Non-recurrent dynamics in the exponential family

A Dissertation presented

by

Anna Benini

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of Doctor of Philosophy

in

Mathematics

Stony Brook University

August 2010

Stony Brook University

The Graduate School

Anna Benini

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Mikhail Lyubich - Dissertation Advisor
Professor, Institute of Mathematical Sciences, Stony Brook University

Marco Martens - Chairperson of the Defense
Associate Professor, Institute of Mathematical Sciences, Stony Brook University

John Milnor
Professor, Institute of Mathematical Sciences, Stony Brook University

Saeed Zakeri
Associate Professor, Department of Mathematics, CUNY, Queens Coll and CUNY, GSUC

This dissertation is accepted by the Graduate School.

Lawrence Martin
Dean of the Graduate School

Abstract of the Dissertation

Non-recurrent dynamics in the exponential family

by

Anna Benini

Doctor of Philosophy

in

Mathematics

Stony Brook University

2010

This dissertation deals with the dynamics of non-recurrent parameters in the exponential family $\{e^z + c\}$. One of the main open problems in one-dimensional complex dynamics is whether hyperbolic parameters are dense; this conjecture can be restated by saying that all fibers, i.e. classes of parameters with the same ray portrait, are single points unless they contain a hyperbolic parameter. The main goal of this dissertation was to prove some

statements in this direction, usually referred to as rigidity statements.

We prove that fibers are single points for post-singularly finite (Misiurewicz) parameters and for combinatorially non-recurrent parameters with bounded post-singular set. We also prove some slightly different rigidity statement for combinatorially non-recurrent parameters with unbounded postsingular set.

We also add some understanding to the correspondence between combinatorics of polynomials and combinatorics of exponentials and we prove hyperbolicity of the postsingular set for non-recurrent parameters, generalizing a previous statement concerning only non-recurrent parameters with bounded post-singular set.

We finally contribute to another open problem in transcendental dynamics, i.e. understanding whether repelling periodic orbits are landing points of dynamic rays, giving a positive answer to this question in the case on non-recurrent parameters with bounded post-singular set.

The strategy used also gives a new, more elementary proof of the corresponding statement for polynomials, dating back to work of Douady.

Contents

Acknowledgements

I would like to thank my advisor Prof. Misha Lyubich for the enjoyable time spent doing mathematics, and for being a committed and patient guide in this mathematical journey, as well as for always allowing much freedom of choice.

Special thanks are also due to Prof. Nuria Fagella, for collaborating during the ESR position in Barcelona in my last year of PhD studies and her helpfulness on many occasions.

Thanks are also due to Profs. Lasse Rempe, Dierk Schleicher, Carsten Petersen, for many extended discussions and suggestions; to the dynamics group in Stony Brook, and to Davoud Cheraghi, Profs. Mitsuhiro Shishikura, Dennis Sullivan, Folkert Tangerman, and many others for listening and offering comments and ideas. Also, this dissertation would not be here if Prof. Stefano Marmi had not encouraged me to pursue a mathematical career.

I also thank my friends and dance teachers for keeping my soul alive during the intense PhD study.

The pictures are drawn with the computer program ItFun graciously provided by Prof. N. Fagella.

Chapter 1

Introduction

This dissertation originated in the idea of extending the puzzle construction and the renormalization techniques to the exponential family in order to generalize some rigidity results obtained in the families of unicritical polynomials.

The outcome were some rigidity statements concerning Misiurewicz (postsingularly finite) and non-recurrent parameters, some topological statement for non-recurrent parameters and some theorems about accessibility of repelling periodic orbits for non-recurrent parameters with bounded postsingular set, as well as accessibility of the postsingular set itself; this also gives a new more elementary proof of Douady's classical theorem stating that repelling periodic points are landing points of dynamic rays.

We will now embed the questions above in their natural setting and explain them in more detail, assuming the reader to be familiar with the iteration of unicritical polynomials (see e.g. [DH], [Mi1]).

One of the main open problems in the dynamics of rational maps is the so-called Density of Hyperbolicity Conjecture. A map is called hyperbolic if all of its critical values converge to an attracting cycle under iteration; this maps are particularly well understood in many ways.

Conjecture 1.0.1 (Density of hyperbolicity). *Hyperbolic maps are dense in the space of unicritical polynomials* $z^d + c$.

This conjecture can be stated also for rational maps, however it has been most actively worked on especially in the case of quadratic polynomials: one of the breakthrough results has been density of hyperbolicity in the family of real quadratic polynomials ([L1], [GS]).

There are many ways to restate Conjecture 1.0.1, the one we will be mostly interested in is in terms of fibers and combinatorics. To explain their meaning in a more precise way requires a more careful analysis of the role of periodic and preperiodic dynamic rays. We will refer as a *ray pair* to two periodic or preperiodic rays landing at the same point z, together with z. In this way, a ray pair is a curve separating the plane in two regions.

The collection of all periodic rays landing together gives a partition of the plane and encodes part of the topological dynamics: for example q rays landing at a fixed point subdivide $\mathbb{C}$ into q sectors on which the dynamics is determined by the dynamics of the rays. Using the fact that dynamic rays are labeled by angles, it is possible to abstract this topological picture into a combinatorial notion: for example, by considering the unit circle $\mathbb{S}^1$ and considering the equivalence classes of angles corresponding to rays which land at the same point . This set of equivalence classes is often referred to as *combinatorics* of the map, even though the word combinatorics is also used with other meanings. It can be shown that this combinatorial notion is a topological invariant and the natural question is whether this combinatorial invariant is sufficient to determine a polynomial up to conformal conjugacy.

The *fiber* of a parameter should be thought of as the set of parameters which have the same combinatorics (see Chapter 4). In terms of fibers the conjecture of density of hyperbolicity can be restated as follows:

Conjecture 1.0.2 (Rigidity conjecture). *Let* $f_c(z) = z^d + c$. *The fiber of* *any parameter is a single point, unless it contains hyperbolic parameters.*

There has been a great amount of work on combinatorics and on the rigidity conjecture, mostly for quadratic and unicritical polynomials.

As the critical point is the unique branching point, its dynamics influences most of the dynamics of the map as a whole. In particular, some specific kinds of recurrent behavior (i.e. when the critical orbit accumulates on the critical point) is what prevents the whole conjecture to be solved.

The Rigidity conjecture for unicritical polynomials has been proven for non-recurrent parameters, as well as many classes of recurrent ones, parameters on boundaries of hyperbolic components, and under some geometric/growth conditions. Still remain unknown some cases in which the critical value is recurrent in a persistent way. We will be concerned with the exponential family $e^z + c$, for which the natural parameter space is $\mathbb{C}$. There are no critical points, and only one asymptotic omitted value c, which has in many ways the same role as the critical value for unicritical polynomials (see Theorem 2.2.5), as the only branching point of an exponential function can be considered to be $-\infty$. This family has been studied, among others, by Eremenko and Lyubich ([EL]), Baker and Rippon ([BR]), Devaney and collaborators, and by Rempe, Schleicher and Zimmer more recently.

Dynamic and parameter rays have been defined in the exponential family (see [BD1],[SZ1],[S0]; see also Section 3.1). For the moment, they can be thought of an analog of the dynamic and parameter rays in the polynomial families; the two main differences are that dynamic rays lie in the Julia set instead of the Fatou set, and that there is no notion of equipotential curves.

Similarly, the postsingular set is defined as $\mathcal{P} := \overline{\bigcup_{n>0} f^n(c)}$. The notion of fibers generalizes to the exponential family (see [RS2]), and so do the Density of hyperbolicity conjecture and the Rigidity conjecture. This thesis proves the conjecture in the simplest non-recurrent cases, namely the case in which the postsingular set is finite (Misiurewicz case) and in a special non-recurrent case called *combinatorial non-recurrence*, in which the singular value and the postsingular set are required to be separated by a special collection of ray pairs (see Section 6.2).

We now describe in more detail the results we obtained and an idea of their proofs.

Misiurewicz parameters

We call a parameter Misiurewicz or postsingularly finite if the singular value is strictly preperiodic. We show that fibers of Misiurewicz parameters are single points in the exponential family.

Theorem 1.0.3 (Triviality of Misiurewicz fibers.)**.** *Let c_0 be a Misiurewicz parameter in the exponential family. Any parameter c can be separated from c_0 by a parameter ray pair, except for those parameters lying on the parameter rays landing at c_0.*

We follow the outline of a proof of Schleicher ([S1]) for the corresponding result in families of unicritical polynomials. The proof consists in showing that any parameter can be separated from a Misiurewicz parameter using curves formed by parameter rays landing together with their common endpoint.

First pairs of rays landing together are found in the dynamical plane of the Misiurewicz parameter and shown to be persistent in a parameter neighborhood, then they are transfered to the parameter plane using results of Rempe and Schleicher ([RS2], [RS1]) on the structure of bifurcations in the exponential family.

The main originality in the proof lies in tracing a combinatorial correspondence between the space of angles and orbit portraits for unicritical polynomials on one side and exponentials on the other side (see Chapter 3), and using it to transfer pairs of rays landing together from polynomial families to exponential families. Combinatorial similarities between exponentials and polynomials had been pointed out before by Devaney and Schleicher.

It was shown in [SZ2] that the singular value is the landing point of a dynamic ray. The preimages of this dynamic ray give a dynamic partition (see Section 3.1) allowing the explicit statement of the combinatorial correspondence and working as a bridge to translate combinatorial notions into topological notions. A key role is also played by expansivity near the post-singular periodic orbit.

Non-recurrent parameters

We say that a parameter is non-recurrent if the post-singular set $\mathcal{P}$ does not contain the singular value. In this case we can show that the postsingular set is expansive with respect to the euclidean metric.

Theorem 1.0.4 (Hyperbolicity of the postsingular set.)**.** *Let $f_c(z) = e^z + c$ such that c is non-recurrent. Then the postsingular set $\mathcal{P}$ is expansive with respect to the euclidean metric, in the sense that there are some integer k and some $\eta > 1$ such that for any $x \in \mathcal{P}$, $|(f^n)'(x)| > \eta$ for $n > k$.*

This generalizes a result of Rempe and van Strien ([RvS])(which, however, holds for a much larger class of transcendental functions) to non-recurrent exponential maps with unbounded postsingular set. The proof is similar to the proof of Mañe's Theorem for rational maps given by Shishikura and Tan Lei ([LS]), using the expansivity properties of the exponential map near infinity to compensate for the fact that the postsingular set can be unbounded.

We then give a description of puzzle and parapuzzle construction for the exponential family (see Section 6.2.1); one of the main differences with the polynomial case is that puzzle/parapuzzle pieces are not bounded, as there are no equipotential curves. However, the basic dynamic properties of puzzle pieces are preserved, as well as the fact that parapuzzle pieces of a given level n identify the set of parameters which are combinatorial equivalent up to level n, i.e. having the same puzzle pieces up to that level.

We will say that a parameter is *combinatorially non-recurrent* if some level of the puzzle construction separates the singular value from the postsingular set.

The persistence of puzzle pieces over parapuzzle pieces allows us to use holomorphic motions to construct conjugacies on the rays of two functions belonging to the same parapuzzle pieces. This conjugacy on a subset of the rays can be extended to a quasiconformal map form $\mathbb{C} \to \mathbb{C}$ using a version of the Lambda Lemma due to Slodkowski ([Sl]), and pulled back to a conjugacy using relatively standard techniques under the assumption of combinatorial non-recurrence.

The final statement says that any two combinatorially non-recurrent exponential maps which belong to the same parapuzzle piece at all levels are quasi-conformally conjugate.

Theorem 1.0.5. *Let $f_c, f_{c'}$ be two maps belonging to the same parapuzzle piece at all levels, such that c is combinatorially non-recurrent under f_c and neither c nor c' are escaping. Then there is a quasiconformal conjugacy Ψ between f_c and $f_{c'}$.*

For non-recurrent parameters with bounded postsingular set, a theorem proven by Rempe and van Strien ([RvS]) states absence of invariant line fields; together with Theorem 1.0.5 this shows that fibers are single points for combinatorially non-recurrent parameters with bounded postsingular set.

The corresponding statement in the case of unbounded postsingular set, as well as showing that combinatorial non-recurrence implies non-recurrence at least in the bounded case, is still work in progress.

Accessibility

We will say that a point is *accessible* if there is a dynamic ray landing at it. We show that repelling periodic points are landing points of dynamic rays for non-recurrent parameters with bounded post-singular set.

Theorem 1.0.6 (Accessibility of periodic orbits for non-recurrent parameters)**.** *Let $f_c(z) = e^z + c$ be non-recurrent with bounded postsingular set; then any periodic point is the landing point of some periodic ray.*

In this case, we also show that each point in the postsingular set, and hence the singular value itself, is accessible.

Theorem 1.0.7 (Accessibility of the postsingular set)**.** *Let $f_c(z) = e^z + c$ be non-recurrent with bounded postsingular set; then any point in the postsingular set is landing point of some ray.*

This gives a dynamic partition (see Section 3.1), a tools which in the past has been particularly useful in the study of hyperbolic, parabolic and Misiurewicz parameters. To show this we first prove that there is a uniform bound on the length of some appropriate ray pieces (fundamental domains), which does not depend on the address for a specific set of addresses. We translate it into a bound on the hyperbolic length and use the Schwarz lemma and the boundary behavior of the hyperbolic metric to show that the length of such fundamental domains shrinks to zero under appropriate pullbacks.

We then use a linearizing neighborhood in the case of repelling periodic points, and the expansive neighborhood of the postsingular set to construct a sequence of dynamic rays converging to a dynamic ray and show that it lands at the desired point. The same strategy simplifies considerably in the case of polynomials, giving a new proof of Douady's theorem about accessibility of repelling periodic orbits for polynomials with connected Julia set.

Theorem 7.2.1 also gives a new proof of a theorem of Schleicher and Zimmer ([SZ2]) that the singular value is accessible for Misiurewicz parameters.

Chapter 2

Preliminaries

We will assume the reader to be familiar with the Julia-Fatou dichotomy and with the iteration of unicritical polynomials.

2.1 Notation

We will denote the unit disk by $\mathbb{D}$, the right half plane by $\mathbb{H}$, the complex plane by $\mathbb{C}$ and by $\hat{\mathbb{C}}$ the Riemann sphere. Given a set $A \subset \mathbb{C}$, $\overline{A}, \partial A$ are its closure and its boundary in $\mathbb{C}$ respectively.

We indicate with $B_r(z)$ the euclidean disk of radius r centered at z.

Given a non-constant, non-linear holomorphic entire function f, f^n is its n-th iterate,

$$F(f) = \{z_0, \text{ the family } \{f^n(z)\} \text{ is normal in a neighborhood of } z_0\}$$

is its *Fatou set* and

$$J(f) = \mathbb{C} - F(f)$$

is its *Julia set*.

The *set of escaping points* is

$$I(f) = \{z \in \mathbb{C}, \lim_{n\to\infty} |f^n(z)| = \infty\}.$$

Given a one parameter family $\{f_c\}$, Π_P will be its parameter plane and Π_c will be the dynamical plane for the function f_c.

2.2 Iteration of transcendental functions with bounded set of singular values

We call a function *transcendental* if infinity is an essential singularity. One of the other main differences with respect to rational dynamics is the presence of asymptotic values: we call z_0 an *asymptotic value* for a function f if there is a curve $\gamma : [0, \infty) \to \mathbb{C}$ such that $\gamma(t) \to \infty$ and $f(\gamma(t)) \to z_0$ as $t \to \infty$. An asymptotic value can be *omitted*, like in the case of c for the exponential family, it is has no preimage in $\mathbb{C}$.

A transcendental function is a local isomorphism outside its *singular set*

$$S(f) := \{\text{singularities for } f^{-1}\}$$

Remark 1. The following facts are equivalent:

- $S(f)$ is the singular set of f
- $S(f) = \overline{\{z \in \mathbb{C}, z \text{ is a critical value or an asymptotic value}\}}$
- $S(f)$ =$\{z$, for any neighborhood U of z , there is a component V of $f^{-1}(U)$ such that $f : U \to V$ is not univalent$\}$

Remark: a transcendental entire functions might have infinitely many asymptotic values accumulating on a finite point $x \in \mathbb{C}$, so that it is necessary to take the closure in the second item.

Points in $S(f)$ are called *singular values.*

Let us denote by B the class of all transcendental functions for which the singular set is bounded, and by S the class of all transcendental functions for which the singular set is finite.

It is proven in [EL] that class S there are no wandering components of the Fatou set ([BR], [EL]).

Theorem 2.2.1 (Classification of Fatou components)**.** *For $f \in S$, there are no wandering Fatou components. Every Fatou component is mapped to a cycle of attracting components, parabolic components or Siegel disks.*

Corollary 2.2.2. *If there are no attracting or parabolic orbits, nor cycles of Siegel disks, the Julia set is equal to $\mathbb{C}$.*

The Julia set is well known to be perfect, non-empty and completely invariant, and that it is either nowhere dense or equal to $\mathbb{C}$. There are other equivalent ways to define the Julia set.

Theorem 2.2.3. *Let J be the Julia set of $e^z + c$. Then*

$$J = \overline{\{\textit{Repelling periodic points}\}} \tag{2.2.1}$$

$$= \partial I(f) \tag{2.2.2}$$

$$= \overline{I(f)} \tag{2.2.3}$$

The first equality is due to Baker ([Ba1]) and the last equality follows from the second one and the proposition below ([EL], Theorem 1).

Theorem 2.2.4. *Let $f \in B$ be a transcendental function. The set of escaping points is contained in the Julia set.*

We define the *postsingular set* as

$$\mathcal{P} = \overline{\bigcup_{n>0} f^n S(f)}.$$

We will say that a singular value c is *non-recurrent* if $\mathcal{P} \cap \{c\} = \emptyset$, and that a map f is non-recurrent if all $x \in S(f)$ are non-recurrent.

From now on unless explicitly stated we will fix the one-parameter family $f_c(z) := e^z + c, c \in \mathbb{C}$.

The next theorem relates the behavior of the singular value to the presence of non-repelling periodic orbits.

Theorem 2.2.5 (Behavior of the singular value)**.** *Let $f = e^z + c$. If f has an attracting or parabolic orbit $\{z_i\}$, the singular value belongs to the immediate attracting basin and its iterates converge to $\{z_i\}$; if f has a cycle of Siegel disks $\{D_i\}$, the singular value is recurrent and its orbit accumulates on the union of the boundaries ∂D_i; if f has a cycle of Cremer points $\{w_i\}$, the singular value is recurrent and $\mathcal{P} \ni w_i$ for each w_i.*

If $\mathcal{P}$ is finite, then z is strictly preperiodic to a repelling periodic orbit.

