# Dynamic Spectrum Allocation in Cellular Networks

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Abstract of the Dissertation

# Dynamic Spectrum Allocation in Cellular Networks

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In cellular networks, a recent trend is to make spectrum access dynamic in the spatial and temporal dimensions, for the sake of efficient utilization of spectrum. In such a model, the spectrum is divided into channels and is periodically allocated to the base stations in both centralized and distributed manners with different goals in mind for each approach.

For the centralized approach, an auction-based market mechanism is favored due to its simplicity, efficiency and high utilization of the spectrum. The model consists of a centralized spectrum broker who owns a part of the spectrum, divides it into channels and issues short-term dynamic spectrum leases of these channels to competing base stations in the region it controls. The base stations, on the other hand, bid for channels depending on their spectrum demands. Subject to wireless interference between base stations, the broker allocates channels to them with various objectives in mind. These objectives include maximizing the generated revenue, optimizing social-choice functions like the social-welfare and/or controlling the strategic behavior of the base stations. In this dissertation, we address the above problem and show how to optimize the solution for these different objectives.

As for the distributed approach, the focus is shifted towards more stable allocation that can maintain certain properties with minimal cost and human intervention even when faced by frequent network topology changes. This is demonstrated by problems such as self-configuration of fractional frequency

reuse (FFR) patterns for LTE/WiMAX networks. In this dissertation, we present distributed algorithms that provide the network designer a flexible tool to tune different objectives like efficiency, stability and near-optimal spectrum utilization. For each possible choice made by the system designer, our tool delivers a near-optimal spectrum utilization with specific guarantees on the rest of the desired properties.

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# Chapter 1

# Introduction

Usage of wireless spectrum by radio communication devices has long been governed by governmental regulatory authorities (e.g., FCC in USA or Ofcom in UK) that divide the spectrum into fixed size chunks to be used strictly for specific purposes, such as broadcast radio/TV, cellular/PCS services, wireless LAN/PANs, public safety related communication, etc. This allocation is very long-term and space-time invariant, and is often based on peak usage per provider. Many recent observations have shown that such long-term static allocation of spectrum introduces significant inefficiencies in utilization [11]. To improve spectrum utilization, there is a new policy trend [54] to make spectrum allocation more dynamic in both spatial and temporal dimensions and more responsive to end-user demands. The allocation process can be performed in both centralized and distributed manners with different goals in mind for each approach.

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## 1.1 Network and Interference Models

Our model of a cellular network consists of a set of geographically distributed base stations. Each base station is associated with a region around it called its *cell*; each base stations serves its clients in its cell. To communicate, the base station and the client must operate "interference-free" on the same channel. In cellular networks, wireless interference at a client may arise due to multiple near-by base stations operating on the same channel. Several interference models have been proposed in the literature. In this dissertation, we consider the two most widely-used models, viz., the *pairwise* model and the *physical* model. Below, we discuss both models formally.

### 1.1.1 Pairwise Interference Model

In the pairwise interference model, pairs of base stations with intersecting cells are said to *interfere* with each other if operating on the same channels. These pairs of interfering base stations can be represented by simple edges over base stations as vertices in an interference graph, as defined below.

**Definition 1** (Interference Graph $G_t$.) The interference graph $G_t = (N_t, E_t)$

is an undirected graph where each vertex represents a base station and there is an edge $(i, j) \in E_t$ between $i$ and $j$ if the corresponding base stations "interfere".

As mentioned before, two base stations are said to interfere when their corresponding cells intersect. Note that interfering base stations should not be allocated a common channel. □

**The Unit-Disk Model.** In the *unit-disk* model, the coverage region of each base station is assumed to be a disk of uniform radius $d$. For simplicity of presentation, we assume distances to be normalized, i.e., $d = 1$. Thus, two base stations interfere if they are within two-unit distance from each other.

**The Non-Uniform Disks Model.** In the *non-uniform disk* model, the cells of the base stations are disks of possibly different radii. Let the maximum and the minimum disk radii in the network be $d_{\max}$ and $d_{\min}$ respectively. For simplicity of presentation, we assume that the distances are normalized, i.e., $d_{\min} = 1$.

**The Pseudo-Disk Model.** Finally, in the *pseudo-disk model* model, the cells of the base stations may have irregular shapes, but are contained within a disk of radius $d_1$ while containing a disk of radius $d_2 \leqslant d_1$, both disks being centered at the base station. See Figure 1. For simplicity of presentation, we assume that $d_1$ and $d_2$ are the same for all base stations. We later present techniques that can be used to extend results for this model to the case wherein $d_1$ and/or $d_2$ may be different for different cells. Also, for clarity of presentation, we use $d_2 = 1$.

### 1.1.2 Physical Interference Model

In the physical interference model, a reception at a certain distance from a base station is successful, if the "signal to noise plus interference ratio" (SINR) at the receiver is greater than a threshold $\beta$. More formally, a reception from a base station $i$ is successful at a point $p$ if and only if,

$$\frac{P/d_i^\alpha}{\mathcal{N} + \sum_{j \in B'} P/d_j^\alpha} \geqslant \beta, \quad (1)$$

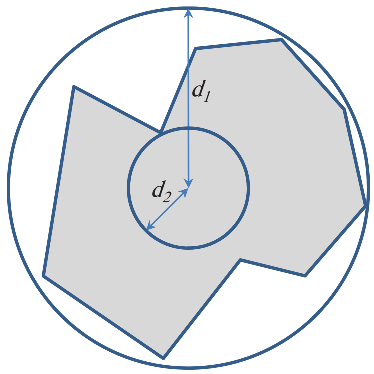



Figure 1: An example of the pseudo-disk model.

where $P$ is the transmission power, $B'$ is the set of other base stations operating on the same channel as $i$, $d_x$ is the distance of the point $p$ from a base station $x$, $\mathcal{N}$ is the background noise, and $\alpha$ is the path loss exponent based on the terrain propagation model. Initially, we assume that each base station operates using the same transmission power $P$. We later present techniques that can be used to extend results for this model to the case wherein each base station has its own transmission power.

**Communication Radius ($r$).** The communication radius $r$ of a base station $i$ is the maximum distance from $i$ within which we *want* the SINR from $i$ to be at least as large as $\beta$. Essentially, the above is based on the stipulation that the coverage cell of base station $i$ is a disk of radius $r$. In our context, the value of $r$ can be arbitrarily large (but finite), since the approximation ratio and time complexity of our designed algorithms are independent of $r$. Thus, the concept of communication radius must not be looked upon as an assumption.

## 1.2 Dissertation Organization

The rest of the dissertation is organized as follows. In Chapter 2, we present our first centralized spectrum allocation algorithm. It is based on a general market-based auction mechanism with provable performance bounds. The algorithm is limited in its treatment of the base stations' behavior. This is the main motivation for our algorithms in Chapters 3 and 4 which deals with different economical aspects of the auction mechanism. We address the issue of distributed spectrum allocation in Chapter 5. Finally, Chapter 6 concludes this dissertation.

# Chapter 2

# Centralized Spectrum Allocation: A Simplistic Auction-Based Approach

## 2.1 Introduction

Usage of wireless spectrum by radio communication devices has long been governed by governmental regulatory authorities (e.g., FCC in USA or Ofcom in UK) that divide the spectrum into fixed size chunks to be used strictly for specific purposes, such as broadcast radio/TV, cellular/PCS services, wireless LAN/PANs, public safety related communication, etc. This allocation is very long-term and space-time invariant, and is often based on peak usage per provider. Many recent observations have shown that such long-term static allocation of spectrum introduces significant inefficiencies in utilization [11]. To improve spectrum utilization, there is a new policy trend [54] to make spectrum allocation more dynamic in both spatial and temporal dimensions and more responsive to end-user demands.

There can be several different architectures for providing dynamic spectrum access (DSA) that can widely vary depending on the technological limitation and usage models. Buddhikot et al. [11], explored the application of a centralized architecture for dynamic spectrum access in cellular networks

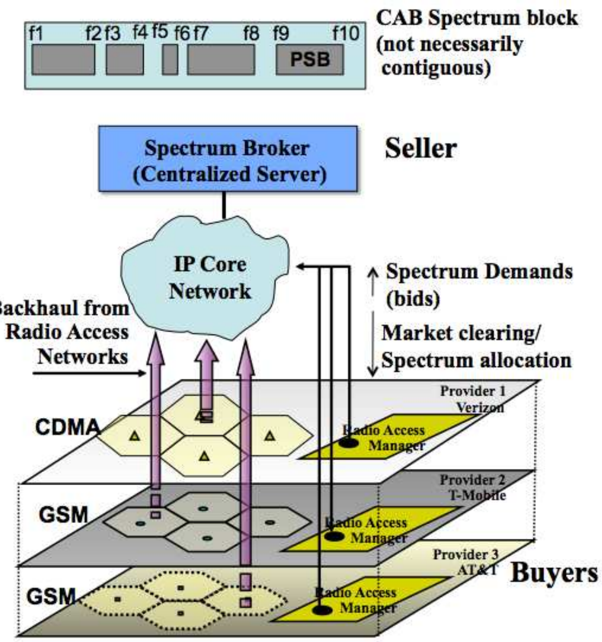



Figure 2: Coordinated dynamic spectrum access architecture.

by introducing the *coordinated dynamic spectrum access* (CDSA) model. In the CDSA model (see Figure 2), there is a centralized entity known as the *spectrum broker* who owns a part of the spectrum called the *coordinated access band* (CAB) and dynamically allocates them to base stations in the region it controls. Indeed, centralized architectures [10, 22, 52] for dynamic spectrum access have gained a lot of interest in the research community due to their practicality and potential impact. However, success of the CDSA model hinges on the design of scalable and efficient spectrum brokers. We address this issue in this chapter by designing efficient spectrum allocation algorithms that deliver near-optimal solutions.

**Problem Addressed.** We consider a dynamic auction-based approach to allocate spectrum to competing base stations. The centralized spectrum broker acts as the *seller* and the base stations (in the region controlled by the broker) act as the *buyers*[1] of the CAB. The spectrum broker divides the CAB into channels (contiguous or non-contiguous blocks of frequency) and the base

[1]We use the terms buyers, bidders, base stations and nodes interchangeably.

stations bid for these channels based on their spectrum demands. The base stations express their bids using a *bidding function* that specifies the price they are willing to pay for a given set of allocated channels. Periodically, the spectrum broker allocates available channels to the base stations (based on the received bids) under the "wireless interference constraint" such that the total revenue (total price paid by the base stations)[2] is maximized. The above auction-based approach allows the base stations to bid according to the spectrum demands, and the spectrum broker to maximize the revenue generated from allocation of spectrum.

The above spectrum allocation problem is known to be NP-hard and has been addressed before [22, 52] in limited contexts; e.g., [22] assumes a unit-disk interference model, piece-wise linear bidding functions, and homogeneous set of non-overlapping channels, while [52] considers very primitive forms of bids and interference models. In contrast, we consider general network graphs and interference models (pairwise and physical), overlapping channels, and arbitrary non-complementary[3] bidding functions. For the above general context, we present approximation algorithms that deliver allocations with near-optimal revenue.

**Chapter Organization:** The rest of the chapter is organized as follows. In Section 2.2, we describe the system architecture of the CDSA model and give details of its components. In Section 2.3 and 2.4, we formally define and present efficient approximation algorithms for the spectrum allocation problem under pairwise and physical interference models, respectively. In Section 2.5, we present detailed simulation results comparing performance of the proposed algorithms. In section 2.6, we discuss related work. Section 2.7 concludes this chapter.

[2]In this mechanism, bidders are charged a payment of equal amount to their bids, which maximizes the revenue without having negative utilities for bidders. Such a payment scheme may lead to "untruthful" bidding which we handle in later chapters.

[3]A bidding function is said to be *non-complementary* when it is defined on a set of items that do not complement each other. For example, the bid for choosing two items together should not be more than the sum of the bids for choosing the items individually.

## 2.2 System Architecture

In this section, we describe the reference system architecture (Figure 2) of our coordinated dynamic spectrum access model and give details of each important component of the model. Due to its simplicity and practicality, we will continue to use this model for all of our centralized algorithms (Chapters 3 and 4) with minor modifications.

### 2.2.1 Spectrum Broker (Seller)

In the CDSA model, a centralized entity called the *spectrum broker* [11, 12] owns and coordinates access to the CAB in a given region and assigns short term spectrum leases to competing wireless service providers. Regulatory authorities like FCC can conduct one-time or long-term periodic auctions to give spectrum licenses to the broker on a regional basis. However, in contrast to existing cellular spectrum licenses, the spectrum broker can in turn grant spectrum leases that are for small geographical regions (e.g., per base station) and valid for short durations (e.g., tens of minutes) [12]. Such a spectrum lease gives the lessee exclusive rights to use the spectrum in the designated region for the duration of the lease without exceeding the maximum power limit. In this chapter, we mainly address the challenge of how to assign these dynamic spectrum leases to various service providers and design fast and scalable spectrum allocation algorithms.

### 2.2.2 Base Stations or Nodes (Buyers)

The region under the control of the spectrum broker can be as large as a single state having a large number (up to hundreds or even thousands) of base stations. These base stations are owned by different Radio Infrastructure Providers (RIP). The Wireless Service Providers (WSP) (e.g., AT&T, Verizon) are customers of the RIPs and use their infrastructure to provide wireless services like voice, data, etc. to end-users. Each base station in the region can be used to operate different types of networks by the WSPs. For example, some base stations can be used to operate a GSM network, some for a CDMA or WCDMA network, and some for a WiMAX network. In a more general model, multiple types of networks can be operated on the same base station. Interference between different base stations depends on the location of the base stations, the frequency band used and the terrain propagation model [59]. We assume[4] that the spectrum broker is aware of all the details of each base station in its region ranging from their exact location, and other characteristics like frequency range of operations, power levels, number of transmitters etc. It also knows the terrain propagation model in the region and can estimate the level of interference between base stations given their location and transmission power used. This knowledge forms part of essential inputs to our spectrum allocation algorithm.

### 2.2.3 Coordinated Access Band (Items Sold)

The portions of the spectrum that are highly underutilized or unused in spatial or temporal dimension qualify as prime candidates to be used as CAB. At the current time, good examples are Specialized Mobile Radio (SMR) (851-854/806-809 MHz, 861-866/816-821 MHz), Public Safety Bands (PSB) (764-776, 794-806 MHz), and unused broadcast UHF TV channels (450-470 MHz,470-512 MHz (channels 14-20), 512-698 MHz (channels 21-51), 698-806 MHz (channels 52-69)). The CAB spectrum is to be shared between different cellular services with macro-cellular infrastructures. Some of the current technology examples that can use the above CAB spectrum are 1xRTT/1xEV-DO that use 1.25 MHz channels, GSM networks that use 200KHz channels, IS-136 legacy TDMA that uses 30 KHz channels, W-CDMA networks that use 5 MHz channels, WiMAX networks that can use 1.75 MHz to 20 MHz channels. Note that different technologies often provide different forms of services. Spectrum sharing between different services is advantageous as they provide the benefit of statistical multiplexing – the services use spectrum differently and have

[4]As discussed by Buddhikot et al. [11], in this brokering model, the service providers or operators of radio access network interested in obtaining the spectrum, register with the broker and provide information on the transmitter location, capabilities (such as frequency, power, number of interfaces, preferred waveforms supported (as in CDMA, OFDM etc.)). This registration happens via a spectrum-leasing protocol that must be run on the base stations and the broker.

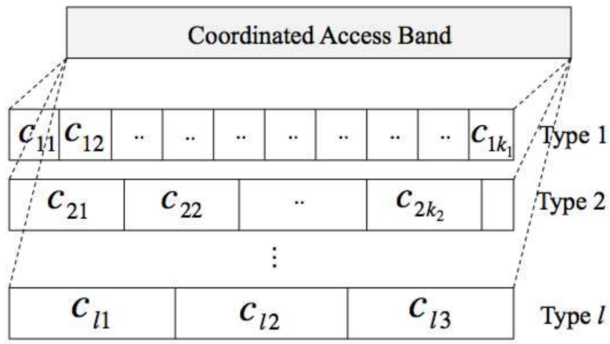



Figure 3: Channels of different types (widths) in the CAB.

different load factors that vary largely on a spatio-temporal scale. It is also reasonable to use existing cellular bands (450 Mhz, 800 MHz or the 1.9 GHz band) as part of the CAB, giving a guaranteed access to incumbent WSPs who already hold licenses and on-demand access to other WSPs that do not conflict with the license holders. As can be realized from the above numbers, the CAB spectrum might include hundreds or thousands of channels.

Since different types of networks use channels of different widths, the spectrum broker has to make the decision on how to divide the available spectrum into channels of different widths and allocate them to different base stations. In our model, we assume that the spectrum broker divides the available spectrum into some *finite* number of channels for each type of network. *This channelization can be quite general.* For example, the spectrum broker may decide to create channels of varying width as shown in Figure 3; here, if $H$ is the total width of CAB, then for $i^{th}$ network/type, $H/h_i$ non-overlapping channels of width $h_i$ are created. Note that channels of different types that overlap with each other, cannot be assigned to the same or interfering base stations. Overlapping of channels makes the spectrum allocation problem very challenging compared to only using homogeneous channels as assumed in prior work [22, 52, 13].

### 2.2.4 Spectrum Demands, Bids, and Bidding Functions

The WSPs aggregate end-user demands at each of the base stations it operates and generate spectrum demands to the broker. Spectrum demand aggregation at each base station can be done using a predictive model based on historical traffic measurements or from end-users' bandwidth inputs. The above demands are then used to generate bids for various combination of number of channels and channel types. In general, the bids are specified using a bidding function, which may be different for different base stations. Basically, the *bidding function* for any base station specifies the price the base station is willing to pay for each set of channels $C$.In general, the complexity of such a bidding function can be exponential in the number of channels, since the number of possible sets of channels is exponential. However, in simpler contexts, each base station may have a separate bidding function for each channel type, and the bidding function may specify a price depending on the *number* of channels of that type. In this chapter, we do not make any assumptions about the complexity of the bidding function,[5] unlike [22] where the authors assume the bidding functions to be piece-wise linear. The time complexity of our allocation algorithms is polynomial in the size of the bidding function.

### 2.2.5 Auction Setting

Our model corresponds to the First-Price Sealed-Bid Auctions, where the auctioned items are the channels. At the beginning of each allocation period, the bidders (base stations) submit their private bidding functions to the broker who chooses the "winners" based on the submitted bids. The winners are selected in a way that there is no interference (as defined later), and the total revenue (sum of "winning" bids) is maximized.

[5]Later, we do assume the bidding function to be non-complementary, to prove the performance guarantee of our designed algorithms.

## 2.3 Spectrum Allocation under Pairwise Interference

In this section, we address the spectrum allocation problem under the pairwise interference model. To give a formal definition of the problem, we need to define a few terms in addition to the definitions given in Section 1.1.1. Later, we design a greedy algorithm, and prove that it delivers a near-optimal spectrum allocation. We use the term *node* to refer to a base station.

**Channel Graph ($G_c$).** The overlapping nature of the channels in the CAB is modeled using a channel graph $G_c$ defined similarly to the base stations interference graph (see Definition 1).

**Definition 2** (Channel Graph $G_c$.) A channel graph $G_c = (V_c, E_c)$ is an undirected graph over channels as vertices, and there is an edge $(c_i, c_j)$ between two channels $c_i$ and $c_j$ if they overlap with each other. For example, the channel graph corresponding to Figure 3 will have an edge between $c_{23}$ and $c_{15}$. An empty (with no edges) channel graph means that the channels are mutually non-overlapping (as in the model of [22]). □

**Valid Spectrum Allocation.** Informally, our spectrum allocation problem is to allocate channels to base stations so as to maximize the total revenue (total price paid by the base stations). However, the allocation of channels should be done without violating the interference constraints. We formalize the above by defining a concept of valid spectrum allocation, in terms of conflicting (base station, channel) pairs.

**Definition 3** (Conflicting (base station, channel) pairs.) Consider two (base station, channel) pairs $(i, c_k)$ and $(j, c_l)$ where $i$ and $j$ are base stations and $c_k$ and $c_l$ are channels. The (base station, channel) pairs $(i, c_k)$ and $(j, c_l)$ are said to be *conflicting* if the following is true: (i) $i = j$, or $(i, j)$ is an edge in the interference graph, *and* (ii) $c_k = c_l$, or $(c_k, c_l) \in E_c$ (i.e., $c_k$ and $c_l$ overlap). □

**Definition 4** ((Valid) Spectrum Allocation.) A *spectrum allocation* is a set of (base station, channel) pairs, i.e., a spectrum allocation is a set

$$\{(i, c_k) | i \text{ is a base station}, c_k \text{ is a channel}\},$$

where a pair $(i, c_k)$ signifies that channel $c_k$ has been allocated to the base station $i$.

A spectrum allocation $\mathcal{A}$ is considered *valid* if no two (base station, channel) pairs in $\mathcal{A}$ are conflicting. □

**Bidding Functions and Revenue.** In general, a bidding function for a base station $i$ gives the price that $i$ is willing to pay for a set of mutually non-overlapping channels.For the sake of simplicity, we use an equivalent notion of total revenue generated by a given valid spectrum allocation. Below, we formally define both the terms bidding functions and revenue.

**Definition 5** (Bidding Function.) A *bidding* (or *valuation*) function $\mathbf{v}_i$ for a base station $i$ is a function $\mathbf{v}_i : P(\mathcal{C}) \mapsto \mathbb{R}$, where $P(\mathcal{C})$ is the power set of all channels $\mathcal{C}$and $\mathbb{R}$ is the set of real numbers. □

**Definition 6** (Revenue $R(\mathcal{A})$). Given the bidding functions of base stations, the *revenue* generated by a *valid* spectrum allocation $\mathcal{A}$ is denoted by $R(\mathcal{A})$ and is defined as the sum of the bids of the base stations for the channels allocated to them by the spectrum allocation $\mathcal{A}$. More formally,

$$R(\mathcal{A}) = \sum_{i \in V_t} \mathbf{v}_i(C_i),$$

where $\mathbf{v}_i$ is the bidding function of $i$, and $C_i = \{c_k | (i, c_k) \in \mathcal{A}\}$ is the set of channels allocated to $i$ by $\mathcal{A}$. Revenue is defined only for valid spectrum allocations. □

In the above definition, we have implicitly enforced that the base stations are asked to pay what they bid. This could lead to "untruthful" behavior, which is left for formal treatment in later chapters.

**Spectrum Allocation Problem.** Based on the above definitions of valid spectrum allocation and revenue, the spectrum allocation problem under the pairwise interference model can be defined as follows.

**Definition 7** (Spectrum Allocation Problem.) Given an interference graph, a channel graph, and the bidding functions for base stations, the spectrum allocation problem is to find a valid spectrum allocation $\mathcal{A}$that maximizes the total revenue $R(\mathcal{A})$. □

### 2.3.1 Greedy Algorithm (GA)

For the above spectrum allocation problem, we design a greedy algorithm that constructs a valid spectrum allocation by iteratively adding the "best" (base station, channel) pair at each stage. We will show that such a greedy strategy results in a valid spectrum allocation with near-optimal revenue. A more formal description of our Greedy Algorithm for the spectrum allocation problem is as follows.

Let $\mathcal{A}$ be the valid spectrum allocation being constructed by the algorithm. Initially, $\mathcal{A} = \emptyset$. In each iteration, the algorithm picks a (base station, channel) pair $(i, c_k)$ to add to $\mathcal{A}$such that

- $\mathcal{A} \cup (i, c_k)$ remains a valid spectrum allocation, and
- $R(\mathcal{A} \cup \{(i, c_k)\}) - R(\mathcal{A})$, the "incremental revenue" is maximum (among all choices of (base station, channel) pairs).

The algorithm terminates when $\mathcal{A}$cannot be extended any further.

If $N$ is the number of base station, $M$ is the number of channels, and $\Delta_t$ and $\Delta_c$ are the maximum vertex-degree in the interference and channel graphs respectively, then, the overall time complexity of the above algorithm can be shown to be bounded by $O(NM\Delta_t\Delta_c \log(NM))$ if we use a heap data structure to compute the maximum at each stage.

**Performance Guarantee of GA.** In the following theorem, we will show that the Greedy Algorithm returns a near-optimal valid spectrum allocation. However, to prove the approximation bound, we need to assume a certain "non-complementary" property of the revenue function. Given the bidding functions, we say that the revenue satisfies the *non-complementary property* if the following condition holds for any two valid spectrum allocations $\mathcal{A}_1$ and

$\mathcal{A}_2$ such that $\mathcal{A}_1 \cup \mathcal{A}_2$ is also a valid spectrum allocation.

$$\max(R(\mathcal{A}_1), R(\mathcal{A}_2)) \leqslant R(\mathcal{A}_1 \cup \mathcal{A}_2) \leqslant R(\mathcal{A}_1) + R(\mathcal{A}_2). \tag{2}$$

Recall that revenue is only defined for *valid* spectrum allocations. It is easy to see that revenue is non-complementary if and only if the bidding functions are non-complementary. The above non-complementary property is commonly assumed in the auction literature [37], and signifies a common assumption signifying that no two valid spectrum allocations "complement" one another. More importantly, the above property entails that the incremental revenue of any particular (base station, channel) pair never increases as the Greedy Algorithm progresses (i.e., with the selection of other (base station, channel) pairs). Such a property is indeed essential for the Greedy Algorithm to have a bounded performance guarantee. Later in this section, we discuss scenarios where the non-complementary property may not be satisfied, but the Greedy Algorithm can still be modified appropriately to preserve the performance guarantee. We now prove that the revenue generated by the Greedy Algorithm is at least $1/(\delta_t(\Delta_c + 1) + 1)$ of the optimal revenue, for non-complementary revenue functions.

**Theorem 1** *For a non-complementary revenue function, the above Greedy Algorithm (GA) returns a $(\delta_t(\Delta_c + 1) + 1)$-approximate valid spectrum allocation. Here, $\delta_t$ is the size of the maximum independent set in the neighborhood of any vertex in the interference graph, and $\Delta_c$ is the maximum degree of a vertex in the channel graph.*

*Proof:* Let $b_i$ be the $i^{th}$ (base station, channel) pair selected by GA in its $i^{th}$ iteration, $a_i$ be the corresponding incremental revenue of $b_i$, and $l$ be the total number of iterations of GA for the given input. We use $\mathcal{A}_i$ to denote $\{b_1, b_2, \ldots, b_i\}$; thus, $a_i = R(\mathcal{A}_i) - R(\mathcal{A}_{i-1})$. Let $\mathcal{O}$ be the optimal solution and let $\mathcal{O}_l$ be the set of (base station, channel) pairs in $\mathcal{O}$ that conflict with some pair in $\mathcal{A}_l$. Below, we use the notation $R(\mathcal{A}_1|\mathcal{A}_2)$ to denote $R(\mathcal{A}_1 \cup \mathcal{A}_2) - R(\mathcal{A}_2)$ where $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_1 \cup \mathcal{A}_2$ are all some valid spectrum allocations.

We make the following three claims.

- For each $j \in \mathcal{O}_l$, let $h(j)$ be the smallest integer such that $j$ conflict with $b_{h(j)}$. Informally, selection of $b_{h(j)}$ by GA is the reason why $j$ is not considered by GA for selection in later iterations. Note, by the greedy choice of $b_i$'s we have,

$$R(\{j\}|\mathcal{A}_{h(j)-1}) \leqslant a_{h(j)}. \tag{3}$$

- By definition of $\delta_t$ (the maximum size of an independent set in the neighborhood of any vertex in the interference graph), it is easy to see that the maximum number of mutually non-conflicting (base station, channel) pairs that conflict with a particular $b_i$ is $\delta_t(\Delta_c + 1)$. Here, $\Delta_c$ is the maximum degree of any vertex in the channel graph. Thus, for any integer $z$, there are at most $\delta_t(\Delta_c + 1)$ (base station, channel) pairs $j$ in $\mathcal{O}_l$ such that $h(j) = z$. Thus, we have,

$$\sum_{j \in \mathcal{O}_l} a_{h(j)} \leqslant \delta_t(\Delta_c + 1) \sum_{z=1}^{l} a_z = \delta_t(\Delta_c + 1) R(\mathcal{A}_l). \tag{4}$$

- Using induction on $l$, we will later show that:

$$R(\mathcal{O}) \leqslant R((\mathcal{O} - \mathcal{O}_l) \cup \mathcal{A}_l) + \sum_{j \in \mathcal{O}_l} R(\{j\}|\mathcal{A}_{h(j)-1}). \tag{5}$$

Without loss of generality, assume that the Greedy and optimal solutions are disjoint. Then, GA continues till $\mathcal{O}_l = \mathcal{O}$. For $\mathcal{O}_m = \mathcal{O}$, the above Equation 5 becomes:

$$R(\mathcal{O}) \leqslant R(\mathcal{A}_l) + \sum_{j \in \mathcal{O}_l} R(\{j\}|\mathcal{A}_{h(j)-1}). \tag{6}$$

Now using Equations 3 and 4 in the above Equation 6, we get

$$R(\mathcal{O}) \leqslant (\delta_t(\Delta_c + 1) + 1) R(\mathcal{A}_l),$$

yielding the approximation ratio.

