} H_a H_b(i),$$

and suppose that $\mathrm{e}^{\phi(i,\cdot)+\phi(\cdot,i)} \neq 1, i \in \mathcal{N}$.

Let $\mathcal{A} = \mathcal{A}(\phi, \psi)$ be the universal, associative, unital $\mathbb{C}$-algebra with generators $\{e_i, f_i\}_{i \in \mathcal{N}}$ and $\{H_a\}_{a \in \mathcal{M}}$ and relations

$$[H_a, H_b] = 0, \quad a, b \in \mathcal{M},$$
$$[H_a, e_i] = H_a(i)e_i, \quad [H_a, f_i] = -H_a(i)f_i,$$
$$[e_i, f_j] = \delta_{ij}\left(\mathrm{e}^{\phi(i,\cdot)} - \mathrm{e}^{-\phi(\cdot,i)}\right).$$

Then the generalized quantum group $\mathcal{A}' = \mathcal{A}'(\phi, \psi)$ is the quotient $\mathcal{A}' = \mathcal{A}/\mathcal{I}(\mathcal{A})$, where $\mathcal{I}(\mathcal{A})$ is the Serre ideal.

There is a set of parameters, some of which are less significant in that they are basis dependent. What really matters are the numbers

$$q_{ij} = \phi^{ab} H_a(i) H_b(j).$$

Our concern is, above all, with the Serre ideal, that is, the ideal generated by the Serre relations.

The Serre relations, here exactly as in the original setting, are polynomials $C(e_1, \ldots, e_n)$ with the property that

$$[f_i, C] = 0, \quad i = 1, \ldots, n. \qquad (1)$$

This property allows one to define a quotient algebra, defined by replacing all such polynomials by zero. And this quotient algebra is the object that is called a quantum group.

So what are these relations? Drinfel'd is concerned with direct deformations of simple Lie algebras obtainable by passing to the limit when all the parameters q_{ij} tend to one. In that narrow context, there are powerful restrictions, on the parameters q_{ij}, and in consequence of these restrictions, one finds that the polynomials that satisfy Eq. (1) take the form

$$[e_i, [e_i, \ldots [e_i, e_j] \ldots].$$

Here, the bracket stands for a q-commutator, for example, $[e_i, e_j] = e_i e_j - q_{ji} e_j e_i$; otherwise, these are the same Serre relations as characterize the underlying or limiting Lie algebra.

But in general, before any special conditions are imposed on q_{ij}, there are no Serre relations at all, no polynomials that satisfy Eq. (1).

Now we can state the project at hand, though still somewhat loosely: for every choice of the parameters, to determine the Serre ideal, and by a classification of the ideals, to arrive at a classification of a very large class of quantum groups.

Let us put away all these complicated formulas to consider a much simpler family of algebras, called here q-algebras, with natural and straightforward relations. These algebras have generators e_i and differential operators ∂_i. It turns out that the determination of the "constants" of these algebras are precisely the Serre relations of the above generalized quantum groups, through the identification of the generators e_i that appear in both. I shall not present the proof of this assertion for lack of time and also because I hope to convince you that the q-algebras would be worthy of study even without the connection to quantum groups.

q-ALGEBRAS

Let $\mathcal{B} = \mathbb{C}[e_1, \ldots, e_N]$ be the complex, unital algebra freely generated by letters $e_1, \ldots, e_n$ and a unit written 1. Multiplication is just the formation of words, it is associative but not commutative, and there are no relations. Introduce differential operators $\partial_1, \ldots, \partial_N$ with the action defined by $\partial_i e_j = \delta_{ij}$ and

$$\partial_i(e_j x) = \delta_{ij} x + q_{ij} e_j \partial_i x, \quad x \in \mathbb{C}[e_1, \ldots, e_N].$$

Let $\mathcal{B}_q$ be the same algebra $\mathcal{B}$ with this differential structure.

This differential structure is a curious one. For example, the following set of equations,

$$\partial_i f = A_i, \quad i = 1, \ldots, n,$$

where $A_1, \ldots, A_N$, can always be solved for f, whatever the polynomials (= words) $A_1, \ldots, A_N$. To illustrate, consider

$$\partial_1 f = e_2, \quad \partial_2 f = 0. \tag{2}$$

Evidently, the first relation is solved by some $f = ae_1e_2 + be_2e_1$, $a, b \in \mathbb{C}$, and we find

$$\partial_1 f = (a + q_{12}b)e_2, \quad \partial_2 f = (aq_{21} + b)e_1.$$

If $q_{12}q_{21} = 1$, then the vanishing of one implies the vanishing of the other, and our system (2) has no solution. But a solution exists in general for all parameters such that $q_{12}q_{21} \neq 1$.

On the other hand, we may consider the system

$$\partial_1 f = 0, \quad \partial_2 f = 0, \tag{3}$$

which is solvable only if $q_{12}q_{21} = 1$.

Definition. *A "constant" in $\mathcal{B}_q$ is a polynomial $C \in \mathcal{B}$, having no term of order 0, such that $\partial_i C = 0$, $i = 1, \ldots, N$. Let $\mathcal{I}_q$ denote the ideal in $\mathcal{B}_q$ that is generated by the constants.*

A connection between these algebras and quantum groups is assured by the following.

Theorem. *The ideal $\mathcal{I}_q$ of $\mathcal{B}$, via the identification of $\mathcal{B}$ with the subalgebra $\mathcal{B}_+ \subset \mathcal{A}$, is precisely the component $\mathcal{I}_+$ of the Serre ideal of $\mathcal{A}$.*

Examples. There are no constants of order 1. Constants of order 2 are

$$e_1e_1, \quad \text{constant if } q_{11} = -1,$$

$$[e_1, e_2]_{q_{21}} := e_1e_2 - q_{21}e_2e_1, \quad \text{constant if } \sigma_{12} := q_{12}q_{21} = 1.$$

Some constants of order 3:

$$[e_1, [e_1, e_2]_{q_{21}}]_{q_{11}q_{21}}, \quad \text{constant if } q_{11}\sigma_{12} = 1,$$

$$\left(q_{13} - \frac{1}{q_{31}}\right)(e_1e_2e_3 + q_{21}q_{31}q_{32}e_3e_2e_1) + \text{cyclic}, \quad \text{if } \sigma_{123} = 1. \tag{4}$$

The existence of constants depends on algebraic relations between the parameters; in general, there are no constants. In the special case that constants exist, they generate an ideal $\mathcal{I}_q$, and we can define a new algebra by passing to the quotient,

$$\mathcal{B}'_q := \mathcal{B}_q/\mathcal{I}_q.$$

Thus, if $q_{ii} = 1$, we get a Grassmann algebra; if $q_{ij}q_{ji} = 1$, the quantum plane, characterized by $e_ie_j = q_{ji}e_je_i$.

The Matrix S

Two algebras $\mathcal{B}_q$ and $\mathcal{B}_{q'}$ have similar Serre ideals if $q_{ii} = q'_{ii}$ for all $i = 1, \ldots, N$, and if for all pairs i, j, $q_{ij}q_{ji} = q'_{ij}q'_{ji}$. If a change of parameters does not change the dimension of the Serre ideal, or if, more precisely, the family of constants remains essentially the same, then we shall say that the two algebras are of the same type. By classification, we shall mean classification by distinct types. We proceed to give a precise meaning to this idea.

We use multi-index notation, $\underline{i} := i_1 \ldots i_n$, $\quad \underline{i}' = i_n \ldots i_1$ and

$$\partial_{\underline{i}'} = \partial_{i_n} \ldots \partial_{i_1}, \quad e_{\underline{j}} = e_{j_1} \ldots e_{j_n}.$$

A matrix $S = (S_{\underline{i}\underline{j}})$ is defined by

$$S_{\underline{i}\underline{j}} = \partial_{\underline{j}'} e_{\underline{i}}|_0,$$

where $x|_0$ is the term of total order 0 in the polynomial $x \in \mathcal{B}$. This matrix is actually a direct sum of finite matrices.

A polynomial $x \in \mathcal{B}$ is homogeneous if it is a linear combination of monomials that differ only by the order of factors. For any homogeneous polynomial x, we define the degree $G(x)$ as the collection of indices (including repetitions) of the factors. Thus, $e_1e_2e_3$ and $e_2e_3e_1$ both have degree $\{1, 2, 3\}$. The degrees form an Abelian semigroup under the set union, and a partial ordering is defined by inclusion. Thus, $\{1, 3\} < \{1, 2, 3\}$. This partial order gives sense to the term "lower degree" that will be used often.

The matrix S commutes with the grading,

$$S = \oplus_G S_G, \quad (S_G)_{\underline{i}\underline{j}} = \partial_{\underline{j}'} e_{\underline{i}},$$

where $\underline{i}$, $\underline{j}$ run over the orderings of the unordered set G.

The matrix S is singular if and only if there is a constant in $\mathcal{B}_q$, and S_G is singular if and only if there is a constant of degree G. The existence of constants can thus be decided by inspection of the determinants. For example, if $\sigma_{12} := q_{12}q_{21} = 1$, then there is a constant of degree $G = \{1, 2\}$, namely, $e_1e_2 - q_{21}e_2e_1$, and

$$S_G = \begin{pmatrix} 1 & q_{12} \\ q_{21} & 1 \end{pmatrix}, \quad \det S_G = 1 - \sigma_{12} = 0.$$

Important Theorem. *The projection S' of S on $\mathcal{B}'_q$ is nonsingular; there are no constants in $\mathcal{B}'_q$.*

The Determinant

The family $\{\mathcal{B}_q\}$ of algebras is parametrized by $q = \{q_{ij}\}_{i,j=1,\ldots,N} \in V := \mathbb{C}^{N^2}$. There is an open subset V_{gen} of V such that for $q \in V_{\text{gen}}$ there are no constants in $\mathcal{B}_q$, namely, the subspace defined by $\det S \neq 0$. We shall say that parameters in this open set are in general position. Until further notice, suppose that the parameters are in general position.

Let $\mathcal{B}_G$ be the subspace of $\mathcal{B}_q$ that consists of all polymials of degree G. From now on, $G = \{1, \ldots, n\}$, n fixed. Set

$$w_{n,k} = u_{n,k} v_k,$$

where

$$u_{n,k} = (n + 1 - k)! \tag{5}$$

and

$$v_k = (k - 2)! \tag{6}$$

Then it is a result of Varchenko that

$$\det S_G = \prod_k \prod_{i_1,\ldots,i_k} (1 - \sigma_{i_1 \ldots i_k})^{w_{n,k}}.$$

The inner product is over all subsets of cardinality $k \geq 2$ of the set $\{1, \ldots, n\}$. The numbers (5) and (6) have the following interpretation. Fix the integer $k \leq n$ and let $G_k = \{1, \ldots, k\}$. Let the parameters approach a portion of the boundary of V_{gen}, where $\sigma_{1\ldots k} = 1$ but $\sigma_{\underline{i}} \neq 1$ for all $\underline{i} \neq 1 \ldots k$ (as unordered sets). Then constants appear in $\mathcal{B}_{G_k}$; v_k is the dimension of the space of (primitive) constants in $\mathcal{B}_{G_k}$ and $u_{n,k}$ is the dimension of the ideal in $\mathcal{B}_G$ generated by a constant in $\mathcal{B}_{G_k}$.

Example. Let $G = \{1, 2, 3\}$ and suppose that there are no constants of lower degree; then,

$$\det S_G = (1 - \sigma_{12})^2 (1 - \sigma_{23})^2 (1 - \sigma_{13})^2 (1 - \sigma_{123}).$$

The surface on which S_G is singular has four components, and, in particular, S_G is singular on the surface $\sigma_{123} = 1$. On an open subset of this surface, the algebra $\mathcal{B}_q$ is characterized by the existence of a constant of degree $G = \{123\}$.

Cell Decomposition of Parameter Space

The space of parameters is the space $V = \mathbb{C}^{N^2}$ in which the N^2 parameters q_{ij} take their values, with the natural analytic structure defined by these parameters. This space is the disjoint union of its G-cells (G fixed), defined as follows.

Definition. *A G-cell in V is a connected subset of V on which the rank of each matrix $S_{G'}$, $G' \leq G$, is constant. A regular function on a G-cell is the restriction to the G-cell of a polynomial on V.*

This concept of a regular function may be used to give a precise sense to regular fields of constants (being a family of constants, the coefficients of which are regular functions of the parameters) and regular fields of algebras.

Fix a G-cell C. The constants in $\mathcal{B}_q$, for $q \in C$, are polynomials with coefficients that are regular functions on C; we have a space of regular fields of constants and regular fields $\mathcal{B}$ and $\mathcal{B}'$ on each cell. The subspaces $\mathcal{B}_G$ and $\mathcal{B}'_G$ are regular fields defined by restriction to degree G. The matrix field S'_G and $\det S'_G$ are also regular fields, with $\det S'_G \neq 0$ on C.

A "cell" in V is a connected subset of V on which all the matrices S_G have constant rank; it is an intersection of determinantal varieties. Each cell carries a regular field of q-algebras and the "type" of an algebra is synonymous with the cell to which it belongs. The program of classification is thus concretely identified with the enumeration of the cells.

The classification proceeds inductively. I shall limit the discussion to the case of polynomials that contain at most one factor of each generator.

Classification to order 2. In degree $G = \{1, 2\}$, we have

$$S_G = \begin{pmatrix} 1 & q_{12} \\ q_{21} & 1 \end{pmatrix}, \quad \det S_G = 1 - \sigma_{12} = 0.$$

The G-cells are

$$C_1 : \sigma_{12} \neq 1, \quad C_2 = dC_1 : \sigma_{12} = 1.$$

Classification to order 3. In degree $G = \{1, 2, 3\}$, we find, for parameters in general position,

$$\det S_G = (1 - \sigma_{12})^2 (1 - \sigma_{23})^2 (1 - \sigma_{13})^2 (1 - \sigma_{123}).$$

To fix the rank we must first decide on the G'-type for each of the three lower degrees.

(a) All $\sigma_{ij} \neq 1$. Then the formula for the determinant tells us that there are just two cells, where $\sigma_{1233} \neq 1$, $\sigma_{1233} = 1$, respectively. The submatrices have rank 2 and the matrix $S_{\{123\}}$ has rank 6 or 5, respectively. [The constant that appears when $\sigma_{123} = 1$ was given above, Eq. (4).]

(b) One or more of the parameters $\sigma_{12}, \sigma_{13}, \sigma_{23}$ is equal to 1. Then there are constants of order 2 that generate an ideal $\mathcal{I}$. We must project the matrix $S_{\{123\}}$ on the quotient algebra and calculate the determinant. We find in each case that the rank is constant; there is only one cell.

The nature of the problem may become clear if we go just one step further.

Classification to order 4. A number of different cases must be considered, for we must first fix the G'-cells for all $G' < \{1, 2, 3, 4\}$.

(a) If there are no lower order constraints, we find

$$\det S_G = \prod_{i<j}(1-\sigma_{ij})^6 \prod_{i<j<k}(1-\sigma_{ijk})^2(1-\sigma_{1234})^2.$$

The only factor that is not different from zero by assumption is the last one; it may vanish or not, giving rise to two cells.

(b) In three cases (and others related to these three by renaming the generators), namely:

(i) $\sigma_{12}=1$,

(ii) $\sigma_{12}=\sigma_{34}=1$,

(iii) $\sigma_{123}=1$,

it is found that the determinant of the projection of S_G has a factor $1-\sigma_{1234}$. In each of these cases, there are two cells distinguished by $\sigma_{1234}=1$ or $\neq 1$.

(c) In all other cases, there is only one cell, that is, no bifurcation of algebraic types at this order. For more details, see my paper [1].

The essential point of each step of the induction is thus to calculate the determinant of S_G after projection on a quotient algebra determined by constraints of lower orders. Some general results have been obtained; among them must be included the rather special case that is the quantum Gabber–Kac theorem. Here is the most recent result.

Suppose that σ_{1n} is not constrained. There is a basis for $\mathcal{B}'_{\{1\ldots n\}}$ that consists of monomials of the type xe_1ye_2z. Filter the polynomials by the order of y. A basic word of the type e_1ye_2 is called a "long" word. Then we have the following theorem.

Theorem. *The exponent of* $1-\sigma_{1\ldots n}$ *in* $\det S_{\{1\ldots n\}}$ *is equal to the number of long basic words in* $\mathcal{B}'_G$.

Example. If $\sigma_{12}=\sigma_{13}=1$, then e_1 commutes with e_2 and with e_3 and there are no long words of degree $\{1234\}$. If $\sigma_{12}=\sigma_{34}=1$, then e_1 commutes with e_2 and e_3 commutes with e_4; there is one long word $e_1e_3e_2e_4$. For details and proofs, see my paper [1].

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Self-Similar Random Processes and Infinite-Dimensional Configuration Spaces*

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Abstract—We discuss various infinite-dimensional configuration spaces that carry measures quasi-invariant under compactly supported diffeomorphisms of a manifold M corresponding to a physical space. Such measures allow the construction of unitary representations of the diffeomorphism group, which are important to nonrelativistic quantum statistical physics and to the quantum theory of extended objects in $M = \mathbb{R}^d$. Special attention is given to measurable structure and topology underlying measures on generalized configuration spaces obtained from self-similar random processes (both for $d = 1$ and $d > 1$), which describe infinite point configurations having accumulation points. © *2005 Pleiades Publishing, Inc.*

1. INTRODUCTION

Let M be the manifold of physical space, usually taken to be d-dimensional Euclidean space $\mathbb{R}^d$. Let $Diff^c(M)$ be the (infinite-dimensional) group of compactly supported diffeomorphisms of M, under composition. The local current algebra approach to nonrelativistic quantum mechanics led to the understanding that a wide variety of quantum systems could be described by constructing the continuous unitary representations (CURs) of $Diff^c(M)$, the group of compactly supported diffeomorphisms of M (under composition) [1–7].

To say that the diffeomorphism ϕ of M has compact support means that for all points $\mathbf{x} \in \mathbf{M}$ that are outside some compact (and therefore bounded) region of M, the diffeomorphism acts as the identity operator: $\phi(\mathbf{x}) \equiv \mathbf{x}$. Our convention here will be to define the group product $\phi_1\phi_2 = \phi_2 \circ \phi_1$, where $\circ$ denotes the composition of $\phi_1, \phi_2 \in Diff^c(M)$, so that $[\phi_1\phi_2](\mathbf{x}) = \phi_2(\phi_1(\mathbf{x}))$ for $\mathbf{x} \in M$. Thus, we have a "right action" of the diffeomorphism group on the manifold.

In a very general framework, the Hilbert space where the unitary representation of $Diff^c(M)$ can be realized is the space of square-integrable functions, $\mathcal{H} = L^2_\mu(\Delta, \mathcal{W})$, where Δ is some configuration space on which the diffeomorphism group naturally acts (with a right action), μ is a measure on Δ satisfying appropriate technical conditions, $\mathcal{W}$ is an inner product space, and the elements of $\mathcal{H}$ are μ-measurable functions $\Psi(\gamma)$ on Δ taking values in $\mathcal{W}$. The inner product of two such functions in $\mathcal{H}$ is given by

$$(\Psi_1, \Psi_2) = \int_\Delta \langle \Psi_1(\gamma), \Psi_2(\gamma)\rangle_{\mathcal{W}} d\mu(\gamma) < \infty, \quad (1)$$

where $\langle \Psi_1(\gamma), \Psi_2(\gamma)\rangle_{\mathcal{W}}$ denotes the inner product in $\mathcal{W}$. Then the operators $V(\phi)$ defining a CUR are given by

$$[V(\phi)\Psi](\gamma) = \chi_\phi(\gamma)\Psi(\phi\gamma)\sqrt{\frac{d\mu_\phi}{d\mu}(\gamma)}, \quad (2)$$

where $\phi\gamma$ refers to the action of the diffeomorphism ϕ on $\gamma \in \Delta$ and where $\chi_\phi : \mathcal{W} \to \mathcal{W}$ is a family of unitary operators acting in $\mathcal{W}$ satisfying a certain cocycle equation (see below).

In this article, we shall consider various candidates for a "large" configuration space, within which different choices of the space Δ may be situated, that permit the construction of measures having the necessary property of quasi-invariance under diffeomorphisms. We then focus on the generalized configuration space Σ_M whose elements are finite or countably infinite subsets of M and discuss ways of endowing it with a σ algebra and a topology. The results underlie the construction of measures on generalized configuration spaces obtained from self-similar random processes in $\mathbb{R}^d$ (both for $d = 1$ and $d > 1$), which describe infinite point configurations having accumulation points.

In Section 2, we briefly discuss the meaning of Eq. (2), reviewing the necessary concepts. Section 3

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surveys some aspects of several possible choices of "large" configuration spaces, while Section 4 focuses on topology and measurable structure in Σ_M. In Section 5, we give a rapid overview of the construction of certain families of quasi-invariant measures in $\Sigma_{\mathbb{R}^d}$ making use of self-similar random processes.

2. MEASURES AND COCYCLES

The measure μ that appears in Eq. (2) and in the definition of $\mathcal{H}$ is, as usual, a countably additive, positive real-valued function defined on a σ algebra $\mathcal{M}$ of subsets of Δ. It is required to have the key property of quasi-invariance under the action of diffeomorphisms on Δ.

In general, let G be a group of transformations of a measurable space $(X, \mathcal{M})$, where $\mathcal{M}$ is a G-invariant σ algebra of subsets of X. A measure μ on $\mathcal{M}$ is said to be invariant under G if and only if for all $E \in \mathcal{M}$ and for all $g \in G$, $\mu(gE) = \mu(E)$. It is said to be quasi-invariant under G if and only if for all $E \in \mathcal{M}$ such that $\mu(E) > 0$ and for all $g \in G$, $\mu(g(E)) > 0$. That is, $g \in G$ acts on X in such a way as to preserve the class of sets that have μ-measure zero.

Quasi-invariance is a fortiori a consequence of invariance, but not conversely. For example, the Lebesgue measure $d\mathbf{x}$ on $X = \mathbb{R}^d$ is invariant under the group of rigid motions (translations and rotations). It is quasi-invariant, but not invariant, under the group of compactly supported diffeomorphisms of $\mathbb{R}^d$.

For $\phi \in G = \mathit{Diff}^c(M)$ acting on $X = \Delta$, define the transformed measure μ_ϕ by setting $\mu_\phi(E) = \mu(\phi(E))$ for any $E \in \mathcal{M}$. Because of the group structure and the G invariance of $\mathcal{M}$, the quasi-invariance of μ under G is equivalent to the absolute continuity of μ_{ϕ_1} with respect to μ_{ϕ_2} for any $\phi_1, \phi_2 \in G$. In particular, the quasi-invariance of μ is necessary and sufficient for the existence of the Radon–Nikodym (RN) derivative $(d\mu_\phi/d\mu)(\gamma)$ appearing in Eq. (2), for all elements $\phi \in \mathit{Diff}^c(M)$. For example, with $M = \mathbb{R}^d$, $\Delta = \mathbb{R}^d$, and $d\mu = d\mathbf{x}$, we have $(d\mu_\phi/d\mu)(\mathbf{x}) = \mathcal{J}_\phi(\mathbf{x})$, the Jacobian of ϕ at $\mathbf{x}$. Since ϕ has compact support, we have $\mathcal{J}_\phi(\mathbf{x}) \equiv 1$ outside some bounded region of $\mathbb{R}^d$.

The square root of the RN derivative in Eq. (2) is precisely the factor necessary to make the operators $V(\phi)$ unitary in $\mathcal{H}$, since $\chi_\phi(\gamma)$ is to be taken as acting unitarily in $\mathcal{W}$ (see below). That is, the diffeomorphism ϕ moves the argument of the wave function Ψ, and the square-root factor corrects it so that, when we calculate the inner product $(V(\phi)\Psi_1, V(\phi)\Psi_2)$ using Eq. (1), we find that we have merely made the change of variable $\gamma' = \phi\gamma$ under the integral sign.

Let $\mathcal{D}(M)$ be the space of real-valued, compactly supported C^∞ functions f on M. We have then the natural semidirect product group $\mathcal{D}(M) \times \mathit{Diff}^c(M)$, with the group law given by

$$(f_1, \phi_1)(f_2, \phi_2) = (f_1 + f_2 \circ \phi_1, \phi_1\phi_2). \qquad (3)$$

Now it may sometimes be the case that $V(\phi)$ is a subrepresentation of a CUR of $\mathcal{D}(M) \times \mathit{Diff}^c(M)$, which we write $U(f)V(\phi)$. Then the operators $U(f), f \in \mathcal{D}(M)$, typically act in $\mathcal{H}$ as multiplication operators, consistently with Eq. (2):

$$[U(f)\Psi](\gamma) = \exp[i\langle\gamma, f\rangle]\Psi(\gamma), \qquad (4)$$

where $\langle\gamma, f\rangle$ denotes an action of $\gamma \in \Delta$ on $f \in \mathcal{D}(M)$ as a continuous linear functional. That is, the configuration γ is here identified with a distribution, and Δ is identified with a subset of the dual space $\mathcal{D}'(M)$. This is one of the possibilities discussed in Section 3.

In Eq. (2), $\chi_\phi : \mathcal{W} \to \mathcal{W}$ is a family of unitary operators in $\mathcal{W}$ satisfying the cocycle equation

$$\chi_{\phi_1}(\gamma)\chi_{\phi_2}(\phi_1\gamma) = \chi_{\phi_1\phi_2}(\gamma), \qquad (5)$$

which holds almost everywhere (a.e.) in Δ for each pair of diffeomorphisms ϕ_1, ϕ_2. That is, Eq. (5) holds outside a μ-measure zero set that, in general, may depend on ϕ_1 and ϕ_2.

The cocycle equation follows directly from the condition that V respect the group law, $V(\phi_1)V(\phi_2) = V(\phi_1\phi_2)$. The trivial cocycle $\chi_\phi(\gamma) \equiv I$ is always permitted, and in the case of a CUR describing N identical particles, this choice corresponds to Bose–Einstein statistics. Inequivalent choices of χ_ϕ (noncohomologous cocycles) are associated with Fermi–Dirac statistics, nontrivial phase effects, and anion statistics in two space dimensions [8–12], as well as with certain nonlinear variations of quantum mechanics [13–15]. In the simplest cases, $\mathcal{W}$ is just the one-dimensional space of complex numbers $\mathbb{C}$, so that we have complex-valued wave functions on Δ. Then the χ_ϕ act through multiplication by complex numbers of modulus 1. Higher dimensional choices for $\mathcal{W}$ are associated with paraparticles in $\mathbb{R}^3$ and plektons in $\mathbb{R}^2$ [16–18].

3. GENERAL CONFIGURATION SPACES

Up to this point, no universal configuration space for the representation theory of $\mathit{Diff}^c(M)$ has been agreed upon. Consequently, we have no one universal configuration space for the physics of systems with infinitely many degrees of freedom in $\mathbb{R}^d$, within which specific choices of configuration spaces for particular systems are situated. This very likely reflects a gap in our present level of understanding. Let us describe

here some choices that have been made, choices that allow the convenient description and interpretation of certain classes of unitary representations.

3.1. Locally Finite Point Configurations

The standard configuration space for statistical physics is the space $\Gamma_M^{(\infty)}$ of countably infinite but locally finite subsets of M, where usually $M = \mathbb{R}^d$. Frequently, one considers the disjoint union of this space with the spaces of N-point subsets; thus, $\Gamma_M = \bigsqcup_{N=1}^{\infty} \Gamma_M^{(N)} \bigsqcup \Gamma_M^{(\infty)}$ is the space of all locally finite subsets of M. Measures on the configuration space $\Gamma_{\mathbb{R}^d}^{(\infty)}$ describe equilibrium states in $\mathbb{R}^d$ in statistical mechanics, while $\Gamma_{\mathbb{R}^d}^{(\infty)}$ also enters quantum theory in the description of infinite gases of quantum particles in $\mathbb{R}^d$.

Let $|\gamma|$ denote the cardinality of the set γ. A configuration $\gamma \subset \mathbb{R}^d$ in $\Gamma_{\mathbb{R}^d}^{(\infty)}$ has the properties that $|\gamma| = \aleph_0$, while for any compact set $K \subset \mathbb{R}^d$, $|\gamma \cap K| < \infty$. Then the diffeomorphism $\phi \in \mathit{Diff}^c(\mathbb{R}^d)$ acts naturally on any configuration $\gamma \in \Gamma_{\mathbb{R}^d}$ by its action on the individual elements of γ. This clearly respects the property of being finite or locally finite. Measures on $\Gamma_{\mathbb{R}^d}^{(\infty)}$ that are quasi-invariant under diffeomorphisms have been extensively studied, and include Poisson measures and Gibbsian measures [3, 5–7, 19].

In particular, the choice of a Poisson measure $d\mu^\sigma$ on $\Gamma_{\mathbb{R}^d}^{(\infty)}$, with intensity $\sigma > 0$, together with the trivial cocycle $\chi_\phi \equiv 1$, gives a CUR of $\mathit{Diff}^c(\mathbb{R}^d)$ via Eq. (2). This representation describe the infinite, free quantum Bose gas having σ as its average particle number density [3]. Here, we have, for any choice of σ,

$$\frac{d\mu_\phi^\sigma}{d\mu^\sigma}(\gamma) = \prod_{\mathbf{x}\in\gamma} \mathcal{J}_\phi(\mathbf{x}). \tag{6}$$

Since ϕ has compact support and γ is locally finite, it is evident that all but a finite number of terms in the infinite product of Jacobians in Eq. (6) are equal to 1. Thus, this product gives a finite, nonzero result for the value of the RN derivative—expressing the fact that Poisson measures on $\Gamma_{\mathbb{R}^d}^{(\infty)}$ are quasi-invariant under compactly supported diffeomorphisms of $\mathbb{R}^d$.

3.2. Configuration Spaces of Closed Subsets

A much larger configuration space, introduced in early work by Ismagilov [20–23], is the space Ω_M of all (nonempty) closed subsets of the manifold M. For any closed set $C \in \Omega_M$, define the natural action of a diffeomorphism $\phi \in \mathit{Diff}^c(M)$ on Ω_M by $\phi C = \{\phi(\mathbf{x}) | \mathbf{x} \in C\}$. Evidently, ϕC also belongs to Ω_M, and we have a (right) group action.

A σ algebra for Ω_M is generated by the family of sets in Ω_M consisting of all closed subsets of a given closed set. Thus, for $C \in \Omega_M$ (i.e., for $C \subseteq M$ closed), let $\Omega_C = \{C' \in \Omega_M | C' \subseteq C\}$. Then let $\mathcal{B}_{\Omega_M}$ be the smallest σ algebra containing the family of sets $\{\Omega_C\}_{C\subseteq M\text{closed}}$. This σ algebra can also be obtained as the algebra of Borel sets with respect to a topology on Ω_M, for which a subbase is the family of sets $\{C | C \cap \mathcal{O} \neq \emptyset\}_{\mathcal{O}\subseteq M\text{open}}$; i.e., the family of subsets of Ω_M whose elements meet a given open set $\mathcal{O} \subseteq M$.

Evidently, any locally finite configuration $\gamma \in \Gamma_M$ is also a closed subset of M, so that in general we have $\Gamma_M \subset \Omega_M$.

3.3. Configuration Spaces of Generalized Functions

Another possibility is to work with the dual space $\mathcal{D}'(M)$, as suggested by the CURs of the semidirect product group mentioned in Section 1. That is, a configuration $\gamma \in \mathcal{D}'(M)$ is a continuous, linear, real-valued functional on $\mathcal{D}(M)$—a distribution or generalized function on M. This is especially convenient for representing Eq. (4), as we can immediately write $\langle \gamma, f \rangle$ for the value taken by γ on the function $f \in \mathcal{D}(M)$.

Diffeomorphisms act on $\mathcal{D}'(M)$ by the dual to their action on $\mathcal{D}(M)$; i.e., $\phi\gamma$ is defined for $\gamma \in \mathcal{D}'(M)$ by $\langle \phi\gamma, f \rangle = \langle \gamma, f \circ \phi \rangle$ for all $f \in \mathcal{D}(M)$. [With this definition and our earlier convention, we have $(\phi_1\phi_2)\gamma = \phi_2(\phi_1\gamma)$, so that the group action is a right action as desired.] A σ algebra in $\mathcal{D}'(M)$ may be built up directly from cylinder sets with Borel base [24], or $\mathcal{D}'(M)$ can be endowed with the weak dual topology and measures constructed on the corresponding Borel σ algebra.

Evidently Γ_M, or more specifically $\Gamma_{\mathbb{R}^d}$, may be identified naturally with a subset of $\mathcal{D}'(M)$, or $\mathcal{D}'(\mathbb{R}^d)$, by the correspondence

$$\gamma \to \sum_{\mathbf{x}\in\gamma} \delta_{\mathbf{x}}, \tag{7}$$

where $\delta_{\mathbf{x}} \in \mathcal{D}'(M)$ is the evaluation functional (i.e., the Dirac δ function) defined by $\langle \delta_{\mathbf{x}}, f \rangle = f(\mathbf{x})$, $\mathbf{x} \in M$.

The so-called "vague topology" in Γ_M is in fact the topology that Γ_M inherits from the weak dual topology in $\mathcal{D}'(M)$. While Γ_M is not a linear space, the larger space $\mathcal{D}'(M)$ is. In addition to linear combinations of evaluation functionals (with possibly distinct real coefficients), $\mathcal{D}'(M)$ contains other kinds of configurations of physical importance that do not belong

to Γ_M and in some cases are not easily identified with elements of Ω_M. For example, configurations may include terms that are derivatives of δ functions, as well as generalized functions with support on embedded submanifolds of M.

3.4. Configuration Spaces of Embeddings and Immersions

Still another characterization of a "large" space of configurations in M begins with some other manifold (or manifold with boundary) L, together with a set of maps $\alpha : L \to M$ that obey some specified regularity and continuity properties (for which there are numerous possible choices). Then we call L the parameter space for the corresponding class of configurations and M the target space. When α is injective (so that self-intersection of the image of L in the target space is not permitted), we have a configuration space of embeddings, while without any such restriction we have a larger space of immersions.

We have at the outset the choice of considering parametrized or unparametrized maps. A space of parametrized C^k immersions consists of mappings $\alpha(\theta), \theta \in L$, that are C^k for some fixed integer $k \geq 0$. For $\phi \in \mathit{Diff}^c(M)$, the formula $[\phi\alpha](\theta) = \phi(\alpha(\theta))$ (i.e., $\phi\alpha = \phi \circ \alpha$) defines the desired (right) group action on the space of parametrized immersions. In addition, the group $\mathit{Diff}(L)$ acts on the space of immersions (as a left action) by reparametrization, so that, for $\psi \in \mathit{Diff}(L)$, $\psi : \alpha \to \alpha \circ \psi$.

Then an unparametrized immersion is just the image set $K = \alpha(L) \subset M$, where the parametrization of K has been disregarded. Alternatively, we can think of the unparametrized immersion as an equivalence class of parametrized immersions modulo reparametrization. Note that the action of $\mathit{Diff}^c(M)$ on the space of (parametrized or unparametrized) immersions leaves the corresponding space of embeddings invariant as a subset and preserves the continuity properties of configurations in the space.

If L is the circle S^1, for instance, configurations are C^k loops in M. The embeddings are the nonselfintersecting loops. The action of the diffeomorphism group also respects the knot class of the loop. If L is the closed interval $[0, 2\pi]$, configurations are finite arcs in M. Further possibilities include ribbons, tubes, or higher dimensional submanifolds of M.

The configuration space of unparametrized immersions of L in M is a subset of the configuration space Ω_M, invariant (as a set) under the action of $\mathit{Diff}^c(M)$. This description thus allows us to refine Ω_M as sensitively as desired, according to the topological and continuity properties of extended configurations.

For example, quantized vortex configurations in ideal, incompressible fluids are obtained from representations of groups of (area- and volume-preserving) diffeomorphisms of $\mathbb{R}^2$ and $\mathbb{R}^3$. For planar fluids, pure point vortices are not permitted quantum-mechanically, but one-dimensional filaments of vorticity are allowed. Similarly, in $\mathbb{R}^3$, pure filaments are kinematically forbidden, while two-dimensional vortex surfaces, e.g., ribbons or tubes, can occur [25–28]. But a major gap is the construction of measures, quasi-invariant under diffeomorphisms, directly on spaces of filaments or tubes. One approach to the filament case has been suggested by Shavgulidze [29].

Naturally, a nonrelativistic quantum theory of strings, with $\mathbb{R}^d$ as the target space, also depends on quasi-invariant measures on the space of loops.

In addition, we remark that diffeomorphism-invariant measures are important to the long-standing problem of finding consistent theories for quantized gravity; for instance, Ashtekar and Lewandowski have constructed a faithful, diffeomorphism-invariant measure on a compactification of the space of gauge-equivalent connections [30, 31].

Reparametrization invariance has nice consequences for quantum mechanics, when expressed in terms of diffeomorphism group representations. Note in particular that we can consider the N-particle configuration space $\Gamma_M^{(N)}$ as a special case of embeddings modulo reparametrization, with the discrete manifold $L = \{1, \ldots, N\}$. The group $\mathit{Diff}(L)$ in this case is the symmetric group S_N. The corresponding configuration space of parametrized embeddings is the space of ordered N tuples $(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ of distinct points, $\mathbf{x}_j \neq \mathbf{x}_k$ for $j \neq k$. The space of parametrized immersions of L in M includes the N tuples with coincident points.

3.5. The Configuration Space of Countable Subsets of $\mathbb{R}^d$

The idea pursued in the balance of this article is the construction of measures, quasi-invariant under diffeomorphisms of $\mathbb{R}^d$, on the space $\Sigma_{\mathbb{R}^d}^{(\infty)}$ of countably infinite subsets of the physical space $\mathbb{R}^d$ that are not necessarily locally finite. Alternatively, we may work on the space $\Sigma_{\mathbb{R}^d}$ whose elements are subsets $\gamma \subset \mathbb{R}^d$ that are finite or countably infinite, with

$$\Sigma_{\mathbb{R}^d} = \bigsqcup_{N=1}^{\infty} \Gamma_{\mathbb{R}^d}^{(N)} \bigsqcup \Sigma_{\mathbb{R}^d}^{(\infty)}. \tag{8}$$

We call this the space of generalized configurations.

Our main mathematical motivation for working with this space is that measures on it can be constructed by means of random point processes on

spaces of infinite sequences of points in $\mathbb{R}^d$. We shall project the measure μ on $[\mathbb{R}^d]^\infty$ that results from such a point process to define the corresponding measure $\hat{\mu}$ on $\Sigma_{\mathbb{R}^d}$, thus obtaining a measure on the space $\Sigma^{(\infty)}_{\mathbb{R}^d}$.

A physical motivation for this direction of work is the goal of constructing quasi-invariant measures for spatially extended systems, which is in general an unsolved problem. Since $\mathbb{R}^d$ is separable, any closed set in $\mathbb{R}^d$ can be obtained as the closure of an element of $\Sigma_{\mathbb{R}^d}$, so that the closure map $\gamma \to \bar{\gamma}$ from $\Sigma_{\mathbb{R}^d}$ to $\Omega_{\mathbb{R}^d}$ is surjective. Thus, our present approach—which puts us into a still larger configuration space than that of Ismagilov—may permit pointlike approximations to embedded manifold configurations.

Apart from this general consideration, the specific measures we can construct appear to have a direct interpretation as descriptive of idealized quantum or statistical configurations forming "particle clouds" about a locus of condensation. These allow for a kind of "phase transition" from a rarefied to a condensed phase, as the self-similarity parameter passes through a critical value.

Let us write $\omega = (\mathbf{x}_j) \in [\mathbb{R}^d]^\infty$ to denote an infinite sequence, with $j = 1, 2, 3, \ldots$.

Now, generalized configurations, like infinite sequences, can have accumulation points. A point $\mathbf{x} \in \mathbb{R}^d$ is an accumulation point of a set $\gamma \subset \mathbb{R}^d$—or, respectively, of an infinite sequence $\omega = (\mathbf{x}_j) \in [\mathbb{R}^d]^{(\infty)}$—if, for any neighborhood $\mathcal{U}$ of $\mathbf{x}$, the set $\mathcal{U} - \{\mathbf{x}\}$ contains infinitely many points of γ (respectively, ω). An accumulation point of γ may or may not itself be an element of γ. Evidently, diffeomorphisms of $\mathbb{R}^d$ act naturally on generalized configurations, respecting accumulation points: if $\mathbf{x} \in \mathbb{R}^d$ is an accumulation point of $\gamma \in \Sigma^{(\infty)}_{\mathbb{R}^d}$, then $\phi(\mathbf{x})$ is an accumulation point of $\phi\gamma$. The points belonging to configurations in $\Sigma^{(\infty)}_{\mathbb{R}^d}$ can cluster in such a manner as to yield fractals or even more complicated objects.

The set of sequences containing coincident points is called the "diagonal" D in $[\mathbb{R}^d]^\infty$; that is, $D = \{(\mathbf{x_j}) \in [\mathbb{R}^d]^\infty | \mathbf{x_k} = \mathbf{x}_\ell \text{ (for some } k \neq \ell)\}$. Typically, D is of measure zero for the point processes of interest, and for technical reasons, it will often be convenient to exclude it. We have the natural projection from the sequence space to the configuration space, $p : [\mathbb{R}^d]^\infty \to \Sigma_{\mathbb{R}^d}$, given by $p[(\mathbf{x}_j)] = \{\mathbf{x}_j\}$. The image of $[\mathbb{R}^d]^\infty$ under p is all of $\Sigma_{\mathbb{R}^d}$, since the possibility of repeated entries in elements of $[\mathbb{R}^d]^\infty$ permits the corresponding configurations to be finite as well as infinite. Then $[\mathbb{R}^d]^\infty$ can also be thought of as a fiber space over $\Sigma_{\mathbb{R}^d}$. It is natural to consider also the restriction of p to sequences without repeated entries, $p : [\mathbb{R}^d]^\infty - D \to \Sigma^{(\infty)}_{\mathbb{R}^d}$ (which is surjective).

Note that the space $\Sigma^{(\infty)}_{\mathbb{R}^d}$ may also be regarded as a special case of the space of unparametrized embeddings discussed in the preceding subsection. The target space M is $\mathbb{R}^d$; the parameter space L is $\mathbb{N}$ (the set of natural numbers); and $Diff(L)$ is the group S^∞ of bijections of $\mathbb{N}$. Of course, $[\mathbb{R}^d]^\infty - D$ is then seen as the space of parametrized embeddings of L into M, while $[\mathbb{R}^d]^\infty$ itself is the space of parametrized immersions.

For any diffeomorphism ϕ of $\mathbb{R}^d$, we have $\phi p[(\mathbf{x}_j)] = \{\phi(\mathbf{x}_j)\} = p[(\phi(\mathbf{x}_j))]$. Thus, we can project a probability measure on the sequence space $[\mathbb{R}^d]^\infty$ or $[\mathbb{R}^d]^\infty - D$, constructed as is usual from an infinite sequence of conditional probability densities on $\mathbb{R}^d$, to a probability measure on the configuration space $\Sigma^{(\infty)}_{\mathbb{R}^d}$, consistent with the action of $Diff^c(\mathbb{R}^d)$.

In earlier work, it was shown how for the one-dimensional manifolds $\mathbb{R}^1$ or S^1, self-similar point processes in the manifold lead quite generally through such a construction to quasi-invariant measures on the configuration space of countably infinite subsets [32–36]. The quasi-invariance is intimately related to the self-similarity. In Section 4, we shall discuss further the relevant σ algebra on this configuration space, which lays the foundation for completing the rigorous proofs of earlier conjectures. Then we shall indicate how the generalization to $d > 1$ is carried out [37].

4. TOPOLOGY AND MEASURABLE STRUCTURE ON $\Sigma_{\mathbb{R}^d}$

There are at least two possible approaches to defining a σ algebra on the generalized configuration space $\Sigma^{(\infty)}_{\mathbb{R}^d}$.

4.1. Indirect Approach through $[\mathbb{R}^d]^\infty$

The indirect approach makes use of the sequence space $[\mathbb{R}^d]^\infty$, which is endowed with the well-known weak product topology τ_w. Let us write $\mathbf{x}_j(\omega)$ for the jth entry of $\omega \in [\mathbb{R}^d]^\infty$. The weak topology is then the coarsest topology for which all the natural projections $\pi_j : [\mathbb{R}^d]^\infty \to \mathbb{R}^d$ given by $\omega \to \mathbf{x}_j(\omega)$ are continuous. This topology is inherited by $[\mathbb{R}^d]^\infty - D$.

Let $\mathcal{B}([\mathbb{R}^d]^\infty)$ denote the σ algebra of Borel sets in $[\mathbb{R}^d]^\infty$ with respect to τ_w. This naturally induces a σ algebra in $\Sigma_{\mathbb{R}^d}$—namely, the largest σ algebra with the property that the projection $p : [\mathbb{R}^d]^\infty \to \Sigma_{\mathbb{R}^d}$ is

measurable [33, 34]. More precisely, we introduce in $\Sigma_{\mathbb{R}^d}$ the σ algebra

$$\mathcal{P}_w(\Sigma_{\mathbb{R}^d}) := \{A \subseteq \Sigma_{\mathbb{R}^d} \mid p^{-1}(A) \in \mathcal{B}([\mathbb{R}^d]^\infty)\}, \quad (9)$$

to which each of the subsets $\Gamma^{(N)}_{\mathbb{R}^d}$, $N = 1, 2, 3, \ldots$, as well as $\Sigma^{(\infty)}_{\mathbb{R}^d}$, belongs.

Evidently, the set of accumulation points of an infinite sequence in $\mathbb{R}^d$ or $\mathbb{R}^d - D$ may be empty, finite and nonempty, countably infinite, or uncountably infinite. Since accumulation points in $\mathbb{R}^d$ depend only on the set $\gamma = \{\mathbf{x}_j\}$, and not specifically on the sequence $(\mathbf{x}_j)$, all the distinct elements of $p^{-1}(\gamma)$ have precisely the same accumulation points.

Now it is straightforward to demonstrate that various sets of interest in $\Sigma^{(\infty)}_{\mathbb{R}^d}$ belong to $\mathcal{P}_w$, by showing that the corresponding sets of sequences belong to $\mathcal{B}([\mathbb{R}^d]^\infty - D)$. A series of lemmas in earlier work [38, 39] shows that the set $[\mathbb{R}^d]^\infty - D$ itself belongs to $\mathcal{B}([\mathbb{R}^d]^\infty)$ and that the following subsets of $[\mathbb{R}^d]^\infty - D$ are likewise Borel: the set of all nonrepeating sequences having precisely n elements in a given compact set $K \subset \mathbb{R}^d$, the set of all locally finite nonrepeating sequences, and the set of all nonrepeating sequences having precisely N accumulation points in K. Each of these sets is the inverse image in $[\mathbb{R}^d]^\infty - D$ (under the projection p) of a set in $\Sigma^{(\infty)}_{\mathbb{R}^d}$; hence, the corresponding sets in $\Sigma^{(\infty)}_{\mathbb{R}^d}$ are measurable.

In fact, $\mathcal{P}_w(\Sigma^{(\infty)}_{\mathbb{R}^d})$ is sufficiently rich to permit us to count the numbers of accumulation points of configurations that are located in arbitrary Borel sets of $\mathbb{R}^d$ (not just compact sets). In particular, the subsets $\Sigma^{(\infty)}_{\mathbb{R}^d,N} \subset \Sigma^{(\infty)}_{\mathbb{R}^d}$ consisting of generalized configurations having exactly N accumulation points in $\mathbb{R}^d$ are measurable. The inverse image $p^{-1}(\Sigma^{(\infty)}_{\mathbb{R}^d,N})$ is the set of infinite sequences having precisely N accumulation points, which we denote by $[\mathbb{R}^d]^\infty_N \subset [\mathbb{R}^d]^\infty$ (for $N = 0, 1, 2, \ldots$).

Suppose that we have a probability measure μ on $[\mathbb{R}^d]^\infty$ or $[\mathbb{R}^d]^\infty - D$. Then we obtain a probability measure $\hat{\mu}$ on $\Sigma_{\mathbb{R}^d}$ by defining, for all $A \in \mathcal{P}_w(\Sigma_{\mathbb{R}^d})$, $\hat{\mu}(A) = \mu(p^{-1}(A))$. The most straightforward way to construct a countably additive measure μ on $[\mathbb{R}^d]^\infty$ [with the σ algebra $\mathcal{B}([\mathbb{R}^d]^\infty)$] is to specify a compatible family of measures on the finite-dimensional spaces from which $[\mathbb{R}^d]^\infty$ is constructed as the projective limit. The existence of the corresponding measure μ is then assured by Kolmogorov's theorem. If μ is quasi-invariant under diffeomorphisms of $\mathbb{R}^d$, then our construction ensures that $\hat{\mu}$ is also quasi-invariant as desired.

4.2. Direct Approach

The more direct approach to constructing a σ algebra on $\Sigma^{(\infty)}_{\mathbb{R}^d}$ is simply to specify a generating set of subsets of $\Sigma_{\mathbb{R}^d}$ or $\Sigma^{(\infty)}_{\mathbb{R}^d}$ for the σ algebra or else to introduce a topology in $\Sigma_{\mathbb{R}^d}$ or $\Sigma^{(\infty)}_{\mathbb{R}^d}$ and to take as our σ algebra the Borel sets with respect to that topology.

For instance, we may begin with Ismagilov's σ algebra on $\Omega_{\mathbb{R}^d}$ described above and lift it to a σ algebra $\mathcal{I}(\Sigma_{\mathbb{R}^d})$ using the closure map. The generating family for $\mathcal{I}(\Sigma_{\mathbb{R}^d})$ becomes all sets of the form $\{\gamma \in \Sigma_{\mathbb{R}^d} | \gamma \subseteq F\}$, where $F \in \Omega_{\mathbb{R}^d}$ is closed. Because F is closed, $\gamma \subseteq F$ if and only if $\bar{\gamma} \subseteq F$. The complement of a set in this generating family is the set $\mathcal{O}_U = \{\gamma \in \Sigma_{\mathbb{R}^d} |\ \gamma \cap U \neq \emptyset\}$, the set of all configurations that meet the open set $U \subseteq \mathbb{R}^d$, where U is $\mathbb{R}^d - F$. The collection of sets $\{\mathcal{O}_U | U \subseteq \mathbb{R}^d \text{open}\}$ likewise serves as a generating family for $\mathcal{I}(\Sigma_{\mathbb{R}^d})$ [37]. The subsets $\Gamma^{(N)}_{\mathbb{R}^d}$ and $\Sigma^{(\infty)}_{\mathbb{R}^d}$ of $\Sigma_{\mathbb{R}^d}$ belong to $\mathcal{I}(\Sigma_{\mathbb{R}^d})$.

We can make use of these families of sets to introduce a natural topology on $\Sigma_{\mathbb{R}^d}$. Define a subbase of open sets for a topology τ_o in $\Sigma_{\mathbb{R}^d}$ to be $\{\mathcal{O}_U | U \subseteq \mathbb{R}^d \text{ open}\}$. Note that, for any index set I, $\cup_{\alpha \in I} \mathcal{O}_{U_\alpha} = \mathcal{O}_{[\cup_{\alpha \in I} U_\alpha]}$, while $\cap_{j=1,\ldots,n} \mathcal{O}_{U_j} \supset \mathcal{O}_{[\cap_{j=1,\ldots,n} U_j]}$. The finite intersections of sets in the subbase form a base for τ_o.

In the topology τ_o, the subsets $\Gamma^{(n)}_{\mathbb{R}^d} \subset \Sigma_{\mathbb{R}^d}$ (for $n > 1$) and $\Sigma^{(\infty)}_{\mathbb{R}^d} \subset \Sigma_{\mathbb{R}^d}$ are neither open nor closed. However, for each $N \geq 0$, $\{\gamma | \ |\gamma| \leq N\} = \bigsqcup_{n=1}^N \Gamma^{(n)}_{\mathbb{R}^d}$ is closed. Of course, we may also consider separately the topology induced in $\Sigma^{(\infty)}_{\mathbb{R}^d}$ by τ_o.

Now, the σ algebra $\mathcal{I}(\Sigma_{\mathbb{R}^d})$, that we obtained by lifting Ismagilov's σ algebra to $\Sigma_{\mathbb{R}^d}$ by the inverse image of the closure map is precisely the Borel σ algebra $\mathcal{B}_o(\Sigma_{\mathbb{R}^d})$ with respect to the topology τ_o. Indeed, we noted already that the complement of $\mathcal{O}_U$ in $\Sigma_{\mathbb{R}^d}$ is just $\{\gamma \in \Sigma_{\mathbb{R}^d} | \gamma \subseteq \mathbb{R}^d - U\}$. Thus, we have immediately that $\mathcal{B}_o(\Sigma_{\mathbb{R}^d})$ contains $\mathcal{I}(\Sigma_{\mathbb{R}^d})$, and the closure map is τ_o-Borel measurable with respect to Ismagilov's σ algebra on $\Omega_{\mathbb{R}^d}$. Conversely, let $\{U_j | j = 1, 2, 3, \ldots\}$ be a countable base for the topology in $\mathbb{R}^d$. Then $\{\mathcal{O}_{U_j}\}$ is a countable subbase for τ_o, and the finite intersections of such sets form a countable base for τ_o whose elements are obtained directly from the generating family for $\mathcal{I}(\Sigma_{\mathbb{R}^d})$. Hence, $\mathcal{B}_o(\Sigma_{\mathbb{R}^d}) = \mathcal{I}(\Sigma_{\mathbb{R}^d})$.

Sakuraba constructs and discusses a related topology τ_s on Σ_M (here, $M = \mathbb{R}^d$), obtained as a quotient of the product topology on the disjoint union of $M^n, n = 1, 2, 3, \ldots$, and M^∞ with respect to the symmetric groups S_n and the infinite symmetric group [37]. In this construction, the topology on Σ_M is the sum of topologies on the components $\Gamma_M^{(n)}$ and $\Sigma_M^{(\infty)}$, and each of the subsets $\Gamma_M^{(n)}$ is both closed and open. Restricted to each component, τ_s coincides with the topology induced by τ_o. Thus, the family of Borel sets of τ_s coincides with the family of Borel sets of τ_o.