Proof. The first part is a combination of classical theorems, see e.g. [Mi].

The last statement about postsingularly finite maps follows by remarking that c is an omitted value, hence cannot be periodic, and that none of its iterates f^n can belong to a non-repelling orbit by the previous part of the theorem. □

Definition 2.2.6 (Types of exponential maps)**.** We will call a parameter c *escaping* if $|f_c^n(c)| \to \infty$ as $n \to \infty$.

We will say that a parameter c (and the corresponding map f_c) is *attracting, parabolic, Siegel* or *Cremer* if it has a non repelling orbit of the corresponding type; we will say that is is *Misiurewicz* or *postsingularly finite* if $\mathcal{P}$ is finite.

Attracting parameters or maps are also referred to as *hyperbolic* parameters or maps.

We will define the *set of escaping parameters*

$$\mathcal{I} := \{c, f_c^n(c) \to \infty \text{ as } n \to \infty\}.$$

2.3 Structural Stability

We call a parameter c *structurally stable* if for every c' close to c the functions f_c and $f_{c'}$ are topologically conjugate.

We call a parameter c *J-stable* if for every c' close to c the function f_c and $f_{c'}$ are topologically conjugate on their Julia set.

Let $\mathcal{R}$ be the *set of structurally stable parameters*.

There are many equivalent characterizations of structural stability ([EL], [Ye]).

Theorem 2.3.1. *Let $c \in \mathbb{C}$. The following are equivalent:*

- *c is structurally stable.*
- *c is J-stable.*
- *Either $J(f_{c'}) = \mathbb{C}$ for all c' in a neighborhood of c, or $J(f_{c'}) \neq \mathbb{C}$ for all c' in a neighborhood of c.*
- *There is a neighborhood of c which contains no indifferent parameters*
- *The family of functions $\{c \mapsto f_c^n(c)\}$ is normal in c.*
- *for every c' close to c the functions f_c and $f_{c'}$ are quasiconformally conjugate.*

Also, structural stability is dense ([EL], Theorem 10).

Theorem 2.3.2. *The set $\mathcal{R}$ of structurally stable parameters for the exponential family is dense in $\mathbb{C}$. The conjugating homeomorphism can be chosen to be quasiconformal.*

The *bifurcation locus* $\mathcal{B} := \mathbb{C} \setminus \mathcal{R}$ is the set of parameters at which the dynamics changes, and is the equivalent of the boundary of the Mandelbrot set. It follows from [MSS] (see for example [R0], Theorem 5.1.5) that $\mathcal{B} = \partial I$.

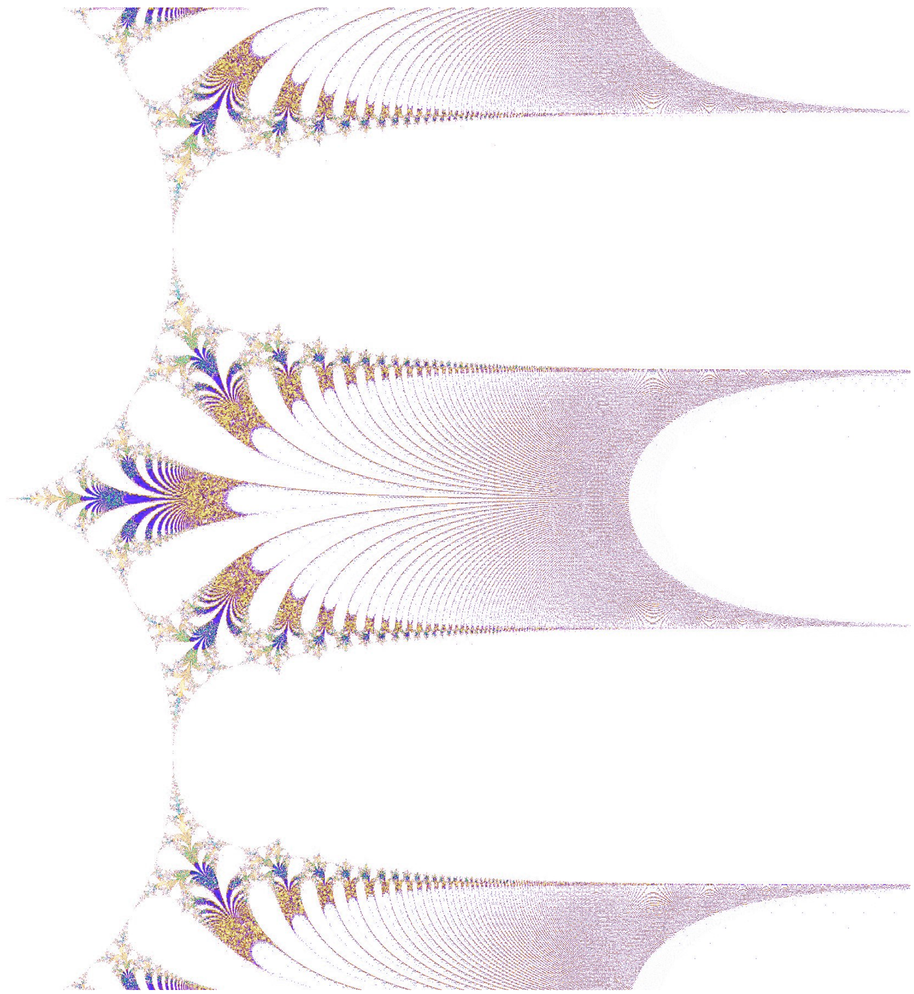

Figure 2.1: Parameter plane for the family $e^z + c$. The period one hyperbolic component can be seen in white on the left, as well as the period two components bifurcating from it. The dynamic rays are the darker curves on the right side. The white regions between hyperbolic components are due to the algorithm.

Very little of its topological properties are known, however we have the following results ([RS2], Theorem 2 and Theorem 5).

Theorem 2.3.3. *$\mathcal{B}$ is a connected subset of $\mathbb{C}$.*

We will say that a set $S \subset \mathbb{C}$ is *locally connected* at $x \in S$ if for each neighborhood U of x there is a smaller neighborhood $V \subset U$ of x such that $V \cap S$ is connected.

Failure of local connectivity for the bifurcation locus of exponentials is due to the fact that $\mathcal{B}$ contains the set of escaping parameters, and that escaping parameters themselves are organized in curves going to infinity and accumulating on each other creating a topological picture similar to the non-locally connected comb space (see Theorem 3.1.5).

In the exponential family the bifurcation locus consists of non-escaping parameters, escaping parameters which are on parameter rays and escaping parameters which are endpoints of such parameter rays (see Section 3.1).

We will state the theorem about non local connectivity of the bifurcation locus in terms of parameter rays (see Section 3.1).

Theorem 2.3.4. *$\mathcal{B}$ is not locally connected at any escaping parameter which is not the endpoint of a parameter ray.*

This theorem follows from the fact that any escaping parameter which is not an endpoint of a parameter ray is in the accumulation set of infinitely many other parameter rays. The construction in [RS2], following ideas from [De], perturbs a small piece L of parameter ray into a countable set of curves approximating L and belonging to hyperbolic components, hence disconnecting L from the other escaping parameters contained in any neighborhood of L.

Like for many other one-parameter families, One of the main open problems is the so called *density of hyperbolicity* conjecture.

Conjecture 2.3.5. *Hyperbolic maps are dense in the exponential family.*

By Theorem 2.3.2, it is enough to show that hyperbolic maps are dense in $\mathcal{R}$. Having an attracting orbit is a topological invariant, and in any connected component of $\mathcal{R}$ all maps are topologically conjugated, so any such connected component can be either *hyperbolic*- in which cases all parameters in the component are hyperbolic- or *non-hyperbolic*, if at least one parameter and hence all parameters in the component do not have attracting orbits.

Non-hyperbolic components are often called *queer* components.

Conjecture 2.3.5 can be restated as follows.

Conjecture 2.3.6. *All connected components in $\mathcal{R}$ are hyperbolic.*

For the Mandelbrot set, the density of hyperbolicity conjecture is equivalent to say that the bifurcation locus is locally connected; luckily for the exponential family this is not the case, as the bifurcation locus is known to be non locally connected.

Discussion on density of hyperbolicity can be restated in terms of rigidity and fibers (see Section 4).

In analogy with [RS2] we will introduce the *reduced bifurcation locus* $\mathcal{B}* := \mathcal{B} - \{\text{parameter rays}\}$. It is then plausible, and would not contradict theorem 2.3.4, that $\mathcal{B}$ is locally connected exactly at points in $\mathcal{B}*$, or at parameters in $\mathcal{B}*$ which are non-escaping.

As a final statement, let us mention that, while hyperbolic components are unbounded and always contain a curve to infinity, non-hyperbolic components do not have access to infinity as a consequence of the *Squeezing Lemma* ([R0], Theorem 5.3.5).

Theorem 2.3.7 (Squeezing Lemma)**.** *Let $\gamma : [0, 1) \to \mathbb{C}$ be a curve in parameter space with $|\gamma(t)| \to \infty$ which does not contain any indifferent parameters. Then either γ is contained in a parameter ray, or γ is contained in a hyperbolic component.*

Chapter 3

Combinatorics

3.1 Dynamic and parameter rays

This section has the purpose of recollecting some of the relevant results about rays and their landing properties. We assume the reader to be familiar with construction and properties of dynamic and parameter rays for polynomials, see for example [McM].

The construction of external rays for the exponential family has been started by Devaney and coauthors ([BD1], [DK]) and completed by Schleicher and Zimmer ([SZ1]).

We will use throughout the paper the concept of itinerary with respect to a partition.

Definition 3.1.1. Let $\mathcal{M} = \{M_a\}_{a \in \mathcal{A}}$ be a countable collection of pairwise disconnected regions of the plane such that each regions is labeled uniquely by a letter in some countable alphabet $\mathcal{A}$. Then whenever the iterates $f^j(z)$ are contained in $\bigcup\limits_{a \in \mathcal{A}} M_a$ for each j we will say that the *itinerary* of z is the sequence $a = a_1 ... a_n ...$ of symbols of $\mathcal{A}$ such that $f^j(z) \in M_{a_j}$. In case that $\mathcal{M}$ is a partition of a forward invariant set X, itineraries are well defined for all points in X.

Rays for the exponential family have been introduced in analogy with the polynomial case in order to construct symbolic dynamics on the set of *escaping points*

$$I(f) := \{z \in \mathbb{C}, |f^n(z)| \to \infty \text{ as } n \to \infty\} \subset \Pi_c.$$

Given an exponential map $f_c(z)$, the preimages of the semiaxis

$$L := \{z \in \mathbb{C}, \text{Re } z < \text{Re } c, \text{Im } z = \text{Im } c\}$$

give a partition the plane (called *static partition*) into strips S_j

$$S_j := \{z, \text{Im } z \in ((2j-1)\pi, (2j+1)\pi)\}.$$

For points whose iterates never belong to L, we can consider itineraries with respect to this partition, i.e.

$$\text{itin}(z) = s_1 s_2 \ldots \text{ if and only if } f^j(z) \in S_{s_j}.$$

For points whose jth iterate belongs to the boundary separating two strips S_{s_j} and $S_{s_{j+1}}$ the corresponding entry in the itinerary will be defined as the boundary symbol $\binom{s_j}{s_{j+1}}$.

For a point z whose itinerary with respect to this partition contains no boundary symbols we will refer to the itinerary as the *external address* of z. By construction external addresses are sequences in $\mathbb{Z}^{\mathbb{N}}$.

We will use the function $F : \mathbb{R} \mapsto \mathbb{R}$, $F : t \mapsto e^t - 1$ to model real exponential growth.

Using the described construction itineraries of points cannot have entries growing faster than iterates of the exponential function. This leads to the following notion.

Definition 3.1.2. A sequence $s = s_1 s_2 \ldots$ is called *exponentially bounded* if $\exists\, x \in \mathbb{R},\ |2\pi s_j| < F^j(x)\ \forall j \in \mathbb{N}$.

This growth condition turns out to be not only necessary but also sufficient [SZ1], so that all sequences s contained in the set

$$\mathcal{S} := \{s \in \mathbb{Z}^{\mathbb{N}} \text{ such that } s \text{ is exponentially bounded}\}$$

are realized as itineraries of some point z.

We will say that an address is *periodic* if it is a periodic sequence, *preperiodic* if it is a strictly preperiodic sequence and *(pre)periodic* if it is either periodic or strictly preperiodic.

The set $\mathcal{S}$ has a natural order induced by the usual order relation on the space of sequences over an ordered set.

If $s = s_1 s_2 \dots$, we will say that $|s| = \sup_i |s_i|$. We will say that s is *bounded* if $|s| < \infty$.

Also, given a an external address s we will define its *minimal potential*

$$t_s := \inf\{t > 0, \limsup_{k \geq 1} \frac{|s_k|}{F^k(t)} = 0\}.$$

Definition, existence and properties of *dynamic rays* for the exponential family are summarized in the following theorem ([SZ1], Proposition 3.2 and Theorem 4.2).

Theorem 3.1.3 (Existence of dynamic rays)**.** *Let c be a parameter such that $|f_c^n(c)|$ does not tend to infinity as $n \to \infty$; then for any $s \in \mathcal{S}$ there exists a unique injective curve $g_s^c : (t_s, \infty) \to \mathbb{C}$ consisting of escaping points such that*

- *$g_s^c(t)$ has external address s for sufficiently large t;*
- *$f_c(g_s^c(t)) = g_{\sigma s}^c(F(t))$;*
- *We have the asymptotics $g_s^c(t) = 2\pi i s_1 + t + Ce^{-t}$ as $t \to \infty$.*

If $s = s_1 s_2 ...$ and $|s_k| < AF^{(k-1)}(x)$, and $|c| < K$ then $C \leq 2(K + 2 + 2\pi|s_2| + 2\pi AC')$ where C' is a universal constant

The next remark follows from Proposition 3.4 and Remark 3.5 in [SZ1] .

Remark 2. Given f_c with $|c| < K$, and s a bounded address, the points on g_s have itinerary s.

Given $s \in \mathcal{S}$, we will call the unique curve g_s^c given by Theorem 3.1.3 the *dynamic ray of address s*. As for polynomials, a dynamic ray is (pre)periodic if and only if its address is (pre)periodic.

We will say that *a dynamic ray g_s lands* at a parameter z if $g_s(t) \to z$ as $t \to t_s$.

The question whether periodic dynamic rays land for the exponential family remained open for some time, and was finally solved by Rempe using the previously known fact that periodic rays land for hyperbolic parameters and an argument about persistence of landing inside wakes. This led to the following Theorem ([R1], Theorem 1).

Theorem 3.1.4 (Landing theorem for periodic dynamic rays)**.** *Let c be such that the singular value c of f_c does not escape to infinity. Then every periodic dynamic ray g_s^c lands at a repelling or parabolic periodic point.*

Note that as preperiodic points are preimages of periodic points, by continuity of the inverse map in a neighborhood of any periodic point Theorem 3.1.4 implies that preperiodic rays also land unless they are preimages of a ray containing the singular value c.

The construction of parameter rays is also done keeping in mind the fundamental property of parameter rays that we have for polynomials: a point c belongs to some parameter ray G_s in Π_P if and only if it belongs to the dynamic ray g_s^c in Π_c. It is carried out by Forster and Schleicher and is summarized in the following theorem about existence of parameter rays ([FS],Theorem 3.7).

Theorem 3.1.5 (Existence of parameter rays)**.** *Let $s \in \mathcal{S}$, t_s its minimal potential. Then there is a unique injective curve $G_s : (t_s, \infty) \to \mathbb{C}$, called a parameter ray, such that, for all $t > t_s$, $c = G_s(t)$ if and only if $c = g_s^c(t)$.*

The map $G_s : (t_s, \infty) \to \mathbb{C}$ is continuous, and $|G_s(t) - (t + 2\pi i s_1)| \to 0$ as $t \to \infty$.

Given an external address $s \in \mathcal{S}$, we will call the unique curve G_s given by Theorem 3.1.5 the *parameter ray of address s*.

Not much is known about the landing properties of parameter rays. Landing of (pre)periodic parameter rays it was shown in [S0] and [SZ2].

Theorem 3.1.6. *Each parameter rays of periodic address lands at a parabolic parameter; each parameter ray with preperiodic address lands at a Misiurewicz parameter.*

We will say that a parameter or dynamic ray is periodic if its address is periodic, preperiodic if its address is strictly preperiodic and (pre)periodic if it is either periodic or strictly preperiodic.

About parametrization of rays and asymptotics

Having dynamic rays parametrized with potential starting at t_s instead of 0 makes this following continuity lemma true ([R2], Lemma 4.7).

Proposition 3.1.7 (Convergence of rays)**.** *Let f_c be an exponential map, s_n a sequence of external addresses converging to an external address s_0 such that also t_{s_n} converges to t_{s_0} , and let $t_0 > 0$ such that g_{s_0} (t) is defined for all $t > t0$. Then g_{s_n} converges uniformly to g_{s_0} on $[t_0, \infty)$.*

Corollary 3.1.8. *Lemma 3.1.7 holds if $\{s_n\}$ is a sequence of addresses all of which are (not necessarily uniformly) bounded, as in this case $t_{s^n} = 0$ for all n.*

This difference with respect to polynomials reflects the fact that in the polynomial case rays only accumulate on a compact set (the Julia set) while for exponentials rays can accumulate on points with arbitrarily large modulus. Observe that when restricted to bounded addresses, $t_s = 0$, and Lemma 3.1.7 becomes more similar to the analogous property for polynomial rays.

Also the asymptotics in Theorem 3.1.3 depend on the bound on the address.

Proposition 3.1.9 (Convergence of parameter rays)**.** *Let s_n be a sequence of external addresses converging to an external address s_0 such that also t_{s_n} converges to t_{s_0} , and let $t_0 > 0$ such that G_{s_0} (t) is defined for all $t > t0$. Then G_{s_n} converges uniformly to G_{s_0} on $[t_0, \infty)$.*

The regions S_j can be labeled so that S_0 is either the region containing the singular value or the region containing asymptotically the positive real axis. In fact, whenever studying the family $e^z + c$ by vertical periodicity we could restrict to singular values with imaginary part between $-\pi$ and π. Choosing S_0 to be the region containing asymptotically the real positive axis has the advantage of being consistent with the parametrization of parameter rays, while choosing S_0 to contain c has the advantage of labeling in the same way dynamic rays for two parameters differing by $2\pi i$.