**Proof of Equation 5.** We use induction on $l$. For $l = 0$, the equation is trivially true since $\mathcal{A}_0 = \{\}, \mathcal{O}_0 = \{\}$. By inductive hypothesis, let us assume Equation 5 to be true for $l = k$. Thus, we have

$$R(\mathcal{O}) \quad \leqslant \quad R((\mathcal{O} - \mathcal{O}_k) \cup \mathcal{A}_k) + \sum_{j \in O_k} R(\{j\}|\mathcal{A}_{h(j)-1}). \tag{7}$$

As before, let $\mathcal{A}_{k+1} = \mathcal{A}_k \cup \{b_{k+1}\}$. Since, $\mathcal{O}_{k+1}$ is defined as the set of (base station, channel) pairs in $\mathcal{O}$ that conflict with some pair in $\mathcal{A}_{k+1}$, there are two cases:

- $b_{k+1}$ does not conflict with any pair in $(\mathcal{O} - \mathcal{O}_k)$. Then, $\mathcal{O}_{k+1} = \mathcal{O}_k$, and Equation 5 holds since

$$R((\mathcal{O} - \mathcal{O}_k) \cup \mathcal{A}_k) \leqslant R((\mathcal{O} - \mathcal{O}_{k+1}) \cup \mathcal{A}_{k+1}).$$

- $b_{k+1}$ conflicts with some (base station, channel) pairs in $(\mathcal{O} - \mathcal{O}_k)$; let $\mathcal{O}'$ be the set of such conflicting pairs in $(\mathcal{O} - \mathcal{O}_k)$. Thus, we have $\mathcal{O}_{k+1} = \mathcal{O}_k \cup \mathcal{O}'$, where $\mathcal{O}' \subseteq (\mathcal{O} - \mathcal{O}_k)$. We prove induction for this case in detail below.

Consider the second case above. We have,

$$\begin{aligned}
&R((\mathcal{O} - \mathcal{O}_k) \cup \mathcal{A}_k) \\
&\quad = R((\mathcal{O} - \mathcal{O}_{k+1}) \cup \mathcal{A}_k \cup \mathcal{O}') \\
&\quad = R((\mathcal{O} - \mathcal{O}_{k+1}) \cup \mathcal{A}_k) + R(\mathcal{O}' | (\mathcal{O} - \mathcal{O}_{k+1}) \cup \mathcal{A}_k) \qquad (8) \\
&\quad \leqslant R((\mathcal{O} - \mathcal{O}_{k+1}) \cup \mathcal{A}_k) + R(\mathcal{O}' | \mathcal{A}_k) \\
&\quad \leqslant R((\mathcal{O} - \mathcal{O}_{k+1}) \cup \mathcal{A}_{k+1}) + R(\mathcal{O}' | \mathcal{A}_k) \\
&\quad \leqslant R((\mathcal{O} - \mathcal{O}_{k+1}) \cup \mathcal{A}_{k+1}) + \sum_{j \in \mathcal{O}'} R(\{j\} | \mathcal{A}_k). \qquad (9)
\end{aligned}$$

Above, Equation 8 is true since $R(\mathcal{A}_1 \cup \mathcal{A}_2) = R(\mathcal{A}_1) + R(\mathcal{A}_2 | \mathcal{A}_1)$ for any valid allocation $\mathcal{A}_1 \cup \mathcal{A}_2$.

Now, for each $j \in \mathcal{O}'$, let $h(j)$ be as defined before, i.e., the smallest integer such that $j$ conflicts with $b_{h(j)}$. Since, elements in $\mathcal{O}'$ do not conflict with any pair in $\mathcal{A}_k$, we have that:

$$\forall\ j \in \mathcal{O}', \quad h(j) = k + 1.$$

Now, applying the above to Equation 9, we get

$$R((\mathcal{O} - \mathcal{O}_k) \cup \mathcal{A}_k) \leqslant R((\mathcal{O} - \mathcal{O}_{k+1}) \cup \mathcal{A}_{k+1}) + \sum_{j \in \mathcal{O}'} R(\{j\} | \mathcal{A}_{h(j)-1}).$$

Using the above in the inductive hypothesis (Equation 7) and noting that $\mathcal{O}_{k+1} = \mathcal{O}_k \cup \mathcal{O}'$, we get

$$R(\mathcal{O}) \leqslant R((\mathcal{O} - \mathcal{O}_{k+1}) \cup \mathcal{A}_{k+1}) + \sum_{j \in \mathcal{O}_{k+1}} R(\{j\} | \mathcal{A}_{h(j)-1}),$$

which proves the inductive step. Thus, Equation 5 holds. ∎

It should be noted here that the bound of the above theorem is tight, which can be easily seen in simple scenarios like the one mentioned in the second remark below.

**Remarks.** We make the following remarks, as special cases of the above result.

- In the case of a unit-disk model (see Section 1.1.1), then $\delta_t$ is at most 5 [44]. In that case, the approximation ratio becomes $5\Delta_c + 6$.

- If we consider non-overlapping channels and a unit-disk interference graph, then the above theorem states that GA returns a 6-approximate solution. This is a direct generalization of the result in [22], for arbitrary revenue functions.

- For the non-uniform disks model (see Section 1.1.1), we can bound $\delta_t$ by $((2d_{\max}/d_{\min}) + 1)^2$, where $d_{\max}$ and $d_{\min}$ are the maximum and the minimum disk radii respectively.

  The approximation-ratio can be further improved as follows. We divide the interference graph into subgraphs $G_0, \ldots, G_{\log(d_{\max}/d_{\min})}$, where subgraph $G_j$ contains any base station with radius in $[2^j, 2^{j+1})$, and then use our techniques on each subgraph separately. Using the result from previous paragraph, the $\delta_t$ in each of these subgraphs is 25. Thus, the overall approximation ratio is $25(\Delta_c + 1)\log(d_{\max}/d_{\min})$.

**Handling Complementary Bidding Functions.** We have so far assumed that the revenue function satisfies the non-complementary property (Equation 2). However, there may be scenarios where the bidding functions (and hence, the revenue function) may not satisfy the non-complementary property.

In many such scenarios, our Greedy Algorithm (and similarly, the algorithms designed in next section) can be modified to ensure the approximation ratio.

For instance, consider the case where a base station may bid for "groups" of channels, i.e., the base station is willing to pay a high price for a group of channels $C$ but bids zero price for any of the individual channels in $C$. More specifically, a base station may pay a price of 100 units for channels 5 and 10 *together*, but pays nothing for either channel 5 or 10 individually. Such a bidding function is *complementary*. However, we can have our Greedy Algorithm handle the above case by creating *super-channels* corresponding to each such group of channels; we also have the set of super-channels include the singleton sets of individual channels. Then, the channel-interference graph is constructed over super-channels as vertices, and allocation is done in terms of such (base station, super-channel) pairs. The modified GA, which selects a (base station, super-channel) pair at each stage, still yields the same approximation ratio.

In a more general scenario of "packaged bids," a service provider (owning multiple base stations) may bid for a channel $c_1$ at a base station $u_1$ only if a base station $u_2$ is also allocated a channel $c_2$. In essence, a service provider may pay certain price for a *group* of (base station, channel) pairs, but none for any individual pair. For the above case, the bidding functions cannot be defined independently for each base station, but must be defined for each service provider (i.e., for the group of base stations owned by it). However, the revenue function can be easily computed from such bidding functions. But, the resulting revenue function is no longer non-complementary. Fortunately, our Greedy Algorithm can still be appropriately modified (by having it allocated in terms of groups of (base station, channel) pairs) to handle the above case, while ensuring its approximation ratio.

For explicitly represented bidding functions (where a price is specified for *each* super-channel or package), GA still runs in time which is polynomial in the size of the input (including the representation of the bidding functions).

## 2.4 Spectrum Allocation under Physical Interference

In this section, we use the physical interference model to capture interference between base stations in the network, and present two approximation algorithms for the spectrum allocation problem in this context. We start by redefining the concept of valid spectrum allocation.

**Valid Spectrum Allocation.** In the context of physical interference model, a spectrum allocation $\mathcal{A}$ is considered valid if it satisfies the following two conditions:

- For any base station $i$, the set $\{c_k | (i, c_k) \in \mathcal{A}\}$ of channels allocated to $i$ consists of mutually non-overlapping channels.

- For a (base station, channel) pair $(i, c_k)$ in $\mathcal{A}$, let $V_{i,k}$ denote the set of base stations that have been allocated in $\mathcal{A}$ some channel $c_l$ that overlaps with $c_k$. More formally, let $V_{i,k} = \{j | (j, c_l) \in \mathcal{A}$ and $c_l = c_k$ or $(c_k, c_l) \in E_c\}$. Now, for $\mathcal{A}$ to be valid, for every $(i, c_k)$ in $\mathcal{A}$ and every point $p$ within a distance of $r$ from $i$, SINR at $p$ due to $i$ should be greater than $\beta$; i.e., the following should hold:

$$\frac{P/d_i^\alpha}{\mathcal{N} + \sum_{j \in V_{i,k}} P/d_j^\alpha} \geqslant \beta,$$

  where $d_x$ is the distance of base station $x$ from the point $p$.

**Spectrum Allocation Problem.** The spectrum allocation problem under the physical interference model is as follows. Given a set of base stations, the channel graph, and the bidding functions, the spectrum allocation problem is to select a valid spectrum allocation $\mathcal{A}$ that maximizes $R(\mathcal{A})$, the total revenue generated by $\mathcal{A}$.

**Distance-2 Neighbor Channels.** For clarity of presentation of the algorithm description and their approximation proofs, we define the following concept.

**Definition 8** (Distance-2 Neighbor Channels.) Two channels are *distance-2 neighbors* if they are at most two hops away from each other in the channel-interference graph. □

In the following paragraphs, we describe two greedy algorithms, viz. GAHT and GACP, for the spectrum allocation problem under physical interference model. For both algorithms, we present worst-case guarantees on their performance depending on the interference model parameters ($\alpha$ and $\beta$). Our simulations results (presented in Section 2.5) show that, in general, GACP generates higher revenue than GAHT for most values of $\alpha$ and $\beta$.

**Greedy Algorithm Based on Hexagonal Tiling (GAHT).** The basic idea of GAHT is as follows. We start with partitioning the entire region into hexagons of certain length (see Figure 4), and color them using three colors[6] such that no two adjacent hexagons have the same color. See Figure 5. Then, we construct *three* valid spectrum allocations, one for each color. For a particular color $c$, we consider only $c$-colored hexagons and pick (base station, channel) pairs iteratively (as in GA). However, we impose the condition that within each hexagon, no two allocated channels are distance-2 neighbors; this condition ensures the validity of the spectrum allocation (as shown in Lemma 1). Finally, we pick the best of the three spectrum allocations (one for each color) thus constructed. We will prove that the above algorithm yields a near-optimal spectrum allocation (Theorem 2).

Formal Description of GAHT. GAHT consists of the following steps.

- Partition the entire region into hexagons of side $\mu r$ each, where $r$ is the communication radius and $\mu$ is defined as:

$$\mu = 4 \sqrt[\alpha]{\frac{2\beta(3\alpha - 5)}{3(\alpha - 1)(\alpha - 2)}}. \tag{10}$$

- Next, color the hexagons using three colors, such that adjacent hexagons are colored differently. See Figure 5.

[6] The coloring here is just to partition the base stations into easier-to-handle groups; it has nothing to do with channels.

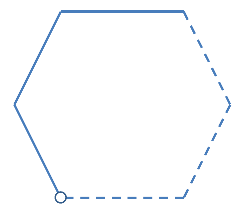


Figure 4: To ensure validity of our algorithms, the hexagons we use are open from side and closed from the other. For simplicity, we will use closed hexagons in the remaining figures.

- For each color $c$, consider only the $c$-colored subregions and construct spectrum allocation $\mathcal{A}^c$ as below.
  - Initially $\mathcal{A}^c = \varnothing$.
  - Pick a (base station, channel) pair $(i, k)$ to add to $\mathcal{A}^c$ such that the revenue of $\mathcal{A}^c \cup \{(i, k)\}$ is maximized, and $\mathcal{A}^c \cup \{(i, k)\}$ satisfies the following *condition*.[7] In $\mathcal{A}^c \cup \{(i, k)\}$, there should be no two elements $(j, l)$ and $(j', l')$ in the *same* hexagon such that $l$ and $l'$ are distance-2 neighbors.
  - Terminate when $\mathcal{A}^c$ cannot be extended any further.
- From the three spectrum allocations $\mathcal{A}^1$, $\mathcal{A}^2$, and $\mathcal{A}^3$ thus constructed, pick the one that has the highest revenue and return it as the solution.

Validity of GAHT. We now prove that each of the three spectrum allocations

[7] For sake of clarity of presentation, we have chosen this conservative condition. However, the correctness and approximation proof of GAHT is preserved even if we use the following less conservative condition: for every (base station, channel) pair $(j, l)$ in $\mathcal{A}^c \cup \{(i, k)\}$, there exist at most one pair $(j', l') \in \mathcal{A}^c \cup \{(i, k)\}$ in each hexagonal subregion (including the one containing $j$) such that $l$ overlaps with $l'$. Here, $j$ may be equal to $j'$, and $l$ may be equal to $l'$.

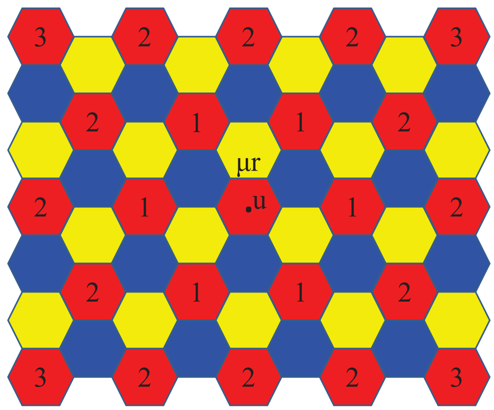



Figure 5: Hexagons colored using three colors such that adjacent hexagons have different colors. The red-colored hexagons around the hexagons containing base station $u$ have been partitioned into hierarchical levels; the numbers denote the hierarchical level. At $l^{th}$ level, there are $6l$ red-colored hexagons.

constructed by GAHT, the above algorithm, are valid. Intuitively, the spectrum allocations are valid because the total interference at any point $p$ due to "far away" (in non-adjacent hexagons) interferers is less than the signal received due to a base station $i$ in the hexagon of $p$.

**Lemma 1** *GAHT returns a valid spectrum allocation.*

*Proof:* Consider a base station $u$ in a hexagon $H$ of color $c$. As shown in Figure 5, partition all $c$-colored hexagons surrounding $H$ into hierarchical levels. The first level contains 6 hexagons and each such hexagon $H'$ is at a distance of $\mu r$ from $H$; here, by distance between two hexagons, we mean that the distance between *any* point in $H'$ and *any* point in $H$ is at least $\mu r$. Similarly, the second level contains 12 hexagons at a distance of at least $2\sqrt{3}\mu r$ from $H$. In general, the $l^{th}$ level contains $6l$ hexagons at a distance of at least $\frac{\sqrt{3}}{2}(3l-2)\mu r$ from $H$.

Now consider a point $p$ within a distance of $r$ from $u$. Let $u$ be operating on a channel $k$. Since GAHT does not allow two elements $(j,l),(j',l')$ of $\mathcal{A}^c$ in the same hexagon such that $l$ and $l'$ are distance-2 neighbors, there will be at most one (base station, channel) pair in each $c$-colored hexagon that "interferes" with $(u,k)$ in each $c$-colored hexagon. Thus, the total signal received at $p$ due to all base stations possibly operating in $k$ or an overlapping channel in the $c$-colored hexagons other than $H$ is at most:

$$\sum_{i=1}^{\infty} 6i \frac{P}{((\frac{\sqrt{3}}{2}(3i-2)\mu - 1)r)^\alpha} < \frac{2P(3\alpha-5)}{3(\alpha-1)(\alpha-2)(\frac{1}{4}\mu r)^\alpha}.$$

Thus, ignoring the noise (we relax this assumption later), the SINR of channel $k$ at $p$ due to $u$ is at least:

$$\frac{\frac{P}{r^\alpha}}{\frac{2P(3\alpha-5)}{3(\alpha-1)(\alpha-2)(\frac{1}{4}\mu r)^\alpha}} = \beta.$$

The above follows from the value of $\mu$ in Equation 10. ∎

Approximation Ratio of GAHT. We now prove the approximation ratio of GAHT. First, we show in Lemma 2 that a valid spectrum allocation (in particular, the optimal) cannot have more than a certain number ($q$, as defined in Equation 11 below) of base stations within a hexagon allocated the same channel. The approximation ratio then follows using similar techniques as in Theorem 1.

**Lemma 2** *No particular channel can be assigned to more than q base stations in any hexagon, by a valid spectrum allocation, where q is:*

$$q = \min(q_1, q_2), \text{with} \tag{11}$$

$$q_1 = \frac{3\sqrt{3}(4\mu^2 + 4\mu(\sqrt[\alpha]{\beta}+1) + (\sqrt[\alpha]{\beta}+1)^2)}{2\pi(\sqrt[\alpha]{\beta}+1)^2}, \text{and} \tag{12}$$

$$q_2 = \frac{(2\mu-1)^\alpha}{\beta}. \tag{13}$$

*Proof:* The number of base stations that can be assigned the same channel in any hexagon, by a valid spectrum allocation is limited by two factors: (i) the size of this hexagon, and (ii) the base stations' mutual interference.

To show the effect of the first factor (captured by $q_1$), we use a simple packing argument. First, we note that the minimum distance between any two base stations in a valid spectrum allocation is $(\sqrt[\alpha]{\beta} + 1)r$. This is directly implied by the SINR equation (Equation 1). Now, the maximum number of non-overlapping circles of radius $(\sqrt[\alpha]{\beta}+1)r/2$ whose centers lie inside a hexagon of side $\mu r$ is given by Equation 12.

Now, to show the effect of the second factor (captured by $q_2$), we assume its contrary, i.e., a valid spectrum allocation assigns a particular channel to $q_2 + 1$ base stations in same hexagon. Now consider a point $p$ at a distance of $r$ from one of these base stations $u$. Then, the SINR at $p$ due to $u$ is at most:

$$\frac{\frac{P}{r^\alpha}}{\frac{q_2 P}{((2\mu-1)r)^\alpha}} < \beta.$$

∎

The above lemma can be used to show that GAHT returns a constant-factor approximate solution.

**Theorem 2** *GAHT returns a valid spectrum allocation whose revenue is at least* $1/(3(q(\Delta_c^2 - \Delta_c + 1) + 1))$ *of the optimal revenue, where* $\Delta_c$ *is the maximum degree of a vertex in the channel graph and* $q$ *is as defined above in Equation 11.*

*Proof:* The proof of this theorem is similar to that of Theorem 1. For sake of clarity of presentation, we define two (base station, channel) pairs $(i, k)$ and $(i', k')$ to be *conflicting* if they are in the same hexagon and $k$ and $k'$ are distance-2 neighbors. Now, among the three different colored hexagons, let us consider the hexagons colored with one color at a time. Using the same notation as in Theorem 1's proof, we make the following two claims.

- For each $j \in \mathcal{O}_l$, let $h(j)$ be the smallest integer such that $j$ conflicts with $b_{h(j)}$. Informally, selection of $b_{h(j)}$ by GAHT is the reason why $j$ is not considered by GAHT for selection in later iterations. Note, by the greedy choice of $b_i$'s we have,

$$R(\{j\}|\mathcal{A}_{h(j)-1}) \leqslant a_{h(j)}. \tag{14}$$

- Lemma 2 shows that there cannot be more than $q$ base stations within each hexagon that can be assigned the same channel by any optimal algorithm. Based on the above, it is easy to see that the maximum number of (base station, channel) pairs in the optimal set that conflict with $b_i$ is at most $q(\Delta_c^2 - \Delta_c + 1)$. Here, $\Delta_c$ is the maximum degree of any vertex in the channel graph. Thus for any integer $z$, there are at most $q(\Delta_c^2 - \Delta_c + 1)$ (base station, channel) pairs $j$ in $\mathcal{O}_l$ such that $h(j) = z$. Thus we have,

$$\begin{aligned} \sum_{j \in \mathcal{O}_l} a_{h(j)} &\leqslant q(\Delta_c^2 - \Delta_c + 1) \sum_{z=1}^{m} a_z \\ &= q(\Delta_c^2 - \Delta_c + 1) R(\mathcal{A}_l). \end{aligned} \tag{15}$$

Using similar proof technique as the one used to prove Equation 5, we can show that

$$R(\mathcal{O}) \leqslant R((\mathcal{O} - \mathcal{O}_l) \cup \mathcal{A}_l) + \sum_{j \in \mathcal{O}_l} R(\{j\}|\mathcal{A}_{h(j)-1}). \tag{16}$$

Without loss of generality, assume that the Greedy and optimal solutions are disjoint for the considered color. Then, GAHT continues till $\mathcal{O}_l = \mathcal{O}$. For $\mathcal{O}_l = \mathcal{O}$, the above Equation 16 becomes:

$$R(\mathcal{O}) \leqslant R(\mathcal{A}_l) + \sum_{j \in \mathcal{O}_l} R(\{j\}|\mathcal{A}_{h(j)-1}). \tag{17}$$

Now using Equations 14 and 15 in the above Equation 17, we get

$$R(\mathcal{O}) \leqslant (q(\Delta_c^2 - \Delta_c + 1) + 1) R(\mathcal{A}_l)$$

for the color being considered.

Now let us denote the set of (base station, channel) pairs chosen by the greedy algorithm and optimal algorithm for the base stations lying in the three different colored hexagons as $\mathcal{A}^c$ and $\mathcal{O}^c$ respectively where $c = 1, 2$, and 3. Let $OPT = \mathcal{O}^1 \cup \mathcal{O}^2 \cup \mathcal{O}^3$. Now we have,

$$\begin{aligned} R(OPT) &= \sum_{c=1}^{3} R(\mathcal{O}^c) && (18)\\ &\leqslant (q(\Delta_c^2 - \Delta_c + 1) + 1) \sum_{c=1}^{3} R(\mathcal{A}^c) && (19)\\ &\leqslant (q(\Delta_c^2 - \Delta_c + 1) + 1)3[\max_{c=1,2,3}(R(\mathcal{A}^c))]. && (20) \end{aligned}$$

■

It should be noted here that the bound of the above theorem is not tight, due to many simplifying assumptions made throughout the analysis.

Time Complexity of GAHT. GAHT can be implemented in a way similar to GA, where the partitioning into hexagons and coloring steps are used to create a "virtual" interference graph. Then, the overall time complexity of GAHT can be shown to be bounded by $O(NMN_H\Delta_c^2 \log(NM))$ if we use a heap data structure to compute the maximum at each stage, where $N$ is the number of base stations, $M$ is the number of channels, $N_H$ is the maximum number of base stations inside any hexagon, and $\Delta_c$ is the maximum vertex-degree in the channel graph.

**Greedy Algorithm Based on Circular Packing (GACP).** We now present another algorithm (GACP) whose approximation proof is based on a circular packing argument.[8]

Formal Description of GACP. In short, GACP works by first constructing a "virtual" interference graph over the (base station, channel) pairs where two (base station, channel) pairs $(i, k)$ and $(i', k')$ are connected by a simple edge if $k$ and $k'$ are distance-2 neighbors and the distance between $i$ and $i'$ is less than $\mu' r$, where $r$ is the communication radius and $\mu'$ is as defined below.

$$\mu' = 2\sqrt[\alpha]{\frac{8(3\alpha - 4)\beta}{(\alpha - 1)(\alpha - 2)}}. \qquad (21)$$

[8]Concurrently, [25] has used similar ideas to solve the problem of local broadcasting in the physical interference model.

Then, GACP works exactly as the GA for the pairwise interference model, except that GACP uses the above constructed virtual interference graph as the pairwise interference graph. It can be shown that the resulting spectrum allocation is valid in the context of *physical* interference model, and its revenue is at least $1/(q'(\Delta_c^2 - \Delta_c + 1) + 1)$ of the optimal revenue possible, where $\Delta_c$ is the maximum degree of a vertex in the channel graph and $q'$ is as defined below.

$$q' = \min(q_1', q_2'), \text{with} \tag{22}$$

$$q_1' = \frac{4\mu'^2 + 4\mu'(\sqrt[\alpha]{\beta} + 1) + (\sqrt[\alpha]{\beta} + 1)^2}{(\sqrt[\alpha]{\beta} + 1)^2}, \text{and} \tag{23}$$

$$q_2' = \frac{(2\mu' - 1)^\alpha}{\beta}. \tag{24}$$

**Theorem 3** *GACP returns a valid spectrum allocation under physical interference model whose revenue is at least* $1/(q'(\Delta_c^2 - \Delta_c + 1) + 1)$ *of the optimal revenue, where* $\Delta_c$ *is the maximum degree of a vertex in the channel graph and* $q'$ *is as defined above in Equation 22.*

*Proof:* The proof of the above theorem is similar to that of Theorem 1. Consider an arbitrary base station $u$ whose receivers within the communication radius $r$ should have an SINR value at least $\beta$. We divide the area around the base station into circular annuluses of width $\mu' r$ as shown in Figure 6. The $l^{th}$ annulus lies between circles of radii $l \cdot \mu' r$ and $(l+1) \cdot \mu' r$. Note that GACP ensures that the minimum distance between any two base stations is more than $\mu' r$ for assigning same or overlapping channels. Thus, using a packing argument, the number of base stations in the $l^{th}$ annulus which are at least at a distance of $\mu' r$ apart for each other is at most

$$\frac{((l+1)\mu' r + \frac{\mu' r}{2})^2 \pi - (l\mu' r - \frac{\mu' r}{2})^2 \pi}{(\frac{\mu' r}{2})^2 \pi} = 8(2l+1).$$

The maximum interference $I$ (due to other active base stations operating at the same channel as $u$) at any point $p$ within a distance of $r$ (the communication radius) from $u$ can now be bounded as:

$$I \leqslant \sum_{l=1}^{\infty} 8(2l+1) \cdot \frac{P}{((l\mu' - 1)r)^\alpha} \leqslant \frac{2^{\alpha+3} P (3\alpha - 4)}{\mu'^\alpha r^\alpha (\alpha - 1)(\alpha - 2)}.$$

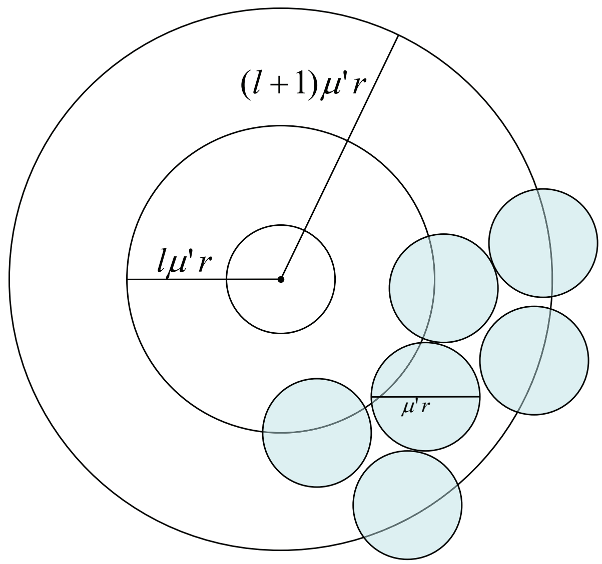



Figure 6: Circular annuluses of width $\mu' r$ around a base station. In each annulus we pack circles of radius $\frac{\mu' r}{2}$ to find the number of interfering base stations that can potentially cause interference at a receiver within the communication radius of the sender base station.

Thus, ignoring the noise, the SINR at $p$ is at least:

$$\frac{\frac{P}{r^{\alpha}}}{I} \geqslant \frac{(\alpha-1)(\alpha-2)}{2^{\alpha+3}(3\alpha-4)} \cdot \mu'^{\alpha} = \beta.$$

This shows that the GACP algorithm returns a valid spectrum allocation in the context of physical interference model.

Using a proof similar to Lemma 2's proof, we can show that no particular channel can be assigned to more than $q'$ base stations in a circle of radius $\mu' r$, by a valid spectrum allocation. Finally, proving the worst-case guarantees of GACP can be completed using similar proof technique as the one used in Theorem 2's proof. ∎

It should be noted here that the bound of the above theorem is not tight, due to many simplifying assumptions made throughout the analysis.