The fact is that $\mathcal{I}(\Sigma_{\mathbb{R}^d}) \subset \mathcal{P}_w(\Sigma_{\mathbb{R}^d})$ is straightforward: since

$$p^{-1}(\mathcal{O}_U) = \bigcup_{j=1}^{\infty} \{\omega \in [\mathbb{R}^d]^\infty \mid \mathbf{x}_j(\omega) \in U\}, \quad (10)$$

the inverse image of $\mathcal{O}_U$ is open in the weak topology of $[\mathbb{R}^d]^\infty$, and therefore $\mathcal{O}_U$ belongs to $\mathcal{P}_w(\Sigma_{\mathbb{R}^d})$. But $\mathcal{I}(\Sigma_{\mathbb{R}^d})$ is in fact smaller than $\mathcal{P}_w(\Sigma_{\mathbb{R}^d})$, and too small for certain purposes. Indeed, by our previous result, any τ_o-Borel set B is the inverse image under the closure map of a set in the σ algebra on $\Omega_{\mathbb{R}^d}$ generated by the sets Ω_F; thus, it has the property that, if $\gamma \in B$, $\bar{\gamma} \in B$.

But it is easy to construct sets in $\mathcal{P}_w(\Sigma_{\mathbb{R}^d})$ that do not have this property. For example, define the set $\mathcal{O}^V$ of all configurations $\gamma \in \Sigma_{\mathbb{R}^d}$ that are subsets of a given open set V. Evidently, there exist countably infinite subsets of V whose closures are no longer subsets of V, so $\mathcal{O}^V$ does not belong to $\mathcal{I}(\Sigma_{\mathbb{R}^d})$. However, $\mathcal{O}^V$ does belong to $\mathcal{P}_w(\Sigma_{\mathbb{R}^d})$, which follows from the fact that

$$p^{-1}(\mathcal{O}^V) = p^{-1}(\{\gamma \in \Sigma_{\mathbb{R}^d} | \gamma \subset V\}) \quad (11)$$
$$= \bigcap_{j=1}^{\infty} \{\omega \mid \mathbf{x}_j(\omega) \in V\}.$$

Thus, $\mathcal{I}(\Sigma_{\mathbb{R}^d}^{(\infty)}) \neq \mathcal{P}_w(\Sigma_{\mathbb{R}^d}^{(\infty)})$. The σ algebra $\mathcal{I}(\Sigma_{\mathbb{R}^d})$ is just not large enough for us to be able to count the number of points in a configuration that belong to a given open set in $\mathbb{R}^d$.

This example suggests consideration of the Vietoris topology on subsets of $\mathbb{R}^d$, restricted to $\Sigma_{\mathbb{R}^d}$ or to $\Sigma_{\mathbb{R}^d}^{(\infty)}$. Let us call this topology τ_v. A subbase for τ_v is given by sets of the form $\mathcal{O}^V \cap \mathcal{O}_U$, where U and V are open, so that $\mathcal{O}^V$ is itself open in τ_v. The Vietoris topology has many nice properties [40–42]. Considering then the σ algebra $\mathcal{B}_v(\Sigma_{\mathbb{R}^d})$ of Borel sets with respect to τ_v, we have $\mathcal{I}(\Sigma_{\mathbb{R}^d}) \subset \mathcal{B}_v(\Sigma_{\mathbb{R}^d})$, but $\mathcal{I}(\Sigma_{\mathbb{R}^d}) \neq \mathcal{B}_v(\Sigma_{\mathbb{R}^d})$. Furthermore, $\mathcal{B}_v(\Sigma_{\mathbb{R}^d}) \subset \mathcal{P}_w(\Sigma_{\mathbb{R}^d})$. To show this, consider again a countable base $\{U_j, j = 1, 2, 3, \ldots\}$ for the topology in $\mathbb{R}^d$. A countable subbase for τ_v is then $\{\mathcal{O}^{U_j} \cap \mathcal{O}_{U_k}, j, k = 1, 2, 3, \ldots\}$, and a countable base for τ_v consists of finite intersections of such sets. Since $p^{-1}(\mathcal{O}^{U_j})$ and $p^{-1}(\mathcal{O}_{U_k})$ are both Borel in $[\mathbb{R}^d]^\infty$, the inverse image of any open set in τ_v is Borel in $[\mathbb{R}^d]^\infty$, which suffices for the result.

We have not, however, determined whether $\mathcal{B}_v(\Sigma_{\mathbb{R}^d})$ is or is not strictly smaller than $\mathcal{P}_w(\Sigma_{\mathbb{R}^d})$.

5. SELF-SIMILAR RANDOM POINT PROCESSES IN $\mathbb{R}^d$ AND QUASI-INVARIANT MEASURES

Now we are prepared to construct measures on the σ algebra $\mathcal{B}([\mathbb{R}^d]^\infty)$ by means of random point processes, using sequences of conditional probability densities. When we do so, it turns out that the RN derivatives under transformations by diffeomorphisms take the form of an infinite product,

$$\frac{d\mu_\phi}{d\mu}(\omega) = \prod_{j=1}^{\infty} u_{j,\phi}(\omega). \quad (12)$$

Here, $\omega \in [\mathbb{R}^d]^\infty$, and the $u_{j,\phi}(\omega)$ are measurable functions that depend only on the first j entries of ω.

Quasi-invariance of μ then requires that (12) converge to a nonzero, noninfinite limit almost everywhere in μ for each ϕ. This means that the individual terms $u_{j,\phi}(\omega)$ must approach 1 sufficiently rapidly as $j \to \infty$. Under conditions that in fact hold for the measures discussed here, these convergence properties have also been proven sufficient to ensure the quasi-invariance of μ [37] and, as a direct consequence, the quasi-invariance of the projected measure $\hat{\mu}$ on $\Sigma_{\mathbb{R}^d}^{(\infty)}$.

Let $f(\mathbf{x}_j|\mathbf{x}_1, \ldots, \mathbf{x}_{j-1})$ be a nonsingular probability density on $\mathbb{R}^d$ for selection of the point $\mathbf{x}_j$, conditioned on the previously selected points $\mathbf{x}_1, \ldots, \mathbf{x}_{j-1}$ in some random sequence. Then $d\mu_j(\mathbf{x}_j) = f(\mathbf{x}_j|\mathbf{x}_1, \ldots, \mathbf{x}_{j-1})d\mathbf{x}_j$ defines a conditional (Borel) probability measure μ_j on $\mathbb{R}^d$ that depends measurably on the $j-1$ real parameters $\mathbf{x}_1, \ldots, \mathbf{x}_{j-1}$ (the positions of the first $j-1$ particle coordinates) and is absolutely continuous with respect to the Lebesgue measure $d\mathbf{x}_j$. We can interpret the joint probability measure for the first k points, specified by $d\mu^{(k)} = \prod_{j=1}^{k} d\mu_j$, as a measure on $[\mathbb{R}^d]^\infty$; and the sequence $(\mu^{(k)}), k = 1, 2, 3, \ldots$, is then a compatible family of probability measures.

By Kolmogorov's theorem, there is a unique measure μ on $[\mathbb{R}^d]^\infty$ determined by the sequence

$(\mu^{(k)})$. Under transformation by $\phi \in \mathit{Diff}^c(\mathbb{R}^d)$, the RN derivative for $\mu^{(k)}$ (when it exists) is given by the finite product

$$\frac{d\mu_\phi^{(k)}}{d\mu^{(k)}}(\omega) = \prod_{j=1}^{k} \frac{d\mu_{j,\phi}}{d\mu_j}(\omega), \tag{13}$$

where

$$\frac{d\mu_{j,\phi}}{d\mu_j}(\omega) = \frac{f(\phi(\mathbf{x}_j)|\phi(\mathbf{x}_1),\ldots,\phi(\mathbf{x}_{j-1}))}{f(\mathbf{x}_j|\mathbf{x}_1,\ldots,\mathbf{x}_{j-1})}\mathcal{J}_\phi(\mathbf{x}_j). \tag{14}$$

The quasi-invariance of $\mu^{(k)}$ is assured as long as the RN derivative in Eq. (13) is almost everywhere positive and finite. Now, as anticipated, in the infinite-dimensional case, quasi-invariance of the measure μ under diffeomorphisms turns out to depend on the behavior of the infinite product in Eq. (12), with $u_{j,\phi}(\omega) = [d\mu_{j,\phi}/d\mu_j](\omega)$.

Of course, not every measure so constructed will be quasi-invariant. The idea that leads to an interesting class of quasi-invariant measures is to scale the probability distribution of the jth particle's position according to the outcomes for the previously chosen particle positions. This establishes a self-similar random process, where in the vicinity of accumulation points the ratio of probability density functions in Eq. (14) approaches the inverse of the Jacobian as $j \to \infty$. The resulting physical systems behave like an interacting gas of particles with one or more loci of condensation. However, our approach differs from the usual one in that our probability measures are constructed directly, rather than by means of an interaction Hamiltonian.

In general, if the positions of the particle coordinates $\mathbf{x}_j(\omega)$, or the successive difference coordinates $\mathbf{y}_{j+1}(\omega) = \mathbf{x}_{j+1}(\omega) - \mathbf{x}_j(\omega)$, distribute independently but nonidentically—so that points can accumulate with nonzero probability—the resulting measure will not be quasi-invariant. However, Ismagilov did demonstrate quasi-invariance under diffeomorphisms of the measures resulting from a particular class of processes of this type, in one space dimension [20].

Sakuraba [37] showed that the quasi-invariant measures constructed by Goldin and Moschella from self-similar random processes and the quasi-invariant measures of Ismagilov are mutually singular.

5.1. Example for $d = 1$

Let us illustrate with the examples based on Gaussian probability densities. Working first with $d = 1$, choose an initial point x_0 from a nowhere vanishing probability density f_0 on $\mathbb{R}$. For $j = 1, 2, 3, \ldots$, let $x_j = x_{j-1} + y_j$, where y_j are a sequence of deviation values. Choose the value y_1 from a unit normal distribution g_1 with mean 0. Given the values $y_1, \ldots, y_j$, choose y_{j+1} from a normal distribution with mean 0 and standard deviation $\sigma_j = \kappa|y_j|$, where $\kappa > 0$ is a fixed correlation parameter independent of j. Small values of κ correspond to more tightly bound systems. Thus, we have the conditional probability densities for y_j,

$$g_{j+1}^{\kappa}(y_{j+1}|y_j) = \frac{(2\pi)^{-1/2}}{\kappa|y_j|}\exp\left[-\frac{1}{2\kappa^2}\left(\frac{y_{j+1}}{y_j}\right)^2\right]. \tag{15}$$

For sufficiently small values of κ, (y_j) converges to 0 (with probability one), while $\sum_{j=1}^{\infty}|y_j| < \infty$.

Let $\mathit{Diff}_0^c(\mathbb{R})$ denote the stability subgroup of $\mathit{Diff}^c(\mathbb{R})$ consisting of the compactly supported diffeomorphisms of $\mathbb{R}$ that leave the origin fixed. The measure on the space of sequences (y_j) resulting from the densities in Eq. (15) is then quasi-invariant under the action of elements of $\mathit{Diff}_0^c(\mathbb{R})$. We thus obtain the random sequence $\omega = (x_k)$, with $x_k = x_0 + \sum_{j=1}^{k} y_j$, and the corresponding random configuration $\gamma = \{x_k\}$.

Defining the terms $u_{j,\phi}$ in Eq. (12) accordingly, we obtain $u_j \to 1$ sufficiently rapidly to ensure convergence of the infinite product. More precisely, there exists a critical value κ_0 such that if $0 < \kappa < \kappa_0$, sequences (x_j) converge to an accumulation point with probability one, while if $\kappa_0 < \kappa$, sequences diverge geometrically with probability one. In both cases, the associated measures on $\Sigma_{\mathbb{R}}^{\infty}$ are quasi-invariant under compactly supported diffeomorphism of $\mathbb{R}$ [32–34, 37]. The proofs make use of the strong law of large numbers.

The above is not tied essentially to the use of normal distributions; all that is really necessary is the scaling property. Thus, for a whole class of models, there exists a critical value κ_0 of the scaling parameter κ. For $0 < \kappa < \kappa_0$, the generalized configuration $\{x_j\}$ has an accumulation point with probability one; we call this the condensed phase. For $\kappa_0 < \kappa$, $\{x_j\}$ has zero average density; we call this the rarefied phase. For each value of κ (except for the critical value itself), we have a bona fide unitary representation of $\mathrm{Diff}^c(\mathbb{R})$, describing the associated quantum system.

5.2. Generalization to $d > 1$

It was suggested earlier that a procedure similar to that suggested by Eq. (15) would work in d space dimensions, $d > 1$, to yield measures on the space

of generalized configurations quasi-invariant under $Diff^c(\mathbb{R}^d)$, with the conditional probability density for $\mathbf{y}_{j+1}$ dependent on the preceding d outcomes $(\mathbf{y}_{j-d+1}, \ldots, \mathbf{y}_j)$ through the covariance matrix of a multivariate normal distribution [33, 34]. The generalization obtained by Sakuraba [37] achieves this, but also involves some new aspects.

Consider a random process where, at each stage, d vectors in $\mathbb{R}^d$ are to be selected. Thus, at each stage, we choose a $d \times d$ random matrix V, and it is appropriate to think of $\omega \in [\mathbb{R}^d]^\infty$ as the sequence of square matrices $([\mathbf{x}_1, \ldots, \mathbf{x}_d], [\mathbf{x}_{d+1}, \ldots, \mathbf{x}_{2d}], \ldots)$.

For the square matrix $Y = [y_{ij}]$, define the norm $||Y|| = \left[\sum_{i,j=1}^d y_{ij}^2\right]^{1/2}$. Note that $||Y||$ is a vector norm, not the operator norm of the matrix. For $Y \in GL(d, \mathbb{R})$, define the condition number $k(Y) = ||Y|| \cdot ||Y^{-1}||$. We may write $Y = P|Y|$, where P is an orthogonal matrix and where $|Y| = \sqrt{Y^t Y}$ is positive. Let $\tau_1, \ldots, \tau_d$ be eigenvalues of the matrix $|Y|$. Then $||Y|| = |||Y|||$, and

$$||Y|| = \left[\sum_{i=1}^d \tau_i^2\right]^{1/2}, \quad ||Y^{-1}|| = \left[\sum_{j=1}^d \tau_j^{-2}\right]^{1/2}, \tag{16}$$

$$k(Y) = \left[\sum_{i,j=1}^d (\tau_i/\tau_j)^2\right]^{1/2}.$$

Evidently $k(Y)$ characterizes the amount of deformation under linear transformation by Y. If Y is not invertible, then $k(Y)$ is undefined (or infinite). Such matrices belong to measure zero sets in the constructions that follow.

We next construct a measure on $[GL(d, \mathbb{R})]^\infty$ and thus on $[\mathbb{R}^d]^\infty$ quasi-invariant under $Diff_0^c(\mathbb{R}^d)$. Define the probability density function

$$f(Y) = C \exp\left\{-\frac{1}{2\kappa^2}[||Y||k(Y)]^2\right\} \tag{17}$$

on the set of $d \times d$ matrices, where C is a normalization constant chosen so that $\int f(Y)dY = 1$; here $dY = d\mathbf{y}_1 \cdots d\mathbf{y}_d$. Let $\mu^{(k)}$ be the probability measure defined by

$$d\mu^{(k)} = f(Y_1)\frac{f(Y_1^{-1}Y_2)}{|\det Y_1|^d} \cdots \frac{f(Y_{k-1}^{-1}Y_k)}{|\det Y_{k-1}|^d} dY_1 \cdots dY_k, \tag{18}$$

where $d\mu^{(k)} = d\mu^{(k)}(Y_1, \ldots, Y_k)$. Then $\mu^{(k)}$ is concentrated on $[GL(d, \mathbb{R})]^k$; i.e., the set of sequences with one or more noninvertible matrices is of measure zero.

Then we again have a critical value κ_0. For $\kappa < \kappa_0$, the sequence (Y_j) of matrices—and thus the sequence of component vectors $(\mathbf{y}_i)$—converges to 0 with probability one, while for $\kappa_0 < \kappa$, it diverges with probability one. Furthermore, the projective limit measure μ on $[GL(d, \mathbb{R})]^\infty$ has the desired property of quasi-invariance under $Diff_0^c(\mathbb{R}^d)$. The presence of the condition number $k(Y)$ in Eq. (17) is essential for the estimates required in demonstrating convergence of the infinite product in the resulting expression for the RN derivative. The proof here again uses the strong law of large numbers.

Equation (17) can be generalized, replacing $k(Y)$ by $k(Y)^\alpha$ $(\alpha \geq 1)$ and replacing the Gaussian density by a more general probability density function.

Finally, we may begin with a matrix of positions $X_0 = [\mathbf{x}_1, \ldots, \mathbf{x}_d]$ chosen from a nowhere vanishing probability density. Let $\bar{\mathbf{x}}_0$ be the center of position of the d vectors comprising X_0. Now, we may treat each new matrix Y_j as a set of deviations from the center of position of the preceding set of vectors X_{j-1}, so that, with obvious notation, $X_j = \bar{\mathbf{x}}_{j-1} + Y_j$. In this manner, we obtain a measure on $[\mathbb{R}^d]^\infty$ quasi-invariant under $Diff^c(\mathbb{R}^d)$ that projects to a quasi-invariant measure on the space $\Sigma_{\mathbb{R}^d}^\infty$ of generalized configurations.

More details about the preceding results may be found in the thesis of Sakuraba [37] and in forthcoming publications.

6. CONCLUSION

We believe the work summarized here strengthens the case for basing a theory of statistical physics in the manifold M on the configuration space Σ_M of countable subsets of M endowed with the Vietoris topology. Measures obtained from random point processes in M project to measures on Σ_M, and when we consider self-similar random processes, we obtain measures quasi-invariant under the group of compactly supported diffeomorphisms of M. The problem of relating these measures to Hamiltonians on a classical phase space remains open.

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The Invariance of Order Parameter and Temperature Redefinition in Helix-Coil Transition Theory of Circular Closed DNA*

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Abstract—The order–disorder (helix-coil) transition in circular closed DNA (ccDNA) is described on the basis of the open chain DNA (ocDNA) model, proposed earlier, which considers the transition as loop formation. The Hamiltonian of the ccDNA model is constructed on the basis of the open chain model taking into account topological restrictions. These restrictions are taken into account through hydrogen bond reduced energy dependence on the fraction of broken hydrogen bonds in the macromolecule. The invariance of the order parameter (helicity degree) has been shown for ocDNA and ccDNA. This invariance results in the interdependence between temperatures of ocDNA and ccDNA with the same value of helicity degree. The dependence can be obtained with the help of the derivative of reduced energy of hydrogen bonding dependence on instantaneous denaturation degree. Thus, it has been shown that the melting curve of ccDNA can be obtained from the consequent curve of ocDNA through the redefinition of temperature scale. The calculated and experimentally measured melting curves have been compared under inversion conditions and qualitative agreement between them is found. © *2005 Pleiades Publishing, Inc.*

1. INTRODUCTION

In 1963, R. Dulbecco and M. Vogt and R. Weil and J. Vinograd discovered that double-stranded DNA of the polyoma virus exists in a closed circular form. At present, it is generally acknowledged that this form is typical of bacterial DNA and of cytoplasmic DNA in animals. Furthermore, giant DNA molecules in higher organisms form loop structures held together by protein fasteners in which each loop is largely analogous to circular closed DNA (ccDNA). The distinctive feature of circular closed molecules is that its topological state cannot be altered by any conformational rearrangement short of breaking DNA strands. This topological constraint is the basis for the characteristic properties of ccDNA, which have fascinated biologists, physicists, and mathematicians for the past 35 years. The first melting experiment of ccDNA was carried out by Vinograd *et al.* [1]. They revealed essential differences in the processes of denaturation of ccDNA and open chain DNA (ocDNA). First of all, it is striking that there is considerable widening of melting interval. Apart from that, the melting temperature of ccDNA exceeds the one obtained for ocDNA by about 30°. Taking into account the above mentioned, we can conclude that the process of melting of ccDNA needs theoretical substantiation. The first attempt was made by the group of Frank-Kamenetskii [2]. The model which was considered by M.D. Frank-Kamenetskii's group was in good agreement with the above-mentioned experimental data [1, 3]. But then experimental data contradicting the model of this group were obtained. Experiments on ccDNA denaturation in so-called "inverted conditions" were carried out [4, 5]. Under these conditions, the melting interval of ocDNA was found to be very narrow, while the melting interval of ccDNA remained unchanged, that is, very wide. These experimental results were at variance with the theoretical model of Frank-Kamenetskii's group. The following theoretical investigations of ccDNA were based on the mean field theory [6–11]. In our last works [12, 13], we solved the melting of heterogeneous ccDNA in the presence of a competing solvent on the basis of the microscopic model. The goal of the present work is the analysis of experimental data [4, 5] on the basis of the obtained model [12, 13].

2. THE MODEL FOR THE OPEN CHAIN AND CLOSED CIRCULAR DNA

We construct the model based on the following Hamiltonian [14]:

$$-\beta H = J\sum_{i=1}^{N}\delta_1^{(i)}, \qquad (1)$$

where $\beta = T^{-1}$ is inverse temperature, N is the number of repeated units, and $J = U/(kT)$ is the reduced

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energy of internal hydrogen bonding. The notation $\delta_1^{(i)} = \delta\left(\sum_{k=1}^{i}\vec{\gamma}_k, 0\right)$ is introduced, with $\delta(x,0)$ being the Kronecker symbol. $\vec{\gamma}_l = \vec{\gamma}^1, \vec{\gamma}^2, \ldots, \vec{\gamma}^Q$ is the vector which describes the conformation of the lth repeated unit, and Q is the number of possible values of this vector. We can explain Eq. (1) as

$$-\beta H = J_0 \times \sum_{i=1}^{N} \begin{cases} 1, & \text{if there is a loop between 0 and } i\text{th repeated units,} \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Due to flexibility of the chain, the statistical weight of conformation with hydrogen bonds connecting any of the base pairs between the first and lth one strictly depends on the bonded base pair position along the chain. Thus, the cooperativity (the interdependence of conformations) of the system is implicitly included through real geometrical restrictions on loop formation. So we construct the Hamiltonian for ccDNA as the Hamiltonian of ocDNA (1), but the reduced energy of the hydrogen bonds is a function of the instantaneous fraction of broken hydrogen bonds in the molecule: $J = J(P)$, where

$$P = 1 - \frac{1}{N}\sum_{i=1}^{N}\delta_1^{(i)}.$$

This means that the denaturation rate at each repeated unit of the macromolecule will depend on the conformation of the whole chain. Let us expand J over P (following [12, 13]):

$$J = J_0 + \sum_{k=0}^{M} a_k P^k. \quad (3)$$

The conformational partition function has the form

$$Z = \sum_{\{\gamma_k\}} \exp\left[J_0 \sum_{i=1}^{N}\delta_1^{(i)} + N\sum_{k=0}^{M} b_k P^k\right], \quad (4)$$

where $b_k = a_k - a_{k-1}$. The first term of this relation is the ocDNA term. The second term is related to ccDNA and depends on the expansion of $J(P)$ over P. Here, M reflects the precision of expansion, b_k are coefficients independent of repeated unit number, and the remaining notation is as above. We can linearize the last term of Eq. (4) using the Dirac delta function as follows:

$$\exp\left[Nb_l P^l\right] = \int_{-\infty}^{+\infty} dy_l \delta(y_l - P)\exp\left(Nb_l y_l^l\right). \quad (5)$$

Substituting Eq. (5) into Eq. (4) and using the integral representation of the Dirac delta function and saddle-point method, one can rewrite the partition function for ccDNA in the case of large N as [13]

$$Z \propto \exp\left[N\left(\sum_{k=0}^{M}(1-k)\,b_k\alpha^k + \ln\lambda_1\right)\right], \quad (6)$$

where $\alpha = \langle P\rangle$ is the thermal average denaturation degree of the basic (open chain) model. Here, λ_1 is the reduced free energy for the ocDNA model, which for ccDNA becomes the function of energy expansion over the averaged denaturation degree α of the open chain model as

$$\lambda_1 = \lambda_1\left(\sum_{k=0}^{M} k b_k \alpha^{k-1} + J_0\right).$$

We think it is necessary to repeat that, for ocDNA, λ_1 is a function dependent only on hydrogen bond energy J_0, as $\lambda_1 = \lambda_1(J_0)$. So we can write the partition function and describe the problem of ccDNA melting using ocDNA parameters.

Using expression (6), it is possible to calculate $\alpha_{\rm cc}$, which is the denaturation degree for ccDNA, and to show that it is equal to $\alpha_{\rm oc}$, which is the denaturation degree for ocDNA. Because of the fact that the denaturation degree has the same arguments as λ_1, we can write

$$\alpha_{\rm cc}\left(\sum_{k=0}^{M} k b_k \alpha^{k-1} + J_0\right)_{T=T_{\rm cc}} = \langle P\rangle = -\frac{\partial \ln Z}{NJ_0\partial b_1} = \alpha_{\rm oc}(J_0)_{T=T_{\rm oc}}. \quad (7)$$

Because of monotonic dependence of the denaturation degree on the temperature parameter, the equality of denaturation degrees leads to the equality of arguments, so

$$\left(J_0 + \sum_{k=0}^{M} k b_k \alpha^{k-1}\right)_{T=T_{\rm cc}} = (J_0)_{T=T_{\rm oc}}. \quad (8)$$

It is necessary to note that the left- and right-hand sides of Eq. (8) are written at different $T_{\rm oc}$ and $T_{\rm cc}$ temperatures.

3. MELTING CURVES FOR ccDNA AND ocDNA

From Eq. (8), we obtain the relationship between the temperatures corresponding to equal values of denaturation degrees for ocDNA and ccDNA:

$$T_{\rm cc} = T_{\rm oc}\left[1 + \sum_{k=0}^{M} k\frac{b_k}{J_0}\alpha^{k-1}\right]. \quad (9)$$

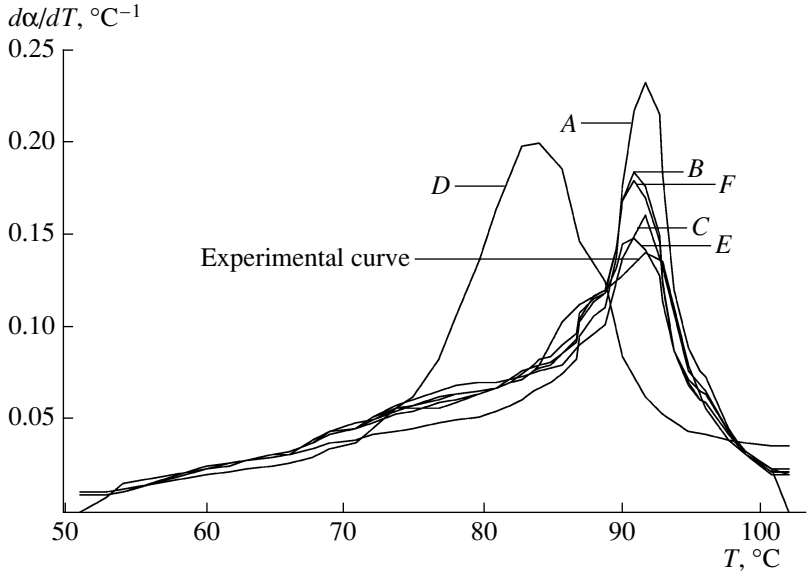

Fig. 1. The derivative melting curve (DMC) of $\varphi X174$ phages under inverted conditions (Experimental curve) [4, 5] and its description according to (9).

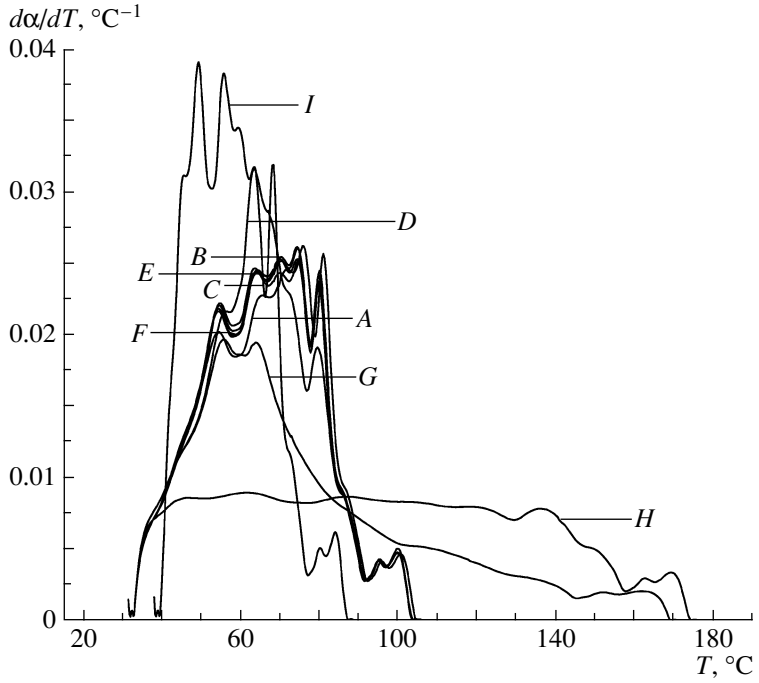

Fig. 2. The experimental DMC of native ocDNA of Calf Thymus in 10^{-3} NaCl ("I") and DMC of hypothetical DMC of ccDNA under the same conditions.

Equation (9) allows one to calculate the denaturation curve for ccDNA from the melting curves of corresponding ocDNA, and, what is more, we have found in [12, 13] that expression (9) is right in the case of heterogeneous ccDNA in the presence of a competing solvent.

4. THE EXPERIMENTAL DATA ANALYSIS OF HELIX-COIL TRANSITION OF ccDNA ON THE BASIS OF STATISTICAL MODEL

In Fig. 1, we present the comparison of the experimental data with the calculated data of obtained polynomials with different terms (the cases $A-F$). Then,

on the basis of the experimental derivative melting curve (DMC) of native ocDNA, we have obtained the DMC of ccDNA using the $A-F$ transformations which were used in Fig. 1. They are presented in Fig. 2. As we can see from Fig. 2, the melting of all ccDNA begins earlier and the transition intervals are very large. In addition, the DMC of ccDNA appeared to have fine structure when using the $A-F$ transformation. In contrast, in the case when we use the temperature transformation formula with expansion up to lower terms, the DMC of ccDNA do not have fine structure. For quantitative comparison of the model and the experiment, it is necessary to have experimental data of melting of a single topoisomerical fraction.

There is also the possibility of describing the melting experiment of several topoisomer mixtures. We think that it can be done by presenting the experimental curve in Fig. 1 as a sum of curves which correspond to the different transformation temperature formulas. We plan to do this in our next paper.

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All Possible Cayley–Klein Contractions of Quantum Orthogonal Groups*

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Abstract—Spaces of constant curvature and their motion groups are described most naturally in the Cartesian basis. All these motion groups, also known as CK groups, are obtained from an orthogonal group by contractions and analytical continuations. On the other hand, quantum deformation of orthogonal group $SO(N)$ is most easily performed in the so-called symplectic basis. We reformulate its standard quantum deformation to the Cartesian basis and obtain all possible contractions of quantum orthogonal group $SO_q(N)$ for both untouched and transformed deformation parameters. It turned out that, similar to the undeformed case, all CK contractions of $SO_q(N)$ are realized. An algorithm for obtaining nonequivalent (as Hopf algebra) contracted quantum groups is suggested. Contractions of $SO_q(N)$, $N = 3, 4, 5$, are regarded as examples. © *2005 Pleiades Publishing, Inc.*

1. INTRODUCTION

Systematic definitions of quantum deformations of classical simple Lie groups and algebras, as well as a description of their properties, were given in [1]. Simple Lie groups and algebras are transformed by the contraction operation first introduced by Wigner and Inönü [2] to nonsemisimple ones. Quantum analogs of the nonsemisimple low-dimension Lie algebras were obtained by contractions of quantum algebras $so_q(3)$, $su_q(2)$ [3–7] and contractions of low-dimension quantum groups were discussed in [8–10]. Two types of contractions were discovered: with untouched deformation parameter (in [3, 6] for quantum algebras and in [9, 10] for quantum groups) and with transformed deformation parameter [4, 5, 7, 8]. For the latter case, the quantum deformations of the algebras of the maximal symmetric motion groups of the N-dimensional flat spaces were constructed in [11]. The γ-Poincare quantum group was obtained by contractions of the orthogonal quantum group $SO_q(N)$ [12]. The quantum Euclid group $E_\kappa(2)$ was described both by contraction of $SU_q(2)$ [13] and by direct quantization of the Lie–Poisson structure [14]. A separate line of investigation is presented by the R-matrix approach to the quantum analogs of Euclid, Heisenberg, and inhomogeneous groups [15–18].

It is well known [19] that the motion groups of all 3^{N-1} $(N-1)$-dimensional constant curvature spaces may be obtained by contractions and analytic continuations of the classical orthogonal group $SO(N)$. Cayley–Klein groups is the short name for this set of groups. The fundamental orthogonal $A^tA = I$ matrix $A \in SO(N)$ is replaced by the matrix $A(j)$ whose elements $(A(j))_{kp} = (k,p)a_{kp}, (k,p) = \prod_{l=\min\{k,p\}}^{\max\{k,p\}-1} j_l, k, p = 1, \ldots, N$ are subject to additional j-orthogonality relations $A(j)^tA(j) = 1$, where the parameters j_k take three values each $j_k = 1, \iota_k, i$. The commutative $\iota_k\iota_p = \iota_p\iota_k \neq 0, k \neq p$ nilpotent $\iota_k^2 = 0$ units ι_k correspond to contractions and the imaginary unit $i^2 = -1$ to analytic continuations.

In the case of the quantum orthogonal group $SO_q(N)$, additionally the deformation parameter $q = \exp z$ is transformed as follows [20]: $z = Jv$, $J = (1, N)$, where v is the new deformation parameter. At the same time, the quantum group contractions with an untransformed deformation parameter are known [9, 10]. For unification of both such cases in one approach, the concept of different couplings of Cayley–Klein and Hopf structures was suggested [21, 22]. It is well known that quantum groups are Hopf algebras and the Cayley–Klein structure is defined by the distribution of the contraction parameters j among the elements of the generating matrix. For the quantum orthogonal group in the so-called "symplectic" basis (where the invariant quadratic form for $q = 1$ is defined by the matrix C_0 with all null elements except units on the secondary diagonal), this concept was realized in [23–25] by the substitution, in the standard machinery of quantum group, of the generating matrix $T_\sigma(j) = D_\sigma A(j) D_\sigma^{-1}$, $D_\sigma = DV_\sigma$, where the matrix D is the solution to the

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equation $D^tC_0D = I$ and describes transformation from the Cartesian basis to the symplectic one. The matrix V_σ, $(V_\sigma)_{ik} = \delta_{\sigma_i,k}$, where $\sigma \in S(N)$ is a N-order permutation, defines the distribution of the contraction parameters in $T_\sigma(j)$. In this case, the transformation of the deformation parameter depends on permutation σ. All permutations which lead to an untouched ($J = 1$) deformation parameter and some permutations which correspond to transformed ones are enumerated in [23–25]. The contracted quantum groups $SO_v(N; j; \sigma)$ in these papers were regarded as a Hopf algebra over Pimenov algebra $D(\iota)$ generated by nilpotent commutative generators. It turned out that not all Cayley–Klein contractions are admissible for quantum groups in this assumption, which therefore is too restrictive.

The main statement of the algebraic structure contraction method is to take into account in all relations only principal parts with respect to contraction parameter tending to zero and to neglect all others. Therefore, in this paper, in all relations of quantum group theory, only principal (complex) terms are taken into account and all other terms with nilpotent multipliers are neglected. In addition, contractions of orthogonal quantum groups $SO_v(N; j; \sigma)$ are regarded in the more usual Cartesian basis. For an untouched deformation parameter, the results are the same as in [23–25], and for all other permutations, the deformation parameter is multiplied by $J = \bigcup_{k=1}^{n}(\sigma_k, \sigma_{k'})$, where n is integral part of $N/2$. The unification of multipliers $(\sigma_k, \sigma_p) \cup (\sigma_m, \sigma_r)$ is understood as the first-power product of all parameters j_k which appear in at least one multiplier (σ_k, σ_p) or (σ_m, σ_r). For example, $(j_1j_2) \cup (j_2j_3) = j_1j_2j_3$. It turned out that the full scheme of Cayley–Klein contractions is realized for the quantum group $SO_q(N)$. Not all identically contracted quantum groups corresponding to different permutations σ are nonisomorphic. Quantum group isomorphism is connected with the notion of equivalent distributions of nilpotent parameters in the generating matrix. Nonisomorphic contracted quantum groups correspond in the first place to generating matrices with nonequivalent distributions of nilpotent parameters and secondly to equivalent generating matrices but with different transformations of the deformation parameter ($J_1 \neq J_2$). As an example, quantum groups $SO_v(3; j; \sigma)$ are considered in detail and nonisomorphic contractions are given for quantum groups $SO_v(N; j; \sigma)$, $N = 4, 5$.

2. DEFINITION OF QUANTUM GROUP $SO_v(N; j; \sigma)$

Let us start with an algebra $\mathbf{D}\langle(U(j;\sigma))_{ik}\rangle$ of noncommutative polynomials of N^2 variables, which are elements of the generating matrix $(U(j;\sigma))_{ik} = (\sigma_i, \sigma_k)u_{\sigma_i\sigma_k}$. Let us introduce the transformation of the deformation parameter $q = e^z$ as follows: $z = Jv$, where v is a new deformation parameter and J is some product of parameters j for the present unknown. Let $\tilde{R}_v(j)$, $\tilde{C}_v(j)$ be matrices which are obtained from $\tilde{R}_q, \tilde{C}$, respectively, by the replacement of deformation parameter z with Jv. The commutation relations of the generators $U(j;\sigma)$ are defined by

$$\tilde{R}_v(j)U_1(j;\sigma)U_2(j;\sigma) = U_2(j;\sigma)U_1(j;\sigma)\tilde{R}_v(j), \tag{1}$$

where

$$U_1(j;\sigma) = U(j;\sigma)\otimes I, \quad U_2(j;\sigma) = I \otimes U(j;\sigma),$$
$$U(j;\sigma) = V_\sigma U(j) V_\sigma^{-1}, \quad (V_\sigma)_{ik} = \delta_{\sigma_i k},$$
$$\tilde{R}_v(j) = (D\otimes D)^{-1} R_v(j)(D\otimes D),$$
$$R_v(j) = R_q(z \to Jv),$$
$$D^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} I & 0 & \tilde{C}_0 \\ 0 & \sqrt{2} & 0 \\ i\tilde{C}_0 & 0 & -iI \end{pmatrix}, \quad N = 2n+1,$$

$\tilde{C}_0$ is the $n \times n$ matrix with all null elements except units on the secondary diagonal, and the explicit form of the matrix $\tilde{R}_q$ in the Cartesian basis is given in the Appendix. The following additional relations of (v, j) orthogonality hold:

$$U(j;\sigma)\tilde{C}_v(j)U^t(j;\sigma) = \tilde{C}_v(j), \tag{2}$$
$$U^t(j;\sigma)\tilde{C}_v^{-1}(j)U(j;\sigma) = \tilde{C}_v^{-1}(j),$$

where $C = C_0q^\rho$, and $\rho = \mathrm{diag}(\rho_1, \ldots, \rho_N)$, $(C_0)_{ik} = \delta_{i'k}$, $i, k = 1, \ldots, N$, $i' = N + 1 - i$, that is, $(C)_{ik} = q^{\rho_{i'}}\delta_{i'k}$ and $(C^{-1})_{ik} = q^{-\rho_i}\delta_{i'k}$, $\tilde{C}_v(j) = D^{-1}C_v(j)(D^t)^{-1}$,

$$(\rho_1, \ldots, \rho_N) = \begin{cases} \left(n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, -n + \frac{1}{2}\right), & N = 2n+1, \\ (n-1, n-2, \ldots, 1, 0, 0, -1, \ldots, -n+1), & N = 2n. \end{cases}$$

The quantum orthogonal Cayley–Klein group $SO_v(N;j;\sigma)$ is defined as the quotient algebra of $\mathbf{D}\langle(U(j;\sigma))_{ik}\rangle$ by relations (1), (2).

Formally, $SO_v(N;j;\sigma)$ is a Hopf algebra with the following coproduct Δ, counit ϵ, and antipode S:

$$\epsilon(U(j;\sigma)) = I, \tag{3}$$
$$\Delta U(j;\sigma) = U(j;\sigma)\dot{\otimes}U(j;\sigma),$$
$$S(U(j;\sigma)) = \tilde{C}_v(j)U^t(j;\sigma)\tilde{C}_v^{-1}(j),$$

where $(A\dot{\otimes}B)_{ik} = \sum_p A_{ip}\otimes B_{pk}$.

Remark. All relations for the quantum group $SO_v(N;j;\sigma)$ may be obtained from the corresponding relations for $SO_q(N)$ in the Cartesian basis [20] by the replacement $z \to Jv$ and $u_{ik} \to (\sigma_i,\sigma_k)u_{\sigma_i\sigma_k}$.

3. THE BASIC THEOREM

According to the algebraic structure contraction method, in all relations of the previous section for nilpotent values of j, only principal (complex) terms are taken into account and all other terms with nilpotent multipliers are neglected. The relation is called *admissible* if it is possible to select principal terms. Otherwise, the relation is called *inadmissible*. For example, the equation $a+\iota_1 b+\iota_2 c = a_1+\iota_1 d$ is an admissible equation and is equivalent to $a=a_1$, whereas the equation $\iota_1 b+\iota_2 c = \iota_1\iota_2 d$ is inadmissible.

The formal definition of the quantum group $SO_v(N;j;\sigma)$ should be a real definition of the contracted quantum group if the proposed construction is a consistent Hopf algebra structure for the principal terms of all relations under nilpotent values of some or all parameters j, in other words, if all relations of the previous section are admissible. The following theorem holds.

Theorem. *If commutation relations (1) of $SO_v(N;j;\sigma)$ are defined for nilpotent values of some or all parameters j and $J=\bigcup_{k=1}^n(\sigma_k,\sigma_{k'})$, then the contracted quantum group is Hopf algebra.*

Proof. Let us prove the consistency of our construction for the most singular case when all parameters j are nilpotent. Counit $\varepsilon(u_{\sigma_i\sigma_k})=0,\ i\neq k,\ \varepsilon(u_{\sigma_k\sigma_k})=1,\ k=1,\ldots,n$ do not restrict the values of j. Multiplier $C_{ikr} = (\sigma_i,\sigma_r)(\sigma_r,\sigma_k)(\sigma_i,\sigma_k)^{-1}$ in coproduct $\Delta(u_{\sigma_i\sigma_k}) = \sum_{r=1}^N C_{ikr}u_{\sigma_i\sigma_r}\otimes u_{\sigma_r\sigma_k}$ is equal to 1 if $\sigma_i<\sigma_r<\sigma_k$, is equal to $(\sigma_k,\sigma_r)^2$ if $\sigma_i<\sigma_k<\sigma_r$, and is equal to $(\sigma_r,\sigma_k)^2$ if $\sigma_r<\sigma_k<\sigma_i$; therefore, all expressions for the coproduct are admissible for nilpotent values of all j. Because of symmetry $(\sigma_i,\sigma_k)=(\sigma_k,\sigma_i)$, it is sufficient to examine the case $\sigma_k<\sigma_i$.

From the analysis of explicit expressions of antipode $S(U(j;\sigma))$ and (v,j)-orthogonality relations [26, 27], it follows that all expressions are admissible for any permutations and for nilpotent values of any parameters; therefore, they do not restrict contractions of the quantum group.

Remark 1. To examine the existence of the commutation relations (1) of $SO_v(N;j;\sigma)$ under contraction for arbitrary σ, it is necessary to have their explicit form at least for identical permutation σ_0. The system (1) is overdetermined and cumbersome, so we obtain its solution only for $N=3$.

Remark 2. For identical permutation σ_0 all Cayley–Klein contractions of quantum group $SO_v(N;j;\sigma_0)$ are allowed. Indeed, commutation relations for $j_1=\iota_1$ are given in [12] and they do not restrict all other contractions $j_r=\iota_r, r=2,3,\ldots,N-1$.

4. NONISOMORPHIC CONTRACTED QUANTUM GROUPS

If all parameters $j_k=1$, then the map $u_{ik}\to(\sigma_i,\sigma_k)u_{\sigma_i\sigma_k}$ is invertible and all quantum groups $SO_v(N;j;\sigma)$ for any $\sigma\in S_N$ are isomorphic as Hopf algebras. Nonisomorphic quantum groups may appear under contractions when all or some parameters j take nilpotent values. It is clear that nonisomorphic quantum groups appear under contractions with different numbers of parameters. Contractions on the same parameters, but with different transformations of deformation parameter (with different J) naturally give in result nonisomorphic quantum groups. Isomorphic quantum groups may appear under contractions of $SO_v(N;j;\sigma)$ with different σ by equal numbers of parameters, when multiplier J includes equal numbers of parameters (but not necessarily the same) or when $J=1$. In our approach, contractions of quantum groups (even on equal numbers of parameters) are distinguished by the distributions of nilpotent parameters j in generating matrix $U(j;\sigma)$. Really, all relations of quantum group theory (commutators, (v,j) orthogonality, antipode, coproduct, and counit) depend on permutation σ by means of a generating matrix, while matrices $R_v(j), C_v(j)$ depend on σ via transformations of a deformation parameter, that is, via J. Isomorphism of contracted quantum orthogonal groups is described by the following theorem.

Theorem. *Quantum groups $SO_v(N;j;\sigma_1)$ and $SO_w(N;j;\sigma_2)$ are isomorphic if the following relations for their generators hold:*

$$U(j;\sigma_1) = V_\sigma U(j;\sigma_2)V_\sigma^{-1}, \tag{4}$$

where matrix $V_\sigma, \sigma \in S_N$ satisfies

$$(V_\sigma \otimes V_\sigma)\tilde{R}_w(j)(V_\sigma \otimes V_\sigma)^{-1} = \tilde{R}_v(j), \qquad (5)$$
$$V_\sigma \tilde{C}_w(j) V_\sigma^t = \tilde{C}_v(j)$$

for $w = \pm v$ and $J_1 = J_2$ with possible replacement of j_k on j_{N-k}, $k = 1, \ldots, N-1$.

Proof. Commutation relations (1) of $SO_v(N; j; \sigma_1)$ after transformation (4) take the form

$$\tilde{R}_v(j)(V_\sigma \otimes V_\sigma)U_1(j;\sigma_2)U_2(j;\sigma_2)(V_\sigma \otimes V_\sigma)^{-1}$$
$$= (V_\sigma \otimes V_\sigma)U_2(j;\sigma_2)U_1(j;\sigma_2)(V_\sigma \otimes V_\sigma)^{-1}\tilde{R}_v(j)$$

or after left multiplying on $(V_\sigma \otimes V_\sigma)^{-1}$ and right multiplying on $V_\sigma \otimes V_\sigma$, in the form

$$(V_\sigma \otimes V_\sigma)^{-1}\tilde{R}_v(j)(V_\sigma \otimes V_\sigma)U_1(j;\sigma_2)U_2(j;\sigma_2)$$
$$= U_2(j;\sigma_2)U_1(j;\sigma_2)(V_\sigma \otimes V_\sigma)^{-1}\tilde{R}_v(j)(V_\sigma \otimes V_\sigma),$$

which give the first equation in (5). Antipode (3) after transformation (4) takes the form

$$V_\sigma S(U(j;\sigma_2))V_\sigma^{-1}$$
$$= \tilde{C}_v(j)\left(V_\sigma^{-1}\right)^t U^t(j;\sigma_2)V_\sigma^t\tilde{C}_v^{-1}(j)$$

or

$$S(U(j;\sigma_2))$$
$$= V_\sigma^{-1}\tilde{C}_v(j)\left(V_\sigma^{-1}\right)^t U^t(j;\sigma_2)V_\sigma^t\tilde{C}_v^{-1}(j)V_\sigma.$$

The last equation is just the antipode of $SO_v(N; j; \sigma_2)$ if one takes into account the second equation in (5). Finally, (v, j)-orthogonality relations (2) after (4) take the form

$$V_\sigma U(j;\sigma_2)V_\sigma^{-1}\tilde{C}_v(j)\left(V_\sigma^{-1}\right)^t U^t(j;\sigma_2)V_\sigma^t = \tilde{C}_v(j)$$

or

$$U(j;\sigma_2)V_\sigma^{-1}\tilde{C}_v(j)\left(V_\sigma^{-1}\right)^t U^t(j;\sigma_2)$$
$$= V_\sigma^{-1}\tilde{C}_v(j)\left(V_\sigma^t\right)^{-1},$$

which evidently is condition (5) for matrix $\tilde{C}_v(j)$.

As a consequence of the theorem is the following algorithm of obtaining of nonisomorphic contracted quantum groups. One calls two distributions of nilpotent parameters among elements of generating matrices $U(j;\sigma_1), U(j;\sigma_2)$ equivalent if they are connected by two operations: (i) they pass in each other by the permutations of the same columns and rows of generating matrices, that is, by (4); (ii) matrices pass in each other by reflection relative to the secondary diagonal with possible simultaneous replacement of j_k with j_{N-k}, $k = 1, \ldots, N-1$. Nonisomorphic contracted quantum groups correspond, in the first place, to the nonequivalent generating matrices and, secondly, to equivalent generating matrices, but with different transformations of deformation parameters ($J_1 \neq J_2$). For illustration of the algorithm, all nonequivalent contractions of quantum groups $SO_v(N; j; \sigma)$, $N = 3, 4, 5$ shall be considered in the following sections.

5. QUANTUM GROUPS $SO_v(3; j; \sigma)$

Quantum group $SO_q(3)$ has four nonisomorphic contracted groups: two Euclid groups $E_v^0(2) = SO_v(3; \iota_1, j_2; \sigma_0)$, $J = \iota_1$, $E_z(2) = SO_z(3; \iota_1, 1; \sigma)$, $J = 1$, where $\sigma_0 = (1, 2, 3)$, $\sigma = (2, 1, 3)$, and two Galilei groups $G_v^0(2) = SO_v(3; \iota_1, \iota_2; \sigma_0)$, $J = \iota_1\iota_2$, $G_v(2) = SO_v(3; \iota_1, \iota_2; \sigma)$, $J = \iota_2$. For comparison, nondeformed complex rotation group $SO(3)$ has two nonisomorphic Cayley–Klein contracted groups: Euclid group $E(2)$ and Galilei group $G(2)$.