Dynamic partition

We will say that we have a *dynamic partition* for an exponential map f_c whenever there is a curve starting at the singular value contained in the Fatou set, or a dynamic ray landing at c. The preimages of such a curve or ray partition the Julia set into countably many strips L_j such that each dynamic ray is fully contained in one of the L_j.

3.2 Combinatorial spaces and cyclic order

Theorems 3.1.3 and 3.1.5 establish a correspondence between the set of dynamic/parameter rays and the set of exponentially bounded addresses $\mathcal{S}$ for the exponential family. Moreover, the equation $f_c(g^c_s(t)) = g^c_{\sigma s}(F(t))$ in Theorem 3.1.3 tells that the dynamics of an exponential function f_c on its set of escaping points I is conjugate to the dynamics of the left-sided shift map σ on $\mathcal{S}$. The asymptotic estimates in Theorems 3.1.3, 3.1.5 show that dynamic and parameter rays have a well defined vertical order at infinity and that this order coincides with the order of their addresses in $\mathcal{S}$. For all this reasons we will refer to $\mathcal{S}$ as the *combinatorial space* for the family $e^z + c$.

For the family of unicritical polynomials P_D, the dynamic/parameter rays are in correspondence with the sequences over D symbols (angles in D-adic expansion), that we can represent as

$$\mathcal{S}_D = \{\frac{-D+1}{2}, \dots, 0, \dots, \frac{D-1}{2}\}^{\mathbb{N}} \qquad \text{for } D \text{ odd}$$
$$\mathcal{S}_D = \{\frac{-D+2}{2}, \dots, 0, \dots, \frac{D}{2}\}^{\mathbb{N}} \qquad \text{for } D \text{ even}$$

As for the exponential family, the dynamics of a unicritical polynomial of degree D on the set of dynamic rays is conjugate to the the dynamics of the shift map σ on $\mathcal{S}_D$; also, dynamic and parameter rays have a cyclic order at infinity which corresponds to the cyclic order on $\mathcal{S}_D$ if we identify the sequences modulo D.

If $l, s \in \mathbb{Z}^{\mathbb{N}}$ are two sequences, $l = l_1 l_2 \dots$ and $s = s_1 s_2 \dots$, we define the following distance:

$$\text{dist}(l, s) = \sum_{s_k \neq l_k} \frac{1}{2^k}.$$

This turns $\mathcal{S}$, $\mathcal{S}_D$ into metric spaces.

The space $\mathcal{S}_D$ embeds naturally in $\mathcal{S}$ via the identity map; similarly, if $A \subset \mathcal{S}$ is such that $|s| < N$ for each $s \in A$, A embeds in $\mathcal{S}_D$ via the identity map for each $D > 2N + 2$. We will refer to $\mathcal{S}_D$ as the *combinatorial space* for the family P_D.

We will refer to this description as *combinatorial correspondence* between the exponential family and polynomials of sufficiently high degree D.

The following proposition is certainly well known to people working in the field, but we prefer to state it and prove it explicitly.

Proposition 3.2.1 (Inverse of the shift map)**.** *Let g_{s_1}, g_{s_2} be two dynamic rays landing together with $s_1 < s_2$, and Γ be the curve formed by g_{s_1} and g_{s_2} together with their common landing point w. Γ separates the plane into two regions, one that we will call V_1 and which is vertically bounded (in the sense that it contains only points with bounded imaginary parts) and one that we will call V_2 and which is vertically unbounded. Then either $c \in V_1$ and the rays $g_{(a+1)s_1}, g_{as_2}$ land together for any $a \in \mathbb{Z}$, or $c \in V_2$ and g_{as_1}, g_{as_2} land together for any $a \in \mathbb{Z}$.*

Proof. As all the dynamic rays of addresses g_{as} are curves congruent to g_{0s} translated by $2\pi ai$, it is enough to prove the statement for $a = 0$. Give an orientation to Γ so that V_1 is to the left of Γ. The preimage of Γ clearly consists of a preimage of g_{s_1}, a preimage of g_{s_2} and a preimage of w, oriented so as to start with the preimage of g_{s_2}, then the preimage of w and eventually the preimage of g_{s_2}. As rays cannot intersect, g_{0s_2} can only be connected to g_{0s_1} or g_{1s_1}. In the first case, as $0s_1 < 0s_2$, the preimage of V_1 is the vertically bounded region enclosed by g_{0s_1},g_{0s_2} and their common landing point, and it is mapped univalently to V_1, hence V_1 did not contain the singular value.

In the second case, as $1s_1 > 0s_2$, the preimage of V_1 contains a left half plane, hence V_1 contains c. □

3.3 Orbit portraits and ray portraits

This section introduces orbit portraits for the exponential family (in analogy with [Mi1, RS1]) and presents some theorems about the correspondence between (pre)periodic parameter rays for polynomials and (pre)periodic parameter rays for exponentials.

Definition 3.3.1. Let g_{s_0}, g_{s_1} be two dynamic or parameter rays with (pre)periodic addresses, landing together at some point z. We call the *ray pair* formed by g_{s_0}, g_{s_1} the curve Γ formed by $g_{s_0} \cup g_{s_1} \cup \{z\}$. We will denote as the region enclosed by the ray pair the connected component of $\mathbb{C} - \Gamma$ containing the rays of addresses s, $s_0 < s < s_1$.

Definition 3.3.2. Let $\{z_i\}_{i=1\cdots n}$ be a repelling or parabolic periodic orbit of period n in Π_c, and $\mathcal{A}_i := \{r \in \mathcal{S}, r$ is periodic and g_r^c lands at $z_i\}$. Then $\mathcal{P} = \{\mathcal{A}_1, \cdots, \mathcal{A}_n\}$ is said to be the *combinatorial orbit portrait* for $\{z_i\}$.

Similarly,

Definition 3.3.3. Let $\{z_i\}_{i=1\cdots n}$ be a repelling or parabolic periodic orbit of period n in Π_c, and $A_i := \{g_r^c, g_r^c$ lands at $z_i\}$. Then $P = \{A_1, \cdots A_n\}$ is said to be the *orbit portrait* for $\{z_i\}$.

Theorem 3.3.4 (Basic properties of combinatorial orbit portraits [RS1])**.** *Given a combinatorial orbit portrait $\mathcal{P}$, every $\mathcal{A}_i \in \mathcal{P}$ consists of a finite number of periodic addresses, and the shift map sends $\mathcal{A}_i$ bijectively onto $\mathcal{A}_{i+1}$. All addresses share the same period qn.*

Theorem 3.3.5 (Basic properties of orbit portraits [RS1])**.** *Given an orbit portrait P, every $A_i \in P$ consists of a finite number of dynamic rays, and f maps A_i bijectively onto A_{i+1}. All dynamic rays in the portrait are periodic with the same period qn.*

Remark: in a more abstract way, we will speak of a combinatorial orbit portrait without specifying a periodic orbit $\{z_1\}$; such a combinatorial object is not necessarily *realized* (i.e occurs for some parameter) as an actual orbit portrait for some polynomial or exponential map.

Definition 3.3.6. The *combinatorial ray portrait* of a function f is the collection of combinatorial orbit portraits realized by f.

The following theorems will show the relation between the combinatorial orbit portraits which are realized for exponential maps and the ones which are realized for unicritical polynomials. As there are necessary and sufficient conditions for a combinatorial orbit portrait to be realized for polynomials, this gives unique and sufficient conditions for a combinatorial portrait to be realized for exponentials.

The following proofs refer to properties of Misiurewicz parameters proven in Chapter 5.1. We chose to do so in order to collect all properties related to orbit portraits in the same chapter. Of course, the proofs in Chapter 5.1 do not rely on the following results.

Let us first state a correspondence between Misiurewicz parameters for exponential maps and for unicritical polynomials.

Theorem 3.3.7 (Misiurewicz addresses for exponentials and polynomials)**.** *The parameter rays $G_{s_1}, \dots, G_{s_q}$ land together at some exponential Misiurewicz parameter in the exponential parameter plane if and only if for each family of unicritical polynomials of sufficiently high degree D the parameter rays with the same addresses land together at some polynomial Misiurewicz parameter.*

Proof. Let $G_{s_1}, \dots, G_{s_q}$ be the parameter rays landing together at some Misiurewicz parameter c_0 in exponential parameter plane. Then the dynamic rays with the corresponding addresses $g_{s_1}, \dots, g_{s_q}$ all have the same itineraries with respect to the dynamical partition induced by g_{s_1} in Π_{c_0} (see Section 5.1), because together with c_0 they form a connected set whose orbit cannot intersect the boundaries of the dynamical partition. The address s_1 is preperiodic, so it is a sequence over finitely many values, so for polynomials of sufficiently high degree D it represents the D-adic expansion of the angle of some parameter ray. As (pre)periodic parameter rays are well known to land for unicritical polynomials, there is a Misiurewicz parameter c_1 depending on D which is the landing point of the corresponding parameter ray.

By 5.1.1, the dynamic ray g_{s_1} lands at c_1 in the polynomial dynamical plane for f_{c_1}.

All the polynomial dynamic rays $g_{s_2}, \dots, g_{s_q}$ also have the same itinerary with respect to the partition induced by g_{s_1} so by Theorem 5.1.4 they all land together in the dynamical plane for f_{c_1}. Then by 5.1.1 the corresponding parameter rays land together at c_1 in the polynomial parameter plane. □

We will now define characteristic rays. We will state the definitions for exponentials, the corresponding definitions for polynomials are analogous (see [Mi1]).

Definition 3.3.8. Given an orbit portrait, the *characteristic rays* are the rays g_{s_1}, g_{s_2} which, together with their common endpoint, separate the singular value from all other rays in the portrait; compare with Lemma 3.3 in [RS1] for existence and uniqueness.

The *characteristic sector* of an orbit portrait is given by the set of points enclosed between g_{s_1} and g_{s_2}.

Definition 3.3.9. A *characteristic ray pair* is a pair of parameter rays G_{s_1}, G_{s_2} with periodic addresses landing together in parameter plane. We will say that the region enclosed by the ray pair formed by G_{s_1} and G_{s_2} is the *parabolic wake* defined by G_{s_1}, G_{s_2}.

Definition 3.3.10. Let $s < 1... < s_q$ be preperiodic addresses such that G_{s_i} lands at a Misiurewicz parameter c_0 for $i = 1...q$. The region enclosed by the ray pair formed by G_{s_1} and G_{s_2} is called the *Misiurewicz wake* defined by G_{s_1}, G_{s_q}.

Theorem 3.3.11 (Correspondence of bifurcations)**.** *A parameter c belongs to the parabolic wake defined by two parameter rays G_{s_1}, G_{s_2} together with their common landing point if and only if the dynamic rays g_{s_1}, g_{s_2} land together in the dynamical plane Π_c and are the characteristic rays for some orbit portrait in Π_c. We will call s_1, s_2 a pair of* characteristic addresses.

Corollary 3.3.12. *G_{s_1}, G_{s_2} is a characteristic ray pair, landing together at some parameter c, if and only if the dynamic rays g_{s_1}, g_{s_2} are the characteristic rays of some orbit portrait.*

Remark: Theorem 3.3.11 is not explicitly stated in this way. It follows from Proposition 5.4 in [RS1] when c is hyperbolic or parabolic, and can be extended by holomorphic motions to all other parameters similarly as in [R1].

Corollary 3.3.13 (Persistence of orbit portraits)**.** *An orbit portrait persists in the wake defined in parameter plane by the parameter rays identified by its characteristic addresses.*

Theorem 3.3.14 (Misiurewicz wakes)**.** *Let $s_1 < ... < s_q$ be preperiodic addresses such that G_{s_i} lands at a Misiurewicz parameter c_0 for $i = 1...q$. Then the dynamic rays $g^c_{s_1}...g^c_{s_q}$ land together in the dynamical plane for f_c if and only if c belongs to the Misiurewicz wake defined by G_{s_1}, G_{s_q}.*

Proof. The dynamic rays $g^{c_0}_{s_1}...g^{c_0}_{s_q}$ land together in the dynamical plane for f_{c_0} by Theorem 5.1.1. As the addresses s_i are preperiodic, the image of the dynamic rays $g^{c_0}_{s_1}...g^{c_0}_{s_q}$ under finitely many iterates defines an orbit portrait

P, and c_0 belongs to some parabolic wake $\mathcal{W}$ identified by the characteristic addresses of P, in which the orbit portrait P is defined.

Let $\tilde{s}_1..\tilde{s}_q$ be the addresses of the rays in P. The pullback of P is well defined for all all parameters c such that $f^k(c)$ never belongs to P, i.e. for all parameters c which do not belong to the parameter rays of address s such that $\sigma^x(s)$ belongs to the set $\{\tilde{s_i}\}$ and which are not the landing point of one of those rays.

If m is the period of the dynamic rays in P, the first m pullbacks of P are always well defined. The $m+1$th pullback depends on whether $f^m(c)$ belongs to the region enclosed by P or not. In the the first case, c is inside the Misiurewicz wake defined by $f^{-m(c)}$, while in the second case c is outside. □

Theorem 3.3.15 (Correspondence of characteristic rays)**.** *A pair of addresses is characteristic for exponentials if and only if it is characteristic for some unicritical polynomial of some degree D.*

Remark: by the definitions and Theorem 3.3.11 this will show that if the parameter rays with periodic address G_{s_1}, G_{s_2} land together in the parameter space Π_P^D for some D then the parameter rays with the corresponding addresses land together in the parameter space of exponential maps Π_P; on the other side, if the parameter rays G_{s_1}, G_{s_2} land together in Π_P, they land together in Π_P^D for all sufficiently high D.

Proof. Let $\mathcal{P} = \{\mathcal{A}_i\}$ be a combinatorial orbit portrait. The inverse of the shift map brings each non-characteristic sector to a sector bounded by rays whose addresses have the same first entry, and the characteristic sector to a sector bounded by rays for whose addresses the first entry differs by one.

So, being a characteristic sector is encoded in the topological orbit portrait, and the claim will follow if we can show that every combinatorial portrait is realized in the exponential family if and only if it is realized for P_D of sufficiently high degree D.

If $\mathcal{P}$ is realized for some polynomial in $\{P_c^D\}$, it persists in the whole wake bounded by its characteristic addresses so in particular it is realized for some polynomial Misiurewicz parameter c as well.

This Misiurewicz parameter is the landing point of a dynamic ray of angle s, inducing a dynamical partition as described in the section 5.1. The rays whose angles belong to the same $\mathcal{A}_i$ land together, so they have the same itinerary with respect to this partition by 5.1.4; in particular, their angles have the same itineraries under the shift map with respect to the partition induced by s.

By the combinatorial correspondence between polynomials and exponentials the angle s in D-adic expansion can be seen as an address s which identifies a Misiurewicz parameter $\tilde{c}$ in the exponential family by 5.1.2. All the dynamic rays whose addresses belong to $\mathcal{P}$ exist in the dynamical plane of $\tilde{c}$, and by Lemma 5.1.4 they land together as they have the same itinerary with respect to the partition induced by the ray landing at the Misiurewicz parameter.

If $\mathcal{P}$ is realized for an exponential parameter, it persists in a wake by Theorem 3.3.11, so it is realized for some Misiurewicz parameter and can be transfered to a polynomial Misiurewicz parameter whose degree is sufficiently high to ensure the existence of the dynamic rays whose addresses belong to $\mathcal{P}$ and of the dynamic ray landing at the Misiurewicz parameter. □

3.4 Combinatorial tuning

The description of combinatorial tuning in this section follows the ideas from [Do] and [R0]. We will describe the tuning procedure for unicritical polynomial of degree D. For simplicity we will label angles as sequences over $\mathcal{S}_D = \{1, ..., D\}^{\mathbb{N}}$; the combinatorial space obtained is clearly isomorphic to the combinatorial space described is Section 3.2. This discussion can be generalized to the exponential family with a little extra care for the fact that the space of addresses does not have a cyclic order, resulting in a discontinuity of the tuning map.

Lets us consider the family $P_D = \{f_c(z) = z^D + c\}$. The parabolic wake of a period m component W_m is bounded by two characteristic addresses, and contains in its interior $D-2$ other parameter rays of period m landing at cusps of ∂W_m. This gives a total of D parameter rays of angles $s_1 < ... < s_D$ of period m subdividing the wake of W_m into $D-1$ sectors $S_1, ..., S_{D-1}$, labeled such that S_i is the sector enclosed by G_{s_i} and $G_{s_{i+1}}$. Observe that G_{s_1} and G_{s_D} are the characteristic rays of the wake. Also each s_i is a sequence of period m, which we can write as $s_i = \overline{u_i}$, where u_i are finite sequences of m symbols.

Proposition 3.4.1 (Combinatorial width)**.** *Given a hyperbolic component W_m, if s_1 and s_D are its characteristic angles,* $\text{dist}(s_1, s_2) < 1/m$

Proof. The addresses of the parameter rays defining a parabolic wake are the same as the addresses of the characteristic dynamic rays in the corresponding orbit portrait, which contains at least m dynamic rays. As the characteristic sector is the one with smallest width among the sectors in the portrait, its combinatorial width is less than $1/m$. □

Given an angle $\ell = a_1 a_2 ...$, and a hyperbolic component W_m, we will define the *tuning of ℓ with respect to W_m* as

$$\tau(\ell) = u_{a_1} u_{a_2} ...,$$

i.e. a new angle where each symbol in the dyadic expansion is replaced by a string of m symbols $u_1, ...u_D$ defined by W_m. This map is well defined for any angle ℓ, is injective, its range is contained in the wake of W_m and preserves cyclic order.

We say that an angle ℓ is *an angle tuned by a hyperbolic component* W_m if $\ell = u_{a_1}u_{a_2}...$ where u_i are the angles defined by the wake of W_m. We call the sequence $\rho(\ell) = a_1a_2... \in \{1,...,D\}^{\mathbb{N}}$ the renormalization of ℓ (with respect to W_n).

Definition 3.4.2. Let σ be the shift map. We say that an angle s is *non recurrent* if $s \notin O(s) := \{\sigma^n(s)\}$.

Proposition 3.4.3 (Recurrence and tuning)**.** *If s is tuned by a hyperbolic component of period m,* $\text{dist}(\sigma^{mk_1}s, \sigma^{mk_2}s) < C_m \to 0$ *as $m \to \infty$, for any integers k_1, k_2. In particular, if s is non recurrent, it cannot be an angle tuned by infinitely many hyperbolic components.*

Proof. If s is tuned by a hyperbolic component of period m, $s = u_{a_1}u_{a_2}...$ where u_{a_i} are sequences of m symbols and $\text{dist}(u_{a_i}, u_{a_j}) < C_m$ by Proposition 3.4.1. The first claim follows. The second claim follows from the fact that if s is tuned by infinitely many components, there exists $n_i \to \infty$, and $k_i \to \infty$, such that

$$\text{dist}(s, \sigma^{n_ik_i}s) < C_m,$$

hence s is recurrent. □

3.5 Properties of conjugacies between exponential maps

For completeness and further reference let us collect some properties of conjugacies between non-escaping exponential maps. They can be seen as additions to the work of Rempe in [R2].