Since GACP is similar to GA, its time complexity can be bounded in a similar manner. If $N$ is the number of base stations, $M$ is the number of channels, and $\Delta_{t'}$ and $\Delta_c$ are the maximum vertex-degree in the "virtual" interference and channel graphs respectively, then the overall time complexity of GACP can be shown to be bounded by $O(NM\Delta_{t'}\Delta_c^2 \log(NM))$ if we use a heap data structure to compute the maximum at each stage.

**Incorporating Non-Zero Noise.** Non-zero noise can be incorporated using techniques similar to [2]. To do so, we need to redefine the $\mu$ and $\mu'$ values for GAHT and GACP schemes as follows:

$$\mu = 4\sqrt[\alpha]{\frac{2(3\alpha-5)\beta}{3(\alpha-1)(\alpha-2)(1-(\mathcal{N}\beta r^\alpha/P))}},$$

$$\mu' = 2\sqrt[\alpha]{\frac{8(3\alpha-4)\beta}{(\alpha-1)(\alpha-2)(1-(\mathcal{N}\beta r^\alpha/P))}}.$$

The equations defining $q$ (Equation 11) and $q'$ (Equation 22) remain unchanged. The proofs of Theorem 2 and Theorem 3 can be easily extended for non-zero noise, using the above redefined values.

**Greedy Heuristic (GH).** Finally, we present a simple greedy heuristic algorithm. This algorithm works exactly as the GA for the pairwise interference model, except that it does not use an interference graph to ensure the validity of the generated spectrum allocation. Instead, in each iteration, GH adds the best (base station, channel) pair that does not "invalidate" the current spectrum allocation. The validity-check is done using the below equation at *each base station* $u$.

$$\frac{P/r^\alpha}{\mathcal{N} + \sum_{i \in V'} P/(d_i - r)^\alpha} \geqslant \beta. \tag{25}$$

Above, $V'$ is the set of base stations assigned the same channel as $u$, $d_i$ is the distance between $i$ and $u$, and $r$ is the communication radius. It is easy to see that the above validity-check *ensures* an SINR greater than $\beta$ at *any point* within a distance of $r$ from $u$, due to the set of interferers $V'$.

Thus, the above heuristic is guaranteed to deliver a valid spectrum allocation. However, there is no performance guarantee on the total revenue of the solution delivered by GH, as shown by the example below. Despite this shortcoming of GH, we compare its results with GACP and GAHT in Section 2.5 and observe that it greatly outperforms both algorithms in almost all simulated settings.

**EXAMPLE 1** Consider a single-channel network, where only one channel is available. Thus, bids can be associated with base stations rather than (base station, channel) pairs. Let two particular base stations $i$ and $j$ have bids of 2 units each, while all the remaining base stations have a bid of 1 unit each. Also, let the distance between $i$ and $j$ be $(\sqrt[\alpha]{\beta}+1)r$, and the distance between any other pair of base stations be at least $\mu' r$.

In the above example, GH will start by adding both $i$ and $j$ to its allocation and stop since the addition of any other base station will cause the SINR value at both $i$ and $j$ to drop below $\beta$. In contrast, GACP will add either $i$ or $j$ (but not both) and *all* of the remaining base stations, achieving an arbitrarily higher revenue than that of GH. □

## 2.5 Simulation

In this section, we present detailed simulation results comparing the performance of the proposed algorithms. We first compare the performance of the various algorithms (GAHT, GACP, and GH) designed for the physical interference model, under different network topologies and parameters. Next, we examine how well a spectrum allocation returned by the greedy algorithm (GA) under the pairwise interference model work under the physical interference model. We start by describing the simulation parameters and then present the results.

**Network Topology.** In order to examine the impact of network topology, we consider two types of networks.

- *Random Networks:* We consider a fixed area of $1000 \times 1000$ units and randomly place base stations within this area. We vary the network density by changing the number of base stations from 100 to 1500. We assume a communication radius $r$ of 25 units in this scenario.

- *Real Networks:* We use locations of real cellular base stations available in FCC public GIS database [1] and choose base stations deployed in 4 different regions of increasing size and number of base stations.

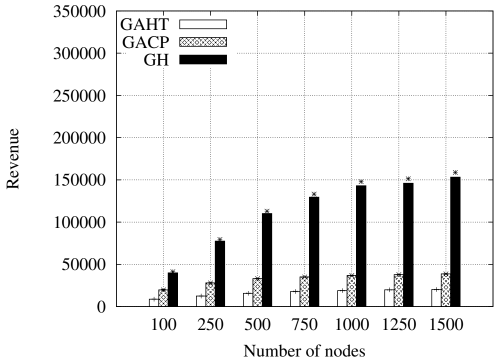
- R1 - 843 base stations in the state of MA.
- R2 - 2412 base stations in New England area (MA, ME, NH, VT, RI, CT).
- R3 - 4467 base stations in New England and New York.
- R4 - 8618 base stations in North East USA (New England, NY, NJ, PA).

Here, the regions are progressively supersets of the previous ones. We assume the communication radius $r$ to be 25 meters in this scenario.

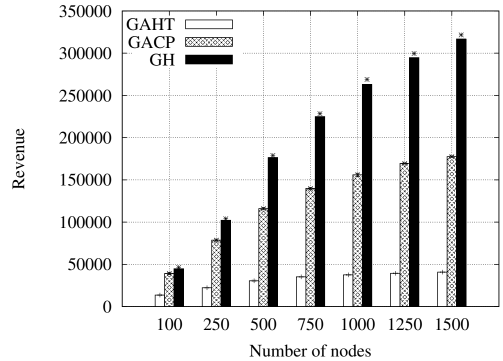
Each experiment is repeated 20 times and the averages are reported.

**CAB.** We consider a CAB with a bandwidth of 50 MHzand assume that each base station in the region can operate one or more of the following types of networks: GSM (200 KHz), CDMA (1.25 Mhz) and W-CDMA (5 Mhz). We assume the CAB is divided into channels of different types as described in Figure 3 and so we have 250 GSM channels, 40 CDMA channels and 10 W-CDMA channels in total.

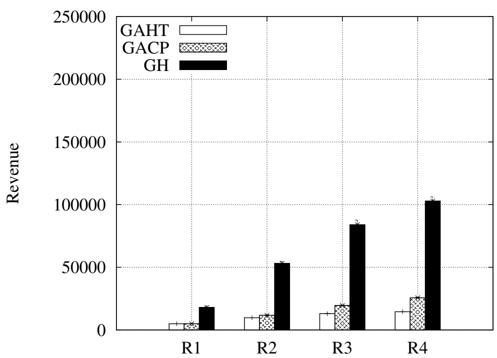
**Bidding Functions.** We generate bidding functions for each base station as follows. First, we randomly assign the number (between one and three) of types of channels/networks operated at the base station. For each network type, we generate a separate bidding function as follows. Let $m$ be the number of channels in the given network type. We generate $m$ random numbers from a predetermine range; let the generated numbers in the non-increasing order be: $\{p_1, p_2, p_3, \ldots, p_m\}$. Now, for any set of channels of size $k$, we assign the bidding price to be $\sum_{i=1}^{i=k} p_i$. Such a bidding function essentially gives higher value to channels that are allocated earlier. In terms of price ranges, for GSM networks, we generate prices from the interval 1–20, for CDMA networks we use the interval 1–125 and for W-CDMA networks we use the interval of 1–500. The intervals are chosen as above so that channels that have higher width (e.g. W-CDMA compared to GSM) are valued at a higher price by each base station.



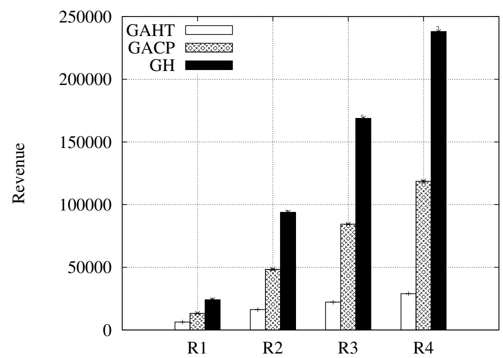

(a) Random Networks, $\alpha = 2.5$, $\beta = 5$ dB




(b) Random Networks, $\alpha = 4$, $\beta = 5$ dB




(c) Real Networks, $\alpha = 2.5$, $\beta = 5$ dB




(d) Real Networks, $\alpha = 4$, $\beta = 5$ dB

Figure 7: Comparison of overall revenue generated by GAHT, GACP and GH algorithms for various network topologies.

### 2.5.1 Comparing GAHT, GACP, and GH (Algorithms Designed for Physical Interference)

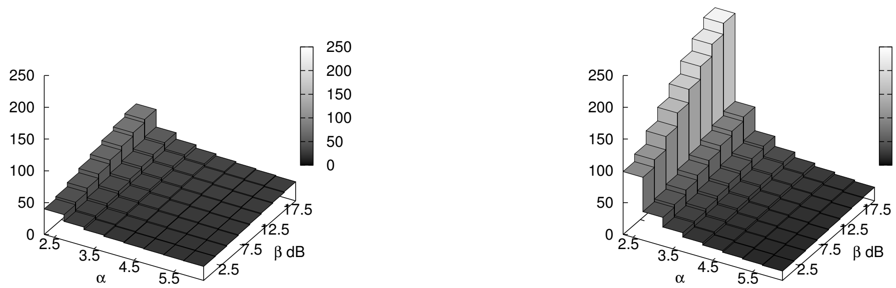
In our first set of experiments, we compare the performance of the two approximation algorithms (GAHT and GACP) with the heuristic GH under the physical interference model. We observe that, in general, the revenue generated by GH is higher than the one generated by GACP which in turn is higher than that of GAHT.

Revenue Comparison for Fixed $\beta$ and Two $\alpha$ Values. In Figure 7, we show the revenue generated using the three algorithms for different network sizes in both random and real network topologies for two different values of $\alpha$ and a fixed value of $\beta = 5$ dB. For the random network case (see Figures 7(a) and 7(b)), we see that in all algorithms, the revenue generated increases with the network size. This is mainly due to the non-increasing nature of the bidding functions. So with more base stations, the spectrum broker tries to allocate channels to base stations that are willing to pay more (those base stations with less number of channels allocated) and thereby generating higher revenue. We also see that when the network size is small (about 100 base station), the difference between the revenue generated by all algorithms is small. As the network size increases, the difference between the revenue generated by GH and the revenues generated by GAHT and GACP becomes increasingly high. We see similar behavior in the real network scenarios (see Figures 7(c) and 7(d)). The more conservative nature of GAHT and GACP is the reason behind their poor performance compared to GH. On the other hand, the amount of time consumed by GH to ensure the validity of the resulting spectrum allocation is very large. For example, the average running time of GH on a 500-base station network of Figure 7(b) is more than 617 seconds, while GAHT and GACP never took more than 2 seconds.

It is interesting to note that when $\alpha = 2.5$, the revenues generated by GAHT and GACP are relatively close to each other and both are very poor compared to the revenue generated by GH. However, increasing $\alpha$ to 4 lead to varying degrees of improvements in each of the three algorithms. The most interesting variation involves GACP and GAHT. This is mainly due to the difference in each algorithm's dependence on $\alpha$ and $\beta$ values.

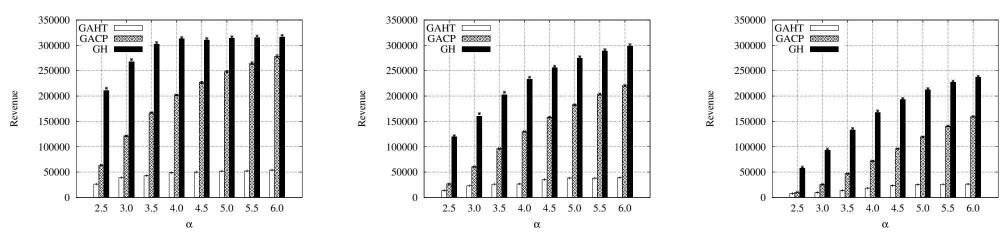
Comparison of Theoretical Bounds of GACP and GAHT. To understand this behavior clearly, we show the actual values of the theoretical bounds of both GAHT and GACP for various values of $\alpha$ and $\beta$ in Figure 8. The figures show that for $\alpha \leqslant 4$, GAHT has a better approximation ratio than GACP, while for $\alpha > 4$, GACP has a better approximation ratio than GAHT. It can also be seen that the variation of the two values due to $\beta$ is relatively small compared to the variation due to $\alpha$. As $\alpha$ increases and $\beta$ decreases, both GAHT and GACP become closer to the optimal performance.

Revenue Comparison for Various $\alpha$ and $\beta$ Values. We now compare the revenues generated by GAHT, GACP, and GH in a 1000-base station random

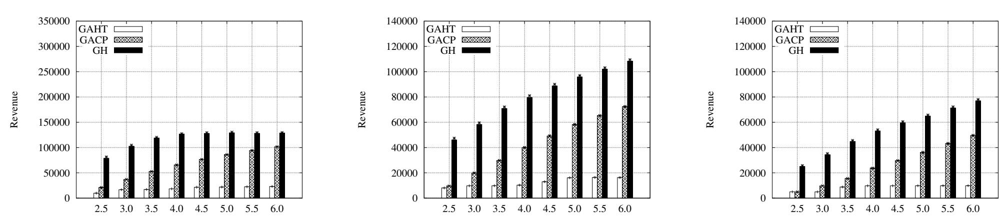



(a) Approximation ratio values of GAHT (b) Approximation ratio values of GACP

Figure 8: Approximation ratio values for different values of $\alpha$ and $\beta$.

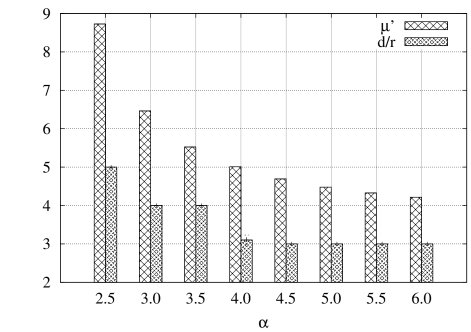



(a) Random Networks, $\beta$ = 1 dB (b) Random Networks, $\beta$ = 7.5 dB (c) Random Networks, $\beta$ = 15 dB

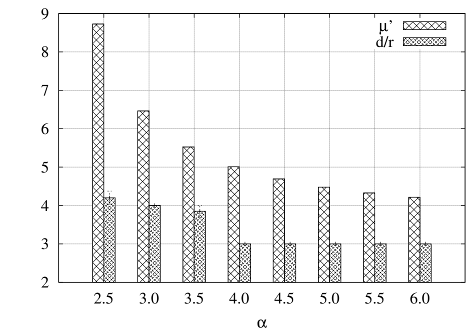



(d) Real Networks, $\beta = 1$ dB (e) Real Networks, $\beta$ = 7.5 dB (f) Real Networks, $\beta = 15$ dB

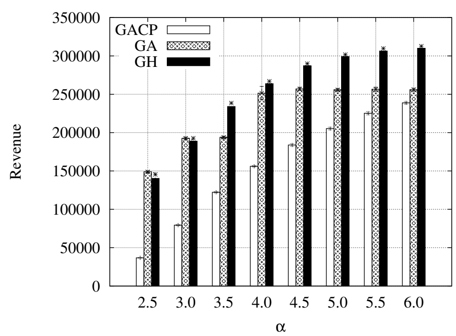
Figure 9: Comparison of overall revenue generated by GAHT, GACP and GH algorithms for varying $\alpha$ and $\beta$ values.



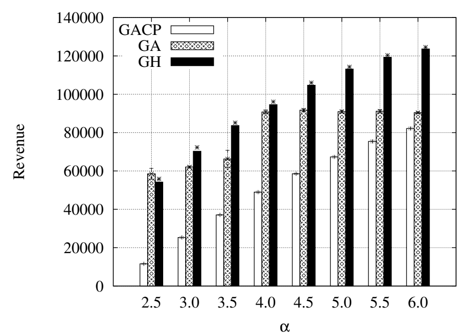

(a) Random Networks, $d/r$ Vs. $\mu'$, $\beta = 5$ dB




(b) Real Networks, $d/r$ Vs. $\mu'$, $\beta = 5$ dB




(c) Random Networks, Revenue Comparison, $\beta = 5$ dB




(d) Real Networks, Revenue Comparison, $\beta = 5$ dB

Figure 10: Comparing performance of GA, GACP, and GH algorithms.

network and a real network (R2) with varying values of $\alpha$ and $\beta$ in Figure 9. Note that, in theory, increasing $\alpha$ means increasing the degree of deterioration in signal strength over distance traveled, which leads to more spatial reuse, and hence, higher revenue generated by all algorithms. Furthermore, decreasing $\beta$ means that the receivers are less sensitive to noise and interference and are more capable of successfully receiving a transmission. This also allows for more spatial reuse, and again, higher revenue generated by all algorithms. The above patterns can be easily observed in the plots of Figure 9.

As noted earlier, we see that increasing $\alpha$ had varying degrees of improvements in each of the three algorithms. For small values of $\alpha$, GAHT and GACP generated poor and relatively similar revenues, while GH was several times better. As $\alpha$ was increased, GACP's performance greatly improved and became close to the performance of GH.

### 2.5.2 Using GA for *Physical* Interference Model

In this subsection, we modify GA (originally defined for the pairwise interference model) to work in the context of physical interference model, and then, compare the modified-GA with GACP and GH algorithms. Recall that in the original GA, two base stations are said to interfere if they are within a certain distance $d$. To tailor GA to work in the context of physical interference model, we increase the distance $d$, until we get a "valid" spectrum allocation (based on the conservative validity-check method used in GH). In our experiments, we vary d from $2r$ to $8r$, where $r$ is the communication radius. We compare the revenue of a valid spectrum allocation returned by the above modified-GA algorithm with the other two efficient algorithms for physical interference model, viz., GACP and GH. We repeat this test for different values of $\alpha$ for a 1000-base station random network and one real network (R2). The value of $\beta$ is fixed at 5 dB in all experiments.

In Figure 10(a) and 10(b), we show the values of $d/r$ compared to $\mu'$ when the spectrum allocation obtained using GA is valid under the physical interference model. In the case of random networks, $d/r$ is smaller than $\mu'$ by about 53% on average. In the case of real networks, we see its is smaller by about 59% on average. Note that when $d/r$ is same as $\mu'$, the revenue generated by GA and GACP should be same as both algorithms are similar. Due to the smaller value of $d/r$, GA can exploit much higher spatial reuse of spectrum and generate more revenue. The revenue generated by both the algorithms are shown in Figures 10(c) and 10(d). The difference between the two algorithms is higher in the case of real networks due to the larger difference between $d/r$ and $\mu'$ compared to the random network case. This clearly demonstrates that while the pairwise interference model is simplistic, it can be used to generate efficient spectrum allocations (with an appropriate choice of $d$) which are valid in the real physical interference model. But in order to prove good theoretical properties for the spectrum allocation algorithm under the physical interference model, we need to use the definition of $\mu'$. We also see from Figures 10(c) and 10(d) that for small values of $\alpha$, the revenues generated by GA and GH are similar and, in general, are much higher than the revenue generated by GACP. For larger values of $\alpha$, the difference between

GACP and GA becomes smaller.

## 2.6 Related Work

**Traditional auctions.** Auctions have been traditionally used for efficiently allocating scarce resources [19, 8, 6]. The auctioneer can maximize its revenue by selling the goods to buyers who are willing to pay the most. At the same time, the buyers also get benefited as auctions tend to assign items to buyers who need them the most based on their valuation. Some examples where auction systems have been successfully used include energy markets [8], treasury bonds [6], and selling commercial goods online [19]. In general, the goods on sale can either be a single item [37], bundle of multiple units of single items [37, 20] or bundles of multiple units of multi-items [17, 7] and the complexity of the auction mechanisms increase in this order.

The spectrum allocation problem in the CDSA model differs from traditional periodic sealed bid multi-unit auctions in the following two important aspects. First, in a conventional multi-unit auction, every buyer competes with every other buyer participating in the auction. In the problem considered in this chapter, there is a network of base stations, and each base station competes only with base stations with which it interferes. This increases the complexity of the auction problem significantly as the way in which the base stations interfere depends on external constraints that include complexities such as radio propagation model, frequency used and transceiver design etc. While traditional multi-unit auctions can be solved optimally in polynomial time, this class of auction problems are known to be NP-hard even when specific restricted class of bidding functions [22] are used. The second major difference is the overlapping nature of channels of different types which puts an additional constraint on assigning channels to base stations. In traditional auctions, any item can be assigned to any buyer; this is not true here.

**Spectrum allocation without revenue models.** Spectrum allocation algorithms, both centralized [12, 57] and distributed [66, 13] in the context of dynamic spectrum access networks have been proposed previously without any revenue model. In these works, the authors propose algorithms to allocate spectrum to different nodes thereby optimizing one or more network properties like network interference or network capacity. All these algorithms only consider pairwise interference models and do not consider heterogeneous channels of varying widths that can overlap. In context of ad hoc networks, Yuan et al. in [65] propose centralized and distributed allocation of variable width frequency blocks to nodes in the network in a time-slotted fashion. Our work in this chapter differs from theirs in two main aspects. We have a general revenue model associated with the channels and try to maximize the overall revenue while they optimize a proportionally fair throughput metric. The second difference is that we propose efficient algorithms using both pairwise and physical interference models, while they use only pairwise interference model. In [14], the authors propose a spectrum allocation algorithm intergraded with interference-aware statistical admission control. Here also, they do not consider any revenue model and use only pairwise interference model to capture interference between access points.

**Revenue maximizing spectrum allocation.** Two previous works that are directly related to our work are [52] and [22]. In [52], Sengupta et al. formulate the spectrum allocation problem as a modification of the knapsack problem. Here, they assume a very primitive revenue model where they consider a constant price for each channel and specify spectrum demands as a fixed number of channels. The spectrum broker should either allocate all channels demanded or it cannot allocate any channel. This kind of spectrum demand is too restrictive to support efficient allocation. Also, they only consider homogeneous type of channels. In [22], Gandhi et al. propose solutions for the revenue maximizing spectrum allocation problem under pairwise interference model. Here, the authors only consider a specific class of revenue function which is piecewise linear in nature and use only homogeneous channels. Their algorithms cannot be extended to work for any general class of revenue functions and heterogeneous types of overlapping channels as we consider here. If we consider homogeneous channels only, our approximation is still better considering the fact we can solve the problem for any general revenue function. In addition, we address the spectrum allocation problem under physical interference model.

**Other Works.** Inspired by traditional work in economics, recent papers have addressed the dynamic spectrum allocation using auction-based [67, 68, 63, 33, 64] or pricing-based [31] techniques. Such papers tend to focus more on the economic aspects of the allocation process like the strategic behavior of the bidders, which are handled by the later chapters of this dissertation. Moreover, they assume simple interference settings. Another paper that uses auctions to solve a related problem is [64]. Unfortunately, their restrictive interference model prohibits the use of their ideas in our setting. Other works worth mentioning here include [21] where a similar problem is addressed but with the goal of throughput maximization. Finally, a concurrent work by Goussevskaia et al. [25] used ideas similar to the ones we use in GACP to solve the problem of local broadcasting in the physical interference model.

## 2.7 Conclusion

In this chapter, we proposed efficient approximation algorithms that give near-optimal solutions for the spectrum allocation problem in cellular network under the coordinated dynamic spectrum access model. We addressed the spectrum allocation problem in a very general context where (i) interference in the network is modeled using pairwise and physical interference models and (ii) base stations can bid for heterogeneous channels of different widths using generic bidding functions. For the specific case of non-overlapping channels and a unit-disk interference graph, our greedy algorithm GA returns a 6-approximate solution which is a direct generalization of the results in [22] for arbitrary revenue functions. Our simulations studies show that the proposed algorithms scale very well for large network topologies. Among the two algorithms proposed for the physical interference model, we see their performance primarily depends on the interference model parameters and the appropriate algorithm can be chosen based on the actual value of $\alpha$ and $\beta$. We also see that the simple pairwise interference model can be used to come up with efficient spectrum allocations that are valid under the physical interference model by appropriately choosing the interference region around the base stations.

# Chapter 3

# Truthful Auctions With Approximate Social-Welfare

## 3.1 Introduction

In the previous chapter, we presented an efficient auction-based market mechanism for dynamic spectrum allocation where we ignored many economic aspects of the mechanism design. Flawed market designs for a precious commodity like spectrum can lead to significant market inefficiencies and adverse economic impacts. This happened in the restructured electricity market in California in 2000 that made international headlines, leading to many academic studies [58, 62, 32, 5, 9].

A natural objective of auction-based mechanism is to maximize the generated *revenue* (the sum of the bids or payments by the buyers) [56, 22, 52, 31]. However, such an objective can encourage the spectrum buyers to lie about their real valuations leading to an "untruthful" auction, fear of market manipulation, and indirectly possibly lowered revenue. Moreover, in a competitive environment, buyers may spent a lot of time/effort in predicting the behavior of other buyers and planning against them. Three recent papers address the problem of designing truthful spectrum auctions [63, 67, 33]. Our focus in this chapter is on auction-based mechanisms that not only encourage truthful behavior but also allocate the spectrum to the bidders who value it the most.

The latter goal of maximizing the total valuation is justified in many settings and is extensively studied in economics [50, 49]. Moreover, the bidders with higher valuations are more capable of making good use of the spectrum to build up a viable cellular phone network [16, 3]. Note that, as discussed in Section 3.2.1, it is not possible to design *truthful* auction mechanisms with optimal/approximate revenue.

**Problem Addressed.** In this chapter, we consider a dynamic auction-based approach to allocate spectrum to competing base stations. Similar to the model of Chapter 2, the centralized auctioneer acts as the *seller* and the base stations act as the *buyers* of the available spectrum. The items being sold are various channels corresponding to certain (contiguous or non-contiguous) blocks of frequency. The base stations bid for these channels, based on their valuations of these channels.

In the above context, we address the problem of designing a spectrum auction mechanism (i.e., an allocation algorithm) with the following *dual* objectives, viz., (i) encourage truthful behavior from the buyers (i.e., ensure that the buyers "benefit" the most when their bids corresponds to their true valuations), and (ii) at the same time, maximize the "social-welfare," i.e., the total valuation of the allocated channels (by allocating them to the buyers who value them the most).

<u>Closest Prior Works.</u> The above problem of truthful spectrum auction design has been recently addressed in [67] by Zhou et al. under limited interference and bidding models. In particular, they design a spectrum auction mechanism that is truthful, but does not have any performance guarantee on the social-welfare. In other closely related work, Wu et al. [63] focusses on preventing collusion attacks and better revenue (sum of payments from the bidders) in a VCG-like spectrum auction. However, their mechanism requires solving an NP-hard optimization problem, does not guarantee truthfulness, and is limited to simple interference and bidding models. Finally, in a recent work, Jia et al. [33] address the problem of designing spectrum auctions in a Bayesian setting, wherein the broker is aware of the probability distributions of the private valuations of the bidders. In this setting, [33] designs truthful mechanisms, while attempting to maximize the *expected* revenue, for simple bidding and

interference models.

**Our Contributions.** In this chapter, we design a spectrum auction mechanism that yields an allocation (i) that encourages truthful behavior by buyers, and (ii) has an approximate (within a constant-factor of optimal) total valuation. We consider general (pairwise and physical) interference and bidding models. To the best of our knowledge, ours is the first work to design a spectrum auction mechanism satisfying the above dual objectives.

**Chapter Organization:** The rest of the chapter is organized as follows. In the next section, we present the background of our work before going into the details of the related works. In Sections 3.3 and 3.4, we formally define and present efficient spectrum auction mechanisms under pairwise and physical interference models, respectively. Section 3.6 concludes the chapter.

## 3.2 Background, Related Work, and Our Contributions

In this section, we present some background material related to our work, and introduce basic terms and definitions from both the spectrum allocation and the auction theory literature. We also discuss related work in more detail, and our contributions in this chapter.

**Dynamic Spectrum Access.** Similar to the model of Chapter 2, a centralized entity known as the *spectrum broker* owns a part of the spectrum called the *coordinated access band* (CAB). The spectrum broker divides the CAB into channels (contiguous or non-contiguous blocks of frequency) and dynamically allocates them to the competing base stations (the buyers) in the region it controls. The base stations express their bids for the available channels using a *bidding function* which specifies the price they are willing to pay for a given set of allocated channels. Periodically, the spectrum broker allocates available channels to the base stations (based on the received bids) under the "wireless interference constraint" such that the total revenue (total price paid by the base stations) is maximized. The above auction-based approach allows the base stations to bid according to the spectrum demands, and the spectrum broker to maximize the revenue generated from allocation of spectrum. However, as mentioned before, this goal is far from ideal in many settings, especially in spectrum auctions, and may cause several problems like market manipulation. To eliminate the fear of market manipulation and allow the bidders to have simple bidding strategies, truthful auction mechanisms are desired.