5.1. Quantum Groups $SO_v(3; j; \sigma_0)$, $\sigma_0 = (1,2,3)$

Let $C_1 = \cosh Jv$, $S_1 = \sinh Jv$, $J = j_1 j_2$. The generating matrix

$$U(j) = \begin{pmatrix} u_{11} & j_1 u_{12} & j_1 j_2 u_{13} \\ j_1 u_{21} & u_{22} & j_2 u_{23} \\ j_1 j_2 u_{31} & j_2 u_{32} & u_{33} \end{pmatrix} \qquad (6)$$

satisfies (v, j)-orthogonality relations: (i) $U(j) \times C_v(j)U^t(j) = C_v(j)$, i.e.,

$$iJS_1[u_{13}, u_{11}] = C_1(u_{11}^2 + J^2u_{13}^2 - 1) + j_1^2u_{12}^2, \quad (7)$$

$$iJS_1[u_{23}, u_{21}] = C_1(j_1^2u_{21}^2 + j_2^2u_{23}^2) + u_{22}^2 - 1,$$
$$iJS_1[u_{33}, u_{31}] = C_1(J^2u_{31}^2 + u_{33}^2 - 1) + j_2^2u_{32}^2,$$
$$u_{11}u_{21}j_1C_1 - iu_{13}u_{21}j_1JS_1 + j_1u_{12}u_{22}$$
$$+ u_{13}u_{23}j_2JC_1 + iu_{11}u_{23}j_2S_1 = 0,$$
$$u_{11}u_{31}JC_1 - iu_{13}u_{31}J^2S_1 + Ju_{12}u_{32}$$
$$+ u_{13}u_{33}JC_1 + iu_{11}u_{33}S_1 = iS_1,$$
$$u_{21}u_{31}j_1JC_1 - iu_{23}u_{31}j_2JS_1 + j_2u_{22}u_{32}$$
$$+ j_2u_{23}u_{33}C_1 + iu_{21}u_{33}j_1S_1 = 0,$$
$$u_{21}u_{11}j_1C_1 - iu_{23}u_{11}j_2S_1 + j_1u_{22}u_{12}$$
$$+ u_{23}u_{13}j_2JC_1 + iu_{21}u_{13}j_1JS_1 = 0,$$
$$u_{31}u_{11}JC_1 - iu_{33}u_{11}S_1 + Ju_{32}u_{12}$$
$$+ u_{33}u_{13}JC_1 + iu_{31}u_{13}J^2S_1 = -iS_1,$$
$$u_{31}u_{21}j_1JC_1 - iu_{33}u_{21}j_1S_1 + j_2u_{32}u_{22}$$
$$+ u_{33}u_{23}j_2C_1 + iu_{31}u_{23}j_2JS_1 = 0,$$

and (ii) $U(j)^tC_v^{-1}(j)U(j) = C_v^{-1}(j)$, i.e.,

$$iJS_1[u_{11}, u_{31}] = C_1(u_{11}^2 + J^2u_{31}^2 - 1) + j_1^2u_{21}^2, \quad (8)$$

$$iJS_1[u_{12}, u_{32}] = C_1(j_1^2u_{12}^2 + j_2^2u_{32}^2) + u_{22}^2 - 1,$$

$$iJS_1[u_{13},u_{33}] = C_1(u_{33}^2 + J^2u_{13}^2 - 1) + j_2^2u_{23}^2,$$
$$j_1u_{11}u_{12}C_1 + iu_{31}u_{12}j_1JS_1 + j_1u_{21}u_{22} + Ju_{31}u_{33}C_1 - iu_{11}u_{33}S_1 = 0,$$
$$Ju_{11}u_{13}C_1 + iu_{31}u_{13}J^2S_1 + Ju_{21}u_{23} + Ju_{13}u_{33}C_1 - iu_{11}u_{33}S_1 = -iS_1,$$
$$j_1Ju_{12}u_{13}C_1 + iu_{32}u_{13}j_2JS_1 + j_2u_{22}u_{23} + j_2u_{32}u_{33}C_1 - iu_{12}u_{33}j_1S_1 = 0,$$
$$j_1u_{12}u_{11}C_1 + iu_{32}u_{11}j_2S_1 + j_1u_{22}u_{21} + j_2Ju_{32}u_{31}C_1 - iu_{12}u_{31}j_1JS_1 = 0,$$
$$Ju_{13}u_{11}C_1 + iu_{33}u_{11}S_1 + Ju_{23}u_{21} + Ju_{33}u_{31}C_1 - iu_{13}u_{31}J^2S_1 = iS_1,$$
$$j_1Ju_{13}u_{12}C_1 + iu_{33}u_{12}j_1S_1 + j_2u_{23}u_{22} + j_2u_{33}u_{32}C_1 - iu_{13}u_{32}j_2JS_1 = 0.$$

There are three independent generators, for example, u_{12}, u_{13}, u_{23}, which are situated above the diagonal. Their commutators are obtained from RUU relations $\tilde{R}_v(j)U_1(j)U_2(j) = U_2(j)U_1(j)\tilde{R}_v(j)$ and are in the form

$$[u_{12},u_{23}] = i\frac{\sinh Jv}{J}u_{22}(u_{11}-u_{33}), \qquad (9)$$

$$[u_{13},u_{23}] = u_{23}\left\{(\cosh Jv - 1)u_{13} - i\frac{\sinh Jv}{J}u_{33}\right\},$$

$$[u_{12},u_{13}] = \left\{(\cosh Jv - 1)u_{13} + i\frac{\sinh Jv}{J}u_{11}\right\}u_{12}.$$

An associative algebra $SO_v(3;j;\sigma_0)$ is a Hopf algebra with counit $\epsilon(U(j)) = I$, i.e., $\epsilon(u_{ik}) = 0$, $\epsilon(u_{kk}) = 1$, coproduct $\Delta U(j) = U(j)\dot{\otimes}U(j)$ in the form

$$\Delta u_{12} = u_{11}\otimes u_{12} + u_{12}\otimes u_{22} + j_2^2u_{13}\otimes u_{32}, \quad (10)$$
$$\Delta u_{21} = u_{21}\otimes u_{11} + u_{22}\otimes u_{21} + j_2^2u_{23}\otimes u_{31},$$
$$\Delta u_{23} = u_{22}\otimes u_{23} + u_{23}\otimes u_{33} + j_1^2u_{21}\otimes u_{13},$$
$$\Delta u_{32} = u_{32}\otimes u_{22} + u_{33}\otimes u_{32} + j_1^2u_{31}\otimes u_{12},$$
$$\Delta u_{13} = u_{11}\otimes u_{13} + u_{12}\otimes u_{23} + u_{13}\otimes u_{33},$$
$$\Delta u_{31} = u_{31}\otimes u_{11} + u_{32}\otimes u_{21} + u_{33}\otimes u_{31},$$
$$\Delta u_{11} = u_{11}\otimes u_{11} + j_1^2u_{12}\otimes u_{21} + J^2u_{13}\otimes u_{31},$$
$$\Delta u_{22} = u_{22}\otimes u_{22} + j_1^2u_{21}\otimes u_{12} + j_2^2u_{23}\otimes u_{32},$$
$$\Delta u_{33} = u_{33}\otimes u_{33} + j_2^2u_{32}\otimes u_{23} + J^2u_{31}\otimes u_{13},$$

and antipode $S(u(j)) = C_v(j)U^t(j)C_v^{-1}(j)$, where

$$S(u_{12}) = u_{21}\cosh\left(\frac{Jv}{2}\right) + ij_2^2u_{23}\frac{1}{J}\sinh\left(\frac{Jv}{2}\right), \qquad (11)$$

$$S(u_{21}) = u_{12}\cosh\left(\frac{Jv}{2}\right) + ij_2^2u_{32}\frac{1}{J}\sinh\left(\frac{Jv}{2}\right),$$
$$S(u_{23}) = u_{32}\cosh\left(\frac{Jv}{2}\right) - ij_1^2u_{12}\frac{1}{J}\sinh\left(\frac{Jv}{2}\right),$$
$$S(u_{32}) = u_{23}\cosh\left(\frac{Jv}{2}\right) - ij_1^2u_{21}\frac{1}{J}\sinh\left(\frac{Jv}{2}\right),$$
$$S(u_{13}) = u_{31}\cosh^2\left(\frac{Jv}{2}\right) + u_{13}\sinh^2\left(\frac{Jv}{2}\right) + i\frac{1}{2}(u_{33}-u_{11})\frac{1}{J}\sinh(Jv),$$
$$S(u_{31}) = u_{13}\cosh^2\left(\frac{Jv}{2}\right) + u_{31}\sinh^2\left(\frac{Jv}{2}\right) + i\frac{1}{2}(u_{33}-u_{11})\frac{1}{J}\sinh(Jv),$$
$$S(u_{11}) = u_{11}\cosh^2\left(\frac{Jv}{2}\right) - u_{33}\sinh^2\left(\frac{Jv}{2}\right) + i\frac{1}{2}(u_{13}+u_{31})J\sinh(Jv),$$
$$S(u_{33}) = u_{33}\cosh^2\left(\frac{Jv}{2}\right) - u_{11}\sinh^2\left(\frac{Jv}{2}\right) - i\frac{1}{2}(u_{13}+u_{31})J\sinh(Jv), \quad S(u_{22}) = u_{22}.$$

Remark. Coproduct and counit of $SO_v(3;j;\sigma)$ are the same for any permutation σ. Only antipode, commutation, and $(v;j)$-orthogonality relations depend on σ.

For $j_1 = \iota_1$ quantum Euclid group $E_v^0(2) = SO_v(3;\iota_1,j_2;\sigma_0)$, $J = \iota_1$ is obtained. From $(v;j)$-orthogonality relations, it follows that $u_{11} = 1$, $u_{22} = u_{33}$, $u_{23} = -u_{32}$, and from RUU equations, it follows that all these generators commute and generate rotation group $SO(2)$. Therefore, it is natural to introduce new notation $u_{22} = u_{33} = \cos\varphi, u_{23} = \sin\varphi = -u_{32}$, and rewrite the generating matrix as

$$U(\iota_1;\sigma_0) = \begin{pmatrix} 1 & \iota_1u_{12} & \iota_1u_{13} \\ \iota_1u_{21} & \cos\varphi & \sin\varphi \\ \iota_1u_{31} & -\sin\varphi & \cos\varphi \end{pmatrix} \sim \begin{pmatrix} \cdot & \circ & \circ \\ & \cdot & \cdot \\ & & \cdot \end{pmatrix}, \qquad (12)$$

where from $(v;j)$-orthogonality relations it follows that

$$u_{21} = -\left(u_{12}\cos\varphi + u_{13}\sin\varphi + i\frac{v}{2}\sin\varphi\right), \quad (13)$$

$$u_{31} = u_{12}\sin\varphi - u_{13}\cos\varphi + i\frac{v}{2}(1-\cos\varphi).$$

Here and later, the distribution of nilpotent parameters among elements of the generating matrix is shown with the help of some notation: $\circ = \iota_1$, $\bullet = \iota_2$,

$\times = \iota_1\iota_2$. (Let us recall that this distribution is symmetric relatively diagonal.) Dots denote complex elements. Commutation relations of independent generators are as follows:

$$[u_{12}, \sin\varphi] = iv\cos\varphi(1-\cos\varphi), \tag{14}$$

$$[\sin\varphi, u_{13}] = iv\sin\varphi\cos\varphi, \quad [u_{12}, u_{13}] = ivu_{12}.$$

The coproduct of quantum Euclid group is given by

$$\Delta u_{12} = 1\otimes u_{12} + u_{12}\otimes\cos\varphi - j_2^2 u_{13}\otimes\sin\varphi, \tag{15}$$

$$\Delta u_{13} = 1\otimes u_{13} + u_{12}\otimes\sin\varphi + u_{13}\otimes\cos\varphi,$$
$$\Delta\sin\varphi = \cos\varphi\otimes\sin\varphi + \sin\varphi\otimes\cos\varphi,$$
$$\Delta\varphi = 1\otimes\varphi + \varphi\otimes 1;$$

the antipode is as follows:

$$S(u_{12}) = -u_{12}\cos\varphi - u_{13}\sin\varphi, \tag{16}$$
$$S(u_{13}) = -u_{13}\cos\varphi + u_{12}\sin\varphi, \quad S(\varphi) = -\varphi;$$

and the counit of independent generators is equal to zero: $\epsilon(u_{12}) = \epsilon(u_{13}) = \epsilon(\varphi) = 0$.

If u_{21}, u_{31}, φ are taken as independent generators, then Eqs. (13)–(16) are rewritten in the following way: from $(v;j)$-orthogonality relations

$$u_{12} = -u_{21}\cos\varphi + u_{31}\sin\varphi - i\frac{v}{2}\sin\varphi, \tag{17}$$

$$u_{13} = -u_{21}\sin\varphi - u_{31}\cos\varphi - i\frac{v}{2}(1-\cos\varphi),$$

commutation relations

$$[u_{21}, \sin\varphi] = iv\cos\varphi(1-\cos\varphi), \tag{18}$$
$$[\sin\varphi, u_{31}] = -iv\sin\varphi\cos\varphi, \quad [u_{31}, u_{21}] = ivu_{21};$$

coproduct

$$\Delta u_{21} = u_{21}\otimes 1 + \cos\varphi\otimes u_{21} + \sin\varphi\otimes u_{31}; \tag{19}$$

$$\Delta u_{31} = u_{31}\otimes 1 - \sin\varphi\otimes u_{21} + \cos\varphi\otimes u_{31},$$
$$\Delta\varphi = 1\otimes\varphi + \varphi\otimes 1;$$

antipode

$$S(u_{21}) = -u_{21}\cos\varphi + u_{31}\sin\varphi - iv\sin\varphi, \tag{20}$$

$$S(u_{31}) = -u_{31}\cos\varphi - u_{21}\sin\varphi + iv(\cos\varphi - 1),$$
$$S(\varphi) = -\varphi;$$

and counit $\epsilon(u_{21}) = \epsilon(\varphi) = \epsilon(u_{31}) = 0$.

Under contraction $j_2 = \iota_2$ quantum analog $N_v^0(2) = SO_v(3; j_1, \iota_2; \sigma_0)$, $J = \iota_2$ of cylindrical group or Newton group $N(2)$ is obtained. Similarly to previous case, with the help of (v,j)-orthogonality relations, the generating matrix may be written in the form

$$U(\iota_2;\sigma_0) = \begin{pmatrix} \cos\psi & \sin\psi & \iota_2 u_{13} \\ -\sin\psi & \cos\psi & \iota_2 u_{23} \\ \iota_2 u_{31} & \iota_2 u_{32} & 1 \end{pmatrix} \sim \begin{pmatrix} \cdot & \cdot & \bullet \\ & \cdot & \bullet \\ & & \cdot \end{pmatrix}, \tag{21}$$

where

$$u_{31} = u_{23}\sin\psi - u_{13}\cos\psi + i\frac{v}{2}(1-\cos\psi), \tag{22}$$

$$u_{32} = -u_{23}\cos\psi - u_{13}\sin\psi - i\frac{v}{2}\sin\psi,$$

and independent generators are subject to commutation relations

$$[\sin\psi, u_{23}] = iv\cos\psi(\cos\psi - 1), \tag{23}$$

$$[u_{23}, u_{13}] = ivu_{23}, \quad [\sin\psi, u_{13}] = iv\sin\psi\cos\psi.$$

The Hopf algebra is defined by coproduct

$$\Delta(\sin\psi) = \cos\psi\otimes\sin\psi + \sin\psi\otimes\cos\psi, \tag{24}$$
$$\Delta(\psi) = 1\otimes\psi + \psi\otimes 1,$$

$$\Delta u_{13} = u_{13}\otimes 1 + \cos\psi\otimes u_{13} + \sin\psi\otimes u_{23},$$

$$\Delta u_{23} = u_{23}\otimes 1 + \cos\psi\otimes u_{23} - j_1^2\sin\psi\otimes u_{13};$$

by antipode

$$S(u_{13}) = u_{31} + i\frac{v}{2}(u_{33} - u_{11}) \tag{25}$$
$$= u_{23}\sin\psi - u_{13}\cos\psi + iv(1-\cos\psi),$$

$$S(u_{23}) = u_{32} - i\frac{v}{2}j_1^2 u_{12} = -u_{23}\cos\psi$$
$$- u_{13}\sin\psi - iv\sin\psi, \quad S(\psi) = -\psi;$$

and by counit $\epsilon(\psi) = \epsilon(u_{13}) = \epsilon(u_{23}) = 0$.

The distribution of ι_1 in matrix (12) is passed to the distribution of ι_2 in matrix (21) under reflection on secondary diagonal and simultaneous substitution $J = \iota_1$ by $J = \iota_2$. This means that the quantum Euclid group $E_v^0(2) = SO_v(3;\iota_1,1;\sigma_0)$ is isomorphic to the quantum Newton group $N_v^0(2) = SO_v(3;1,\iota_2;\sigma_0)$ as well as in the nondeformed case. Under substitution u_{31} on u_{13}, u_{21} on u_{23}, φ on $-\psi$, and v on $-v$, commutation relations (18) are transformed in (23), coproduct (19) is transformed in (24), and antipode (20) is transformed in (25).

Two-dimensional contraction $j_1 = \iota_1$, $j_2 = \iota_2$ gives quantum Galilei group $G_v^0(2) = SO_v(3;\iota_1,\iota_2;\sigma_0)$, $J = \iota_1\iota_2$. With the help of $(v;j)$-orthogonality

relations, the generating matrix may be written in the form

$$U(\iota;\sigma_0) = \begin{pmatrix} 1 & \iota_1 u_{12} & \iota_1\iota_2 u_{13} \\ -\iota_1 u_{12} & 1 & \iota_2 u_{23} \\ \iota_1\iota_2 u_{31} & -\iota_2 u_{23} & 1 \end{pmatrix} \sim \begin{pmatrix} \cdot & \circ & \times \\ & \cdot & \bullet \\ & & \cdot \end{pmatrix}, \tag{26}$$

where $u_{31} = -u_{13} + u_{12}u_{23}$, and independent generators satisfy commutation relations

$$[u_{12}, u_{23}] = 0, \quad [u_{23}, u_{13}] = ivu_{23}, \quad [u_{12}, u_{13}] = ivu_{12}. \tag{27}$$

The Hopf algebra structure is given by coproduct

$$\begin{aligned} \Delta u_{12} &= 1 \otimes u_{12} + u_{12} \otimes 1, \\ \Delta u_{23} &= 1 \otimes u_{23} + u_{23} \otimes 1, \\ \Delta u_{13} &= 1 \otimes u_{13} + u_{13} \otimes 1 + u_{12} \otimes u_{23}; \end{aligned} \tag{28}$$

antipode

$$S(u_{12}) = -u_{12}, \quad S(u_{13}) = -u_{13} + u_{12}u_{23}, \quad S(u_{23}) = -u_{23}; \tag{29}$$

and standard counit $\epsilon(u_{12}) = \epsilon(u_{13}) = \epsilon(u_{23}) = 0$.

5.2. Quantum Groups $SO_v(3;j;\sigma)$, σ = (2,1,3)

The deformation parameter is transformed by multiplication on $J = (\sigma_1, \sigma_3) = (2,3) = j_2$. Commutators, (v,j)-orthogonality relations, and antipode are easily obtained from corresponding formulas of $SO_z(3) = SO_v(3; j = 1; \sigma_0)$ by interchange of indices 1 and 2 and then by standard reconstruction of contraction parameters j. In particular, the generating matrix is as follows:

$$U(j;\sigma) = \begin{pmatrix} u_{22} & j_1 u_{21} & j_2 u_{23} \\ j_1 u_{12} & u_{11} & j_1 j_2 u_{13} \\ j_2 u_{32} & j_1 j_2 u_{31} & u_{33} \end{pmatrix}. \tag{30}$$

The commutation relations of independent generators are

$$j_1^2[u_{21}, u_{13}] = i\frac{1}{j_2}\sinh(j_2 v)u_{11}(u_{22} - u_{33}), \tag{31}$$

$$\begin{aligned} [u_{23}, u_{13}] &= u_{13}\left\{\frac{1}{j_2}(\cosh j_2 v - 1)u_{23} \right. \\ &\left. - i\frac{1}{j_2}\sinh(j_2 v)u_{33}\right\}, \\ [u_{21}, u_{23}] &= \left\{\frac{1}{j_2}(\cosh j_2 v - 1)u_{23} \right. \\ &\left. + i\frac{1}{j_2}\sinh(j_2 v)u_{22}\right\}u_{21}. \end{aligned}$$

The antipode is easily obtained by the transformations of (11)

$$S(u_{21}) = u_{12}\cosh\left(j_2\frac{v}{2}\right) + ij_2^2 u_{13}\frac{1}{j_2}\sinh\left(j_2\frac{v}{2}\right), \tag{32}$$

$$\begin{aligned} S(u_{12}) &= u_{21}\cosh\left(j_2\frac{v}{2}\right) + ij_2^2 u_{31}\frac{1}{j_2}\sinh\left(j_2\frac{v}{2}\right), \\ S(u_{13}) &= u_{31}\cosh\left(j_2\frac{v}{2}\right) - iu_{21}\frac{1}{j_2}\sinh\left(j_2\frac{v}{2}\right), \\ S(u_{31}) &= u_{13}\cosh\left(j_2\frac{v}{2}\right) - iu_{12}\frac{1}{j_2}\sinh\left(j_2\frac{v}{2}\right), \\ S(u_{23}) &= u_{32}\cosh^2\left(j_2\frac{v}{2}\right) + u_{23}\sinh^2\left(j_2\frac{v}{2}\right) \\ &+ i\frac{1}{2}(u_{33} - u_{22})\frac{1}{j_2}\sinh(j_2 v), \\ S(u_{32}) &= u_{23}\cosh^2\left(j_2\frac{v}{2}\right) + u_{32}\sinh^2\left(j_2\frac{v}{2}\right) \\ &+ i\frac{1}{2}(u_{33} - u_{22})\frac{1}{j_2}\sinh(j_2 v), \\ S(u_{22}) &= u_{22}\cosh^2\left(j_2\frac{v}{2}\right) - u_{33}\sinh^2\left(j_2\frac{v}{2}\right) \\ &+ \frac{i}{2}(u_{23} + u_{32})j_2\sinh(j_2 v), \\ S(u_{33}) &= u_{33}\cosh^2\left(j_2\frac{v}{2}\right) - u_{22}\sinh^2\left(j_2\frac{v}{2}\right) \\ &- \frac{i}{2}(u_{23} + u_{32})j_2\sinh(j_2 v), \quad S(u_{11}) = u_{11}. \end{aligned}$$

The coproduct and counit are not changed and are given by (10), which correspond to identical permutation σ_0.

Contraction $j_1 = \iota_1$ leaves the deformation parameter fixed since $J = j_2 = 1$ and gives new quantum Euclid group $E_z(2) = SO_z(3; \iota_1, 1; \sigma)$ with the matrix

$$U(\iota_1;\sigma) = \begin{pmatrix} \cos\varphi & \iota_1 u_{21} & \sin\varphi \\ \iota_1 u_{12} & 1 & \iota_1 u_{13} \\ -\sin\varphi & \iota_1 u_{31} & \cos\varphi \end{pmatrix} \sim \begin{pmatrix} \cdot & \circ & \cdot \\ & \cdot & \circ \\ & & \cdot \end{pmatrix}, \tag{33}$$

where the generators are

$$u_{11} = 1, \quad u_{22} = u_{33} = \cos\varphi, \quad u_{23} = -u_{32} = \sin\varphi, \tag{34}$$

$$u_{12}\cos\left(\varphi - i\frac{v}{2}\right) = -\left(u_{21} + u_{13}\sin\left(\varphi - i\frac{v}{2}\right)\right),$$

$$u_{31}\cos\left(\varphi - i\frac{v}{2}\right) = -\left(u_{13} + u_{21}\sin\left(\varphi - i\frac{v}{2}\right)\right),$$

and the commutation relations

$$[u_{21}, u_{13}] = 0, \tag{35}$$
$$[u_{13}, \sin\varphi] = 2i\sinh\frac{z}{2}u_{13}\cos\left(\varphi - i\frac{z}{2}\right),$$
$$[u_{21}, \sin\varphi] = 2i\sinh\frac{z}{2}\cos\left(\varphi + i\frac{z}{2}\right)u_{21}$$

hold. The antipode is given by

$$S(u_{21}) = u_{12}\cosh\frac{z}{2} + iu_{13}\sinh\frac{z}{2}, \tag{36}$$
$$S(u_{13}) = u_{31}\cosh\frac{z}{2} - iu_{21}\sinh\frac{z}{2},$$
$$S(\varphi) = -\varphi,$$

and the coproduct is in the form

$$\Delta u_{13} = 1\otimes u_{13} + u_{13}\otimes\cos\varphi + u_{12}\otimes\sin\varphi, \tag{37}$$
$$\Delta u_{21} = \cos\varphi\otimes u_{21} + u_{21}\otimes 1 + \sin\varphi\otimes u_{31},$$
$$\Delta\varphi = 1\otimes\varphi + \varphi\otimes 1.$$

Quantum Newton group $N_v(2) = SO_v(3; 1, \iota_2; \sigma)$, $J = \iota_2$ is described by relations $u_{33} = 1$, $u_{11} = u_{22} = \cos\psi$, $u_{21} = \sin\psi = -u_{12}$; i.e., the generating matrix is in the form

$$U(\iota_2;\sigma) = \begin{pmatrix} \cos\psi & \sin\psi & \iota_2 u_{23} \\ -\sin\psi & \cos\psi & \iota_2 u_{13} \\ \iota_2 u_{32} & \iota_2 u_{31} & 1 \end{pmatrix} \sim \begin{pmatrix} \cdot & \cdot & \bullet \\ & \cdot & \bullet \\ & & \cdot \end{pmatrix}, \tag{38}$$

where

$$u_{31} = -u_{13}\cos\psi - u_{23}\sin\psi - i\frac{v}{2}\sin\psi, \tag{39}$$
$$u_{32} = -u_{23}\cos\psi + u_{13}\sin\psi + i\frac{v}{2}(1-\cos\psi),$$

and the commutation relations

$$[\sin\psi, u_{13}] = iv\cos\psi(\cos\psi - 1), \tag{40}$$
$$[\sin\psi, u_{23}] = iv\cos\psi\sin\psi, \quad [u_{23}, u_{13}] = -ivu_{13}$$

hold for independent generators. The antipode is given by

$$S(u_{13}) = -u_{13}\cos\psi - u_{23}\sin\psi - iv\sin\psi, \tag{41}$$
$$S(\psi) = -\psi,$$
$$S(u_{23}) = -u_{23}\cos\psi + u_{13}\sin\psi + iv(1-\cos\psi),$$

and the coproduct is

$$\Delta\psi = 1\otimes\psi + \psi\otimes 1, \tag{42}$$
$$\Delta u_{23} = u_{23}\otimes 1 + \cos\psi\otimes u_{23} + \sin\psi\otimes u_{13},$$
$$\Delta u_{13} = u_{13}\otimes 1 + \cos\psi\otimes u_{13} - \sin\psi\otimes u_{23}.$$

Generating matrices (38) and (21) are equal from the viewpoint of nilpotent unit distribution, while formulas (39)–(42) pass to (22)–(25) under substitution u_{13} on u_{23} and u_{23} on u_{13}. Thus, both quantum groups are isomorphic $N_v(2) \simeq N_v^0(2) \simeq E_v^0(2)$.

For quantum Galilei group $G_v(2) = SO_v(3; \iota_1, \iota_2; \sigma)$, $J = \iota_2$, it follows from (v, j)-orthogonality relations that $u_{11} = u_{22} = u_{33} = 1$ and the generating matrix takes the form

$$U(\iota;\sigma) = \begin{pmatrix} 1 & \iota_1 u_{21} & \iota_2 u_{23} \\ -\iota_1 u_{12} & 1 & \iota_1\iota_2 u_{13} \\ -\iota_2 u_{23} & \iota_1\iota_2 u_{31} & 1 \end{pmatrix} \sim \begin{pmatrix} \cdot & \circ & \bullet \\ & \cdot & \times \\ & & \cdot \end{pmatrix}, \tag{43}$$

where $u_{31} = -u_{13} - u_{21}u_{23} + i\frac{v}{2}u_{21}$; the commutation relations are

$$[u_{21}, u_{13}] = 0, \quad [u_{23}, u_{13}] = -ivu_{13}, \tag{44}$$
$$[u_{21}, u_{23}] = ivu_{21};$$

the antipode may be written as

$$S(u_{21}) = -u_{21}, \quad S(u_{23}) = -u_{23}, \tag{45}$$
$$S(u_{13}) = -u_{13} - u_{21}u_{23};$$

and the coproduct is

$$\Delta u_{21} = 1\otimes u_{21} + u_{21}\otimes 1, \tag{46}$$
$$\Delta u_{23} = 1\otimes u_{23} + u_{23}\otimes 1,$$
$$\Delta u_{13} = 1\otimes u_{13} + u_{13}\otimes 1 + u_{21}\otimes u_{23}.$$

Let us stress that $G_v(2)$ is not isomorphic to $G_v^0(2)$, in spite of the fact that both matrices (43), (26) are equivalent from the viewpoint of nilpotent unit distribution, but deformation parameters are transformed in a different way, namely, with multipliers $J = \iota_2$ and $J = \iota_1\iota_2$, respectively. Therefore commutation relations (27), (44), antipodes (29), (45), and counits pass in each other under substitution u_{13} on u_{23} and vice versa, but in coproduct (28), $\Delta(u_{13})$ does not pass in $\Delta(u_{23})$ from (46).

5.3. Quantum Groups $SO_v(3; j; \sigma)$, $\sigma = (1,3,2)$

The deformation parameter is multiplied by $J = (\sigma_1, \sigma_3) = (1, 2) = j_1$. Commutators, (v, j)-orthogonality relations and antipode are easily obtained from corresponding formulas of $SO_z(3) = SO_v(3; 1, 1; \sigma_0)$ by interchange of indices 2 and 3 and then by standard reconstruction of contraction parameters j. In particular, the generating matrix is as follows:

$$U(j;\sigma) = \begin{pmatrix} u_{11} & j_1 j_2 u_{13} & j_1 u_{12} \\ j_1 j_2 u_{31} & u_{33} & j_2 u_{32} \\ j_1 u_{21} & j_2 u_{23} & u_{22} \end{pmatrix}. \tag{47}$$

For $j_1 = \iota_1$, quantum Euclid group $\tilde{E}_v(2) = SO_v(3; \iota_1, 1; \sigma)$ is obtained with generators

$$U(\iota_1;\sigma) = \begin{pmatrix} 1 & \iota_1 u_{13} & \iota_1 u_{12} \\ \iota_1 u_{31} & \cos\varphi & \sin\varphi \\ \iota_1 u_{21} & -\sin\varphi & \cos\varphi \end{pmatrix} \sim \begin{pmatrix} \cdot & \circ & \circ \\ & \cdot & \cdot \\ & & \cdot \end{pmatrix}. \tag{48}$$

As far as the generating matrix (48) is equal to (12), then $\tilde{E}_v(2)$ is isomorphic with $E_v^0(2)$ and therefore does not represent a new quantum group.

Quantum Newton group $\tilde{N}_z(2) = SO_z(3; 1, \iota_2; \sigma)$ is described by untouched deformation parameter z and generators $u_{33} = 1$, $u_{11} = u_{22} = \cos\psi$, $u_{12} = \sin\psi = -u_{21}$, which are arranged in matrix form

$$U(\iota_2;\sigma) = \begin{pmatrix} \cos\psi & \iota_2 u_{13} & \sin\psi \\ \iota_2 u_{31} & 1 & \iota_2 u_{32} \\ -\sin\psi & \iota_2 u_{23} & \cos\psi \end{pmatrix} \sim \begin{pmatrix} \cdot & \bullet & \cdot \\ & \cdot & \bullet \\ & & \cdot \end{pmatrix}. \tag{49}$$

This quantum group as a Hopf algebra is isomorphic to quantum Euclid group $E_z(2)$ with untouched deformation parameter ($J = 1$), since the generating matrix (49) is equal to (33), if we put ι_1 instead of ι_2. Finally, quantum Galilei group $\tilde{G}_v(2) = SO_v(3; \iota_1, \iota_2; \sigma)$ is characterized by $J = \iota_1$; the diagonal generators are equal to one, $u_{11} = u_{22} = u_{33} = 1$; and the generating matrix is as follows:

$$U(\iota;\sigma) = \begin{pmatrix} 1 & \iota_1\iota_2 u_{13} & \iota_1 u_{12} \\ \iota_1\iota_2 u_{31} & 1 & \iota_2 u_{32} \\ -\iota_1 u_{12} & -\iota_2 u_{32} & 1 \end{pmatrix} \sim \begin{pmatrix} \cdot & \times & \circ \\ & \cdot & \bullet \\ & & \cdot \end{pmatrix}. \tag{50}$$

The nilpotent parameter distribution of (50) passes in (43) under exchange ι_1 and ι_2 and simultaneous reflection with respect to the secondary diagonal. Therefore, $\tilde{G}_v(2)$ is isomorphic to $G_v(2)$. Thus, the permutation $\sigma = (1, 3, 2)$ does not lead to new contracted quantum groups.

6. QUANTUM GROUPS $SO_v(4; j; \sigma)$

In this section, all nonisomorphic contractions of $SO_q(4)$ are enumerated. The deformation parameter is multiplied by $J = (\sigma_1, \sigma_4) \cup (\sigma_2, \sigma_3)$, which is equal to $J = j_1 j_2 j_3$ for permutation $\sigma_0 = (1, 2, 3, 4)$ and $J = j_1 j_3$ for $\sigma' = (1, 3, 4, 2)$. There are no other values of J. The above-mentioned values of J correspond to nonisomorphic on the equal parameter number contracted quantum groups which have nonequivalent generating matrices for permutations σ_0 and σ'.

One-dimensional contractions. For $j_1 = \iota_1$, $J = \iota_1$ quantum Euclid group $E_v(3) = SO_v(4; \iota_1; \sigma_0)$ is obtained. For $j_2 = \iota_2$, there are two nonisomorphic quantum Newton groups: $N_v(3) = SO_v(4; \iota_2; \sigma_0)$, $J = \iota_2$, and $N_z(3) = SO_z(4; \iota_2; \sigma')$ with $J = 1$.

Two-dimensional contractions. For $j_1 = \iota_1$, $j_2 = \iota_2$ two nonisomorphic quantum Galilei groups $G_v(3) = SO_v(4; \iota_1, \iota_2; \sigma_0)$, $J = \iota_1\iota_2$ and $G_w(3) = SO_w(4; \iota_1, \iota_2; \sigma')$, $J = \iota_1$ are obtained. Contractions $j_1 = \iota_1, j_3 = \iota_3$ give in result quantum groups $SO_v(4; \iota_1, \iota_3; \sigma_0)$, $J = \iota_1\iota_3$, which has no special name.

Under maximal three-dimensional contractions $j_1 = \iota_1$, $j_2 = \iota_2$, $j_3 = \iota_3$, two nonisomorphic quantum flag groups $F_v(4) = SO_v(4; \iota; \sigma_0)$, $J = \iota_1\iota_2\iota_3$ and $F_w(4) = SO_w(4; \iota; \sigma')$, $J = \iota_1\iota_3$ are obtained.

$$E_v(3) \sim \begin{pmatrix} \cdot & \circ & \circ & \circ \\ & \cdot & \cdot & \cdot \\ & & \cdot & \cdot \\ & & & \cdot \end{pmatrix}, \quad N_v(3) \sim \begin{pmatrix} \cdot & \cdot & \bullet & \bullet \\ & \cdot & \bullet & \bullet \\ & & \cdot & \cdot \\ & & & \cdot \end{pmatrix},$$

$$N_z(3) \sim \begin{pmatrix} \cdot & \bullet & \bullet & \cdot \\ & \cdot & \cdot & \bullet \\ & & \cdot & \bullet \\ & & & \cdot \end{pmatrix}, \quad G_v(3) \sim \begin{pmatrix} \cdot & \circ & \times & \times \\ & \cdot & \bullet & \bullet \\ & & \cdot & \cdot \\ & & & \cdot \end{pmatrix},$$

$$G_w(3) \sim \begin{pmatrix} \cdot & \times & \times & \circ \\ & \cdot & \cdot & \bullet \\ & & \cdot & \bullet \\ & & & \cdot \end{pmatrix}, \quad F_v(4) \sim \begin{pmatrix} \cdot & \circ & \times & \otimes \\ & \cdot & \bullet & \diamondsuit \\ & & \cdot & \star \\ & & & \cdot \end{pmatrix},$$

$$SO_v(4; \iota_1, \iota_3; \sigma_0) \sim \begin{pmatrix} \cdot & \circ & \circ & \triangle \\ & \cdot & \cdot & \star \\ & & \cdot & \star \\ & & & \cdot \end{pmatrix},$$

$$F_w(4) \sim \begin{pmatrix} \cdot & \times & \otimes & \circ \\ & \cdot & \star & \bullet \\ & & \cdot & \diamondsuit \\ & & & \cdot \end{pmatrix},$$

where $\triangle = \iota_3$, $\star = \iota_1\iota_3$, $\diamondsuit = \iota_2\iota_3$, and $\otimes = \iota_1\iota_2\iota_3$.

Thus, for the quantum case, there are eight different contracted groups, while for classical group $SO(4)$ there are only five nonisomorphic contracted Cayley–Klein groups.

7. QUANTUM GROUPS $SO_v(5;j;\sigma)$

The deformation parameter is multiplied by $J = (\sigma_1,\sigma_5)\cup(\sigma_2,\sigma_4)$, which is equal to $J = j_1j_2j_3j_4$ for permutation $\sigma_0 = (1,2,3,4,5)$, equal to $J = j_1j_2j_3$ for permutation $\sigma^1 = (1,2,5,3,4)$, equal to $J = j_1j_2j_4$ for permutation $\sigma^2 = (1,4,2,5,3)$, equal to $J = j_1j_3$ for permutation $\sigma^3 = (1,3,5,4,2)$, equal to $J = j_1j_4$ for permutation $\sigma^4 = (1,4,3,5,2)$, equal to $J = j_2j_4$ for permutation $\sigma^5 = (2,4,1,5,3)$, equal to $J = j_1j_3j_4$ for permutation $\sigma^6 = (1,3,4,5,2)$, and equal to $J = j_2j_3j_4$ for permutation $\sigma^7 = (2,3,1,4,5)$.

If contractions only on parameters j_1, j_2 are considered, then there are two quantum Euclid groups $E_v(4) = SO_v(4;\iota_1;\sigma_0)$, $J = \iota_1$ and $E_z(4) = SO_z(4;\iota_1;\sigma^5)$, $J = 1$ with distribution of nilpotent parameters in the form

$$E_v(4)\sim\begin{pmatrix}\cdot&\circ&\circ&\circ&\circ\\&\cdot&\cdot&\cdot&\cdot\\&&\cdot&\cdot&\cdot\\&&&\cdot&\cdot\\&&&&\cdot\end{pmatrix},\quad E_z(4)\sim\begin{pmatrix}\cdot&\circ&\cdot&\cdot&\cdot\\&\cdot&\circ&\circ&\circ\\&&\cdot&\cdot&\cdot\\&&&\cdot&\cdot\\&&&&\cdot\end{pmatrix},$$

two quantum Newton groups: $N_v(4) = SO_v(4;\iota_2;\sigma_0)$, $J = \iota_2$ and $N_z(4) = SO_z(4;\iota_2;\sigma^3)$, $J = 1$ with generating matrices

$$N_v(4)\sim\begin{pmatrix}\cdot&\cdot&\bullet&\bullet&\bullet\\&\cdot&\bullet&\bullet&\bullet\\&&\cdot&\cdot&\cdot\\&&&\cdot&\cdot\\&&&&\cdot\end{pmatrix},\quad N_z(4)\sim\begin{pmatrix}\cdot&\bullet&\cdot&\bullet&\bullet\\&\cdot&\bullet&\cdot&\cdot\\&&\cdot&\bullet&\bullet\\&&&\cdot&\cdot\\&&&&\cdot\end{pmatrix},$$

and two quantum Galilei groups: $G_v(4) = SO_v(4;\iota_1\iota_2;\sigma_0)$, $J = \iota_1\iota_2$ and $G_z(4) = SO_v(4;\iota_1\iota_2;\sigma^3)$, $J = \iota_1$ with generating matrices

$$G_v(4)\sim\begin{pmatrix}\cdot&\circ&\times&\times&\times\\&\cdot&\bullet&\bullet&\bullet\\&&\cdot&\cdot&\cdot\\&&&\cdot&\cdot\\&&&&\cdot\end{pmatrix},$$

$$G_z(4)\sim\begin{pmatrix}\cdot&\times&\circ&\times&\times\\&\cdot&\bullet&\cdot&\cdot\\&&\cdot&\bullet&\bullet\\&&&\cdot&\cdot\\&&&&\cdot\end{pmatrix}.$$

As compared with the case $N = 3$ two quantum Newton groups are added.

In all discussed examples for $N = 3,4,5$ the number of nonisomorphic quantum analogs of the corresponding classical groups equals two. One may think that this number for any contractions does not exceed two. But this is not so. The number of nonisomorphic quantum analogs of the classical Cayley–Klein groups is increased when the number of nilpotent valued contraction parameters is increased. For example, under maximal contraction $j_k = \iota_k$, $k = 1,\ldots,4$ five quantum analogs of the flag group $F(5) = SO(5;\iota)$ are obtained, namely, $F_v(5) = SO_v(5;\iota;\sigma_0)$, $J = \iota_1\iota_2\iota_3\iota_4$; $F_{v_1}(5) = SO_{v_1}(5;\iota;\sigma^1)$, $J = \iota_1\iota_2\iota_3$; $F_{v_2}(5) = SO_{v_2}(5;\iota;\sigma^2)$, $J = \iota_1\iota_2\iota_4$; $F_{v_3}(5) = SO_{v_3}(5;\ \iota;\ \sigma^3)$, $J = \iota_1\iota_3$; $F_{v_4}(5) = SO_{v_4}(5;\iota;\sigma^4)$, $J = \iota_1\iota_4$. They all have generating matrices with nonequivalent distributions of nilpotent parameters.

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Appendix

R MATRIX OF QUANTUM GROUP $SO_q(N)$ IN CARTESIAN BASIS

$$\tilde R_q = (D\otimes D)R(D\otimes D)^{-1} = I + \frac{1}{2}(q-1)$$

$$\times (1-q^{-1})\sum_{\substack{k=1\\k\neq k'}}^{N}(e_{kk}\otimes e_{kk}+e_{kk}\otimes e_{k'k'})$$

$$+\frac{\lambda}{2}\sum_{\substack{k=1\\k\neq k'}}^{N}(e_{k'k}\otimes e_{kk'}-e_{k'k}\otimes e_{k'k})$$

$$+\frac{\lambda}{2}\sum_{k=1}^{n}(e_{k',n+1}\otimes e_{n+1,k'}-ie_{k',n+1}\otimes e_{n+1,k}$$
$$+ie_{k,n+1}\otimes e_{n+1,k'}+e_{k,n+1}\otimes e_{n+1,k}$$
$$+e_{n+1,k}\otimes e_{k,n+1}+ie_{n+1,k}\otimes e_{k',n+1}$$
$$-ie_{n+1,k'}\otimes e_{k,n+1}+e_{n+1,k'}\otimes e_{k',n+1})$$

$$-\frac{\lambda}{2}\sum_{k=1}^{n}q^{-\rho_k}(-ie_{k',n+1}\otimes e_{k,n+1}$$
$$+e_{k',n+1}\otimes e_{k',n+1}+e_{k,n+1}\otimes e_{k,n+1}$$
$$+ie_{k,n+1}\otimes e_{k',n+1}+ie_{n+1,k}\otimes e_{n+1,k'}$$
$$+e_{n+1,k}\otimes e_{n+1,k}+e_{n+1,k'}\otimes e_{n+1,k'}$$
$$-ie_{n+1,k'}\otimes e_{n+1,k})+\frac{\lambda}{4}\sum_{\substack{k,p=1\\k>p,\,k,p\neq n+1}}^{N}(e_{kp}\otimes e_{pk}$$
$$+e_{kp}\otimes e_{p'k'}+ie_{kp}\otimes e_{p'k}-ie_{kp}\otimes e_{pk'}$$
$$+e_{k'p'}\otimes e_{pk}+e_{k'p'}\otimes e_{p'k'}+ie_{k'p'}\otimes e_{p'k}$$
$$-ie_{k'p'}\otimes e_{pk'}+ie_{k'p}\otimes e_{pk}+ie_{k'p}\otimes e_{p'k'}$$
$$-e_{k'p}\otimes e_{p'k}+e_{k'p}\otimes e_{pk'}-ie_{kp'}\otimes e_{pk}$$
$$-ie_{kp'}\otimes e_{p'k'}+e_{kp'}\otimes e_{p'k}-e_{kp'}\otimes e_{pk'})$$

$$-\frac{\lambda}{4}\sum_{\substack{k,p=1\\k>p,\,k,p\neq n+1}}^{N}q^{\rho_k-\rho_p}(e_{kp}\otimes e_{k'p'}+e_{kp}\otimes e_{kp}$$
$$+ie_{kp}\otimes e_{kp'}-ie_{kp}\otimes e_{k'p}+e_{k'p'}\otimes e_{k'p'}$$
$$+e_{k'p'}\otimes e_{kp}+ie_{k'p'}\otimes e_{kp'}-ie_{k'p'}\otimes e_{k'p}$$
$$+ie_{k'p}\otimes e_{k'p'}+ie_{k'p}\otimes e_{kp}-e_{k'p}\otimes e_{kp'}$$
$$+e_{k'p}\otimes e_{k'p}-ie_{kp'}\otimes e_{k'p'}-ie_{kp'}\otimes e_{kp}$$
$$+e_{kp'}\otimes e_{kp'}-e_{kp'}\otimes e_{k'p}),\quad \lambda=q-q^{-1}.$$

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Radial Coherent States—From the Harmonic Oscillator to the Hydrogen Atom*

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Abstract—We construct spectrum generating algebras of $SO(2,1) \sim SU(1,1)$ in arbitrary dimension for the isotropic harmonic oscillator and the Sturm–Coulomb problem in radial coordinates. Using these algebras, we construct the associated radial Barut–Girardello coherent states for the isotropic harmonic oscillator (in arbitrary dimension). We map these states into the Sturm–Coulomb radial coherent states and show that they evolve in a fictitious time parameter without dispersing. © *2005 Pleiades Publishing, Inc.*

1. INTRODUCTION

Since their introduction by Schrödinger in 1926 [1], coherent states have become a tool to discuss the classical limit of a quantum system. The states which Schrödinger constructed were associated with the one-dimensional harmonic oscillator (HO) and regarded as the most classical ones. The HO coherent states evolve according to the classical equations of motion and the states do not disperse as they evolve with time. As the subject developed, new methods to construct coherent states were suggested. The common definitions are minimum uncertainty states, eigenstates of the lowering operator, and states constructed with a displacement operator acting on a system's fiducial state. For the one-dimensional HO, these three definitions result in the same set of states. They minimize the uncertainty relation in position and momentum (which equals the uncertainty relation of the position and momentum in the HO ground state). As for other systems, Schrödinger suggested looking for coherent states of the hydrogen atom as well. Since then, there have been many attempts to construct classical states for the hydrogen atom. Many of these attempts, as found in [2–5], involve mapping of the Coulomb problem into the four-dimensional harmonic oscillator using the Kustaanheimo–Stiefel (KS) transformation [6]. This transformation involves the introduction of a new time parameter. This time parameter is proportional to the eccentric anomaly [7]. Therefore, all the states obtained this way evolve with the new time parameter rather than in ordinary time. Other attempts consider temporally stable coherent states and their time evolution [8–11].

In 1994, Zlatev, Zhang, and Feng [12] constructed the most general coherent states for the hydrogen atom using $SO(4,2)$, which is the maximal group for this case. They showed that these states cannot be localized and cannot follow the classical orbits. However, they suggested that coherent states of $SU(1,1)$, which is a subgroup of $SO(4,2)$, and the spectrum generating algebra for this atom may have a classical limit for large l, where l is the angular momentum. A few years earlier, Gerry and Kiefer [13] presented a work on radial coherent states for the Coulomb problem, where they used the $SO(2,1) \sim SU(1,1)$ algebra. Their states are composed of superposition of Sturm functions, all scaled by the same factor. The method they used to construct their coherent states is Perelomov's method. They showed that the wave packets constructed this way evolve with their new time parameter and do not disperse as they evolve with that parameter. However, they change their shape periodically as they move from the apogee to the perigee. Radial coherent states for the Coulomb problem and other central potentials were obtained as well by Nieto and collaborators (e.g., [14]). They used the minimum uncertainty states method. The states obtained that way lose their coherence as they evolve with time [15].

In this work, we extend the discussion on radial coherent states to Barut–Girardello (BG) states, based on the $SO(2,1) \sim SU(1,1)$ approach. To our knowledge, these states have never been discussed in the literature in this context. Our discussion is a little more general since the calculations are in arbitrary dimensions.

The plan of the paper is as follows. In Sections 2 and 3, we construct the $SO(2,1)$ algebra for the

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isotropic harmonic oscillator (IHO) and the hydrogen atom (in arbitrary dimensions), respectively. In Section 4, we present the mapping between the HO radial functions and the Coulomb radial functions. BG radial coherent states for the HO in arbitrary dimensions are calculated in Section 5. In Section 6, we map the BG coherent states of the four-dimensional IHO into the BG coherent states of the three-dimensional hydrogen atom. The time evolution and the classical limit of these states is discussed in Section 7. Final conclusions are drawn in Section 8.

2. THE d-DIMENSIONAL ISOTROPIC HARMONIC OSCILLATOR

The dynamical group for the d-dimensional IHO is the real, symplectic noncompact group $Sp(2d,R)$ (e.g., [16]). A subgroup of $Sp(2d,R)$ is the direct product group $SO(2,1)\times SO(d)$ which corresponds to separation of variables of the HO wave functions into the d-dimensional radial functions $R^d_{n_r l}(r)$ and the d-dimensional spherical harmonics functions $Y_{l,m_1,\ldots,m_{d-2}}(\theta_1,\theta_2,\ldots,\theta_{d-1})$. These are basis functions for the $SO(2,1)$ and $SO(d)$ representations, respectively.

Since we are dealing with radial coherent states, we concentrate only on the group $SO(2,1)$ and its representations. $SO(2,1)$ is known as the radial group and its algebra is known as the spectrum generating algebra (SGA) of the IHO. It is a semisimple, noncompact Lie group and is locally isomorphic to $SU(1,1)$ and $Sp(2,R)$.

These groups have three generators, which we denote as k_0, k_1, and k_2. They obey the following commutations relations [17]:

$$[k_0,k_1]=ik_2, \tag{1}$$

$$[k_1,k_2]=-ik_0, \tag{2}$$

$$[k_2,k_0]=ik_1. \tag{3}$$

Alternatively, we can define raising and lowering operators as

$$k_\pm=k_1\pm ik_2, \tag{4}$$

where together with k_0 the commutations relations are

$$[k_0,k_\pm]=\pm k_\pm, \tag{5}$$

$$[k_-,k_+]=2k_0. \tag{6}$$

The generators act on the general group basis $|m,k\rangle$ as follows (e.g., [18]):

$$k_0|m,k\rangle=(m+k)|m,k\rangle, \tag{7}$$

$$k_-|m,k\rangle=\sqrt{m(m+2k-1)}|m-1,k\rangle, \tag{8}$$

$$k_+|m,k\rangle=\sqrt{(m+1)(m+2k)}|m+1,k\rangle, \tag{9}$$

where k is the Bargmann index [19].

In order to obtain the relevant operators for the radial symmetry, we use the d-dimensional radial functions $R^d_{n_r l}(r)$ which are given by [20]

$$R^d_{n_r l}(r) \tag{10}$$
$$=\left[\frac{2\Gamma(n_r+1)}{\Gamma(n_r+l+\frac{d}{2})}\right]^{1/2} r^l e^{-r^2/2}L^{l+(d-2)/2}_{n_r}(r^2),$$

where the $L^{l+(d-2)/2}_{n_r}(r^2)$ are the associated Laguerre functions [21].

These functions are orthonormal with respect to the measure r^{d-1}:

$$\int_0^\infty dr R^d_{n_r l}(r)R^d_{n_r' l}(r)r^{d-1}=\delta_{n_r,n_r'}. \tag{11}$$

Using the recursion relation and the differential equation for the Laguerre functions [21] (since the Laguerre functions in our case are functions of r^2, we performed a change of variables on the original equations)

$$(p+1)L^q_{p+1}(r^2)+(r^2-q-2p-1)L^q_p(r^2) \tag{12}$$
$$+(p+q)L^q_{p-1}(r^2)=0,\quad p=1,2,3,\ldots,$$

$$\frac{1}{2}r\frac{d}{dr}L^q_p(r^2)=pL^q_p(r^2)-(p+q)L^q_{p-1}(r^2), \tag{13}$$
$$p\geq 1,$$

and using the relation for arbitrary operators A and B

$$e^B Ae^{-B}=A+[B,A]+\frac{1}{2}[B,[B,A]]+\cdots, \tag{14}$$

we obtain raising and lowering operators for the quantum number n_r

$$K_\pm=\frac{1}{2}\left(\pm r\partial_r-r^2+H\pm\frac{d}{2}\right), \tag{15}$$

where H is the d-dimensional IHO radial Hamiltonian [20]:

$$H=\frac{1}{2}\left(-\frac{1}{r^{d-1}}\partial_r(r^{d-1}\partial_r)+\frac{l(l+d-2)}{r^2}+r^2\right). \tag{16}$$

Defining $K_0=\frac{1}{2}H$, we find that K_+, K_-, and K_0 obey the $SO(2,1)$ algebra commutations relations presented in Eqs. (5), (6).

The raising, lowering, and eigenvalue relations for the normalized functions $R^d_{n_r l}$ are given by

$$K_+ R^d_{n_r l} = \sqrt{(n_r+1)\left(n_r + l + \frac{d}{2}\right)} R^d_{n_r+1 l}, \quad (17)$$

$$K_- R^d_{n_r l} = \sqrt{n_r\left(n_r + l + \frac{d}{2} - 1\right)} R^d_{n_r-1 l}, \quad (18)$$

$$K_0 R^d_{n_r l} = \left(n_r + \frac{l}{2} + \frac{d}{4}\right) R^d_{n_r l}. \quad (19)$$

By comparing Eqs. (7), (9) with Eqs. (17)–(19) we find that $k = l/2 + d/4$ and m is actually the quantum number n_r. In order to obtain k in a rigorous way, rather than comparing equations, we use the Casimir operator.

The Casimir operator for this group is [17]

$$\hat{C} = K_0^2 - K_1^2 - K_2^2 \quad (20)$$
$$= K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+)$$

and its eigenvalues are $k(k-1)$.

Calculating the Casimir eigenvalue in our case we obtain

$$\hat{C} R^d_{n_r l}(r) = \left(\frac{l}{2} + \frac{d}{4}\right)\left(\frac{l}{2} + \frac{d}{4} - 1\right) R^d_{n_r l}(r). \quad (21)$$

Hence, we have two possible solutions $k = l/2 + d/4$ or $k = -(l/2 + d/4 - 1)$. We are interested only in the positive discrete representations of $SO(2,1)$, $\mathcal{D}^+(k)$; hence,

$$k = \frac{l}{2} + \frac{d}{4}. \quad (22)$$

3. RADIAL GROUP FOR THE HYDROGEN ATOM

Unlike the case of the IHO, the Coulomb problem Hamiltonian cannot be expressed in terms of the $SO(2,1)$ group generators. The group $SO(2,1)$ is related to the radial group or spectrum generating group for the Coulomb problem, when its basis functions are taken as the Sturmian functions which are related to the three-dimensional radial Coulomb functions $R_{N_r L}(r)$ by a tilting (squeezing) transformation [16, 20, 22]. In order to generalize this relation for the D-dimensional case, we begin by generalizing the Sturm basis given in [13] to D dimensions,

$$S^D_{N_r L}(\rho) = 2\sqrt{\frac{\Gamma(N_r+1)}{\Gamma(N_r+2L+D-1)}} \quad (23)$$
$$\times (2\rho)^l e^{-\rho} L^{2L+D-2}_{N_r}(2\rho).$$

These functions satisfy the orthonormality condition (note the measure ρ^{D-2})

$$\int_0^\infty d\rho S^D_{N_r L}(\rho) S^D_{N'_r L}(\rho)\rho^{D-2} = \delta_{N_r, N'_r}. \quad (24)$$

As in the IHO case, we would like to construct for this basis the relevant realization of raising and lowering operators. We use again the Laguerre recursion relations and Eq. (14) to obtain

$$K^{\mathrm{C}}_\pm = \pm\rho\partial_\rho - \rho + K^{\mathrm{C}}_0 \pm \frac{D-1}{2}, \quad (25)$$

where the superscript "C" stands for "Coulomb" and where

$$K^{\mathrm{C}}_0 = -\frac{1}{2}\rho\left(\partial^2_\rho + \frac{D-1}{\rho}\partial_\rho - \frac{L(L+D-2)}{\rho^2} - 1\right), \quad (26)$$

satisfying the eigenvalue equation

$$K^{\mathrm{C}}_0 S^D_{N_r L} = \left[N_r + L + \frac{1}{2}(D-1)\right] S^D_{N_r L}. \quad (27)$$

K^{C}_+ and K^{C}_- act on the group basis as follows:

$$K^{\mathrm{C}}_+ S^D_{N_r L} = \sqrt{(N_r+1)(N_r+2L+D-1)} S^D_{N_r+1 L}, \quad (28)$$

$$K^{\mathrm{C}}_- S^D_{N_r L} = \sqrt{N_r(N_r+2L+D-2)} S^D_{N_r-1 L}. \quad (29)$$

From $K^{\mathrm{C}}_\pm = K^{\mathrm{C}}_1 \pm iK^{\mathrm{C}}_2$, we obtain

$$K^{\mathrm{C}}_1 = K^{\mathrm{C}}_0 - \rho, \quad (30)$$

$$K^{\mathrm{C}}_2 = -i\left(\rho\partial_\rho + \frac{D-1}{2}\right), \quad (31)$$

where K^{C}_2 is the D-dimensional tilting, squeezing, or dilatation generator. Taking $D = 3$, we obtain the realization of [13, 20].

The operators K^{C}_+ and K^{C}_- are adjoint operators with respect to the $S^D_{N_r L}$ basis, with a measure ρ^{D-2} (note the difference from K_+ and K_- of Section 2, which are adjoint with respect to $R^d_{n_r l}$ with measure r^{d-1}). Therefore, under this condition, K^{C}_1 and K^{C}_2 are Hermitian operators.