Suppose we have a topological conjugacy between two exponential maps f and g, whose singular values are c_f and c_g respectively. We have that $h(c_f) = c_g$ because it is the unique omitted value.

On both planes, label the strips of the static partition so that the strip containing the singular value is called S_0. Note that if we label the rays in this way we do not have the usual correspondence between parameter and dynamical plane.

Our goal will be to show that dynamic rays are mapped to dynamic rays, and that the labeling above is preserved by h. The next two theorems are Theorem 1.1 and Theorem 8.2 in [R2].

Theorem 3.5.1 (Existence of conjugacy)**.** *Let f_{c_1} and f_{c_2} be two exponential maps, R be large enough and*

$$A_R := \{z \in \mathbb{C}, |f_{c_1}^n(z)| \geq R \;\forall\; n \in \mathbb{N}\}$$

. Then there exists a quasiconformal map ϕ: $\mathbb{C} \to \mathbb{C}$ conjugating f_{c_1} to f_{c_2} on A.

$|f_{c_1}^n(z) - f_{c_2}^n(\phi(z))| \to 0$ on $A \cap I(f_{c_1})$.

If neither c_1 nor c_2 are escaping, we can extend $\phi|_{A_R \cap I(f_{c_1})}$ to a bijection (not necessarily continuous) between escaping sets satisfying the properties above.

Theorem 3.5.2 (Uniqueness of conjugacy)**.** *Suppose f_1 and f_2 are exponential maps with non escaping singular value, which are conjugate on their set of escaping points by a conjugacy h which sends each dynamic ray of f to the corresponding dynamic ray of f^2. Then h is the standard conjugacy.*

Theorem 3.5.3 (Correspondence of rays)**.** *Let f_1, f_2 be two exponential maps conjugate by a homeomorphism h. Let g_s^1 be the dynamic ray of address s in the f_1 plane and g_s^2 be the dynamic ray of address s in the f_2 plane. Then $h(g_s^1) = g_s^2$.*

Proof. Escaping points are mapped to escaping points, because h maps compact sets to compact sets, so if a point escapes any compact set in the f_1

plane, its image under h escapes all compact sets in the f_2 plane. Analogously, rays are mapped to rays, because path connected sets are mapped to path connected sets under a continuous map and rays are the only path connected components of the escaping set. So the only question is whether a ray of a given address is mapped to a ray of the same address.

Let us start showing that $h(z+2\pi i) = h(z) + 2\pi i$. By the functional equation $h(f_1(z)) = f_2(h(z))$ we have that

$$h(f_1(z+2\pi i)) = f_2(h(z+2\pi i)) \tag{3.5.1}$$

$$LHS = h(f_1(z)) = f_2(h(z)) \text{ by the functional equation, so} \tag{3.5.2}$$

$$f_2(h(z)) = f_2(h(z+2\pi i)); \tag{3.5.3}$$

by injectivity of the exponential map f_2 on horizontal strips of width $2\pi i$ we have

$$h(z+2\pi i) = h(z) + 2n\pi i. \tag{3.5.4}$$

Note that n, being integer, is constant by continuity of h.

Suppose that $n \neq 1$, and consider the set $F(z)$ given by any point z together with all its $2\pi i$ translates; the image of this set would only consist of $h(z)$ and all translates which are multiples of n, so their image under h^{-1} is surjective in $F(z)$. On the other side, by 3.5.4, the image of any point of the form $h(z) + k\pi i$ is also contained in $F(z)$, contradicting injectivity of h unless $n = 1$.

This proves that if the ray of address $\overline{0}$ is mapped to the ray of address s_0, any ray of address s is mapped to the ray of address $s + s_0$, so we only have to show that $s_0 = \overline{0}$ and it is enough to show that the ray of address $\overline{0}$ goes to the ray of address $\overline{0}$.

The ray $g^1_{\overline{0}}$ for large potentials has to belong to the strip S_0 in the f_1 plane, as it is the strip of the singular value, $g^2_{s+s_0}$ also has to belong to the singular strip S_0 in the f_2-plane (otherwise connect the tail of the first ray to c_1 by a curve contained in S_0, by $2\pi i$ periodicity it has to go to a curve connecting c_2 to $g^2_{s+s_0}$ which only intersects one strip), so the first entry of s_0 has to be zero. Repeating the same argument for preimages of g^1_0 gives that s_0 is identically zero. □

Note that the labeling we use is not the same as the labeling in [R2], but Theorem 3.5.2 still holds up to pre-post composition by a translation from the model space into itself (i.e. by redefining the canonical conjugacy with a different labeling)

Lemma 3.5.4. *Suppose that h is a topological conjugacy between two holomorphic functions f and g which is K-quasiconformal on some set $E \subset \mathbb{C}$. Then h is K-quasiconformal on $f_1^{-1}(E)$.*

Proof. Draw a picture of the commutative diagram. We obtain that $h|_{f^{-1}(E)} = g^{-1} \circ h|_E \circ f$. By holomorphicity of f and g the quasiconformality constant of $h|_E$ is preserved on $f^{-1}(E)$. □

Chapter 4

Fibers and Rigidity

4.1 Fibers and Rigidity

One of the main problems in one-dimensional complex dynamics is to show that the set of structurally stable parameters $\mathcal{R}$ consists only of hyperbolic components. The statement for the exponential family is

Conjecture 4.1.1. *Hyperbolic maps are dense in the family* $f_c(z) = e^z + c$.

One way for studying non-hyperbolic components is to concentrate on topological invariants. For example, repelling periodic orbits are stable under perturbations, as well as the set of dynamic rays landing at them (not only topologically but also combinatorially- as rays are labeled by sequences over the integers).

In any connected component of $\mathcal{R}$, all the maps are topologically conjugate, hence all have the same combinatorial ray portraits (see Definition 3.3.6). By Corollary 3.3.13, two maps which are topologically conjugate are also contained in the same set of wakes, which implies that they cannot be separated by any pair of periodic rays landing together (and hence forming a wake). The collection of all orbit portraits of an exponential maps is often referred to as *combinatorics* of the map, hence all maps in the same structurally stable component have the same combinatorics. This leads to the following definitions.

Definition 4.1.2. The *parameter fiber* of a parameter c_0 is the set of parameters which cannot be separated from c_0 by some pair of (pre)periodic parameter rays landing together at a parabolic or Misiurewicz parameter.By analogy, the *dynamical fiber* of a point c_0 is the set of points which cannot be separated from c_0 by some pair of (pre)periodic rays landing together at some (pre)periodic point.

Remark: in [RS2], this definition corresponds to the definition of *extended fibers.*

Definition 4.1.3. We will say that the fiber of a point c_0 in dynamical/parameter space is *trivial*, if any point $c \neq c_0$ can be separated from c_0 via a pair of (pre)periodic dynamic/parameter rays landing together (a ray pair), except for the points belonging to the dynamic/parameter rays which might land at the point c_0 itself.

The following conjecture in terms of fibers implies the Density of Hyperbolicity Conjecture.

Conjecture 4.1.4 (Rigidity conjecture)**.** *Fibers of non-hyperbolic parameters are trivial.*

Note that the definitions of fiber in dynamical and in parameter space are analogous by replacing dynamic rays with parameter rays. We will call any result about triviality of fibers a *rigidity result.* This comes from the fact that any map whose singular value does not escape and whose fiber is trivial cannot be conjugate to any other map in a neighborhood because two maps with different ray portrait cannot be topologically conjugate.

We will call *reduced fiber* a fiber intersected with the reduced bifurcation locus $\mathcal{B}* := \mathcal{B} - \{\text{parameter rays}\}$.

Some properties of fibers, coming from Theorem 7 in [RS2] are summarized in the next theorem.

Theorem 4.1.5. *Every fiber is a closed connected subset of parameter space. Moreover, if the fiber does not contain hyperbolic parameters, than it intersects $\mathcal{B}*$. A reduced fiber, and hence a fiber, is either trivial or uncountable.*

The next result follows immediately from the previous definitions. (See again [RS2] for a slightly different formulation of this discussion.)

Theorem 4.1.6. *If the fiber of every non hyperbolic parameter with non-escaping singular value is trivial, then every components in the set of structurally stable parameters is hyperbolic.*

There are two main advantages in considering fibers to study density of hyperbolicity: for the exponential case, periodic parameter rays for exponentials are closely related to parameter rays for unicritical polynomials (see Theorem 3.3.15), so that it is possible to infer results about exponentials using known results about polynomials; the second one, and more general one, is that fibers are a way to localize the global conjecture, and select specific classes of parameters which are easier to study.

Our main results will concern triviality of fibers for Misiurewicz and non-recurrent parameters with bounded postsingular set.

Chapter 5

Misiurewicz parameters

5.1 Misiurewicz parameters

Given the exponential family $f_c = e^z + c$, or a family of degree D unicritical polynomials $f_c = z^D + c$, we call a parameter c_0 *Misiurewicz* (or *postsingularly finite*) if the orbit of the singular value is preperiodic.

Note that the singular value is an omitted value and hence for the exponential family the postsingular orbit cannot be periodic; note also that such an orbit has to be repelling, otherwise the unique singular/critical value would belong to the immediate attracting basin by an old theorem of Fatou contradicting the fact that it is preperiodic.

So being postsingularly finite is equivalent to say that the singular value c_0 lands at some repelling orbit $\{z_i\}$ of period m after k iterations, for some integers k, m.

From the definition above and the discreteness of solutions of the equation $f_c^{k+m}(c) = f_c^k(c)$, it follows immediately that Misiurewicz parameters belong to the bifurcation locus. There cannot be hyperbolic or parabolic Fatou components because c belongs to the Julia set, nor Siegel disks because the orbit of c accumulates on a finite set, so for an exponential Misiurewicz map the Julia set is equal to $\mathbb{C}$.

We will say that an exponential or polynomial map f_{c_0} is *Misiurewicz* (or *postsingularly finite*) if c_0 is a Misiurewicz parameter.

There are a few reasons why proving triviality of fibers (see section 4 for definition of fibers and a discussion on rigidity) for Misiurewicz parameters is easier than the other cases. Among them, there is a theorem proven by Schleicher and Zimmer in [SZ1].

Theorem 5.1.1 (Correspondence between dynamical and parameter plane at Misiurewicz points)**.** *A Misiurewicz parameter c_0 is the landing point of finitely many parameter rays $G_{s_1}, .., G_{s_q}$ whose addresses $s_1 < ... < s_q$ are preperiodic of period mq and preperiod k; moreover, the dynamic rays $g_{s_1}, ..., g_{s_q}$ with the corresponding addresses land at c_0 in Π_{c_0}.*

This theorem, well known for unicritical polynomials, expresses a form of combinatorial similarity between parameter and dynamical plane at Misiurewicz points. Together with the generalization of Thurston's rigidity theorem for exponentials ([HSS]) and a subsequent work ([LSV]), it gives a combinatorial classification of postsingularly finite exponential maps in the following Theorem ([LSV], Theorem 2.6):

Theorem 5.1.2 (Classification of Misiurewicz exponential maps)**.** *For every preperiodic address s, there is a unique postsingularly finite exponential map such that the dynamic ray of address s lands at the singular value. Every postsingularly finite exponential map is associated in this way to a finite number of preperiodic addresses.*

For unicritical polynomials it is well known that preperiodic parameter rays land at Misiurewicz parameters and that Misiurewicz parameters are landing points of preperiodic parameter rays, so the previous theorem offers a natural correspondence between exponential Misiurewicz parameters and polynomial Misiurewicz parameters through the addresses of the parameter rays landing at them.

Before exploring further the consequences of the combinatorial classification of Misiurewicz exponential maps, let us mention that the second main ingredient in proving triviality of fibers is offered by the linearizing coordinates which give contraction under the inverse map in a neighborhood of the postsingular periodic orbit.

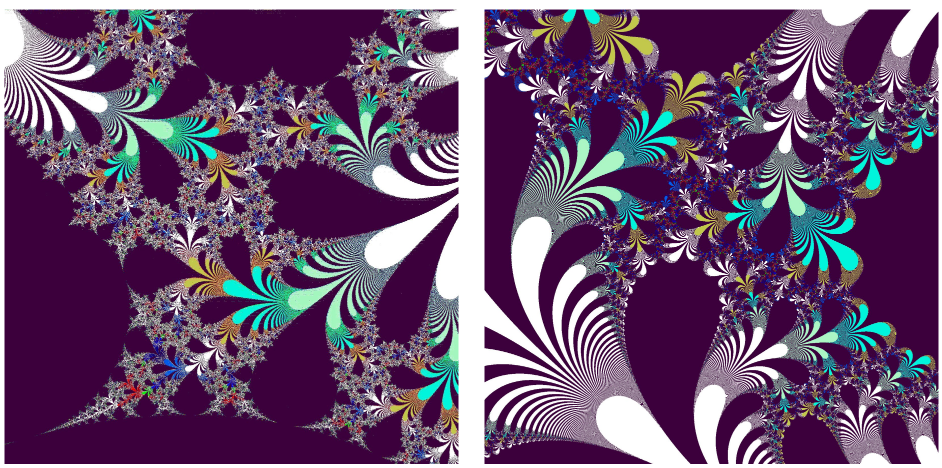

Figure 5.1: Parameter (on the left) and dynamical plane near the Misiurewicz parameter 1.81507+4.70945i; the spiralling of the two rays landing at it can be inferred from the picture.

A combinatorial property of Misiurewicz parameters One of the features of Misiurewicz parameters that we are going to use in the proof of our main theorem is a lemma connecting topology to combinatorics. It is proved in [SZ1] for exponentials and seems to be well known for unicritical poynomials of degree D; for completeness we will include a proof following the outline of [SZ2].

Dynamical partition Let $f(z) = e^z + c_0$ or $f(z) = z^D + c_0$ where c_0 is a Misiurewicz parameter, and choose one of the finitely many dynamic rays landing at c_0 given by Theorem 5.1.1, say g_{s_1}, where s_1 is the address/angle respectively. The preimage of g_{s_1} under f is a set of countably many curves going to $-\infty$ in the case of exponentials, and a set of D curves connecting at 0 for a polynomial of degree D. In both cases, the preimages of g_{s_1} partition the plane into open regions W_j.

Similarly, the preimages of s_1 under the shift map partition into the same number of sectors the combinatorial spaces $\mathcal{S}$ and $\mathcal{S}_D$. Label with the entry 0 the dynamical and the combinatorial sector containing c_0 and s_1 respectively, and label all other sectors using consecutive integers respecting cyclic order at infinity.

Any non-escaping point and any ray g_s which is not a preimage of g_{s_1} has a well defined itinerary whose entries keep track of the sectors visited by iterates of s under the shift map. We call this partition of the plane into the open regions W_i a *dynamical partition* for f_{c_0}.

Remark: depending on the choice of the dynamic ray landing at c_0 we will obtain different dynamical partitions. However, this choice will not matter to us.

In order to use a hyperbolic contraction argument we need a picture which is forward invariant, so we also need to remove from $\mathbb{C}$ the finitely many forward images of g_{s_1}. We obtain a new partition of the plane into domains $\widehat{W}_{i,j}$ where for each fixed i $\widehat{W}_{i,j}$ denotes a connected component of $\subset W_i$.

For convenience of the reader, let us recall that any hyperbolic surface S, and in particular any region of $\mathbb{C}$ whose complement contains at least two points, has for universal covering the unit disk; by pushing forward the hyperbolic metric on the unit disk via the covering map, we obtain a well defined hyperbolic metric ρ_S on S.

We will use the following basic theorem about hyperbolic contraction ([Mi], Theorem 2.11):

Theorem 5.1.3 (Schwarz-Pick Lemma)**.** *If $f : S \to S'$ is a holomorphic map between Riemann surfaces, either f is a local isometry, or f strictly decreases all nonzero distances in the hyperbolic metrics of S, S' respectively.*

We are now ready to prove the following lemma:

Lemma 5.1.4 (Significance of dynamical partition for Misiurewicz parameters)**.** *Two (pre)periodic rays which are not preimages of the dynamic rays landing at the singular value land together if and only if they have the same itinerary with respect to the dynamical partition described above.*

Proof. Endow each of the regions W_i and of the domains $\widehat{W}_{i,j}$ with the corresponding hyperbolic metric. Given a region W_i, we can choose a branch of f_i^{-1} mapping $\mathbb{C} - g_{s_1}$ into W_i, so that the same branch restricted to any $W_{j'}$ contracts the hyperbolic metric from $W_{j'}$ to W_i. Similarly each $\widehat{W}_{i,j}$ carries its own hyperbolic metric which is bigger than the metric of W_i, and the restriction of f_i^{-1} to any $\widehat{W}_{i,j}$ contracts the hyperbolic metric of that $\widehat{W}_{i,j}$.

Let us start by considering any two periodic rays which have the same itinerary; they land at two points w_1 and w_2 which are periodic, so that up to selecting branches they are both fixed under some M-th iterate f^{-M} of the inverse of f.

The periodic points w_1 and w_2 have the same itinerary under $f^k,\ k = 1...m$ so at each step we can select the same branch of f_{-1} and get hyperbolic contraction along the backward orbits until we get back to w_1 and w_2 decreasing hyperbolic distance, which is a contradiction unless $w_1 = w_2$ to start with.

This proves the theorem for periodic rays unless the iterates of w_1 and w_2 always belong to different connected components of $W_j - \cup_n f^n(g_{s_1})$ (i.e to different $\widehat{W}_{i,j}$). So suppose that w_1 and w_2 belong to the same W_i but to different $\widehat{W}_{i,j}$; then at least one of them, say w_1, belongs to one of the internal sectors defined in the section about orbit portraits and originating at some point z of the postsingular periodic orbit: the dynamics permutes those sectors transitively, so each image $f^n(w_1)$ belongs to the same W_i as $f^n(z)$, hence w_1 has the same itinerary as z and, as w_2 has the same itinerary as w_1, it also has the same itinerary as z.

Remains to prove that no periodic point w_i can have the same itinerary as some postsingular periodic point z. The family of inverse iterates is normal in a neighborhood of z, and is defined in a connected set containing w_i because the two points have the same itinerary. As the iterates converge to the constant map contracting everything to z in a linearizing neighborhood for z, they converge to the same map in the entire domain where they are defined, contradicting the fact that w_i is fixed under some appropriate iterate of the inverse function.