### 3.2.1 Truthful Auction Mechanisms

In this subsection, we formally define the concepts of auction mechanisms and truthful auction mechanisms. We also discuss VCG auction mechanisms, the only general form of auction mechanism that guarantees truthfulness.

**Auction Mechanism.** In an auction [49], a set of rational bidders compete over one or more items through a bidding system. An auction is described by the following:

- A finite set $O$ of *allowed outcomes*.
- Each bidder $i$ has a privately-known real function $\mathbf{v}_i : O \mapsto \mathbb{R}$ called its *valuation function*, which quantifies the bidder's benefit from each outcome.
- Bidders are asked to declare their valuation functions in the form of *bidding functions* $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_N)$. The bidders may lie about their valuation functions; thus, $\mathbf{w}_i$ may not be equal to $\mathbf{v}_i$.
- An *auction mechanism* chooses an outcome $o$ based on some criteria over the declared valuation functions.
- In addition to choosing an outcome, the auction mechanism also charges each bidder $i$ a certain amount of money $p_i$.
- The utility $u_i$ of each bidder $i$ is the difference between its true valuation of the outcome $o$ and its payment $p_i$, i.e., $u_i = \mathbf{v}_i(o) - p_i$. Each bidder's goal is to maximize its utility.

Based on the above model and setting, we define the concepts of auction mechanism and social-welfare.

**Definition 9** (Auction Mechanism.) Let $O$ be the set of possible outcomes of an auction. An auction mechanism is a pair of functions $(\mathbf{x}, p)$ such that:

- The winner determination function $\mathbf{x}$ accepts as input a vector $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_N)$ of bidding (declared valuation) functions and returns an output $\mathbf{x}(\mathbf{w}) \in O$.

- The payment function $p(\mathbf{w}) = (p_1(\mathbf{w}), \ldots, p_N(\mathbf{w}))$ returns a real vector quantifying the payment charged by the mechanism to each of the bidders.

□

**Definition 10** (Social-Welfare; Revenue) Social-welfare of an outcome $o$ is defined as the sum of the valuations, i.e., $\sum_i \mathbf{v}_i(o)$. Social-welfare may also be defined over declared valuations, i.e., as $\sum_i \mathbf{w}_i(o)$.

The revenue of an auction mechanism $(\mathbf{x}, p)$ is the sum of the payments $\sum_i p_i(\mathbf{w})$ charged to the bidders for a given declared valuation vector $\mathbf{w}$. □

Generally, the goal of the auction mechanisms is to maximize the total social-welfare, and not necessarily the revenue. The goal, also known as social efficiency, is justified in many settings and is extensively studied in economics [50, 49].

**EXAMPLE 2** Let us illustrate the above concepts using the well-known Vickrey's Second-Price Sealed-Bid Auction [61]. Consider an auction wherein a *single* item is up for sale. Each bidder has a certain valuation for the item, and makes a bid accordingly. Here,

- The set of outcomes are $o_1, \ldots, o_N$ where $o_i$ is the outcome in which the item is sold to the $i^{th}$ bidder.

- Valuation function $\mathbf{v}_i$ of a bidder $i$ defines the value the bidder assigns to each outcome. Thus, $\mathbf{v}_i(o_i)$ is equal to the value of the item for the bidder $i$, and $\mathbf{v}_i(o_j) = 0$ for all $j \neq i$ since a bidder does not get any benefit in an outcome where it does not get the item.

- In the Vickrey's auction mechanism, the item is sold to the bidder with the highest bid, i.e., the winner determination function $\mathbf{x}$ chooses an outcome $o$ with maximum social-welfare on $\mathbf{w}$, $\sum_i \mathbf{w}_i(o)$. In addition, the payment charged to the highest-bidder is an amount equal to the *second-highest* bid.

Now, consider a two-bidder scenario, where $\mathbf{v}_1(o_1) = 5$ and $\mathbf{v}_2(o_2) = 20$. Note that $\mathbf{v}_1(o_2)$ and $\mathbf{v}_2(o_1)$ are zero. Let us assume that the bidders are truthful, i.e., $\mathbf{w}_i = \mathbf{v}_i$ for all $i$. Then, the Vickrey's mechanism picks the outcome $o_2$ (i.e., sells the item to the second bidder), and charges the payment of 5 to the second bidder. In this case, the total revenue is 5, and the utilities of the first and second bidders are zero and 15 respectively. $\square$

**Truthful Auction Mechanisms.** In a selfish environment, bidders may not declare their valuation functions truthfully, if it were to their advantage (result in increase of their utility). Such a behavior may severely damage the resulting welfare and force each bidder to have complex bidding strategies based on its belief/knowledge about the strategies of other bidders. A *truthful* (also known as *incentive-compatible* or *strategy-proof*) mechanism enforces bidders to behave truthfully by offering them incentives (in the form of reduced payments) for such a behavior, or at least, by giving them no incentive for untruthful behavior. These incentives are based on the presumption that each bidder is selfish, and thus, only interested in maximizing its own utility. We now formally define the notion of truthful auction mechanism.

**Definition 11** (Truthful Auction Mechanisms.) *Given* the valuation functions, in a truthful auction mechanism, each bidder's utility is maximized when it truthfully declares its valuation function $\mathbf{v}_i$.

More formally, let the true valuation functions of the bidders be $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_N)$. Consider two declared valuation function vectors, viz., (i) $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_{i-1}, \mathbf{v}_i, \mathbf{w}_{i+1}, \ldots, \mathbf{w}_N)$, and (ii) $\mathbf{w}' = (\mathbf{w}_1, \ldots, \mathbf{w}_{i-1}, \mathbf{w}_i, \mathbf{w}_{i+1}, \ldots, \mathbf{w}_N)$ (where $\mathbf{w}_i \neq \mathbf{v}_i$). A mechanism $(\mathbf{x}, p)$ is considered *truthful* if $\mathbf{v}_i(\mathbf{x}(\mathbf{w})) - p_i(\mathbf{w}) \geqslant \mathbf{v}_i(\mathbf{x}(\mathbf{w}')) - p_i(\mathbf{w}')$ for all $i$ and $\mathbf{w}_i$. $\square$

It is easy to see that Vickrey's auction mechanism (see Example 2) is indeed truthful. Essentially, by lying about its valuation, a bidder may only hurt its chances of winning, while at the same time, not changing its payment if it wins (since the payment does not depend on its bidding function).

Truthfulness and Revenue Maximization. Informally speaking, an attempt to maximize the revenue without enforcing truthfulness may backfire, since in a non-truthful auction, bidders may bid much lower (than their actual valuations). For instance, in the simple case of single-item auction, a truthful Vickrey's auction (i.e., the second-price auction) may actually yield more revenue than a revenue-maximizing non-truthful auction (if the bidders were to bid too low, in the non-truthful case). Therefore, revenue maximization is not a feasible objective for truthful auction mechanisms [3, 24]. Basically, even in the case of an auction of a single copy of a single item, there is no way to deal with an astronomical bidder [3, 24]. In this paper, our main focus is on the design of truthful spectrum auction mechanisms, while also maximizing the social-welfare, which is justified in many settings and is extensively studied in economics [50, 49]. Through simulations, we show that the revenue of our designed mechanism is close to that delivered by the best-known revenue-approximation algorithm, and is an order of magnitude better than a naive truthful spectrum auction.

Truthful Auctions with Maximum Social-Welfare. In a truthful auction, since the bidders' bids are equal to their true valuations, the social-welfare (sum of the valuations) is also equal to the sum of the bids. Thus, the maximum possible social-welfare is an upper bound on the revenue (since, bidders are never asked to pay more than what they bid, to ensure positive utilities). However, in the case of a truthful auction mechanism, the winning bidders are asked to pay *less* than their bids, as an "incentive" to bid truthfully, which is the reason why the revenue yielded by a truthful auction cannot be optimal, in general.

**VCG Auction Mechanisms.** The only general mechanism that guarantees truthfulness is due to Vickrey-Clarke-Groves (VCG) [61, 15, 27], and in some scenarios, it is known that no other method exists [49]. In restricted settings, however, other approaches [41, 45] may exist. Informally, the celebrated VCG mechanism finds the outcome $o$ with maximum social-welfare, and charges each winner $i$ an amount equal to the total "damage" that it causes to the other bidders, i.e., the difference between the social welfare of the others with and without i's participation. The key property of such a mechanism is that it "aligns" each bidder's utility with the social-welfare [50]. Thus, by maximizing its own utility, each bidder is also maximizing the social-welfare. Below, we give a formal definition of the VCG mechanism.

**Definition 12** (VCG Mechanism.) A VCG mechanism is an auction mechanism $(\mathbf{x}, p)$ (see Definition 9) that satisfies the following two conditions, for any given declared valuation functions $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_N)$.

- $\mathbf{x}(\mathbf{w}) \in \arg\max_o \sum_i \mathbf{w}_i(o)$, i.e., the winner determination function $\mathbf{x}$ chooses an outcome that maximizes the social-welfare according to $\mathbf{w}$.
- The payment functions are determined by the VCG formula $p_i(\mathbf{w}) = (-\sum_{j\neq i} \mathbf{w}_j(\mathbf{x}(\mathbf{w}))) + h_i(\mathbf{w}_{-i})$, where each $h_i(\mathbf{w}_{-i})$ is an arbitrary function of $\mathbf{w}_{-i} = (\mathbf{w}_1, \ldots, \mathbf{w}_{i-1}, \mathbf{w}_{i+1}, \ldots, \mathbf{w}_N)$. For non-negative declared valuations, the function $h_i(\cdot)$ is usually chosen according to the Clarke pivot rule which suggests $h_i(\mathbf{w}_{-i}) = \max_{o\in O} \sum_{j\neq i} \mathbf{w}_j(o)$, the maximum social-welfare due to others. It can be shown that for non-negative valuations, the Clarke pivot rule ensures non-negative unities and payments, which are desirable properties [50].

□

One of the main shortcomings of VCG mechanisms is that they may result in low (even zero) revenue in some cases. But, VCG's payment function is key to ensuring its truthfulness, and altering its payment scheme may destroy its truthfulness property.

<u>VCG Mechanisms for NP-hard problems.</u> Unfortunately, a VCG mechanism requires solving an optimization problem of maximizing social-welfare, which can be NP-hard in many settings. Furthermore, Nisan and Ronen [49] showed that choosing an allocation with even approximate social-welfare destroys the truthfulness of the mechanism. However, to circumvent the above obstacle,

they introduce maximal-in-range (MIR) mechanisms (formally defined in the next subsection) where a suboptimal allocation can be used while maintaining the truthfulness property.

### 3.2.2 Related Works, Our Approach and Contributions

In this section, we discuss related works, our approach to design truthful spectrum auction mechanisms, and the main contributions of our work.

**Related Works**. As discussed in Section 2.6, traditional auction mechanism are not directly applicable to spectrum auctions due to the "multi-winner" property of each item (due to spatial reuse of spectrum channels) and wireless interference constraints. Moreover, the corresponding optimization problem of maximizing social-welfare in the context of spectrum auctions is known to be NP-hard [56], which makes VCG auction mechanisms inapplicable. Since the truthfulness property is key to our work, below we discuss recent works on truthful spectrum auctions in detail.

Truthful Spectrum Auctions. To the best of our knowledge, there has been only two works till date, viz., [67, 33], that have designed truthful mechanisms for spectrum auction. The truthful mechanism designed by Zhou et al. [67] however does not address the goal of maximizing social-welfare. Moreover, their approach is limited to only simple (single-minded or range) bidding functions and pairwise interference model. As observed in [67, 3], it is rather straight-forward to design a truthful auction mechanism without any regard for social-welfare. But, the authors in [67] show through simulations that their mechanism returns better social-welfare and revenue compared to a simple truthful mechanism. Recently, this work has been extended to consider double auctions [68].

In another recent work, Jia et al. [33] design a spectrum auction mechanism under the Bayesian setting, wherein the broker is aware of the probability distribution of the (private) valuation of each bidder. In this setting, [33] designs truthful spectrum auction mechanisms, while attempting to maximize *expected* revenue. In contrast, in this chapter, we consider the traditional auction model, wherein the bidder valuations are completely private. Moreover, [33] considers a special type of pairwise interference model, and is only limited to single-minded bidding functions. Finally and most importantly, the mechanisms designed by [33] either take exponential time or provide no guarantees on the expected revenue.

In another closely related work, Wu et al. [63] design a spectrum auction mechanism based on VCG mechanism. They focus on modifying the VCG payment function to eliminate colluding attacks by losing bidders and to improve the total revenue. However, their altered payment scheme destroys the truthfulness property of the VCG scheme. In addition, their mechanism requires solving an integer linear programming (NP-hard) problem, which makes their approach impractical for large networks. Note that in practice cellular networks may have thousands of base stations [55]. Finally, they assume either a single-channel system or that each bidder is interested in only one channel in a multi-channel system.

<u>Other Works.</u> Recently there have been lots of works on dynamic spectrum allocation using auctions, but most of the works have focussed on the goal of revenue maximization [56, 22, 52, 31] without worrying about the truthfulness of the auction mechanism. However, as mentioned before, an untruthful auction mechanism can encourage the bidders to lie about their valuations, which may lead to market manipulation and lowered revenues. In addition, in a competitive environment, buyers may be forced to spent a lot of time/effort in predicting the behavior of other buyers and planning against them. Finally, [53] uses budget-balanced mechanism to achieve truthfulness for the power allocation problem in single-cell CDMA networks.

**Our Approach: Truthfulness with Approximate Social-Welfare.** In this chapter, we focus on designing a spectrum auction mechanism that is truthful and selects an outcome with approximate social-welfare, for general interference models and bidding functions. In our approach, we make use of the recent result by Dobzinski and Nisan [18] on the design of truthful auction mechanisms with approximate social-welfare for multi-unit auctions (MUA) using a maximal-in-range (MIR) mechanism. Thus, we start with formally defining MIR mechanisms and multi-unit auctions. Then, we give an outline of our approach.

Maximal-In-Range Mechanisms. In [49], the authors provide a computationally-efficient way to overcome the problem of finding an allocation with *optimal* social-welfare in VCG mechanisms, since finding such an allocation may be NP-hard in some cases. In particular, they show that an auction mechanism is truthful if it (i) chooses an outcome that optimizes social-welfare over a fixed *subset* of the outcomes, and (ii) uses VCG payments (as defined in Definition 12). Such mechanisms are termed *Maximal-In-Range (MIR)* and formally defined below.

**Definition 13** (Maximal-In-Range (MIR) Mechanism.) Let $V_i$ be the set of all possible valuation functions of bidder $i$, and $V = \prod_i^N V_i$ be the space of all possible valuation functions. Let $O'$ denote the range of the winner determination function $\mathbf{x}$ at $V$, i.e., $O' = \{\mathbf{x}(\mathbf{v}) | \mathbf{v} \in V\}$. We say that $\mathbf{x}$ is *maximal in its range* if for every $\mathbf{v} \in V$, $\mathbf{x}(\mathbf{v})$ maximizes the social-welfare over $O'$. □

Multi-Unit Auctions (MUA). Multi-unit auctions (MUA) have been heavily studied in economics due to their practical implications. In a MUA, a set of $M$ identical items are up for auction among bidders, and each bidder expresses interest for certain *quantities* of the items, without any preference to any specific item. Thus, the valuation function of a bidder $i$ can be represented[1] as $\mathbf{v}_i : \{1, \ldots, M\} \mapsto \mathbb{R}$, where $\mathbf{v}_i(q)$ is the value for obtaining $q$ items. In [18], the authors design an MIR mechanism for multi-unit auctions that is truthful and yields an allocation with approximate social-welfare.

Our Approach. As mentioned above, traditional auction mechanisms cannot be directly used on spectrum auctions due to the "multi-winner" property of each auctioned item in the spectrum auctions. Our approach utilizes the geographical nature of the spectrum auction problem to re-formulate it as a *set* of multi-unit auction instances. Then, we use the MIR mechanisms for multi-unit auctions from [18] to solve each instance, i.e., independently determine

[1]Note that such a representation can be easily mapped to the original form wherein a valuation function maps outcomes to real numbers.

spectrum allocation with approximate social-welfare for each instance. We ensure truthfulness by using VCG payments. Finally, we combine the allocations over these independent instances in a way that preserves truthfulness and the approximation ratios of the social-welfare.

**Our Contributions.** Basically, we present a simple and general approach to dynamically allocate spectrum to competing base stations with the goal of maximizing the social-welfare while maintaining truthfulness. We consider general bidding functions and interference models. For the pairwise interference model, we consider the unit-disk, non-uniform disk, and the pseudo-disk models. For the physical interference model, we consider uniform as well as non-uniform power transmission models. Our contributions can be summarized as follows:

- For the pairwise interference with unit-disk model and general bidding functions, we present a truthful auction mechanism that yields an allocation whose social-welfare is within a constant-factor of the optimal. We extend the truthful mechanism and its approximation result to (i) non-uniform disk and pseudo-disk pairwise interference models, (ii) $k$-minded bidding function (where the bidder expresses its valuations for at most $k$ quantities of channels), (iii) non-orthogonal channels, and (iv) multi-type channel auctions.

- For the physical interference model with uniform power transmissions and general bidding functions, we present a truthful auction mechanism that yields an allocation whose social-welfare is within a constant-factor of the optimal. The result is extended to $k$-minded bidding functions, and non-uniform power transmission model.

## 3.3 Spectrum Auction Under Pairwise Interference

In this section, we address our problem of designing truthful spectrum auctions with approximate social-welfare under the pairwise interference model. In the first subsection, we will consider the unit-disk model, wherein the coverage region of each base station is assumed to be a disk of uniform radius. In later subsections, we extend our techniques to non-uniform disk and pseudo-disk interference models (see Section 1.1.1). We start with defining our network model, the core concepts, and formulating the problem formally.

**Network Model** Our model of a cellular network consists of a set of geographically distributed base stations. Spectrum is divided into *orthogonal* channels of the *same type*, and the spectrum auction involves each base station bidding for certain *quantities* of channels (as in basic multi-unit auctions). In a later subsection, we will consider non-orthogonal channels and multi-type channel networks.

**Valid Spectrum Allocation.** Informally, our spectrum auction problem is to allocate channels to base stations so as to maximize the social-welfare and maintain truthfulness. However, the allocation of channels should be done without violating the interference constraints. Again, we formalize this by using the concept of valid spectrum allocation (see Definition 4), which essentially represent the possible outcomes of the spectrum auction.

**Representation of Valuation and Bidding Functions.** In a spectrum auction of channels of same type, a bidder $i$'s valuation of an outcome/allocation $o$ depends only on the number of channels $i$ is getting in $o$. Thus, we represent bidder $i$'s valuation function $\mathbf{v}_i$ as $\mathbf{v}_i : \{1, \ldots, M\} \mapsto \mathbb{R}$, where $M$ is the total number of channels and $\mathbf{v}_i(q)$ denotes bidder $i$'s value for obtaining $q$ channels. Recall that the bidding function $\mathbf{w}_i$ for a bidder $i$ is a declaration of its privately-known valuation function $\mathbf{v}_i$. Thus, the bidding function $\mathbf{w}_i$ is represented similarly as $\mathbf{w}_i : \{1, \ldots, M\} \mapsto \mathbb{R}$. We assume free disposal (i.e., valuation for higher number of channels is larger than smaller number of channels), and that valuation of zero channels is zero.

<u>General-Minded and $k$-minded Bidding Functions.</u> In the most general model, a bidder has a valuation for any number of channels, and thus, the bidding functions is represented by $M$ real numbers – one for each quantity of channels. For efficiency and practicality issues, another model is commonly assumed in

the literature, viz., the $k$-minded bidding function, wherein the bidder expresses its valuations for at most $k$ quantities of channels.

**TSA-MSW (Truthful Spectrum Auctions with Maximum Social-Welfare) Problem.** Given an interference graph, *number* of channels, and the bidding functions for the base stations, the *TSA-MSW problem* is to design a truthful auction mechanism that returns a valid spectrum allocation with maximum social-welfare.

Thus, the TSA-MSW problem involves determining (i) a *valid* spectrum allocation with optimal social-welfare, and (ii) payments by each bidder, so that the overall mechanism is truthful. TSA-MSW problem is NP-hard even without the truthfulness objective [56]. Thus, we focus on designing a mechanism that is truthful and yields a valid spectrum allocation with approximate social-welfare.

Input and Output Sizes. If $N$ and $M$ denote the number of base stations and channels respectively, then note that the size of the input is polynomial in $N$ and $\log M$ (since the input only includes the *number* of channels). On the other hand, the size of the output as defined in Definition 4 may be polynomial in $N$ and $M$, and thus, exponential in the input size. However, it is easy to modify Definition 4 so that valid spectrum allocation is polynomial in $N$ and $\log M$, by associating a *number* of channels with each base station. We have defined valid spectrum allocation as in Definition 4 for simplicity of presentation.

### 3.3.1 TSA-MSW Problem in Unit-Disk Model

As defined in Section 1.1.1, in the unit-disk model, the coverage region of each base station is assumed to be a disk of uniform radius $d$. For simplicity of presentation, we assume distances to be normalized, i.e., $d = 1$. Thus, two base stations interfere if they are within two-unit distance from each other. We start with giving an outline of our allocation algorithm (i.e., the winner determination function) for the unit-disk model. Then, we will discuss various parts of the algorithm in detail, and prove its truthfulness and social-welfare approximation ratio.

**Outline of the Allocation Algorithm.** Our approach utilizes the geographical nature of the spectrum auction problem to divide it into smaller and more tractable subproblems. Then, we solve each subproblem independently and "combine" the allocations, without sacrificing much on the overall approximation factor of the final social-welfare. At a high-level, our algorithm consists of the follows steps.

1. Divide the entire network region into small hexagons[2] of side-length one unit each. See Figure 4. This division ensures that any pair of base stations in the same hexagon interfere with each other (due to the unit-disk interference model).

2. Uniformly-color the hexagons with enough colors, such that base stations in co-colored hexagons are more than two-unit distance away and hence do not interfere.

3. Allocate channels to base stations in each hexagon *independently*, treating it as a multi-unit auction (MUA) and using techniques similar to [18]. Note that the interference subgraph in each hexagon is actually a complete graph.

4. For each color, combine the results from all hexagons of that color.

5. Pick the color that has the highest total social-welfare.

The above gives a $(7\gamma)$-approximate solution, where $\gamma$ is the approximation factor in Step (3), and 7 is the number of colors used to color the hexagons. The value of $\gamma$ is 2 for general-minded bidding, and $(1+\epsilon)$ for $k$-minded bidding functions for any $\epsilon > 0$. Furthermore, the above algorithm, and all the generalizations discussed afterwards, run in time polynomial in the size of the input, i.e., in $N$, the number of base stations, and $\log M$, where $M$ is the number of channels.

**Hexagonal Division, and 7-Coloring of Hexagons.** Our Algorithm starts by diving the plane into hexagons of side-length one unit each (creating a

[2]This hexagonal division is not to be confused with the actual *cells* associated with the base stations.

hexagonal division of the plane), and proceed to uniformly coloring these hexagons using 7 colors. See Figure 11. In such a coloring, the following two properties hold.

(P1) Every pair of base stations in the same hexagon interfere with each other (i.e., are connected by an edge in the interference graph).

(P2) Base stations in different hexagons with same color do not interfere with each other (i.e., are *not* connected by an edge in the interference graph).

Property (P1) follows directly from the definition of unit-disk interference, while Property (P2) follows from the fact that the distance between base stations in different hexagons with the same color will be at least $(\sqrt{3(7)}-2) > 2$.

**Allocation in Each Hexagon.** The above properties imply that the channels cannot be re-used inside the same hexagon, but can be fully re-used across different hexagons of the same color. Thus, allocation in each hexagon can be treated as an MUA, and using techniques similar to [18], we can design an MIR mechanism with approximate social-welfare. Below, we describe their technique in detail for general-minded and $k$-minded functions.

General-Minded Bidding Function. In case of general-minded bidding model, the available $M$ channels are split into $N_H^2$ bundles of size $\left\lfloor \frac{M}{N_H^2} \right\rfloor$ each, where $N_H$ is the number of base stations in hexagon $H$, and a single bundle of remaining channels. Using dynamic programming, we can *optimally* allocate these bundles to the $N_H$ bidders in time polynomial in $N_H$. The above approach yields an allocation whose social-welfare is at least 1/2 of the optimal possible (see [18]).

$k$-minded Bidding Function. In the case of $k$-minded bidding function, a restricted form of allocation known as the *$t$-round* allocation is used. For a given $t$ (where $t$ is a PTAS parameter), a $t$-round allocation allocates $l$ ($l \leqslant M$) channels to a subset $T$ of the bidders where $|T| \leqslant t$; this part of the allocation is done optimally by exhaustive search for each $l$ and $T$. Also, for each $l$ and $T$, the remaining $(M - l)$ channels are divided into equi-sized bundles and distributed optimally to the remaining bidders using dynamic programming,

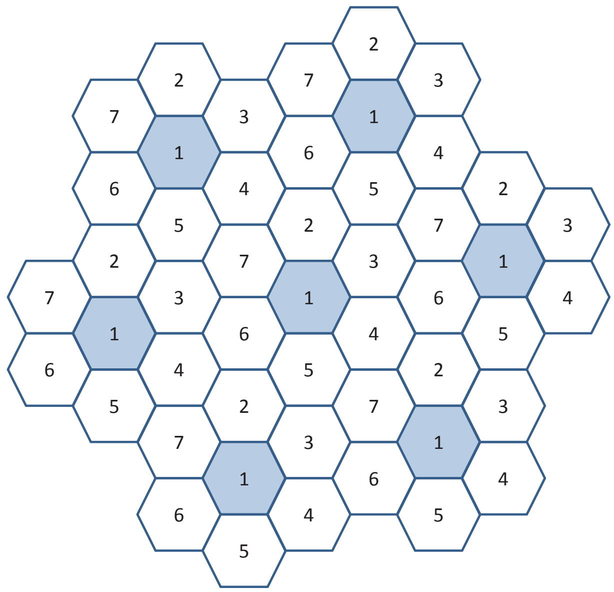


Figure 11: Hexagons uniformly-colored using 7 colors.

as in the case of general-minded bidding model. Finally, the best allocation among the $tN_H^t k^t$ such allocations is picked as the optimal $t$-round allocation. The above allocation algorithm runs in $O(tN_H^t k^t (N_H - t)^4)$ time, which is polynomial for a fixed $t$, and yields a $\left(1 - \frac{1}{t+1}\right)$-approximate allocation [18].

Note that instead of the above bundling approach, we could also use the optimal dynamic programming approach which runs in $O(M^4 N_H)$ time; since the size of the input is $O(\log M)$, this optimal dynamic programming algorithm has a pseudo-polynomial time complexity. In our simulations, we observe that this optimal algorithm does not perform any better than our above polynomial algorithms based on creating bundles.

**Combining The Results.** Since base stations in different hexagons of same color do not interfere with each other (Property (P2)), we can combine allocations of co-colored hexagons to form one single allocation. Thus, we get seven allocations, one for each color. Among these seven allocations, we pick the allocation with the highest social-welfare, as our final solution.

**Proof of Truthfulness and Approximability.** Our overall spectrum auction mechanism consists of the above described allocation algorithm (winner determination function) combined with VCG payments (as described in Definition 12). In the below theorem, we prove that this overall auction mechanism is truthful and returns a valid spectrum allocation with approximate social-welfare.

**Theorem 4** *For the TSA-MSW problem under the pairwise interference with unit-disk model, the above described auction mechanism is truthful and returns a valid spectrum allocation whose social-welfare is 14-approximate for the general-minded bidding model and is* $7(1+\epsilon)$*-approximate for the* $k$*-minded bidding model for a given* $\epsilon > 0$*.*

*Proof:* Truthfulness. Our allocation algorithm picks a $t$-round allocation with the highest social-welfare, for a given $t$ (for the case of general-bidding functions, $t$ can be considered to be zero). Thus, our allocation algorithm is maximal in its range, where the range of allocations/outcomes is restricted to $t$-round allocations. Thus, our auction mechanism is truthful since MIR allocations with VCG payments are truthful [49].