In order to make the connection between the Sturmian functions and the radial Coulomb functions in D dimensions, we write the D-dimensional radial Schrödinger equation for the hydrogen atom (in atomic units):

$$(H^{\mathrm{C}} - E) R^D_{N_r L}(\rho) = 0, \quad (32)$$

where

$$H^{\mathrm{C}} = -\frac{1}{2}\left(\partial_\rho^2 + \frac{D-1}{\rho}\partial_\rho - \frac{l(l+D-2)}{\rho^2} + \frac{2Z}{\rho}\right). \tag{33}$$

Upon left-multiplying Eq. (32) by ρ, the equation becomes a pseudoeigenvalue equation for Z,

$$(\tilde{H^{\mathrm{C}}} - Z)R^D_{N_rL}(\rho) = 0, \tag{34}$$

where $\tilde{H}^{\mathrm{C}}$ is the pseudo-Hamiltonian which can be expressed in terms of the group generators,

$$\tilde{H}^{\mathrm{C}} = \left(\frac{1}{2} - E\right)K_0^{\mathrm{C}} + \left(\frac{1}{2} + E\right)K_1^{\mathrm{C}}. \tag{35}$$

Since neither K_0^{C} nor K_1^{C} is diagonal in the $R^D_{N_rL}(\rho)$ basis, we use instead the $S^D_{N_rL}(\rho)$ basis, related to the $R^D_{N_rL}(\rho)$ by the tilting transformation [16, 22]

$$R^D_{N_rL}(\rho) = Ce^{i\theta K_2^{\mathrm{C}}}S^D_{N_rL}(\rho), \tag{36}$$

where C is the normalization constant.

Multiplying Eq. (34) from the left by $e^{-i\theta K_2^{\mathrm{C}}}$, we obtain

$$(\mathcal{H} - Z)S^D_{N_rL}(\rho) = 0, \tag{37}$$

where

$$\mathcal{H} \equiv e^{-i\theta K_2^{\mathrm{C}}}\tilde{H}^{\mathrm{C}}e^{i\theta K_2^{\mathrm{C}}} \tag{38}$$
$$= \left[\left(\frac{1}{2} + E\right)\cosh\theta + \left(\frac{1}{2} - E\right)\sinh\theta\right]K_1^{\mathrm{C}}$$
$$+ \left[\left(\frac{1}{2} + E\right)\sinh\theta + \left(\frac{1}{2} - E\right)\cosh\theta\right]K_0^{\mathrm{C}}.$$

The right-hand side of this equation was obtained by performing the similarity transformation on both the compact generator K_0^{C} and the noncompact one K_1^{C}.

Obviously, by a proper choice of θ, it is possible to eliminate either K_0^{C} or K_1^{C}. However, eliminating K_0^{C} will yield the continuous part of the spectrum [16, 22]. In order to obtain the discrete part of the spectrum, we would like to eliminate K_1^{C}. Taking $\theta = \frac{1}{2}\ln(-2E)$, we obtain

$$\mathcal{H} = \sqrt{-2E}K_0^{\mathrm{C}}. \tag{39}$$

Remembering Eq. (27), we obtain the D-dimensional bound-state energy spectrum

$$E = E_N = -\frac{Z^2}{2\left[N + (D-3)/2\right]^2}, \tag{40}$$

where N is the principal quantum number $N = N_r + L + 1$. Thus, the expression for θ is actually N-dependent.

To obtain the normalization constant C and the radial functions, we use the fact that, while the Sturmians are orthonormal with respect to ρ^{D-2}, the radial functions are orthonormal with respect to ρ^{D-1}. Since $\rho = K_0^{\mathrm{C}} - K_1^{\mathrm{C}}$, we require the following condition (the bracket represents the Sturmian measure ρ^{D-2}):

$$\int_0^\infty d\rho\rho^{D-1}|R^D_{N_rL}|^2 \tag{41}$$
$$\equiv \langle R^D_{N_rL}|K_0^{\mathrm{C}} - K_1^{\mathrm{C}}|R^D_{N_rL}\rangle = 1$$
$$= C^2e^{-\theta}\langle S^D_{N_rL}|K_0^{\mathrm{C}} - K_1^{\mathrm{C}}|S^D_{N_rL}\rangle,$$

where we used Eq. (36). Using Eqs. (27)–(29), we obtain $C = C_N = \sqrt{Z/M^2}$, where $M = N + (D-3)/2$.

In Eq. (36), we made the connection between the Sturmians and the physical radial functions, so now we are able to write the expression for $R^D_{N_rL}(\rho)$:

$$R^D_{NL}(\rho) = \sqrt{\frac{Z^D}{M^{D+1}}}S^D_{NL}\left(\frac{2Z\rho}{M}\right) \tag{42}$$
$$= 2\sqrt{\frac{Z^D}{M^{D+1}}}\sqrt{\frac{\Gamma(N-L)}{\Gamma(N+L+D-2)}}\left(\frac{2Z\rho}{M}\right)^l$$
$$\times e^{-Z\rho/M}L^{2L+D-2}_{N-L-1}\left(\frac{2Z\rho}{M}\right),$$

where we have used the property of the dilatation operator

$$\exp\left[\ln\theta\rho\partial_\rho\right]f(\rho) = f(\theta\rho). \tag{43}$$

In a similar way to Section 2, we obtain the Casimir eigenvalue with respect to the Sturm functions basis

$$k = L + \frac{D-1}{2}. \tag{44}$$

4. RADIAL FUNCTION MAPPING

Since the IHO radial functions and the Sturm functions are two different representations of $SO(2,1)$ basis functions, there are certain mapping conditions which transform the IHO radial functions into the Sturm functions.

By comparing the Casimir eigenvalues, Eqs. (22) and (44), we obtain the most general relation between those two representations:

$$d = 2D - 2 - 2\lambda, \tag{45}$$
$$\lambda = l - 2L.$$

This map implies that there is a relation between the D-dimensional Coulomb radial states (42) and even-dimensional IHO radial states (10).

To be more specific, D and λ are integers and therefore $d \geq 2$. This implies the condition $\lambda \leq D-2$. The fact that the eigenvalues in Eqs. (19) and (27) should coincide and using Eq. (45), yields another condition

$$n = 2N - 2 + \lambda, \tag{46}$$

where n and N are the principal quantum numbers for the IHO and the hydrogen atom, respectively. Recalling that $N = N_r + L + 1 \geq 1$ and that $n = 2n_r + l \geq 0$ implies that, if we wish to map the hydrogenic states into the IHO states starting from the ground state, then $\lambda \geq 0$. Taking, for example, the case $\lambda = 0$ with $L = 0$ and $l = 0$, we see that there is a mapping between the hydrogenic ground state $N = 1$ and the oscillator ground state $n = 0$. The next states which map are $N = 2$ and $n = 2$, etc. Taking $\lambda = -1$, the minimum allowed angular-momentum values are $L = 1$ and $l = 1$ (recall that $L \geq 0$ and $l \geq 0$) and therefore the lower states which map are the $N = 2$ state of the hydrogen atom and $n = 1$ of the oscillator. The ground states are excluded. As λ is more negative, more states are excluded. Since we wish to map all the hydrogenic states including the ground state, we take the lower bound of λ to be zero. Therefore,

$$0 \leq \lambda \leq D - 2. \tag{47}$$

Considering the three-dimensional Coulomb problem, $D = 3$, the valid values for λ are either $\lambda = 0$ or $\lambda = 1$. Taking first $\lambda = 1$ we obtain $l = 2L + 1$ and $d = 2$. Therefore, in this case, the two-dimensional IHO states with odd principal quantum number are mapped into the three-dimensional Coulomb radial states. Taking $\lambda = 0$, we obtain $d = 4$ and $l = 2L$; therefore, the four-dimensional IHO states with even principal quantum number are mapped into the three-dimensional hydrogenic states.

In order to find the coordinate mapping [for any D and d related by Eq. (45)], we compare Eq. (25) with Eq. (15) to obtain

$$\rho = \frac{1}{2} r^2. \tag{48}$$

Mapping the IHO radial states into the Coulomb states, we should take care of the normalization constant. This is done by multiplying the IHO states by the factor $2\sqrt{Z^D/M^{D+1}}$. This factor comes from the fact that the IHO radial states and the Sturm functions are normalized with respect to a different measure [cf. Eq. (41)], and from the tilting operation (36) which takes the Sturm functions into the physical Coulomb radial functions.

The relations between these two systems were discussed by many other authors (e.g., [23–27]).

5. $SO(2,1)$ BARUT–GIRARDELLO RADIAL COHERENT STATES FOR THE IHO

To calculate this set of radial states for the IHO, we use the BG formalism for noncompact Lie groups [28]. These states are the eigenfunctions of the lowering operator K_- with complex eigenvalues α:

$$K_-|\alpha, k\rangle_{\mathrm{BG}} = \alpha|\alpha, k\rangle_{\mathrm{BG}}. \tag{49}$$

Expanding $|\alpha, k\rangle$ in terms of the $R^d_{n_r l}(r)$ and using Eqs. (18) and (49), we obtain

$$\begin{aligned}\langle r|\alpha, k\rangle_{\mathrm{BG}} &= \frac{\alpha^{k-1/2}}{\sqrt{I_{2k-1}(2|\alpha|)}} \\ &\times \sum_{n_r=0}^{\infty} \frac{\alpha^{n_r}}{\sqrt{\Gamma(n_r+1)\Gamma(n_r+2k)}} R^d_{n_r l}(r) \\ &= \sqrt{\frac{2}{I_{2k-1}(2|\alpha|)}} e^{\alpha} r^{1-d/2} e^{-r^2/2} J_{2k-1}(2\sqrt{\alpha r^2}),\end{aligned} \tag{50}$$

where J_l is the Bessel function of the first kind, I_l is the modified Bessel function of the first kind [21], and $k = l/2 + d/4$. The sum was calculated with the help of the Laguerre generating function [29]

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\alpha^n}{\Gamma(n+z+1)} L_n^z(r) \\ = J_z(2\sqrt{r\alpha}) e^{\alpha} (r\alpha)^{-z/2}.\end{aligned} \tag{51}$$

For $d = 1$, we obtain the result of Agarwal and Chaturvedi [30] for the Calogero–Sutherland oscillator.

For any value of k, these states are nonorthogonal, satisfying

$$\langle \alpha_1, k|\alpha_2, k\rangle = \frac{I_{2k-1}(2\sqrt{\bar{\alpha_1}\alpha_2})}{\sqrt{I_{2k-1}(2|\bar{\alpha_1}|) I_{2k-1}(2|\alpha_2|)}}. \tag{52}$$

Also, they satisfy the completeness relation

$$\int d\mu(\alpha, k)|\alpha, k\rangle\langle\alpha, k| = I, \tag{53}$$

where

$$d\mu(\alpha, k) = \frac{2}{\pi} K_{2k-1}(2|\alpha|) I_{2k-1}(2|\alpha|) d^2\alpha \tag{54}$$

is the measure which enables the resolution of the identity (K_ν is the modified Bessel function of the second kind [21]).

Useful expectation values with respect to the BG coherent states are given by

$$\langle K_1\rangle_{\mathrm{BG}} = \mathrm{Re}\alpha, \tag{55}$$

$$\langle K_2\rangle_{\mathrm{BG}} = \mathrm{Im}\alpha, \tag{56}$$

where we used the fact that $\langle K_-\rangle_{\mathrm{BG}} = \alpha$ and $\langle K_+\rangle_{\mathrm{BG}} = \bar{\alpha}$.

6. BARUT–GIRARDELLO STATES FOR THE HYDROGEN ATOM

Here, we discuss only the three-dimensional case. Hence, all the operators in this section are the three-dimensional forms of the operators in Section 3. We follow the same notation. The generalization to any D is easy using the formalism developed in previous sections.

Following the mapping from Section 4, we take Eq. (50) with $d = 4$ and $l = 2L$, consistent with the case $\lambda = 0$, to obtain

$$\langle \rho|\alpha, L\rangle_{\rm BG} = \sqrt{\frac{2}{I_{2L+1}(2|\alpha|)}} \times e^{\alpha}\frac{1}{\sqrt{\rho}}e^{-\rho}J_{2L+1}(2\sqrt{2\rho\alpha}), \tag{57}$$

where we mapped the coordinates according to (48) and multiplied the solution by the factor $\sqrt{2}$ to adjust the normalization. The measure for normalization here is ρ.

Obviously, Eq. (57) is a generating function for the three-dimensional Sturm functions ($S^{D=3}_{N_rL}$) and an eigenfunction of $K^{\rm C}_{-}$ with eigenvalue α.

Recalling Eq. (37), the expectation value of the pseudo-Hamiltonian with respect to the Sturm functions is

$$\langle S^D_{N_rL}|\mathcal{H}|S^D_{N_rL}\rangle = Z. \tag{58}$$

We require the same expectation value for $\mathcal{H}$ with respect to the BG coherent states basis,

$$\langle \alpha, L|\mathcal{H}|\alpha, L\rangle_{\rm BG} = \sqrt{-2E}\langle \alpha, L|K^{\rm C}_0|\alpha, L\rangle_{\rm BG} = Z. \tag{59}$$

Using the expectation value of $K^{\rm C}_0$ in the $|\alpha, L\rangle_{\rm BG}$ basis

$$\langle K^{\rm C}_0\rangle = |\alpha|\frac{I_{2L+2}(2|\alpha|)}{I_{2L+1}(2|\alpha|)} + L + 1, \tag{60}$$

we easily determine E

$$E = -\frac{Z^2}{2\langle K^{\rm C}_0\rangle^2} = -\frac{Z^2 I_{2L+1}(2|\alpha|)^2}{2\left[|\alpha| I_{2L+2}(2|\alpha|) + (L+1)I_{2L+1}(2|\alpha|)\right]^2}. \tag{61}$$

Similarly to [13, 31], we define a tilted coherent state as

$$|\widetilde{\alpha, L}\rangle_{\rm BG} = \tilde{C}e^{i\theta K^{\rm C}_2}|\alpha, L\rangle_{\rm BG}, \tag{62}$$

where $\theta = \ln\sqrt{-2E}$ and E is given by Eq. (61). The tilted state obeys the following energy relation:

$$\langle\widetilde{\alpha, L}|H^{\rm C}|\widetilde{\alpha, L}\rangle_{\rm BG}/\langle\widetilde{\alpha, L}|\widetilde{\alpha, L}\rangle_{\rm BG} = Z = \langle \alpha, L|\mathcal{H}|\alpha, L\rangle_{\rm BG}. \tag{63}$$

The normalization measure for the tilted states should be ρ^2; therefore, in order to determine $\tilde{C}$, we use the normalization requirement $\langle\widetilde{\alpha, L}|\rho|\widetilde{\alpha, L}\rangle_{\rm BG} = 1$ (the bracket denotes the measure ρ),

$$1 = \langle\widetilde{\alpha, L}|K^{\rm C}_0 - K^{\rm C}_1|\widetilde{\alpha, L}\rangle_{\rm BG} = \tilde{C}^2 e^{-\theta}\langle \alpha, L|K^{\rm C}_0 - K^{\rm C}_1|\alpha, L\rangle_{\rm BG}. \tag{64}$$

Thus,

$$\tilde{C} = \frac{(-2E)^{1/4}}{\sqrt{\langle K^{\rm C}_0 - K^{\rm C}_1\rangle_{\rm BG}}}, \tag{65}$$

and using Eqs. (55) and (60), we obtain

$$\langle \alpha, L|K^{\rm C}_0 - K^{\rm C}_1|\alpha, L\rangle_{\rm BG} = |\alpha|\frac{I_{2L+2}(2|\alpha|)}{I_{2L+1}(2|\alpha|)} + L + 1 - {\rm Re}(\alpha). \tag{66}$$

Finally, we get the expression for the radial BG coherent states for the hydrogen atom

$$\langle\rho|\widetilde{\alpha, L}\rangle_{\rm BG} = \frac{(-2E)^{1/2}}{\sqrt{\langle K^{\rm C}_0 - K^{\rm C}_1\rangle_{\rm BG}}}\sqrt{\frac{2}{I_{2L+1}(2|\alpha|)}}e^{\alpha}\rho^{-1/2} \times \exp^{-\sqrt{-2E}\rho} J_{2L+1}\left(2\sqrt{2\sqrt{-2E}\rho\alpha}\right). \tag{67}$$

This set of coherent states is different from the solution obtained in [13] using Perelomov's method.

7. TIME EVOLUTION

We consider the resolvent operator for the Coulomb problem using the pseudo-Hamiltonian $\mathcal{H} = \sqrt{-2E}K^{\rm C}_0$ [E is given by (61)]

$$G(Z) = \frac{i}{Z - \mathcal{H}} = \int_0^\infty e^{iZ\tau}U(\tau)d\tau, \tag{68}$$

where $U(\tau)$ is the corresponding evolution operator

$$U(\tau) = \exp(-i\tau\mathcal{H}) \tag{69}$$

with respect to the new parameter τ which can be treated as a fictitious time and was discussed extensively in [7, 13, 32].

Acting with the evolution operator on the state $|\widetilde{\alpha, L}\rangle_{\rm BG}$, we obtain

$$e^{-2i\omega K^{\rm C}_0\tau}|\widetilde{\alpha, L}\rangle_{\rm BG} = e^{-2i\omega(L+1)\tau}|\widetilde{\alpha(\tau), L}\rangle_{\rm BG}, \tag{70}$$

where

$$\alpha(\tau) = \alpha_0 e^{-2i\omega\tau}, \quad \alpha_0 = \alpha(0), \tag{71}$$

and $\omega = \sqrt{-E/2}$.

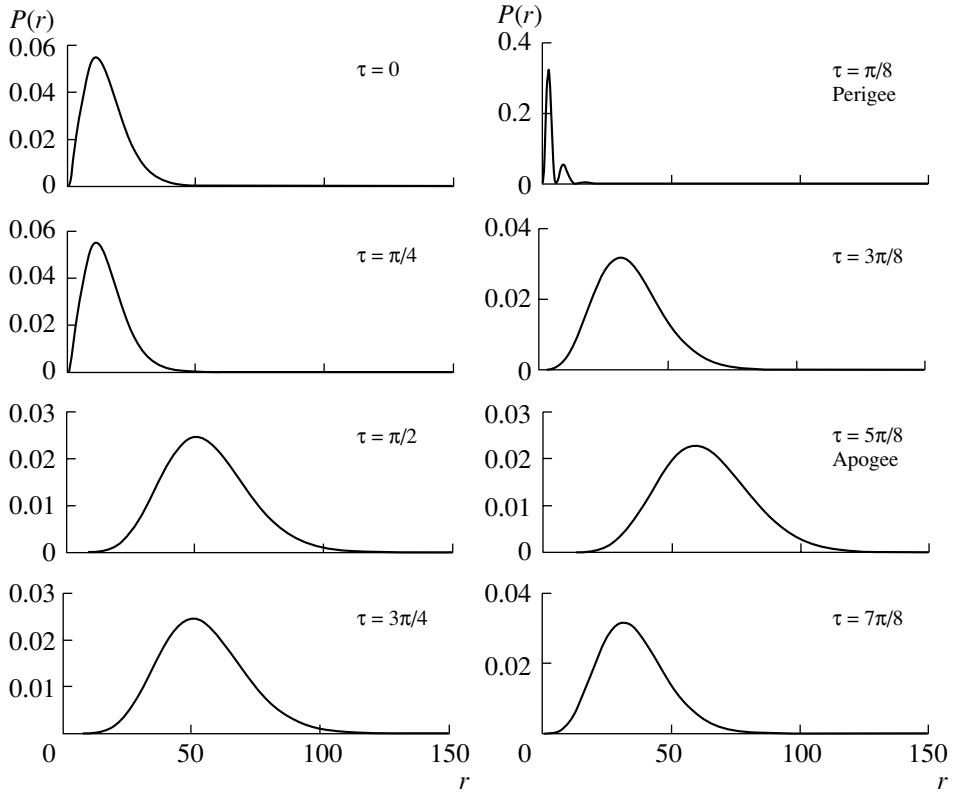

Fig. 1. Time evolution of BG coherent states for the Coulomb problem for $L = 1$ and $|\alpha| = 5$ (see discussion in the text).

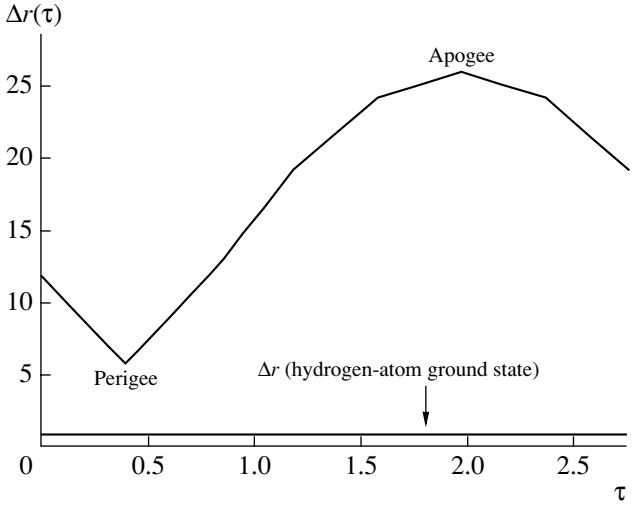

Fig. 2. Time evolution of BG coherent states for the Coulomb problem. Δr as function of τ for $L = 5$ and $|\alpha| = 5$ [$\Delta r = \sqrt{\langle r^2\rangle - \langle r\rangle^2}$, where the expectation values are calculated with respect to the BG coherent-state functions (67)].

Therefore, the explicit expression for the fictitious time dependence of the BG coherent states (67) is

$$\langle\rho|\widetilde{\alpha, L, \tau}\rangle_{\mathrm{BG}} = e^{-i\omega(L+1)\tau}\sqrt{\frac{-2E}{\langle K_0^{\mathrm{C}} - K_1^{\mathrm{C}}\rangle_{\mathrm{BG}}}} \quad (72)$$

$$\times \sqrt{\frac{2}{I_{2L+1}(2|\alpha|)}}e^{\alpha(\tau)}\rho^{-1/2}$$

$$\times \exp^{-\sqrt{-2E}\rho} J_{2L+1}\left(2\sqrt{2\sqrt{-2E}\rho\alpha(\tau)}\right).$$

Clearly, this function is periodic in τ.

Relation (63) is satisfied also by the time-dependent coherent states

$$\widetilde{\langle\alpha,L,\tau}|\tilde{H}^{\mathrm{C}}|\widetilde{\alpha,L,\tau\rangle}_{\mathrm{BG}}/\widetilde{\langle\alpha,L,\tau}|\widetilde{\alpha,L,\tau\rangle}_{\mathrm{BG}} \quad (73)$$
$$= Z = \langle\alpha,L,\tau|\mathcal{H}|\alpha,L,\tau\rangle_{\mathrm{BG}}.$$

In order to explore the fictitious time-dependence behavior of this function, we consider the radial probability distribution $P(\rho,\alpha,\tau) = |\langle\rho|\alpha,\tilde{L},\tau\rangle_{\mathrm{BG}}|^2\rho^2$.

As shown in Fig. 1, the distribution oscillates between two turning points, r_p, which corresponds to the perigee, where at that point the function becomes narrow (at its minimum width), and r_a, which corresponds to the apogee, where at that point the function is at its maximum width. Although the distribution changes its width, it is not dispersing, so in this manner this wave packet behaves classically. Similar behavior was observed before for the Perelomov coherent states [13, 33]. This motion of the wave packet between two turning points is actually analogous to a bounded motion of a classical particle in the effective Coulomb potential.

By comparing the width of the distribution Δr_{BG} to that of the hydrogen-atom ground state, we observe that, although the distribution becomes narrow at the perigee, we always have $\Delta r_{\mathrm{BG}} > \Delta r_{\mathrm{g.s}}$, where $\Delta r_{\mathrm{g.s}}$ corresponds to the hydrogen atom ground state. This is true for any value of L; in Fig. 2, it is shown for $L = 5$.

As shown in the table, keeping the same value of $|\alpha|$ and taking larger values of L, we find that, as L increases, the orbit becomes more circular (less elliptic). As the orbit becomes more circular, the width of the distribution increases, and the perigee and the apogee become more distant from the center (we get higher values of r_p and r_a). In the limit $L \gg 1$, the motion is circular, the distribution does not change its shape as it evolves with fictitious time, and it is located at one value of r which is the radius of the circular motion.

Performing the calculations with different values of $|\alpha|$ and the same value of L shows that $|\alpha|$ changes the eccentricity (ε) of the orbit: for larger values of $|\alpha|$, the orbit is more elliptic, while being more circular for small values of $|\alpha|$.

The orbit of the Coulomb BG coherent states as function of L, for $|\alpha| = 5$ [Δr_p and Δr_a correspond to the width of the distribution at the perigee and the apogee, respectively; r_p (r_a) corresponds to the maximum of the distribution at the perigee (apogee); ε is the eccentricity of the orbit, $\varepsilon = 1 - 2r_p/(r_a + r_p)$]

L	1	2	5	10
r_p	0.36	0.79	2.68	6.82
Δr_p	2.82	3.05	5.75	17.69
r_a	9.73	10.14	12.1	16.42
Δr_a	17.89	19.23	26	53.43
ε	0.93	0.86	0.64	0.41

8. SUMMARY

In this paper, we constructed arbitrary dimension realizations of $SO(2,1)$ algebra for the harmonic oscillator and the hydrogen atom. We constructed Barut–Girardello radial coherent states for the IHO associated with this algebra and mapped them into the Coulomb problem coherent states. We showed that these states evolve periodically in the fictitious time parameter and change their shape as they move between the two turning points of an elliptic orbit. Although they change their shape, they do not disperse. In the limit of high angular momentum or $|\alpha| \ll 1$, the orbit of the BG coherent states becomes circular. It was shown before [34, 35] that, considering the Coulomb radial distributions, classical circular orbits occur for the maximum value of L ($L = N - 1$) and large principal quantum number. It was shown also that the width of the distribution diverges in these limits. Thus, the BG coherent states behave similarly in the classical limit $L \gg 1$. However, it is interesting to explore whether the second set of coherent states which corresponds to $SO(2,1) \sim SU(1,1)$, Perelomov coherent states, possesses the same behavior. It is interesting also to explore how these behaviors agree with the suggestion by Zlatev, Zhang, and Feng [12] mentioned earlier.

Radial Coulomb states (Rydberg states) have been investigated both theoretically and experimentally for a long time (e.g., [36, 37]). However, these states disperse and show wave packet revivals, which is a quantum phenomenon [38–42]. A question which should be asked is whether it is possible to produce in the laboratory radial coherent states with classical properties or whether one should consider the BG coherent states we constructed as only a tool to discuss quantum-classical correspondence.

Finally, we mention that this method of using spectrum generating algebras to construct coherent states can be extended to other models (e.g., the Morse potential whose SGA is $SO(2,1) \sim SU(1,1)$ [43]).

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Quantum Oscillator as 1D Anyon*

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Abstract—It is shown that, in one spatial dimension, the quantum oscillator is dual to the charged particle situated in the field described by the superposition of Coulomb and Calogero–Sutherland potentials.

1. INTRODUCTION

In one spatial dimension, a particle moving in the Calogero–Sutherland potential $V_{\rm CS} = -\hbar^2\nu(1-\nu)/2\mu x^2$ has a very unusual property. Unlike the potential $V_{\rm CS}$, the wave function is not invariant under the replacement $\nu \to (1-\nu)$. It describes a boson for even ν and a fermion for odd ν. Statistics corresponding to the other values of ν is called fractional statistics [1], and the system influenced along with $V_{\rm CS}$ by a potential binding the particle to the center is called the 1D anyon [2–4]. Nobody has observed a 1D anyon yet, but nevertheless it is of both theoretical [5] and experimental [6] interest. The purpose of the present note is to prove that such an extraordinary object can be constructed from a 1D quantum oscillator.

2. ANYON–OSCILLATOR DUALITY

Consider the Schrödinger equation

$$\partial_u^2\Psi + \frac{2\mu}{\hbar^2}\left(E - \frac{\mu\omega^2u^2}{2}\right)\Psi = 0, \qquad (1)$$

which describes the 1D quantum oscillator. Introduce the quantum number $s = 0, 1/2$ and write $N = 2n + 2s$, with N numerating the energy levels $E = \hbar\omega(N + 1/2)$ and n being integer and nonnegative. Without loss of information we can assume u to belong to the region $0 \le u < \infty$. We interpret s as a spin of the reduced oscillator. The corresponding wave function is denoted by $\Psi_n^{(s)}$.

Let us look for the function $\Psi_n^{(s)}$ in the form

$$\Psi_n^{(s)}(u) = Cu^{2s}\bar\Psi_n, \qquad (2)$$

where $\bar\Psi_n$ is subordinate to the condition $\bar\Psi_n(0) \ne 0$, and C is a normalization constant. Equation (1) is easily seen to take the form

$$\partial_u^2\bar\Psi_n + \frac{4s}{u}\frac{\partial}{\partial u}\bar\Psi_n + \frac{2\mu}{\hbar^2}\left(E - \frac{\mu\omega^2u^2}{2}\right)\bar\Psi_n = 0. \qquad (3)$$

After change of the variable $x = u^2$, we arrive at the equation ($2\nu = 2s + 1/2$)

$$\partial_x^2\bar\Psi_n + \frac{2\nu}{x}\frac{\partial}{\partial x}\bar\Psi_n + \frac{2\mu}{\hbar^2}\left(-\frac{\mu\omega^2}{8} + \frac{E}{4x}\right)\bar\Psi_n = 0. \qquad (4)$$

Now we set

$$\bar\Psi_n = x^{-\nu}\Phi_n^{(\nu)}, \qquad (5)$$

then cancel the undesirable term with first derivative in (4) and obtain

$$\partial_x^2\Phi_n^{(\nu)} + \frac{2\mu}{\hbar^2}\left(\varepsilon - V_c - V_{\rm CS}\right)\Phi_n^{(\nu)} = 0, \qquad (6)$$

where $V_c = -\alpha/x$, $V_{\rm CS}$ is the Calogero–Sutherland potential with $\nu = 1/4$ or $3/4$ and

$$\varepsilon = -\frac{\mu\omega^2}{8}, \quad \alpha = \frac{E}{4}. \qquad (7)$$

Equation (6) describes a system which we call the 1D Coulomb anyon.

This equation realizes a special case of a more general equation that has a relation to $(2+1)$-dimensional anyons [7].

Comparing Eq. (1) with Eqs. (6) and (7), we summarize that there are two alternative possibilities connected with Eq. (1)—explicit and hidden. In the first case, the parameter ω is fixed ($\omega = \text{fix.} > 0$) and plays a role of a coupling constant, the parameter E is

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quantized and has a meaning of energy, and the system is a 1D quantum oscillator. For the hidden possibility, the parameter E is fixed ($E = \text{fix.} > 0$), the coupling constant is equal to $E/4$, ω is quantized, the value of energy takes the quantity $\varepsilon = -\mu\omega^2/8$, and the system is the 1D Coulomb anyon. In the abovementioned sense, the 1D quantum oscillator is dual to the 1D Coulomb anyon.

3. ENERGY LEVELS AND WAVE FUNCTIONS

Let us return to Eq. (6) and make the substitution

$$\Phi_n^{(\nu)} = y^\nu e^{-y/2} Q(y), \tag{8}$$

where $y = x(-8\mu\varepsilon/\hbar^2)^{1/2}$ and $Q(0) \neq 0$ and is finite. The function $Q(y)$ can diverge at infinity but not higher than a finite power of y. Using (8) and (6), we come to the equation

$$y\partial_y^2 Q + (2\nu - y)\frac{\partial}{\partial y}Q - (\nu - \lambda)Q = 0, \tag{9}$$

with $\lambda = (-\mu\alpha^2/2\hbar^2\varepsilon)^{1/2}$. Equation (9) is the equation for a confluent hypergeometric function, it has a general solution [8]

$$Q(y) = C_1 F(\nu - \lambda, 2\nu, y) + C_2 y^{1-2\nu} F(1 - \lambda - \nu, 2 - 2\nu, y) \tag{10}$$

convergent for all finite y. For large y, the asymptotic formula [8] is valid,

$$F(a, b, y) \sim \frac{\Gamma(b)}{\Gamma(b-a)}(-y)^{-a} + \frac{\Gamma(b)}{\Gamma(a)} e^y (y)^{a-b}. \tag{11}$$

The second term in (10) for $\nu = 3/4$ is singular at $y = 0$, and hence C_2 has to be taken equal to zero. The first term in (10), as is evident from (11), is "well-behaved" at infinity under the condition $3/4 - \lambda = -n$, where n is an integer number greater than or equal to zero. For $\nu = 1/4$, both terms in (10) are regular at $y = 0$, but the satisfactory behavior at infinity needs the simultaneous requirements $1/4 - \lambda = -n, 3/4 - \lambda = -m$, or $n - m = 1/2$, which is impossible. Hence, either $C_1 = 0$ or $C_2 = 0$. But for $C_1 = 0$, the function $Q(y)$ will become zero at $y = 0$. This contradicts the condition $Q(0) \neq 0$ and, therefore, we set $C_2 = 0$ and $1/4 - \lambda = -n$. Thus, we conclude that $\nu - \lambda = -n$, i.e.,

$$\varepsilon_n^{(\nu)} = -\frac{\mu\alpha^2}{2\hbar^2(n+\nu)^2}, \quad n = 0, 1, 2, \ldots. \tag{12}$$

Returning to the corresponding eigenfunctions, we set

$$\Phi_n^{(\nu)} = C_n^{(\nu)} y^\nu e^{-y/2} F(-n, 2\nu, y). \tag{13}$$

It is known [9] that

$$F(-n, 2\nu, y) = \frac{n!\Gamma(2\nu)}{[\Gamma(n+2\nu)]^2} L_n^{2\nu-1}(y)$$

where $L_n^{2\nu-1}(y)$ is an associated Laguerre polynomial. Using the integration properties of Laguerre polynomials and taking into account the relation

$$\left(-\frac{8\mu\varepsilon}{\hbar^2}\right)^{1/4} = \frac{1}{\hbar}\left(\frac{2\mu\alpha}{n+\nu}\right)^{1/2},$$

we find the normalization constant $C^{(\nu)}$ and summarize that

$$\Phi_n^{(\nu)} = \frac{\sqrt{\mu\alpha}}{\hbar}\frac{1}{n+\nu}\frac{1}{\Gamma(2\nu)} \times \sqrt{\frac{\Gamma(n+2\nu)}{n!}} y^\nu e^{-y/2} F(-n, 2\nu, y). \tag{14}$$

Thus, we have two types of 1D Coulomb anyons with $\nu = 1/4$ and $\nu = 3/4$. They are dual to reduced oscillators with $s = 0$ and $s = 1/2$, respectively.

4. DUALITY FOR SOLUTIONS

Now we will calculate the energy levels ε_n and wave functions $\Phi_n^{(\nu)}$ in another, more straightforward, way. For energy levels, we have

$$\varepsilon = -\frac{\mu\omega^2}{8} = -\frac{\mu}{8}\left[\frac{E}{2\hbar(n+\nu)}\right]^2 = -\frac{\mu}{8}\left[\frac{2\alpha}{\hbar(n+\nu)}\right]^2 = -\frac{\mu\alpha^2}{2\hbar^2(n+\nu)^2}.$$

It follows from Eqs. (2) and (5) that

$$\Phi_n^{(\nu)} = \frac{1}{C} x^{1/4} \Psi_n^{(\nu)}$$

and, therefore,

$$1 = \int_0^\infty |\Phi_n^{(\nu)}|^2 dx = \frac{1}{|C|^2}\int_0^\infty x^{1/2}|\Psi_n^{(s)}|^2 dx,$$

where Ψ_n is the normalized wave function of a 1D quantum oscillator. Thus,

$$\Phi_n^{(\nu)} = \frac{(-1)^n}{2}\sqrt{\frac{\mu\omega}{\hbar(n+\nu)}} x^{1/4}\Psi_n^{(s)}. \tag{15}$$

Recall that, according to the theory of quantum oscillator [9],

$$\Psi_n^{(s)} = \sqrt{2}\left(\frac{\mu\omega}{\pi\hbar}\right)^{1/4}\frac{1}{2^N N!} e^{-\mu\omega u^2/2} H_N\left(u\sqrt{\frac{\mu\omega}{\hbar}}\right). \tag{16}$$

Further, it is known [10] that Hermite polynomials could be expressed in terms of confluent hypergeometric functions. For our case,

$$H_{2n+2s}(\sqrt{y}) = (-1)^n \frac{(2n+2s)!}{n!} \times (2\sqrt{y})^{2s} F(-n, 2s+1/2, y). \quad (17)$$

Using the identification $y = x\mu\omega/\hbar$ and the relations $2s + 1/2 = 2\nu$ and $\mu\omega/\hbar = 2\mu\alpha/\hbar^2(n+\nu)$ and taking into account Eqs. (15)–(17), we get

$$\Phi_n^{(\nu)} = \tilde{C}_n^{(\nu)} y^\nu e^{-y/2} F(-n, 2\nu, y), \quad (18)$$

where

$$\tilde{C}_n^{(\nu)} = \sqrt{\frac{\mu\alpha}{\hbar^2}} \frac{1}{2^{n-\nu+1/4}} \frac{\sqrt{\Gamma(2n+2\nu+1/2)}}{\pi^{1/4} n!(n+\nu)}. \quad (19)$$

From the duplication formula for a gamma function

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+1/2)$$

and taking into account that $\Gamma(1/2) = \pi^{1/2}$ and $\Gamma(3/2) = \pi^{1/2}/2$, we conclude that $\tilde{C}_n^{(\nu)} = C_n^{(\nu)}$ for $\nu = 1/4, 3/4$ and, consequently, Eqs. (18) and (14) are identical.

5. CONCLUSIONS

(a) The 1D oscillator has only a discrete energy spectrum and, therefore, is a model provided by the property which is known in QCD as confinement. A particle situated in the confinement potential cannot be removed from the center and transferred to infinity. On the other hand, the 1D Coulomb anyon is a system possessing both a discrete and a continuous part in the energy spectrum. At the same time, it includes $1/x^2$ interaction and, therefore, pretends to be a magnetic monopole in one spatial dimension. All these ideas confirm that our result can be interpreted in the spirit of the Seiberg–Witten duality [11]: the theories with strong coupling (i.e., including confinement) are equivalent to the theories with weak coupling (i.e., without confinement) accompanied by magnetic monopoles. We conclude that the Seiberg–Witten duality has its prototype in 1D quantum mechanics.

(b) The anyon–oscillator duality is a simple example of a more complicated dyon–oscillator duality [12–21]. The latter connects the isotropic oscillator with a charge–dyon bound system (a dyon is a hypothetical object which has both electric and magnetic charge [22]). The passage from an oscillator to a charge–dyon system is realized by nonbijective bilinear transformations [23] (for the mapping of the 1D Coulomb system into the oscillator, refer to [24]).

(c) The wave function (13) of a 1D Coulomb anyon can formally be extended to the region $-\infty < y < 0$. Such a continuation is an arbitrary-rule operation and we choose the following one. First, still being in the region $0 < y < \infty$, we change y in the exponent and confluent hypergeometric function by $|y|$ and leave unchanged the factor y^ν. Then, we extend the expression to the region $-\infty < y < 0$. These steps allow us to get rid of divergence in the exponent for large negative values of y and conserve the normalization condition in $-\infty < y < \infty$ by multiplying the function $\Phi_n^{(\nu)}$ by the factor $1/\sqrt{2}$. The obtained wave function $\bar{\Phi}_n^{(\nu)}(y)$ satisfies Eq. (6) in the region $-\infty < y < \infty$ and has the parity $(-1)^\nu$, i.e., describes the 1D anyon [4].

(d) Equation (6) for $-\infty < x < \infty$ and $\nu = 0$ corresponds to the so-called 1D hydrogen atom [25] (for later references, see [26]), which has some mysterious properties. For example, the ground state corresponds to an infinite negative value of the energy and the excited levels are doubly degenerate. The reason is that the potential $(-1/|x|)$ is singular in 1D space and the system is provided by hidden symmetry [27–29] and supersymmetry [30, 31]. As follows from (6) and (12), the Calogero–Sutherland potential transforms the 1D hydrogen atom into two modified atoms with the statistical parameter $\nu = 1/4$ and $\nu = 3/4$. This transformation leads to the formation of the ground states with a finite energy level and removes the problem of degeneracy (replacement $n \to n + \nu$).

(e) It is easy to be convinced that Eq. (4) is identical to the Schrödinger equation with the Hamiltonian

$$\hat{H} = \frac{1}{2\mu}\left(-i\hbar\partial_x - \frac{e}{c}A\right)^2 - \frac{\alpha}{x} - \frac{\hbar^2}{2\mu}\frac{\nu(1-\nu)}{x^2}, \quad (20)$$

where $\alpha = e^2$, $A = g/x$, and $g = i\nu\hbar c/e$. So, we deal with a charged particle moving in the field created by the 1D Coulomb dyon of the electric charge e and purely imaginary magnetic charge g. The Calogero–Sutherland potential acquires the meaning of the Goldhaber term typical of the theory of magnetic monopoles [32, 33].

Note that the Hamiltonian in (20) is not Hermitian, but it could be transformed into a Hermitian one if we do the following: (i) consider, instead of the semiaxis $x \in (0, \infty)$, the axis $x \in (-\infty, \infty)$; (ii) replace α/x with $\alpha/|x|$; (iii) introduce the Yang–Dunkl operator [36] $\hat{D} = -i\hbar\partial_x - eA\hat{R}/c$ for the Calogero model, where $\hat{R}$ is the reflection operator.

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AdS Branes from Partial Breaking of Superconformal Symmetries*

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Abstract—It is shown how the static-gauge world-volume superfield actions of diverse superbranes on the AdS_{d+1} superbackgrounds can be systematically derived from nonlinear realizations of the appropriate AdS supersymmetries. The latter are treated as superconformal symmetries of flat Minkowski superspaces of the bosonic dimension d. Examples include the $N = 1$ AdS_4 supermembrane, which is associated with the 1/2 partial breaking of the $OSp(1|4)$ supersymmetry down to the $N = 1$, $d = 3$ Poincaré supersymmetry, and the T-duality related L3-brane on AdS_5 and scalar 3-brane on $AdS_5 \times S^1$, which are associated with two different patterns of 1/2 breaking of the $SU(2,2|1)$ supersymmetry. Another (closely related) topic is the AdS/CFT equivalence transformation. It maps the world-volume actions of the codimension-one AdS_{d+1} (super)branes onto the actions of the appropriate Minkowski (super)conformal field theories in the dimension d. © *2005 Pleiades Publishing, Inc.*

1. INTRODUCTION

A view of superbranes as theories explicitly exhibiting the phenomenon of partial spontaneous breaking of global supersymmetry (PBGS) [1, 2] has received considerable attention (see, e.g, [3–5] and references therein). In the approach with PBGS as the guiding principle, the manifestly world-volume supersymmetric superbrane actions (in a static gauge) emerge as the Goldstone superfield actions associated with nonlinear realizations of some global spacetime supersymmetry groups spontaneously broken down to smaller supersymmetries.

Until now, the PBGS approach was applied to spontaneously broken Poincaré supersymmetries in diverse dimensions, in general, properly extended by some central-charge generators. All systems of this kind amount to p or Dp superbranes on flat Minkowski backgrounds. It is tempting to generalize the PBGS approach to the case of branes on curved backgrounds. In view of the famous AdS/CFT correspondence [6–8], the natural first step is to look at the $AdS_n \times S^m$-type backgrounds. The Green–Schwarz-type world-volume actions for superbranes on such backgrounds were intensively discussed in the literature (see, e.g., [9–12]). However, explicit examples of world-volume superfield actions were given quite recently. Such actions were constructed for the $N = 1$ supermembrane in AdS_4 [13] (and some of its dimensional reductions [14–15]), as well as for the L3 superbrane on AdS_5 and a scalar $N = 1$ superbrane on $AdS_5 \times S^1$, which are related to each other via T duality [16]. In all these cases, the partially broken supersymmetries are the $N = 1$ superconformal symmetries of the relevant superworld-volumes, namely, $OSp(1|4)$ in the former case and $SU(2,2|1)$ in the latter two. The PBGS actions of the AdS superbranes were derived from the special nonlinear realizations of these superconformal groups, such that the only unbroken symmetries are the $N = 1$ Poincaré supersymmetries of the superworld-volume (and, generically, some of the original R symmetries). The nonlinearly realized half of the superconformal symmetries act as AdS superisometries mixing the superworld-volume coordinates with the brane transverse coordinates. One of the aims of the present talk is to review this recent progress in generalizing the PBGS ideas to AdS superbranes.

The PBGS approach to AdS (super)branes allows one to reveal a new aspect of the AdS/CFT duality, the existence of the so-called AdS/CFT transform [15, 17, 18]. It relates conventional superconformal theories containing a Goldstone field of a dilaton among their fields and living in the standard Minkowski (super)space to the superbranes of the bosonic codimension 1 evolving on AdS supermanifolds for which the given superconformal group defines superisometries. The AdS/CFT transform maps the Minkowski superspace onto the AdS brane superworld-volume and the dilaton onto the brane transverse coordinate. The second half of the present talk is devoted to explaining the origin of this AdS/CFT transform and reviewing some implications of it.

*The text was submitted by the author in English.

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2. AdS_4 MEMBRANE FROM THE COSET APPROACH

We start with the case of the bosonic AdS_4 membrane. Whereas it was known how to derive the static-gauge Nambu–Goto action for the branes in the d-dimensional flat Minkowski background from the nonlinear realizations (coset) approach applied to the relevant Poincaré group [5, 19], no such self-contained derivation existed for AdS branes. The algebra of the AdS_4 group $SO(2,3)$ in the $d=3$ spinor notation reads

$$\begin{gathered}[M_{ab}, M_{cd}] = \varepsilon_{ac}M_{bd} + \varepsilon_{ad}M_{bc} + \varepsilon_{bc}M_{ad} \qquad (1)\\ + \varepsilon_{bd}M_{ac} \equiv (M)_{ab,cd},\\ [K_{ab}, K_{cd}] = -(M)_{ab,cd}, \quad [M_{ab}, K_{cd}] = (K)_{ab,cd},\\ [M_{ab}, P_{cd}] = (P)_{ab,cd},\\ [K_{ab}, D] = -2P_{ab} + 2mK_{ab}, \quad [P_{ab}, D] = -2mP_{ab},\\ [P_{ab}, P_{cd}] = 0,\\ [K_{ab}, P_{cd}] = -2(\varepsilon_{ac}\varepsilon_{bd} + \varepsilon_{bc}\varepsilon_{ad})D - m(M)_{ab,cd},\\ a, b, c, d = 1, 2.\end{gathered}$$

The contraction parameter m is proportional to the inverse AdS_4 radius, and

$$\begin{gathered}P_{ab}^\dagger = P_{ab}, \quad M_{ab}^\dagger = -M_{ab}, \qquad (2)\\ K_{ab}^\dagger = -K_{ab}, \quad D^\dagger = D, \quad m^\dagger = -m.\end{gathered}$$

The $SO(1,2)$ generators M_{ab} together with K_{ab} form the algebra of $SO(1,3)$. As $m \to 0$, (1) becomes the $d+1=4$ Poincaré algebra. Another basis may be defined as

$$\tilde{K}_{ab} = \frac{1}{m}K_{ab} - \frac{1}{2m^2}P_{ab}, \quad \tilde{D} = \frac{1}{m}D, \qquad (3)$$

which are the standard $d=3$ special conformal and dilatation generators:

$$\begin{gathered}[\tilde{K}_{ab}, \quad \tilde{K}_{cd}] = 0, \qquad (4)\\ [M_{ab}, \tilde{K}_{cd}] = (\tilde{K})_{ab,cd}, \quad [\tilde{K}_{ab}, \tilde{D}] = 2\tilde{K}_{ab},\\ [P_{ab}, \tilde{D}] = -2P_{ab},\\ [\tilde{K}_{ab}, P_{cd}] = -2(\varepsilon_{ac}\varepsilon_{bd} + \varepsilon_{bc}\varepsilon_{ad})\tilde{D} - (M)_{ab,cd}.\end{gathered}$$

In the basis (1), the $d=3$ Poincaré subalgebra $\propto(P_{ab}, M_{ab})$ is manifest (together with the manifest $so(1,3)$). The generators (P_{ab}, D) form the maximal solvable subalgebra of $so(2,3)$. Any AdS_{d+1} algebra $so(2,d)$ can be written in the basis where the d-dimensional Poincaré algebra is manifest, the d-dimensional translation operator together with the dilatation generator forms a solvable subalgebra, and the $(d+1)$-dimensional Lorentz algebra $so(1,d)$ is manifest [20]. This basis, the particular case of which is (1), is very advantageous for treating AdS branes in the nonlinear realization approach.

Now we consider the coset $SO(2,3)/SO(1,2)$ parametrized by

$$g = e^{x^{ab}P_{ab}}e^{q(x)D}e^{\Lambda^{ab}(x)K_{ab}}. \qquad (5)$$

The parameters $x^{ab} = -(x^{ab})^\dagger$ and $q(x) = -q^\dagger(x)$ provide a specific parametrization of the coset $SO(2,3)/SO(1,3) \sim AdS_4$, just adapted to the above solvable-subgroup basis of $so(2,3)$. The vector field $\Lambda^{ab}(x) = (\Lambda^{ab}(x))^\dagger$ parametrizes the coset $SO(1,3)/SO(1,2)$. Its inclusion is necessary for deducing the AdS_4 membrane action from the coset approach. Taking into account that the parameters associated with P_{ab} are the $d=3$ spacetime coordinates, the resulting nonlinear realization actually describes the spontaneous breaking of $SO(2,3)$ down to its $d=3$ Poincaré subgroup as the only linearly realized one.

The full set of the $SO(2,3)$ transformations of the coset parameters in (5) can be found by acting on (5) from the left by various $SO(2,3)$-group elements. The $d=3$ conformal transformations of the AdS_4 coordinates $(x^{ab}, q(x))$ are generated by $g_0 = e^{b^{ab}\tilde{K}_{ab}}$:

$$\begin{gathered}\delta x^{ab} = 4(x^2 b^{ab} - 2x^{cd}b_{cd}x^{ab}) - \frac{1}{2m^2}e^{4mq}b^{ab}, \qquad (6)\\ \delta q = -\frac{4}{m}x^{ab}b_{ab}.\end{gathered}$$

These transformations provide a specific nonlinear realization of the $d=3$ conformal group algebra, such that the Goldstone field $q(x)$ is present in the conformal transformation of x^{ab}. Just this realization underlies the AdS_4 membrane. The building blocks in constructing the action are left-invariant Cartan one-forms:

$$g^{-1}dg = \omega_P P + \omega_D D + \omega_K K + \omega_M M. \qquad (7)$$

For our purposes, it suffices to know the expressions for ω_P^{ab} and ω_D:

$$\begin{gathered}\omega_P^{ab} = e^{-2mq}\left(dx^{ab} + \frac{4\lambda^{ab}\lambda_{cd}dx^{cd}}{1-2\lambda^2}\right) \qquad (8)\\ + \frac{2\lambda^{ab}dq}{1-2\lambda^2} \equiv E_{cd}^{ab}(q,\lambda)dx^{cd},\\ \omega_D = \frac{1+2\lambda^2}{1-2\lambda^2}\left(dq + \frac{4e^{-2mq}\lambda_{ab}dx^{ab}}{1+2\lambda^2}\right), \qquad (9)\\ \lambda^{ab} \equiv \frac{\tanh\sqrt{2\Lambda^2}}{\sqrt{2\Lambda^2}}\Lambda^{ab}, \quad \lambda^2 = \lambda^{ab}\lambda_{ab}.\end{gathered}$$

The field λ^{ab} can be traded for $q(x)$ by the covariant constraint [21]

$$\omega_D = 0 \Rightarrow \lambda_{ab} = -\frac{1}{2}e^{2mq}\frac{\partial_{ab}q}{1+\sqrt{1-\frac{1}{2}e^{4mq}(\partial q)^2}}, \qquad (10)$$

$$E^{ab}_{cd}(q) = e^{-2mq}\delta^{(a}_{(c}\delta^{b)}_{d)} - \frac{1}{2}e^{2mq}\frac{1}{1+\sqrt{1-\frac{1}{2}e^{4mq}(\partial q)^2}}\partial^{ab}q\partial_{cd}q. \quad (11)$$

The simplest invariant is the covariant volume of the $d=3$ space, $\int d^3x \det E(q)$, and the correct invariant action vanishing for a constant q reads (up to a normalization factor)

$$S = \int d^3x[e^{-6mq} - \det E(q)] = \int d^3x e^{-6mq}\left(1-\sqrt{1-\frac{e^{4mq}}{2}\partial^{ab}q\partial_{ab}q}\right). \quad (12)$$

By construction, it possesses all symmetries of the AdS_4 space and, in the limit $m=0$, goes into the static-gauge Nambu–Goto action for a membrane in $d=4$ Minkowski space. The term $\sim\int d^3x e^{-6mq}$ is $SO(2,3)$ invariant in its own right.

To see that the action (12) indeed describes a membrane embedded into the AdS_4 background, let us look at the induced distance defined as the square of $\omega^{ab}_P = E^{ab}_{cd}(q)dx^{cd}$:

$$ds^2 = \omega^{ab}_P\omega_{Pab} = e^{-4mq}(dx^{ab}dx_{ab}) - \frac{1}{2}dqdq. \quad (13)$$

Introducing $U = e^{-2mq}$ and rescaling

$$x^{ab} = \frac{1}{2\sqrt{2}m}\tilde{x}^{ab},$$

one can rewrite (13) and (12), up to some overall constant factors, as

$$ds^2 = U^2(d\tilde{x}^{ab}d\tilde{x}_{ab}) - \left(\frac{dU}{U}\right)^2, \quad (14)$$

$$S = \int d^3\tilde{x}U^3\left(1-\sqrt{1-\frac{(\tilde{\partial}U\tilde{\partial}U)}{U^4}}\right).$$

Thus, ds^2 is recognized as the standard invariant interval on AdS_4, while S is recognized as the $d=3$ analog of the Maldacena scale-invariant brane action on AdS_5 [6] (actually, of the scalar fields piece of the D3-brane action). The derivation of this AdS_4 interval from the coset $SO(2,3)/SO(1,3)$ parametrized by coordinates associated with the solvable subgroup generators (and a generalization to the generic case of AdS_{d+1}), as well as deducing the field-dependent conformal transformations (6), was given in [20] (see also [22]). A novel point is the explicit derivation of the AdS_4 membrane action from the coset approach. It can be straightforwardly extended to the case of the $(d-1)$ brane in AdS_{d+1} [9, 12] (see Section 5).