If the rays are preperiodic and have the same itinerary their periodic images also have the same itineray, hence land together by previous part; and since our preperiodic rays are not preimages of the rays landing at the singular value, and they have the same itinerary, we can take pullbacks using the same branch for both, so that they keep landing together.

On the other side if two rays land together they form a connected set, which never intersects the original partition under iterates of f, so they always belong to the same region of the partition. □

5.2 Triviality of Misiurewicz fibers

In this section we prove Theorem 5.2.4; for the remaining of the section, let c_0 be a Misiurewicz parameter.

The proof follows the outline of the corresponding result for polynomials (Lemma 7.1 and Theorem 7.3 in [S1]), using Theorem 3.3.7 to establish a bridge between the combinatorics for polynomials and the combinatorics for exponentials.

In particular, we will prove the exponential version of Lemma 7.1 in [S1], whose statement is exactly the same except for replacing polynomial maps with exponential maps:

Proposition 5.2.1 (Combinatorial approximation of parameter rays)**.** *Let c_0 be a Misiurewicz parameter for the exponential family or for a family of unicritical polynomials, and let $G_{s_1}, ..., G_{s_q}$ be the parameter rays landing at c_0. Then for any $\varepsilon > 0$ there exist parameter ray pairs P_i of angles $(\alpha_i,\ \alpha'_i)$ such that $s_i < \alpha_i < \alpha'_i < s_{i+1}$ for $i = 1...q-1$ and dist$(\alpha_i, s_i) < \varepsilon$, dist $(\alpha'_i, s_{i+1}) < \varepsilon$; moreover there is a parameter ray pair P_0 of angles (α_0, α'_0) such that $\alpha_0 < s_1 < s_q < \alpha'_0$ and dist$(\alpha_0, s_1) < \varepsilon$, dist $(\alpha'_0, s_q) < \varepsilon$.*

This is Lemma 7.1 in [S1] for polynomials and will be proven below for exponentials. There is a crucial point here: at first sight it might seem that this proposition would solve our problem, but the relation between the "combinatorial topology" and the topology on $\mathbb{C}$ are far from clear, so we still have to show that those rays which approximate the Misiurewicz rays combinatorially actually converge to them with respect to the standard topology on $\mathbb{C}$ in a neighborhood of c_0. We will derive this from the following propositions:

Proposition 5.2.2 (Triviality in dynamical plane)**.** *Let c_0 be a Misiurewicz parameter for the exponential family . Then dynamical fibers of the postsingular periodic orbit $\{z_i\}$ are trivial.*

Proposition 5.2.3 (Persistence of dynamical triviality)**.** *Let c_0 be as above. The postsingular periodic orbit has a well defined analytic continuation $\{z_i(c)\}$ for c in a neighborhood of c_0, and the dynamical fibers of points in this analytic continuation are trivial up to restricting to a smaller parameter neighborhood.*

At this point we will be able to prove our final theorem.

Theorem 5.2.4 (Triviality of Misiurewicz fibers)**.** *Any parameter c can be separated from c_0 by a parameter ray pair, except for those parameters lying on the rays G_{s_i} landing at c_0.*

Proof of Proposition 5.2.1: Combinatorial approximation of parameter rays. The core of the proof relies on the correspondence between combinatorial spaces for polynomials and for exponential parameters described in the end of Section 3.1; when the angles labeling rays for polynomials of degree D are written in D-adic expansion as sequences with D symbols, they can be seen as a subset of the exponentially bounded sequences encoding the combinatorics for exponential maps.

Consider the dynamic rays of addresses $s_1, \dots, s_q$ landing at the Misiurewicz parameter c_0. As noted in Lemma 5.1.4, each g_{s_i} defines a partition with respect to which dynamic rays which are never mapped to g_{s_i} have the same itinerary if and only if they land together in the dynamical plane.

Also, c_0 is the landing point of the parameter rays $G_{s_1}, ..., G_{s_q}$. As $s_1, \dots, s_q$ include only finitely many symbols because they are finitely many preperiodic addresses, by taking a sufficiently high degree the parameter rays of angles $s_1, ..., s_q$ exist. If we choose one of them, say s_1, G_{s_1} lands at some polynomial Misiurewicz parameter c_D.

For c_D the dynamic ray at angle s_1 also lands at the singular value. All other angles $s_2, \dots, s_q$ have the same itinerary as s_1 with respect to the partition induced by s_1 because they land together in Π_{c_0}, so by Lemma 5.1.4

the dynamic rays $g_{s_1}, ..., g_{s_q}$ all land together at c_0^P. No other dynamic ray can land together with them, otherwise its angle would be an admissible sequence for exponentials and would have the same itinerary, so the corresponding ray would land together with $g_{s_1}, \ldots, g_{s_q}$ in the exponential dynamical plane as well.

In the dynamical plane $\Pi_{c_D}^D$ by Lemma 7.1 in [S1] we have characteristic dynamic ray pairs approximating each sector arbitrarily close, and the two rays in each ray pair have the same itinerary again by Lemma 5.1.4; this is a purely combinatorial notion, so that it carries over to exponentials and the ray pairs with the same addresses keep landing together in the dynamical plane for $e^z + c_0$, giving the wanted approximating couples of rays in the dynamical plane.

Let us transfer those approximating ray pairs to the parameter plane for exponential maps. By Proposition 3.3.11 as the approximating rays are characteristic the parameter rays with the corresponding addresses land together in the exponential parameter plane giving the wanted approximation for the sector defined by G_{s_1} and G_{s_q}. To approximate the other parameter sectors as well, fix a sector, say the sector between G_{s_1} and G_{s_2}, call it $\widehat{s_1 s_2}$.

Let $V \subset \Pi_P$ be a neighborhood of c_0 such that there is an analytic continuation $\tilde{z}(c)$ of c_0 which keeps all the rays landing at c_0, and pick a Misiurewicz parameter c in $V \cap \widehat{s_1 s_2}$. In Π_c we will have the same relative position between $\tilde{z}$ and c as we have in parameter plane between c_0 and c, in the sense that c in Π_c belongs to the sector defined by the rays of addresses s_1 and s_2: this follows from the fact that rays respect the vertical order induced by their addresses both in dynamical and in parameter plane.

Lemma 7.1 in [S1] gives us characteristic dynamic ray pairs approximating $g_{s_1}^c$ and $g_{s_2}^c$ for polynomials (now $g_{s_1}^c$ and $g_{s_2}^c$ are landing at the repelling point $\tilde{z}(c)$, not at the singular value c); the corresponding rays can be obtained in the exponential dynamical plane by the same technique described above, and they can be transfered in parameter plane by Proposition 3.3.11.

□

Note that this proposition proves that we can separate a Misiurewicz parameter from all other Misiurewicz parameters, and from any parameter which is described combinatorially, for example parabolic and escaping parameters and landing points of parameter rays.

Remark 3. By the correspondence of characteristic ray pairs between polynomials and exponentials as stated in Theorem 3.3.15, we could have obtained the combinatorial approximation directly in the parameter plane, but we need it also in dynamical plane in order to prove that dynamical fibers of the postsingular orbit are trivial and to proceed with the topological part of the proof.

Proof of Proposition 5.2.2: Triviality in dynamical plane. Let z be the first periodic point in the postsingular orbit, and L be a linearizing neighborhood. Recall that k was the preperiod of the singular orbit, so taking the kth image of the approximating dynamic ray pairs found in the proof of Proposition 5.2.1 we obtain dynamic ray pairs which approximate combinatorially the q rays $g_{\sigma^k(s_1)}, \dots, g_{\sigma^k(s_q)}$ landing at z. We want to show that this combinatorial separation corresponds to an actual separation of all points in L from z.

So for each sector defined by the $g_{s'_i}$ consider an approximating ray pair which enters L. Note that such a ray pair must exist: as $J = \overline{I}$ and $J = \mathbb{C}$, any open set contains escaping points, so at least one ray must enter each sector, and once we have a ray inside we can surround it by one of the combinatorially approximating ray pairs.

So each sector contains a ray pair, and the region between that ray pair and the boundaries of the sector is uniformly contracted under f^{-m}, where m

was the period of the orbit, so that the region left out by the approximating ray pairs shrinks to $\{z\}$ under iterates of f^{-m}.

$\square$

Proof of Proposition 5.2.3: Persistence of dynamical triviality. Let $\{p_i\}_{i=1\ldots q}$ be the landing points of the ray pairs which enter the linearizing neighborhood in the proof of Propositon 5.2.2; then we can find a parameter neighborhood V of c_0 in which we can continue analytically both c_0, the p_i's and the postsingular periodic orbit $\{z_i\}$ with the same rays landing at them.

Up to shrinking V, we can also assume that the rays enter the new linearizing neighborhood, and by contraction under the inverse map the neighborhoods between the approximating rays and the actual rays landing at the analytic continuation of the z_i shrink to points. By carrying the approximating rays forward we obtain that the dynamical fiber of every $z_i(c)$ is trivial. $\square$

Proof of Theorem 5.2.4: Triviality of Misiurewicz fibers. We want to find a parameter neighborhood V of c_0 so that every $c \in V$ can be separated from c_0 by some parameter ray pair.

Note that it is enough to separate from c_0 any parameter c in the bifurcation locus, as rays cannot cross non-hyperbolic components.

We use Propositions 5.2.2 and 5.2.3 to show that the combinatorially approximating ray pairs found in Proposition 5.2.1 converge on the rays landing at c_0 in the complex plane, so that the regions which we can separate combinatorially actually fill $V - \cup G_{s_i}$.

Like we did before in dynamical plane, let us distinguish the cases in which the parameter c that we want to separate from c_0 is in the external sector which contains $-\infty$ (the one bounded by G_{s_1} and G_{s_q}) and the case in which c belongs to some of the other internal sector.

If c belongs to the external sector, consider the dynamical plane for c_0, and separate c from c_0 there by a preperiodic dynamic ray pair as from Proposition 5.2.2 follows directly that the dynamical fiber of c_0 is trivial. Now separate this preperiodic ray pair by by one of the approximating characteristic ray pairs found in Proposition 5.2.1 and then transfer this characteristic ray pair into parameter plane by remark 3.3.11. Note that c in general does not have a ray landing at it. However the parameter ray pair and c keep the same relative position in parameter plane that had c and the corresponding dynamic rays in Π_{c_0} by Proposition 3.3.11

If c belongs to one of the internal sectors, say $\widehat{s_1 s_2}$, and also belongs to the neighborhood V as in Proposition 5.2.3 then consider the dynamical plane Π_c. There, c belongs to the corresponding dynamical sector $\widehat{s_1 s_2}$ defined at the analytic continuation $\tilde{z}(c)$. The dynamical fiber of $\tilde{z}$ is trivial by Proposition 5.2.3, so we can separate c and $\tilde{z}(c)$ by some periodic ray pair (α, α'). This ray pair is persistent over a parameter neighborhood U of c. This means that, for the parameters in this neighborhood, in dynamical plane the singular value will be inside the sector bounded by the dynamical rays (α, α'). In particular, by vertical order, escaping parameters in this neighborhood lie on a dynamic ray of address between α and α' in dynamical plane, so they lie on a parameter ray of address between α and α'. By the combinatorial approximation given by Proposition 5.2.1, such a parameter is separated from c_0 by any of the ray pairs whose addresses are closer to s_1 and s_2 than α and α'. This means that we can separate all those escaping parameters simultaneously from c_0 using the same ray pair (β, β'). By density of escaping points in the bifurcation locus, we can approximate c by escaping parameters, so the ray pair (β, β') also separates c_0 from c unless c lies on β or β' in which case it has a well defined address and can be separated from c_0 by any ray pair closer than β or β'. $\square$

Remark 4. It was observed by Schleicher and then proven by Devaney and Jarque that there are rays which do not land and form indecomposable continua.

Chapter 6

Non-recurrent parameters

In this section we will restrict ourselves to parameters with modulus of the imaginary part bounded by π. As any two functions with singular value c and $c + 2\pi i$ are conformally conjugate, this will be enough.

Two exponential maps are called *combinatorially equivalent* whenever they have the same ray portrait in the sense of definition 3.3.6, i.e. the same set of periodic and preperiodic rays landing together as identified by their addresses.

Then the rigidity conjecture can be restated as

Conjecture 6.0.5. *Two non-escaping non-hyperbolic exponential maps can not be combinatorially equivalent.*

We will study *non-recurrent* parameters, that is those parameters c for which the postsingular set

$$\mathcal{P} = \overline{\bigcup_{n>0}\{f^n(c)\}}$$

does not intersect a sufficiently small neighborhood of c itself. In this case we will first show expansivity of the postsingular set $\mathcal{P}$ generalizing a theorem of Mañe (see e.g [MS] for rational functions and [RvS] in the transcendental setting).

We will say that a forward invariant set K is *hyperbolic* (with respect to the Euclidean metric) if there exists $\overline{k}, \eta > 1$ such that for any $k > \overline{k}$, $|(f^k)'(z)| > \eta$ for any $z \in K$.

The first main theorem, which will be proven in Section 6.1, states that

Theorem 6.0.6 (Hyperbolicity of the postsingular set)**.** *Let $f_c(z) = e^z + c$ such that c is non-recurrent. Then the postsingular set $\mathcal{P}$ is hyperbolic.*

The second main theorem holds for the smaller class of *combinatorially non-recurrent* parameters(see Section 6.2.1): we say that a parameter c is *combinatorially non-recurrent* if there is a suitable collection (see Definition 6.2.10) of periodic or preperiodic rays landing together in dynamical plane and which, together with their common endpoint, separate the singular value from the postsingular set.

A map $f_c : z \to e^z + c$ is called *(combinatorially) non-recurrent* if its singular value c is (combinatorially) non-recurrent.

Theorem 6.0.7. *Let f_c be such that c is combinatorially non-recurrent and non-escaping. Assume also that f_c is neither parabolic, nor hyperbolic, nor Siegel. If c' is non-escaping and $f_{c'}$ is combinatorially equivalent to f_c, then $f_c = f_{c'}$.*

In the first part (Section 6.1) we take a topological approach, proving a theorem (see Theorem 6.1.7) about the expansivity of the postsingular set under the assumption that the singular value is non-recurrent. This generalizes a theorem by Rempe and van Strien ([RvS]) for the exponential family to the case in which the postsingular set is not bounded.

The second part deals with rigidity results. We will start by defining Yoccoz puzzle and combinatorial non-recurrence in the exponential setting.

We will then restrict to the class of parameters for which the singular value is combinatorially non-recurrent, and show that in this case the combinatorics is unique by constructing a quasiconformal conjugacy between any two maps which are combinatorially equivalent and one of which is combinatorially non-recurrent.

Together with a theorem in [RvS] about absence of invariant line fields, this shows that there is a unique combinatorially non-recurrent exponential map with a given combinatorics.

6.1 Expansivity of the postsingular set

Expansivity is a crucial property in relating the dynamics on a small neighborhood with the dynamics of its preimages at smaller and smaller scale, and, as the contraction happens around the preimages of the singular value, it is usually related to the accumulating behavior around the singular value itself. In particular, expansivity of the postsingular set is related to non-recurrence of the singular value.

The case in which the postsingular set is compact has been studied extensively by various authors in the context of rational and transcendental functions ([RvS], [MS]).

This section is devoted to prove that the postsingular set is expansive in a more general setting in which the postsingular set is not bounded. When the postsingular set is bounded, the problem of expansivity has been solved for a very general class of functions in [RvS], Theorem 1.2.

Theorem 6.1.1 (Hyperbolic sets)**.** *Let f be a nonlinear non-constant meromorphic function with compact postsingular set. Let K be a forward invariant compact subset of the Julia set J which contains no parabolic points, no critical points and no accumulation points for any recurrent critical point, nor a singular value contained in wandering domains ; then K is hyperbolic.*

One of our first tasks will be to prove an unbounded version for the exponential family (See Theorem 6.1.7).

Let us setup some notation and results following [RvS]. Many of the results proven in [RvS] hold for more general classes of functions; for simplicity we will reformulate them in terms of exponential maps with non-recurrent singular value.

Definition 6.1.2. A point x is called *regular* if there is a $\delta > 0$ such that any connected component U of $f^{-n}(B_\delta(x))$ intersecting the postsingular set is simply connected, and $f^n : U \to B_\delta(x)$ is univalent.

The following is Theorem 2.7 in [RvS].

Theorem 6.1.3. *Let $f_c(z) = e^z + c$. Suppose that $z \in J$ is not a parabolic point, and that the singular value c is non-recurrent. Then z is a regular point.*

Remark: The additional hypothesis in the original theorem in [RvS] are satisfied for a non-recurrent exponential map.

Definition 6.1.4. A forward invariant set $\Lambda \supset \mathcal{P}$ is *locally expansive* if for all $z \in \Lambda$ and for all $\varepsilon > 0$, there exists $U(z)$ such that for all $n \geq 0$ and V' connected component of $f^{-n}(U(z))$ which intersects $\mathcal{P}$ we have the two following properties:

1. diam(V')$\leq \varepsilon$, and $f^n : V' \to U$ is univalent;

2. $\forall \varepsilon' > 0 \; \exists \; n_0 > 0$, if $n \geq n_0$, diam$(V') \leq \varepsilon'$.

The proof of the next theorem follows ideas from [LS].

Theorem 6.1.5 (Local to global)**.** *Let Λ be a forward invariant closed set such that Re $(z) > -M$ $\forall z \in \Lambda$, for some $M > 0$. If Λ is locally expansive, then Λ is hyperbolic.*

Proof. Suppose not. Then $\exists\, n_k \to \infty$, and $z_k \in \Lambda$, such that $|(f^{n_k})'(z_k)| \leq 1$. As Λ is closed any finite accumulation point of $\{f^{n_k}(z)\}$ belongs to Λ.

- Case 1: $\{f^{n_k}(z_k)\}$ has a finite accumulation point x.
- Case 2: $f^{n_k}(z_k) \to \infty$.

Case 1: As $x \in \Lambda$ by local expansivity there is a neighborhood $U_\varepsilon = U_\varepsilon(x)$ satisfying the two conditions in definition 6.1.4. So $\operatorname{diam}(V') < \varepsilon$ for any V' connected component of $f^{-n}(U_\varepsilon)$, and f^n is a univalent map from V' to U_ε.