Approximate Social-Welfare. First, note that by the properties (P1) and (P2) of the hexagonal division, the allocation returned by our algorithm is valid. Now, let us prove the approximation factor for the general-minded bidding model; the proof for $k$-minded bidding model is similar. Consider a particular color $c$, and for the set of all hexagons colored $c$, let $\mathcal{A}_c$ be the allocation constructed by our algorithm and $\mathcal{O}_c$ be the allocation with optimal social-welfare. We show that the social-welfare of $\mathcal{A}_c$ is within a factor of 2 of that of $\mathcal{O}_c$. Note that, for any particular hexagon cell, our algorithm constructs an allocation whose social-welfare is within a factor of 2 of the optimal for that hexagon. Since $\mathcal{A}_c$'s ($\mathcal{O}_c$'s) social-welfare is the sum of the social-welfares of the constructed (optimal) allocations for the individual $c$-colored hexagons, we get that the social-welfare of $\mathcal{A}_c$ is within a factor of 2 of that of $\mathcal{A}_c$. Now, since there are seven colors and we pick the best of the seven allocations, the social-welfare of the returned allocation is within a factor of 14 of the overall optimal social-welfare. ∎

### 3.3.2 TSA-MSW Problem in Non-Uniform Disks Model

We now extend our techniques of previous subsection to the *non-uniform disk* model, wherein cells of the base stations are disks of possibly different radii. As before, two base stations are considered to interfere if their cells intersect. Let the maximum and the minimum disk radii in the network be $d_{\max}$ and $d_{\min}$ respectively. For simplicity of presentation, we assume that the distances are normalized, i.e., $d_{\min} = 1$.

For the above disk model, we divide the base stations into classes depending upon their cell's radius, and then solve the spectrum allocation problem for each class independently. Finally, we pick the allocation of the class that has the highest social-welfare. Thus, our algorithm consists of the following steps.

1) Classify the base stations into $\lfloor \log(d_{\max}) \rfloor$ *radius-classes*, based on their cell's radius. In particular, class $L$ contains base stations whose cell's radius $d_i$ lie in the range $d_i \in [2^L, 2^{L+1})$.

2) For each radius-class $L$:

   (a) Divide the network region into hexagons of side-length $2^L$ each.

   (b) Uniformly-color the hexagons using 12 colors as shown in Figure 12.

   (c) Independently, for each hexagon $H$, allocate channels to base stations of radius-class $L$ contained in $H$. The interference subgraph induced by these base stations is a complete graph, and thus, we can use the same technique as for the unit-disk model.

   (d) For each color, combine the results from all hexagons of that color. Note that base stations of class $L$ in different co-colored hexagons do not interfere with each other.

   (e) Pick the color that has the highest total social-welfare.

3) The above gives an allocation for each radius-class. Pick the allocation for the radius-class that has the highest social-welfare.

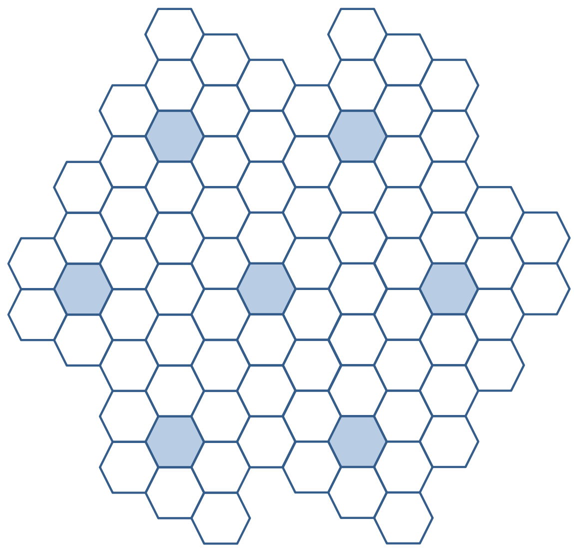


Figure 12: Hexagons uniformly-colored using 12 colors.

**Theorem 5** *For the TSA-MSW problem under non-uniform disks interference model, the auction mechanism based on the above allocation algorithm and VCG payments is truthful and returns a valid spectrum allocation whose social-welfare is* $24\lfloor\log(d_{\max})\rfloor$*-approximate for the general-minded bidding model and is* $12\lfloor\log(d_{\max})\rfloor(1+\epsilon)$*-approximate for the* $k$*-minded bidding model for a given* $\epsilon > 0$.

*Proof:* The truthfulness of the auction mechanism follows from the same arguments as in the proof of Theorem 4.

Validity. Validity of the returned allocation follows from the fact that for each radius-class $L$, the base stations of the radius class $L$ satisfy the Property (P2) of previous subsection. Note that with 12-coloring of hexagons of side-length $2^L$ each, the distance between base stations in different co-colored hexagons is at least $(\sqrt{3(12)} - 2)2^L = 2(2^{L+1})$, which is not close enough to create interference between base stations in radius-class $L$.

Approximate Social-Welfare. The proof of approximation follows from similar arguments as in the proof of Theorem 4, except for the fact that we use 12 colors here (instead of 7) and the extra $\lfloor \log(d_{\max}) \rfloor$ factor comes due to the number of radius-classes considered independently. ∎

### 3.3.3 TSA-MSW Problem in Pseudo-Disk Model

We now extend our techniques to the most general case of *pseudo-disk model* model, wherein cells of the base stations have irregular shapes but are contained within a disk of radius $d_1$ while containing a disk of radius $d_2 \leqslant d_1$. See Figure 1. For simplicity of presentation, we assume that $d_1$ and $d_2$ are the same for all base stations. Techniques of previous subsection can be used to extend our results below to the case wherein $d_1$ and/or $d_2$ may be different for different cells. Also, for clarity of presentation, we use $d_2 = 1$.

The allocation algorithm for the pseudo-disk model is similar to the one for unit-disk model, except that the side-length of the hexagons and the coloring scheme are different. To ensure the correctness of the unit-disk approach in the context of pseudo-disk model, we need to do the division and coloring appropriately to ensure that Properties (P1) and (P2) of Section 3.3.1 hold. To ensure Property (P1), we divide the network region into hexagons of side-length one unit, as in the case of unit-disk model. Below, we compute the number of colors required to *uniformly* color the hexagons, in order to satisfy Property (P2).

Required Number of Colors. To satisfy Property (P2), i.e., to ensure that base stations in different hexagons with the same color do not interfere, we must color the hexagons in a way that the distance between any two points in different hexagons of the same color is greater than $2d_1$. To estimate the number of colors required, we make use of the following two lemmas from [40, 55].

**Lemma 3** *In a hexagonal division with side-length $s$ and uniformly-colored*[3] *with $x$ colors, the distance between the centers of two hexagons of the same color is at least $\sqrt{3x}s$.* □

[3] *Informally, in a uniform-coloring of hexagons, the distance between the "closest" hexagons with the same color is uniform.*

**Lemma 4** *A hexagonal division can be uniformly colored using $c$ colors if and only if $c$ is of the form $i^2 + j^2 + ij$ for some positive integers $i$ and $j$.* □

Now, by Lemma 3 above, to ensure a distance of $2d_1$ between co-colored hexagons, the number of colors must be at least $4d_1^2/3$. Then, by Lemma 4 above, the minimum number of colors required would be given by:

$$q_w = \min\{x | x \geqslant 4d_1^2/3 \text{ and } x = i^2 + j^2 + ij \text{ where } i, j \in \mathbb{Z}^+\}. \qquad (26)$$

Approximate Social-Welfare. Using arguments similar to before, the above hexagonal division and coloring yields an allocation algorithm with approximation factor of $2q_w$ and $q_w(1 + \epsilon)$ for the general-minded and $k$-minded bidding models respectively.

### 3.3.4 Non-Orthogonal Channels; Multiple Types of Channels

Thus far in this chapter, we have assumed that the channels in our network model are orthogonal and of the same type. Here, we discuss relaxation of these two assumptions.

**Non-Orthogonal Channels.** As discussed in Section 2.3, the non-orthogonal (overlapping) nature of channels can be modeled using a channel graph $G_c$ over channels as vertices, wherein there is an edge between two channels $c_i$ and $c_j$ if they are non-orthogonal. Let $I$ be a maximum independent set in $G_c$. If we can somehow compute $I$, then we can just use $I$ as the set of channels to allocate in *each* hexagon of our technique; this will maintain our approximation ratios because of the following two facts. First, reuse of $I$ in different hexagons can be done without any changes to our techniques, since either they have no interference edges between them or they are colored differently. Second, within any hexagon, using $I$ is sufficient, since all the channels are of the same type and no two base stations within a hexagon can share a channel. Thus, the computation of a maximum independent set in $G_c$ is sufficient to use our techniques and preserve the approximation guarantees. Now, if the channels are contiguous blocks of spectrum, then the channel graph $G_c$ is an "interval graph" wherein the maximum independent set (MIS) can be easily computed using a simple greedy approach. For non-contiguous channels, our techniques can still be used, but the performance guarantees do not hold.

**Orthogonal Channels of Multiple Types.** We have assumed till now in this chapter that the available channels are of the same type; thus, each base station had valuations for *quantities* of channels without any preference for specific channels. Such a model allowed us to treat the allocation subproblems in each hexagon as multi-unit auctions (MUA). However, if we have multiple types of channels, and bidders have valuations for various *sets* of channels, then the allocation subproblem in each hexagon must be instead treated as a Combinatorial Auction (CA) problem, which is more difficult than the MUA problem. For the CA problem, the best known result is by Holzman et al. [30] who give a truthful MIR mechanism that achieves $O(\frac{M}{\sqrt{\log M}})$ of the maximum social-welfare for general-minded bidding model, where $M$ is the total number of items. Thus, for spectrum auction of multi-type channels, we can use [30]'s allocation algorithm to allocate channels in each hexagon, and combine results as before. The resulting auction mechanism will be truthful and yield a $O(\frac{M}{\sqrt{\log M}})$-approximate social-welfare, where $M$ is the total number of channels.

## 3.4 TSA-MSW Problem Under Physical Interference

In this section, we extend our techniques from the previous section to the case of physical interference. We start with considering the uniform-power transmission model.

### 3.4.1 Uniform Transmission Powers

In this subsection, we assume that each base station operates using the same transmission power $P$; we relax this assumption in the next subsection. We start by redefining the concept of valid spectrum allocation in this context.

**Valid Spectrum Allocation.** In the context of physical interference model, a spectrum allocation $\mathcal{A}$ is considered valid if it satisfies the following condition. For each pair $(i, c)$ in $\mathcal{A}$, let $V_{i,c}$ denote the set of base stations that have been allocated the channel $c$ in $\mathcal{A}$. Now, for $\mathcal{A}$ to be valid, for every $(i, c)$ in $\mathcal{A}$ and every point $p$ within a distance of $r$ from $i$, SINR at $p$ due to $i$ and $c$ should be greater than $\beta$; i.e, the following should hold:

$$\frac{P/d_i^\alpha}{\mathcal{N} + \sum_{j \in V_{i,c}} P/d_j^\alpha} \geqslant \beta$$

where $d_x$ is the distance of base station $x$ from the point $p$.

**Allocation Algorithm.** The allocation algorithm for physical interference model is similar to the one for pairwise interference model in the previous section, except for the chosen side-length of the hexagons and the number of colors used for uniform-coloring of the hexagons. To ensure correctness of our approach in the context of physical interference, we need to do the hexagonal division and coloring in such a way that the following two properties are satisfied.

(P'1) Every pair of base stations $b_1$ and $b_2$ in the same hexagon must "interfere" with each other when operating on the same channel, even if no other base station is active. In other words, no valid spectrum allocation must assign the same channel to $b_1$ and $b_2$.

(P'2) If in each hexagon with the same color there is at most one active base station, then the transmission from each of these base stations must be successful within their communication radius.

To ensure Property (P'1), we can just divide the network region into hexagons of side-length

$$R = \frac{(\sqrt[\alpha]{\beta} + 1)r}{2}. \tag{27}$$

It is easy to see that Property (P'1) is satisfied for the above hexagonal division.

Coloring Hexagons to Satisfy Property (P'2). In the below Lemma 5, we will show that Property (P'2) can be satisfied by ensuring that the minimum distance between hexagons with the same color is at least $\sqrt{3q_h'}R$, where $R$ is as defined above and $q'_h$ is as defined below.

$$q'_h = \left(\frac{4\sqrt{7}}{(3\sqrt{7}-6)(\sqrt[\alpha]{\beta}+1)}\right)^2 \left(\frac{6\beta}{(\alpha-2)}\right)^{\frac{2}{\alpha}}.$$

Then, using arguments similar to Section 3.3.3, the number of colors required to satisfy Property (P'2) is given by:

$$q_h = \min\{x | x \geqslant q'_1, x \geqslant 7, \text{ and } x = i^2 + j^2 + ij \text{ where } i, j \in \mathbb{Z}^+\} \tag{28}$$

Note that in our context we should use at least 7 colors, irrespective of the values of $\alpha$ and $\beta$. We now state and prove the Lemma 5 used in the above argument.

**Lemma 5** *Property (P'2) is satisfied if the minimum distance between hexagons with the same color is at least $\sqrt{3q'_h}R$, where $R$ and $q'_h$ are as defined above.*

*Proof:* Consider a base station $i$ in a hexagon $H$ of color $\mathcal{C}$. Partition all $\mathcal{C}$-colored hexagons surrounding $H$ into hierarchical levels. In a uniform-coloring, the first level will contain 6 hexagons of color $\mathcal{C}$ and each such hexagon $H'$ is at distance[4] of at least $(\sqrt{3q_h} - 2)R$ from $H$ (from Lemma 3). Similarly, the second level contains 12 hexagons at a distance of at least $(3\sqrt{q_h} - 2)R$ from $H$. In general, the $l^{th}$ level contains $6l$ hexagons at a distance of at least $(\frac{3}{2}\sqrt{q_h}l - 2)R$ from $H$.

Now consider a point $p$ within the communication radius $r$ from the base station $i$. Then, the total signal received at the point $p$ due to all other base stations (at most one per $\mathcal{C}$-colored hexagon) active on the same channel as $i$

[4]By distance between two hexagons we mean that the distance between *any* point in $H'$ and *any* point in $H$.

is at most:

$$\begin{aligned}
&\sum_{l=1}^{\infty} 6l \cdot \frac{P}{\left(\frac{(\frac{3}{2}\sqrt{q_h}l-2)(\sqrt[\alpha]{\beta}+1)}{2} - 1\right)^{\alpha} r^{\alpha}} \\
\leqslant\ & \frac{6P}{r^{\alpha}} \sum_{l=1}^{\infty} \frac{l}{\left(\frac{l\sqrt{q_h}(3\sqrt{7}-4)(\sqrt[\alpha]{\beta}+1)}{4\sqrt{7}} - 1\right)^{\alpha}} \\
\leqslant\ & \frac{6P}{r^{\alpha}} \sum_{l=1}^{\infty} \frac{l}{\left(\frac{l\sqrt{q_h}(3\sqrt{7}-6)(\sqrt[\alpha]{\beta}+1)}{4\sqrt{7}}\right)^{\alpha}} \\
=\ & \frac{6P}{\alpha-2} \cdot \left(\frac{4\sqrt{7}}{\sqrt{q_h}(3\sqrt{7}-6)(\sqrt[\alpha]{\beta}+1)r}\right)^{\alpha}.
\end{aligned}$$

Above, the second equation follows from the following two facts: (i) $\frac{3}{2}\sqrt{q_h}l \geqslant 3\sqrt{7}/2$ (since $q_h \geqslant 7$), and (ii) for $x \geqslant 3\sqrt{7}/2$, we have $(x-2) \geqslant \frac{x}{3\sqrt{7}/(3\sqrt{7}-4)}$. And, the third equation follows from the following facts: (i) $l\sqrt{q_h}(3\sqrt{7}-4)(\sqrt[\alpha]{\beta}+1)/4\sqrt{7} \geqslant (3\sqrt{7}-4)/2$, since $q_h \geqslant 7$ and $\sqrt[\alpha]{\beta} \geqslant 1$, and (ii) for $x \geqslant (3\sqrt{7}-4)/2$, $(x-1) \geqslant \frac{x}{(3\sqrt{7}-4)/(3\sqrt{7}-6)}$.

For simplicity, we assume ambient noise to be zero; non-zero noise can be incorporated using techniques similar to Section 2.4. Now, using the value of $q_h$ from Equation 28, the SINR at point $p$ due to the transmission at base station $i$ is at least:

$$\frac{P}{r^{\alpha}} \cdot \frac{\alpha-2}{6P} \cdot \left(\frac{\sqrt{q_h}(3\sqrt{7}-6)(\sqrt[\alpha]{\beta}+1)r}{4\sqrt{7}}\right)^{\alpha} \geqslant \beta$$

∎

Overall Allocation Algorithm. In the above paragraph, we discussed hexagonal division and its coloring in a uniform way so as to satisfy Properties (P'1) and (P'2). Now, note that Property (P'1) ensures that allocation in each hexagon can be treated as a multi-unit auction, while Property (P'2) allows us to re-use channels across different hexagons with same color. Thus, we can use the same allocation algorithm as for the unit-disk model, with the above hexagonal division and coloring. Use of VCG payments yields an overall truthful auction mechanism. Thus, we have the following theorem.

**Theorem 6** *For the TSA-MSW problem under the physical interference model with uniform transmission power, the auction mechanism based on the above described allocation algorithm and VCG payments is truthful and returns a valid spectrum allocation whose social-welfare is $2q_h$-approximate for the general-minded bidding model and is $q_h(1+\epsilon)$-approximate for the k-minded bidding model for a given $\epsilon > 0$. Here, $q_h$ is as defined in Equation 28.* □

### 3.4.2 Non-Uniform Transmission Power

We now consider the model wherein different base stations may operate on different transmission powers. Let the maximum and the minimum transmission power levels used in the network be $P_{\max}$ and $P_{\min}$ respectively. For simplicity of presentation, we assume that transmission powers are normalized, i.e., $P_{\min} = 1$. For this non-uniform transmission model, the physical interference model and valid spectrum allocations can be appropriately defined. For simplicity, we assume uniform communication radius; *non-uniform communication radii can be handled in the similar manner as non-uniform disks in Section 3.3.2.*

**Allocation by Division into Power Classes.** Our technique is somewhat a combination of the technique for the uniform-power physical interference model and the non-uniform disk model. Basically, we divide the base stations into power-classes depending upon their transmission power, and then, solve the allocation problem for each class independently. Finally, we pick the power-class that has the allocation with the highest social-welfare. The outline of our allocation algorithm is as follows.

1) Classify the base stations into $\lfloor \log(P_{\max}) \rfloor$ *power-classes*, based on the associated transmission power. In particular, power-class $J$ contains base stations whose transmission power $P_i$ lies in the range $P_i \in [2^J, 2^{J+1})$.

2) For each power-class $J$,

   (a) Divide the network region into hexagons of side-length $R = (\sqrt[\alpha]{\beta} + 1)r/2$ each.

(b) Uniformly-color the hexagons using $q_g$ colors, where $q_g$ is as defined in Equation 29 later.

(c) Independently, for each hexagon $H$, allocate channels to base stations of power-class $J$ contained in $H$. These set of base stations satisfy Property (P'1), and hence, this allocation subproblem is a multi-unit auction and we can use techniques described in Section 3.3.1.

(d) For each color, combine the results from all hexagons of that color, since it can be shown that base stations of class $J$ in different co-colored hexagons satisfy Property (P'2).

(e) Pick the color that has the highest total social-welfare.

3) The above gives an allocation for each power-class. Pick the allocation for the power-class that has the highest social-welfare.

We need to define $q_g$ as:

$$q_g = \min\{x | x \geqslant q'_g, x \geqslant 7, \text{ and } x = i^2 + j^2 + ij \text{ where } i, j \in \mathbb{Z}^+\}, \tag{29}$$

where $q'_g$ is

$$q'_g = \left(\frac{4\sqrt{7}}{(3\sqrt{7}-6)(\sqrt[\alpha]{\beta}+1)}\right)^2 \left(\frac{12\beta}{(\alpha-2)}\right)^{\frac{2}{\alpha}}.$$

Now, using arguments similar to the uniform-power physical interference and non-uniform disk model, the following result follows.

**Theorem 7** *For the TSA-MSW problem under the physical interference model with non-uniform transmission power, the auction mechanism based on the above described allocation algorithm and VCG payments is truthful and returns a valid spectrum allocation whose social-welfare is $2q_g$-approximate for the general-minded bidding model and is $q_g(1+\epsilon)$-approximate for the $k$-minded bidding model for a given $\epsilon > 0$. Here, $q_g$ is as defined in Equation 29.* □

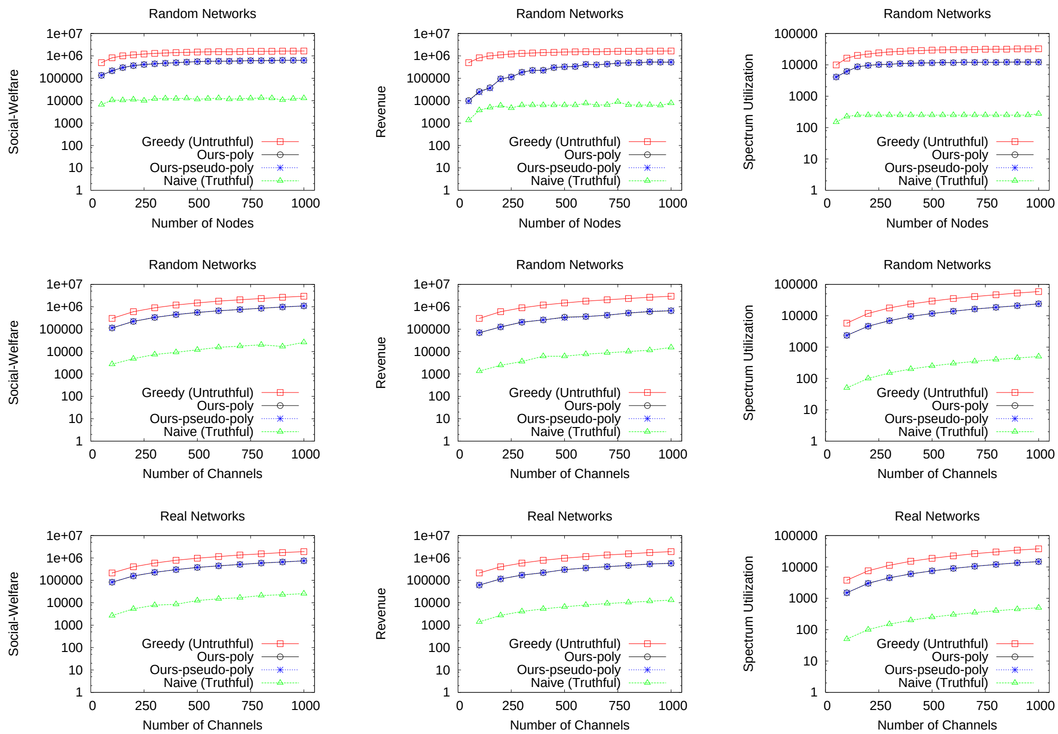



Figure 13: Performance comparison of various auction mechanisms. The first six plots (in the first two rows) are for random networks with varying number of nodes (with 500 channels) and varying number of channels (with 500 nodes). The last three plots are for the cellular network in Massachusetts with 843 base stations and varying number of channels. Recall that `Ours-poly` and `Ours-pseudo-poly` refer to the polynomial-time and the optimal pseudo-polynomial time versions of our auction mechanism.

## 3.5 Simulation

In this section, we present our simulation results. The main purpose of our simulations is to demonstrate the efficiency of our designed auction mechanism in terms of multiple performance metrics. We start by describing our simulations set-up.

**Network Topology and Model.** In our simulations, we consider only unit-disk pairwise interference model because for physical interference model, there is no simple truthful auction mechanism known that we could use for comparison. We consider two types of networks, as described below.

- *Random Networks:* We randomly place base stations within a fixed area of $1000 \times 1000$ square units. We vary the network density by varying the number of base stations from 50 to 1000 (with the default being 500). We use cells of uniform radius of 50 units.

- *Real Networks:* We use locations of real cellular base stations available in FCC public GIS database [1] and choose the 843 base stations deployed in the state of Massachusetts. Here, we choose a realistic cell radius of 10 kilometers.

In both networks, we set up an auction of up to 1000 orthogonal single-type channels with the default being 500 channels; this is a reasonable range based on the past FCC auctions [16, 3].

**Bidding Functions.** We generate general-minded bidding functions for each base station $i$ as follows. First, we randomly choose a non-zero *demand* for $i$, which is the maximum number of channels $i$ is interested in. Then, we randomly generate $i$'s bid for the first channel and "marginal" bids for each additional channel until the demand is satisfied. Beyond the demand, marginal bids for each additional channel is assigned zero (to satisfy the free-disposal property). Each marginal bid is chosen from the range $[0, 100]$. The above scheme results in valid general-minded bidding functions.

**Auction Mechanisms Compared.** We compare our auction mechanism with two auction mechanisms, viz., (i) Greedy, the best known (non-truthful) approximation spectrum allocation algorithm for maximizing social-welfare and/or revenue, and (ii) Naive, a simple truthful spectrum auction mechanism. Note that the only work on truthful spectrum auction mechanism is by Zhou et al. [67] which is restricted to simple single-minded or range bidding functions. In addition, we also consider *two* versions of our auction mechanisms, viz., one based on a polynomial-time allocation algorithm in each hexagon, and the other based on the optimal pseudo-polynomial time algorithm. We refer to these two versions as `ours-poly` and `ours-pseudo-poly` respectively in the plots.

Greedy Auction Mechanism. Greedy (presented in Section 2.3.1) is a non-truthful auction mechanism whose social-welfare as well as revenue is within a factor of 6 of the respective optimals, for the unit-disk model and non-complementary bidding function – thus, it is the best and most general approximation algorithm known. Greedy's winner determination function allocates channels iteratively to the highest available bid without violating the interference constraint; this allocation results in a 6-approximate social-welfare [56]. If we charge each bidder a payment equal to its bid (declared valuation) for the allocated number of channels, then Greedy's revenue is also within a factor of 6 of the optimal revenue possible.[5] Note that Greedy's social-welfare is equal to its revenue.

Naive Auction Mechanism. We now describe a simple auction mechanism (called Naive) that is truthful, but has no performance guarantee on the social-welfare and revenue. The Naive auction mechanism is loosely based upon the Naive auction mechanism suggested by Zhou et al. [67] and used as a comparison-benchmark in their work. Naive's allocation algorithm divides the entire network region into square grid of unit side-length.[6] Next, the algorithm uniformly colors the resulting square cells using 4 colors, and assigns each color $(1/4)^{th}$ of the available channels. This means that all square cells of the same color will use the same channels. Now, for each square cell $H$, Naive allocates all the channels usable in $H$ to the bidder with the maximum bid for that many channels, and charges it a payment equal to the second highest bid in $H$. This is a simple generalization of Vickrey's auction [61] (see Section 3.2) in each square cell. Finally, we note here that Greedy is a pseudo-polynomial algorithm since its running time is polynomial in $M$, the number of channels, while our algorithm and Naive are polynomial in $\log(M)$.

**Simulation Results.** In our simulation, we compare Greedy, Naive, and Our (based on hexagonal division and coloring) auction mechanisms for the following three performance metrics: (i) social-welfare, (ii) revenue, and (iii)

[5] For computing the optimal revenue, we assume that bidder's payment in an outcome must not be more than its declared valuation for the outcome.

[6] To ensure validity of the resulting allocation, the square cells are open from one side and closed from the other (similar to Figure 4).

spectrum utilization. The *spectrum utilization* [67] is defined as the total number of allocation pairs in the spectrum allocation (see Definition 4). Spectrum utilization gives a measure of the spatial reuse of a spectrum allocation.

In Figure 13, we plot results for the above three metrics. For the random network, we vary the number of base stations (nodes) as well as the number of available channels, while for the fixed real network we only vary the number of available channels. We observe that Greedy performs the best in all three performance metrics, but is only within a factor of 2 to 3 of that of our auction mechanism. Note that both Greedy and ours deliver an approximate social-welfare, and Greedy also delivers an approximate revenue, but is untruthful. Secondly, our auction mechanism outperforms the Naive mechanism by an order of magnitude, in all three performance metrics. Finally, we note that the difference in performance of the polynomial-time and pseudo-polynomial time algorithms is negligible.

Thus, apart from the key properties of truthfulness and provably approximate social-welfare, our auction mechanism also delivers near-optimal revenue in practice. The simulation results also show that a Naive truthful auction mechanism can perform very badly.

## 3.6 Conclusion

The recent trend of dynamic spectrum access in cellular networks creates a setting for auctioning of pieces of wireless spectrum to competing base stations. To mitigate market manipulation, a truthful spectrum auction is highly desired, so that bidders can simply bid their true valuations. For economic efficiency, we also want to allocate channels to bidders that value them the most. Thus, in this chapter, we have designed a truthful spectrum auction that delivers an allocation with approximate social-welfare, for general interference and bidding models. Through simulations, we show that the revenue generated by our auction mechanism is also within a factor of 2-3 of the best-known approximation algorithm. In general, our mechanism performs an order of magnitude better than a Naive truthful spectrum auction mechanism.