3. AdS_4 SUPERMEMBRANE

Our starting point will be the $N=1$ AdS_4 superalgebra $osp(1|4)$ in the following basis:

$$\begin{gathered}\{Q_a,Q_b\} = 2P_{ab}, \quad (15)\\ \{S_a,S_b\} = 2P_{ab} - 4mK_{ab},\\ \{Q_a,S_b\} = 2\varepsilon_{ab}D - 2mM_{ab},\\ [M_{ab},Q_c] = \varepsilon_{ac}Q_b + \varepsilon_{bc}Q_a \equiv (Q)_{ab,c},\\ [M_{ab},S_c] = (S)_{ab,c}, \quad [K_{ab},Q_c] = (S)_{ab,c},\\ [K_{ab},S_c] = -(Q)_{ab,c}, \quad [P_{ab},Q_c] = 0,\\ [P_{ab},S_c] = -2m(Q)_{ab,c}, \quad [D,Q_a] = mQ_a,\\ [D,S_a] = -mS_a.\end{gathered}$$

The generators Q_a, P_{ab}, M_{ab} form the $N=1$, $d=3$ super-Poincaré algebra. The passing to the conformal basis, besides the redefinitions (3), implies the rescaling $S_a = m\tilde{S}_a$ such that $\tilde{S}_a$ is the $d=3$ conformal supersymmetry generator. The advantage of the basis (15) is that it manifests the $N=1$, $d=3$ super-Poincaré subalgebra of $osp(1|4)$ and still yields the $N=1$, $d=4$ super-Poincaré algebra in the contraction limit $m=0$. The $N=1$, $d=3$ Poincaré supertranslations $\propto(Q_a,P_{ab})$ together with D form the maximal solvable supersubalgebra of $osp(1|4)$.

We wish to construct an $OSp(1|4)$ extension of the AdS_4 membrane action (12) such that it possesses a manifest $N=1$, $d=3$ supersymmetry extending the manifest $d=3$ Poincaré world-volume invariance of (12) and reproduces the action of the flat $N=1$, $d=4$ supermembrane [23] in the limit $m=0$.

The construction of the AdS_4 supermembrane action as a Goldstone superfield action is not so straightforward as in the bosonic case. The only known way of constructing such actions proceeds from a linear realization of the partially broken supersymmetry in some appropriate superspace. The nonlinear realization is recovered by imposing proper covariant constraints on the corresponding superfields (see, e.g., [24, 25]). The correct Goldstone superfield actions then arise from some simple invariants of the initial linear realization. There is a systematic way of searching for such covariant constraints [4, 26–28]. We shall apply these techniques to construct the PBGS action of the AdS_4 supermembrane.

As a first step, we need to define the appropriate analog of the aforementioned linear realization. It turns out that, in the AdS case, it is already a sort of nonlinear realization, but with weaker nonlinearities as compared to the final nonlinear realization. As a natural superextension of the bosonic coset element (5), we choose

$$g = e^{x^{ab}P_{ab}}e^{\theta^aQ_a}e^{\psi^aS_a}e^{u(z)D}e^{\Lambda^{ab}(z)K_{ab}}. \quad (16)$$

Here, the parameters $z \equiv (x^{ab}, \theta^a, \psi^a)$ are $N = 2$, $d = 3$ superspace coordinates, while $u = u(z)$ and $\Lambda^{ab}(z)$ are Goldstone superfields given on this superspace. The subspace spanned by the coordinate set $\zeta \equiv (x^{ab}, \theta^a)$ is the flat $N = 1$, $d = 3$ superspace in which $N = 1$, $d = 3$ Poincaré supertranslations $\propto (Q_a, P_{ab})$ are realized in a standard way:

$$\delta x^{ab} = a^{ab} - \frac{1}{2}(\epsilon^a \theta^b + \epsilon^b \theta^a), \qquad (17)$$
$$\delta \theta^a = \epsilon^a.$$

These transformations correspond to the left shift of (16) by $g_0 = e^{a^{ab} P_{ab}} e^{\epsilon^a Q_a}$. The rest of the $OSp(1|4)$ transformations except for the $SO(1,2)$ rotations are nonlinearly realized on the coset coordinates, mixing the $N = 2$ superspace coordinates with the Goldstone superfield $u(z)$. Acting on (16) from the left by the element $g_0 = e^{\eta^a S_a}$, we find the explicit form of the broken supersymmetry transformations

$$\delta x^{ab} = 2m(\theta^a x^{bc} + \theta^b x^{ac})\eta_c + \frac{1}{2} e^{4mu} \qquad (18)$$
$$\times (\psi^a \eta^b + \psi^b \eta^a) + \frac{3}{2} m e^{4mu} \psi^2 (\theta^a \eta^b + \theta^b \eta^a),$$
$$\delta \theta^a = 4m x^{ac} \eta_c + m\theta^2 \eta^a - 3m e^{4mu} \psi^2 \eta^a,$$
$$\delta u = 2\theta^a \eta_a,$$
$$\delta \psi^a = \eta^a - 2m(\eta^b \theta_b \psi^a - \eta^a \theta^b \psi_b - \eta^b \theta^a \psi_b).$$

As follows from (15), all bosonic transformations are actually contained in the closure of the supersymmetry transformations.

What we have at this stage is a nonlinear realization of the $N = 1$ AdS_4 supergroup on the $N = 2$, $d = 3$ Goldstone superfield $u(x, \theta, \psi)$:

$$\delta^* u(x, \theta, \psi) = -(\delta x^{ab} \partial_{ab} + \delta \theta^a \partial^\theta_a + \delta \psi^a \partial^\psi_a) \qquad (19)$$
$$\times u(x, \theta, \psi) + 2\theta^a \eta_a.$$

The first component in the θ, ψ expansion of u can be regarded as the Goldstone dilaton field discussed in the previous section. The spinor derivative $D_a u$, where

$$D_a = \frac{\partial}{\partial \theta^a} + \theta^b \partial_{ab}, \quad \{D_a, D_b\} = 2\partial_{ab}, \qquad (20)$$

is shifted by η_a under the S supersymmetry. This suggests that we actually face the $1/2$ spontaneous breaking of the AdS_4 supersymmetry, with $D_a u|_{\psi=0}$ as the corresponding Goldstone fermionic $N = 1$ superfield. However, u contains extra component fields having no Goldstone interpretation. To construct the minimal Goldstone multiplet, we resort to the method which was applied in [28] to $d = 2$ PBGS systems and, in [4], to the flat-space $N = 1$, $d = 4$ supermembrane. Following the reasonings of [4] and keeping in mind that the scalar multiplets of the $N = 1$ AdS_4 supergroup are represented by chiral $N = 1, d = 4$ (or $N = 2$, $d = 3$) superfields, we regard the Goldstone superfield $u(z)$ to be complex and subject it to the covariant chirality constraint

$$(\nabla^Q_a - i\nabla^S_a)u = 0, \qquad (21)$$

where $\nabla^Q_a u$ and $\nabla^S_a u$ are the $OSp(1|4)$ covariant spinor derivatives of $u(z)$ with respect to θ^a and ψ^a. For our purpose, there is no need to know their precise structure; what actually matters is that all the coefficients in the ψ expansion of $u(z)$ can be expressed by (21) in terms of $u(z)|_{\psi^a=0}$ and derivatives thereof. For example, the $\psi^a = 0$ component of (21) expresses the first coefficient as

$$\left.\frac{\partial u}{\partial \psi^a}\right|_{\psi=0} = -ie^{2mu} D_a u\Big|_{\psi=0}. \qquad (22)$$

Thus the complex $N = 1$, $d = 3$ superfield

$$u_0(x, \theta) \equiv q(x, \theta) + i\Phi(x, \theta), \qquad (23)$$
$$q^\dagger = -q, \quad \Phi^\dagger = -\Phi,$$

incorporates the full irreducible field content of the $N = 2$, $d = 3$ Goldstone chiral superfield $u(x, \theta, \psi)$. Its S-supersymmetry transformation reads

$$\delta q = Lq - e^{2mq} \eta^a [\sin(2m\Phi) D_a q \qquad (24)$$
$$+ \cos(2m\Phi) D_a \Phi] + 2\eta^a \theta_a,$$
$$\delta \Phi = L\Phi + e^{2mq} \eta^a [\cos(2m\Phi) D_a q$$
$$- \sin(2m\Phi) D_a \Phi],$$

where Lq and $L\Phi$ denote the variations caused by the corresponding coordinate shifts.

The nonlinear realization that we have at this step is still nonminimal. Besides the $N = 1$ superfield $q(x, \theta)$ which contains all Goldstone fields required by the $1/2$ breaking of $OSp(1|4)$ down to its $N = 1$, $d = 3$ Poincaré subgroup ($q|_{\theta=0}$ for the dilatations, $(D_a q)|_{\theta=0}$ for the broken S transformations and $\partial_{ab} q|_{\theta=0}$ for the broken $SO(1,3)/SO(1,2)$ transformations), there is an extra non-Goldstone $N = 1$, $d = 3$ superfield $\Phi(x, \theta)$. The last step is to eliminate the latter in terms of q and its derivatives by imposing some nonlinear covariant constraint on $u_0(x, \theta)$, analogous to the constraints imposed in the flat case [23]. It reads

$$\Phi = \frac{e^{2mq} D^a q D_a q}{4 + e^{2mq} D^2 \Phi} \Longleftrightarrow \Phi \qquad (25)$$
$$= \frac{e^{2mq} D^a q D_a q}{2 + \sqrt{4 + e^{4mq} D^2 (D^b q D_b q)}}.$$

It can be directly checked to be covariant with respect to the transformations (24). From our superfield u_0,

we can construct the invariant

$$S_2 = -\frac{1}{2im}\int d^3x d^2\theta (e^{-4mu_0} - e^{4mu_0^\dagger}) \quad (26)$$
$$= \frac{1}{m}\int d^3x d^2\theta e^{-4mq}\sin(4m\Phi).$$

In view of the nilpotency of Φ defined by Eq. (25), the final action takes the form

$$S_2 \sim \int d^3x d^2\theta \frac{e^{-2mq} D^a q D_a q}{2+\sqrt{4+e^{4mq}D^2(D^b q D_b q)}}. \quad (27)$$

The action S_2 contains the kinetic term of $q(\zeta)$ and, in the limit $m \to 0$, reduces to the flat $N=1$, $d=4$ supermembrane PBGS action of [23]. After eliminating the auxiliary field $B = D^2 q|_{\theta=0}$, the bosonic part of S_2 coincides with (12).

We come to the conclusion that the Goldstone superfield action (27) is the natural superextension of the conformally invariant AdS_4 membrane action (12). Besides being manifestly invariant under $N=1$, $d=3$ Poincaré supersymmetry, it is invariant under the nonlinearly realized part of $N=1$ AdS_4 supersymmetry $OSp(1|4)$ which acts on the $N=1$, $d=3$ superworld-volume as the Goldstone superfield-modified $d=3$ superconformal transformations. Thus, it is a PBGS superfield form of the world-volume action of the $N=1$ AdS_4 supermembrane.

4. 3-BRANES IN SUPER AdS_5 AND $AdS_5 \times S^1$ BACKGROUNDS

We start with recalling how the PBGS $N=1$ L3-brane action and (related to it via T duality) $N=1$ scalar 3-brane action in the flat Minkowski backgrounds can be deduced as the Goldstone superfield actions describing the one-half partial breaking of global $N=2$ Poincaré supersymmetry in $d=4$.

The first option corresponds to the $N=1$ tensor multiplet as the Goldstone one [24, 25, 29]. The starting point is the $N=2$, $d=4$ Poincaré superalgebra with a real central charge D

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2P_{\alpha\dot\alpha}, \quad \{S_\alpha, \bar{S}_{\dot\alpha}\} = 2P_{\alpha\dot\alpha}, \quad (28)$$
$$\{Q_\alpha, S_\beta\} = -\varepsilon_{\alpha\beta}D, \quad \{\bar{Q}_{\dot\alpha}, \bar{S}_{\dot\beta}\} = -\varepsilon_{\dot\alpha\dot\beta}D.$$

Here, Q_α, $\bar{Q}_{\dot\alpha}$ and S_α, $\bar{S}_{\dot\alpha}$ are generators of the unbroken and broken $N=1$ supersymmetries, respectively. These generators and the 4-translation generator $P_{\alpha\dot\alpha}$ possess the standard commutation relations with the Lorentz $so(1,3)$ generators $(M_{\alpha\beta}, \bar{M}_{\dot\alpha\dot\beta})$:

$$i[M_{\alpha\beta}, M_{\rho\sigma}] = \varepsilon_{\alpha\rho}M_{\beta\sigma} + \varepsilon_{\alpha\sigma}M_{\beta\rho} \quad (29)$$
$$+ \varepsilon_{\beta\rho}M_{\alpha\sigma} + \varepsilon_{\beta\sigma}M_{\alpha\rho} \equiv (M)_{\alpha\beta,\rho\sigma},$$
$$i[\bar{M}_{\dot\alpha\dot\beta}, \bar{M}_{\dot\rho\dot\sigma}] = (\bar{M})_{\dot\alpha\dot\beta,\dot\rho\dot\sigma},$$
$$i[M_{\alpha\beta}, P_{\rho\dot\rho}] = \varepsilon_{\alpha\rho}P_{\beta\dot\rho} + \varepsilon_{\beta\rho}P_{\alpha\dot\rho},$$
$$i[\bar{M}_{\dot\alpha\dot\beta}, P_{\rho\dot\rho}] = \varepsilon_{\dot\alpha\dot\rho}P_{\rho\dot\beta} + \varepsilon_{\dot\beta\dot\rho}P_{\rho\dot\alpha},$$
$$i[M_{\alpha\beta}, Q_\gamma] = \varepsilon_{\alpha\gamma}Q_\beta + \varepsilon_{\beta\gamma}Q_\alpha \equiv (Q)_{\alpha\beta,\gamma},$$
$$i[M_{\alpha\beta}, S_\gamma] = (S)_{\alpha\beta,\gamma}, \quad i[\bar{M}_{\dot\alpha\dot\beta}, \bar{Q}_{\dot\gamma}] = (\bar{Q})_{\dot\alpha\dot\beta,\dot\gamma},$$
$$i[\bar{M}_{\dot\alpha\dot\beta}, \bar{S}_{\dot\gamma}] = (\bar{S})_{\dot\alpha\dot\beta,\dot\gamma}.$$

Then one introduces two $N=1$ superfields: a real one $L(x,\theta)$ subjected to the constraint

$$D^2 L = \bar{D}^2 L = 0 \quad (30)$$

and so describing a tensor $N=1$ supermultiplet, and a complex chiral superfield F, $\bar{F}$,

$$D_\alpha F = \bar{D}_{\dot\alpha}\bar{F} = 0. \quad (31)$$

Here,

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}, \quad (32)$$
$$\bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\partial_{\alpha\dot\alpha}, \quad D^2 = D^\alpha D_\alpha,$$
$$\bar{D}^2 = \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}.$$

On these $N=1$ superfields, one implements [24] the following off-shell representation of the full $N=2$ supersymmetry (28):

$$\delta L = -i(\eta^\alpha\theta_\alpha - \bar\eta_{\dot\alpha}\bar\theta^{\dot\alpha}) + \eta^\alpha D_\alpha\bar{F} - \bar\eta^{\dot\alpha}\bar{D}_{\dot\alpha}F, \quad (33)$$
$$\delta F = -\eta^\alpha D_\alpha L, \quad \delta\bar{F} = \bar\eta^{\dot\alpha}\bar{D}_{\dot\alpha}L,$$

where $\eta_\alpha, \bar\eta_{\dot\alpha}$ are the infinitesimal transformation parameters associated with the generators S_α, $\bar{S}_{\dot\alpha}$. It is a modification of the transformation law of the $N=2$ tensor multiplet [30] written in terms of its $N=1$ superfield components.

One can construct the simplest invariant "action" as follows:

$$S = \frac{1}{4}\int d^4x d^2\bar\theta F + \frac{1}{4}\int d^4x d^2\theta\bar{F}. \quad (34)$$

To make it meaningful, one should express the chiral supermultiplet $F, \bar{F}$ in terms of the Goldstone tensor multiplet L by imposing proper covariant constraints [24, 25]:

$$F = -\frac{D^\alpha L D_\alpha L}{2 - D^2\bar{F}}, \quad \bar{F} = -\frac{\bar{D}_{\dot\alpha}L\bar{D}^{\dot\alpha}L}{2-\bar{D}^2F} \quad \Rightarrow \quad (35)$$
$$F = -\psi^2 + \frac{1}{2}D^2 \quad (36)$$
$$\times \left[\frac{\psi^2\bar\psi^2}{1+\frac{1}{2}A_+ + \sqrt{1+A_+ + \frac{1}{4}(A_-)^2}}\right],$$
$$\psi_\alpha \equiv D_\alpha L, \quad \bar\psi_{\dot\alpha} \equiv \bar{D}_{\dot\alpha}L, \quad (37)$$
$$A_\pm = \frac{1}{2}(D^2\bar\psi^2 \pm \bar{D}^2\psi^2).$$

Finally, the action (34) becomes

$$S = -\frac{1}{4}\int d^4xd^2\theta\bar{\psi}^2 - \frac{1}{4}\int d^4xd^2\bar{\theta}\psi^2 \qquad (38)$$
$$+\frac{1}{4}\int d^4xd^4\theta\frac{\psi^2\bar{\psi}^2}{1+\frac{1}{2}A_+ + \sqrt{1+A_+ + \frac{1}{4}(A_-)^2}}.$$

It is a nonlinear extension of the standard $N = 1$ tensor multiplet action. In the bosonic sector, it gives rise to the static-gauge Nambu–Goto action for the L3-brane in $d = 5$ Minkowski space, with one physical scalar of L being the transverse brane coordinate and another one represented by the notoph field strength. After dualizing L into a pair of conjugated chiral and antichiral $N = 1$ superfields (the notoph strength is dualized into a scalar field), the PBGS form of the world-volume action of the super 3-brane in $d = 6$ is reproduced [31].

Let us point out that the constraints (35), which play the central role in deriving the action (38), are intimately related to the five-dimensional nature of the brane under consideration. They guarantee five-dimensional Lorentz covariance [16].

Now we wish to generalize this flat superspace construction to the case of partial spontaneous breaking of the simplest AdS_5 supersymmetry $SU(2,2|1)$, that is, the $N = 1$ superconformal group in $d = 4$.

The superalgebra $su(2,2|1)$ contains the $so(2,4)\oplus u(1)$ bosonic subalgebra with the generators $\{P_{\alpha\dot{\alpha}}, M_{\alpha\beta}, \bar{M}_{\dot{\alpha}\dot{\beta}}, K_{\alpha\dot{\alpha}}, D\}$ and $\{J\}$ and eight supercharges $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}, S_\alpha, \bar{S}_{\dot{\alpha}}\}$. We choose the basis in such a way that the generators $K_{\alpha\dot{\alpha}}$ form the $so(1,4)$ subalgebra together with the $d = 4$ Lorentz generators $\{M_{\alpha\beta}, \bar{M}_{\dot{\alpha}\dot{\beta}}\}$. The rest of the nontrivial (anti)commutators reads

$$i[D, P_{\alpha\dot{\alpha}}] = mP_{\alpha\dot{\alpha}}, \qquad (39)$$
$$i[D, K_{\alpha\dot{\alpha}}] = 2P_{\alpha\dot{\alpha}} - mK_{\alpha\dot{\alpha}},$$
$$i[P_{\alpha\dot{\alpha}}, K_{\beta\dot{\beta}}] = \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}D - \frac{m}{2}(\varepsilon_{\alpha\beta}\bar{M}_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}}M_{\alpha\beta}),$$
$$\{Q_\alpha, S_\beta\} = -\varepsilon_{\alpha\beta}(D + imJ) + mM_{\alpha\beta},$$
$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2P_{\alpha\dot{\alpha}}, \quad \{S_\alpha, \bar{S}_{\dot{\alpha}}\} = 2P_{\alpha\dot{\alpha}} - 2mK_{\alpha\dot{\alpha}},$$
$$i[D, Q_\alpha] = \frac{m}{2}Q_\alpha, \quad i[D, S_\alpha] = -\frac{m}{2}S_\alpha,$$
$$[J, Q_\alpha] = \frac{3}{2}Q_\alpha, \quad [J, S_\alpha] = -\frac{3}{2}S_\alpha,$$
$$i[K_{\alpha\dot{\alpha}}, Q_\beta] = -\varepsilon_{\alpha\beta}\bar{S}_{\dot{\alpha}}, \quad i[K_{\alpha\dot{\alpha}}, S_\beta] = \varepsilon_{\alpha\beta}\bar{Q}_{\dot{\alpha}},$$
$$i[P_{\alpha\dot{\alpha}}, S_\beta] = m\varepsilon_{\alpha\beta}\bar{Q}_{\dot{\alpha}}.$$

This basis is another example of the "AdS basis" of conformal superalgebras [13, 17, 20, 32]. The parameter m has the meaning of the inverse AdS_5 radius, $m = R^{-1}$. In the limit $m = 0$ ($R = \infty$), one recovers from (39) the $N = 1$, $d = 5$ Poincaré superalgebra, with D becoming the fifth component of momenta. The generators J and $K_{\alpha\dot{\alpha}}, M_{\alpha\beta}, \bar{M}_{\dot{\alpha}\dot{\beta}}$ decouple and generate outer $u(1)\oplus so(1,4)$ automorphisms.

Our goal is to construct an AdS_5 version of the nonlinear realization (33), (35). The main hints which allowed us to do this are as follows. First, we assert that this realization involves some modification of $N = 1$ tensor multiplet L and, as before, a pair of mutually conjugated $N = 1$ chiral and antichiral superfields F, $\bar{F}$ subjected to some generalization of (35). Second, in a close analogy with the flat case, we require that the following "action"

$$S \sim \int d^4xd^2\bar{\theta}F + \int d^4xd^2\theta\bar{F} \qquad (40)$$

be an invariant of the AdS_5 supersymmetry. Third, in the limit $m = 0$, our construction should reproduce the flat case outlined above. Finally, it is sufficient to find the realization of conformal S supersymmetry, since the rest of the $SU(2,2|1)$ transformations appear in the closure of S transformations with themselves and with $N = 1$ Poincaré supersymmetry.

It turns out that this reasoning almost uniquely fixes the sought transformation laws and constraints (more details of the derivation are given in [16]). These are

$$\delta^*\bar{F} = 6im\theta^\alpha\eta_\alpha\bar{F} - \Delta x^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\bar{F} \qquad (41)$$
$$+\Delta\theta^\alpha D_\alpha\bar{F} + ie^{-2mL}\bar{\eta}^{\dot{\alpha}}\bar{D}_{\dot{\alpha}}L,$$
$$\delta^*F = -6im\bar{\theta}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}}F - \Delta x^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}}F$$
$$-\Delta\bar{\theta}^{\dot{\alpha}}\bar{D}_{\dot{\alpha}}F + ie^{-2mL}\eta^\alpha D_\alpha L,$$
$$\delta^*L = -i(\theta^\alpha\eta_\alpha - \bar{\theta}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}}) - \Delta x^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}}L$$
$$+\Delta\theta^\alpha D_\alpha L - \Delta\bar{\theta}^{\dot{\alpha}}\bar{D}_{\dot{\alpha}}L - ie^{2mL}[\eta^\alpha D_\alpha(e^{2mL}\bar{F})$$
$$+\bar{\eta}^{\dot{\alpha}}\bar{D}_{\dot{\alpha}}(e^{2mL}F)],$$
$$\frac{1}{m}D^2e^{-2mL} = \frac{1}{m}\bar{D}^2e^{-2mL} = 0, \qquad (42)$$
$$D_\alpha F = \bar{D}_{\dot{\alpha}}\bar{F} = 0,$$
$$F = -\frac{e^{-2mL}D^\alpha LD_\alpha L}{2 - e^{4mL}D^2\bar{F}}, \qquad (43)$$
$$\bar{F} = -\frac{e^{-2mL}\bar{D}_{\dot{\alpha}}L\bar{D}^{\dot{\alpha}}L}{2 - e^{4mL}\bar{D}^2F}.$$

Here,

$$\Delta x^{\alpha\dot{\alpha}} = 2im(\eta_\beta x^{\beta\dot{\alpha}}\theta^\alpha + \bar{\eta}_{\dot{\beta}}x^{\alpha\dot{\beta}}\bar{\theta}^{\dot{\alpha}}) \qquad (44)$$
$$-m(\theta^2\eta^\alpha\bar{\theta}^{\dot{\alpha}} - \bar{\theta}^2\bar{\eta}^{\dot{\alpha}}\theta^\alpha),$$
$$\Delta\theta^\alpha = m\bar{\eta}_{\dot{\alpha}}x^{\alpha\dot{\alpha}} + im(\theta^2\eta^\alpha - \bar{\theta}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}}\theta^\alpha),$$
$$\Delta\bar{\theta}^{\dot{\alpha}} = m\eta_\alpha x^{\alpha\dot{\alpha}} - im(\bar{\theta}^2\bar{\eta}^{\dot{\alpha}} - \theta^\alpha\eta_\alpha\bar{\theta}^{\dot{\alpha}})$$

are the standard transformations of the $N = 1$ superspace coordinates with respect to the conformal supersymmetry.

In the limit $m = 0$, Eqs. (41), (42), and (43) go, respectively, into (33), (30), (31), and (35). It can be checked that, on the surface of the nonlinear constraints (43), the off-shell transformations (41) are, first, compatible with the differential constraints (42) and, second, produce the whole $SU(2,2|1)$ symmetry when commuted among themselves and with $N = 1$ Poincaré supersymmetry. It is just due to the presence of the nonlinear mixed terms that the transformations (41) constitute a realization of $SU(2,2|1)$ as the superisometry group of super AdS_5 background and correctly generalize the flat superspace realization (33). A striking difference between Eqs. (33) and (41) lies in the fact that Eqs. (33) close on the $N = 2$ Poincaré superalgebra before imposing the constraints (35), while Eqs. (41) define a closed supergroup structure only provided the constraints (43) are imposed from the very beginning. It is easy to check that Eqs. (43) are covariant under (41).

Inspecting (41), one can be convinced that this realization corresponds to a half-breaking of the $SU(2,2|1)$ supersymmetry: the spinor derivatives of L are shifted by spinor parameters under the action of S supersymmetry, thus signaling that the latter is spontaneously broken. Broken also are D transformations (with $L|$ as the Goldstone field) and the $SO(1,4)/SO(1,3)$ transformations (with $\partial_{\alpha\dot\alpha}L|$ as the relevant "Goldstone field").

Like their flat counterparts, the constraints (43) can be easily solved:

$$F = -e^{-2mL}\psi^2 + \frac{1}{2}D^2 \times \left[\frac{\psi^2\bar\psi^2}{1+\frac{1}{2}A_+ + \sqrt{1+A_+ + \frac{1}{4}(A_-)^2}}\right], \quad (45)$$

$$\psi_\alpha \equiv D_\alpha L, \quad \bar\psi_{\dot\alpha} \equiv \bar D_{\dot\alpha} L, \quad A_\pm = \frac{1}{2}e^{2mL}(D^2\bar\psi^2 \pm \bar D^2\psi^2). \quad (46)$$

Finally, the action (40) can be written in the form

$$S = -\frac{1}{4}\int d^4x d^2\theta e^{-2mL}\bar\psi^2 - \frac{1}{4}\int d^4x d^2\bar\theta e^{-2mL}\psi^2 + \frac{1}{4}\int d^4x d^4\theta \times \frac{\psi^2\bar\psi^2}{1+\frac{1}{2}A_+ + \sqrt{1+A_+ + \frac{1}{4}(A_-)^2}}. \quad (47)$$

The first two terms in (47) are recognized as the action of the improved tensor $N = 1$ superfield [33]. In the limit $m = 0$, (47) converts into the flat superspace action (38).

With the bosonic components defined as

$$\phi = L|_{\theta=0}, \quad [D_\alpha, \bar D_{\dot\alpha}]e^{-2mL}|_{\theta=0} = -2mV_{\alpha\dot\alpha}, \quad (48)$$

where in virtue of (43)

$$\partial_{\alpha\dot\alpha}V^{\alpha\dot\alpha} = 0, \quad (49)$$

the bosonic part of (47) proves to be

$$S_B = \int d^4x e^{-4m\phi}\left[1 - \sqrt{1 + \frac{1}{2}e^{6m\phi}V^2 - 2e^{2m\phi}(\partial\phi)^2 - e^{8m\phi}(V^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\phi)^2}\right]. \quad (50)$$

It is a conformally invariant extension of the static-gauge Nambu–Goto action for the L3 brane in $d = 5$: the dilaton ϕ can be interpreted as a radial brane coordinate, while $V^{\alpha\dot\alpha}$ is the field strength of the notoph which contributes one more scalar degree of freedom on shell. As is well known, $V^{\alpha\dot\alpha}$ can be dualized into an off-shell scalar by introducing the constraint (49) into the action with a Lagrange scalar multiplier and then eliminating $V^{\alpha\dot\alpha}$ by its algebraic equation of motion. Extending (50) as

$$S_B \quad \Rightarrow \quad S_B^{\text{dual}} = S_B + \int d^4x\lambda\partial_{\alpha\dot\alpha}V^{\alpha\dot\alpha} \quad (51)$$

and eliminating $V^{\alpha\dot\alpha}$, after some algebra, we get

$$S_B^{\text{dual}} = \int d^4x|Z|^4 \times \left[1 - \sqrt{-\det\left(\eta_{\mu\nu} - \frac{2}{m^2}\frac{\partial_\mu Z^n\partial_\nu Z^n}{|Z|^4}\right)}\right], \quad (52)$$

where

$$Z^1 = r\cos\vartheta, \quad Z^2 = r\sin\vartheta, \quad r \equiv e^{-m\phi}, \quad \vartheta \equiv m\lambda, \quad \eta_{\mu\nu} = \text{diag}(+---). \quad (53)$$

The action (52) is recognized as the $S^5 \to S^1$ reduction of the scalar part of the D3-brane action

on $AdS_5 \times S^5$ [6], that is, the static-gauge Nambu–Goto action of the scalar 3 brane on $AdS_5 \times S^1$. The field ϑ can be shown to undergo a shift under the action of the $U(1)$ generator J, which justifies its interpretation as the S^1 angular variable.

The above duality transformation can be performed at the full superfield level. This results in $SU(2,2|1)$ invariant action of the Goldstone chiral $N = 1$ superfield which generalizes the action of [25, 29, 31] and describes a super 3 brane on $AdS_5 \times S^1$ superbackground. In its basic steps, this dualization procedure is similar to the flat superspace one of [29].

We start with the superfield action (47) and relax the constraints for L in (42) by adding a Lagrange multiplier term to the superfield Lagrangian,

$$S^{\rm dual} = \frac{1}{4}\int d^4x d^2\theta d^2\bar\theta\Big[-\frac{1}{2m^2}Y(\ln Y - 1) + \frac{Y^{-4}}{(2m)^4}(DY)^2(\bar D Y)^2 f + \frac{Y}{2m}(\varphi+\bar\varphi)\Big], \quad (54)$$

$$Y \equiv e^{-2mL}, \quad \bar D_{\dot\alpha}\varphi = D_\alpha\bar\varphi = 0, \quad (55)$$

$$f = \frac{1}{1+\frac{1}{2}A_+ + \sqrt{1+A_+ + \frac{1}{4}(A_-)^2}}.$$

Next, we vary the action (54) with respect to Y in order to obtain an algebraic equation that trades Y for φ, $\bar\varphi$. Skipping the details, we obtain the dual action

$$S^{\rm dual} = \frac{1}{8}\int d^4x d^4\theta\Bigg(\frac{e^{m(\varphi+\bar\varphi)}}{m^2} + \frac{\frac{1}{8}(D\varphi)^2(\bar D\bar\varphi)^2}{1-e^{-m(\varphi+\bar\varphi)}\partial\varphi\partial\bar\varphi + \sqrt{(1-e^{-m(\varphi+\bar\varphi)}\partial\varphi\partial\bar\varphi)^2 - e^{-2m(\varphi+\bar\varphi)}(\partial\varphi)^2(\partial\bar\varphi)^2}}\Bigg). \quad (56)$$

This action goes into the flat $N = 2 \to N = 1$ chiral Goldstone superfield action of [25, 29, 31] in the limit $m = 0$ and is obviously $SU(2,2|1)$ invariant as it was obtained by dualizing the $SU(2,2|1)$ invariant action (47). It is noteworthy that the standard $U(1)$ isometry associated with the duality transformation, viz. $\delta\varphi = i\alpha$, $\delta\bar\varphi = -i\alpha$, now appears in the closure of the Q and S transformations on these Goldstone superfields, with the imaginary part of $\varphi|$ being the related extra Goldstone field. It is just the J (or γ_5) symmetry of $SU(2,2|1)$. The bosonic core of the action (56) coincides with (52) after the identification

$$\phi = -\frac{1}{2}(\varphi+\bar\varphi), \quad \lambda = \frac{i}{2}(\varphi-\bar\varphi). \quad (57)$$

Thus, we conclude that the Goldstone superfield action (56) describes the option when the internal $U(1)$ R symmetry with the generator J is also broken in addition to the (super)symmetries broken in the action (47). The bosonic coset is basically $AdS_5 \times S^1 \propto \{x^{\alpha\dot\alpha}, \phi\} \times \{\lambda\}$ and the bosonic part of the action (56) is just the static-gauge Nambu–Goto action of a 3 brane on this manifold. This solves the problem of constructing an invariant Goldstone superfield action for such a PBGS option, as was posed in [34].

5. AdS/CFT EQUIVALENCE TRANSFORMATION

The group-theoretical origin of the AdS/CFT map to be discussed lies in the existence of two different nonlinear realizations of the conformal group in d dimensions.

The algebra of conformal group $SO(2,d)$ of $d = p+1$-dimensional Minkowski space reads

$$[M_{\mu\nu}, M^{\rho\sigma}] = 2\delta^{[\rho}_{[\mu}M^{\sigma]}_{\nu]}, \quad (58)$$

$$[P_\mu, M_{\nu\rho}] = -\eta_{\mu[\nu}P_{\rho]}, \quad [K_\mu, M_{\nu\rho}] = -\eta_{\mu[\nu}K_{\rho]},$$

$$[P_\mu, K_\nu] = 2(-\eta_{\mu\nu}D + 2M_{\mu\nu}), \quad [D, P_\mu] = P_\mu,$$

$$[D, K_\mu] = -K_\mu.$$

Its standard nonlinear realization [35] is defined in the coset $SO(2,d)/SO(1,d-1)$:

$$g = e^{y^\mu P_\mu}e^{\varphi D}e^{\Omega^\mu K_\mu}. \quad (59)$$

The left shifts with the parameters a^μ, b^μ, and c related to the generators P_μ, K_μ, and D induce the familiar conformal transformations of the coset coordinates

$$\delta y^\mu = a^\mu + cy^\mu + 2(yb)y^\mu - y^2b^\mu, \quad (60)$$

$$\delta\varphi = c + 2yb.$$

The left-covariant Cartan 1-forms are defined as follows:

$$g^{-1}dg = \omega^\mu_P P_\mu + \omega_D D + \omega^{\mu\nu}_M M_{\mu\nu} \quad (61)$$

$$+\omega_K^\mu K_\mu = e^{-\varphi}dy^\mu P_\mu + (d\varphi - 2e^{-\varphi}\Omega_\mu dy^\mu)D - 4e^{-\varphi}\Omega^\mu dy^\nu M_{\mu\nu} + [d\Omega^\mu - \Omega^\mu d\varphi + e^{-\varphi}(2\Omega_\nu dy^\nu \Omega^\mu - \Omega^2 dy^\mu)]K_\mu.$$

The vector Goldstone field $\Omega^\mu(x)$ can be covariantly expressed through the dilaton $\varphi(x)$ [21]

$$\omega_D = 0 \Rightarrow \Omega_\mu = \frac{1}{2}e^{\varphi}\partial_\mu\varphi, \quad \omega_P^\mu = e^{-\varphi}dy^\mu, \quad (62)$$
$$\omega_K^\mu = d\Omega^\mu - e^{-\varphi}\Omega^2 dy^\mu.$$

The covariant derivative of Ω^μ is defined by the relation

$$\omega_K^\mu = \omega_P^\nu \mathcal{D}_\nu \Omega^\mu \Rightarrow \mathcal{D}_\nu\Omega^\mu \quad (63)$$
$$= \frac{1}{2}e^{2\varphi}\left[\partial_\nu\partial^\mu\varphi + \partial_\nu\varphi\partial^\mu\varphi - \frac{1}{2}(\partial\varphi\partial\varphi)\delta_\nu^\mu\right].$$

The conformally invariant measure of integration over $\{y^\mu\}$ is defined as

$$S_1 = \int \mu(y) = \int d^{(p+1)}y e^{-(p+1)\varphi} \quad (64)$$

and the covariant kinetic term of φ reads

$$S_\varphi^{\text{kin}} = \int d^{(p+1)}y e^{-(p+1)\varphi}\mathcal{D}_\mu\Omega^\mu \quad (65)$$
$$= \frac{1}{4}(p-1)\int d^{(p+1)}y e^{(1-p)\varphi}\partial\varphi\partial\varphi.$$

In any field theory with spontaneously broken conformal symmetry, it is always possible to make a field redefinition which splits the full set of scalar fields of the theory into the dilaton φ with the transformation law (60) and the subset of fields which are scalars of zero conformal weight. In this sense, the above nonlinear realization is universal.

There is another nonlinear realization of the same group [13] which is relevant to the description of codimension-one branes on AdS_{d+1}. In this realization, $SO(2,d)$ acts as the group of motion of AdS_{d+1}. It is related to the existence of the AdS basis in (58).

In the AdS basis, we introduce the following generators:

$$\hat{K}_\mu = mK_\mu - \frac{1}{2m}P_\mu, \quad \hat{D} = mD, \quad (66)$$

where m is the inverse AdS radius. The basic relations of the $SO(2,d)$ algebra become

$$[\hat{K}_\mu, \hat{K}_\nu] = 4M_{\nu\mu}, \quad (67)$$
$$[P_\mu, \hat{K}_\nu] = 4mM_{\mu\nu} - 2\eta_{\mu\nu}\hat{D},$$
$$[\hat{D}, P_\mu] = mP_\mu, \quad [\hat{K}_\mu, \hat{D}] = P_\mu + m\hat{K}_\mu.$$

The main difference between (67) and (58) is that the generators $(\hat{K}^\mu, M_{\rho\nu})$ generate the semi-simple subgroup $SO(1,d) \subset SO(2,d)$, while the subgroup with $(K^\mu, M_{\rho\nu})$ has the structure of a semidirect product. As a result, in the coset element (59) in the new basis

$$g = e^{x^\mu P_\mu}e^{q\hat{D}}e^{\Lambda^\mu \hat{K}_\mu}, \quad (68)$$

x^μ and $q(x)$ parametrize $SO(2,d)/SO(1,d) \sim \text{AdS}_{d+1}$ [20]. Equation (62) now yields

$$\omega_{\hat{D}} = 0 \Rightarrow \lambda_\mu = e^{mq}\frac{\partial_\mu q}{1 + \sqrt{1 - \frac{1}{2}e^{2mq}(\partial q\partial q)}}, \quad (69)$$
$$\lambda^\mu \equiv \Lambda^\mu \frac{\tanh\sqrt{\Lambda^2/2}}{\sqrt{\Lambda^2/2}}.$$

The Cartan form ω_P^μ is then given by the expression

$$\omega_P^\mu = e^{-mq}\left(\delta_\nu^\mu - \frac{\lambda^\mu\lambda_\nu}{1+\lambda^2/2}\right)dx^\nu \quad (70)$$
$$\equiv E_\nu^\mu dx^\nu = e^{-mq}\hat{E}_\nu^\mu dx^\nu.$$

The transformation laws of x^μ, $q(x)$ under the left shifts of (68) are as follows:

$$\delta x^\mu = a^\mu + cx^\mu + 2(xb)x^\mu - x^2 b^\mu \quad (71)$$
$$+ \frac{1}{2m^2}e^{2mq}b^\mu, \quad \delta q = \frac{1}{m}(c + 2xb).$$

After a field redefinition, they are recognized as the field-dependent conformal transformations [6, 9, 12, 36] representing AdS isometries in the solvable-subgroup parametrization.

The simplest invariant of the nonlinear realization considered is again the covariant volume of x space obtained as an integral of wedge product of $(p+1)$ 1-forms ω_P^μ. It is basically the static-gauge Nambu–Goto action for the p-brane in AdS_{p+2}

$$S_{\text{NG}} = \int d^{(p+1)}x[e^{-(p+1)mq} - \det E] \quad (72)$$
$$= \int d^{(p+1)}x e^{-(p+1)mq}\left[1 - \sqrt{1 - \frac{1}{2}e^{2mq}(\partial q\partial q)}\right],$$

where we have subtracted 1 to obey the standard requirement of absence of vacuum energy [6]. The subtracted term is also invariant under (71). The action (72) is universal; it describes the radial (pure AdS) part of any $(n-2)$-brane action on $\text{AdS}_n \times S^m$.

In both nonlinear realizations described above, we deal with the same coset manifold $SO(2,d)/SO(1, d-1)$, the parameters of which are separated into the spacetime coordinates and Goldstone fields in two different ways. Hence, there should exist a relation between these two parametrizations. It can be read off by comparing (59) and (68):

$$y^\mu = x^\mu - \frac{e^{mq}}{2m}\lambda^\mu, \quad (73)$$

$$\varphi = mq + \ln\left(1 - \frac{\lambda^2}{2}\right), \quad \Omega^\mu = m\lambda^\mu.$$

It is invertible at any finite and nonzero $m = 1/R$ and maps the Minkowski space conformal transformations (60) onto the field-dependent ones (71). This AdS/CFT transform can be defined only in the framework of extended coset manifolds $\{y^\mu, \varphi, \Omega^\mu\}$ and $\{x^\mu, q, \lambda^\mu\}$. In [18], (73) is interpreted in a setting where all coset parameters are independent.

Using (73), any Minkowski space conformal field theory with a dilaton among its basic fields can be projected onto the variables of the AdS brane and vice versa. Making, e.g., in (72) the change of variables inverse to (73), we find

$$S_{\rm NG} = \frac{1}{4m^2}\int d^{(p+1)}y e^{(1-p)\varphi} \times \frac{(\partial\varphi\partial\varphi)}{1 - \frac{1}{8m^2}e^{2\varphi}(\partial\varphi\partial\varphi)}\det\left(I + \frac{1}{2m^2}\mathcal{D}\Omega\right). \quad (74)$$

It is an equivalent form of the static-gauge action (72) of the p-brane in AdS_{p+2} as a nonlinear extension of the conformally invariant dilaton action in $\mathcal{M}_{p+1}$. In [17], the conformal field theory image of the full bosonic part of D3-brane action on $AdS_5 \times S^5$ was found.

6. SUPERCONFORMAL MECHANICS REVISITED

Conformal mechanics (CM) [37] and its superconformal extensions (SCM) [38, 39] are the simplest models of (super)conformal field theory. Recently, it was suggested [36] that the so-called "relativistic" generalizations of these $d = 1$ models are candidates for the conformal field theory dual to AdS_2 (super)gravity in the AdS_2/CFT_1 framework. The simplest model of this kind is a charged particle evolving on the $AdS_2 \times S^2$ background (the Bertotti–Robinson (BR) metric). It describes a near-horizon geometry of the extreme $d = 4$ Reissner–Nordström black hole.

Since the "old" and "new" (super)conformal mechanics models respect the same (super)conformal symmetry, these models can be expected to be equivalent to each other. The $d = 1$ version of the map (73) allows one to explicitly prove this conjecture [15]. Here, we illustrate this by the example of $N = 2$ superconformal mechanics.

The $N = 2$, $d = 1$ superconformal algebra $su(1, 1|1)$ includes as a subalgebra the $d = 1$ conformal algebra $so(2, 1)$

$$[P, D] = -P,\ [K, D] = K,\ [P, K] = -2D, \quad (75)$$

in parallel with the generators of Poincaré $\{Q, \bar{Q}\}$ and conformal $\{S, \bar{S}\}$ supersymmetries and the $U(1)$ generator U. The nonvanishing (anti)commutators of the latter read

$$\{Q, \bar{Q}\} = 2iP, \quad \{Q, \bar{S}\} = 2iD - 2iU, \quad (76)$$
$$\{S, \bar{S}\} = 2iK, \quad \{S, \bar{Q}\} = 2iD + 2iU,$$
$$\left[P, \begin{pmatrix} S \\ \bar{S} \end{pmatrix}\right] = -\begin{pmatrix} Q \\ \bar{Q} \end{pmatrix}, \quad \left[K, \begin{pmatrix} Q \\ \bar{Q} \end{pmatrix}\right] = \begin{pmatrix} S \\ \bar{S} \end{pmatrix},$$
$$\left[D, \begin{pmatrix} Q \\ \bar{Q} \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} Q \\ \bar{Q} \end{pmatrix},$$
$$\left[D, \begin{pmatrix} S \\ \bar{S} \end{pmatrix}\right] = -\frac{1}{2}\begin{pmatrix} S \\ \bar{S} \end{pmatrix},$$
$$\left[U, \begin{pmatrix} Q \\ \bar{Q} \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} Q \\ -\bar{Q} \end{pmatrix},$$
$$\left[U, \begin{pmatrix} S \\ \bar{S} \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} S \\ -\bar{S} \end{pmatrix}.$$

The standard nonlinear realization of $SU(1, 1|1)$ as the $d = 1$, $N = 2$ superconformal group is set up as left multiplications of the coset

$$g = e^{tP}e^{\theta Q + \bar{\theta}\bar{Q}}e^{qD}e^{\lambda K}e^{\psi S + \bar{\psi}\bar{S}}, \quad (77)$$

where $(t, \theta, \bar{\theta}) \equiv z$ are coordinates of the $N = 2$, $d = 1$ superspace and the remaining coset parameters are superfields given on this superspace. The transformation rules of the supercoset parameters and the structure of the related left-covariant Cartan superforms can be found in [39]. We only note that, on the $N = 2$, $d = 1$ superspace coordinates, one recovers the standard $N = 2$ superconformal transformations, while all the superfield coset parameters are expressed through $q(z)$ by the appropriate inverse Higgs constraints:

$$\lambda = \frac{1}{2}e^q\dot{q}, \quad \bar{\psi} = -\frac{i}{2}e^{q/2}Dq, \quad (78)$$
$$\psi = -\frac{i}{2}e^{q/2}\bar{D}q,$$
$$D = \frac{\partial}{\partial\theta} + i\bar{\theta}\partial_t, \quad \bar{D} = \frac{\partial}{\partial\bar{\theta}} + i\theta\partial_t,$$
$$\{D, \bar{D}\} = 2i\partial_t.$$

The invariant action of $N = 2$ SCM reads

$$S_{N=2} = \int dt d^2\theta\left[\frac{\mu}{2}DY\bar{D}Y \right. \quad (79)$$

$$+\sqrt{\mu\gamma}\ln Y\Big], \quad Y=e^{q/2}.$$

Its bosonic core coincides with the action of the standard "old" conformal mechanics [37]

$$S=\frac{1}{2}\int dt\left(\frac{1}{4}\mu e^{u}\dot{u}^2-\gamma e^{-u}\right) \tag{80}$$
$$=\frac{1}{2}\int dt(\mu\dot{x}^2-\frac{\gamma}{x^2}),$$

upon the identifications $q|_{\theta=0}=u(t)$, $x(t)=e^{(1/2)u(t)}$ and eliminating the auxiliary field $[D,\bar{D}]q|_{\theta=0}$ by its equation of motion.

Now we pass in (75), (76) to the AdS basis

$$\hat{K}=mK-\frac{1}{m}P, \quad \hat{D}=mD, \tag{81}$$
$$\hat{S}=mS, \quad \hat{\bar{S}}=m\bar{S}.$$

We define the realization of $SU(1,1|1)$ in the AdS basis by its left action on the coset $SU(1,1|1)/U(1)$ in the following parametrization:

$$g=e^{yP}e^{\theta Q+\bar{\theta}\bar{Q}}e^{\Phi\hat{D}}e^{\Omega\hat{K}}e^{\xi\hat{S}+\bar{\xi}\hat{\bar{S}}}. \tag{82}$$

As in the case of a standard nonlinear realization, one can directly find the transformation rules of the superspace coordinates and Goldstone superfields. As distinct from the standard case, the transformation laws of coordinates now essentially include Goldstone superfields; i.e., we face a field-dependent realization of the $N=2$ superconformal group. The only essential Goldstone superfield is Φ; the remaining ones are eliminated by the inverse Higgs constraints:

$$\Lambda=e^{m\Phi}\partial_y\Phi\frac{1}{1+\sqrt{1-e^{2m\Phi}(\partial_y\Phi)^2}}, \tag{83}$$
$$\xi=-\frac{i}{2}\frac{1+\Lambda^2}{\sqrt{1-\Lambda^2}}e^{(m/2)\Phi}\bar{D}_y\Phi.$$

By comparing two different parametrizations of the same coset $SU(1,1|1)/U(1)$, Eqs. (77) and (82), one finds the $N=2$ extension of the $d=1$ version of the map (73)

$$t=y-\frac{1}{m}e^{m\Phi}\Lambda, \quad q=m\Phi+\ln(1-\Lambda^2), \tag{84}$$
$$\lambda=m\Lambda, \quad \psi=m\xi, \quad \bar{\psi}=m\bar{\xi}.$$

Now we are prepared to obtain the invariant superfield action which is pertinent to the above AdS realization of the $N=2$, $d=1$ superconformal group and so is expected to describe the $N=2$ superextension of the bosonic BR particle action. One should perform the transformation (84) in the "old" $N=2$ SCM action (79). For simplicity, we choose $\gamma=0$, which amounts to requiring zero vacuum energy. We obtain

$$S=\frac{\mu m^2}{8}\int dyd^2\theta e^{m\Phi} \tag{85}$$
$$\times\left(\frac{1-\Lambda^2}{1+\Lambda^2}-\frac{1}{m}e^{m\Phi}\partial_y\Lambda\right)\frac{(1+\Lambda^2)^2}{1-\Lambda^2}D_y\Phi\bar{D}_y\Phi,$$

where Λ is expressed through Φ according to (83).

It is straightforward to pass to the component fields in (85) and to show that, when all fermions are discarded, $F=0$ on shell. After substituting this into the pure bosonic part of the component action, the latter, up to a total derivative in the Lagrangian, becomes

$$S_{\rm bos}=\frac{\mu m^2}{4}\int dye^{-m\phi}\left(1-\sqrt{1-e^{2m\phi}(\partial_y\phi)^2}\right), \tag{86}$$

which is just the bosonic worldline action of BR particle in a static gauge, with the angular S^2 variables being "frozen."

Thus, (85) provides a manifestly $N=2$ supersymmetric off-shell form of the $N=2$ superconformal extension of the "new" CM action (86) which describes the radial (AdS_2) motion of the charged particle in the BR $AdS_2\times S^2$ background. By construction, it is related by the equivalence transformation (84) to the $\gamma=0$ case of the "old" $N=2$ SCM action (79).

The classical equivalence between the "old" and "new" (S)CM models can hopefully be extended to the quantum case and used to solve the quantum mechanics of the charged AdS_2 (super)particles in terms of (super)conformal quantum mechanics. In the classical Hamiltonian approach, this equivalence, both for the radial motion and with the angular S^2 variables taken into account, was proved in a recent paper [40].

7. CONCLUSIONS

In this paper, I reviewed the recent progress in generalizing the PBGS approach to the case of superbranes on the super AdS backgrounds. The off-shell world-volume actions of superbranes on supermanifolds with the AdS_4, AdS_5, and $AdS_5\times S^1$ even parts were derived from the appropriate nonlinear realizations of $N=1$, $d=3$ and $N=1$, $d=4$ superconformal symmetries associated with the supergroups $OSp(1|4)$ and $SU(2,2|1)$ [13, 14, 16]. The nonlinear realizations constructed describe the partial one-half breaking of these superconformal symmetries, and the superbrane actions are the corresponding Goldstone superfield actions. I also described a new aspect of the AdS/CFT correspondence. It

consists in existence of the coordinate transformation which relates the standard realizations of (super)conformal groups in field theories with spontaneously broken scale invariance and their realizations in superbranes as groups of (super)isometries of AdS (super)backgrounds [15, 17, 18]. One of the corollaries of this map is an equivalence of various "old" and "new" (super)conformal mechanics models, which could be helpful in solving the second type of models in terms of the simpler first type, both at the classical and at the quantum level [15, 40].

Among problems for further study, let me mention generalizing the PBGS approach to more complicated superconformal groups and superbackgrounds, such as $N \geq 2$, $d = 4$ supergroups $SU(2,2|N)$, and the related super $\mathrm{AdS}_5 \times S^m$-type backgrounds. In this way, one can hope to construct manifestly world-volume supersymmetric actions for AdS D3 branes which should amount to superconformal extensions of supersymmetric Born–Infeld actions. The corresponding versions of the AdS/CFT map (73), (84) could be of help in establishing these equivalences. Another interesting related problem is to generalize the manifestly world-volume supersymmetric PBGS approach to branes on PP-wave type backgrounds.

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Optical Transitions in Parabolic Quantum Dot*

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Abstract—Direct optical absorption of light is theoretically investigated in a spherical quantum dot from GaAs. The confinement potential of the dot is approximated as parabolic. Three regimes of size quantization are discussed: weak, strong, and intermediate. The corresponding threshold frequencies of absorption are determined. A comparison with the case of a spherical quantum dot with rectangular infinitely high confinement potential is performed. © *2005 Pleiades Publishing, Inc.*

1. INTRODUCTION

The investigation of spectra of optical absorption of different semiconductor structures still remains one of the most powerful tools to determine their band structure. From the viewpoint of theoretical investigation of absorption processes in semiconductors, the most convenient is the study of the absorption threshold of the considered sample. The theory of both direct and indirect (with third body) optical transitions is evolved for such absorption [1]. A lot of articles are devoted to theoretical and experimental investigation of optical absorption in both massive and size-quantized semiconductors (see, e.g., [2–5]).

The presence of size quantization essentially affects the nature of absorption. Indeed, the formation of size-quantization levels near extrema of bands makes possible the appearance of new interband transitions. The influence of size quantization on the optical transitions in semiconductor films and wires is considered in detail in [6]. At the same time, zero-dimensional structures or quantum dots (QDs), in which the particle spectrum is completely quantized, are the most interesting objects from the viewpoint of effects of size quantization. Such objects in many respects are like real atoms; therefore, not occasionally, they are also called "artificial atoms" [6].