As $f^{n_k}(z_k) \to x$, for large k $f^{n_k}(z_k) \in U_\varepsilon$. Let V_k be the component of $f^{-n_k}(U)$ containing z_k. By local expansivity, $\operatorname{diam}(f^j(V_k)) \leq \varepsilon$ for $j = 0 \dots n_k$, and since $f^j(z_k) \in \Lambda$ for all j's we have that $f^j(V_k)$ does not contain c, so $f^{n_k} : V_k \to U$ is bijective and we can define its local inverse $\phi_k : U \to V_k$. Then the family $\{\phi_k\}$ is normal, and any limit function ϕ has to be constant because $\operatorname{diam}(V_k) \to 0$ as $k \to \infty$. This contradicts the initial assumption that

$$|\phi'(x)| = \lim_{k\to\infty} |\phi_k'(f^{n_k}(z_k))| = \lim_{k\to\infty} \left| \frac{1}{(f^{n_k}(z_k))} \right| \geq 1.$$

Case 2: $f^{n_k}(z_k) \to \infty$. To treat this case we will find another subsequence $f^{j_k}(z_k)$ such that $j_k \to \infty$, $|f^{j_k}(z_k)'| \leq 1$, and $\operatorname{Re} f^{j_k}(z_k) < 0$, so that $\{f^{j_k}(z_k)\}$ is bounded and we can reduce ourselves to case 1.

Note that as Λ is forward invariant $f^{j_k}(z_k) \in \Lambda \forall j$, and that the j_k's are not a subsequence of the n_k; however for any n_k we will find $j_k \leq n_k, j_k \to \infty$ with the properties above.

Let $w_k^j = f^{j_k}(z_k)$, $j = 0 ... n_k$. If $f^{n_k}(z_k) \to \infty$ in a right half plane, $\operatorname{Re}\, f^{-1}(f^{n_k}(z_k)) \to \infty$, so without loss of generality we can assume $\operatorname{Re}\, f^{n_k}(z_k) \to \infty$.

$$\begin{aligned}
1 &\geq |f^{n_k}(z_k)'| = \\
&= |f'(z_k) \cdot f'(f(z_k)) \cdot \ldots \cdot f'(f^{n_k}(z_k))| = \\
&= e^{\mathrm{Re} z_k} \cdot e^{\mathrm{Re} f(z_k)} \cdot \ldots \cdot e^{\mathrm{Re} f^{n_k}(z_k)} \Rightarrow \\
&\Rightarrow \prod_{i=0}^{n_k - 1} e^{\mathrm{Re} w_k^i} \leq \frac{1}{e^{\mathrm{Re} w_k^{n_k}}}.
\end{aligned}$$

Also,

$$\begin{aligned}
& e^{\mathrm{Re} w_k^i} \geq e^{-M} \forall w_k^i \\
& \Rightarrow \text{ if } N \text{ is the cardinality of the set } \{w_k^i, \mathrm{Re} w_k^i < 0\}, \\
& \prod_{i=0}^{n_k - 1} e^{\mathrm{Re} w_k^i} \geq e^{-MN} \Rightarrow \\
& \Rightarrow e^{-MN} \leq e^{-\mathrm{Re} w_k^{n_k}} \to 0 \text{ because } \mathrm{Re} w_k^{n_k} \to \infty \\
& \Rightarrow N \to \infty
\end{aligned}$$

So, up to selecting a subsequence of the z_k's, we can always find arbitrarily large iterates

$$f^{j_k}(z_k) = w_k^{j_k} \text{ such that Re } w_k^{j_k} < 0.$$

To check that $|f^{j_k}(z_k)'| \leq 1$ as required, note that this is always true up to selecting a bigger $j_k < n_k$, as $|f^{n_k}(z_k)| \leq 1$ and the derivative of $f(w_k^i)$ is bigger than 1 as soon as Re $w_k^i > 0$. □

Theorem 6.1.6 (Local expansivity of P)**.** *The set $\mathcal{P}$ is locally expansive in the sense of definition 6.1.4*

Proof. Claim 1: For any $z \in \mathcal{P}$, $\varepsilon > 0$ there exists $U = U(z)$ such that for each $n \in \mathbb{N}$ and any V' connected component of $f^{-n}(U)$ intersecting $\mathcal{P}$, diam$V' < \varepsilon$.

The family of inverse iterates $\{f^{-n}\}$ is normal in a neighborhood U of z, and as z is a regular point by 6.1.3 $f^n : V' \to U$ is univalent. By normality of $\{f^{-n}\}$ and Ascoli-Arzela's Theorem, $|(f^{-n})'| \leq B$ in U for some constant $B > 0$ independent of n. By Koebe 1/4-Theorem and univalency of $f^n : \; V' \to U$,

$$\mathrm{diam}(V') \leq 4B\,\mathrm{diam}(U) \leq \varepsilon \text{ if } \mathrm{diam}(U) \leq \varepsilon/(4B).$$

Now let us prove that the diameters of the pullbacks tend to zero when the number of backward iterates tends to infinity, if we only consider pullbacks intersecting the postsingular set.

Claim 2: For any $\varepsilon' > 0$ there exists $n_0 > 0$ such that $\mathrm{diam}(V') \leq \varepsilon'$ if $n \geq n_0$.

Fix $\varepsilon > 0$ and let U be the neighborhood that we obtain from the first property in definition 6.1.4, and suppose by contradiction that $V_k := V_{n_k}$ is a subsequence of preimages of $U(z)$ such that $\mathrm{diam}(V_k) \geq \varepsilon$ for some $\varepsilon > 0$. If $|f^{-n_k}(z)|$ does not tend to ∞ we can find a finite accumulation point ζ for $f^{-n_k}(z)$. As the diameter of the V_k does not go to zero, and $f^{n_k} : V_k \to U$ is univalent, there is a definite neighborhood which is mapped univalently into U under the infinitely many iterates f^{n_k}, contradicting the fact that $\zeta \in J$.

If $|f^{-n_k}(z)| \to \infty$, let us find another subsequence of pullbacks $V_{n_j} = f^{-n_j}(U)$ such that $|f^{-n_j}(z)|$ is bounded and $\mathrm{diam}(V_j) > \varepsilon''$ for some $\varepsilon'' > 0$, in order to reduce the proof to the previous case.

Let us first note that if $|f^{-n_k}(z)| \to \infty$ then $\mathrm{Re}\, f^{-n_k-1}(z) \to \infty$. Also, $\mathrm{diam}\, V_{n_k-1} > \varepsilon' e^{-M+2\varepsilon}$ because $f^{-n_k}(z) > -M - \varepsilon$ on all branches of the function intersecting the postsingular set (as $\mathrm{Re}\, x > -M$ for all $x \in \mathcal{P}$, and $\mathrm{diam}\, V_{n_k} < \varepsilon$).

Note that whenever $\mathrm{Re}\, z > 0$ $\mathrm{diam} V_{n_{k+1}} \leq \mathrm{diam} V_{n_k}$, so if $\mathrm{diam} V_k > \varepsilon_1$ for $z_{n_k} \to \infty$ we must have a sequence V_{n_j}, $\mathrm{diam} V_{n_j} \geq \varepsilon_1 e^{-2M+2\varepsilon}$ contained in the vertical strip $\{-M - 2\varepsilon < \mathrm{Re}\,(z) < 2\varepsilon\}$.

The $V_{n_{j}-1}$ are a countable family of congruent disks shifted vertically by

$2m\pi$ over $m \in \mathbb{N}$, with diameter bigger or equal to $\varepsilon' e^{-3M+3\varepsilon}$. For each n_j we can choose branches so as to get disks of imaginary part between -2π and 2π; as we already have a bound on the real part, this gives us a family of bounded disks with diameter $> \varepsilon'$ hence we can reduce ourselves to the previous case.

Note that the preimages of imaginary part between -2π and 2π do not need to intersect the postsingular set in order to deduce a contradiction: the intersection property was used only when assuming that the contraction in the diameter could not be bigger than $e^{-M+\varepsilon}$.

□

Theorem 6.1.7 (Hyperbolicity of the postsingular set)**.** *Let $f_c(z) = e^z + c$ such that c is non-recurrent. Then the postsingular set $\mathcal{P}$ is hyperbolic.*

Proof of Theorem 6.1.7. $\mathcal{P}$ is locally expansive by Theorem 6.1.6, hence it is hyperbolic by Theorem 6.1.5. □

6.2 Using puzzles to construct conjugacies

Let c be a parameter contained in one of the parabolic wakes attached to the period one component, and $\alpha = \alpha(c)$ be the analytic continuation of the attracting fixed point; then α is the landing point of at least two periodic dynamic rays.

We will say that the singular value c is *combinatorially non-recurrent* if under finitely many pullbacks the singular value and the postsingular set are separated by preimages of the rays landing at α together with their endpoint.

We will show that two combinatorially non-recurrent maps $f_c, f_{c'}$ with the same combinatorics and non-escaping singular value are quasiconformally conjugate.

We will start by constructing a quasiconformal conjugacy on the postsingular set and then use pullback arguments to extend the conjugacy on the whole plane.

Once we have a quasiconformal conjugacy on the whole plane, if the postsingular set is bounded we can use a theorem in [RvS] on absence of invariant line fields to conclude that the two maps are conformally conjugate, hence the two parameters either coincide or differ by $2\pi i$.

The strategy for constructing a quasiconformal conjugacy between two combinatorially non-recurrent maps consists in using holomorphic motions of rays and puzzle pieces to construct a quasiconformal map between the dynamical planes of f_1 and f_2, and pull it back using non-recurrence so as to get a sequence of quasiconformal maps converging to a conjugacy.

6.2.1 Holomorphic motions, puzzles and conjugacies

This section will describe holomorphic motions and construction of Yoccoz's puzzle in the exponential family.

Holomorphic motions. Holomorphic motions are widely used in 1-dimensional complex dynamics. For a reference see e.g. [L], Chapter 3.

Definition 6.2.1. Let $(\Lambda, *)$ be a topological open disk in $\mathbb{C}$ with a marked point $*$, and let X be an arbitrary subset of the Riemann sphere.

A *holomorphic motion.* $\mathbf{h}$ of X over Λ is a family of injections $h_\lambda : \mathbb{C} \to \mathbb{C}$, $\lambda \in \Lambda$, depending holomorphically on λ for each fixed $x \in X$, and such that h_* is the identity. We will use the notation $X_\lambda := h_\lambda(X_*)$. We will say that a set X_* *moves holomorphically* on Λ if such a holomorphic motion exists.

The following theorem, due to [MSS] and [Sl] collects the most relevant results about holomorphic motions.

Theorem 6.2.2 (Advanced Lambda Lemma)**.** *A holomorphic motion* **h** *of any set X extends to a continuous holomorphic motion of* $\mathbb{C}$.

The maps h_λ are K-quasiconformal with dilatation K depending only on the hyperbolic distance between λ and $*$. However, K tends to infinity as $\lambda \to \delta\Lambda$.

Yoccoz's puzzle. Given a parameter c which belongs to a parabolic wake attached to the period one hyperbolic component, there is a repelling fixed point $\alpha = \alpha(c)$, defined as the analytic continuation of the attracting fixed point, which is the landing point of q periodic rays permuted transitively by the dynamics (see for example Section 3 in [RS1]). Let $s_1...s_q$ be the addresses of the rays landing at α.

We will show in Chapter 7 (Corollary 7.2.12) that a non-recurrent parameter with bounded postsingular set and which is not on the boundary of the period one hyperbolic component belongs to such a wake.

Roughly speaking, a puzzle is a collection of subsequently finer partitions of the plane (whose elements are the puzzle pieces) for which we have a good knowledge of the dynamics restricted to puzzle pieces.

We will describe the construction of puzzles for the exponential family, in analogy to the construction described for polynomials e.g. in [H], [L1] and [Mi]. Various versions of Yoccoz puzzle have been constructed for other maps, for example rational maps.

Consider the set G formed by α and the dynamic rays landing at α: G is forward invariant and separates the plane into finitely many connected components. Then the collection of connected components of $\mathbb{C} - f^{-n}G$ is called the *puzzle of level* n and is denoted by $V^{(n)}$; the connected components themselves are called *puzzle pieces of level* n and are denoted by $Y_j^{(n)}$, where j is some labeling to be defined later.

The next lemma follows directly from the definition of puzzle pieces.

Lemma 6.2.3. *For each $n \in \mathbb{N}$, $Y_i^{(n)} \in V^{(n)}$ and $Y_j^{(n+1)} \in V^{(n+1)}$, either $Y_j^{(n+1)}$ is contained in $Y_i^{(n)}$ or the two puzzle pieces are disjoint.*

For any fixed n, $Y_i^{(n)}$ is mapped to some $Y_j^{(n-1)}$, either univalently or as an infinite degree covering with only one branching point.

We will now describe a way of labeling puzzle pieces respecting the dynamics, providing a tool which will be useful later to relate how two take inverse branches for two functions having the same puzzle.

Labeling of puzzle pieces. Consider the puzzle pieces of level 0.Let us call $Y_*^{(0)}$, or branching puzzle piece of level 0, the region containing $-\infty$, and $Y_i^{(0)}$ the other sectors defined by the rays landing at α, with the convention that $Y_0^{(0)}$ is the puzzle piece containing the singular value and the other pieces are labeled respecting the vertical order.

Let us introduce a labeling for puzzle pieces of level n which keeps track of the dynamics; to do so we need to introduce some kind of partition in order to keep track of the branches under which we take the preimages. Let s_1 be the address of the lowermost ray landing at α, and let us define *combinatorial sectors*

$$S_i := \{s \in \mathcal{S}, s_1 + i < s < s_1 + i + 1\}.$$

This defines a partition of the combinatorial space $\mathcal{S}$ into strips.

In $\mathbb{C}$, this partition only has a meaning near infinity, so it can be only used to encode the position of objects which tend to infinity to the right like other dynamic rays or puzzle pieces. Consider a vertical line $\mathcal{L} := \{z, \text{Re } z = L\}$ sufficiently far to the right so that the q rays landing at α all intersect only once every vertical line to the right of $\mathcal{L}$. Given a point z such that Re $z > L$, let t_z such that Re $g_{s_1}(t_z) =$ Re z and $\Phi(z) = \operatorname{Im} g_{s_1}(t_z)$. $\Phi(z)$ is well defined because g_{s_1} intersects only once any vertical line wit real part bigger than L.

Let $\tilde{S}$ be the semi-strips

$$\tilde{S}_j = \{z, \text{Re } (z) > L, \operatorname{Im} z \text{ between } g_{s_1} + 2\pi i j \text{ and } g_{s_1} + 2\pi i(j+1)\}.$$

As the puzzle pieces of level $n+1$ are defined as pullbacks under f_c of the puzzle pieces of level n, we can label them by a string of increasing length by adding to the left the symbol corresponding to the combinatorial sector they belong to, where we say that a puzzle piece *belongs to a combinatorial sector* if the uppermost ray on its boundary belongs to that combinatorial sector. This is well defined unless we have a branching puzzle piece, in which case we will add to the left the symbol $*$.

This construction labels every puzzle piece of level N by a string in $\{\mathbb{Z} \cup \{*\}\}^{\mathbb{N}}$.

Example 6.2.4. At the first level, the puzzle pieces are given by sectors based at preimages of α and by a branching puzzle piece; we will call this last one $Y_{*0}^{(1)}$ because it is mapped to the singular piece of level 0, which was labeled as $Y_0^{(0)}$, and we will call each one of the other sectors $Y_{nm}^{(1)}$, where n is the combinatorial sector they belong to and m is the label of the piece of level 0 they are mapped to. Note that $n \in \mathbb{Z}$, but $m = 0, ..., q-1$.

By construction, the piece of level N labeled by a string ℓ is mapped univalently to the piece of level $N-1$ labeled by the string $\sigma\ell$, unless it is the branching puzzle piece (i.e. the first symbol in ℓ is $*$) in which case it is mapped ∞-to-1 to the singular puzzle piece of level $N-1$.

6.2.2 Parapuzzles and combinatorial equivalence

Parapuzzle pieces. The topology of the puzzle at the level $N+1$ is determined only by the piece on level N which contains the singular value, which is the same as saying the piece of level 1 which contains $f^N(c)$. In particular, the topology of the puzzle is allowed to change every q iterates, when the orbit of c comes back to the singular sector of level 0.

As the rays landing at the α fixed point move holomorphically inside their characteristic wake, puzzle pieces at any finite level move holomorphically inside the regions delimited by the Misiurewicz wakes corresponding to preimages of the rays landing at α and contained in the characteristic wake; note that by vertical order, Misiurewicz wakes contained in the characteristic sector in dynamical plane- which is when the singular value comes back to the singular puzzle piece of level 0 and changes the topology of the next level- correspond to Misiurewicz wakes contained in the characteristic wake in parameter plane.

The discussion about puzzles in dynamical plane leads to the following definition and lemmas in analogy with polynomials (see for example [L1]).

Definition 6.2.5. A *parapuzzle piece* of level N is a region in parameter space over which the boundaries of puzzle pieces of level N move holomorphically.

Proposition 6.2.6 (Parapuzzle pieces)**.** *The boundary of a parapuzzle piece consists of infinitely many parameter ray pairs landing at Misiurewicz parameters and possibly the wake in which the parapuzzle piece is contained; the addresses of the parameter rays delimiting a parapuzzle are preimages under the shift map of the addresses of the dynamic rays landing at* α.

Parapuzzle pieces of different levels are either disjoint or contained one inside the other.

Proof. A periodic ray pair belongs to an orbit portrait and moves holomorphically on the wake defined by the characteristic sector of the orbit portrait. A preperiodic ray pair moves holomorphically on the wake defined by the parameter rays with the same addresses. □

Definition 6.2.7. We say that two maps f_1, f_2 are *combinatorially equivalent* or *have the same combinatorics* if two periodic or preperiodic dynamic rays land together in the dynamical plane for f_1 if and only if the rays with the same address land together in the dynamical plane for f_2.

Proposition 6.2.8 (Combinatorial equivalence)**.** *If two exponential maps f_1, f_2 are combinatorially equivalent, they belong to the same parapuzzle pieces at all levels, hence have the same puzzle pieces at all levels, in the sense that each puzzle piece moves holomorphically.*

Also, at each level the singular values have the same itinerary with respect to the partition induced in the two dynamical plane by puzzle pieces of that level. (actually, it is enough to say by puzzle pieces of level 1.)

In particular, being combinatorially equivalent implies that each puzzle piece of level N for f_1 labeled by a string ℓ moves holomorphically to the puzzle piece for f_2 labeled by the same string ℓ over the parapuzzle piece of level N.

Proof. Topology of puzzle pieces is determined uniquely by the vertical order of dynamic rays at infinity and by the pattern in which the rays forming the boundaries of puzzle pieces land together. The vertical order at infinity is independent of the specific parameter under consideration, so two maps which are combinatorially equivalent, having the same pattern for the landing of (pre)periodic rays, have the same puzzle pieces at all levels.