# Chapter 4

# Truthful Auctions With Approximate Revenue

## 4.1 Introduction

A natural objective of auction-based mechanism is to maximize the generated *revenue* (the sum of the payments by the buyers) [56, 22, 52, 31]. However, as discussed in Chapter 3, such an objective alone can encourage the spectrum buyers to lie about their real valuations leading to an untruthful auction, fear of market manipulation, and indirectly possibly lowered revenue. Moreover, in a competitive environment, buyers may spent a lot of time/effort in predicting the behavior of other buyers and planning against them. In this chapter, our focus is on design a spectrum auction mechanism that not only encourages truthful behavior but also provides some form of guarantee on the revenue. Since it is not possible (see Section 3.2.1) for a *truthful* auction mechanism to give *any* guarantees on revenue, when the bidder valuations are completely private, we consider the relaxed *Bayesian* setting wherein the bidder valuations are independently drawn from publicly known probability distributions.

**Problem Addressed.** As in Chapters 2 and 3, we consider a dynamic auction-based approach to allocate spectrum to competing base stations. The centralized auctioneer acts as the *seller* and the base stations act as the *buyers* of the available spectrum. The items being sold are various channels corresponding to certain blocks of frequency. The base stations bid for these channels, based on their valuations of these channels. In the Bayesian setting [50], the valuations are assumed to be independently drawn from known distributions.

In the above context, we address the problem of designing an efficient (polynomial-time) spectrum auction mechanism in the Bayesian setting with the following *dual* objectives, viz., (i) encourage truthful behavior from the buyers (see Section 3.1), and (ii) maximize the expected revenue. The auction mechanism designed in this chapter is truthful and yields an allocation with $O(1)$-approximate expected revenue. We consider general (pairwise and physical) interference and bidding models.

**Chapter Organization:** The rest of the chapter is organized as follows. In the next section, we present the background of our work before going into the details of the related works. In Section 4.3, we formally define and present computationally efficient spectrum auction mechanisms under simple settings/assumptions. We discuss various extensions in Section 4.4 before concluding the chapter with Section 3.6.

## 4.2 Background and Related Works

In Section 3.2, we introduce basic terms and definitions from both the spectrum allocation and the auction theory literature. Here, we add more background material with more relevance to this chapter. Specifically, we discuss the Bayesian Setting and the classical works related to it. We start with basic definitions.

**Single-parameter Auctions.** In a *single-parameter* auction, each bidder $i$ has a publicly-known set of outcomes $O_i \subset O$ known as its *winning alternatives* and a private value $v_i$ such that $\mathbf{v}(o) = v_i$ for every $o \in O_i$ and $\mathbf{v}(o) = 0$ for every $o \notin O_i$.

(Valid) Allocation Vector. Also, in a single-parameter auction, each outcome can be represented by an *allocation vector* of $n$ binary variables $x =$

$(x_1, \ldots, x_n)$, where $x_i$ is 1 if the bidder $i$ wins and zero otherwise. However, not all 0-1 vectors of length $n$ may correspond to an outcome of the mechanism. The 0-1 vectors that correspond to an outcome are referred to as *valid allocation vectors*. For instance, in a single-item auction with 4 bidders, wherein the item is given to one of the 4 bidders, (0,0,0,1) is a valid allocation vector while (0,1,1,0) is not a valid allocation vector.

**Bayesian Setting.** In traditional auction setting, the bidders' valuations are privately-known information which makes it impossible for *truthful* auctions to make any guarantees on the generated revenue [3, 24]. To circumvent this, researchers have considered the *Bayesian setting* wherein each bidder's valuation $v_i$ is drawn from a known probability distribution $F_i$ [50].

**Myerson's Optimal Mechanism.** In a seminal work [46], Myerson presents a truthful optimal mechanism for a single-item auction under the Bayesian setting. Here, we briefly present the key points of Myerson's mechanism applied to the more general single-parameter auctions (based on Chapter 13 of [50]). Given, for each bidder $i$, the winning alternatives $O_i$, declared valuation value $w_i$ for outcomes in $O_i$, and the distribution $F_i$ of the private valuation value $v_i$, the mechanism finds an allocation vector and payments such that truthfulness is maintained and the expected revenue is optimal where the expectation is taken over the randomness in bidders' valuations [50]. Myerson's mechanism is based on the following characterization of truthful mechanisms for single-parameter auctions.

**Theorem 8 ([50, Theorem 13.6])** *Consider an auction with single-minded bidders, wherein the losers pay nothing (i.e., $x_i = 0 \rightarrow p_i = 0$). Under the Bayesian setting, a mechanism is truthful if and only if, for any bidder $i$ and any fixed choice of bids by the other bidders,*

*(i) $x_i$ is monotone nondecreasing with the increase in $w_i$.*

*(ii) The payment $p_i$ for any winning bidder $i$ is set to the* critical value $t_i$, *which is the minimum amount $i$ needs to bid in order to win. Note that, in general, $t_i$ depends upon the bids of the other bidders.*

■

Given the above theorem, to specify a truthful mechanism, we need to only specify a winner determination function that satisfies the monotone allocation rule (the first condition of the theorem), and the payments can be derived from the second condition. In [46], Myerson specifies the winner-determination function based on "virtual bids," and shows that it leads to optimal expected revenue, if the payments are determined as described above.

Virtual Bids and Virtual Surplus. Myerson's mechanism [50] starts by replacing each bid $w_i$ with a *virtual bid* $\phi_i(w_i)$ as follows.

$$\phi_i(w_i) = w_i - \frac{1 - F_i(w_i)}{f_i(w_i)}, \tag{30}$$

where $f_i$ is the probability density function for $F_i$, i.e., $f_i(x) = \frac{d}{dx}F_i(x)$.

For a given outcome $o = (x_1, x_2, \ldots, x_n)$, the *virtual surplus* is defined as the sum of winning virtual bids, i.e., $\sum_i x_i \phi_i(w_i)$. The following theorem is key to the design of an optimal truthful mechanism.

**Theorem 9 ([50, Theorem 13.10])** *The expected revenue of any truthful mechanism under the Bayesian setting is equal to its expected virtual surplus. Here, the expectations are taken over the distributions of the valuations.* ∎

Myerson's Mechanism for Single-Item, and its Extensions. Based on the Theorems 8 and 9, Myerson's mechanism essentially involves determining an outcome that maximizes the virtual surplus, and determines payments based on condition (ii) of Theorem 8. By the virtue of the above two theorems, such a mechanism will be truthful and optimal if and only if the virtual bids are monotonically nondecreasing [50].

Myerson's technique can be easily extended to more general single-parameter auctions [43, 38, 23]. Some other works have also extended Myerson's technique to simple multi-parameter settings [4, 39, 47, 28]. We refer the reader to Chapter 13 of [50] for a complete coverage of Myerson's work and its extensions.

**Applying Myerson's Mechanism To Spectrum Auctions.** In a recent work, Jia et al. [33] present a simple extension of Myerson's mechanism to the context of spectrum auctions. However, the extension results in an exponential-time mechanism, since the corresponding virtual-surplus maximizing problem is NP-hard. Realizing the seriousness of this shortcoming, [33] present a polynomial-time mechanism based on the greedy mechanism of Lehmann et al. [41]. However, the expected revenue delivered by such a mechanism can be arbitrarily bad, as shown later in Section 4.3.

Our Contribution. In this chapter, we present a polynomial-time truthful spectrum auction mechanism whose expected revenue is a constant factor of the optimal expected revenue. Our mechanism is based on the above described Myerson's technique, and involves computing an allocation with approximate virtual surplus in polynomial-time. We also show in Section 4.4.2 how to extend the above to a more general setting.

## 4.3 Truthful Spectrum Auction with Approximate Expected Revenue

In this section, we define and address the problem of designing truthful spectrum auctions with approximate expected revenue. For simplicity of presentation, we make two simplifying assumptions, viz., single-mindedness of bidders and single-type channels; we relax these assumptions in the next section. Extensions of our techniques to more general pairwise interference models (like non-uniform disk and pseudo-disk models) as well as the physical interference model can be achieved similarly to Sections 3.3 and 3.4. We start with defining the set-up of spectrum auction, certain core concepts, and then, giving a formal definition of the problem.

**Spectrum Auction Model.** Our model of a cellular network consists of a set of geographically distributed base stations. Spectrum is divided into *orthogonal* channels of the same type (we consider multi-type channels later), and the *spectrum auction* involves each base station bidding for a certain number of channels. In our simple setting of single-minded bidders, each bidder $i$ bids for $D_i$ number of channels; i.e., the winning alternatives for $i$ are the outcomes wherein $i$ gets at least $D_i$ channels. Each bidder reports to the mechanism its bid $w_i$, the declared valuation for the winning alternatives. In addition, the probability distribution $F_i$ from which $v_i$ (the private valuation for the winning alternatives) is drawn, is publicly known. Note that and the losing outcomes (wherein $i$'s demand is not satisfied) are valued at zero by $i$.

**Valid Spectrum Allocation.** Given an interference graph over the base stations (bidders) and the demands $D_i$, the spectrum allocation must be done in such a way that no pair of interfering base stations are allocated a common channel. This *interference constraint* is incorporated in the following definition of a valid spectrum allocation.

**Definition 14** (Valid Spectrum Allocation.) Let $V_t$ and $V_c$ be the set of base stations and available channels, respectively and let $P(V_c)$ denote the power set of $V_c$. A binary allocation vector $(x_1, \ldots, x_N)$, where $N$ is the number of nodes, is considered *valid* if and only if there is an assignment $a : V_t \mapsto P(V_c)$ from the set of nodes to the power set of the set of channels such that (i) $|a(i)| = D_i$, for all $i \in V$ where $x_i = 1$, and (ii) $a(i) \cap a(j) = \emptyset$, if $(i, j)$ is an edge in the interference graph. □

It can be shown it is NP-complete to test whether a given allocation vector is valid, through a reduction from the problem of partitioning a graph into minimum number of independent sets. Thus, it is desirable for the auction mechanism to output the assignment function $a$ *in addition* to the allocation vector. The auction mechanism designed in this article does output the assignment function $a$, in addition to the spectrum allocation vector.

**TSA-MER (Truthful Spectrum Auctions with Maximum Expected Revenue) Problem.** Given an interference graph, the number of channels, and the bid-demand pair of each base station along with the distribution from which the valuation was drawn, the *TSA-MER problem* is to design a truthful auction mechanism that returns a valid spectrum allocation with maximum expected revenue.

Thus, the TSA-MER problem involves determining (i) a *valid* spectrum allocation, and (ii) payments by each bidder, so that the overall mechanism is truthful and the expected revenue is optimal. The TSA-MER problem can be shown to be NP-hard, by a reduction from the maximum independent set problem, since maximizing expected revenue is equivalent to maximizing virtual surplus (sum of virtual bids).

Recent Work on TSA-MER. In a recent work, Jia et al. [33] extended Myerson's mechanism for the TSA-MER problem. However, since maximization of virtual surplus is NP-hard, due to the interference constraint, Myerson's technique only yields an exponential-time mechanism. Thus, [33] designed a Greedy-heuristic mechanism for the TSA-MER problem as follows. First, the Greedy algorithm sorts the bidders in decreasing order of their virtual-bid per channel (i.e., $\phi_i(w_i)/D_i$). Then, the algorithm considers each bidder in the sorted order, and adds it to the allocation vector if the interference constraint is not violated. Note that to check the violation of interference constraint efficiently, we need to maintain the channels-to-node assignment function. Finally, the payments by the winners are determined as suggested in Theorem 8. By Theorem 8, it is easy to show that such a mechanism is truthful. However, the revenue yielded by such a Greedy mechanism can be arbitrarily bad. (see Figure 14).

Below, we design a polynomial-time mechanism that is truthful and yields a valid spectrum allocation with *approximate* expected revenue.

**Outline of the Truthful Mechanism with Approximate Expected Revenue.** Based on Theorems 8 and 9 of Subsection 4.2, our method for designing a truthful spectrum auction mechanism with approximate expected revenue is outlined in the following two steps:

1) Determine a valid spectrum allocation with approximate virtual surplus.

2) Determine payments based on condition (ii) of Theorem 8.

We discuss the above two steps in the following two paragraphs.

**Valid Allocation with Approximate Virtual Surplus.** Given a network with base stations, the unit-disk interference graph, the bids and the probability distributions of the valuations of the bidders (base stations), we determine a valid allocation with approximate virtual surplus as follows. Similarly to the technique of Chapter 3, we divide the entire networks into small hexagonal regions, solve the simpler optimization problem in each hexagon independently,

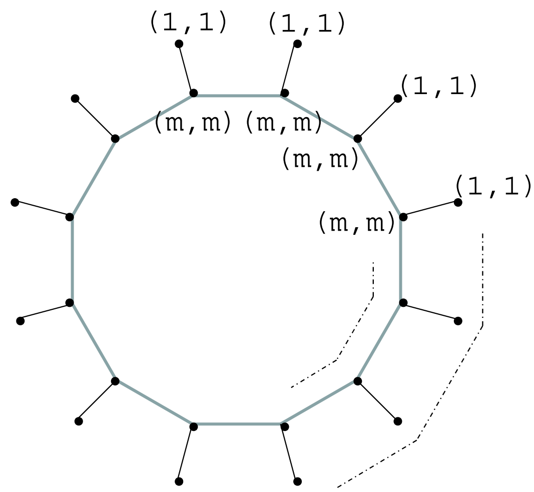



Figure 14: Counter example for the greedy algorithm. The figure shows the interference graph of $n$ bidders. The (demand, bid) pair for the inner nodes is $(m, m)$, while for the outer nodes is $(1, 1)$. The bids are constant, and hence, virtual-bid of each node is equal to its bid. Since all the nodes have the same rank (= virtual-bid/demand), the greedy mechanism may pick all the outer nodes and yield a total revenue of $m/2$, while the optimal revenue is $m^2/4$.

and then, "combine" the solutions. At a high-level, our algorithm consists of the follows steps.

1) Replace each bid $w_i$ with a *virtual bid* $\phi_i(w_i)$ as defined by Equation 30.

2) Divide the entire network region into small hexagonsof unit side-length.

3) Uniformly-color the hexagons with seven colors.

4) Allocate channels to base stations in each hexagon *independently*, treating it as a Knapsack problem where the virtual bids are the "values" of items to be placed in the knapsack and the demands are their "weights." The well-known fully polynomial-time approximation scheme (FPTAS) [36] can be used to get a $(1+\epsilon)$-approximate virtual surplus of each hexagon for any $\epsilon > 0$. Note that the interference subgraph in each hexagon is actually a complete graph.

5) For each color, combine the results from all hexagons of that color.

6) Pick the color that has the highest total virtual surplus and allocate the channels to the winners accordingly.

7) Perform a post-processing step to greedily satisfy the demands of more base stations.

The resulting allocation is guaranteed to have at least a $1/7(1+\epsilon)$-factor of the optimal virtual surplus for any $\epsilon > 0$. Moreover, its running time is polynomial in $1/\epsilon$ and the size of the input, i.e., in $N$ and $\log M$, where $N$ is the number of nodes and $M$ is the number of channels. Similarly to Chapter 3, the above hexagonal division and coloring guarantees that Properties (P1) and (P2) hold.

Allocation in Each Hexagon. Properties (P1) and (P2) imply that the channels cannot be re-used inside the same hexagon, but can be fully re-used across different hexagons of the same color. Thus, allocation in each hexagon can be treated as a Knapsack problem where the virtual bids are the "values" of items to be placed in the knapsack and the demands are their "weights." The well-known FPTAS [36] can be used to get a $(1+\epsilon)$-approximate virtual surplus of each hexagon for any $\epsilon > 0$.

Combining The Results. Since base stations in different hexagons of same color do not interfere with each other (Property (P2)), we can combine allocations of co-colored hexagons to form one single allocation. Thus, we get seven allocations, one for each color. Among these seven allocations, we pick the allocation with the highest virtual surplus. If needed, the derived allocation can be easily converted into a channels-to-node assignment function, since allocation in each hexagon is independent of each other and is over a *complete* graph.

Post-Processing Step. Our above described allocation algorithm returns an allocation with approximate virtual surplus, and with appropriate payments (as discussed below), can be turned into a truthful auction mechanism with approximate expected revenue. Incidentally, we can further improve the above allocation algorithm, by allocating more bidders in a greedy manner as in the Greedy mechanism described before. In particular, we sort the *remaining* bidders by their virtual-bids per demand (i.e., $\phi_i(w_i)/D_i$), and consider them for allocation in that order without violating the interference constraint. As noted before, we would need to maintain the channels-to-nodes assignment function, to efficiently implement this greedy post-processing part. Also, we note that the above greedy post-processing step does not however improve the worst-case approximation bound of our allocation algorithm. In Theorem 10, we show that our overall allocation algorithm (with the above post-processing step) does satisfy the monotonicity of $x_i$'s (i.e., the first condition of Theorem 8).

**Determining Payments.** The payments are determined according to Theorem 8 as follows. For each winner $i$, we use a binary search to find its critical value $t_i$ (for the given fixed bids of other bidders) such that $i$ wins if $w_i \geqslant t_i$ and loses otherwise. Note that such a value $t_i$ is guaranteed to exist, since our allocation algorithm results in monotonically increasing $x_i$'s (i.e., satisfies the first condition of Theorem 8). Then, for each such winning bidder, we set its payment $p_i$ as $t_i$. The payments of losing bidders are set to zero.

The critical values for bidders who win in the post-processing step can be determined using ideas based on the "critical neighbor" technique of [33]. The critical value for a bidder $i$ who wins in the first step (involving coloring of hexagon cells) can be computed using at most $\log w_{\max}$ runs of the allocation within its hexagon cell[1] followed by the above "critical neighbor" technique. The latter part may be needed to determine the critical value for $i$'s win due to the post-processing step; note that even if lowering the bid of $i$ makes its hexagon color a loser in the first step, bidder $i$ can still win due to the post-processing step.

**Proof of Truthfulness and Approximability.** In the below theorem, we show that the auction mechanism, based on the above described allocation algorithm and payment determination, is truthful and returns a valid spectrum allocation with approximate expected revenue.

**Theorem 10** *For the TSA-MER problem under the Bayesian setting and the pairwise interference with unit-disk model, the above described auction mechanism is truthful and returns a valid spectrum allocation whose expected revenue*

[1]Note that the allocation within other hexagon cells does not change with the variation in $i$'s bid.

*is at least* $\frac{1}{7(1+\epsilon)}$ *of the optimal expected revenue, for a given* $\epsilon > 0$.

*Proof:* Truthfulness. By Theorem 8, we need to only show that our allocation algorithm results in monotonically nondecreasing $x_i$'s. We start by showing the monotonicity of the FPTAS algorithm used in each hexagon which follows immediately from the facts that the FPTAS algorithm is an optimal algorithm on scaled-down values and that the optimal algorithm is certainly monotonic. To show that our algorithm is monotonic overall, we need to consider two cases, viz., (i) when a bidder $i$ is selected as a winner in the first step, and (ii) when a bidder $i$ is selected as a winner in the post-processing step. In the first case, if the bids of all other bidders remain fixed, then an increase in the bid of $i$ would not change (a) the presence of $i$ in the FPTAS knapsack-solution (due to its monotonicity), and (b) the winning of the color of $i$'s hexagon. In the second case, increasing the bid of $i$ can cause the color of $i$'s hexagon to become the winning color. However, in such a case, $i$ must remain a winner in its hexagon (else, its color should not have become a winning color). If increasing the bid of $i$ does not cause the color of $i$'s hexagon to become a winning color, then $i$ must remain a winner due to the greedy method of the post-processing step.

Approximate Expected Revenue. By Theorem 9, we only need to show that the virtual surplus of our delivered allocation is within a $7(1 + \epsilon)$-factor of the optimal virtual surplus. Since the post-processing step only improves the total virtual surplus without violating the interference constraint, we can show that the allocation before the post-processing step is valid and has a $7(1 + \epsilon)$-approximate virtual surplus using an argument similar to that of Theorem 4's proof. ∎

## 4.4 Extensions

In the previous section, we made two simplifying assumptions, viz., single-mindedness of bidders and single-type channels. We now show how to relax these assumptions.

### 4.4.1 Multi-type Channels

Till now, we have assumed that all available channels are of the same type and bidders bid for a certain number of channels. However, our techniques can also be generalized to multi-type channels, as long as the bidders are single-minded as follows.

Single Channel of Each Type. We start by considering the case when there are $m$ channels, each of a different type, and each bidder $i$ bids for a set $s_i$ of channels. In such a case, a valid spectrum allocation allocates *disjoint* set of channels to the winners. A truthful auction mechanism with approximate expected revenue can be devised similar to the one for the single-type channels, except that within each hexagon we would need to solve the *Weighted Maximum Set Packing* (WMSP) problem [29] (instead of the Knapsack problem). Unfortunately, the general WMSP problem is as hard to approximate as the maximum clique problem (and thus, is only approximable within a factor of $O(\sqrt{m})$), but if the demand sets $\{s_i\}$ are each bounded in size by a constant $k$, then the WMSP problem can be approximated within a factor of $2(k+1)/3$ [29], yielding a mechanism with a $14/3(k+1)$-approximate expected revenue. Finally, in order to prove truthfulness, we only need to show that the MWSP approximation algorithm of [29] is indeed monotonic. The following lemma states and proves this.

**Lemma 6** *The approximation algorithm of [29] for the MWSP problem is monotonic.*

*Proof:* We start by giving a brief sketch of the algorithm from which the monotonicity will be fairly obvious. The algorithm starts by creating a vertex-weighted graph where every bidder $i$ is represented by a vertex whose weight is $\phi_i(w_i)$, and two vertices are connected by an edge if their demands contain at least one common channel. The problem is thus reduced to find the Maximum Weighted Independent Set (MWIS) in this special graph, where an independent set is one which does not contain neighboring vertices. The algorithm starts with a greedy solution and iteratively looks for the best possible "local improvement." A local improvement on a set $I$ is achieved by removing one vertex $i \in I$, adding an independent set $S$ from the direct neighbors of $v$, and removing any vertex from $I$ with an edge to a vertex in $S$. To be called an improvement, this substitution must produce a solution with higher weight. Thus, if a bidder $i$ was chosen as a winner, then it must be the case that it has no local improvement. It is easy to see that by raising $i$'s bid, this fact cannot change. ∎

Multiple Channel of Each Type. In a more general setting, consider the scenario where there are $m_j$ channels of type $j$ and there are $t$ different channel types. In such a case, a single-minded bidder's demand would be a $t$-dimension vector $(D_{i,1}, \ldots, D_{i,t})$, where $D_{i,j}$ is $i$'s demand of the $j^{\text{th}}$ channel type. Here, the number of channel types $t$ is a constant. Our designed mechanism for the single-type channels can easily be extended to this general scenario, by using the PTAS for the *Multi-Dimensional* Knapsack Problem (MDKP) [36] within each hexagon, yielding a $7(1+\epsilon)$-approximate expected revenue. However, the time-complexity of the mechanism is now exponential in $1/\epsilon$, since there is no FPTAS known for MDKP. Similar to the above, in order to prove truthfulness, we only need to show that the PTAS algorithm of [36] for the MDKP problem is monotonic, which is stated by the following lemma.

**Lemma 7** *The PTAS algorithm of [36] for the MDKP problem is monotonic.*

*Proof:* Note first that the MDKP problem can be formulated as an Integer Linear Programming (ILP) problem for which the Linear Programming (LP) relaxation can be easily found. Before discussing it monotonicity, we give an informal sketch of the algorithm. The algorithm constructs many possible solutions by considering all subsets $L$ of size at most $l$ (which depends on $\epsilon$). For each subset $L$ of size exactly $l$, the algorithm "pads" the solution with additional winners as follows. It computes the LP solution for the problem instance consisting of the bidders whose virtual bids do not exceed the minimum virtual bid in $L$. Each LP returns three sets ($J_1$, $J_F$ and $J_0$) of bidders who are assigned their entire demands, a fraction of the their demands or nothing, respectively. Then, the algorithm either chooses all bidders in $J_1$ or one of the bidders in $J_F$ depending on which choice has higher total virtual bids. Finally, among the set of constructed solutions, the algorithm picks the solution with maximum total virtual bids. Now, if the optimal solution contains less than $l$ bidders, then the algorithm would find it and the monotonicity trivially holds. Otherwise, consider a bidder $i$ who was picked as a winner when bidding $w_i$ and we show that it will remain winner if it bids $w'_i > w_i$ assuming the remaining bids are fixed. Note first, that between the two settings, the set of constructed solutions have the same subsets $L$ and only differ in the LP solutions and the greedy choice made thereafter. Thus, raising $w_i$ to $w'_i$ only affects the set of solutions in which $i$ is either (i) in $L$ or (ii) in the LP solution. Now, in case (i), $i$ will end-up in the final set of winners, and in case (ii), the solution can only increase by at most $\phi_i(w'_i) - \phi_i(w_i)$. But, any solution in which $i$ remains a winner would be increased by $\phi_i(w'_i) - \phi_i(w_i)$ and we already know that previous maximum solution is among these solutions. Thus, no new solution which excludes $i$ can be better. ∎

### 4.4.2 Beyond Single-Minded Bidding

We now extend our technique beyond single-minded bidding by handling *fractional demands*, wherein each bidder $i$ is willing to buy any number of channels from a certain range $\{\breve{D}_i, \ldots, \widehat{D}_i\}$, and the valuation for winning alternatives is expressed in terms of per-channel valuation. We assume single-type channels here. More formally, a bidder $i$'s declared demand-bid is of the form $(\breve{D}_i, \widehat{D}_i, w_i)$, signifying that the bidder would accept any number of channels between $\breve{D}_i$ and $\widehat{D}_i$ at a price of at most $w_i$ per channel. For simplicity, we first assume that $\breve{D}_i = 0$ for every bidder $i$; we relax this assumption later.

For the above setting, the mechanism's output is an allocation vector $(x_1, \ldots, x_n)$ wherein $x_i \in [0, 1]$ represents the fraction of demand satisfied, i.e., for a given $x_i$, the number of channels allocated is $x_i \widehat{D}_i$. Also, for a given allocation vector, the virtual surplus is defined as $\sum_i \phi_i(w_i) \widehat{D}_i x_i$. For this setting of fractional demands, Theorems 8 and 9 can be generalized (based on [50]) as follows. Below, we use the notation $x_i(w_i)$ to denote $x_i$ for a given $w_i$ and fixed bids of other bidders.

**Theorem 11** *A mechanism (wherein losing bidders pay zero, i.e., $x_i = 0$ implies $p_i = 0$) is truthful iff for any bidder $i$ and any fixed choice of bids by* *other bidders,*

- *$x_i(w_i)$ is monotonically nondecreasing in $w_i$.*
- *The payment is set as follows*

$$p_i(w_i) = w_i\widehat{D_i}x_i(w_i) - \int_0^{w_i} \widehat{D_i}x_i(t)dt. \tag{31}$$

*Proof:* For simplicity, we drop the subscript $i$. To show truthfulness, we only need to show that the utility of truthful bidding $v$ is no smaller than bidding any other value $w$. I.e.,

$$\begin{aligned} v\widehat{D}x(v) - p(v) &\geqslant v\widehat{D}x(w) - p(w) \\ \int_0^v \widehat{D}x(t)dt &\geqslant v\widehat{D}x(w) - w\widehat{D}x(w) + \int_0^w \widehat{D}x(t)dt. \end{aligned}$$

For $w > v$, the above is true since $(w - v)(\widehat{D}x(w)) \geqslant \int_v^w \widehat{D}x(t)dt$ follows from the monotonicity of $x$, while for $w < v$, the above is true since $(v - w)(\widehat{D}x(w)) \leqslant \int_w^v \widehat{D}x(t)dt$ also follows from the monotonicity of $x$.

Now to show the other direction, we take the truthfulness constraints at $v$, $v\widehat{D}x(v) - p(v) \geqslant v\widehat{D}x(w) - p(w)$, and at $w$, $w\widehat{D}x(v) - p(v) \leqslant w\widehat{D}x(w) - p(w)$. Rearranging these inequalities gives

$$v\widehat{D}(x(w) - x(v)) \leqslant p(w) - p(v) \leqslant w\widehat{D}(x(w) - x(v)). \tag{32}$$

From this, we get $(w - v)(x(w) - x(v)) \geqslant 0$ which implies the monotonicity of $x$.