Optical transitions in QDs were discussed by different authors. The article [7] is one of the first articles devoted to these problems. In this article, light absorption in semiconductor spheres disseminated into a dielectric matrix was considered. In the scope of the model of an infinitely high spherical dot, interband transitions at different regimes of size quantization were considered. Further, the authors of [8] investigated the influence of anisotropy of band structure on the nature of optical transitions in spherical QDs (with the same confinement potential) made of lead sulfide and selenide of lead. It was shown that the strong anisotropy of the band structure of PbS and PbSe results in appearance of optical transitions, forbidden in an isotropic approximation. On the other hand, beginning from 1990s, in connection with the interpretation of some magneto-optical experiments on QDs [9–11], the attention towards QDs with parabolic confinement potential sharply increased. The question is in the fact that the shape of a potential well is determined by the variation of composition of isovalent components. As a result of some mixing of components during the growth of QD, smoothing of the shape of the sample confinement potential takes place. Therefore, as the first approximation, the usage of the parabolic dot approximation is quite reasonable [12, 13]. In this connection, the consideration of optical transitions in QDs with a parabolic confinement potential is interesting.

In the article, direct optical transitions in spherical QD with a parabolic confinement potential are investigated.

2. THEORY

Let us consider a spherical symmetrical QD with confinement potential of the form

$$V_{\text{conf}}(r) = \frac{\mu\omega^2 r^2}{2},$$

where ω is the frequency of the QD confinement potential ($\omega \sim \hbar/(\mu r_0^2)$, r_0 is QD radius), and μ is electron effective mass (or of the hole, moreover, $\mu_e \ll \mu_h$). By analogy with [7], direct optical transitions in the considered system will be discussed for three cases of size quantization.

*The text was submitted by the authors in English.

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(a) Regime of strong size quantization $a_{\rm B}^e,\ a_{\rm B}^h \gg r_0$ ($a_{\rm B}^e, a_{\rm B}^h$ are effective Bohr radii of electron and hole);

(b) Regime of weak size quantization $a_{\rm B}^e,\ a_{\rm B}^h \ll r_0$;

(c) Regime of intermediate size quantization $a_{\rm B}^h \ll r_0 \ll a_{\rm B}^e$.

2.1. Regime of Strong Size Quantization

In this case the Coulomb interaction between electron and hole can be neglected and, accordingly, the wave function of these particles in QD can be written as [14]

$$\Psi_{n_r,l,m}^{e(h)}(r,\theta,\varphi) = C_n^{e(h)} r^l e_1^{-\lambda_{e(h)} r^2/2} \times {}_1F_1[-n_r, l+3/2; \lambda_{e(h)} r^2] Y_{lm}(\theta,\varphi), \quad (1)$$

where $C_n^{e(h)}$ is the normalizing constant, $n = 2n_r + l$ is the principal quantum number, $\{n_r, l, m\}$ are quantum numbers, $\lambda_{e(h)}$ are oscillator length of electron and hole, ${}_1F_1[a,b;x]$ is a confluent hypergeometrical function of the first kind, and $Y_{lm}(\theta,\varphi)$ are spherical functions. The energy levels corresponding to these wave functions are determined in the following way [14]:

$$E_n^{e(h)} = \hbar\omega_{e(h)}\left(n + \frac{3}{2}\right), \quad \omega_{e(h)} \sim \frac{\hbar}{\mu_{e(h)} r_0^2}.$$

According to [7], the absorption coefficient may be written as

$$K = A \sum_{n_r,n_r',l,l',m,m'} \left| \int \Psi_{n_r,l,m}^{e}(r,\theta,\varphi) \times \Psi_{n_r',l',m'}^{h}(r,\theta,\varphi) dV \right|^2 \delta(\hbar\tilde{\omega} - \varepsilon_g - E_n^e - E_{n'}^h),$$

where A is the quantity, proportional to the square of the modulus of dipole matrix element, taken on Bloch functions; ε_g is the width of the forbidden band; and $\tilde{\omega}$ is the frequency of incident light. After the integration (using selection rules $m = -m'$, $l = l'$), one can obtain the expression

$$K = A \sum_{n_r,n_r',l} B_{n_r,n_r',l} |I_{n_r,n_r',l}|^2 \times \delta(\hbar\tilde{\omega} - \varepsilon_g - E_n^e - E_{n'}^h),$$

where $B_{n_r,n_r',l}$ is some new constant, and

$$I_{n_r,n_r',l} = \frac{1}{2}\Gamma\left(l + \frac{3}{2}\right)\left(\frac{\lambda_e + \lambda_h}{2}\right)^{-(n_r+n_r'+l+3/2)} \times \left(\frac{\lambda_e - \lambda_h}{2}\right)^{n_r}\left(\frac{\lambda_e - \lambda_h}{2}\right)^{n_r'} \times {}_2F_1\left[-n_r, -n_r', l+\frac{3}{2}; -\frac{4\lambda_e\lambda_h}{(\lambda_h - \lambda_e)^2}\right],$$

where $\Gamma(x)$ is the Euler gamma function, and ${}_2F_1[a,b,c;x]$ is a hypergeometrical function.

2.2. Regime of Weak Size Quantization

At weak size quantization, system energy is mainly conditioned by Coulomb interaction between the electron and the hole. Therefore, the wave function of the system may be written in the form

$$f(\mathbf{r}_e, \mathbf{r}_h) = \varphi(\mathbf{r})\Psi_{n_r,l,m}(\mathbf{R}),$$

where $\mathbf{r} = \mathbf{r}_e - \mathbf{r}_h$, $\mathbf{R} = \dfrac{\mu_e\mathbf{r}_e + \mu_h\mathbf{r}_h}{\mu_e + \mu_h}$, $\varphi(\mathbf{r})$ is the wave function of relative motion, and $\Psi_{n_r,l,m}(\mathbf{R})$ is the wave function of exciton center of mass determined by Eq. (1), where $\mu = \mu_e + \mu_h$ is inserted instead of $\mu_{e(h)}$. The system energy will be written as

$$E = \hbar\Omega\left(n + \frac{3}{2}\right) - E_{\rm ex},$$

where $\Omega \sim \dfrac{\hbar}{(\mu_e + \mu_h) r_0^2}$, and $E_{\rm ex}$ is exciton energy.

Due to the electron localization in a quite close neighborhood of the QD center, for K one can write

$$K = A \sum_{n_r,l,m} |\varphi(0)|^2 \left| \int \Psi_{n_r,l,m}(\mathbf{R}) d\mathbf{R} \right|^2 \times \delta(\hbar\tilde{\omega} - \varepsilon_g + E_{\rm ex} - \hbar\Omega(n + 3/2)). \quad (2)$$

Taking into account that $\varphi(0) \neq 0$ only for the ground state ($l = m = 0$), after integration, one can obtain

$$K = \frac{32}{(\lambda a_{\rm ex})^3}\Gamma\left(\frac{3}{2}\right)\sum_{n_r} {}_2F_1^2\left[n_r, \frac{3}{2}, \frac{3}{2}; 2\right] \times D_{n_r}\delta\left(\hbar\tilde{\omega} - \varepsilon_g + E_{\rm ex} - \hbar\Omega\left(n + \frac{3}{2}\right)\right), \quad (3)$$

where D_{n_r} is some constant, and $\lambda = (\mu_e + \mu_h)\Omega/\hbar$, $a_{\rm ex}$ is exciton radius. In (3), the circumstance is taken into account that, in integral in (2), only the states $l = m = 0$ are nonzero, as follows from the properties of spherical functions $Y_{lm}(\theta,\varphi)$.

2.3. Regime of Intermediate Size Quantization

Taking into account that, in this case, an electron moves much faster than a heavy hole, the adiabatic approximation can be used [7]. Then the motion of the hole will take place under the influence of the spherical symmetrical potential of the form

$$V_{n_r,l,m}(\mathbf{r}) = -\frac{e^2}{\chi}\int \frac{|\Psi_{n_r,l,m}(r,\theta,\varphi)|^2}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}', \quad (4)$$

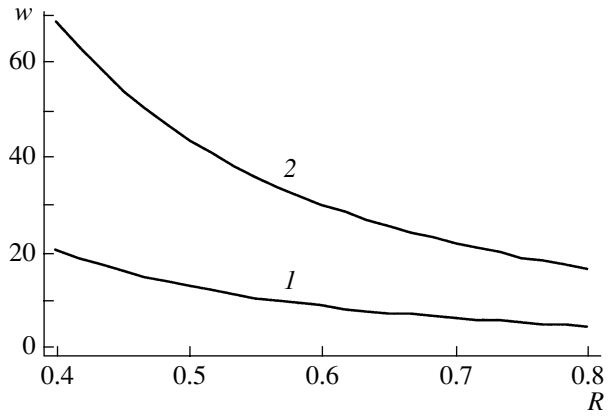

Fig. 1. The dependences of absorption threshold frequencies on QD radius in units for the regime of strong size quantization. Curve *1* corresponds to the case of parabolic confinement potential, and curve *2* corresponds to the rectangular infinitely high potential well.

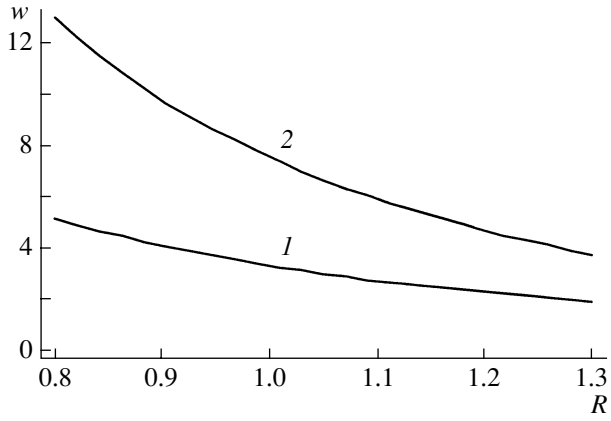

Fig. 2. The same as in Fig. 1, but for the regime of intermediate size quantization.

where χ is the dielectric constant of the QD. After integration, for the potential, one can obtain

$$V_0(r) = -\frac{e^2\sqrt{\lambda_e}}{4\pi^{3/2}\chi} + \frac{\mu_h\omega_0^2 r^2}{2},$$

where

$$\frac{\mu_h\omega_0^2}{2} = \frac{2e^2\lambda_e^{3/2}}{3\pi^{3/2}\chi}.$$

Due to the spherical symmetry of potential (4), the quantum numbers of the new problem are orbital l', azimuthal m', and radial n'_r numbers. Therefore, the hole energy levels can be written as

$$E_0^{n'_r,l'} = -\frac{e^2\sqrt{\lambda_e}}{4\pi^{3/2}\chi} + \hbar\omega_0\left(2n'_r + l' + \frac{3}{2}\right).$$

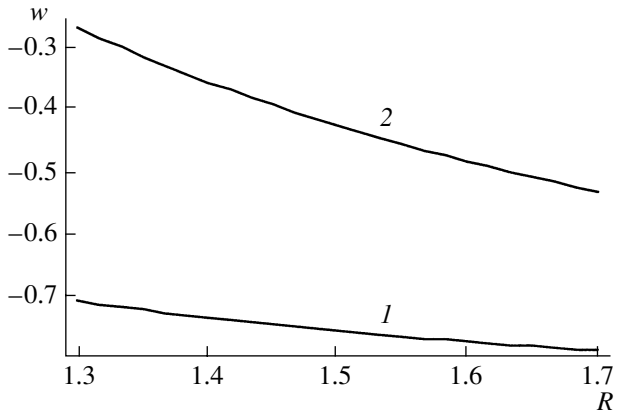

Fig. 3. The same as in Fig. 1, but for the case of weak size quantization.

In this case, for the transition connected with the electron level $n = 1, m = 0$ at different $n'_r (l' = 0, m' = 0)$, the expression for the quantity K is obtained:

$$K = \frac{AC_1^2}{4\pi^{1/2}}\left(\frac{\hbar}{\mu_h\omega_0}\right)^{3/2}\sum_{n'_r}\frac{(2n'_r)!}{2^{2n'_r}(n'_r!)^2}$$
$$\times\,\delta\left(\hbar\tilde{\omega} - \varepsilon_g - \hbar\omega_e\frac{5}{2} + \frac{e^2\sqrt{\lambda_e}}{4\pi^{3/2}\chi}\hbar\omega_0\left(2n'_r + \frac{3}{2}\right)\right).$$

3. CONCLUSION

The expressions for QD light absorption coefficients at different regimes of size quantization obtained above allow one to determine the corresponding threshold frequencies of incident light beginning from which this quantization takes place. Thus, in the case of strong size quantization, the following expression holds for absorption threshold frequency:

$$\hbar\tilde{\omega}_0^1 = \varepsilon_g + \frac{3\hbar}{2}(\omega_e + \omega_h). \qquad (5)$$

At weak size quantization, this frequency is determined from the equality

$$\hbar\tilde{\omega}_0^2 = \varepsilon_g - E_{\rm ex} + \frac{3\hbar\Omega}{2}. \qquad (6)$$

Finally, in the case of intermediate size quantization, this frequency looks like

$$\hbar\tilde{\omega}_0^3 = \varepsilon_g + \frac{3\hbar}{2}(\omega_e + \omega_0) - \frac{e^2\sqrt{\lambda_e}}{4\pi^{3/2}\chi}. \qquad (7)$$

Let us also mention that, with the help of Eqs. (5)–(7), one can easily determine the dependences of absorption threshold frequencies on QD sizes. Therefore, one needs to express $\omega_e, \omega_h, \Omega$ frequencies through the QD radius. For the case of strong size

quantization, this dependence is presented in Fig. 1 (in $w = (\hbar\tilde{\omega}_0 - \varepsilon_g)/E_R$, $R = r_0/a_B$ units). Curve *1* corresponds to the case of parabolic confinement potential, and curve *2* corresponds to the rectangular infinitely high potential well. In calculations, we took into account the equality $\omega = \zeta\hbar/(\mu r_0^2)$, where ζ is some constant for the value of the parameter $\zeta = 1$ which corresponds to the frequency of the confinement potential, e.g., for an electron in QD of GaAs, $\omega = 1.6 \times 10^{13}$ cm^{-1}. As follows from Fig. 1, at this value of ζ, the curve of the $w(R)$ dependence for the parabolic case is lower than the curve for the rectangular case. Along with increasing r_0, when the role of size quantization decreases, curves *1* and *2* decrease and come close to each other. Vice versa, at small r_0, the role of size quantization sharply increases and, therefore, the effective width of the forbidden band increases. This fact conditions the growth of values of absorption threshold frequencies. The analogous dependences corresponding to two other regimes of size quantization are presented in Figs. 2 and 3 (Fig. 2 corresponds to the regime of intermediate size quantization, and Fig. 3 corresponds to the regime of weak size quantization). In these figures, one can see that the curve corresponding to QD parabolic confinement is lower than the curve corresponding to the case of the rectangular infinitely high well. At this, the qualitative behavior of $w(R)$ dependence curves does not change. Negative values in the graph of Fig. 3 are conditioned by the fact that, in case of weak size quantization, the input into the energy of the system is mainly due to the Coulomb interaction. In conclusion, let us mention that, in the case of strong size quantization, the selection rules change. In contrast to the case considered by the authors of [7], in our case the transitions may take place between levels with different n_r and n_r'.

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Noncommutative Versions of Born–Infeld Theory*

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Abstract—A new generalization of nonlinear Born–Infeld electrodynamics is proposed. It is inspired by the noncommutative geometry and a new interpretation of gauge theories. The variational principle introduced here leads to quite complicated nonlinear equations, which can be solved numerically in certain cases. The spherically symmetric ansatz is analyzed, and static finite-energy solutions are obtained via numerical integration. Then a pure Higgs sector Lagrangian is introduced by analogy with the non-Abelian Born–Infeld generalization. A spherically symmetric configuration and a time-dependent homogeneous field are investigated and qualitative behavior of solutions are discussed. © *2005 Pleiades Publishing, Inc.*

1. INTRODUCTION

After Coulomb's law had been formulated in the 18th century, it was clear that the electric forces become infinite for pointlike particles. Later on, when Maxwell found his final and elegant mathematical formulation of electrodynamics, with the introduction of the energy–momentum tensor of electromagnetic field, the energy remained infinite for pointlike charges. After the discovery of the electron, physicists started to look for models able to represent it as an extended, finite-dimensional particle, endowed with finite distribution of charge and energy densities. The model proposed by Mie [1] could be considered as the most successful one at the time it was published. It was based on the idea that Maxwell's electrodynamics should be considered as a linear approximation of a certain nonlinear theory; as long as the field strength is not too high, the linear theory describes almost perfectly its behavior far away from the source, which can be considered pointlike as seen from great distance; the nonlinear effects should become dominant at small distances, where the extended nature of elementary charges must be taken into account.

To this end, Mie [1] introduced the notion of maximal field strength $\mathbf{E}_0$, and in order to make it impossible for any electric field to go beyond this value, he modified Maxwell's theory by introducing the following nonlinear Lagrangian density for a pure electric field:

$$\mathcal{L} = \sqrt{1 - \frac{\mathbf{E}^2}{\mathbf{E}_0^2}}. \tag{1}$$

Although the nonlinear theory derived from this Lagrangian enabled Mie to obtain a nonsingular, finite-energy solution, it was clear that such a Lagrangian cannot represent a Lorentz-invariant theory, especially since the magnetic field contribution was absent. This is why Born and Infeld (BI) [2] have introduced a Lorentz-invariant Lagrangian density, defined as follows:

$$\begin{aligned}\mathcal{L}_{\rm BI}(g,F) &= L_{\rm BI}(g,F)\sqrt{|g|} = \beta^2\left(\sqrt{|\det(g_{\mu\nu})|}\right.\\ &\left. - \sqrt{|\det(g_{\mu\nu}+\beta^{-1}F_{\mu\nu})|}\right)\\ &= \beta^2\left(1 - \sqrt{1+\frac{1}{\beta^2}(\mathbf{B}^2-\mathbf{E}^2) - \frac{1}{\beta^4}(\mathbf{E}\cdot\mathbf{B})^2}\right)\sqrt{|g|}.\end{aligned} \tag{2}$$

The constant β appears for dimensional reasons and plays the same role here as the limiting value of the electric field in Mie's nonlinear electrodynamics. Defining

$$P = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \text{ and } S = \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu},$$
$$\text{with } \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho},$$

we can write

$$L_{\rm BI} = \beta^2\left[1 - \sqrt{1+2P-S^2}\right]. \tag{3}$$

Since Dirac introduced his equation for the electron, the interest in classical models of charged particles has considerably faded. Only in 1970 did Boillat [3] consider BI electrodynamics in order to study its propagation properties. Investigating general nonlinear theories derived from a Lagrangian depending on two Lorentz invariants $\mathcal{L}(P,S)$, he discovered that

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the BI electrodynamics is the only one ensuring the absence of birefringence, i.e., propagation along a single light cone, and the absence of shock waves. In this respect, the BI theory is unique (except for another singular Lagrangian $\mathcal{L} = P/S$). A beautiful discussion of these properties can be found in Bialynicki-Birula's paper [4]; see also [5].

An unexpected new impulse for the revival of interest in BI electrodynamics and in its non-Abelian generalizations came from string theories, in which BI-type Lagrangians appear in effective action for D branes.

Another motivation for studying BI-type theories is the possibility of existence of solitonic solutions in nonlinear field theories. In a pure Yang–Mills theory in flat spacetime, with the usual Lagrangian density $\mathcal{L}_{\rm YM} = -\frac{1}{4} g_{AB} F^A_{\mu\nu} F^{B\mu\nu}$, there are no finite-energy static nonsingular solutions. This fact can be explained qualitatively by the conformal invariance of the theory and the tracelessness of the energy–momentum tensor,

$$T^\mu_\mu = -T_{00} + \sum_{i=1}^{3} T_{ii} = 0. \qquad (4)$$

Given the positivity of energy, $T_{00} > 0$, this means that the sum of principal pressures is positive, $\sum T_{ii} > 0$, leading to the conclusion that Yang–Mills "matter" is naturally subjected to repulsive forces only.

In the presence of a Higgs field, the conformal invariance is broken, and this leads to the existence of solitonic solutions like 't Hooft's [6] and Prasad–Sommerfield's [7] magnetic monopoles. In what follows, we are looking for soliton-like solutions arising in other nonlinear theories, including non-Abelian versions of BI theory, which are not conformally invariant, as well as the pure Higgs field model with a generalized BI-type Lagrangian.

2. NON-ABELIAN GENERALIZATIONS OF BORN–INFELD THEORY

2.1. Basic Properties of Abelian Born–Infeld Theory

In their original paper [2], Born and Infeld considered the now famous least action principle:

$$S_{\rm BI}[g,F] = \int_{R^4} \beta^2 \Big(\sqrt{|\det(g_{\mu\nu})|} - \sqrt{|\det(g_{\mu\nu} + \beta^{-1} F_{\mu\nu})|} \Big) d^4x.$$

This action can be defined not only on the Minkowskian spacetime but also on any locally Lorentzian curved manifold.

It is useful to recall here the three basic properties of this Lagrangian which we want to maintain in the case of a non-Abelian generalization:

(i) Maxwell's theory (respectively, usual gauge theory with quadratic Lagrangian density) is found in the limit $\beta \to \infty$:

$$S_{\rm BI} = -\int_{R^4} \frac{1}{2} (\mathbf{B}^2 - \mathbf{E}^2) \sqrt{|g|} d^4x + o\left(\frac{1}{\beta^2}\right). \qquad (5)$$

(ii) There exists an upper limit for the electric field intensity, equal to β when the magnetic components of the field vanish:

$$L_{\rm BI}|_{B=0} = \beta^2 \left(1 - \sqrt{1 - \beta^{-2}\mathbf{E}^2}\right). \qquad (6)$$

Due to this fact, the energy of a pointlike charge is finite, and the field remains finite even at the origin. This was the main goal pursued by Mie [1] in suggesting the choice of nonlinear generalization of Maxwell's theory. Indeed, for a point charge e, one has

$$\mathbf{E} = \frac{e\mathbf{n}}{\sqrt{e^2 + r^4}} \quad (\mathbf{n} = \mathbf{r}/r), \qquad (7)$$

$$\text{mass} = \int_0^\infty (\sqrt{e^2 + r^4} - r^2) dr.$$

(iii) The BI action principle is invariant under the diffeomorphisms of R^4 and gauge transformations. In this respect, this theory can be viewed as a covariant generalization (in the sense of General Relativity) of Mie's theory, as well as an extension of the usual volume element $\sqrt{|g|} d^4x$.

It is also well known that the BI electromagnetism has good causality properties as well as interesting dual symmetries (electric–magnetic duality, Legendre duality). Here, we shall not consider these properties, our main interest being focused on static solutions.

2.2. First Non-Abelian Generalizations of Born–Infeld Theory

The idea of non-Abelian generalization of the BI theory Lagrangian has been in the air since the end of the 1970s. Hagiwara discussed various possibilities in [8]; however, he did not try to find soliton-like solutions. In 1997, Tseytlin [9] argued in favor of the symmetrized trace prescription which reproduced in the first four orders the string effective action for gauge potential. Finally, Park [10] introduced yet another non-Abelian generalization and investigated qualitative behavior of instanton-like solutions. Also supersymmetric generalization was proposed for Abelian and non-Abelian versions [11, 12].

Only instanton-like solutions were discussed in the aforementioned papers. The first solitons in Minkowskian spacetime were found in [13], which we briefly recall in this section.

A straightforward generalization of BI theory in four dimensions can be achieved by replacing the quadratic invariants of $U(1)$ theory by the non-Abelian ones:

$$F_{\mu\nu}F^{\mu\nu} \to F^a_{\mu\nu}F^{\mu\nu}_a \quad \text{and} \quad F_{\lambda\rho}\tilde{F}^{\lambda\rho} \to F^a_{\lambda\rho}\tilde{F}^{\lambda\rho}_a. \tag{8}$$

The corresponding action is

$$S = \frac{\beta^2}{4\pi}\int (1-\mathcal{R})d^4x, \tag{9}$$

$$\mathcal{R} = \sqrt{1 + \frac{1}{2\beta^2}F^a_{\mu\nu}F^{\mu\nu}_a - \frac{1}{16\beta^4}(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a)^2}.$$

It is easy to see that the BI nonlinearity breaks the conformal symmetry ensuring the nonzero trace of the stress–energy tensor

$$T^\mu_\mu = \mathcal{R}^{-1}\left[4\beta^2(1-\mathcal{R}) - F^a_{\mu\nu}F^{\mu\nu}_a\right] \neq 0. \tag{10}$$

This quantity vanishes in the limit $\beta \to \infty$, when the theory reduces to the standard one. For the Yang–Mills field, we assume the usual monopole ansatz:

$$A^a_0 = 0, \quad A^a_i = \epsilon_{aik}\frac{n^k}{r}(1-w(r)), \tag{11}$$

where $n^k = x^k/r$, $r = (x^2+y^2+z^2)^{1/2}$, and $w(r)$ is a real-valued function. After integration over the sphere in (9), one obtains a two-dimensional action from which β can be eliminated by the coordinate rescaling $\sqrt{\beta}t \to t, \sqrt{\beta}r \to r$. The following static action results then:

$$S = \int L dr, \quad L = r^2(1-\mathcal{R}) \tag{12}$$

$$\text{with } \mathcal{R} = \sqrt{1 + 2\frac{w'^2}{r^2} + \frac{(1-w^2)^2}{r^4}},$$

where the prime denotes the derivative with respect to r. The corresponding equation of motion reads

$$\left(\frac{w'}{\mathcal{R}}\right)' = \frac{w(w^2-1)}{r^2\mathcal{R}}. \tag{13}$$

A trivial solution $w \equiv 0$ corresponds to the point-like magnetic BI monopole with unit magnetic charge (embedded $U(1)$ solution). In the BI theory, it has a finite self-energy. For time-independent configurations, the energy density is equal to minus the Lagrangian, so the total energy (mass) is

$$M = \int_0^\infty (\mathcal{R}-1)r^2 dr. \tag{14}$$

For $w \equiv 0$, one finds the usual BI monopole mass [see Eq. (7)]

$$M = \int \left(\sqrt{r^4+1} - r^2\right) dr \tag{15}$$

$$= \frac{\pi^{3/2}}{3\Gamma(3/4)^2} \approx 1.23604978.$$

Looking now for the essentially non-Abelian solutions of finite mass, we observe that, in order to assure the convergence of the integral (14), the quantity $\mathcal{R}-1$ must fall faster than r^{-3} as $r \to \infty$. Thus, far from the core, the BI corrections have to vanish and Eq. (13) should reduce to the ordinary Yang–Mills equation, equivalent to the following two-dimensional autonomous system:

$$\dot{w} = u, \quad \dot{u} = u + (w^2-1)w, \tag{16}$$

where a dot denotes the derivative with respect to $\tau = \ln r$. This dynamical system has three nondegenerate stationary points $(u=0, w=0,-1)$, from which $u=w=0$ is a focus, while two others $u=0, w=-1$ are saddle points with eigenvalues $\lambda = -1$ and $\lambda = 2$. The separatrices along the directions $\lambda = -1$ start at infinity and, after passing through the saddle points, go to the focus with the eigenvalues $\lambda = (1-i\sqrt{3})/2$. It has been proved in [13] that *the only finite-energy configurations with nonvanishing magnetic charge are the embedded U(1) BI monopoles*. Indeed, such solutions should have asymptotically $w=0$, which does not correspond to bounded solutions unless $w \equiv 0$. The remaining possibility is $w=-1, \dot{w}=0$ asymptotically, which corresponds to zero magnetic charge. Coming back to variable r, one finds from (13)

$$w = -1 + \frac{c}{r} + O(r^{-2}), \tag{17}$$

where c is a free parameter. This gives a convergent integral (14) as $r \to \infty$. The two values $w=-1$ correspond to two neighboring topologically distinct Yang–Mills vacua. Now consider local solutions near the origin $r=0$. For convergence of the total energy (14), w should tend to a finite limit as $r \to 0$. Then, using Eq. (13), one finds that the only allowed limiting values are $w=-1$ again. In view of the symmetry of (16) under reflection $w \to -w$, one can take without loss of generality $w(0)=1$. The following Taylor expansion satisfies Eq. (16):

$$w = 1 - br^2 + \frac{b^2(44b^2+3)}{10(4b^2+1)}r^4 + O(r^6), \tag{18}$$

with b being a unique free parameter. As $r \to 0$, the function $\mathcal{R}$ tends to

$$\mathcal{R} = \mathcal{R}_0 + O(r^2), \quad \mathcal{R}_0 = 1 + 12b^2. \tag{19}$$

Therefore, it is not a solution to the initial system (14). What remains to be done is to find appropriate values of constant b leading to smooth finite-energy solutions by gluing together the two asymptotic solutions between 0 and ∞.

It has been proved in [13] that *any regular solution to* Eq. (13) *belongs to the one-parameter family of local solutions* (18) *near the origin*. It follows that the global finite-energy solution starting with (18) should meet some solution from the family (17) at infinity. Since both of these local solutions are nongeneric, one can at best match them for some discrete values of parameters. This idea was used first in [14].

For some precisely tuned value of b, the solution will remain a monotonic function of τ reaching the value -1 at infinity. This happens for $b_1 = 12.7463$. By a similar reasoning, one shows that, for another fine-tuned value $b_2 > b_1$, the integral curve $w(\tau)$ which has a minimum in the lower part of the strip will be stabilized by the friction term in the upper half of the strip $[-1, 1]$ and tend to $w = 1$. This solution will have two nodes. Continuing this process, we obtain an increasing sequence of parameter values b_n for which the solutions remain entirely within the strip $[-1, 1]$ tending asymptotically to $(-1)^n$. The lower values b_n converge very rapidly to the limit value given by (15).

Some analogous solutions have been found in the symmetrized trace prescription in [15, 16].

2.3. A New Non-Abelian Generalization

In [17], we introduced a new non-Abelian generalization of the BI Lagrangian and found a family of nonsingular soliton-like solutions using 't Hooft's ansatz for the $SU(2)$ gauge potential. As in the case discussed in [13], and in contrast to the usual Yang–Mills case, soliton and magnetic monopole solutions were possible without the presence of a Higgs field or other scalar multiplets.

Our starting point is the gauge-field tensor associated with a compact and semisimple gauge group G, defined as a connection 1-form in the principal fiber bundle over Minkowskian spacetime, with its values in $\mathcal{A}_G$, the Lie algebra of G. We choose the representation of the connection in the tensorial product of a matrix representation of the Lie algebra $\mathcal{A}_G$ and the Grassman algebra of forms over M_4:

$$A = A^a_\mu dx^\mu \otimes T_a, \tag{20}$$

where T_a, $a, b = 1, 2, \ldots, N = \dim(G)$, is the anti-Hermitian basis of the particular representation R of dimension d_R of $\mathcal{A}_G$.

By analogy with the Abelian case, we want the Lagrangian to satisfy the following properties:

(i) One should find the usual Yang–Mills theory in the limit $\beta \to \infty$.

(ii) The (non-Abelian) analog of the electric field strength should be bounded from above when the magnetic components vanish. (To satisfy this particular constraint, we must ensure that the polynomial expression under the root sign should start with terms $1 - \beta^{-2}(E^a)^2 + \ldots$ when $B^a = 0$.)

(iii) The action should be invariant under diffeomorphisms of R^4 and gauge transformations.

The idea is to compute a determinant in the tensor product of endomorphisms of $\mathbb{R}^4$ and $R(\mathcal{A}_G)$. This enables us to introduce the following generalization of the BI Lagrangian density for a non-Abelian gauge field:

$$\mathcal{L} = \sqrt{|g|} - \left| \det_{\mathbb{C}^2 \otimes \mathbb{R}^4 \otimes R} (\mathbb{1}_2 \otimes g_{\mu\nu} \otimes \mathbb{1}_{d_R} + \beta^{-1} J \otimes F^a_{\mu\nu} \otimes T_a) \right|^{1/(4d_R)}. \tag{21}$$

In the expression above, J denotes an $SL(2, \mathbb{C})$ matrix satisfying $J^2 = -\mathbb{1}_2$, thus introducing a quasi-complex structure. This extra doubling of tensor space is necessary in order to ensure that the resulting Lagrangian is real. In the $SU(2)$ case, it is possible to compute the Lagrangian and we obtain

$$L = 1 - \sqrt[4]{(1 + 2P - Q^2)^2 + (2K_3)^2}, \tag{22}$$

where

$$\begin{cases} 2P &= \dfrac{1}{2} F^a_{\mu\nu} F^{\mu\nu}_a, \\ Q^2 &= \dfrac{1}{16} F^a_{\mu\nu} \tilde{F}^{b\mu\nu} F^c_{\alpha\beta} \tilde{F}^{d\alpha\beta} K_{acbd}, \\ K_3 &= \dfrac{1}{6} \epsilon_{abc} F^{a\mu}_\nu F^{b\nu}_\alpha F^{c\alpha}_\mu \end{cases} \tag{23}$$

with $K_{abcd} = \delta_{ab}\delta_{bc} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}$. We then study spherically symmetric static configurations by considering the well-known 't Hooft ansatz:

$$A = (1 - k(r))(T_\theta \sin\theta d\varphi - T_\varphi d\theta). \tag{24}$$

Then the action becomes

$$S = \int \left(1 - \sqrt{\left(1 + \left(\frac{1 - k^2}{r^2}\right)^2\right)\left(\left(1 + \frac{2\dot{k}^2}{r^4}\right)^2 + \left(\frac{1 - k^2}{r^2}\right)^2\right)} \right) e^{3\tau} d\tau \tag{25}$$

with $\tau = \ln(r)$.

The equations of motion can be written in the standard form:

$$\begin{cases} \dot{k} &= u, \\ \dot{u} &= \gamma(k,u,\tau)u + k(k^2-1) \end{cases} \tag{26}$$

with

$$\gamma(k,u,\tau) = 1 - 2\frac{u^2 + 2uk(1-k^2) + (1-k^2)^2}{r^4 + (1-k^2)^2} \tag{27}$$
$$+ \frac{6u(1-k^2)\left[ku^2 + 2u(1-k^2) + k(1-k^2)^2\right]\left[r^4 + 2u^2 + (1-k^2)^2\right]}{\left[r^4 + (1-k^2)^2\right]\left[(r^4 + 2u^2)^2 + (1-k^2)^2(r^4 + 6u^2)\right]}.$$

Although the equations display asymptotic expansions analogous to those found in [13, 18, 19], careful analysis shows that solutions of the Bartnik–McKinnon type [20] are excluded here.

Near the origin, there are two types of asymptotic development which satisfy the equations of motion:

$$k = k_0 + ar - k_0\left(\frac{5a^2}{6g} + \frac{g}{12a^2}\right)r^2 \tag{28}$$
$$+ \frac{a^8(52-70g) - 9a^4g^3 + (g-1)g^4}{108a^5g^2}r^3 + O(r^4),$$

where $g = 1 - k_0^2$, $a \neq 0$ and $g \neq 0$, are two free parameters.

A second development depends on only one free parameter b and starts as follows:

$$k = -\left(1 - br^2 + \frac{3b^2 + 92b^4 + 608b^6}{10 + 200b^2 + 1600b^4}r^4 + O(r^6)\right), \tag{29}$$

which corresponds to solutions along the separatrix with $\lambda = 2$ discussed in the previous section.

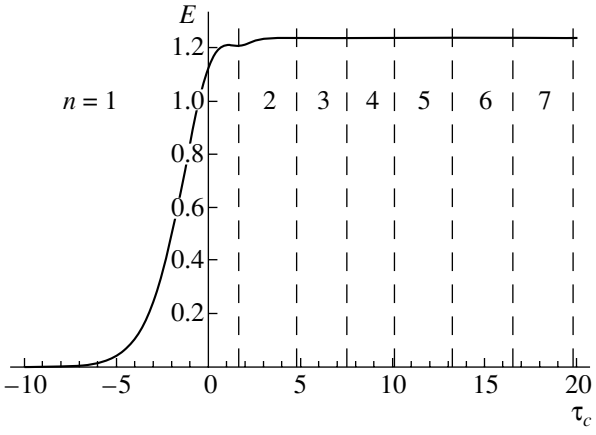

Fig. 1. Energy as function of the parameter τ_c.

At infinity, the Taylor expansion can be made with respect to r^{-1}. It depends on one free parameter, denoted by c:

$$k = -\left(1 - \frac{c}{r} + \frac{3c^2}{4r^2} + O\left(\frac{1}{r^3}\right)\right), \tag{30}$$

which corresponds to solutions along the separatrix with $\lambda = -1$ discussed in the previous section.

Taking these expansions as the first approximation either at $r = 0$ or at $r = \infty$, we then use standard numerical techniques in order to generate solutions valid everywhere. It was interesting to note that, when we started from infinity, no fine-tuning was necessary, and an arbitrarily fixed constant c would lead to a solution which, when extrapolated to $r = 0$, would define a particular pair of values of constants k_0 and a. On the contrary, starting from $r = 0$, arbitrarily chosen values of k_0 and a would not necessarily lead to good extrapolation at $r = \infty$. Therefore, the three parameters occurring in the asymptotic expansions must be interrelated by two constraint equalities. Then the solutions can be labeled by only one real parameter, and then the two parameters k_0 and a of (28) are functions of the parameter $\tau_c = \log(c)$.

We have evaluated the energy E of the solutions found and the values of the parameter k_0 for τ_c varying from -10 to 20. The energy E is represented as a function of the parameter τ_c in Fig. 1. The energies converge to the limit $E_{\tau_c=\infty} = E_{n=\infty} = 1.23605\ldots$, which coincides with the energy of the BI monopole.

Our solutions do not interpolate between the two singular points at $k = 1$ and $k = -1$, but between the singular point at $k = 1$ for $r = \infty$ and a certain value k_0 (related to τ_c) which is always lower than 1 and bigger than -1 (as a matter of fact, $k_0 = 0$ is a solution, which corresponds to the monopole solution). This is radically different from the sphaleron-like solutions or solutions of Bartnik–McKinnon type found in [13, 20].

As in the Bartnik–McKinnon case, we can assign to each solution an integer n, with $n - 1$ denoting

the number of zeros of the function u or the winding number of the corresponding trajectory in the phase plane (k, u), as seen in Fig. 2, where a few solutions are plotted. When the parameter τ_c goes from $-\infty$ to $+\infty$, we observe that this integer n grows from 1 to ∞. At certain special values of the parameter τ_c, this integer increases by 1. Here are the first critical values of τ_c:

$$\tau_c = 1.658, 4.781, 7.510, 10.092, 13.218, 16.530, 19.813.$$

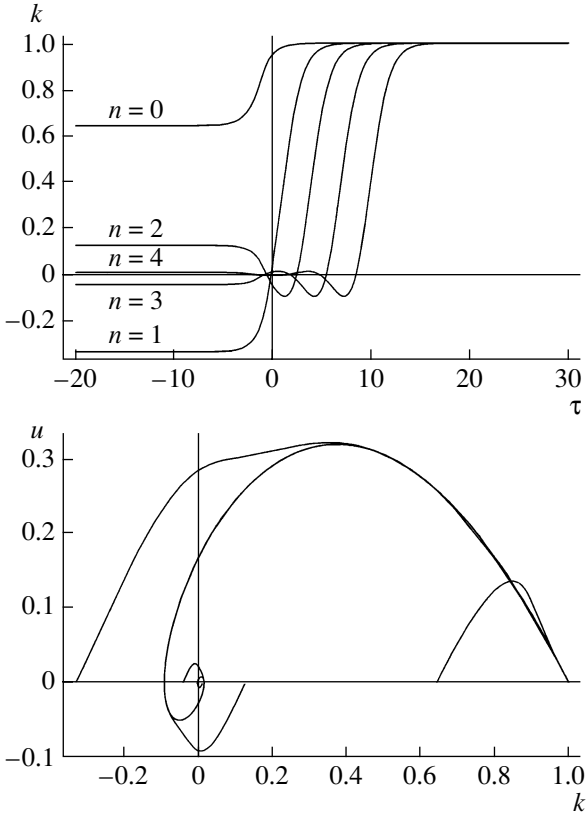

Fig. 2. Plots of solutions for the values of $\tau_c = -3, 1.2, 4, 7, 10$.

3. BORN–INFELD-TYPE LAGRANGIAN FOR HIGGS FIELDS FROM NONCOMMUTATIVE GEOMETRY

In this section, we study the Higgs-like fields which naturally appear in the version of the standard model based on noncommutative geometry [21]. We show that soliton-like solutions with finite energy cannot be obtained with pure Higgs fields obeying this version of generalized BI dynamics in the case when the Higgs multiplet reduces to a single scalar. This situation corresponds to the particular choice of matrix-valued generalized Higgs field when the corresponding matrix is proportional to the identity. This does not exclude the possibility of soliton-like solutions in more complicated cases, with a many-component Higgs field.

3.1. Gauge Fields in Noncommutative Geometry

We shall generalize now the "noncommutative Maxwell theory" developed in [21] in order to obtain a BI-like theory. Let us resume the notation and language of the theory. We consider the algebra $\mathcal{A} = C^\infty(V) \otimes M_n(\mathbb{C})$ with the vector fields spanned by the derivations of $C^\infty(V)$ and inner derivations of $M_n(\mathbb{C})$. The differential algebra is generated by the basis of linear 1-forms acting on the derivations. We can consider $\mathcal{A}$ as a bimodule over itself. Then one defines a gauge by the choice of a unitary element e of $\mathcal{A}$, satisfying $h(e, e) = 1$, with h a Hermitian structure on $\mathcal{A}$. Then any element of $\mathcal{A}$ can be written in the form em with $m \in \mathcal{A}$ and a connection on $\mathcal{A}$ is a map:

$$\nabla : \mathcal{A} \to \Omega^1(\mathcal{A}), \quad em \to (\nabla e)m + edm. \qquad (31)$$

In the gauge e, the connection can be completely characterized by an element ω of $\Omega^1(\mathcal{A})$:

$$\nabla e = e\omega.$$

One can also decompose ω in vertical and horizontal parts:

$$\omega = \omega_h + \omega_v \qquad (32)$$

with

$$\omega_h = A, \quad \omega_v = \theta + \phi.$$

Here, A is like the Yang–Mills connection, whereas θ is the canonical 1-form of the matrix algebra, and plays the role of a preferred origin in the affine space of vertical connections. It satisfies the equation

$$d\theta + \theta^2 = 0.$$

Then ϕ is a tensorial form and can be identified with scalar-field multiplet.

We choose a local basis of derivations of $\mathcal{A}$: $\{e_\mu, e_a\}$, where for convenience e_μ are outer derivations of $C^\infty(V)$, and $e_a = ad(\lambda_a)$, with $\{\lambda_a\}$ a basis of anti-Hermitian matrices of $M_n(\mathbb{C})$, are inner derivations.

The dual basis will be denoted by $\{\theta^\mu, \theta^a\}$. In this particular basis, we have

$$A = A_\mu \theta^\mu, \quad \theta = -\lambda_a \theta^a, \quad \phi = \phi_a \theta^a.$$

If we choose the connection to be anti-Hermitian, we can write $\phi = \phi_a^b \lambda_b \theta^a$. The curvature tensor associated with ω is

$$\Omega = d\omega + \omega^2,$$

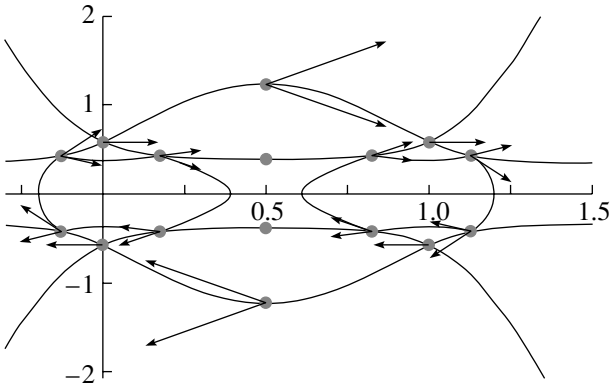

Fig. 3. Characteristic curves and points in the phase space.

we can also define the field strength

$$F = dA + A^2.$$

Then one can identify

$$\Omega_{\mu\nu} = F_{\mu\nu}, \quad \Omega_{\mu a} = D_\mu \phi_a$$
$$\Omega_{a\mu} = -D_\mu \phi_a, \quad \Omega_{ab} = [\phi_a, \phi_b] - C^c_{ab}\phi_c,$$

where C^c_{ab} are the constant structure in the $\{\lambda_a\}$ basis.

A gauge transformation is performed by the choice of a unitary element U of $M_n(\mathbb{C})$, satisfying $h(eU, eU) = 1$. Then, in the gauge $e' = eU$

$$\omega' = U^{-1}\omega U + U^{-1}dU,$$

θ is invariant under gauge transformations; then,

$$A' = U^{-1}AU + U^{-1}dU, \quad \phi' = U^{-1}\phi U.$$

3.2. Noncommutative Born–Infeld Lagrangian

The generalization proposed in our previous article can be adapted to the noncommutative gauge theory. The Lagrangian which we consider is

$$\sqrt{\det|g|} - \{|\det(\mathbb{1} \otimes g + J \otimes \hat{\Omega})|\}^{1/4n}$$

and $\hat{\Omega} = \Omega_{\alpha\beta}\hat{L}^{\alpha\beta}$ with $\hat{L}^{\alpha\beta}$ the generators of the fundamental representation of $SO(4 + n^2 - 1)$. $\Omega_{\alpha\beta}$ are the components of the curvature defined in previous section and then are anti-Hermitian elements of $M_n(\mathbb{C})$. J is an element of $SL(2, \mathbb{C})$ of square $-\mathbb{1}$.

The above Lagrangian contains the contribution of two types of fields: the classical Yang–Mills potential, $A = A_\mu\theta^\mu$, corresponding to the usual spacetime components of the connection 1-form, and the scalar multiplet coming from its matrix components $\phi = \phi_a\theta^a = \phi^b_a\lambda_b\theta^a$. In the case when $\phi = 0$, this Lagrangian coincides with the one studied in [17] and exposed in the previous section. The complete analysis of general solutions seems too tedious for the time being. This is why we shall restrict ourselves to a qualitative analysis of the case when the spacetime components of Ω do vanish $F_{\mu\nu} = 0$, leaving only the contribution of scalar-multiplet degrees of freedom.

3.3. The Reduced Lagrangian for Scalar Fields in the Case $n = 2$

Let us recall the notation which will be used in the subsequent calculations. The basis of matrix representation of the $su(2)$ algebra is chosen as follows:

$$\lambda_a = -i\sigma_a \quad \lambda_a\lambda_b = -\delta_{ab} + \sum_c \epsilon_{abc}\lambda_c, \qquad (33)$$
$$[\lambda_a, \lambda_b] = C^c_{ab} = 2\epsilon_{abc}\lambda_c$$

with σ_a denoting the Pauli matrices. Now we have to evaluate the determinant of the following matrix:

$$\begin{vmatrix} 1 & iD\hat{\phi} \\ -iD\hat{\phi} & 1 + i\hat{H} \end{vmatrix}, \qquad (34)$$

where

$$\hat{H} = \{\Omega_{ab}\}_{a,b=1,2,3}, \qquad (35)$$
$$D\hat{\phi} = \left\{D_\mu\hat{\phi}_a\right\}_{a=1,2,3\ \mu=0,1,2,3}.$$

From now on, we choose the simplest ansatz with one scalar field φ only:

$$\phi = \varphi\theta.$$

After some algebra, we get the following result:

$$L = 1 - \{1 + 6(D\varphi)^2 + 9(D\varphi)^4$$
$$+ 16\varphi^2(\varphi - 1)^2\}^{\frac{1}{4}}\sqrt{1 + 4\varphi^2(\varphi - 1)^2}.$$

3.4. The Absence of Static Configurations of φ

In this subsection, we show that there is no nontrivial static configurations in the present system. We generalize Derrick's theorem [22] to our Lagrangian. The idea of the proof is to use spatial dilatations of the field $\varphi(r) \to \varphi_\lambda(r) = \varphi(\lambda r)$ to generate a one-parameter curve in the space of fields around such a solution. Thus, the variational principle along this curve gives $\partial S[\varphi_\lambda]/\partial\lambda = 0$ at $\lambda = 1$, i.e.,

$$\int 4\pi r^2 dr \left(\frac{\partial\mathcal{L}}{\partial\varphi'}\varphi' - 3\mathcal{L}\right) = 0. \qquad (36)$$

We can show, by algebraic manipulations, that the quantity under the integral is always nonnegative and satisfies (36) if and only if it is zero. The solutions are just the trivial ones $\varphi' = 0$ and $\varphi = 0$ or 1 which exclude other nontrivial solutions.

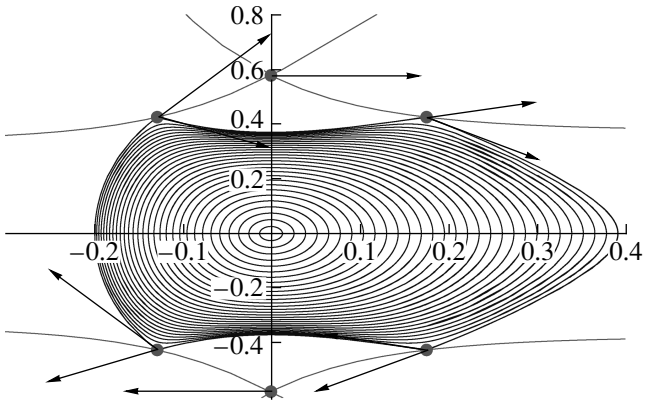

Fig. 4. Trajectories in the confined region of the phase space.

3.5. The Time-Dependent Scalar Field

We have performed a numerical analysis of the time-dependent configurations of the scalar field resulting from the simplest ansatz $\varphi = \varphi(t)$. It gives an interesting phase-space portrait and confirms the idea that BI-like theories set an upper bound on velocities (i.e., time derivatives of φ) and on the field strength as well. Such an ansatz could be of use in cosmology, when coupled with the scale factor $a(t)$ of the Robertson–Friedmann metric. Their investigation will be the subject of our forthcoming paper [23].

The equations of motion in this case take on the following form:

$$\dot{\varphi} = u,$$
$$(1+4X)g(X,Y)\dot{u} + 4ss'h(X,Y) = 0,$$

where

$$s = \varphi(\varphi-1), \quad s' = 2\varphi - 1,$$
$$X = s^2, \quad Y = u^2,$$
$$g(X,Y) = 16X(1-9Y) + (1-3Y)^2,$$
$$h(X,Y) = ((1-3Y)^2 + 16X)(1-Y+8X) - 6(1+4X)(1-3Y)Y.$$

At some points of the phase space, $\dot{u}$ is not well defined. These are the points at which the polynomial g vanishes (four curves in Fig. 3). Nevertheless, in most of the cases, singular behavior is only apparent, because the undetermined ratios $0/0$ prove to have a finite limit. The total number of singular points in the phase space is 16, but only 2 of them display a genuine singularity. In the 14 remaining cases, the function $4ss'h(X,Y)$ vanishes at the same time as the function $g(X,Y)$, but their ratio remains finite. In Fig. 3, one can observe the 16 aforementioned points. The only two (Fig. 3) points with genuine singularity are the ones without any vector attached to them, found on the central vertical line $\varphi = 0.5$ on both sides of the horizontal line and close to it.

The phase-space portrait is symmetric by reflection around the vertical line $\varphi = 0.5$. Cyclic trajectories are contained inside the two pentagon-like areas circumscribed by separatrices. These areas are disposed symmetrically with respect to the vertical line $\varphi = 0.5$. One of these areas is represented in more detailed manner in Fig. 4 below.

One can note that, in a certain region of the phase space, the trajectories are periodic and defined for all values of time t. If one chooses the initial conditions outside this region, the integration ends up after some finite time. This means that the solutions $\varphi(t)$ obtained with these initial conditions have their second derivative divergent after finite time when they hit one of the curves on which $g = 0$.

Nevertheless, some of these curves, with finedtuned initial conditions, can go beyond the singular curve $g = 0$ at points at which the infinite expressions become finite again. These particular trajectories form a special set; they can be extended beyond the limits of the region shown in Fig. 4 and be defined for all values of time $t \in \mathbb{R}$.

4. CONCLUSION AND PERSPECTIVES

Certain generalizations of the Born–Infeld-type Lagrangian for scalar fields have been proposed by several authors [24]. However, in these papers, only a formal analogy was used, usually by inserting a classical scalar field Lagrangian under the square root sign.

The highly nonlinear behavior of the field Φ in this model suggests that, when coupled to gravitation in a standard way, i.e., via minimal coupling resulting from the replacement of ordinary derivatives by their covariant counterparts, and adding the Einstein–Hilbert Lagrangian for gravitational field, it may lead to unusual behavior of cosmological models.

The investigation of cosmological models using this scalar field will be the subject of our forthcoming paper [23].

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The Utility of Coherent States and Other Mathematical Methods in the Foundations of Affine Quantum Gravity*

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Abstract—Affine quantum gravity involves (i) affine commutation relations to ensure metric positivity, (ii) a regularized projection operator procedure to accommodate first- and second-class quantum constraints, and (iii) a hard-core interpretation of nonlinear interactions to understand and potentially overcome nonrenormalizability. In this program, some of the less traditional mathematical methods employed are (i) coherent-state representations, (ii) reproducing kernel Hilbert spaces, and (iii) functional-integral representations involving a continuous-time regularization. Of special importance is the profoundly different integration measure used for the Lagrange multiplier (shift and lapse) functions. These various concepts are first introduced on elementary systems to help motivate their application to affine quantum gravity.

1. INTRODUCTION

The unification of gravity and quantum theory offers a major challenge to theoretical physics. The favored approaches of string theory and loop-plus-spin foam gravity use formulations that are in some sense rather far from the original classical theory of Einstein. Most workers feel this is necessary because of the usual difficulties encountered in quantizing gravity, namely, nonrenormalizability and anomalies, to mention just two. The program of affine quantum gravity, which is relatively new [1], attempts to stay closer to the standard classical theory, so as to provide suitable touchstones along the way. As a consequence, it becomes necessary to deal directly with some of the major problems, such as the two mentioned above. How one deals with such difficult issues, and especially the role played by coherent states in this effort, is part of the story told in this article.

As a pedagogical device, we illustrate our basic methodology on simple quantum-mechanical systems before we discuss the case of quantum gravity.