To see that this is uniquely determined by the itinerary of the singular value, note again that each puzzle piece is completely determined by its boundary, which consists of countably many ray pairs each of which is the preimage of one of the ray pairs in the puzzle pieces of level 0. Each ray in a ray pair has countably many preimages, and how they land together depends on whether the initial ray pair was surrounding or not the singular value.

This means that the puzzle of level $N+1$ is topologically determined by which puzzle piece of level N contains the singular value, and at the same time it is determined by the pattern in which the rays land together. This shows that the pattern in which rays land together determines the itinerary of the singular value with respect to the puzzle.

Correspondence of labeling and homeomorphism between puzzle pieces can be checked by induction. It is true at level 0. Now suppose the labeling between the puzzle pieces for f_1, f_2 corresponds at level N. Fix a puzzle piece Y_ℓ of level N for f_1 and let Y'_ℓ be the corresponding puzzle piece for f_2. By the previous part, $c_1 \in Y_\ell$ if and only if $c_2 \in Y'_\ell$, so the countably many preimages of Y_ℓ, Y'_ℓ are all homeomorphic. Also, the label on each preimage only depends on vertical order near infinity, which does not depend on the parameter under consideration. □

According to this construction, if two maps are in the same fiber, hence cannot be separated by a rational ray pair in parameter plane, holomorphically they have the same puzzle pieces at all levels; so to show that the fiber of a parameter c_0 is trivial it is enough to show that the nest of parapuzzle pieces containing c_0 intersected with the non-escape locus shrinks to a single point.

We will pursue this results by showing that any two combinatorially non-recurrent maps which are also combinatorially equivalent need to be quasiconformally conjugate hence, by absence of invariant line fields (Theorem 6.3.2), conformally conjugate if we assume $\mathcal{P}$ to be bounded.

6.2.3 Initial quasiconformal map

In this section we will construct a quasiconformal map between the dynamical planes of two parameters belonging to the same parapuzzle piece up to some level n.

Proposition 6.2.9 (Initial quasiconformal map)**.** *Let f_c, $f_{c'}$ be two exponential maps who belong to the same parapuzzle piece of level n for some $n > 1$. Then there exists a quasiconformal map $\psi_0 : \Pi_c \to \Pi_{c'}$ such that*

- $\psi_0(c) = c'$

- *For each dynamic ray $g_s \in f_c^{-n}(\{g_{s_1}, ..., g_{s_q}\})$, i.e. the boundaries of puzzle pieces of level n, $\psi_0(g_s(t)) = g'_s(t)$, where g'_s is the dynamic ray for $f_{c'}$ of address s. By the functional equation in Theorem 3.1.3, this implies that ψ_0 is a conjugacy on $f_c^{-n}(\{g_{s_1}, ..., g_{s_q}\})$.*

Proof. The parapuzzle piece of level n containing c is open and connected, so we can find a path γ joining c to c' for any c' which belongs to the same parapuzzle piece of level n. By Theorem 3.1.3 we have that the map

$$g : \mathbb{C} \times \mathcal{S} \times (0, \infty) \to \Pi_c$$
$$g : (c, s, t) \mapsto g_s^c(t)$$

is holomorphic in c for any fixed (s, t). In particular, given a dynamic ray $g_s^{\tilde{c}}$ which is defined for all $\tilde{c} \in \gamma$, we have that for any $z \in g^c(s, t)$ the map $g_s^{c'} \circ (g_s^c)^{-1}$ is holomorphic in c', injective in (s, t) and the identity for $c' = c$, hence it is a holomorphic motion of the ray g_s^c. Also, if s is periodic and lands at a periodic point z with a given ray portrait A, the holomorphic motion can be extended to z over the whole wake defined by the characteristic rays of A; and the singular value c' itself also moves holomorphically on the whole plane.

As $f_c, f_{c'}$ belong to the same parapuzzle piece of level n, they have the same topological puzzle pieces up to level n, so the rays defining those puzzle pieces can be moved holomorphically from one dynamical plane to the other along the path connecting c to c' inside the parapuzzle piece.

By noticing that the singular value c itself also moves holomorphically, we get a holomorphic motion H defined on the boundaries of the puzzle pieces of level n and on the singular value.

By the lambda lemma (6.2.2), the holomorphic motion H of the rays of level n and the singular value can be extended to a quasiconformal map ψ_0 :

$\Pi_c \to \Pi_{c'}$; ψ_0 is equivariant on the rays, i.e. $f_{c'} \circ \psi_0 = \psi_0 \circ f_c$ restricted to the dynamic rays $f_c^{-n}(\{g_{s_1}, ..., g_{s_q}\})$, because by definition of the rays themselves $f_i(g_s^i(t)) = g_{\sigma s}^i(F(t))$ for $i = c, c'$ and $g_s^c(t)$ is mapped to $g_s^{c'}(t)$ by ψ_0.

Moreover, ψ_0 maps c to c'. □

6.2.4 Quasiconformal Pullback and Conjugacy on $\mathcal{P}$

In this section we will pullback the map ψ_0 obtained in the previous section in order to obtain a quasiconformal map which is a conjugacy on the postsingular set.

Definition 6.2.10. We will say that a parameter c or the corresponding map f_c is *combinatorially non-recurrent* if there exists a level N in the puzzle construction such that $P^{(N)} := \{Y \in V^{(N)},\ Y \cap \mathcal{P} \neq \emptyset\}$ does not contain c.

Corollary 6.2.11 (Corollary of Theorem 6.2.8)**.** *If two maps $f_c, f_{c'}$ are combinatorially equivalent and f_c is combinatorially non-recurrent, then $f_{c'}$ is also combinatorially non-recurrent.*

Proposition 6.2.12 (Conjugacy on $\mathcal{P}$)**.** *Let $f_c, f_{c'}$ be two combinatorially equivalent maps, such that c is combinatorially non-recurrent under f_c and neither c nor c' are escaping. Then there exists a quasiconformal map ψ which conjugates f_c to $f_{c'}$ on the postsingular set $\mathcal{P}$.*

Proof. Let Y_ℓ be a puzzle piece for f_c; as $f_c, f_{c'}$ are combinatorially equivalent, there is a corresponding puzzle piece Y'_ℓ for $f_{c'}$, in the sense that each boundary ray of Y_ℓ can be moved holomorphically to a corresponding boundary ray of Y'_ℓ (see Proposition 6.2.8).

For each level n, denote by $P^{(n)}$ and $P'^{(n)}$ the union of the puzzle pieces of level n containing the postsingular set for f_c and $f_{c'}$ respectively.

As f_c is combinatorially non-recurrent, there is some level N such that neither the singular puzzle piece nor the branching puzzle piece of level N belong to $P^{(N)}$ (recall that the branching puzzle piece of level N is the pullback of the singular puzzle piece of level $N-1$ and that $\mathcal{P}$ is forward invariant).

Let ψ_N be the map obtained in Proposition 6.2.9 on the puzzle pieces of level N, and define inductively ψ_{n+1} on the puzzle pieces of level $n+1$ as

$$\psi_{n+1} = f_{c'}^{-1} \circ \psi_n \circ f_c \tag{6.2.1}$$

$$= f_{c'}^{-n} \circ \psi_n \circ f_c^n \qquad \text{whenever } V_\ell^{(n+1)} \subset P^{(n)} \tag{6.2.2}$$

$$\psi_{n+1} = \psi_n \qquad \text{whenever } V_\ell^{(n+1)} \subset V^{(n)} - P^{(n)} \tag{6.2.3}$$

$$\psi_{n+1} = \psi_n \qquad \text{on } \partial P^{(n)} \tag{6.2.4}$$

Choose the branch of $f_{c'}^{-1}$ that brings $(\psi_n \circ f_c)V_\ell^{(n+1)}$ to Y'_ℓ; this can be done by Proposition 6.2.8.

Observe that for each n we are redefining the quasiconformal map on all puzzle pieces of level $n+1$ which are contained in $P^{(n)}$, not only on those which are contained in $P^{(n+1)} \subset P^{(n)}$. In this way, by the functional equation in Theorem 3.1.3, we obtain that ψ_{n+1} is a conjugacy on the boundary of all puzzle pieces of level $n+1$ which are contained in P^N; this ensures continuity of Ψ_{n+2}.

The map ψ_{n+1} is well defined because $f(V_\ell^{(n+1)})$, hence $(\psi_n \circ f_c)(V_\ell^{(n+1)})$, does not contain the singular value.

The map ψ_{n+1} is continuous on $\partial P^{(n)}$, hence on $\mathbb{C}$, by continuity of ψ_n and the functional equation in 3.1.3.

As the maps ψ_n are given by pre-post composition of the same quasiconformal map with univalent functions, they are all uniformly quasiconformal; they can be extended as to fix infinity and they all coincide on the boundaries of $P^{(N)}$, hence they converge in the spherical metric to some quasiconformal function $\psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ fixing infinity (see for example [L]).

By the functional equation in 6.2.1, the limit map ψ is a conjugacy between f_c and $f_{c'}$ on $\bigcap_{n \geq N} P^{(n)} \supset \mathcal{P}$. $\square$

6.2.5 Spreading the Conjugacy

In this section we want to subsequently lift ψ to a conjugacy on the preimages of the postsingular set, and use the fact that this last one is dense to obtain by continuity a conjugacy on the entire plane.

Lemma 6.2.13 (Alexander's trick)**.** *Let D, D' be a topological disks. Two functions $f, f' : D \to D'$ which are homotopic on ∂D are homotopic on D.*

Theorem 6.2.14 (Quasiconformal conjugacy)**.** *Let $f_c, f_{c'}, \psi$ be as in 6.2.12; then we can subsequently lift ψ in order to obtain a limit map Ψ to be a quasiconformal conjugacy Ψ between f_c and $f_{c'}$.*

Proof. Let Ψ_0 be ψ as given by Proposition 6.2.12. We want to show that given Ψ_n which is a conjugacy on $f^{-n}(\mathcal{P})$ and sends c to c' it can be lifted to Ψ_{n+1} which is a conjugacy on $f^{-n-1}\mathcal{P}$.

Consider the lifting diagram below :

$$\begin{array}{ccc} & \Psi_{n+1} & \\ (\mathbb{C}, c) & \longrightarrow & (\mathbb{C}, c') \\ f_c \downarrow & & \downarrow f_{c'} \\ (\mathbb{C} - c, f_c(c)) & \longrightarrow & (\mathbb{C} - c', f_{c'}(c')) \\ & \Psi_n & \end{array}$$

Both f_c: $\mathbb{C} - \{c\} \to \mathbb{C}$ and $f^{c'}$: $\mathbb{C} - \{c'\} \to \mathbb{C}$ are covering maps, and all lifts of Ψ_N are defined up to a translation by $2\pi i$.

To show existence of the lift at each stage, it is enough to show that Ψ_N is homotopic to the identity relatively $\mathcal{P}$. This follows from Lemma 6.2.13 and from the fact that Ψ_N is homotopic to the identity on the boundaries of puzzles pieces at the level N given by combinatorial non-recurrence (compare the definition of ψ_n in 6.2.9).

To check that this condition is preserved by the induction step recall that $\Psi_0 = \psi$ is a conjugacy on those rays hence is never modified by the lifting procedure on those rays themselves.

To choose the lift Ψ_{n+1} so that it maps c to c', consider each time a base point w_n in the same puzzle piece of level n as c. Ψ_{n+1} is uniquely defined up to translation by $2\pi ik$, and the puzzle pieces containing c have width less than 2π, there is a unique choice of Ψ_{n+1} sending w_n to a w'_n in the same puzzle piece as c'. By continuity, c will be mapped to c' as well.

By definition of Ψ_{n+1},

$$\Psi_{n+1}|_{\mathcal{P}_n} = f_{c'}^{-1} \circ \Psi_n \circ f_c|_{\mathcal{P}_n} = f_{c'}^{-1} \circ f_{c'} \circ \Psi_n|_{\mathcal{P}_n} = \Psi_n|_{\mathcal{P}_n}$$

, so that Ψ_{n+1} is still a conjugacy on $f^{-n}\mathcal{P}$.

To show that the new map Ψ_{n+1} is a conjugacy on $f^{-n-1}(\mathcal{P})$, note that by definition

$$f_{c'} \circ \Psi_{n+1} = \Psi_n \circ f_c$$

hence

$$f_{c'} \circ \Psi_{n+1}|_{\mathcal{P}_{n+1}} = \Psi_n \circ f_c|_{\mathcal{P}_{n+1}} = \Psi_n|_{\mathcal{P}_n} = \Psi_{n+1}|_{\mathcal{P}_n} = \Psi_{n+1} \circ f_c|_{\mathcal{P}_{n+1}}.$$

As we are lifting via holomorphic maps, all Ψ_n are uniformly quasiconformal and they all coincide on the postsingular set. Extending the ψ_n to fix infinity we get a limit map Ψ which is a conjugacy on the union of the preimages of the postsingular set. As the latter ones are dense, by continuity Ψ is a conjugacy on all of $\mathbb{C}$. □

6.3 Uniqueness of combinatorics for non-recurrent parameters

Theorem 6.3.1 (Uniqueness of combinatorics)**.** *If f_c, $f_{c'}$ are two combinatorially non-recurrent, non-escaping exponential maps with the singular* *value in the Julia set, bounded postsingular set and the same combinatorics, $c = c' + 2\pi i n$ for some $n \in \mathbb{N}$.*

The following theorem is Theorem 1.1 in [RvS].

Theorem 6.3.2 (Absence of line fields)**.** *Let $f_c(z) = e^z + c$ such that c is non recurrent-and the postsingular set is bounded. Then the Julia set of f supports no invariant line fields.*

Proof of Theorem 6.3.1. The quasiconformal conjugacy obtained in Theorem 6.2.14 allows us to pushforward the standard conformal structure σ_0 in the dynamical plane for f_c to an invariant quasiconformal structure σ' in the dynamical plane for $f_{c'}$. The new quasiconformal structure defines an invariant line field, which has to be the constant line field by Theorem 6.3.2. So the quasiconformal conjugacy is conformal by Weyl's Lemma, and $c = c' + 2\pi i n$ for some $n \in \mathbb{N}$. □

Corollary 6.3.3. *If f_c is combinatorially non recurrent with bounded postsingular set, c non escaping, then the parameter fiber of c does not contain any non-escaping parameters, hence its reduced fiber is trivial.*

Proof. Let $f_{c'}$ be such that c' is in the same parameter fiber as c. By Corollary 7.2.12, parapuzzles are well defined for c. As c' is in the same fiber as c, c and c' are in the same parapuzzle piece for all levels, hence they are conformally conjugate by Theorem 6.3.1. Two parameters whose height differs by $2\pi i$ cannot belong to the same fiber because they are separated by one of the parameter rays with constant address, hence $c = c'$. □

Open-closed argument for triviality of fibers

An alternative proof, using an open-closed argument, shows that fibers of combinatorially non-recurrent parameters are trivial also when $\mathcal{P}$ is not bounded.

Lemma 6.3.4 (Quasiconformal classes are open)**.** *Let $Q(c)$ be the connected component containing c of the set of parameters c' such that $f_{c'}$ is quasiconformally conjugate to f_c. If $Q(c) \neq \{c\}$, $Q(c)$ is open.*

Sketch of proof. Let $c, c' \in Q(c)$, $c \neq c'$, ψ the conformal conjugacy between f_c and $f_{c'}$. Let μ be the Beltrami differential for standard conformal structure on Π_c (i.e. $\mu = 0$), $\mu' := \psi_* \mu$ the Beltrami differential obtained on $\Pi_{c'}$, and μ_λ over $\lambda \in \mathbb{D}$, $\mathbb{D}$ the unit disk, be an analytical interpolation between μ and μ', for example $\mu_\lambda = \lambda \mu'$. By the Measurable Riemann Mapping Theorem, to each μ_λ corresponds an exponential map f_λ which is conjugate to f_c via a quasiconformal map ψ_λ. As μ_λ depends holomorphically on λ, the ψ_λ, and hence f_λ, depend holomorphically on λ (see e.g. Theorem 2.41, [L]), hence there is an open neighborhood of c on which all maps are quasiconformally conjugate to f_c. □

Lemma 6.3.5 (Boundaries of non-hyperbolic components)**.** *Consider the exponential family or any family of unicritical polynomials. The boundary if a non-hyperbolic component $\mathcal{Q}$ in parameter space cannot contain escaping parameters which are accessible from the inside of $\mathcal{Q}$. In particular, $\partial \mathcal{Q}$ contains at least a non-escaping parameter in its boundary.*

Proof. Let c_0 be an escaping parameter which is accessible from the inside of $\mathcal{Q}$, i.e. for which there exists a curve $\gamma : [0, 1) \to \mathcal{Q}$ such that $\gamma(t) \to c_0$ as $t \to 1$. Then $c_0 = G_{s_0}(t_0)$ for some parameter ray G_{s_0}. Consider a piece Γ of G_{s_0}, $\Gamma := G_{s_0}(t_0 - e, t_0 + e)$ oriented following increasing t. As γ is a local transversal to Γ at c_0, we can determine whether γ is to the right or to the left of Γ. If γ is to the right (resp. left) of Γ consider a sequence of addresses s_n converging to s_0 from below (resp. above). Then the parameter rays of addresses G_{s_n} converge to G_{s_0} on compact sets by Lemma **??**, obtaining a contradiction from the fact that no parameter rays can intersect $\mathcal{Q}$. □

Theorem 6.3.6 (Triviality of fibers for combinatorially non-recurrent parameters)**.** *Reduced fibers of combinatorially non-recurrent parameters are trivial.*

Proof. Let $F_R(c)$ denote the reduced fiber of a combinatorial non-recurrent parameter c. By definition of reduced fibers any two parameters in $F_R(c)$ have the same combinatorics, hence by Theorem 6.2.14 they are quasiconformally conjugate. As $F_R(c)$ is connected, $F_R(c) = Q(c)$ hence, if $Q(c) \neq \{c\}$, it is open by Lemma 6.3.4. However, by Lemma 6.3.5, $F_R(c)$ contains at least one of its boundary points hence cannot be open, giving a contradiction. □

6.4 Recurrence and combinatorial non-recurrence

The goal of this section is to use the results about accessibility from Chapter 7 to show that, for parameters with bounded postsingular set, non-recurrence implies combinatorial non-recurrence, in order to prove the following theorem.

Theorem 6.4.1 (Combinatorial non-recurrence for bounded $\mathcal{P}$)**.** *Let f_c be an exponential map such that c is non-recurrent and $\mathcal{P}$ is bounded. Then c is combinatorially non-recurrent.*

Definition 6.4.2. We will say that an address or angle s is non-recurrent if $s \notin O(s) := \{\sigma^n(s)\}_{n\in\mathbb{N}}$.