We now derive Equation 31. Let $w = v + \epsilon$, then, by dividing Equation 32 by $\epsilon$ and taking the limit, we get

$$v\widehat{D}\ \frac{dx}{dv} \leqslant \frac{dp}{dv} \leqslant v\widehat{D}\ \frac{dx}{dv}.$$

Now, since $p(w) = 0$ for any $w$ smaller than the critical value, we get

$$p(w) = \int_0^w t\widehat{D}x'(t)dt.$$

Integrating the above equation by parts gives Equation 31. ∎

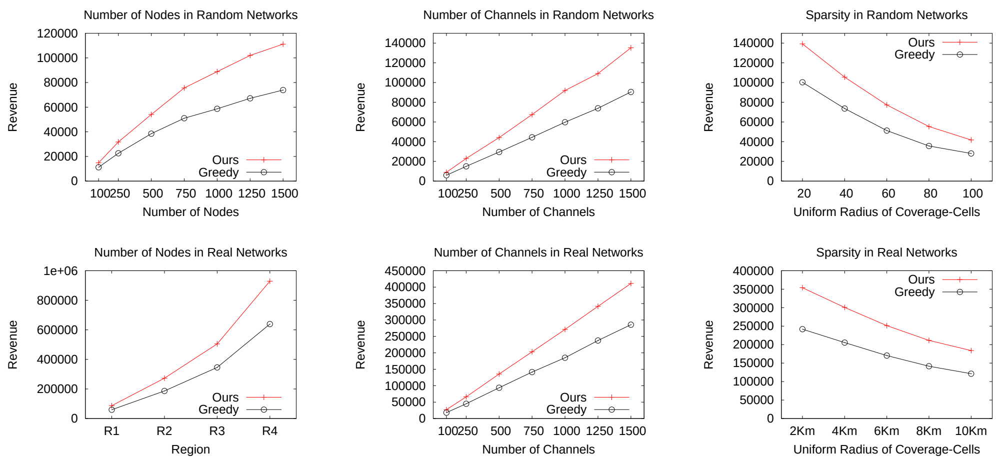



Figure 15: Comparing the generated revenue of our enhanced spectrum auction mechanism with the Greedy mechanism. The plots of first row are for random networks whereas the remaining plots are for real networks. For both topologies, the default number of channels is 1000. For random networks, the default number of nodes is 1000 and the default uniform radius of the coverage-cells is 50 units, whereas for real networks, the default region is R2 and the default uniform radius of the coverage-cells is 5 Km.

**Theorem 12** *Under the Bayesian setting with fractional demands, the expected revenue of any truthful mechanism is equal to its expected virtual surplus, where the virtual surplus is as defined above.* □

**Overall Mechanism.** On the basis of the above two theorems, our mechanism from the previous section can be extended to the case of fractional demands, by solving the appropriate allocation problem within each hexagon. In fact the resulting allocation problem within each hexagon can now be solved *optimally* in polynomial-time using a greedy approach, yielding a truthful auction mechanism with a 7-approximate expected revenue.

**Non-zero Minimum Demands.** We handle non-zero minimum demands $\{\breve{D}_i\}$ by defining a new allocation vector $(y_1, \ldots, y_n)$ wherein $y_i$ is equal to $x_i$ if $x_i \geqslant \breve{D}_i/\widehat{D}_i$ and zero otherwise. The arguments of this subsection straightforwardly apply to the this new allocation vector.

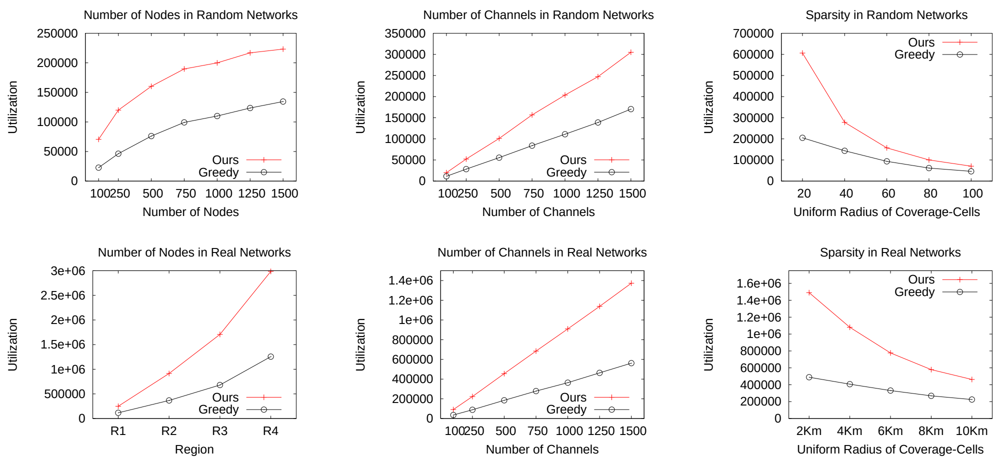



Figure 16: Comparing the spectrum utilization of our enhanced spectrum auction mechanism with the Greedy mechanism. The plots of first row are for random networks whereas the remaining plots are for real networks. For both topologies, the default number of channels is 1000. For random networks, the default number of nodes is 1000 and the default uniform radius of the coverage-cells is 50 units whereas for real networks, the default region is R2 and the default uniform radius of the coverage-cells is 5 Km.

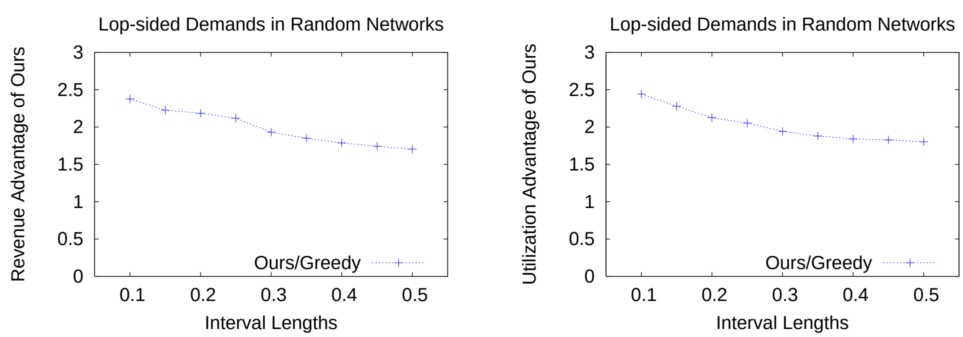



Figure 17: Ratio of the performance of our enhanced mechanism to the Greedy mechanism, for the special case of "lop-sided" demands. The lop-sided demands are randomly chosen, for each base station, from the set $[1, \mathcal{I}m] \cup [m - \mathcal{I}m, m]$ where $\mathcal{I}$ is the interval length (varied on the $x$-axis above). The considered networks are randomly generated with 1500 nodes, 1000 channels and a 50 unit uniform radius of the coverage-cells.

## 4.5 Simulation

In this section, we present our simulation results. The main purpose of our simulations is to compare the performance of our designed auction mechanism with the greedy mechanism of [33] under several settings and performance metrics. We start by describing our simulations set-up.

**Network Topology and Model.** In our simulations, we consider two types of networks, as described below.

- *Random Networks:* We randomly place base stations within a fixed area of $1000 \times 1000$ square units. We vary the network density by varying the number of base stations from 100 to 1500 (with the default being 1000). To generate the interference graph, we use coverage-cells of uniform radius, which is varied from 20 to 100 (with the default being 50).

- *Real Networks:* We use locations of real cellular base stations available in FCC public GIS database [1] and choose base stations deployed in 4 different regions of increasing size and number of base stations.

  - R1 - 843 base stations in the state of MA.
  - R2 - 2412 base stations in New England area (MA, ME, NH, VT, RI, CT).
  - R3 - 4467 base stations in New England and NY.
  - R4 - 8618 base stations in North East USA (New England, NY, NJ, PA).

  Here, the default region is R2 and we choose a realistic coverage-cell radius of 5 kilometers.

In both networks, we set up an auction of up to 1500 orthogonal single-type channels with the default being 1000 channels; this is a reasonable range based on the past FCC auctions [16, 3].

**Demands and Bids.** For simplicity, we based all our experiments on uniformly-distributed valuations. We generate single-minded bids for each base station $i$ as follows. First, we randomly choose a non-zero demand for $i$, then, we randomly choose $i$'s per-channel bid from the range $(0, 1]$.

**Auction Mechanisms Compared.** We compare an enhanced version of our auction mechanism with the greedy mechanism of [33], the only mechanism in the literature for the problem considered here. Below, we discuss the heuristic enhancement we introduce to our mechanism.

Our Enhanced Mechanism. In order to improve the performance of our mechanism in practice, we have modified the way we combine the results of the independent solutions of each hexagon to obtain the overall allocation. Instead of using the conservative approach of simply combining the hexagons of the same color together, we use a greedy approach to construct the set of hexagons as follows. First, we rank the hexagons according to their total virtual surplus, and "pick" the hexagon $H$ with the highest rank. Next, we remove $H$ and any "conflicting" hexagons from further consideration; here, two hexagons $H$ and $H'$ are said to conflict if they contain a pair of interfering base stations. Then, we "pick" the hexagon with the highest rank from the remaining hexagons, and repeat. The overall allocation is the set of all picked hexagons (with the independent allocations within each of them, based on the FPTAS for Knapsack) in the above process. To see that this allocation rule does not affect the monotonicity (and hence, the truthfulness) of our mechanism, consider a winning bidder $i$. Increasing $w_i$ while fixing the remaining bids would not change (a) the presence of $i$ in the FPTAS knapsack-solution (due to its monotonicity), and (b) the winning status of $i$'s hexagon (due to the monotonicity of the greedy choice of winning hexagons).

**Simulation Results.** In our simulations, we compare our enhanced auction mechanisms with Greedy [33] in terms of the generated revenue, and the spectrum utilization. The *spectrum utilization* is defined as the total number of allocation pairs in the spectrum allocation. Spectrum utilization gives a measure of the spatial reuse of a spectrum allocation. To show the effect of different system parameters on the performance of the two algorithms, we vary: (i) the number of nodes, (ii) the number of available channels and (iii) the uniform radius of the coverage-cells.

In Figure 15, we plot the generated revenue for different network topologies and parameters. We observe that our mechanism outperforms Greedy in all settings, by an average factor of 47%. Moreover, it is apparent that with the increase in the numbers of nodes and channels, the gap between our mechanism and Greedy is becoming larger, crossing the level of 50% improvement in each plot. As for the case of varying the uniform radius of the coverage-cells, the performance of both algorithms suffer from the increase of the uniform radius of the coverage-cells which affects the sparsity of the network and hence the spatial-reuse. Figure 16 captures the issues related to spatial-reuse by plotting the spectrum utilization for for different network topologies and parameters. The trends observed for the generated revenue are similar to those for the spectrum utilization, and this shows that our mechanism not only generates more revenue than the Greedy mechanism,, but also increases the spectrum utilization, making it more appealing to practical use.

**Experiments With "Lop-Sided" Demands.** In the above experiments with randomly generated demands and bids, our mechanism outperforms the Greedy mechanism by about 50-60%. However, as noted in Figure 14, Greedy mechanism can perform arbitrary bad compared to our mechanism. We now try to generate quasi-random instances, wherein the advance of our mechanism compared to the Greedy mechanism can be much higher. In particular, we consider randomly generated networks as before, but assign "lop-sided" demands and almost-equal bids to nodes as follows. For each base station $i$, we randomly choose $D_i$ from $[1, \mathcal{I}m] \cup [m - \mathcal{I}m, m]$, where $\mathcal{I}$ is some value between $1/m$ and 1. Then, we give the nodes with low demands an advantage when generating the bids as follows. If $D_i \leqslant m/2$, then the per-channel bid is chosen from [0.95,1]. Otherwise, it is chosen from [0.9,0.95]. Note that, in practice, there is no reason why the bids and demands should have a random distribution. The above specialized setting may reflect a scenario where small start-up concerns compete with large service providers.

In Figure 17, we show the performance ratio of our mechanism to the Greedy mechanism, in random networks with 1500 nodes and coverage cells of uniform radius 50 units in an area of $1000 \times 1000$ units. We use 1000 channels. In Figure 17, we see that the performance ratio is as high as 2.5, for low values of $\mathcal{I}$, and the ratio decreases with increase of $\mathcal{I}$.

## 4.6 Conclusion

The recent trend of dynamic spectrum access in cellular networks creates a setting for auctioning pieces of wireless spectrum to competing base stations. To mitigate market manipulation, a truthful spectrum auction is highly desired, so that bidders can simply bid their true valuations. In this article, we designed a truthful spectrum auction that delivers an allocation with near-optimal expected revenue, in the Bayesian setting, for single-minded bidders. We have considered a simple pairwise interference model, but our techniques can be extended to more general interference models. We have shown the superiority of our mechanism over the only known mechanism in the literature for our problem using both theoretical and empirical analysis. As a future direction, we plan to extend our mechanism to simple multi-parameter settings.

# Chapter 5

# Distributed Spectrum Allocation

## 5.1 Introduction

Till now, we have focused on centralized approaches for spectrum allocation. The general goals were to maximize some social-choice function like social-welfare or revenue while controlling the strategic behavior of the base stations. When it comes to a distributed approach, the focus is shifted towards more stable allocation that can maintain certain properties with minimal cost and human intervention even when faced by frequent network topology changes. This is demonstrated by problems such as self-configuration of fractional frequency reuse (FFR) patterns for LTE/WiMAX networks. We start by giving the different (more practical) perspective associated with spectrum allocation in such a setting.

In recent years, Self-management (Self-X) technologies that fully automate the tasks of managing (i.e. configuring, monitoring, and optimizing) a cellular network are emerging as an important tool in reducing service provider OPEX and CAPEX and will be a distinguishing feature of LTE networks. Examples include the SOCRATES project [60] and the E3 project [42]. In this chapter, we focus on one such Self-X technology, namely, self-configuration of fractional frequency reuse (FFR) patterns for LTE/WiMAX.

We contend that any solution to this problem must meet the following often-conflicting objectives:

1) *Computational efficiency:* The self-assignment procedure should be efficient and use only local neighborhood information for computation.

2) *Controlled cascading and stability:* in the event of cell addition or deletion, the impact of recomputing the FFR should be restricted to a well-defined local neighborhood of the base station and should not cascade over the entire network.

3) *Optimality of solution:* The spectrum utilization resulting from FFR computed should be closest to optimal as possible.

Practical considerations requires that the following additional objectives must be met:

1) *Contiguity:* each base station is assigned a contiguous chunk of the spectrum.

2) *Minimum Demand Satisfaction:* each base station is assigned a minimal part of the spectrum necessary to carry out its basic functionality.

Unfortunately, satisfying *all* of the above objectives/properties at the same time is hard. In fact, a subset of them can be proven to be NP-hard even under simplistic assumptions (see previous chapters). Another difficulty comes from the fact that some of these objectives conflict with each other and satisfying one comes at the expense of the other. Our aim in this chapter is to address this kind of trade-offs. We design a flexible tool that can be tuned to achieve some of these objectives (fully of partially) without sacrificing the others. Basically, for each possible choice of parameter values made by the network designer, our tool delivers a near-optimal spectrum utilization with specific guarantees on the rest of the objectives.

The following two definitions will help us discuss the problem in more details.

- A distributed algorithm is *k-local* (or has a *locality value of* $k$) if each base station uses only information regarding its $k$-neighbors (base stations within $k$ units of distance).

- An algorithm is *k-cascade* (or has a *cascade value of* $k$) if a dynamic change in the network (e.g., addition or deletion of a base station or changes in utility functions) only affects a limited part of the network, specifically the $k$-neighbors of the event.

**Problem Formulation.** As assumed in previous chapters, the spectrum is divided into a set of orthogonal channels $K$. Given a set of $N$ base stations, their interference relationships and the set $K$ of available channels, the problem considered here is to efficiently find an "interference-free" channel assignment to the base stations that maximizes the "spectrum utilization" while satisfying the rest of the objectives discussed above. This should be done in a distributed fashion with bounded local and cascading values.

A maximized spectrum utility reflects assigning channels to the base stations which will make the best use of them. For the purposes of this discussion, we break away from the auction setting we used in previous chapters, and redefine the utility functions to simply be non-increasing functions showing the "incremental" utility of each base stations if more channels were assigned to it. I.e., for a base station $i$, $u_i(k)$ is $i$'s utility for getting its $k^{th}$ channel (assuming $i$ has already been assigned $k-1$ channels). From practical considerations, a base station will always have positive incremental utility for each additional channel although after obtaining a large number of channel, these increments will be minimal.

**Related Works.** Due to the practical considerations, the objective of the above problem is rather complicated. Previous works have only considered subsets of the objectives discussed above under simplistic assumptions. One example is to ignore the dynamicity and contiguity aspects of the solution and consider very simple utility functions. Then, the problem becomes similar to graph multi-coloring with the addition of interference constraints. One example of such formulations is the work of Peng et al. [51].

The closest to our work is the recent work of Karla [35]. The main differences between our work and [35]'s are three folds. Firstly, they consider a very simple objective of assigning exactly one channel to each base station. Secondly, the assume a simple distribution of base stations where every base station lie on a hexagonal grid. Finally, the proposed solution is a heuristic with no bounded guarantees on cascading effects and it cannot be extended to give any guarantees on spectrum utility maximization.

As discussed above, we present a flexible distributed algorithm that not only guarantees near-optimal spectrum utilization, but also imposes tight bounds on local and cascading values.

**Chapter Organization.** In Section 5.2, we present the *Hexagonal Division* approach, a simple efficient, yet static, solution for this problem. In Section 5.3, we show how to improve the solution via the use of *Clustering*. We present simulations evaluating the performance of the two approaches in Section 5.4 before concluding in Section 5.5.

## 5.2 Hexagonal Division Approach

In this section, we present our underlaying basic approach which we extend in the next section to make it more flexible to the system designer's demands. The "centralized static" basic approach is similar to the one used in Chapter 3 except for the following important distinctions which we discuss in greater details later.

- Due to practical considerations, the concept of a "coverage cell" is modified.

- Here, we divide the available channels between colors (i.e., we divide the available channels into 7 equi-sized groups and associate each group with one of the colors used to color the hexagons).

- Spectrum allocation in each hexagon is different.

- We also need to worry about the details related to the objectives mentioned in the previous section like dynamicity and distributed implementation.

For simplicity, we restrict our discussion here to the pairwise interference with unit-disk model. Extension to other models of interference is easy along the lines of our arguments in Chapter 3.

**Coverage Cells.** Each base station is associated with a region around it called its *coverage cell*; each base stations serves its clients in its cell. The cell is divided into two regions, inner and edge regions. In the inner region, the base station uses any channel to serve it clients without worrying about the wireless interference generated by multiple near-by base stations operating on the same channel. In the edge region, on the other hand, this interference might disrupt the communication between the base station and its clients. Ideally, the base station and the client should operate "interference-free" on the same channel, but in case this is not possible, interference must be minimized. When the physical interference model is used, we will not explicitly account for transmissions carried out between a base station and the clients in the close-in region of its cell, since the transmission power for that purpose is low and its effect on clients of the edge region can be built into the SINR equation (Equation 1).

**Allocation in Each Hexagon.** Informally, we use a simple greedy algorithm that assigns each available channel to the base station with the highest utility for that channel. The nature of the utility function suggests that this approach is optimal for the allocation problem in each hexagon. Since we only use a subset (specifically, $(1/7)^{th}$) of the given channels for each hexagon, we get a 7-approximation of the optimal total utility. Finally, note that this algorithm runs in time polynomial in the size of the input, i.e., in $N$, the number of base stations, and $|K|$, where $|K|$ is the number of channels.

**Minimum Demand Satisfaction and Contiguity.** Now, we need to take care of two more issues with this approach, viz., the minimum demand satisfaction and the contiguity of the assignment. For the first issue, it is enough to add a preprocessing step in which every base station is assigned its minimum channel demand, and run the greedy approach on the remaining channels. This does not affect the quality of the solution since this is the best an optimal algorithm can do. As for the issue of contiguity, it is not hard to see that the resulting non-contiguous assignment can be manipulated (through a simple swapping technique) to convert it into a contiguous one since each channel can be assigned to exactly one base station. The greedy algorithm can be thought of as running in "simulation mode" just to figure out the number of channels to be assigned to each base station, whereas another algorithm can simply do the actual assignment by sequentially going through all base stations assigning each one a number of contiguous channels equal to the one computed by the greedy part.

**Distributed Dynamic Version.** As mentioned above, the above algorithm is a centralized static one. This is purely for the sake of simplicity of presentation. The distributed version of this algorithm is rather simple since each base station knows its location and can run the hexagonal division and coloring parts independently. The greedy channel assignment algorithm discussed in the above paragraph can be either handled by a central base station in each hexagon, or each base station can run the same algorithm and get the same result independently (provided that each run of the algorithm consider the base stations in the same order, say based on their unique IDs or locations). Here, only information of the 2-neighborhood is needed to figure out the channel distribution. Thus, this is a 2-local algorithm. The dynamic version of this algorithm is also simple. When a base station $i$ is added or deleted or when a utility function $u_i$ changes, then, only the base stations inside the same hexagon as $i$ are affected. Thus, this is a 2-cascade algorithm.[1]

The following theorem states the properties of our algorithm. The proof of this theorem directly follow from the more complex proofs of the next section and of Chapter 3.

**Theorem 13** *For the unit-disk model, the above algorithm returns an interference-free 7-approximate assignment in time polynomial in the size of*

[1]For the physical interference model, since we use hexagons of side-length $R$ (defined by Equation 27, the resulting algorithm would be $(2R)$-local $(2R)$-cascade.)

*the input (i.e., in $N$, the number of base stations, and $|K|$, the number of channels). Furthermore, it meets the other objectives of contiguity and minimum demand satisfaction while being 2-local 2-cascade.* □

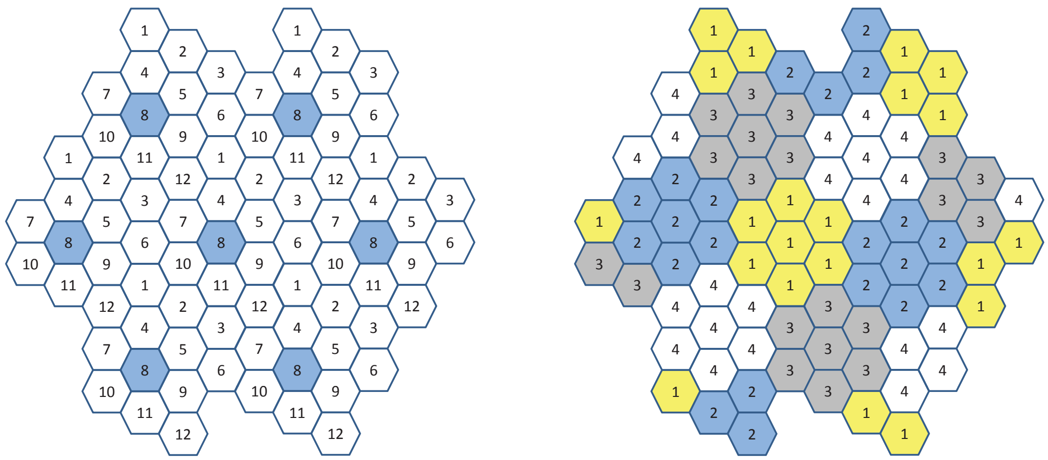
## 5.3 Improvements via Clustering

In this section, we show how to improve the hexagonal division approach and make it more practical and flexible towards the designer requirements (thus, allowing the designer to tune the trade-off properties as discussed in the introduction).

To motivate this in an intuitive manner, consider the many practical scenarios in which it is obvious that the static hexagonal division approach underutilizes the spectrum. For example, in the unit-disk model, consider the case when one hexagon is heavily congested while a neighboring hexagon is empty. Then, $(1/7)^{th}$ of the available spectrum is wasted. Inspired by earlier works in the literature [34, 48], "clustering" of hexagons is a promising approach to handle this issue. For simplicity of presentation, we discuss this idea under the most general pairwise interference model, namely the pseudo-disk model (see Section 1.1.1). As it happens, the pseudo-disk model is where the merits of this idea are emphasized. Further extension to the physical interference follows.

### 5.3.1 Pairwise Models

The basic idea here is to focus on clusters of hexagons rather than single hexagons. Consider for example a cluster of 7 hexagons (similar to the clusters of Figure 18(b)). If a simple unit-disk model is assumed, then 3-coloring the clusters with an assignment algorithm similar to the ones of the previous section would give a valid assignment. Furthermore, if the assignment inside the cluster is handled properly, then the overall solution is likely to be an improvement over the hexagonal division algorithm since now each base station has access to $(1/3)^{rd}$ of the channels whereas previously it had access to only $(1/7)^{th}$. Nonetheless, the situation is trickier here since base stations of the



(a) 12-coloring of hexagons. (b) 4-coloring of 7-hexagon clusters.

Figure 18: Flexible Clustering.

same cluster might not have mutual interference, and hence, a channel may be reuseable among them. In fact, an interference-free set of 7 base stations may exist in a cluster of 7 hexagons. Thus, an inefficient assignment inside each cluster may diminish any benefit of this clustering idea.

As a more concrete example of this idea, consider the non-uniform disks model, where base stations (of the same radius-class) in co-colored hexagons must be at least $4a$ away, where $a$ is the side length of the hexagon (for class $L$, $a = 2^L$), to ensure that assigning common channels to such base stations would not cause interference. As previously discussed, this can be achieved by 12-coloring the hexagons. Another solution is to use clusters of 7 hexagons and 4-color them (see Figure 18). The reduction in number of colors is likely to improve the quality of the returned solution.

As the above example shows, one can achieve the same validity guarantees using different clustering/coloring schemes. The choice of which scheme to use depends highly on the designer. The trade-off here is as follow. Increasing the size of the cluster yields higher spectrum utilization, but with longer running time and higher locality and cascade values.

Our aim here is to provide the designer with enough information to make an informed decision as follows. For any cluster size $\mathcal{C}$, we present an algorithm to find a near-optimal assignment of channels in polynomial time. Not only we present guarantees on the improvement in the utilization over the hexagon division algorithm, we also bound the effect on both the local and cascade values in terms of $\mathcal{C}$.

**Outline of the Clustering Approach.** Again, the basic idea is to handle clusters of hexagons rather than single hexagons. Given a designer's choice of cluster size $\mathcal{C}$ and an appropriate number of colors $q_W$ (we will later show how to compute this number), the steps of the algorithm are as follows.

1) Divide the network region into hexagons of side-length one unit each.
2) Group the hexagons into clusters of $\mathcal{C}$ hexagons.
3) Uniformly-color the hexagons using $q_W$ colors.
4) Divide the available channels into $q_W$ equi-sized groups and associate each group with one of the colors used in the previous step.
5) Assign channels to base stations in each hexagon independently using only the group of channels associated with the color of the hexagon.

**Required Number of Colors.** As discussed in Section 3.3.3, the correctness of the algorithm can be established by finding a number of colors that satisfies Property (P2), i.e., it ensures that base stations in different clusters with the same color do not interfere. Thus, we must color the clusters in a way that the distance between any two points in different clusters of the same color is greater than $2d_1$. This is achieved using

$$q_W = \min\{x | x \geqslant (d+2\sqrt{H(\mathcal{C})})^2/3\mathcal{C}, x \geqslant 3, \text{ and } x = i^2+j^2+ij \text{ where } i,j \in \mathbb{Z}^+\} \tag{33}$$

colors, where

$$H(x) = \min\{y | y \geqslant x \text{ and } y = 3i^2 - 3i + 1 \text{ where } i \in \mathbb{Z}^+\}.$$

For $\mathcal{C} = 1, 3$ and 4 the term $\sqrt{H(\mathcal{C})}$ in Equation 28 is replaced by 1, 2 and 2.5 respectively.

We show the satisfaction of Property (P1) using arguments similar to Section 3.3.3 as follows. Given a clustering/coloring scheme with a cluster size of $\mathcal{C} \geqslant 7$ hexagons and $q_W$ colors, it can be shown that (i) the distance between the centers of two clusters[2] of the same color is at least $\sqrt{3\mathcal{C}q_W}$ and (ii) the maximum distance between any pair of points inside a cluster is at most $2\sqrt{H(\mathcal{C})}a$.[3] This means that the minimum distance between any two points in two different co-colored clusters is $(\sqrt{3\mathcal{C}q_W} - 2\sqrt{H(\mathcal{C})})a$ and Property (P2) follows. A concrete example is given below.