1.1. Coherent State Basics

It is well known that coherent states provide a useful bridge between a classical theory and the corresponding quantum theory. Let us briefly recall how that bridge works with a simple example. Let Q and P denote standard Heisenberg self-adjoint operators satisfying the usual commutation relation $[Q, P] = i\mathbb{1}$ with $\hbar = 1$. Then we define canonical coherent states by the relation

$$|p, q\rangle \equiv e^{-iqP} e^{ipQ} |\eta\rangle, \tag{1}$$

where $|\eta\rangle$ denotes a normalized vector called the fiducial vector, which, in terms of the abbreviation $\langle(\cdot)\rangle \equiv \langle\eta|(\cdot)|\eta\rangle$, is subject to the modest requirements that $\langle Q\rangle = 0$ and $\langle P\rangle = 0$. This condition on $|\eta\rangle$ has been referred to as "physically centered." Here, we add the additional requirement that

$$\lim_{\hbar\to 0} \langle (P^2 + Q^2)\rangle = 0, \tag{2}$$

a relation we refer to as "physically attractive." Given appropriate domain conditions, it follows from (2) that

$$\lim_{\hbar\to 0} \langle (P^2 + Q^2)^m\rangle = 0 \tag{3}$$

for arbitrary $m > 0$. It is clear that the ground state of a harmonic oscillator satisfies these conditions, but so do many other vectors as well.

If $\mathcal{G}$ denotes a quantum "generator" in a wide sense, then we assert that

$$G(p, q) \equiv \langle p, q|\mathcal{G}|p, q\rangle \tag{4}$$

defines the ($\hbar$ augmented) classical generator $G(p, q)$ associated with $\mathcal{G}$. Of course, this connection is not strictly what we usually mean by the classical generator since $\hbar$ has not been set equal to zero—which explains the "$\hbar$ augmented" phrase. In addition, we can also consider the expression

$$G_c(p, q) \equiv \lim_{\hbar\to 0} \langle p, q|\mathcal{G}|p, q\rangle, \tag{5}$$

which corresponds to the complete classical limit. The association between a quantum and classical

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generator illustrated by (4) and (5) is known as the "weak correspondence principle" [2].

To illustrate this situation, let us discuss a few examples. For example, it follows that

$$\langle p,q|Q|p,q\rangle = q, \quad \langle p,q|P|p,q\rangle = p, \tag{6}$$

while

$$\begin{aligned}\langle p,q|Q^2|p,q\rangle &= q^2 + \langle Q^2\rangle, \\ \langle p,q|P^2|p,q\rangle &= p^2 + \langle P^2\rangle.\end{aligned} \tag{7}$$

More generally, if $\mathcal{W}(P,Q)$ denotes an arbitrary polynomial, and subject to suitable domain conditions, then

$$\begin{aligned}W(p,q) &\equiv \langle p,q|\mathcal{W}(P,Q)|p,q\rangle \\ &= \mathcal{W}(p,q) + \mathcal{O}(\hbar;p,q),\end{aligned} \tag{8}$$

where, under the condition (3), it follows that $\mathcal{O}(\hbar;p,q) \to 0$ as $\hbar \to 0$.

A complete characterization of an operator is given in terms of its matrix elements. In particular, it is clear that

$$\langle p,q|\mathcal{W}(P,Q)|p',q'\rangle \tag{9}$$

fully determines the operator in question.

1.2. Reproducing Kernel Hilbert Space

By assumption, the coherent states span the Hilbert space. Therefore, a dense set of vectors may be written in the form

$$|\psi\rangle = \sum_{j=1}^{J} \alpha_j |p_j,q_j\rangle, \tag{10}$$

where $(p_j,q_j) \in \mathbb{R}^2$, $\alpha_j \in \mathbb{C}$, and $J < \infty$. Another such vector may be taken as

$$|\phi\rangle = \sum_{k=1}^{K} \beta_k |p_{(k)},q_{(k)}\rangle, \quad K < \infty. \tag{11}$$

As *functional representatives* of these abstract vectors, let us choose their inner product with arbitrary coherent states which leads to

$$\psi(p,q) \equiv \langle p,q|\psi\rangle = \sum_{j=1}^{J} \alpha_j \langle p,q|p_j,q_j\rangle, \tag{12}$$

$$\phi(p,q) \equiv \langle p,q|\phi\rangle = \sum_{k=1}^{K} \beta_k \langle p,q|p_{(k)},q_{(k)}\rangle. \tag{13}$$

As the *inner product* between two such functional representatives, we take

$$(\psi,\phi) \equiv \langle\psi|\phi\rangle = \sum_{j,k=1}^{J,K} \alpha_j^* \beta_k \langle p_j,q_j|p_{(k)},q_{(k)}\rangle, \tag{14}$$

as follows from (10) and (11). We now have functional representatives, $\psi(p,q)$, $\phi(p,q)$, etc., and an inner product between them; all that remains to make a Hilbert space is to complete the space by including the limits of all Cauchy sequences in the norm $||\psi|| \equiv +\sqrt{(\psi,\psi)}$. The result is the so-called (separable) reproducing kernel Hilbert space in which the continuous function given by the coherent state overlap function serves as the reproducing kernel. For additional information about such Hilbert spaces, see [3].

2. AFFINE FIELD OPERATORS

In a $3+1$ split of space and time, a subset of the basic kinematical operators chosen for affine quantum field theory involves the self-adjoint spatial metric $\hat{g}_{ab}(x)$, where $a,b = 1,2,3$. Moreover, we insist that the spectrum of the spatial metric is restricted so that $u^a \hat{g}_{ab}(x) u^b > 0$ for any set $\{u^a\}$ for which $\sum_a (u^a)^2 > 0$, a requirement that we call "metric positivity." To complete the set of basic kinematical operators we employ the "momentric" field $\hat{\pi}_d^b(x)$. The latter field is the self-adjoint operator associated with the classical momentric field $\pi_d^b(x) \equiv \pi^{bc}(x) g_{cd}(x)$, which involves both the ADM classical momentum $\pi^{bc}(x)$ and spatial metric $g_{cd}(x)$. Promoting the Poisson brackets satisfied by the fields π_d^c and g_{ab} to commutators leads us directly to the *affine commutation relations* (for $\hbar = 1$) given by

$$\begin{aligned}&[\hat{\pi}_b^a(x), \hat{\pi}_d^c(y)] \\ &= \frac{1}{2} i [\delta_b^c \hat{\pi}_d^a(x) - \delta_d^a \hat{\pi}_b^c(x)] \delta(x,y), \\ &[\hat{g}_{ab}(x), \hat{\pi}_d^c(y)] \\ &= \frac{1}{2} i [\delta_a^c \hat{g}_{bd}(x) + \delta_b^c \hat{g}_{ad}(x)] \delta(x,y), \\ &[\hat{g}_{ab}(x), \hat{g}_{cd}(y)] = 0.\end{aligned} \tag{15}$$

Observe that these relations define an infinite dimensional Lie algebra. The reason for choosing these particular kinematical commutators follows directly from the fact that

$$\begin{aligned}&e^{i\int \gamma_b^a(y)\hat{\pi}_a^b(y)d^3y} \hat{g}_{rs}(x) e^{-i\int \gamma_b^a(y)\hat{\pi}_a^b(y)d^3y} \\ &= (e^{\gamma(x)/2})_r^t \hat{g}_{tu}(x) (e^{\gamma(x)/2})_s^u.\end{aligned} \tag{16}$$

The meaning of this relation is clear: Unitary transformations generated by the self-adjoint momentric field manifestly preserve the desired spectral domain of the spatial metric tensor ensuring that $u^a \hat{g}_{ab}(x) u^b > 0$ for any set $\{u^a\}$ that is not identically zero.

2.1. Affine Coherent States

A representation of the basic affine operators is determined by a state on the algebra they generate—or, equivalently, by the set of coherent states

$$|\pi,\gamma\rangle \equiv e^{i\int \pi^{ab}(x)\hat{g}_{ab}(x)d^3x}e^{-i\int \gamma^a_b(x)\hat{\pi}^b_a(x)d^3x}|\eta\rangle \quad (17)$$

for all smooth functions π^{ab} and γ^a_b of compact support. Here, $|\eta\rangle$ is a suitable fiducial vector which, in effect, determines the field operator representation. The appropriate (physical) restriction on this operator representation arises from fully enforcing the gravitational constraints. However, according to Dirac [4], quantization should be carried out first, while the introduction of constraints should come second. Thus, we are obliged to choose an initial—and temporary—field operator representation just to get started. To this end, it proves convenient to choose $|\eta\rangle$ so that

$$\langle\pi'',\gamma''|\pi',\gamma'\rangle \equiv \langle\pi'',g''|\pi',g'\rangle \quad (18)$$
$$= \exp\left[-\int b(x)d^3x\ln\left(\frac{\det\left\{\frac{1}{2}[g''^{ab}(x)+g'^{ab}(x)]+ib(x)^{-1}[\pi''^{ab}(x)-\pi'^{ab}(x)]\right\}}{\{\det[g''^{ab}(x)]\det[g'^{ab}(x)]\}^{1/2}}\right)\right].$$

Here, the symmetry of $|\eta\rangle$ is such that, instead of all nine components of γ, the coherent states depend on only the six components of g, which are defined by

$$g_{ab}(x) \equiv (e^{\gamma(x)/2})^c_a\tilde{g}_{cd}(x)(e^{\gamma(x)/2})^d_b, \quad (19)$$

where

$$\tilde{g}_{ab}(x) \equiv \langle\eta|\hat{g}_{ab}(x)|\eta\rangle \quad (20)$$

arises as a property of $|\eta\rangle$. The scalar density $b(x)$, $0 < b(x) < \infty$, arises as a property of $|\eta\rangle$ as well. As usual, $g^{ab}(x)$ is the inverse metric defined by $g^{ab}(x)g_{bc}(x) = \delta^a_c$ for each x.

Observe that the coherent state overlap (18) is a jointly continuous function of its arguments, e.g., in the topology of the test function space $\mathcal{D}$.

2.2. Reproducing Kernel Hilbert Space

Just as in the elementary example, we can use the coherent state overlap function (18) as a reproducing kernel to construct a reproducing kernel Hilbert space. In particular, functional representatives in a dense set of the Hilbert space may be given by

$$\psi(\pi,g) \equiv \sum_{j=1}^{J}\alpha_j\langle\pi,g|\pi_j,g_j\rangle, \quad J<\infty, \quad (21)$$

$$\phi(\pi,g) \equiv \sum_{k=1}^{K}\beta_j\langle\pi,g|\pi_{(k)},g_{(k)}\rangle, \quad K<\infty, \quad (22)$$

etc. As an inner product between two such vectors, we choose

$$(\psi,\phi) \equiv \sum_{j,k=1}^{J,K}\alpha^*_j\beta_k\langle\pi_j,g_j|\pi_{(k)},g_{(k)}\rangle. \quad (23)$$

We may complete this Hilbert space by introducing all limit elements of Cauchy sequences in the norm $||\psi|| \equiv +\sqrt{(\psi,\psi)}$, in complete analogy to what we did in the elementary example. The result is the separable reproducing kernel Hilbert space with (18) serving as the reproducing kernel.

3. IMPOSITION OF CONSTRAINTS

To explain our procedure for the imposition of constraints, we return to an N degree-of-freedom model, $N<\infty$. Let us suppose there are classical constraints for this model given by the conditions $\phi_\alpha(p,q)=0$ for $1\le\alpha\le A$, where $p=(p^1, p^2,\ldots,p^N)$ and $q=(q^1,q^2,\ldots,q^N)$. Upon quantization, these constraints become self-adjoint operators $\Phi_\alpha(P,Q)$, $1\le\alpha\le A$. Ideally, there should be a nonvanishing subspace $\mathfrak{H}_{\text{phys}}$ of the original Hilbert space $\mathfrak{H}$ for which $\Phi_\alpha|\psi\rangle_{\text{phys}}=0$ for all $|\psi\rangle_{\text{phys}}\in\mathfrak{H}_{\text{phys}}$ [4]. Unfortunately, this ideal situation does not always occur. As a replacement for this criterion, we introduce a *projection operator*

$$\mathbb{E} = \mathbb{E}\left(\sum\Phi^2_\alpha \le \delta(\hbar)^2\right), \quad (24)$$

i.e., a projection operator such that

$$0 \le \mathbb{E}\left(\sum\Phi^2_\alpha\right)\mathbb{E} \le \delta(\hbar)^2\mathbb{1}, \quad (25)$$

where $\delta(\hbar)$ is a regularization parameter. We define $\mathfrak{H}_{\text{phys}} = \mathbb{E}\mathfrak{H}$ as the regularized physical Hilbert space. The general idea is to reduce the regularization parameter $\delta(\hbar)$ to an appropriate value for each situation. For example, if $\sum\Phi^2_a = J_1^2+J_2^2+J_3^2$, the Casimir operator for $SU(2)$, then $\delta(\hbar)^2 = \frac{1}{2}\hbar^2$ (or

any $\delta(\hbar)$, $0 \le \delta(\hbar)^2 < (3/4)\hbar^2$) is sufficient to ensure that $\sum J_k^2 = 0$. If $\sum \Phi_a^2 = P^2 + Q^2$, then $\delta(\hbar)^2 = \hbar$ (or any $\delta(\hbar)$, $\hbar \le \delta(\hbar)^2 < 3\hbar$) ensures that $\mathbb{E}$ projects unto vectors $|\psi\rangle_{\rm phys}$ for which $(Q + iP)|\psi\rangle_{\rm phys} = 0$. This procedure enables a consideration of first- and second-class constraints within the same formulation without any need either to eliminate the second-class constraints before quantization or to introduce Dirac brackets for them. Other constraint situations can also be treated; see [5].

It is noteworthy that for any set of constraint operators, there exists a universal integral representation to construct the projection operator [6]. In particular, it follows that

$$\mathbb{E}\left(\sum \Phi_a^2 \le \delta(\hbar)^2\right) \tag{26}$$
$$\equiv \int \mathrm{T} e^{-i\int_{t_1}^{t_2} \lambda^\alpha(t)\Phi_\alpha dt} \mathcal{D}R(\lambda),$$

where T denotes time ordering, $R(\lambda)$ is a formal measure on c-number functions $\{\lambda^\alpha(t)\}$, and $t_2 - t_1$ corresponds to any positive time interval. The integral in (26) is constructed in a two-step procedure. First, a Gaussian functional integral leads to

$$e^{i\gamma(t_2-t_1)\sum \Phi_\alpha^2} \tag{27}$$
$$= \mathcal{N} \int \mathrm{T} e^{-i\int \lambda^\alpha(t)\Phi_\alpha dt - (i/4\gamma)\int \sum \lambda^\alpha(t)^2 dt} \prod \mathcal{D}\lambda^\alpha.$$

Second, we integrate over γ as follows:

$$\lim_{\zeta\to 0^+} \int \frac{\sin[\gamma(t_2 - t_1)(\delta(\hbar)^2 + \zeta)]}{\pi\gamma} \tag{28}$$
$$\times\, e^{i\gamma(t_2-t_1)\sum \Phi_\alpha^2} d\gamma = \mathbb{E}\left(\sum \Phi_\alpha^2 \le \delta(\hbar)^2\right).$$

The integral representation (26) may be used within a coherent state path integral representation of the propagator. We focus on [5]

$$\langle p'', q''|\mathbb{E} e^{-i(\mathbb{E}\mathcal{H}\mathbb{E})T}\mathbb{E}|p', q'\rangle \tag{29}$$
$$= \lim_{\epsilon\to 0} \langle p'', q''|e^{-i\mathcal{H}\epsilon}\mathbb{E}e^{-i\mathcal{H}\epsilon}\mathbb{E}\cdots e^{-i\mathcal{H}\epsilon}\mathbb{E}|p', q'\rangle,$$

where there are $(L + 1)$ short-time evolution operators $e^{-i\mathcal{H}\epsilon}$, and $(L + 1)\epsilon = T$. Insertion of L coherent state resolutions of unity leads to

$$\lim_{\epsilon\to 0} \int \cdots \int \prod_{l=0}^{L} \langle p_{l+1}, q_{l+1}|e^{-i\mathcal{H}\epsilon}\mathbb{E}|p_l, q_l\rangle \tag{30}$$
$$\times \prod_{l=1}^{L} d^N p_l d^N q_l/(2\pi)^N,$$

where $p_{L+1}, q_{L+1} = p'', q''$ and $p_0, q_0 = p', q'$. In turn, it follows that

$$\langle p'', q''|\mathbb{E} e^{-i(\mathbb{E}\mathcal{H}\mathbb{E})T}\mathbb{E}|p', q'\rangle \tag{31}$$
$$= \lim_{\epsilon\to 0} \int \cdots \int \prod_{l=0}^{L} \langle p_{l+1}, q_{l+1}|e^{-i\epsilon(\mathcal{H} + \lambda_l^\alpha \Phi_\alpha)}|p_l, q_l\rangle$$
$$\times \prod_{l=1}^{L} d^N p_l d^N q_l/(2\pi)^N \mathcal{D}R(\lambda_l)$$
$$= \mathcal{M} \int e^{i\int [p\cdot\dot{q} - H(p,q) - \lambda^\alpha(t)\phi_\alpha(p,q)]dt} \mathcal{D}p\mathcal{D}q\mathcal{D}R(\lambda),$$

where $H(p, q) \equiv \langle p, q|\mathcal{H}|p, q\rangle$, $\phi_\alpha(p, q) \equiv \langle p, q|\Phi_\alpha|p, q\rangle$.

In this fashion, we see how repeated insertions of the projection operator lead to temporal evolution entirely within the physical Hilbert space. Moreover, we see how this evolution can be realized by a suitably interpreted path integral which does not involve the usual flat measure on the Lagrange multipliers but, instead, uses the measure $R(\lambda)$ that is designed to enforce the quantum constraints rather than the classical constraints.

Although this is not the only way the integral representation for the projection operator (26) can be used to formulate a path integral, it is probably the most straightforward construction and readily illustrates the basic principles involved.

3.1. Constraints in Quantum Gravity

For quantum gravity, there are four constraint fields, which, from a classical point of view, comprise an open first-class system. The quantum constraints, however, exhibit an anomaly, and thus they are partially second class in nature. In particular, the diffeomorphism $(\mathcal{H}_a, a = 1, 2, 3)$ and Hamiltonian $(\mathcal{H})$ constraint operator fields fulfill the commutation relations

$$[\mathcal{H}_a(x), \mathcal{H}_b(y)] \tag{32}$$
$$= i[\delta_{,a}(x, y)\mathcal{H}_b(x) - \delta_{,b}(x, y)\mathcal{H}_a(x)],$$
$$[\mathcal{H}_a(x), \mathcal{H}(y)] = i\delta_{,a}(x, y)\mathcal{H}(x), \tag{33}$$
$$[\mathcal{H}(x), \mathcal{H}(y)] \tag{34}$$
$$= i\frac{1}{2}\delta_{,a}(x, y)[\hat{g}^{ab}(x)\mathcal{H}_b(x) + \mathcal{H}_b(x)\hat{g}^{ab}(x)].$$

Ideally, one asks that $\mathcal{H}_a(x)|\psi\rangle_{\rm phys} = 0$ as well as $\mathcal{H}(x)|\psi\rangle_{\rm phys} = 0$ for all a and x and for all $|\psi\rangle_{\rm phys} \in \mathfrak{H}_{\rm phys}$. However, this ideal situation is not possible because it is almost surely the case that $\hat{g}^{ab}(x)|\psi\rangle_{\rm phys} \notin \mathfrak{H}_{\rm phys}$, and therefore it does not follow that $[\mathcal{H}(x), \mathcal{H}(y)]|\psi\rangle_{\rm phys} = 0$ as would be required. This inconsistency of Eq. (34) gives rise

to the gravitational anomaly—a partially second-class behavior—that makes conventional treatments of quantum gravity especially difficult. However, as noted above, the projection operator method treats first- and second-class constraints in the very same manner, the only difference being how small the regularization parameter $\delta(\hbar)^2$ can be made.

By introducing a cutoff to regularize the quantum constraints, we can imagine constructing a projection operator $\mathbb{E}$ onto a regularized physical Hilbert space in which the regularized quantum constraints are fulfilled to a certain degree. Such a cutoff can be introduced in a variety of ways, and for simplicity we will leave this necessary cutoff implicit. At a later point in the calculation, it would be necessary to remove this cutoff as well, but we will not examine this important issue either. Instead, we go straight to the heart of the matter and note that there is a functional integral representation [1] for the coherent-state matrix elements of the projection operator onto the regularized physical Hilbert space given by

$$\begin{aligned}
&\langle \pi'', g''|\mathbb{E}|\pi', g'\rangle \qquad (35)\\
&= \int \langle \pi'', g''|\mathrm{T} e^{-i\int [N^a \mathcal{H}_a + N\mathcal{H}]d^3x dt}|\pi', g'\rangle\\
&\times \mathcal{D}R(N^a, N) = \lim_{\nu\to\infty} \bar{\mathcal{N}}_\nu\\
&\times \int e^{-i\int [g_{ab}\dot{\pi}^{ab} + N^a H_a + NH]d^3x dt}\\
&\times \exp\Big\{ -(1/2\nu)\int [b(x)^{-1} g_{ab} g_{cd} \dot{\pi}^{bc}\dot{\pi}^{da}\\
&\quad + b(x) g^{ab} g^{cd} \dot{g}_{bc}\dot{g}_{da}]d^3x dt\Big\}\\
&\times \Big[\prod_{x,t}\prod_{a\le b} d\pi^{ab}(x,t) dg_{ab}(x,t)\Big]\mathcal{D}R(N^a, N).
\end{aligned}$$

Implicit in these expressions are cutoffs in the constraint operators $\mathcal{H}_a$ and $\mathcal{H}$, and correspondingly in the c-number symbols H_a and H that arise from the constraint operators and which appear in the functional integral as their representatives.

Note the appearance of the measure $\mathcal{D}R(N^a, N)$ on the Lagrange multiplier fields, N_a (the shift) and N (the lapse). It is this measure, in contrast to the usual flat measure on these fields, which leads to the projection operator $\mathbb{E}$ that projects the original Hilbert space $\mathfrak{H}$ onto the regularized physical Hilbert space $\mathfrak{H}_{\mathrm{phys}}$.

Note also the appearance of a limit as $\nu \to \infty$ as well as a ν-dependent factor in the integrand. This factor and the limit are connected with a different kind of regularization of the functional integral that may be used instead of the usual lattice regularization. The indicated form represents a *continuous-time regularization* which involves a Wiener-like measure that controls the nature of the paths. The result of interest arises in the ultradiffusive limit in which the diffusion constant ν diverges. Continuous-time regularization procedures have been well studied for phase-space path integrals appropriate to canonical, spin, and affine variables [7], and they have the virtue that they *automatically* lead to a quantum-mechanical representation of the corresponding expression in terms of coherent states with a fiducial vector given by an extremal weight vector. Recently, additional studies of such path integral representations have been made in the case of weak coherent states for the affine group when a traditional resolution of unity as a local integral fails to exist [8].

Let us add that the quantity $\langle \pi'', g''|\mathbb{E}|\pi', g'\rangle$ may also be used as a reproducing kernel to build the reproducing kernel Hilbert space associated with the regularized physical Hilbert space $\mathfrak{H}_{\mathrm{phys}}$ in exactly the same way that the original coherent state overlap $\langle \pi'', g''|\pi', g'\rangle$ may be used to build the reproducing kernel Hilbert space associated with the original Hilbert space $\mathfrak{H}$.

Equation (35) represents as far as we can presently go in our formal development. Despite the canonical appearance of (35), we emphasize that this functional integral representation has been based on the affine commutation relations (15) and *not* on any canonical commutation relations.

4. HARD-CORE INTERACTIONS IN QUANTUM MECHANICS

Let us again return to the world of quantum mechanics to motivate the next issue of concern. Consider an imaginary-time path integral for a single degree-of-freedom problem formally given by

$$\begin{aligned}
I(\lambda) \equiv \mathcal{N}\int \exp\Big\{ &-\frac{1}{2}\int [\dot{x}(t)^2 + m^2 x(t)^2]dt \qquad (36)\\
&- \lambda \int x(t)^{-4} dt\Big\}\mathcal{D}x,
\end{aligned}$$

where the path integral runs over continuous paths for which $x(0) = x'$ and $x(T) = x''$, namely, all paths are pinned at the initial and final times, $t = 0$ and $t = T$, respectively. This example clearly pertains to an oscillator with a singular potential and a coupling constant λ that we require to be nonnegative, $\lambda \ge 0$. To help interpret (36), it proves useful to first *regularize* the singularity of the inverse quartic interaction. However, no matter how one attempts to regularize the singularity of the inverse quartic interaction, so as to give unambiguous meaning to the path integral, and

subsequently proceeds to remove that regularization, it is known that the result leads to a *discontinuous perturbation* of the oscillator [9]. A discontinuous perturbation has the property that it leaves an indelible imprint on the original system such that, once the interaction is introduced, *it cannot be completely removed* as the coupling constant $\lambda \to 0$. In other words,

$$\lim_{\lambda \to 0} I(\lambda) = I'(0) \neq I(0). \tag{37}$$

Singular interactions with this property are called *hard-core interactions*. What happens is the following. Whenever $\lambda > 0$, those paths allowed by the free action that reach or cross the point of singularity, $x = 0$, lead to a divergent value for the interaction term, with the property that the paths in question are all *projected out* of the integration for any positive value of λ, however small. Thus, those paths make no contribution to the path integral $I(\lambda)$ for any $\lambda > 0$, and, as a consequence, as $\lambda \to 0$, those paths never reappear, and the result $I'(0)$ is based on the oscillator but has a contribution from *only* those continuous paths $x(t)$ for which $x(t) \neq 0$ for all t such that $0 \leq t \leq T$. The evaluation of the resultant path integral with the restricted set of paths defines the expression $I'(0)$ and it clearly gives rise to a different result than if the interaction had never been present in the first place, namely, $I(0)$, which corresponds to the free theory, i.e., the usual imaginary-time oscillator. The theory implicitly defined by $I'(0)$ is called the "pseudofree theory."

4.1. Hard-Core Interactions in Field Theory

The kind of behavior illustrated above is not limited to the inverse quartic interaction but arises for any interaction of the form $|x|^{-\alpha}$ whenever $\alpha > 2$. There are good reasons to make the analogy of such discontinuous perturbations with nonrenormalizable interactions as they are known in quantum field theory. The full story of this analogy is presented in Chapter 8 in [9]. In other words, it is reasonable to suppose that what are regarded as nonrenormalizable interactions in quantum field theory behave as they do because they are in fact discontinuous perturbations that act as hard cores within appropriate functional integral formulations. Moreover, certain specialized nonrenormalizable models exhibit exactly the stated behavior; see, e.g., Chapters 9, 10 in [2]. These models possess enough symmetry so that solutions can be constructed outside of perturbation theory on the basis of generally accepted principles. Based on the experience gained with such models, it is our strong conviction that all nonrenormalizable quantum field theories can be understood as discontinuous perturbations that act as hard cores within functional integrals.

Of course, there is an important difference in the nature of the excluded paths between what happens in the quantum mechanical case and the field-theory case. For quantum mechanics, the interactions exhibit singularities at finite positions (e.g., $x = 0$, as is the case for the interaction x^{-4}), while for the field case, the interactions exhibit singularities for fields that themselves have singular behavior at some point in Euclidean spacetime, e.g., a field having the distributional behavior $|x|^{-\gamma}$ near $x = 0$, where γ is chosen such that this local behavior is acceptable for the free term but unacceptable for the interaction term; see Chapter 8 in [9].

For quantum-mechanical cases, it is quite straightforward to identify which paths should be excluded and which paths are to be retained. On the other hand, in the case of quantum fields, the situation is far more difficult. In addition, it is one thing to say that functions with certain singular behavior are to be excluded, but it is a far more difficult thing to say how, in fact, *operationally* to accomplish that exclusion. For covariant, nonrenormalizable scalar fields, a proposal has recently been put forward [10] that identifies a *novel, nonclassical* ($\propto \hbar^2$) *counterterm*, which, it is conjectured, captures the effect of the hard-core character of the interaction, a counterterm that remains behind—as any hard-core portion of an interaction must certainly do—even after the strength of the interaction is reduced to zero. The proposal offered is at a stage where Monte Carlo computer studies could illuminate this proposal to a considerable degree; unfortunately, such computer studies have yet to be made. If such computer studies were made, however, and they confirmed that the hard-core picture makes good sense and also led to nontrivial results for such nonrenormalizable models as ϕ^4_n, for spacetime dimensions $n \geq 5$, then we would have greater confidence in their possible utility in the study of quantum gravity. Since we do not yet have this additional degree of support, we are obliged to rely on the conjecture that the nonrenormalizable aspect of traditional quantum gravity can be understood—and eventually dealt with—by invoking the hard-core hypothesis, even if at this stage we do not fully know how to actually realize this proposal.

At any rate, we can make a few reasonable conjectures as to how the appearance of the hard-core terms may enter. Just as with covariant scalar fields, we expect the counterterm(s) to be *atypical* and *not* what would be predicted on the basis of perturbation theory. After all, perturbation theory is based on the

assumption that the interacting theory is continuously connected to the noninteracting theory, indeed, explicitly in such a way that, as the coupling constant goes to zero, one passes continuously from the interacting theory to the noninteracting one. But, for hard-core interactions, that is *exactly what cannot happen*. Thus, we are led to expect modifications of the constraint operators $\mathcal{H}_a$ and $\mathcal{H}$, which will then lead to $O(\hbar)$ modifications to their c-number symbols H_a and H that enter into the functional integral (35). Since these all-important modifications are unknown at present, we are not yet in a position to try to use (35) in order to evaluate, even approximately, the coherent-state matrix element of the projection operator, $\langle \pi'', g''|\mathbb{E}|\pi', g'\rangle$.

In conclusion, we expect the next level of understanding in this program to arise from the study of (i) ϕ_n^4, $n \geq 5$, models and (ii) simple models with anomalous constraint behavior. However, predicting the future is known to be fairly risky!

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Spheroidal Analysis of the Generalized MIC–Kepler System*

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Abstract—This paper deals with the dynamical system that generalizes the MIC–Kepler system. It is shown that the Schrödinger equation for this generalized MIC–Kepler system can be separated in prolate spheroidal coordinates. The coefficients of the interbasis expansions between three bases (spherical, parabolic, and spheroidal) are studied in detail. It is found that the coefficients for this expansion of the parabolic basis in terms of the spherical basis, and vice versa, can be expressed through the Clebsch–Gordan coefficients for the group $SU(2)$ analytically continued to real values of their arguments. The coefficients for the expansions of the prolate spheroidal basis in terms of the spherical and parabolic bases are proved to satisfy three-term recursion relations. © *2005 Pleiades Publishing, Inc.*

1. INTRODUCTION

The generalized MIC–Kepler system is described by the equation [1]

$$\frac{1}{2}\left(-i\nabla - s\mathbf{A}\right)^2\psi + \left[\frac{s^2}{2r^2} - \frac{1}{r} + \frac{c_1}{r(r+z)} + \frac{c_2}{r(r-z)}\right]\psi = E\psi, \quad (1)$$

where c_1 and c_2 are nonnegative constants, and

$$\mathbf{A} = \frac{1}{r(r-z)}(y, -x, 0) \quad \text{and} \quad \mathrm{curl}\mathbf{A} = \frac{\mathbf{r}}{r^3}.$$

(We use the system of units for which $\hbar = m = e = c = 1$.) The monopole number s satisfies the Dirac rule of charge quantization $s = 0, \pm 1/2, \pm 1, \ldots$. Each value of s describes its particular generalized MIC–Kepler system. The Schrödinger equation (1) for $c_i = 0$ $(i = 1, 2)$ and $s \neq 0$ reduces to the Schrödinger equation of the MIC–Kepler system [2, 3]. The MIC–Kepler system could be constructed by the reduction of the four-dimensional isotropic oscillator by the use of the so-called Kustaanheimo–Stiefel transformation, both on classical and quantum mechanical levels [4]. In a similar way, reducing the two- and eight-dimensional isotropic oscillator, one can obtain the two- [5] and five-dimensional [6] analogs of the MIC–Kepler system. An infinitely thin solenoid providing the system with the spin 1/2 plays the role of monopole in the two-dimensional case, whereas in the five-dimensional case this role is performed by the $SU(2)$ Yang monopole [7], endowing the system with the isospin. All the above-mentioned systems have Coulomb symmetries and are solved in spherical and parabolic coordinates, both in discrete and continuous parts of energy spectra [8, 9]. There are generalizations of MIC–Kepler systems on a three-dimensional sphere [10] and hyperboloid [11] as well. The MIC–Kepler system has been worked out from different points of view in [12–16].

At $s = 0$, Eq. (1) is reduced to the Schrödinger equation for the generalized Kepler–Coulomb system [17]. In the case when $s = 0$ and $c_1 = c_2 \neq 0$, Eq. (1) reduces to the Hartmann system that has been used for describing axially symmetric systems like ring-shaped molecules [18].

The system described by the Schrödinger equation (1) is one of the superintegrable potentials investigated in [19–21].

In [1], it is shown that the variables in the Schrödinger equation (1) are separated in spherical and parabolic coordinates. In this article, it is shown that the variables in Eq. (1) can be separated in prolate spheroidal coordinates also. The system of spheroidal coordinates is a natural system for investigating many problems in mathematical physics (see [22] and references therein). In quantum mechanics, the spheroidal coordinates play an important role because they are appropriate in describing the behavior of a charged particle in the field of two Coulomb centers. The distance R between the centers is a dimensional parameter characterizing the spheroidal coordinates. These coordinates are changed into spherical and parabolic coordinates as $R \to 0$ and $R \to \infty$, respectively, if the positions of one Coulomb center and the charged particle are fixed when taking the limits. In this sense, the spheroidal

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coordinates are more general than the spherical and parabolic coordinates.

2. SPHERICAL AND PARABOLIC BASES

For completeness, we here present the solutions of the Schrödinger equation (1) found in [1]. Equation (1) in the spherical coordinates becomes

$$\left\{\Delta_{r\theta}+\frac{1}{4r^2\cos^2(\theta/2)}\left(\frac{\partial^2}{\partial\varphi^2}-4c_1\right)+\frac{1}{4r^2\sin^2(\theta/2)}\right. \tag{2}$$
$$\left.\times\left[\left(\frac{\partial}{\partial\varphi}+2is\right)^2-4c_2\right]+2\left(E+\frac{1}{r}\right)\right\}\psi=0,$$

where

$$\Delta_{r\theta}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right).$$

The solution to Eq. (2) has the form

$$\psi^{(s)}_{njm}(r,\theta,\varphi;\delta_1,\delta_2) \tag{3}$$
$$=R^{(s)}_{nj}(r;\delta_1,\delta_2)\,Z^{(s)}_{jm}(\theta,\varphi;\delta_1,\delta_2).$$

The functions $Z^{(s)}_{jm}(\theta,\varphi;\delta_1,\delta_2)$ and $R^{(s)}_{nj}(r;\delta_1,\delta_2)$ are given by the formulas

$$Z^{(s)}_{jm}(\theta,\varphi;\delta_1,\delta_2)=N_{jm}(\delta_1,\delta_2)\left(\cos\frac{\theta}{2}\right)^{m_1} \tag{4}$$
$$\times\left(\sin\frac{\theta}{2}\right)^{m_2}P^{(m_2,m_1)}_{j-m_+}(\cos\theta)e^{i(m-s)\varphi},$$
$$R^{(s)}_{nj}(r;\delta_1,\delta_2)=C_{nj}(\delta_1,\delta_2)(2\varepsilon r)^{j+(\delta_1+\delta_2)/2}e^{-\varepsilon r}$$
$$\times F\left(-n+j+1;2j+\delta_1+\delta_2+2;2\varepsilon r\right),$$

where $P^{(\alpha,\beta)}_n(x)$ are the Jacobi polynomials, $F(a;c;x)$ is the confluent hypergeometric function, and $N_{jm}(\delta_1,\delta_2)$ and $C_{nj}(\delta_1,\delta_2)$ are normalization constants:

$$N_{jm}(\delta_1,\delta_2)=\sqrt{\frac{(2j+\delta_1+\delta_2+1)(j-m_+)!\Gamma(j+m_++\delta_1+\delta_2+1)}{4\pi\Gamma(j-m_-+\delta_1+1)\Gamma(j+m_-+\delta_2+1)}},$$

$$C_{nj}(\delta_1,\delta_2)=\frac{2\varepsilon^2}{\Gamma(2j+\delta_1+\delta_2+2)}\sqrt{\frac{\Gamma(n+j+\delta_1+\delta_2+1)}{(n-j-1)!}}.$$

We assume that

$$\int_0^\pi\int_0^{2\pi}\sin\theta Z^{(s)*}_{j'm'}(\theta,\varphi;\delta_1,\delta_2)$$
$$\times Z^{(s)}_{jm}(\theta,\varphi;\delta_1,\delta_2)\,d\theta d\varphi=\delta_{jj'}\delta_{mm'},$$

$$\int_0^\infty r^2R^{(s)}_{n'j}(r;\delta_1,\delta_2)\,R^{(s)}_{nj}(r;\delta_1,\delta_2)\,dr=\delta_{nn'} \tag{5}$$

and denote by ε the following expression:

$$\varepsilon=\sqrt{-2E}=\frac{1}{n+(\delta_1+\delta_2)/2}.$$

The energy spectrum has the form

$$E\equiv E^{(s)}_n=-\frac{1}{2}\left(n+\frac{\delta_1+\delta_2}{2}\right)^{-2} \tag{6}$$

and the quantum numbers m and j run through the values: $m=-j,-j+1,\ldots,j-1,j$ and

$$j=\frac{|m+s|+|m-s|}{2},\frac{|m+s|+|m-s|}{2}+1,\ldots.$$

We make the following notation also: $m_\pm=(|m+s|\pm|m-s|)/2$ and

$$m_1=|m-s|+\delta_1=\sqrt{(m-s)^2+4c_1},$$
$$m_2=|m+s|+\delta_2=\sqrt{(m+s)^2+4c_2}.$$

The wave functions (3) are the eigenfunctions of commuting operators $\hat{M}$ and $\hat{J}_z$ and

$$\hat{M}\psi^{(s)}_{njm}(r,\theta,\varphi;\delta_1,\delta_2)=\left(j+\frac{\delta_1+\delta_2}{2}\right) \tag{7}$$
$$\times\left(j+\frac{\delta_1+\delta_2}{2}+1\right)\psi^{(s)}_{njm}(r,\theta,\varphi;\delta_1,\delta_2),$$

where

$$\hat{M}=\hat{J}^2+\frac{2c_1}{1+\cos\theta}+\frac{2c_2}{1-\cos\theta}.$$

Here, $\hat{J}^2$ is the square of the angular momentum [2]

$$\hat{\mathbf{J}} = \mathbf{r} \times (-i\nabla - s\mathbf{A}) - s\frac{\mathbf{r}}{r},$$

$\hat{J}_z = s - i\partial/\partial\varphi$ is its z component, and $\hat{J}_z\psi = m\psi$. The operator $\hat{M}$ is written in Cartesian coordinates as

$$\hat{M} = -r^2\Delta + x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} + 2x_i \frac{\partial}{\partial x_i} \quad (8)$$
$$+ \frac{2isr}{r-z}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} - is - i\frac{c_2}{s}\right) + \frac{2c_1 r}{r+z}.$$

Let us consider the generalized MIC–Kepler system in the parabolic coordinates $\xi, \eta \in [0, \infty)$, $\varphi \in [0, 2\pi)$, defined by the formulas

$$x = \sqrt{\xi\eta}\cos\varphi, \quad y = \sqrt{\xi\eta}\sin\varphi, \quad z = \frac{1}{2}(\xi - \eta).$$

In these coordinates, the differential elements of length and volume read

$$dl^2 = \frac{\xi+\eta}{4}\left(\frac{d\xi^2}{\xi} + \frac{d\eta^2}{\eta}\right) + \xi\eta d\varphi^2,$$
$$dV = \frac{1}{4}(\xi+\eta)d\xi d\eta d\varphi,$$

while the Laplace operator looks like

$$\Delta = \frac{4}{\xi+\eta}\left[\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right)\right] + \frac{1}{\xi\eta}\frac{\partial^2}{\partial\varphi^2}.$$

The substitution

$$\psi(\xi,\eta,\varphi) = \Phi_1(\xi)\Phi_2(\eta)\frac{e^{i(m-s)\varphi}}{\sqrt{2\pi}}$$

separates the variables in the Schrödinger equation and we arrive at the following system of equations:

$$\frac{d}{d\xi}\left(\xi\frac{d\Phi_1}{d\xi}\right) + \left[\frac{E}{2}\xi - \frac{m_1^2}{4\xi} + \frac{1}{2}\beta + \frac{1}{2}\right]\Phi_1 = 0, \quad (9)$$

$$\frac{d}{d\eta}\left(\eta\frac{d\Phi_2}{d\eta}\right) + \left[\frac{E}{2}\eta - \frac{m_2^2}{4\eta} - \frac{1}{2}\beta + \frac{1}{2}\right]\Phi_2 = 0, \quad (10)$$

where β is the separation constant.

These equations are analogs to the equations of the hydrogen atom in the parabolic coordinates [23]. Thus, we get

$$\psi^{(s)}_{n_1 n_2 m}(\xi,\eta,\varphi;\delta_1,\delta_2) \quad (11)$$
$$= \sqrt{2}\varepsilon^2\Phi_{n_1 m_1}(\xi)\Phi_{n_2 m_2}(\eta)\frac{e^{i(m-s)\varphi}}{\sqrt{2\pi}},$$

where

$$\Phi_{n_i m_i}(x) = \frac{1}{\Gamma(m_i+1)}\sqrt{\frac{\Gamma(n_i+m_i+1)}{(n_i)!}}$$
$$\times e^{-\varepsilon x/2}(\varepsilon x)^{m_i/2}F(-n_i; m_i+1; \varepsilon x).$$

Here, n_1 and n_2 are nonnegative integers:

$$n_1 = -\frac{|m-s|+\delta_1+1}{2} + \frac{\beta+1}{2\varepsilon},$$
$$n_2 = -\frac{|m+s|+\delta_2+1}{2} - \frac{\beta-1}{2\varepsilon}.$$

From the last relations, taking into account (6), we find that the parabolic quantum numbers n_1 and n_2 are connected with the principal quantum number n as follows:

$$n = n_1 + n_2 + \frac{|m-s|+|m+s|}{2} + 1.$$

Excluding the energy E from Eqs. (9) and (10), we obtain the additional integral of motion

$$\hat{X} = \frac{2}{\xi+\eta}\left[\xi\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right) - \eta\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right)\right]$$
$$+ \frac{\xi-\eta}{2\xi\eta}\frac{\partial^2}{\partial\varphi^2} - is\frac{\xi^2+\eta^2}{\xi\eta(\xi+\eta)}\frac{\partial}{\partial\varphi} - s^2\frac{\xi-\eta}{2\xi\eta}$$
$$+ \frac{2c_1\eta}{\xi(\xi+\eta)} - \frac{2c_2\xi}{\eta(\xi+\eta)} + \frac{\xi-\eta}{\xi+\eta}$$

with the eigenvalues

$$\beta = \varepsilon\left(n_1 - n_2 + \frac{|m-s| - |m+s| + \delta_1 - \delta_2}{2}\right)$$

and eigenfunctions $\psi^{(s)}_{n_1 n_2 m}(\xi,\eta,\varphi;\delta_1,\delta_2)$, i.e.,

$$\hat{X}\psi^{(s)}_{n_1 n_2 m}(\xi,\eta,\varphi;\delta_1,\delta_2) = \beta\psi^{(s)}_{n_1 n_2 m}(\xi,\eta,\varphi;\delta_1,\delta_2). \quad (12)$$

In Cartesian coordinates, the operator $\hat{X}$ can be rewritten as

$$\hat{X} = z\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - x\frac{\partial^2}{\partial x\partial z} \quad (13)$$
$$- y\frac{\partial^2}{\partial y\partial z} - is\frac{r+z}{r(r-z)}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial z}$$
$$- s^2\frac{r+z}{r(r-z)} + c_1\frac{r-z}{r(r+z)} - c_2\frac{r+z}{r(r-z)} + \frac{z}{r},$$

so that it immediately follows that $\hat{X}$ is connected to the z component $\hat{I}_z$ of the analog of the Runge–Lenz vector

$$\hat{\mathbf{I}} = \frac{1}{2}\left[(-i\nabla - s\mathbf{A}) \times \hat{\mathbf{J}} - \hat{\mathbf{J}} \times (-i\nabla - s\mathbf{A})\right] + \frac{\mathbf{r}}{r}$$

via

$$\hat{X} = \hat{I}_z + c_1\frac{r-z}{r(r+z)} - c_2\frac{r+z}{r(r-z)}$$

and coincides with $\hat{I}_z$ when $c_1 = c_2 = 0$.

3. BIORTHOGONALITY OF THE RADIAL WAVE FUNCTIONS

We shall prove that, along with condition (5) the radial wave functions $R_{nj}^{(s)}(r;\delta_1,\delta_2)$ satisfy the following additional orthogonality condition:

$$I_{jj'} = \int_0^\infty R_{nj'}^{(s)}(r;\delta_1,\delta_2)\, R_{nj}^{(s)}(r;\delta_1,\delta_2)\, dr \qquad (14)$$
$$= \frac{2}{(n+(\delta_1+\delta_2)/2)^3}\frac{\delta_{jj'}}{2j+\delta_1+\delta_2+1}.$$

This new relation is used in the next section to derive interbasis expansions. It can be proved as follows.

In the integral appearing in (14), we substitute explicit expressions (4) for $R_{nj}^{(s)}(r;\delta_1,\delta_2)$ and $R_{nj'}^{(s)}(r;\delta_1,\delta_2)$. Then, we take the confluent hypergeometric function in (4) as a finite sum,

$$F(-n+j+1;2j+\delta_1+\delta_2+2;2\varepsilon r)$$
$$= \sum_{p=0}^{n-j-1}\frac{(-n+j+1)_p}{p!(2j+\delta_1+\delta_2+1)_p}(2\varepsilon r)^p,$$

and perform the integration term by term with the help of the formula [23]

$$\int_0^\infty e^{-\lambda x}x^\nu F(\alpha,\gamma;kx)dx \qquad (15)$$
$$= \frac{\Gamma(\nu+1)}{\lambda^{\nu+1}}{}_2F_1\left(\alpha,\nu+1,\gamma;\frac{k}{\lambda}\right).$$

Applying the formula

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad (16)$$

for the hypergeometric function, we obtain

$$I_{jj'} = \frac{\Gamma(j+j'+\delta_1+\delta_2+1)}{\Gamma(2j+\delta_1+\delta_2+2)}\left[\frac{\Gamma(n+j+\delta_1+\delta_2+1))}{(n-j-1)!(n-j'-1)!\Gamma(n+j'+\delta_1+\delta_2+1)}\right]^{1/2}\frac{2}{(n+(\delta_1+\delta_2)/2)^3} \qquad (17)$$
$$\times\sum_{p=0}^{n-j-1}\frac{(-n+j+1)_p(j+j'+\delta_1+\delta_2+1)_p}{p!(2j+\delta_1+\delta_2+2)_p}\frac{\Gamma(n-j-p)}{\Gamma(j'-j-p+1)}.$$

By introducing the formula [24]

$$\frac{\Gamma(z)}{\Gamma(z-n)} = (-1)^n\frac{\Gamma(-z+n+1)}{\Gamma(-z+1)}$$

into (17), the sum over p can be expressed in terms of the ${}_2F_1$ Gauss hypergeometric function of argument 1. We thus obtain

$$I_{jj'} = \frac{1}{j+j'+\delta_1+\delta_2+1} \qquad (18)$$
$$\times\left[\frac{(n-j-1)!\Gamma(n+j+\delta_1+\delta_2+1))}{(n-j'-1)!\Gamma(n+j'+\delta_1+\delta_2+1)}\right]^{1/2}$$
$$\times\frac{2}{(n+(\delta_1+\delta_2)/2)^3}\frac{1}{\Gamma(j-j'+1)\Gamma(j'-j+1)}.$$

Equation (14) then easily follows from (18) since $[\Gamma(j-j'+1)\Gamma(j'-j+1)]^{-1} = \delta_{jj'}$.

The result provided by Eq. (14) generalizes the one for the hydrogen atom [25]. Indeed, orthogonality properties similar to (14) hold for the Kepler–Coulomb system and harmonic oscillator in f-dimensional spaces ($f \geq 2$) [25].

4. INTERBASIS EXPANSION BETWEEN PARABOLIC AND SPHERICAL BASES

The connection between spherical (r,θ,φ) and parabolic (ξ,η,φ) coordinates is

$$\xi = r(1+\cos\theta), \qquad \eta = r(1+\cos\theta), \qquad (19)$$
$$\varphi(\text{parabolic}) = \varphi(\text{spherical}).$$

Now, we can write, for fixed value energy $E_n^{(s)}$, the parabolic bound states (11) as a coherent quantum mixture of the spherical bound states (3):

$$\psi_{n_1n_2m}^{(s)}(\xi,\eta,\varphi;\delta_1,\delta_2) \qquad (20)$$
$$= \sum_{j=m_+}^{n-1} W_{n_1n_2ms}^{j}(\delta_1,\delta_2)\,\psi_{njm}^{(s)}(r,\theta,\varphi;\delta_1,\delta_2).$$

By virtue of Eq. (19), the left-hand side of (20) can be rewritten in spherical coordinates. Then, by substituting $\theta = 0$ into the so-obtained equation and by taking into account that

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!},$$

we get an equation that depends only on the variable r. Thus, we can use the orthogonality relation (14) on the quantum number j. This yields

$$W^{j}_{n_1n_2ms}(\delta_1,\delta_2) = \frac{\sqrt{(2j+\delta_1+\delta_2+1)(j-m_+)!}}{\Gamma(m_1+1)\Gamma(2j+\delta_1+\delta_2+2)} E^{jms}_{n_1n_2} K^{nn_1}_{jms}, \quad (21)$$

where

$$E^{jms}_{n_1n_2} = \left[\frac{\Gamma(j-m_-+\delta_1+1)\,\Gamma(n_1+m_1+1)\Gamma(n_2+m_2+1)\Gamma(n+j+\delta_1+\delta_2+1)}{(n_1)!(n_2)!(n-j-1)!\Gamma(j+m_-+\delta_2+1)\,\Gamma(j+m_++\delta_1+\delta_2+1)}\right]^{1/2}, \quad (22)$$

and

$$K^{nn_1}_{jms} = \int_0^\infty e^{-x}x^{j+m_1+\delta_1+\delta_2}F(-n_1;m_1+1;x)\times F(-n+j+1;2j+\delta_1+\delta_2+2;x)\,dx.$$

To calculate the integral $K^{nn_1}_{jms}$, it is sufficient to write the confluent hypergeometric function $F(-n_1;m_1+1;x)$ as a series, integrate according to (15), and use formula (16) for the summation of the hypergeometric function ${}_2F_1$. We thus obtain

$$K^{nn_1}_{jms} = \frac{(n-m_+-1)!\Gamma(2j+\delta_1+\delta_2+2)\Gamma(j+m_++\delta_1+\delta_2+1)}{(j-m_+)!\Gamma(n+j+\delta_1+\delta_2+1)} \quad (23)$$

$$\times\,{}_3F_2\left\{\begin{matrix}-n_1,-j+m_+,j+m_++\delta_1+\delta_2+1\\ m_1+1,-n+m_++1\end{matrix}\middle|1\right\}.$$

The introduction of (22) and (23) into (21) gives

$$W^{j}_{n_1n_2ms}(\delta_1,\delta_2) = \sqrt{\frac{(2j+\delta_1+\delta_2+1)\,\Gamma(n_1+m_1+1)\Gamma(n_2+m_2+1)}{(n_1)!(n_2)!(n-j-1)!(j-m_+)!\Gamma(j+m_-+\delta_2+1)}}\frac{(n-m_+-1)!}{\Gamma(m_1+1)} \quad (24)$$

$$\times\sqrt{\frac{\Gamma(j-m_-+\delta_1+1)\,\Gamma(j+m_++\delta_1+\delta_2+1)}{\Gamma(n+j+\delta_1+\delta_2+1)}}\,{}_3F_2\left\{\begin{matrix}-n_1,-j+m_+,j+m_++\delta_1+\delta_2+1\\ m_1+1,-n+m_++1\end{matrix}\middle|1\right\}.$$

The next step is to show that the interbasis coefficients (24) are, indeed, a continuation on the real line of the Clebsch–Gordan coefficients for the group $SU(2)$. It is known that the Clebsch–Gordan coefficient $C^{c,\gamma}_{a,\alpha;b,\beta}$ can be written as [26]

$$C^{c\gamma}_{a\alpha;b\beta} = \left[\frac{(2c+1)(a+\alpha)!(c+\gamma)!}{(a-\alpha)!(c-\gamma)!(a+b+c+1)!(a+b-c)!(a-b+c)!(b-a+c)!}\right]^{1/2} \quad (25)$$

$$\times(-1)^{a-\alpha}\delta_{\gamma,\alpha+\beta}\frac{(a+b-\gamma)!(b+c-\alpha)!}{\sqrt{(b-\beta)!(b+\beta)!}}\,{}_3F_2\left\{\begin{matrix}-a-b-c-1,-a+\alpha,-c+\gamma\\ -a-b+\gamma,-b-c+\alpha\end{matrix}\middle|1\right\}.$$

By using the formula [27]

$${}_3F_2\left\{\begin{matrix}s,s',-N\\ t',1-N-t\end{matrix}\middle|1\right\} = \frac{(t+s)_N}{(t)_N}\,{}_3F_2\left\{\begin{matrix}s,t'-s',-N\\ t',t+s\end{matrix}\middle|1\right\},$$

Eq. (25) can be rewritten in the form

$$C^{c\gamma}_{a\alpha;b\beta} = \left[\frac{(2c+1)(b-a+c)!(a+\alpha)!(b+\beta)!(c+\gamma)!}{(b-\beta)!(c-\gamma)!(a+b-c)!(a-b+c)!(a+b+c+1)!}\right]^{1/2} \tag{26}$$
$$\times\, \delta_{\gamma,\alpha+\beta}\frac{(-1)^{a-\alpha}}{\sqrt{(a-\alpha)!}}\frac{(a+b-\gamma)!}{(b-a+\gamma)!}{}_3F_2\left\{\begin{matrix}-a+\alpha, c+\gamma+1, -c+\gamma\\ \gamma-a-b, b-a+\gamma+1\end{matrix}\middle|1\right\}.$$

Finally, comparing (26) and (24), we obtain the representation

$$W^j_{n_1n_2ms}(\delta_1,\delta_1) = (-1)^{n_1} \tag{27}$$
$$\times\, C^{j+\frac{\delta_1+\delta_2}{2},\frac{m_1+m_2}{2}}_{\frac{n+m_-+\delta_2-1}{2},\frac{m_2+n_2-n_1}{2};\frac{n-m_-+\delta_1-1}{2},\frac{m_1+n_1-n_2}{2}}.$$

Equation (27) proves that the coefficients for the expansion of the parabolic basis in terms of the spherical basis are nothing but the analytical continuation, for real values of their arguments, of the $SU(2)$ Clebsch–Gordan coefficients.