**EXAMPLE 3** Consider the pseudo-disk model with $d_1 = 3$. Then, to ensure validity, the minimum distance between any two base stations with common channels must be $4d_1^2/3 = 12$. Using the above equation gives the required number of colors for different cluster sizes as follows.

| Cluster size | 1 | 3 | 4 | 7 | 9 | 12 | 13 | 16 |
|---|---|---|---|---|---|---|---|---|
| Number of colors | 67 | 31 | 25 | 16 | 16 | 12 | 12 | 9 |

□

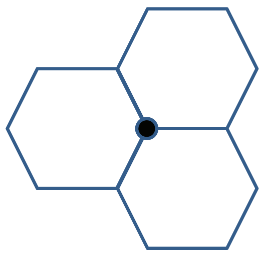
**Clustering the Hexagons.** The process of grouping the hexagons into uniform clusters is simple. We start with an origin (or central) hexagon. We then find central hexagons of the 6 neighboring clusters. Using an iterative trial-an-error technique, the hexagons are distributed among the 7 clusters used in this construction process. The resulting clustering is uniform in the sense that the distance between the centers of neighboring clusters is uniform,[4] where the center of a cluster is actually the center of its central hexagon.[5] For example, in Figure 18(a), consider the 12 hexagons of 12 different colors to be a cluster,

[2]For ease of presentation, we choose a designated hexagon for each cluster size to be the central hexagon of the cluster and define its center to be the center of the cluster.

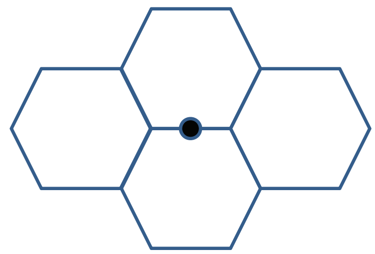
[3]To optimize the solutions, small clusters are treated differently. For clusters constituting of 1, 3 and 4 hexagons, the maximum distance between any pair of points inside a cluster is 1, 2 and 2.5 respectively (see Figure 19).

[4]Thus, it is not hard to see that uniform clusters can be viewed as forming an additional overlay of hexagonal grid themselves and that the techniques used on regular hexagons can be extended to work on them with minor modifications as later discussion shows.

[5]To optimize the solutions, small clusters are treated differently. See Figure 19.



(a) Center of a 3-hexagon cluster.



(b) Center of a 4-hexagon cluster.

Figure 19: Centers of small clusters.

then, hexagon #8 would be the central hexagon. Finally, it should be noted that in order to get a uniform clustering, the cluster size must be from the set

$$\{x | x = i^2 + j^2 + ij \text{ where } i, j \in \mathbb{Z}^+\}.$$

**Assignment in Each Cluster.** The last step of the algorithm abstracts away a lot of details about how to assign a set of channels $K'$ to the $N_C$ base stations of a certain cluster $C$ of size $\mathcal{C}$. Below, we discuss these details while giving proof sketches to the claims leading to the Theorem 14.

The first step is to ensure minimum demand satisfaction. This is the weighted One-Shot Scheduling problem [26] and intuitively speaking, it is equivalent to properly coloring the base stations of $C$ using the minimum number of colors. Although this is an NP-hard problem in general, it can be done here in $O(N_C^{\mathcal{C}})$ time due to the special geometric structure of the interference subgraph induced by $N_C$, for which any "independent" set has at most one base station from each hexagon of the cluster. Let the number of channels used here be $\chi_C$ (this is the chromatic number of the interference subgraph induced by $N_C$). This will introduce a multiplicative factor of

$$\frac{|K'|}{|K'| - \chi_C}$$

to the approximation ratio due to the following two facts:

- Considering the same setting, the utility of an optimal allocation that *does not* ensure that each base station has at least one channel cannot be smaller than the utility of an optimal allocation that *does* ensure this.

- Since the incremental utilities are non-increasing functions, the following holds. Considering the same setting, the utility of an optimal allocation that uses $k$ channels cannot be greater than $k/(k-k')$ times the utility of an optimal allocation that uses $k' < k$ channels.

Nonetheless, this difficult problem can be avoided (and thus, greatly improving the performance) by using a $\delta_C$-approximation for $\chi_C$, where $\delta_C$ is the network interference degree [2] of the interference subgraph induced by $N_C$ (i.e., it is the maximum size of an interference-free set in the neighborhood of a single base station in $C$). This approach is even more favorable when considering the more complicated physical interference model as discussed in Section 5.3.2.

The remaining $|K'| - \chi_C$ are distributed among $N_C$ using the same greedy approach of the hexagonal division, but here it is allowed to assign a channel to more than one base station whenever possible. This part is a special case of the Greedy Algorithm of Chapter 2 and the same approximation ratio of $\min\{\mathcal{C}, \delta_{\max}\} + 1$ can be shown here, where $\delta_{\max}$ is the maximum interference degree in any cluster.

The above results in a non-contiguous assignment. To ensure contiguity, the assignment is scaled-down by a factor of $\min\{\mathcal{C}, q_w\}$ (note that the $q_w$ used here is defined in Equation 26).To see that a scaling-down factor of $\mathcal{C}$ is enough, consider the following remarks.

- Scaling-down the assignment of each base station is equivalent (in terms of feasibility) to scaling-up the total number of available channels by the same factor.

- The interference graph inside every single one of the $\mathcal{C}$ hexagons forming the cluster is a clique, and thus, the following holds about it.

– The total number of assigned channels by a non-contiguous assignment cannot exceed the number of available channels.[6]

– Any non-contiguous assignment can be easily converted into a contiguous one as discussed in the previous section.

Informally speaking, given a non-contiguous assignment of $k$ channels to base stations inside a cluster of size $\mathcal{C}$, it is easy to convert it to a contiguous assignment using $\mathcal{C}k$ channels. The same argument holds for the $q_w$ scaling factor with the exception that if hexagons of the cluster are re-colored using $q_w$ colors, then co-colored hexagons can use the same set of channels. Now, since the incremental utilities are non-increasing functions, scaling-down the assignments means that the approximation ratio is scaled-down by the same factor.

Finally, a greedy maximal packing technique is used to distribute the remaining channels without violating the interference constraints. This is useful because the scaling down factor is rather conservative and will probably leave some "gaps" (in the form of unassigned channels) in the assignment. The packing technique simply keeps track of these channels and greedily assigns them to the base stations with the highest utility. The below theorem follows from this discussion.

**Theorem 14** *For the pseudo-disk model, the clustering algorithm returns an interference-free* $(q_W(\min\{\mathcal{C}, q_w\}+1)|K|/(|K|-\chi_{\max}))$*-approximate assignment in time polynomial in the size of the input, where* $q_W$ *and* $q_w$ *are as defined in Equations 33 and 26 and* $\chi_{\max}$ *is the maximum number of channels needed to optimally satisfy the demands of base stations in any cluster. Furthermore, it meets the other objectives of contiguity and minimum demand satisfaction while being* $2H(\mathcal{C})$*-local* $2H(\mathcal{C})$*-cascade.* □

The above bounds on the cascading effects of the clustering technique is mainly of theoretical importance. In practice, there is no real need to force all base stations of a certain cluster to be involved in the re-assignment process.

[6]In contrast to cases where the interference graph is not complete, and thus, allows for channel reuse.

Instead, smarter heuristics can be designed. One example in the case of base station addition is to run the above algorithm in "simulation mode" to figure out how many channels to assign to the new base station followed by a search trying to do the assignment in a "minimally invasive" way, thus, reducing the number of actually affected base stations. It is worth mentioning that such heuristics will not improve the above bounds.

### 5.3.2 Physical Model

As for the physical interference model, the same techniques apply here but with more complicated analysis. Here, we assume that each base station operates using the same transmission power. Techniques of Section 3.3.2 can be used to extend our results below to the case wherein this assumption is relaxed. For simplicity, we assume ambient noise to be zero; non-zero noise can be incorporated using techniques similar to Section 2.4.

For a cluster of size $\mathcal{C} \geqslant 7$, the number of colors has to be at least

$$q_H = \min\{x | x \geqslant q'_H, x \geqslant 7, \text{ and } x = i^2 + j^2 + ij \text{ where } i, j \in \mathbb{Z}^+\}, \quad (34)$$

$$\text{where } q'_H = \frac{7}{\mathcal{C}(\sqrt[\alpha]{\beta} + 1)^2} \left[ 6\beta\mathcal{C} \left( \frac{4}{\alpha - 2} \left( \frac{1}{\sqrt{7}(9\sqrt{7} - 2\sqrt{19}) - 3} \right)^{\alpha} + \left( \frac{6}{3\sqrt{21} - 2\sqrt{19}} \right)^{\alpha} \right) \right]^{\frac{2}{\alpha}}$$

For smaller cluster sizes, more optimization is needed since the centers of such clusters are chosen differently. For $\mathcal{C} = 3$, the number of colors has to be at least

$$q_3 = \min\{x | x \geqslant q'_3, x \geqslant 3, \text{ and } x = i^2 + j^2 + ij \text{ where } i, j \in \mathbb{Z}^+\}, \text{ where}$$

$$q'_3 = \frac{3}{(\sqrt[\alpha]{\beta} + 1)^2} \left[ 18\beta \left( \frac{2^{2-\alpha}}{\alpha - 2} + \left( \frac{2}{3\sqrt{3} - 5} \right)^{\alpha} \right) \right]^{\frac{2}{\alpha}},$$

while for $\mathcal{C} = 4$, the number of colors has to be at least

$$q_4 = \min\{x | x \geqslant q'_4, x \geqslant 4, \text{ and } x = i^2 + j^2 + ij \text{ where } i, j \in \mathbb{Z}^+\}, \text{ where}$$

$$q'_4 = \frac{4}{(\sqrt[\alpha]{\beta} + 1)^2} \left[ 24\beta \left( \frac{4}{3^{\alpha}(\alpha - 2)} + \left( \frac{1}{2\sqrt{3} - 3} \right)^{\alpha} \right) \right]^{\frac{2}{\alpha}}.$$

Finally, using $\mathcal{C} = 1$ means using the basic hexagonal division technique of the previous section.

Note that using the clustering algorithm requires finding the chromatic number of a given cluster, which in turn uses the notion of independence. Remember that a set of base stations, $V^*$, is considered *independent* if and only if for every base station $u \in V^*$, the SINR value at any point in the coverage region of $u$ is at least as large as $\beta$ even if all base stations are concurrently transmitting on the same channel. A simple conservative way to check this condition at each base station $u \in V^*$ would be to take the maximum level of interference caused by any other base station $v \in V^*$, $v \neq u$, at any point in the coverage region of $u$. This means that if the following equation is satisfied for every $u \in V^*$, then the set $V^*$ is independent.

$$\frac{\frac{P}{r^\alpha}}{\sum_{v \in V^*, v \neq u} \frac{P}{(d_v - r)^\alpha}} \geqslant \beta, \tag{35}$$

where $P$ is the power and $d_v$ is the distance between base stations $u$ and $v$. Obviously, the above constraint is too conservative and it is easy to come up with examples where independent sets fail to satisfy it. To get exact and reliable results, one must use the point $p_u = (x_{p_u}, y_{p_u})$ inside the coverage region of $u$ where the interference from other base stations is maximized. The $xy$-coordinates of this point is the solution of the following non-linear program.

$$\text{Min} \quad \frac{((x_{p_u} - x_u)^2 + (y_{p_u} - y_u)^2)^{-\alpha/2}}{\displaystyle\sum_{v \in V^*, v \neq u} ((x_{p_u} - x_v)^2 + (y_{p_u} - y_v)^2)^{-\alpha/2}} \quad \text{s.t.} \tag{36}$$

$$(x_{p_u} - x_u)^2 + (y_{p_u} - y_u)^2 \leqslant r^2,$$

where the pair $(x_u, y_u)$ denote the $xy$-coordinates of base station $u$. Now, we can use $p_u = (x_{p_u}, y_{p_u})$ in the SINR equation (Equation 1). If this condition is satisfied at every base station $u \in V^*$, then the set $V^*$ is indeed independent. Now, the methods of Chapter 2 can be used to approximate the chromatic number of base stations in any given cluster to a factor of

$$q^* = \min(q_1, q_2, q'_1, q'_2), \tag{37}$$

where $q_1$, $q_2$, $q'_1$ and $q'_2$ are as defined in Equations 12, 13, 23 and 24.

Finally, performing the greedy maximal packing part of the above algorithm requires using a method to check the independence of sets of base stations but it need not be exact. Thus, the conservative constraint of Equation 35 is preferred due to its simplicity and efficiency.

The following theorem states the properties of the clustering algorithm for the physical interference model. The proof is given in the appendix.

**Theorem 15** *For the physical interference model, the clustering algorithm returns an interference-free* $(q_H(\min\{\mathcal{C}, q_h\}+1)|K|/(|K|-q^*\chi_{\max}))$*-approximate assignment in time polynomial in the size of the input, where* $q_H$*,* $q_h$ *and* $q^*$ *are as defined in Equations 34, 28 and 37 respectively and* $\chi_{\max}$ *is the maximum number of channels needed to optimally satisfy the demands of he base stations in any cluster. Furthermore, it meets the other objectives of contiguity and minimum demand satisfaction while being* $2H(\mathcal{C})$*-local* $2H(\mathcal{C})$*-cascade.*

*Proof:* The proof is similar to that of Theorem 14 with the exception of the validity part. To prove that the resulting assignment is valid, it is sufficient to show that the conservative SINR equation (Equation 35) is satisfied. We will only show this for the case where $\mathcal{C} \geqslant 7$; remaining cases can be proven using the same technique.

Consider a base station $i$ in a cluster $C$. Note that each cluster of size $\mathcal{C}$ can have at most $\mathcal{C}$ base stations active at the same time. Nonetheless, the greedy algorithm for channel assignment inside each cluster ensures that no other base station $j$ in $C$ interferes with $i$'s transmission.[7] Let the color of $C$ be $b$. Partition all $b$-colored hexagons surrounding $C$ into hierarchical levels. In a uniform-coloring, the first level will contain $6\mathcal{C}$ clusters of color $b$ and each such hexagon $C'$ is at distance[8] of at least $(\sqrt{3\mathcal{C}q_H} - 2\sqrt{H(\mathcal{C})})R$ from $C$. Similarly, the second level contains $12\mathcal{C}$ hexagons at a distance of at least

[7] This greedy algorithm is of conservative nature. It ensures that the set of base stations assigned a common channel are independent regardless of clusters. Thus, if the greedy algorithm allows two base stations of the same cluster to be transmitting on the same channel, then we need not worry about the interference caused by base stations in other co-colored clusters.

[8] By distance between two clusters we mean that the distance between *any* point in $C'$ and *any* point in $C$.

$(3\sqrt{\mathcal{C}q_H} - 2\sqrt{H(\mathcal{C})})R$ from $C$. In general, the $l^{th}$ level contains $6l\mathcal{C}$ hexagons at a distance of at least $(\frac{3}{2}\sqrt{\mathcal{C}q_H}l - 2\sqrt{H(\mathcal{C})})R$ from $C$.

Now consider a point $p$ within the communication radius $r$ from the base station $i$. Then, the total signal received at the point $p$ due to all other base stations (at most one per $b$-colored hexagon) active on the same channel as $i$ is at most:

$$I = I_1 + I_{2\geqslant},$$

where $I_1$ and $I_{2\geqslant}$ are the interference caused by the first and second and above levels of the hierarchy respectively, and are bounded as follows.

$$\begin{aligned}
I_1 &\leqslant \frac{6\mathcal{C}P}{\left((\sqrt{3\mathcal{C}q_W} - \frac{2}{3}\sqrt{19\mathcal{C}})(\sqrt[\alpha]{\beta}+1)r/2\right)^\alpha} \\
&\leqslant \frac{6\mathcal{C}P}{\left(\frac{\sqrt{3q_W}(3\sqrt{21}-2\sqrt{19})}{3\sqrt{21}}\sqrt{\mathcal{C}}(\sqrt[\alpha]{\beta}+1)r/2\right)^\alpha} \\
&= 6\mathcal{C}P\left(\frac{6\sqrt{7}}{\sqrt{q_W\mathcal{C}}(\sqrt[\alpha]{\beta}+1)(3\sqrt{21}-2\sqrt{19})r}\right)^\alpha .
\end{aligned}$$

The first equation follows from the definition of $R$ and the fact that $H(\mathcal{C}) \leqslant 19\mathcal{C}/9$ for $\mathcal{C} \geqslant 7$ while the second equation follows from the following two facts: (i) $\sqrt{3q_W} \geqslant \sqrt{21}$, and (ii) for $x \geqslant \sqrt{21}$, we have $x - 2\sqrt{19}/3 \geqslant x(3\sqrt{21} - 2\sqrt{19})/3\sqrt{21}$.

$$\begin{aligned}
I_{2\geqslant} &\leqslant \sum_{l=2}^{\infty} \frac{6\mathcal{C}lP}{\left(\left(\frac{3}{4}\sqrt{q_W}l - \frac{\sqrt{19}}{3}\right)\sqrt{\mathcal{C}}(\sqrt[\alpha]{\beta}+1) - 1\right)^\alpha r^\alpha} \\
&\leqslant \frac{6\mathcal{C}P}{r^\alpha}\sum_{l=2}^{\infty} \frac{l}{\left(\left(\frac{3\sqrt{q_W}l}{4} \cdot \frac{2(\frac{9}{2}\sqrt{7}-\sqrt{19})}{9\sqrt{7}}\right)\sqrt{\mathcal{C}}(\sqrt[\alpha]{\beta}+1) - 1\right)^\alpha} \\
&\leqslant \frac{6\mathcal{C}P}{r^\alpha}\sum_{l=2}^{\infty} \frac{l}{\left(\frac{(\frac{9}{2}\sqrt{7}-\sqrt{19})\sqrt{q_W\mathcal{C}}(\sqrt[\alpha]{\beta}+1)l(2\sqrt{7}(\frac{9}{2}\sqrt{7}-\sqrt{19})-3)}{2\sqrt{7}(\frac{9}{2}\sqrt{7}-\sqrt{19})}\right)^\alpha} \\
&\approx \frac{24\mathcal{C}P}{\alpha-2}\left(\frac{\sqrt{7}}{(2\sqrt{7}(\frac{9}{2}\sqrt{7}-\sqrt{19})-3)\sqrt{q_W\mathcal{C}}(\sqrt[\alpha]{\beta}+1)r}\right)^\alpha .
\end{aligned}$$

The second equation follows from the following two facts: (i) $3\sqrt{q_W}l/4 \geqslant 3\sqrt{7}/2$, and (ii) for $x \geqslant \sqrt{19}/3$, we have $x - \sqrt{19}/3 \geqslant x(\frac{9}{2}\sqrt{7} - \sqrt{19})/\frac{9}{2}\sqrt{7}$.

Similarly, the third equation follows from the following two facts: (i) $(\frac{9}{2}\sqrt{7} - \sqrt{19})\sqrt{q_W \mathcal{C}}(\sqrt[\alpha]{\beta}+1)l/6\sqrt{7} \geqslant (\frac{9}{2}\sqrt{7} - \sqrt{19})2\sqrt{7}/3$, and (ii) for $x \geqslant (\frac{9}{2}\sqrt{7} - \sqrt{19})2\sqrt{7}/3$, we have $x - 1 \geqslant x(2\sqrt{7}(\frac{9}{2}\sqrt{7} - \sqrt{19}) - 3)/2\sqrt{7}(\frac{9}{2}\sqrt{7} - \sqrt{19})$. Then,

$$I \leqslant 6\mathcal{C}P\left(\frac{\sqrt{7}}{\sqrt{q_W \mathcal{C}}(\sqrt[\alpha]{\beta}+1)r}\right)^{\alpha} \times$$
$$\left(\left(\frac{6}{3\sqrt{21}-2\sqrt{19}}\right)^{\alpha} + \frac{4}{\alpha-2}\left(\frac{1}{2\sqrt{7}(\frac{9}{2}\sqrt{7}-\sqrt{19})-3}\right)^{\alpha}\right).$$

Substituting in the SINR equation gives $P/(r^{\alpha} + I) \geqslant \beta$. ∎

## 5.4 Simulation

In this section, we present simulation results depicting some trends in the performance of our algorithms. The main comparison here is between the hexagonal division technique of Section 5.2 and the clustering technique of Section 5.3 for various choices of cluster sizes. We start by showing one of the main trade-offs of our algorithms, namely the spectrum utilization vs. cascading effect. Next we examine how well our algorithms perform with increasing network density. We start by describing the simulation parameters and then present the results.

**Network Model.** The following lists the settings we use in our experiment:

- In order to examine the impact of network topology, we consider two types of networks.
  - *Random Networks:* We consider a fixed area of $50000 \times 50000$ units and randomly place base stations within this area. We vary the network density by changing the number of base stations from 250 to 1500 with the default being 1000 base stations.
  - *Real Networks:* We use locations of real cellular base stations available in FCC public GIS database [1] and choose base stations deployed in 4 different regions of increasing size and number of base stations.

    * R1 - 256 base stations in the state of MA
    * R2 - 917 base stations in New England area (MA, ME, NH, VT, RI, CT)
    * R3 - 1727 base stations in New England and New York
    * R4 - 3052 base stations in North East USA (New England, NY, NJ, PA)

  Here the regions are progressively supersets of the previous ones. The default is R3.

- The interference model we choose to demonstrate our work is the unit-disk model, where each base station has a randomly-chosen communication radius $r$ ranging between 500 and 2500 units.

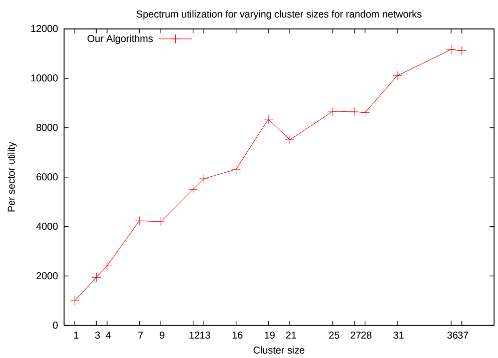
- For each base station, we randomly choose a number of sectors between 1 and 6 to sectorize its coverage region and occasionally "drop" some sectors to model base stations whose coverage regions are not complete disks. Each sector is treated as a separate "base station" and sectors of the same base station may compete with each other over available spectrum.

- The available spectrum we consider here is divided into 500 channels.

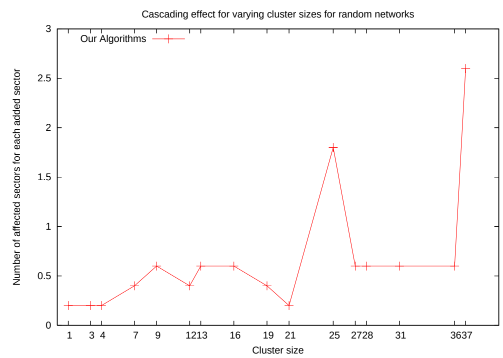
- We generate utility functions for each base station as follows. For base station $i$, we randomly choose a value for the first channel, $u_i(1)$ from the range [1,1000]. For every subsequent number of channels, $k > 1$, we choose $u_i(k)$ from the range $[1, u_i(k-1)]$. Sectors of the same base station use the same utility function.

Each experiment is repeated 5 times and the averages are reported.

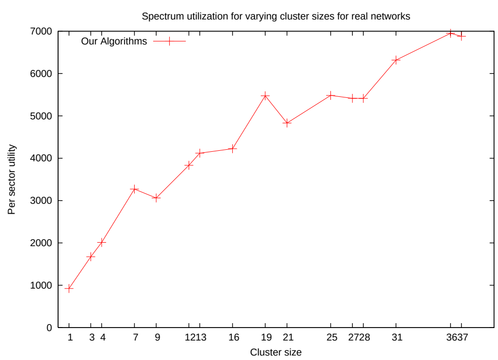
**Spectrum Utilization vs. Cascading Effect.** Our first set of experiments addresses the trade-off our algorithms faces between the spectrum utilization and the cascading effect. For a random network of 1500 base stations (sectorized into a total of more than 5000 sectors), Figure 20(a) shows how increasing



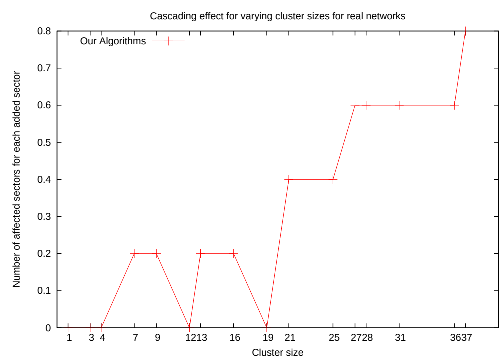

(a) Spectrum utilization for random networks




(b) Cascading effect for random networks




(c) Spectrum utilization for real networks



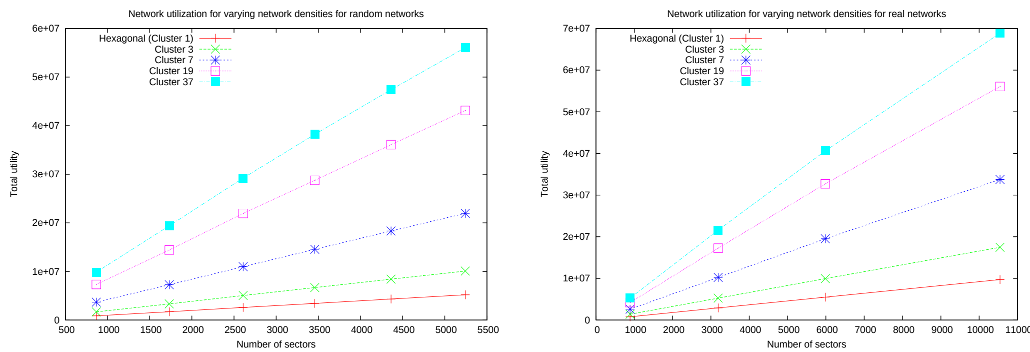

(d) Cascading effect for real networks

Figure 20: Spectrum utilization vs. cascading effect.

the cluster size yield an immediate increase in spectrum utilization.[9] On the other hand, the cascading effect (Figure 20(b)) remained limited even for large cluster sizes. We see similar trends for real networks (Figures 20(c) and 20(d)) as well. The fluctuation in the cascading effects plots (Figures 20(b) and 20(d)) is of minimal importance since the numbers are very small. For example, in random networks, every added sector affects less than 3 other sectors (in the worst case) out of more than 5000 sectors.

**Increasing Network Density.** In this experiment, we examine how well our algorithms perform with increasing network density. We show this for the hexagonal division technique as well as the clustering technique with four

[9]Figures 20(a) and 20(c) use per-sector utility for the $y$-axis. This is due to the random nature of the sectorization process. I.e., for each one of the 5 runs, a different number of sectors is generated, and thus, the total utility cannot be used.




(a) Increasing network density for random networks

(b) Increasing network density for real networks

Figure 21: Increasing network density.

choices of cluster sizes. As mentioned above, we vary the number of base stations from 250 to 1500 base stations for random networks and from 256 to 3052 for real networks. This corresponds to a variation in the number of sectors from 867 to 5243 sectors for random networks and from 890 to 10545 for real networks. Figure 21 shows that with the increase in network density, the total spectrum utility increases for both random and real networks. Furthermore, using larger cluster sizes resulted in a higher increase rate in spectrum utility making our clustering technique even more appealing when the network is scaled up.

## 5.5 Conclusion

For a distributed spectrum allocation approach, the goal is to generally produce more stable allocation that can maintain certain properties with minimal cost and human intervention even when faced by frequent network topology changes. This is demonstrated by problems such as self-configuration of fractional frequency reuse (FFR) patterns for LTE/WiMAX networks. The often-conflicting objectives of this problem (like spectrum utilization, cascading effects, etc.) force the system designer to deal with many trade-offs. We present a flexible tool for this purpose specifically. I.e., our tool gives the designer a choice of parameter values and for every choice, certain bounds are guaranteed on all aspects of the trade-offs. Through simulations, we show how these choices affect the final result in practical scenarios.

# Chapter 6

# Conclusion

To increase spectrum utilization, recent studies suggest using a dynamic spectrum access model in both spatial and temporal dimensions. For such a model, the spectrum is divided into channels and is periodically allocated to the base stations in both centralized and distributed manners with different goals in mind for each approach.

For the centralized approach, we present auction-based algorithms that are simple, efficient and produce high utilization of the spectrum. We start with a simplistic revenue-maximizing algorithm that disregards economical aspects of the allocation process. Next, we present more involved auctions with the goal of maximize some social-choice function like social-welfare or revenue while controlling the strategic behavior of the base stations.

We also presented a distributed approach, wherein the focus is shifted towards more stable allocation that can maintain certain properties with minimal cost and human intervention even when faced by frequent network topology changes. This is demonstrated by problems such as self-configuration of fractional frequency reuse (FFR) patterns for LTE/WiMAX networks. Our distributed algorithms provide the network designer a flexible tool to tune different objectives like efficiency, stability and near-optimal spectrum utilization. For each possible choice made by the system designer, our tool delivers a near-optimal spectrum utilization with specific guarantees on the rest of the desired properties.

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