The inverse of Eq. (20), namely,

$$\psi^{(s)}_{njm}(r,\theta,\varphi;\delta_1,\delta_2) \tag{28}$$
$$= \sum_{n_1=0}^{n-m_+-1}\tilde{W}^{n_1}_{njms}(\delta_1,\delta_2)\,\psi^{(s)}_{n_1n_2m}(\xi,\eta,\varphi;\delta_1,\delta_2),$$

is an immediate consequence of the orthonormality property of the $SU(2)$ Clebsch–Gordan coefficients. The expansion coefficients in (28) are thus given by

$$\tilde{W}^{n_1}_{njms}(\delta_1,\delta_2) = (-1)^{n_1}C^{j+\frac{\delta_1+\delta_2}{2},\frac{m_1+m_2}{2}}_{\frac{n+m_-+\delta_2-1}{2},\frac{n+m_-+\delta_2-1}{2}-n_1;\frac{n-m_-+\delta_1-1}{2},n_1+|m-s|-\frac{n-m_--\delta_1-1}{2}} \tag{29}$$

and may be expressed in terms of the ${}_3F_2$ function through (25) or (26).

5. PROLATE SPHEROIDAL BASIS

We now pass to the prolate spheroidal coordinates

$$x = \frac{R}{2}\sqrt{(\mu^2-1)(1-\nu^2)}\cos\varphi,$$
$$y = \frac{R}{2}\sqrt{(\mu^2-1)(1-\nu^2)}\sin\varphi, \quad z = \frac{R}{2}(\mu\nu+1),$$

where $\mu\in[0;\infty)$, $\nu\in[-1;1]$, $\varphi\in[0,2\pi)$, and $R\in[0;\infty)$. The parameter R is the interfocus distance, and in the limits where $R\to 0$ and $R\to\infty$, the prolate spheroidal coordinates give back the spherical coordinates and the parabolic coordinates, respectively [22, 28].

The Laplace operator in these coordinates has the form

$$\Delta = \frac{4}{R^2(\mu^2-\nu^2)}\left[\frac{\partial}{\partial\mu}\left(\mu^2-1\right)\frac{\partial}{\partial\mu}\right.$$
$$\left.+\frac{\partial}{\partial\nu}\left(1-\nu^2\right)\frac{\partial}{\partial\nu}\right] + \frac{4}{R^2\left(\mu^2-1\right)\left(1-\nu^2\right)}\frac{\partial^2}{\partial\varphi^2}.$$

After the substitution

$$\psi(\mu,\nu,\varphi) = \psi_1(\mu)\psi_2(\nu)\frac{e^{i(m-s)\varphi}}{\sqrt{2\pi}},$$

the variables in the Schrödinger equation (1) are separated

$$\left[\frac{d}{d\mu}\left(\mu^2-1\right)\frac{d}{d\mu}+\frac{m_1^2}{2(\mu+1)}-\frac{m_2^2}{2(\mu-1)}\right. \tag{30}$$
$$\left.+R\mu+\frac{ER^2}{2}\left(\mu^2-1\right)\right]\psi_1 = \lambda(R)\psi_1,$$
$$\left[\frac{d}{d\nu}\left(1-\nu^2\right)\frac{d}{d\nu}-\frac{m_1^2}{2(1+\nu)}-\frac{m_2^2}{2(1-\nu)}\right. \tag{31}$$
$$\left.-R\nu+\frac{ER^2}{2}\left(1-\nu^2\right)\right]\psi_2 = -\lambda(R)\psi_2,$$

where $\lambda(R)$ is a separation constant in prolate spheroidal coordinates. By eliminating the energy E from Eqs. (30) and (31), we produce the operator

$$\hat{\Lambda} = \frac{1}{\mu^2-\nu^2} \tag{32}$$
$$\times\left[\left(1-\nu^2\right)\frac{\partial}{\partial\mu}\left(\mu^2-1\right)\frac{\partial}{\partial\mu}-\left(\mu^2-1\right)\right.$$
$$\left.\times\frac{\partial}{\partial\nu}\left(1-\nu^2\right)\frac{\partial}{\partial\nu}\right]+\frac{2-\mu^2-\nu^2}{\left(\mu^2-1\right)\left(1-\nu^2\right)}\frac{\partial^2}{\partial\varphi^2}$$
$$+2s\frac{(\mu+\nu)^2-(\mu+1)(1+\nu)}{(\mu+\nu)(\mu-1)(1-\nu)}\left(s+i\frac{\partial}{\partial\varphi}\right)$$

$$+2c_1\frac{(\mu+\nu)^2+(\mu-1)(1-\nu)}{(\mu+\nu)(\mu+1)(1+\nu)}$$
$$+2c_2\frac{(\mu+\nu)^2-(\mu+1)(1+\nu)}{(\mu+\nu)(\mu-1)(1-\nu)}+R\frac{\mu\nu+1}{\mu+\nu},$$

the eigenvalues of which are $\lambda(R)$ and the eigenfunctions of which are $\psi(\mu,\nu,\varphi)$. The significance of the self-adjoint operator $\hat{\Lambda}$ can be found by switching to Cartesian coordinates. Passing to Cartesian coordinates in (32) and taking (8) and (13) into account, we obtain

$$\hat{\Lambda}=\hat{M}+R\hat{X}. \tag{33}$$

Therefore,

$$\hat{\Lambda}\psi^{(s)}_{nqm}(\mu,\nu,\varphi;R,\delta_1,\delta_2) \tag{34}$$
$$=\lambda_q(R)\psi^{(s)}_{nqm}(\mu,\nu,\varphi;R,\delta_1,\delta_2),$$

where index q labels the eigenvalues of the operator $\hat{\Lambda}$ and varies in the range $0\le q\le n-m_+-1$.

We are now ready to deal with the interbasis expansions

$$\psi^{(s)}_{nqm}(\mu,\nu,\varphi;R,\delta_1,\delta_2) \tag{35}$$
$$=\sum_{j=m_+}^{n-1}U^j_{nqms}(R;\delta_1,\delta_2)\,\psi^{(s)}_{njm}(r,\theta,\varphi;\delta_1,\delta_2),$$

$$\psi^{(s)}_{nqm}(\mu,\nu,\varphi;R,\delta_1,\delta_2) \tag{36}$$
$$=\sum_{n_1=0}^{n-m_+-1}V^{n_1}_{nqms}(R;\delta_1,\delta_2)\,\psi^{(s)}_{n_1n_2m}(\xi,\eta,\varphi;\delta_1,\delta_2)$$

for the prolate spheroidal basis in terms of the spherical and parabolic bases. {Equation (35) was first considered by Coulson and Joseph [29] in the particular case $s=\delta_1=\delta_2=0$.}

First, we consider Eq. (35). Let the operator $\hat{\Lambda}$ act on both sides of (35). Then, by using Eqs. (33), (34), and (7) as well as the orthonormality property of the spherical basis, we find that

$$\left[\lambda_q(R)-\left(j+\frac{\delta_1+\delta_2}{2}\right)\right. \tag{37}$$
$$\left.\times\left(j+\frac{\delta_1+\delta_2}{2}+1\right)\right]U^j_{nqms}$$
$$=R\sum_{j'=m_+}^{n-1}U^{j'}_{nqms}(\hat{X})_{jj'},$$

where

$$(\hat{X})_{jj'}=\int\psi^{(s)*}_{njm}(r,\theta,\varphi;\delta_1,\delta_2) \tag{38}$$
$$\times\hat{X}\psi^{(s)}_{njm}(r,\theta,\varphi;\delta_1,\delta_2)\,dV.$$

The calculation of the matrix element $(\hat{X})_{jj'}$ can be done by expanding the basis in (38) in terms of parabolic wave functions [see Eq. (28)] and by making use of the eigenvalue equation for $\hat{X}$ [see Eq. (12)]. This leads to

$$(\hat{X})_{jj'}=\frac{2}{2n+\delta_1+\delta_2}\sum_{n_1=0}^{n-m_+-1}\left(2n_1-n\right.$$
$$\left.+|m-s|+\frac{\delta_1+\delta_2}{2}+1\right)\tilde{W}^{n_1}_{njm}\tilde{W}^{n_1}_{nj'm}.$$

Then, by using Eq. (29) together with the recursion relation [26]

$$C^{c\gamma}_{a\alpha;b\beta}=-\left[\frac{4c^2(2c+1)(2c-1)}{(c+\gamma)(c-\gamma)(b-a+c)(a-b+c)(a+b-c+1)(a+b+c+1)}\right]^{1/2}$$
$$\times\left\{\left[\frac{(c-\gamma-1)(c+\gamma-1)(b-a+c-1)(a-b+c-1)(a+b-c+2)(a+b+c)}{4(c-1)^2(2c-3)(2c-1)}\right]^{1/2}\right.$$
$$\left.\times C^{c-2,\gamma}_{a\alpha;b\beta}-\frac{(\alpha-\beta)c(c-1)-\gamma a(a+1)+\gamma b(b+1)}{2c(c-1)}C^{c-1,\gamma}_{a\alpha;b\beta}\right\}$$

and the orthonormality condition

$$\sum_{\alpha+\beta=\gamma}C^{c\gamma}_{a\alpha;b\beta}C^{c'\gamma'}_{a\alpha;b\beta}=\delta_{c'c}\delta_{\gamma'\gamma},$$

we find that $(\hat{X})_{jj'}$ is given by

$$(\hat{X})_{jj'} = \frac{(m_1+m_2)(m_1-m_2)}{(2j+\delta_1+\delta_2)(2j+\delta_1+\delta_2+2)}\delta_{j',j} - \frac{2}{2n+\delta_1+\delta_2}\left(A^{j+1}_{nm}\delta_{j',j+1} + A^{j}_{nm}\delta_{j',j-1}\right), \quad (39)$$

where

$$A^j_{nm} = \left[\frac{(j-m_+)(j+m_++\delta_1+\delta_2)(j-m_-+\delta_1)(j+m_-+\delta_2)(n-j)(n+j+\delta_1+\delta_2)}{(j+(\delta_1+\delta_2)/2)^2(2j+\delta_1+\delta_2-1)(2j+\delta_1+\delta_2+1)}\right]^{1/2}.$$

Now, by introducing (39) into (37), we get the following three-term recursion relation for the coefficient U^j_{nqms}:

$$\left[\lambda_q(R) - \left(j+\frac{\delta_1+\delta_2}{2}\right)\left(j+\frac{\delta_1+\delta_2}{2}+1\right) - \frac{R(m_1+m_2)(m_1-m_2)}{(2j+\delta_1+\delta_2)(2j+\delta_1+\delta_2+2)}\right]U^j_{nqms} + \frac{2R}{2n+\delta_1+\delta_2}\left[A^{j+1}_{nm}U^{j+1}_{nqms} + A^j_{nm}U^{j-1}_{nqms}\right] = 0. \quad (40)$$

The recursion relation (40) provides us with a system of $n-m_+$ linear homogeneous equations which can be solved by taking into account the normalization condition

$$\sum_{j-m_+}^{n-1}\left|U^j_{nqms}(R;\delta_1,\delta_2)\right|^2 = 1.$$

The eigenvalues $\lambda_q(R)$ of the operator $\hat{\Lambda}$ then follow from the vanishing of the determinant for the latter system.

Second, let us concentrate on the expansion (36) of the prolate spheroidal basis in terms of the parabolic basis. By employing a technique similar to the one used for deriving Eq. (37), we get

$$\left[\lambda_q(R) - \frac{2R}{2n+\delta_1+\delta_2}\left(n_1-n_2 + \frac{m_1-m_2}{2}\right)\right]V^{n_1}_{nqms} = \sum_{n'_1=0}^{n-m_+-1} V^{n'_1}_{nqms}(\hat{M})_{n_1n'_1}, \quad (41)$$

where

$$(\hat{M})_{n_1n'_1} = \int \psi^{(s)*}_{n_1n_2m}(\xi,\eta,\varphi;\delta_1,\delta_2) \times \hat{M}\psi^{(s)}_{n'_1n'_2m}(\xi,\eta,\varphi;\delta_1,\delta_2)\,dV.$$

The matrix elements $(\hat{M})_{n_1n'_1}$ can be calculated in the same way as $(\hat{X})_{jj'}$ except that now we must use the relation [30]

$$[c(c+1)-a(a+1)-b(b+1)-2\alpha\beta]\,C^{c,\gamma}_{a,\alpha;b,\beta} = \sqrt{(a+\alpha)(a-\alpha+1)(b-\beta)(b+\beta+1)}C^{c,\gamma}_{a,\alpha-1;b,\beta+1} + \sqrt{(a-\alpha)(a+\alpha+1)(b+\beta)(b-\beta+1)}C^{c,\gamma}_{a,\alpha+1;b,\beta-1}$$

and the orthonormality condition

$$\sum_{c=|\gamma|}^{a+b} C^{c\gamma}_{a\alpha;b\beta}C^{c\gamma}_{a\alpha';b\beta'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}$$

permits deriving the formula for the matrix element $(\hat{M})_{n_1n'_1}$:

$$(\hat{M})_{n_1n'_1} = \Big[(n_1+1)(n_2+m_-) + (n-n_1+\delta_2)(n_1+|m-s|+\delta_2) + m_-(m_++\delta_2) + \frac{1}{4}(\delta_1-\delta_2)(\delta_1-\delta_2-2)\Big]\delta_{n'_1n_1} - \sqrt{n_2(n_1+1)(n_1+|m-s|+\delta_1+1)(n_2+|m-s|+\delta_2)}\delta_{n'_1,n_1+1} - \sqrt{n_1(n_2+1)(n_1+|m-s|+\delta_1+1)(n_2+|m-s|+\delta_2+1)}\delta_{n'_1,n_1-1}. \quad (42)$$

Finally, the introduction of (42) into (41) leads to the three-term recursion relation

$$\left[(n_1+1)(n_2+m_-)+(n-n_1+\delta_2)(n_1+|m-s|+\delta_2)+\frac{1}{4}(\delta_1-\delta_2)(\delta_1-\delta_2-2)+m_-(m_++\delta_2)\right. \quad (43)$$
$$\left.+\frac{2R}{2n+\delta_1+\delta_2}\left(n_1-n_2+\frac{m_1-m_2}{2}\right)-\lambda_q(R)\right]V^{n_1}_{nqms}(R;\delta_1,\delta_2)$$
$$=\sqrt{n_2(n_1+1)(n_1+|m-s|+\delta_1+1)(n_2+|m-s|+\delta_2)}V^{n_1+1}_{nqms}(R;\delta_1,\delta_2)$$
$$+\sqrt{n_1(n_2+1)(n_1+|m-s|+\delta_1+1)(n_2+|m-s|+\delta_2+1)}V^{n_1-1}_{nqms}(R;\delta_1,\delta_2)$$

for the expansion coefficients $V^{n_1}_{nqms}(R;\delta_1,\delta_2)$. This relation can be iterated by taking account of the normalization condition

$$\sum_{n_1=0}^{n-m_+-1}\left|V^{n_1}_{nqms}(R;\delta_1,\delta_2)\right|^2=1.$$

Here again, the eigenvalues $\lambda_q(R)$ may be obtained by solving a system of $n-m_+$ linear homogeneous equations.

It should be mentioned that formulas (27) and (29) and three-term recursion relations (40) and (43) generalize the analogous results for the following systems:

hydrogen atom [28, 29, 31–34], when $s=\delta_1=\delta_2=0$;

generalized Kepler–Coulomb system [17], when $s=0$, $\delta_1\neq\delta_2\neq 0$;

Hartmann system [35], when $s=0$, $\delta_1=\delta_2\neq 0$;

charge–dyon system [8], when $s\neq 0$, $\delta_1=\delta_2=0$.

Finally, it should be noted that the following four limits

$$\lim_{R\to 0}U^j_{jnqms}(R;\delta_1,\delta_2)=\delta_{jq},$$
$$\lim_{R\to\infty}U^j_{nqms}(R;\delta_1,\delta_2)=W^j_{n_1n_2ms}(\delta_1,\delta_2),$$
$$\lim_{R\to\infty}V^{n_1}_{nqms}(R;\delta_1,\delta_2)=\delta_{n_1q},$$
$$\lim_{R\to 0}V^{n_1}_{nqms}(R;\delta_1,\delta_2)=\tilde{W}^{n_1}_{njms}(\delta_1,\delta_2)$$

furnish a useful means for checking the calculations presented in Sections 4 and 5.

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Infinite-Order Symmetries for Quantum Separable Systems*

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Abstract—We develop a calculus to describe the (in general) infinite-order differential operator symmetries of a nonrelativistic Schrödinger eigenvalue equation that admits an orthogonal separation of variables in Riemannian n space. The infinite-order calculus exhibits structure not apparent when one studies only finite-order symmetries. The search for finite-order symmetries can then be reposed as one of looking for solutions of a coupled system of PDEs that are polynomial in certain parameters. Among the simple consequences of the calculus is that one can generate algorithmically a canonical basis for the space. Similarly, we can develop a calculus for conformal symmetries of the time-dependent Schrödinger equation if it admits R separation in some coordinate system. This leads to energy-shifting symmetries.

1. INTRODUCTION

The main point we want to get across in this paper is that, if a Schrödinger equation on a pseudo-Riemannian manifold (real or complex),

$$(\Delta_n + V(x))\,\Psi = E\Psi \quad \text{or} \quad (\Delta_n + V(x))\,\Theta = \Theta_t,$$

admits an orthogonal separation (or R separation) of variables, then the differential symmetry operators for the system, including those of infinite order, can be obtained by solving a strictly finite system of PDEs with parameters. The finite-order symmetry (or conformal symmetry) operators correspond to solutions that are polynomial in the parameters. This point of view exhibits a structure in the space of symmetries that is not apparent when one looks for finite-order symmetries alone. Understanding this structure is of particular importance for superintegrable systems [1–7], where there exist differential symmetries that are not obvious from the separation of the systems in a single-coordinate system.

We will describe the basic ideas by first reviewing the simplest example, the time-dependent Schrödinger equation (with potential) in two-dimensional spacetime [8]. There, we can easily produce infinite-order conformal symmetries and show their relevance to finding energy shifting operators for the time-independent Schrödinger equation. However, the system is so simple that one might not appreciate the vital role of variable separation in the results. This is clarified when we take up the study of the time-independent Schrödinger equation (with potential) on a two-dimensional pseudo-Riemannian space. The approach extends to any number of space variables.

2. INFINITE-ORDER CONFORMAL SYMMETRIES FOR THE TIME-DEPENDENT SCHRÖDINGER EQUATION IN ONE SPATIAL DIMENSION

The basic equation is the heat or time-dependent Schrödinger equation

$$(\partial_t - \partial_{xx} - V(x))\,\Psi(x,t) = 0. \qquad (1)$$

Here, V and Ψ are complex analytic functions of the complex variables x, t. Recall that an operator L, acting on the solution space of (1), is a (conformal) symmetry if

$$[\partial_t - H, L] \equiv \partial_t L - [H, L] = R(\partial_t - H)$$

for some linear operator R. Here, $H = \partial_{xx} + V(x)$.

We have separation of variables for (1), in the coordinates $\{x,t\}$. Indeed, the potential $V(x,t) = V_1(x) + V_2(t)$ also permits separation, but a gauge transformation $\Psi(x,t) = e^{T(t)}\Theta(x,t)$ with $T'(t) = V_2(x)$ leads to Eq. (1) again for Θ.

It should not be thought that (1) refers only to Cartesian coordinates. Indeed, there are three R-separable coordinate systems for this equation:

(i) Cartesian coordinates (x,t), $\Psi_{xx} + V(x)\Psi = \Psi_t$.

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(ii) Heat coordinates (u,τ), $u=x/\sqrt{t}$, $\tau=\ln t$. If we set $\Psi=e^{-u^2/4}\Theta(u,\tau)$, then (1) becomes

$$\Theta_{uu}+\left(-\frac{u^2}{4}-\frac{1}{2}+e^{\tau}V\right)\Theta=\Theta_{\tau},$$

separable if $e^{\tau}V=\tilde{V}(u)$.

(iii) Airy coordinates (u,τ), $u=x-t^2/2$, $\tau=t$. If we set $\Psi=e^{-\tau^3/12-u\tau/2}\Theta(u,\tau)$, then (1) becomes

$$\Theta_{uu}+\left(\frac{1}{2}u+V\right)\Theta=\Theta_{\tau},$$

separable if $V=\tilde{V}(u)$.

This means that the symmetry analysis below applies to potentials of the form

$$V=f(x),\quad V=\frac{f(x/\sqrt{t})}{t},$$
$$\text{or}\quad V=f\left(x-\frac{t^2}{2}\right).$$

We will only consider the action of $L(t)$ on the solution space of (1). Then each term ∂_x^2 in the formal expansion of the (possibly infinite-order) conformal symmetry

$$L(t)=\sum_{n,m=0}^{\infty}\ell(x,t)_{n,m}\partial_x^n\partial_t^m$$

can be replaced successively by $\partial_t-V(x)$, if at each stage the terms in the expansion are reordered so that the derivative terms act directly on the solution space. Thus $L(t)$ can be placed in the canonical form:

$$L(t)=a(x,t,\lambda)\partial_x+b(x,t,\lambda). \tag{2}$$

Here, we consider

$$a(x,t,\lambda)=\sum_{m=0}^{\infty}a_m(x,t)\partial_t^m,\quad \lambda=\partial_t,$$

with a similar interpretation for b. (We could also expand a in a power series in $\lambda-\lambda_0$, so it is only necessary for a to be analytic in λ about some complex number λ_0. We have chosen $\lambda_0=0$ for clarity of exposition.) The action of $L(t)$ on constant energy solutions

$$\Psi(x,t)=e^{Et}\phi(x),\quad H\phi=E\phi,$$

can be made rigorous, even if a and b are not analytic:

$$L\Psi=e^{Et}\{a(x,t,E)\partial_x+b(x,t,E)\}\phi(x).$$

Now let us determine the conditions on a and b so that $L(t)$ is a symmetry. The conditions are

$$b_x=\frac{1}{2}a_t-\frac{1}{2}a_{xx},$$
$$b_t=\frac{1}{2}a_{tx}-\frac{1}{2}a_{xxx}+2a_x\lambda-2a_xV-aV_x.$$

The integrability condition for these equations is

$$a_{tt}-2a_{xxt}+a_{xxxx}+4a_{xx}(V-\lambda)+6a_xV_x+2aV_{xx}=0. \tag{3}$$

Theorem. *Condition* (3) *is necessary and sufficient for* $L(t)=a(x,t,\lambda)\partial_x+b(x,t,\lambda)$ *to be a symmetry.*

It is not difficult to find all solutions of (3) which are of the form

$$a=\exp(t\kappa(\lambda))f(x,\lambda).$$

We obtain the fourth-order ordinary differential equation

$$f_{xxxx}+(4V-4\lambda-2\kappa)f_{xx}+6V_xf_x+(2V_{xx}+\kappa^2)f=0. \tag{4}$$

It is easy to show that these solutions occur in raising-operator/lowering-operator pairs [8].

To solve Eq. (4), we make use of Whittaker's theorem: Let $u(x)$ and $v(x)$ be solutions of the differential equations $u''-p(x)u=0$, $v''-q(x)v=0$. Then $y(x)=u(x)v(x)$ satisfies

$$(p-q)y''''-(p'-q')y'''-2(p^2-q^2)y''+(-pp'+qq'+5p'q-5pq')y'+(p'^2-q'^2-(p-q)(p''+q'')+(p-q)^3)y=0.$$

Now consider the equations

$$\text{(i)}\ \ u''+Vu=(\lambda+\kappa)u,\quad \text{(ii)}\ \ v''+Vv=\lambda v,$$

i.e., $p=\lambda+\kappa-V$, $q=\lambda-V$. Then we get (4) with $f=uv$. Similarly, we can find structure results for the basic Eq. (3).

Although our theorems exhibit clearly the structure of the generalized symmetries, other methods for computing the recurrences may be simpler.

Example (pseudo-Coulomb potential). We compute the possible solutions to (4) of the form $f(x,\lambda)=x$. We find the pseudo-Coulomb potential

$$V(v)=\frac{a^2}{x^2}-b^2x^2,\quad \kappa=\pm 4b.$$

Here, the raising and lowering operators are of finite order, and they raise and lower by a fixed energy. The raising and lowering operators and H generate the Lie algebra $s\ell(2)$ and a standard weight vector argument yields the bound-state energy levels for the hydrogen atom.

Example (Morse potential). We compute the solutions to (4) of the form $f(x,\lambda)=\exp(\mu(\lambda)x)$. We find that μ is independent of λ and

$$V(x)=D[2\exp(-\mu x)-\exp(-2\mu x)],$$

where D and μ are positive parameters. The Schrödinger equation admits the generalized (infinite-order) symmetries

$$L^{\pm} = e^{t\kappa^{\pm}}\left[e^{\mu x}\partial_x + \left(\frac{\kappa^{\pm}}{2\mu} - \frac{\mu}{2}\right)e^{\mu x} - \frac{2\mu D}{\kappa^{\pm}}\right],$$

where $\kappa^{\pm}(\lambda) = \mu^2 \pm 2\mu\sqrt{\lambda}$. Since $\kappa^{+}(\lambda) + \kappa^{-}(\lambda + \kappa^{+}(\lambda)) = 0$ for $\sqrt{\lambda} + \mu \geq 0$ and $\kappa^{-}(\lambda) + \kappa^{+}(\lambda + \kappa^{-}(\lambda)) = 0$ for $\sqrt{\lambda} - \mu \geq 0$, we can easily verify that

$$L^{+}L^{-} \sim D - \frac{4D^2}{(\mu - 2\sqrt{\lambda})^2},$$

$$L^{-}L^{+} \sim D - \frac{4D^2}{(\mu + 2\sqrt{\lambda})^2},$$

where equality is meant in the sense that the two sides agree when applied to a solution to (1). Thus, we have the commutation relations

$$[L^{+}, L^{-}] \sim \frac{-32D^2\mu\sqrt{\lambda}}{(\mu^2 - 4\lambda)^2},$$

$$[\lambda, L^{+}] \sim (\mu^2 + 2\mu\sqrt{\lambda})L^{+},$$

$$[\lambda, L^{-}] \sim (\mu^2 - 2\mu\sqrt{\lambda})L^{-},$$

an analog of the commutation relations for the Lie algebra $s\ell(2)$.

Even though L^{+}, L^{-}, λ do not generate a finite-dimensional Lie algebra, one can easily mimic the (weight vector) approach to the representation theory of $s\ell(2)$ to determine the irreducible representations of the associative algebra generated by these three operators. Note the "Casimir operator" C acting on the solution space of (1):

$$C = L^{+}L^{-} + \frac{4D^2}{(\mu - 2\sqrt{\lambda})^2} \sim L^{-}L^{+} + \frac{4D^2}{(\mu + 2\sqrt{\lambda})^2} \sim D.$$

We look for a "lowest weight vector" Ψ_0 for λ, i.e., a nonzero solution to the equations

$$(\lambda - H)\Psi_0 = 0, \quad \lambda\Psi_0 = E_0\Psi_0, \quad L^{-}\Psi_0 = 0.$$

Evaluating $C\Psi_0 = D\Psi_0$ we find $4D^2/(\mu - 2\sqrt{E_0})^2 = D$ or

$$E_0 = \mu^2\left(\frac{\sqrt{D}}{\mu} - \frac{1}{2}\right)^2,$$

assuming $\mu - 2\sqrt{E_0} \geq 0$. Recursively applying L^{+} to get $\Psi_n = (L^{+})^n\Psi_0$ with eigenvalues E_n satisfying the recurrence $E_{n+1} = E_n + \kappa^{+}(E_n) = (\mu + \sqrt{E_n})^2$, we find the spectrum

$$E_n = \mu^2\left[\frac{\sqrt{D}}{\mu} - \left(n + \frac{1}{2}\right)\right]^2, \quad n = 0, 1, 2, \ldots.$$

As an application of the determining equations

$$a_{tt} - 2a_{xxt} + a_{xxxx} + 4a_{xx}(V - \lambda) + 6a_xV_x + 2aV_{xx} = 0, \tag{5}$$

let us consider the problem of finding those potentials that admit third-order invariants,

$$L(t) = a(x, t\lambda)\partial_x + b(x, t, \lambda),$$

where we consider λ as a second-order invariant. Thus, we look for solutions to (5) of the form

$$a(x, t, \lambda) = A(x, t)\lambda + B(x, t),$$

where $A(x, t) \neq 0$. Substituting this expression into (5) and equating powers of λ, we find

$$A_{xx} = 0 \Longrightarrow A = \alpha(t) + \beta(t)x, \tag{6}$$

$$A_{tt} - 4B_{xx} + 6A_xV_x + 2AV_{xx} = 0, \tag{7}$$

$$B_{tt} - 2B_{xxt} + B_{xxxx} + 4B_{xx}V + 6B_xV_{xx} = 0. \tag{8}$$

Substituting (6) into (7) and integrating, we find

$$B(x, t) = \ddot{\alpha}(t)\frac{x^2}{8} + \ddot{\beta}(t)\frac{x^3}{24} + \beta(t)W(x) + \frac{1}{2}(\alpha(t) + \beta(t)x)W'(x) + \gamma(t)x + \delta(t),$$

where $V(x) = W'(x)$.

Substituting this result into (8), we find the functional equation for the potential:

$$\alpha^{(4)}(t)\left[\frac{x^2}{8}\right] + \alpha^{(3)}(t)\left[-\frac{1}{2}\right] + \ddot{\alpha}(t)\left[\frac{3}{2}W' + \frac{3x}{2}W'' + \frac{x^2}{4}W'''\right] + \dot{\alpha}(t)[-W'''] + \alpha(t)\left[3W'W''' + \frac{1}{2}W''''' + 3(W'')^2\right] + \beta^{(4)}(t)\left[\frac{x^3}{24}\right] + \beta^{(3)}(t)\left[-\frac{x}{2}\right] + \ddot{\beta}(t)\left[\frac{x^3}{12}W''' + \frac{3x^2}{4}W'' + \frac{3x}{2}W' + \frac{1}{2}W\right] + \dot{\beta}(t)[-3W'' - W'''x] + \beta(t)\left[\frac{x}{2}W''''' + 3xW'W''' + WW''' + 3(W'')^2x + 12W'W'' + \frac{5}{2}W'''\right] + \ddot{\gamma}(t)[x] + \gamma(t)[6W'' + 2W'''x] + \ddot{\delta}(t) + \delta(t)[2W'''] = 0. \tag{9}$$

To find all solutions W, we would need to study this functional equation in detail. However, many solutions are obvious. Indeed, if we choose

$$\alpha(t) \equiv \alpha_0, \quad \beta(t) \equiv \beta_0, \quad \gamma(t) \equiv \gamma_0, \quad \delta(t) \equiv \delta_0,$$

i.e., constants, then (9) becomes a nonlinear ODE for the potential $W(x)$, and every solution yields a potential with a third-order differential symmetry operator.

Another very important case is obtained by setting

$$\alpha(t)=\alpha_0 e^{\kappa t},\quad \beta(t)=\beta_0 e^{\kappa t},\quad \gamma(t)=\gamma_0 e^{\kappa t},\quad \delta(t)=\delta_0 e^{\kappa t},$$

where κ is a constant. Then we factor out $e^{\kappa t}$ from (9) and the result is an ODE for W again. For these potentials, $L(t)$ becomes a third-order energy raising operator, increasing the energy from H to $H+\kappa$. Every third-order raising operator is associated with a third-order lowering operator, so all these cases permit ladders of bound-state energy levels, subject to normalization requirements.

3. TWO-DIMENSIONAL SEPARABLE SYSTEMS FOR THE TIME-INDEPENDENT SCHRÖDINGER EQUATION $(\Delta_2+V)\Psi=E\Psi$

If $\{x,y\}$ is an orthogonal separable coordinate system in a general Riemannian space, the corresponding Schrödinger operator has the form [9]

$$H=L_1=\frac{1}{f_1(x)+f_2(y)}(\partial_x^2+\partial_y^2+v_1(x)+v_2(y))$$

and, due to the separability, there is the second-order symmetry operator

$$L_2=\frac{f_2(y)}{f_1(x)+f_2(y)}\left(\partial_x^2+v_1(x)\right)-\frac{f_1(x)}{f_1(x)+f_2(y)}\left(\partial_y^2+v_2(y)\right),$$

i.e., $[L_2,H]=0$, and the operator identities

$$f_1(x)H+L_2=\partial_x^2+v_1(x),\qquad f_2(y)H-L_2=\partial_y^2+v_2(y). \tag{10}$$

We look for a partial differential operator $\tilde{L}(H,L_2,x,y)$ that satisfies

$$[H,\tilde{L}]=0. \tag{11}$$

We require that the symmetry operator take the standard form

$$\tilde{L}=\sum_{j,k}(A_{j,k}(x,y)\partial_{xy}+B_{j,k}(x,y)\partial_x+C_{j,k}(x,y)\partial_y+D_{j,k}(x,y))H^jL_2^k. \tag{12}$$

(Again, only for convenience do we expand about $(H_0,L_0)=(0,0)$. We only require analyticity about some point (H_0,L_0).) Note that, if the formal operators (12) contained partial derivatives in x and y of orders ≥ 2, we could use the identities (10) recursively and rearrange terms to achieve the unique standard form (12).

Using operator identities

$$[\partial_x,H]=-\frac{f_1'}{f_1+f_2}H+\frac{v_1'}{f_1+f_2},\quad [\partial_y,H]=-\frac{f_2'}{f_1+f_2}H+\frac{v_2'}{f_1+f_2},$$
$$[\partial_x,L_2]=-\frac{f_1'f_2}{f_1+f_2}H+\frac{f_2v_1'}{f_1+f_2},\quad [\partial_y,L_2]=\frac{f_1f_2'}{f_1+f_2}H-\frac{f_1v_2'}{f_1+f_2}$$

we see that

$$\begin{aligned}(f_1(x)+f_2(y))[H,A(x,y)\partial_{xy}+B(x,y)\partial_x+C(x,y)\partial_y+D(x,y)]&=(A_{xx}+A_{yy}+2B_y+2C_x)\partial_{xy}+(B_{xx}+B_{yy}-2A_yv_2+2D_x-Av_2')\partial_x+(2A_yf_2+Af_2')\partial_xH-2A_y\partial_xL_2\\&+(C_{xx}+C_{yy}-2A_xv_1+2D_y-Av_1')\partial_y+(2A_xf_1+Af_1')\partial_yH+2A_x\partial_yL_2+(D_{xx}+D_{yy}-2B_xv_1-2C_yv_2-Bv_1'-Cv_2')+(2B_xf_1+2C_yf_2+Bf_1'+Cf_2')H+(2B_x-2C_y)L_2.\end{aligned}$$

The symmetry condition (11) is equivalent to the system of equations

$$\partial_{xx}A_{j,k}+\partial_{yy}A_{j,k}+2\partial_yB_{j,k}+2\partial_xC_{j,k}=0, \tag{13}$$

$$\partial_{xx}B_{j,k}+\partial_{yy}B_{j,k}-2\partial_yA_{j,k}v_2+2\partial_xD_{j,k}-A_{j,k}v_2'+(2\partial_yA_{j-1,k}f_2+A_{j-1,k}f_2')-2\partial_yA_{j,k-1}=0, \tag{14}$$

$$\partial_{xx}C_{j,k}+\partial_{yy}C_{j,k}-2\partial_xA_{j,k}v_1+2\partial_yD_{j,k}-A_{j,k}v_1'+(2\partial_xA_{j-1,k}f_1+A_{j-1,k}f_1')+2\partial_xA_{j,k-1}=0, \tag{15}$$

$$\partial_{xx}D_{j,k}+\partial_{yy}D_{j,k}-2\partial_xB_{j,k}v_1-2\partial_yC_{j,k}v_2-B_{j,k}v_1'-C_{j,k}v_2'+(2\partial_xB_{j-1,k}f_1+2\partial_yC_{j-1,k}f_2+B_{j-1,k}f_1'+C_{j-1,k}f_2')+(2\partial_xB_{j,k-1}-2\partial_yC_{j,k-1})=0. \tag{16}$$

Note that condition (12) makes sense, at least formally, for infinite-order differential equations. Indeed, one can consider H and L_2 as parameters in these equations. Then, once $\tilde{L}$ is expanded as a power series in these parameters, the terms are reordered so that the powers of the parameters are on the right, before they are replaced by explicit differential operators. Alternatively, one can consider the operator $\tilde{L}$ as acting on a simultaneous eigenbasis of the commuting

operators H and L_2, in which case the parameters are the eigenvalues.

In this view, we can write

$$\tilde{L}(H,L_2,x,y) = A(x,y,H,L_2)\partial_{xy} + B(x,y,H,L_2)\partial_x + C(x,y,H,L_2)\partial_y + D(x,y,H,L_2) \quad (17)$$

and consider $\tilde{L}$ as an at most second-order order differential operator in x, y that is analytic in the parameters H, L_2. Then the above system of equations can be written in the more compact form

$$A_{xx} + A_{yy} + 2B_y + 2C_x = 0, \quad (18)$$

$$B_{xx} + B_{yy} - 2A_y v_2 + 2D_x - Av_2' + (2A_y f_2 + Af_2')H - 2A_y L_2 = 0, \quad (19)$$

$$C_{xx} + C_{yy} - 2A_x v_1 + 2D_y - Av_1' + (2A_x f_1 + Af_1')H + 2A_x L_2 = 0, \quad (20)$$

$$D_{xx} + D_{yy} - 2B_x v_1 - 2C_y v_2 - Bv_1' - Cv_2' + (2B_x f_1 + 2C_y f_2 + Bf_1' + Cf_2')H + (2B_x - 2C_y)L_2 = 0. \quad (21)$$

and this system has many solutions.

We start with a very special case

$$A \equiv 0, \quad B = X(x,H,L_2), \quad C = Y(y,H,L_2), \quad D = \tilde{X}(x,H,L_2) + \tilde{Y}(y,H,L_2). \quad (22)$$

Then the above PDEs uncouple into ODEs for X and Y, whose structure we can easily analyze. We write

$$\tilde{L} = M(H,L_2,x,\partial_x) + N(H,L_2,y,\partial_y),$$

where

$$M(H,L_2,x,\partial_x) = \sum_{j,k} \left(X_{j,k}(x)\partial_x + \tilde{X}_{j,k}(x)\right) H^j L_2^k, \quad (23)$$

with a similar equation for N. We immediate obtain the system of equations

$$X_{j,k}'' + 2\tilde{X}_{j,k}' = 0, \quad (24)$$

$$\tilde{X}_{j,k}'' - v_1' X_{j,k} - 2v_1 X_{j,k}' + 2f_1 X_{j-1,k}' + f_1' X_{j-1,k} + 2X_{j,k-1}' = \alpha_{j,k}$$

with a similar system for $Y_{j,k}$.

Equations (24) can be written in the more compact form

$$X''' + 4(v_1 - f_1 H - L_2)X' + 2(v_1' - f_1' H)X = -2P(H,L_2), \quad \tilde{X} = -\frac{1}{2}X', \quad (25)$$

where the arbitrary function $P(H,L_2)$ (a separation parameter that we frequently choose to be a polynomial) is common to the equations for X and for Y. The first of Eqs. (25) always has solutions for any f_1, v_1, say continuously differentiable. Thus, we can always construct M and it will be analytic in the parameters H, L_2. (Of course, a basic question is for what choices of f_1, v_1, P do solutions X exist that are polynomials in the parameters H, L_2?)

Similarly, the equation for $Y(H,L_2,y)$ is

$$Y''' + 4(v_2 - f_2 H + L_2)Y' + 2(v_2' - f_2' H)Y = 2P(H,L_2), \quad \tilde{Y} = -\frac{1}{2}Y'. \quad (26)$$

Once we have obtained M and N, then we see that the operator $L_3 = M + N$ commutes with H:

$$[H,L_3] = \frac{1}{f_1+f_2}P(H,L_2) - \frac{1}{f_1+f_2}P(H,L_2) = 0.$$

Thus, we can view L_3 as an infinite-order differential symmetry operator for H. (In special cases this will be a finite-order operator.)

Theorem. *For any v_1, v_2, f_1, f_2, all solutions to Eqs.* (25), (26) *determine a separated symmetry operator of the form $L_3 = M(x) + N(y) = (X\partial_x + \tilde{X}) + (Y\partial_y + \tilde{Y})$.*

A straightforward computation yields

$$[L_2,M] = \frac{f_2}{f_1+f_2}P(H,L_2), \quad [L_2,N] = \frac{f_1}{f_1+f_2}P(H,L_2),$$

so $[L_2,L_3] = P(H,L_2) \neq 0$. Thus, L_3 is *not* a function of H and L_2.

An exactly analogous construction using the commutators

$$[H,\tilde{M}] = \frac{f_1}{f_1+f_2}P(H,L_2), \quad [H,\tilde{N}] = \frac{f_2}{f_1+f_2}P(H,L_2)$$

yields the operator $L_4 = \tilde{M} + \tilde{N}$, not a symmetry, such that $H = L_1, L_2, L_3, L_4$ satisfy the commutation relations

$$[L_1,L_2] = [L_1,L_3] = [L_2,L_4] = [L_3,L_4] = 0,$$

$$[L_1,L_4] = [L_2,L_3] = P.$$

If we choose $P(H,L_2) = I$, the identity operator, these are just the canonical commutation relations.

Example. Let us consider the quantum Hamiltonian

$$H = \partial_x^2 + \partial_y^2 + x.$$

It is known to be associated with several symmetries, such as

$$\ell_0 = \partial_y, \quad \ell_1 = \{\partial_y, x\partial_y - y\partial_x\}_+ - \frac{1}{2}y^2,$$
$$\ell_2 = \partial_x\partial_y + \frac{1}{2}y,$$

where $\{A, B\}_+ = (AB + BA)$ is the anticommutator of two operators. The occurrence of ℓ_0 is obvious, because y is an ignorable variable for the Hamiltonian. How can we obtain ℓ_1 and ℓ_2, which are associated with the separation of the Schrödinger equation in parabolic and shifted parabolic coordinates, from our Cartesian coordinate construction? The obvious separation in Cartesian coordinates yields the additional second-order symmetry

$$L_2 = \frac{1}{2}(\partial_x^2 - \partial_y^2 + x).$$

Let us now consider the defining equations for a symmetry in the following form:

$$X''' + 4\left(x - \frac{1}{2}H - L_2\right)X' + 2X = \left(\frac{1}{2}H - L_2\right),$$
$$Y''' - 4\left(\frac{1}{2}H - L_2\right)Y' = -\left(\frac{1}{2}H - L_2\right).$$

These equations have the solutions

$$X = \frac{1}{2}\left(\frac{1}{2}H - L_2\right), \quad Y = \frac{y}{4} - \frac{1}{8}.$$

The corresponding symmetry is thus finite and given by

$$L_3 = \frac{1}{2}\left(\partial_y^2\partial_x + \frac{1}{2}y\partial_y\right) - \frac{1}{4}\partial_y^2$$
$$= \{\ell_2, \partial_y\}_+ - \frac{1}{4}\partial_y^2 - \frac{1}{2}.$$

We see that our construction yields reasonably easily the existence of ℓ_2 and thereby ℓ_1. Note also that $[\partial_y, \ell_1] = 2\ell_2$.

4. THE GENERAL CASE IN TWO DIMENSIONS

Up to now we have only considered the special case $A = 0$, $B = X(x)$, $C = Y(y)$, $D = \tilde{X}(x) + \tilde{Y}(y)$ of conditions (18)–(21). Let us now consider the case such that $A \equiv 0$, but otherwise, B, C, and D are arbitrary. Then there is a function $G(x, y, H, L_2)$ such that $B = -\partial_x G$, $C = \partial_y G$, and the determining conditions simplify to

$$\text{(i)}\ G_{xxxy} + G_{xyyy} = 0,$$

$$\text{(ii)}\ \frac{1}{2}G_{xxxx} + 2G_{xx}v_1 + G_x v_1' - (2G_{xx}f_1 + G_x f_1')H - 2G_{xx}L_2 = \frac{1}{2}G_{yyyy} + 2G_{yy}v_2 + G_y v_2' - (2G_{yy}f_2 + G_y f_2')H + 2G_{yy}L_2.$$

The first determining equation means that

$$G(x, y) = K(x, y) + F(x) + J(y),$$

where F and J are arbitrary and K is harmonic: $K_{xx} + K_{yy} = 0$. This representation is unique in K, F, J, up to the addition of the harmonic separable function $\tilde{K}(x, y) = \frac{a}{2}(x^2 - y^2) + bx + cy + d$. Alternatively, we can write

$$G(x, y) = z_1(x + iy) + z_2(x - iy) + F(x) + J(y),$$

where z_1 and z_2 are arbitrary analytic functions. Then only condition (ii) remains to be satisfied. Specific examples are readily apparent.

Example. If we make the ansatz $G = X(x, H, L_2)Y(y, H, L_2)$, then, in addition to the well-known angular momentum invariant given earlier, we find the following polynomial invariants:

$$X = \left(\frac{1}{4} + L_2\right)\cos x + s(1 + \beta H), \qquad (27)$$
$$Y = \left(\frac{1}{4} + L_2\right)\cosh y + t(1 + \xi H),$$

$$v_1(x) = 2s\frac{\sin x}{\cos^2 x} + \frac{a_1}{\cos^2 x},$$
$$f_1(x) = -2s\beta\frac{\sin x}{\cos^2 x} + \frac{a_2}{\cos^2 x},$$

$$v_2(y) = 2t\frac{\sinh y}{\cosh^2 y} + \frac{b_1}{\cosh^2 y},$$
$$f_2(y) = -2t\xi\frac{\sinh y}{\cosh^2 y} + \frac{b_2}{\cosh^2 y},$$

$$D = -\frac{1}{2}\left(\frac{1}{4} + L_2\right)(t\cos x(1 + \xi H) + s\cosh y(1 + \beta H)), \qquad (28)$$

$$\tilde{L} = -2x(y^2 + 4L_2)\partial_x + 2y(x^2 - 4L_2)\partial_y + x^2 - y^2,$$

$$v_1(x) = \frac{1}{8}x^2 + \frac{a_1}{x^2}, \quad f_1(x) = \frac{a_2}{x^2},$$
$$v_2(y) = \frac{1}{8}y^2 + \frac{b_1}{y^2}, \quad f_2(y) = \frac{b_2}{y^2}.$$

Example. Again, we consider the special case of conditions (18)–(21) such that $A \equiv 0$, where now we require

$$G(x, y) = -2\log(X(x) + Y(y)) + \mathcal{F}(x) + \mathcal{J}(y)$$
$$= K(x, y) + F(x) + J(y),$$

where F and J are arbitrary and K is harmonic. Then the harmonic requirement on K implies that

$$K = -2\log(X+Y) + \tilde{F}(x) + \tilde{J}(y),$$

where

$$(X')^2 = \frac{\alpha}{12}X^4 + \frac{\beta}{3}X^3 + \gamma X^2 + 2\delta X + \phi,$$

$$(Y')^2 = -\frac{\alpha}{12}Y^4 + \frac{\beta}{3}Y^3 - \gamma Y^2 + 2\delta Y - \phi,$$

$$X'' = \frac{\alpha}{6}X^3 + \frac{\beta}{2}X^2 + \gamma X + \delta,$$

$$Y'' = -\frac{\alpha}{6}Y^3 + \frac{\beta}{2}Y^2 - \gamma Y + \delta.$$

Further,

$$\tilde{F}(x) = \frac{1}{3}\frac{X'''}{X'}, \quad \tilde{J}(y) = \frac{1}{3}\frac{Y'''}{Y'},$$

and the metric and potential terms have the solution

$$v_1 - f_1 H = \frac{-\frac{a}{12}X^4 - \frac{b}{3}X^3 + \frac{b_1}{2}X^2 + \eta_1 X + \eta_2}{24(X')^2},$$

$$v_2 - f_2 H = \frac{\frac{a}{12}Y^4 - \frac{b}{3}Y^3 - \frac{b_1}{2}Y^2 + \eta_1 Y - \eta_2}{24(Y')^2}.$$

Here, $\alpha, \beta, \gamma, \delta, \phi$ and

$$a = a^{(1)} + a^{(2)}H, b = b^{(1)} + b^{(2)}H,$$

$$b_1 = b_1^{(1)} + b_1^{(2)}H,$$

$$\eta_1 = \eta_1^{(1)} + \eta_1^{(2)}H, \eta_2 = \eta_2^{(1)} + \eta_2^{(2)}H$$

are parameters.

The remaining condition is

$$\frac{1}{2}F'''' + 2F''(v_1 - f_1 H - L_2) + F'(v_1 - f_1' H)$$
$$- \frac{1}{2}J'''' - 2J''(v_2 - f_2 H - L_2) - J'(v_2' - f_2' H)$$
$$= \frac{1}{36}\left(\frac{a}{2}X^2 + bX - \frac{a}{2}Y^2 + bY\right)$$
$$+ \frac{2}{3}\left(\frac{X'''}{X'}(v_1 - f_1 H) - \frac{Y'''}{Y'}(v_2 - f_2 H)\right)$$
$$+ \tilde{F}'(v_1' - f_1' H) - \tilde{J}'(v_2' - f_2' H).$$

The simplest family of solutions is obtained by setting $F \equiv \tilde{F}$, $J \equiv \tilde{J}$, and $\alpha = \beta = a = b = 0$.

Now we consider the general case of conditions (18)–(21). Then there are two functions $F(x, y, H, L_2)$ and $G(x, y, H, L_2)$ such that

$$A = \partial_{xy}F, \quad B = -\frac{1}{2}\partial_{xyy}F - \partial_x G,$$

$$C = -\frac{1}{2}\partial_{xxy}F + \partial_y G,$$

and the determining conditions simplify to

$$\text{(i)}\ 2G_{xyyy} + \frac{1}{2}F_{xyyyyy}$$
$$+ 2F_{xyyy}(v_2 - f_2 H + L_2) + 3F_{xyy}(v_2' - f_2 H)$$
$$+ F_{xy}(v_2'' - f_2'' H) = -2G_{xxxy} + \frac{1}{2}F_{xxxxxy}$$
$$+ 2F_{xxxy}(v_1 - f_1 H - L_2)$$
$$+ 3F_{xxy}(v_1' - f_1' H) + F_{xy}(v_1'' - f_1'' H),$$
$$\text{(ii)}\ \frac{1}{2}F_{xxxxyy} + 2F_{xxyy}(v_1 - f_1 H)$$
$$+ F_{xxy}(v_2' - f_2' H) + \frac{1}{2}G_{xxxx}$$
$$+ 2G_{xx}(v_1 - f_1 H - L_2) + G_x(v_1' - f_1' H)$$
$$= -\frac{1}{2}F_{xxyyyy} - 2F_{xxyy}(v_2 - f_2 H)$$
$$- F_{xyy}(v_1' - f_1' H) + \frac{1}{2}G_{yyyy}$$
$$+ 2G_{yy}(v_2 - f_2 H + L_2) + G_y(v_2' - f_2' H).$$

Theorem. *For any v_1, v_2, f_1, f_2, there are always solutions for the above equations in which $A \not\equiv 0$, $G \equiv 0$, and F factors as $F = \mathcal{X}(x, H, L_2)\mathcal{Y}(y, H, L_2)$, where $\mathcal{X}'\mathcal{Y}' \neq 0$.*

Indeed, with $X = \mathcal{X}', Y = \mathcal{Y}'$, we have a solution to Eqs. (18)–(21) whenever $X'Y' \neq 0$ and X and Y satisfy the ordinary differential equations

$$X''' + 4X'(v_1 - f_1 H - L_2) \qquad (29)$$
$$+ 2X(v_1' - f_1' H) = 0,$$
$$Y''' + 4Y'(v_2 - f_2 H + L_2) \qquad (30)$$
$$+ 2Y(v_2' - f_2' H) = 0.$$

5. FINAL REMARKS

The underlying structure of the solutions to the general equations (18)–(21) is fairly simple. Let $u_1(x, L_2) = u_1[L_2]$, $u_2(x, L_2) = u_2[L_2]$ be a basis of solutions of the separated equation

$$\left(\frac{d^2}{dx^2} + v_1(x) - f_1(x)H - L_2\right)u = 0, \qquad (31)$$

and let $w_1(y, L_2), w_2(y, L_2)$ be a basis of solutions of the separated equation

$$\left(\frac{d^2}{dy^2} + v_2(y) - f_2(y)H - L_2\right)w = 0. \qquad (32)$$

Then, for any parameter $\hat{L}_2$, the operator

$$S(\hat{L}_2) = w_2[\hat{L}_2]u_2[\hat{L}_2](w_1[L_2]u_1[L_2]\partial_{xy}$$

$$- w_1'[L_2]u_1[L_2]\partial_x - w_1[L_2]u_1'[L_2]\partial_y + w_1'[L_2]u_1'[L_2]$$

is a symmetry operator of L_1 that maps any eigenspace of L_2 into another (generally different) eigenspace. The point is that the Wronskian of any two solutions to (31) or to (32) is constant. It is not hard to characterize the space spanned by all linear combinations of functions $w_2[\hat{L}_2]u_2[\hat{L}_2]w_1[L_2]u_1[L_2]$ and this gives the equations for A. Similarly, we can characterize B, C, and D. The details can be complicated, but the principle is simple.

All of these methods in this paper extend to n dimensions. If any of the equations

$$\sum_{i,j=1}^{n} g^{ij}p_1p_j + V(x) = E, \quad n \geq 2,$$
$$(\Delta_n + V(x))\Psi(x) = E\Psi(x), \quad n \geq 2,$$
$$(\Delta_n + V(x))\Psi(x) = \partial_t\Psi(x), \quad n \geq 1,$$
$$(\Delta_n + V(x))\Psi(x) = 0, \quad n \geq 3$$

on a pseudo-Riemannian manifold admits an orthogonal (in the space variables) separable or R-separable coordinate system, then we can develop a similar calculus to describe all differential symmetries and conformal symmetries of the system, even those of infinite order. In the lowest dimensional cases, we have verified the same statements for nonorthogonal separable systems. We will provide all these details in forthcoming papers.

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