Proposition 6.4.3. *If $f_c(z) = e^z + c$ or $f_c(z) = z^D + c$ where c is non-recurrent and the postsingular set is bounded, then s is non-recurrent.*

Proof. If $s \in O(s)$, there is $s_k := \sigma^k(s) \to s$; let $c_k := f^k(c)$ be the landing point of g_{s_k}. By expansivity of the postsingular set, $\ell(g_{s_k}(0,t)) < \varepsilon_t \to 0$ uniformly in k as $t \to 0$.

By Lemma 3.1.7, given ε and t_* there is δ such that $|g_s(t) - g_{s_k}(t)| < \varepsilon$, if $\text{dist}(s, s_k) < \delta$ and $t > t*$.

By non-recurrence there is a disk B of radius R centered at c such that $B \cap \mathcal{P} = \emptyset$. Let $\varepsilon = R/3$, t such that $\ell(g_{s_k}(0,t)) < R/3$ for all k and s_n any address such that $\text{dist}(s, s_n) < \delta$; then $|c_n - c| < R/3$ contradicting $B \cap \mathcal{P} = \emptyset$. □

The next theorem is one of the cornerstones in the theory of rigidity for quadratic and unicritical polynomials (see.. for the definition of renormalization and for a statement of Yoccoz Theorem).

Theorem 6.4.4 (Yoccoz Theorem). *Fibers of parameters which are finitely renormalizable are trivial. Also, the Julia set of such parameters is locally connected.*

Proposition 6.4.5 (Landing of non-recurrent rays). *If G^D_s is a parameter ray of angle s in the parameter plane for the family P^D, and s is non-recurrent, then G^D_s lands at $\tilde{c}$ for some non-recurrent parameter $\tilde{c}$.*

Proof. By Proposition **??**, G^D_s is contained in finitely many wakes, hence any non-escaping parameter in the extended fiber of G^D_s (and, in particular, any parameter in the accumulation set of G^D_s) is at most finitely renormalizable. By Yoccoz Theorem, there is only such parameter c and G^D_s lands at it. □

While proving Theorem 6.4.1, we are using the following lemma.

Lemma 6.4.6. *Let f_c be a polynomial or exponential map. If G_s lands at c and c belongs to its Julia set, then g^c_s belongs to the dynamic fiber of c. Viceversa, if g^c_s lands at c, the parameter ray G_s belongs to the parameter fiber of c.*

Proof. It is enough to say that for any ε there is t_n, $|g_s(t_n) - c| < \varepsilon$. Let $G_s(t_n) =: c_n \to c$. For any n, $g^{c_k}_s$ converges uniformly to g^c_s on $[t_n, \infty]$ as $k \to \infty$. (We can probably not use continuity bc the rays are broken.) □

Proof of Theorem 6.4.1. By Corollary 7.2.12, c belongs to a parabolic wake bounded by two parameter rays of addresses s_1, s_2 and hence the definition of combinatorial non-recurrence is meaningful. Let g_s be the dynamic ray landing at c given by Theorem 7.2.3 and let $D > 2\max(|s|, |s_1|, |s_2|$.

By Proposition 6.4.3, s is non-recurrent, so by Proposition 6.4.5 the parameter ray G_s^D lands at a non-recurrent polynomial parameter $\tilde{c}$. Also, $\tilde{c}$ belongs to the parabolic wake bounded by the parameter rays with addresses s_1, s_2 in the polynomials parameter plane.

In $\Pi_{\tilde{c}}$, $\tilde{c}$ is in the same dynamical fiber of the corresponding dynamic ray $g_s^{\tilde{c}}$. As $\tilde{c}$ is non-recurrent, by Yoccoz Theorem its Julia set is locally connected and puzzle pieces shrink to points, hence there is some level of the puzzle construction for which the singular value and the postsingular set are separated. This means that there are finitely many preperiodic ray pairs (preimages of rays landing at the $\alpha(\tilde{c})$ fixed point) separating c from the postsingular set. By Theorem 3.3.14 c is contained into the Misiurewicz wakes defined by the ray pairs.

By Theorem 3.3.14, the same Misiurewicz wakes exist in the exponential parameter plane, and by vertical order of parameter rays G_s, and hence c, is contained in all of them. By Theorem 3.3.14 again, the same preperiodic ray pairs land together in Π_c. By vertical order of dynamical rays, as they separate $\tilde{c}$ from its forward orbit in the polynomial dynamical plane (and each point on the orbit is the landing point of a dynamic ray $g_{\sigma^k s}^{\tilde{c}}$), they also separate c from its forward orbit in the exponential plane. By continuity, they also separate c from the closure of its forward orbit i.e. from the postsingular set. □

Chapter 7

Accessibility

7.1 Accessibility of repelling periodic orbits for polynomials with connected Julia set

We will give a new proof of a classical theorem by Douady for polynomials with connected Julia set, showing that any repelling periodic orbit is the landing point of finitely many periodic ray. While the original proof uses in an essential way the structure of the basin of infinity near the repelling cycle, we will only use the structure of rays near infinity and the uniform contraction given by linearizing coordinates, opening this result to be generalized to other families of functions (see [H], Theorem IA).

Theorem 7.1.1. *Let f be a polynomial of degree D with connected Julia set, $\{z_i\}$ one of its repelling periodic orbits. Then there is at least one and at most finitely many periodic rays landing at each of the $\{z_i\}$.*

7.1.1 Setting

For all of this section, let f be a polynomial of degree D with connected Julia set. Let K be the filled Julia set for f and let ρ be the hyperbolic density on $\mathbb{C}-K$. As K is full and contains more than three points $\mathbb{C}-K$ is also connected and admits a hyperbolic metric ρ.

Without loss of generality, up to considering iterates of f we can assume $\{z_i\}$ to be a repelling fixed point α with modulus of the multiplier $\mu > 1$ and we can fix a linearizing neighborhood L for α with linearizing coordinates ψ_L.

We will denote by $\ell(\gamma)$ the euclidean length of a curve γ and by ℓ_ρ its hyperbolic length in $\mathbb{C}-K$. We will denote by $|a-b|$ the distance of two points in $\mathbb{C}$ and by $d(X,Y) := \inf_{x\in X}\inf_{y\in Y}|x-y|$ the distance between two compact sets or between a compact set and a point. The Böttcher function B conjugates the dynamics of f on $\mathbb{C}-K$ to the dynamics of z^D on $\mathbb{C}-\mathbb{D}$. The preimage under B of the straight ray of angle s is called the dynamic ray of angle s and is denoted by g_s; if we parametrize g_s as $g_s(t) = B^{-1}(s, e^t)$ (where (s, e^t) indicate the polar coordinates of a point) and we define the radial growth function $F : t \mapsto Dt$ we have

$$f(g_s(t)) = g_{\sigma s}(F(t)), \tag{7.1.1}$$

where s is written in D-adic expansion and σ is the shift map.

We will call a *fundamental domain* starting at t for a ray g_s the piece of curve $g_s([t, F(t)))$, and we will denote it by $I_t(g_s)$. We will call the *equipotential of level* t the curve $\{g_s(t), s \in \mathbb{S}^1\}$.

We will need the following lemma about convergence of angles and rays, following directly by uniform continuity of the Boettcher map on any compact set:

Lemma 7.1.2. *Let f be a polynomial of degree D. For each $\varepsilon, t_*, t^* > 0$ there exists δ, if dist$(\tilde{s}-s) < \delta$ then $|g_{\tilde{s}}(t) - g_s(t)| < \varepsilon$ for any $t_* < t < t^*$. In particular if $s_n \to s$, there exists N such that $|g_{s_n}(t) - g_s(t)| < \varepsilon$ for any $n > N$.*

7.1.2 Proof of Theorem 7.1.1

The first part of the proof studies the length of fundamental domains starting at a fixed potential.

Proposition 7.1.3 (Bounded fundamental domains)**.** *For each t the euclidean length $\ell(I_t(g_s))$ is bounded uniformly in s. By equivalence of euclidean and hyperbolic metric on any compact subset of $\mathbb{C} - K$, the hyperbolic length $\ell_\rho(I_t(g_s))$ is also bounded uniformly in s.*

Proof. The derivative of B^{-1} is bounded by some constant C_t on the annulus $A_t := \{z, t+1 \leq |z| \leq (t+1)^d\}$, hence the euclidean length of $I_t(g_s)$ is bounded by $C_t(F(t) - t)$. □

The following is a standard fact that follows from the boundary behavior of the hyperbolic metric (see Appendix).

Lemma 7.1.4. *Let $\Omega \subset \mathbb{C}$ be an open hyperbolic region. Let $\gamma_n : [0,1] \to \mathbb{C}$ be a family of curves with uniformly bounded hyperbolic length and such that $\gamma_n(0) \to \partial\Omega$. Then $\ell(\gamma_n) \to 0$.*

Proposition 7.1.5 (Fundamental domains shrinking)**.** *The euclidean length $\ell(I_t(g_s))$ tends to 0 uniformly in s as $t \to 0$.*

Proof. The family of curves $\{I_{t_*}(g_s)\}$ has uniformly bounded hyperbolic length for any fixed t_*, hence by the Schwarz Lemma the family of curves $\{I_t(g_s)\}$ has uniformly bounded hyperbolic length for $t < t_*$.

If we can show that the distance $d(g_s(t), J) \to 0$ uniformly in s as $t \to 0$, the estimate follows from Lemma 7.1.4.

Suppose $d(g_s(t), J)$ does not tend to 0 uniformly. Then there is a sequence of points z_n whose potential t_n going to 0 who stay a definite distance away from J. As B is proper, any of their limit points has potential 0 hence has to belong to the Julia set giving a contradiction. □

Theorem 7.1.6. *Let f be a polynomial with connected Julia set, α be a repelling fixed point for P. Then there is at least one dynamic ray g_s landing at α.*

Proof. Let U' be an ε neighborhood of α which is fully contained in the linearizing neighborhood L, and let U be its preimage under the inverse branch ψ of f fixing α. Let μ, C given by the uniform contraction in linearizing coordinates, such that $|\alpha - \psi^n(x)| \leq \frac{C|\alpha - x|}{\mu^n}$ for all $x \in U'$. Let ε be the distance $d(\partial U, \partial U')$.

By Proposition 7.1.5, find t_ε such that

$$\ell(I_t(g_s)) < \varepsilon \text{ for all } s, t < t_\varepsilon. \tag{7.1.2}$$

As α is in the Julia set, α belongs to the boundary of the basin of infinity, hence is approximated by escaping points with arbitrary small potential t, hence there exists a dynamic ray g_0 such that $g_0(t_0)$ belongs to U for some $t_0 < t_\varepsilon$. Then $\ell(I_{t_0}(g_0)) \leq \varepsilon$, hence $I_{t_0}(g_0) \subset U'$.

Let $g_n := \psi^n(g_0)$ be the pullbacks of g_0 along the branches fixing α. Let us define inductively a sequence of curves γ_n as

$$\gamma_0 := I_{t_0}(g_0)$$
$$\gamma_n := \psi\gamma_{n-1} \cup I_{t_0}(g_n)$$

Claim: properties of γ_n.

1. $\gamma_n = g_n(t_n, F(t_0))$ where $t_n := F^{-n}(t_0)$;

2. $\gamma_n \subset U'$ for all n;

3. $|x - \alpha| \leq \frac{C \operatorname{diam} U'}{\mu^n}$ for $x \in I_{t_n}(g_m),\ m \geq n$. (*)

Proof of the Claim: All claims are true for γ_0, so let us suppose they hold for γ_{n-1} and show that they also hold for γ_n. We have that $\psi\gamma_{n-1} =$

$\psi(g_{n-1}(t_{n-1}, F(t_0))) = g_n(t_n, t_0)$ by the functional equation 7.1.1 and by the definition of g_n. Also γ_n is contained in U because $\gamma_{n-1} \subset U'$ and $\psi(U') = U$; as $\ell(I_{t_0}(g_n)) \leq \varepsilon$, $\gamma_n \subset U'$.

Also if $x \in I_{t_n}(g_m)$, $m \geq n$, $x = \psi^n$ for some $y \in I_{t_0}(g_{m-n}) \subset U'$, hence

$$|x - \alpha| \leq \frac{|y - \alpha|}{\mu^n} \leq \frac{C \operatorname{diam} U'}{\mu^n},$$

proving the third part of the Claim.

Let $\{s_n\}$ be the sequence of angles of the rays g_n; as the space of angles for polynomials is homeomorphic to $\mathbb{S}^1$ which is compact, there is a subsequence converging to some angle s. As the Julia set is connected no singular value is escaping, hence the ray g_s of angle s is well defined for all potentials $t > 0$.

Landing of g_s at α. It is enough to show that for each n, and $x \in I_{t_n}(g_s)$,

$$|x - \alpha| \leq \frac{C \operatorname{diam} U'}{\mu^n} + \frac{1}{2^n}. \tag{7.1.3}$$

So fix n and a subsequence of angles s_m converging to s. Let $\varepsilon_n := \frac{1}{2^n}$, $t_* := \frac{t_n}{2}$,and N as given by Lemma 7.1.2 such that $|g_s(t) - g_m t| \leq \frac{1}{2^n}$ for $t > t_* = \frac{t_n}{2}$ and $m > N$. Then if $g_s(t) \in I_{t_n}(g_s)$,

$$|g_s(t) - \alpha| \leq |g_s(t) - g_m(t)| + |g_m(t) - \alpha| \leq \frac{1}{2^n} + \frac{C \operatorname{diam} U'}{\mu^n}$$

for $m > N$. Observe that the estimates depend on n because we consider $t \in [t_n, t_{n-1}]$, but they hold for all sufficiently big m. □

Proposition 7.1.7 (Periodicity of landing ray)**.** *A dynamic ray g_s landing at a repelling periodic point is periodic.*

This proposition follows from a well known lemma about rotation sets ([Mi])

Lemma 7.1.8 (Rotation sets)**.** *Let $A \subset \mathbb{S}^1$ be closed and forward invariant under the shift map $\sigma : \theta \mapsto D\theta$. If $\sigma|_A$ is a homeomorphism, then A is finite.*

Proof. If not, $\sigma^{-1}|_A$ is a locally contracting homeomorphism, which cannot happen since A is compact. □

Proof of Proposition 7.1.7. Without loss of generality, the repelling periodic orbit is a fixed point α with some linearizing neighborhood L.

Let $A := \{\sigma^n(s)\}$ be the orbit of s under the shift map $\sigma : \theta \mapsto D\theta$, where D is the degree of the polynomial under consideration. The map σ_A is a homeomorphism by injectivity of f near α, so if we can prove that A is closed, Lemma 7.1.8 implies the claim.

Let $\tilde{s}_m := \sigma^m(s)$. If we can show a uniform estimate like the ones in equation 7.1.3 for all dynamic rays $g_{\tilde{s}_m}$, it will follow that any of the limiting rays also lands at α using the same estimates used to show that g_s lands at α. Note that $|(f^n)'(x)| = |(\psi_L^{-1} \circ \mu^n \circ \psi_L)'(x)| \leq C'\mu^n$ for some constant C' depending only on ψ_L for any $x \in L$ such that $f^n(x) \in L$. Also, if $x \in I_{t_n}(g_{\tilde{s}_m})$, $\psi^m(x) \in I_{t_{n+m}}(g_s)$, hence

$$\begin{aligned}|x - \alpha| &\leq \mu^m C'|\psi^m(x) - \alpha| \leq \\ &\leq \mu^m C'(\frac{C \operatorname{diam} U'}{\mu^{n+m}} + \frac{1}{\mu^{n+m}}) \to 0 \text{ uniformly in } m.\end{aligned}$$

The fact that for any $\tilde{s} \in \overline{A}$ the dynamic ray $g_{\tilde{s}}$ lands at α follows from the same estimates as in Theorem 7.1.6. □

We say that a set Λ is *hyperbolic* (with respect to the euclidean metric) if it is forward invariant set such that $|(f^k)'x| > \eta > 1$ for all $x \in \Lambda$, $k > \overline{k}$. In a similar way we can prove the following.

Theorem 7.1.9 (Accessibility of hyperbolic sets)**.** *Let f be a polynomial with connected Julia set. Let $\Lambda \subset J$ be a hyperbolic set. Then for each $x \in \Lambda$ there is a dynamic ray landing at x.*

This theorem was originally stated as accessibility of the postsingular set for exponential maps with bounded post-singular set. It has been observed by Lasse Rempe that the proof applies to general hyperbolic sets.

Proof. Let U be a δ-neighborhood of Λ such that $|(f^k)'x| > \eta' > 1$ for $x \in U$, $k > \overline{k}$. Up to considering $f^{\overline{k}}\Lambda$ we can assume $\overline{k} = 0$. Observe that U will in general not be connected. Let $\mathcal{B}$ be a finite covering of Λ formed by balls of radius $\delta/3$ centered at points in Λ.

Fix x in Λ, and let us construct a dynamic ray landing at x. Let $B \in \mathcal{B}$ be a disk containing infinitely many x_n. Observe that $B_\delta(x_n) \supset B$ for any $x_n \in B$, and that $f^{-1}(B_\delta(x_n)) \subset B_{\delta/\eta'}(x_{n-1})$ for any inverse branch of f mapping x_n to x_{n-1}.

Let $\varepsilon := \min(\delta/3, \delta - \delta/\eta')$, t_ε as in Lemma 7.1.5 and $g_0(t_0) \in B$ for some $t_0 < t_\varepsilon$. Let $t_n = F^{-n}(t_0)$. Like in the case of a repelling periodic orbit we will construct a sequence of pieces of rays $g_n(t_n, F(t_0))$ for which we have uniform control on each fundamental domain $I_{t_m}(g_n)$, $n > m$. We will construct one such curve for each $x_n \in B$, hence not for all integers, but we will still get infinitely many curves.

Given $x_n \in B$, let us construct a sequence of curves $\gamma_{n,j} = g_{n,j}(t_j, F(t_0))$, $j = 0, ..., n$ in the following way: $\gamma_{n,0} := I_{t_0}(g_0)$; the pullback of γ_0 under the branch of h^{-1} mapping x_n to x_{n-1} (which is well defined on all of $B_\delta(x_n)$ as $B_\delta(x_n) \cap \mathcal{P} = \emptyset$) coincides with some piece of dynamic ray $g_{n,1}(t_1, t_0)$.

As $g_{n,1}(t_0) \in B_{\delta/\eta'}(x_{n-1})$, by Lemma 7.1.5 we can extend it to $g_{n,1}(t_1, F(t_0)) \subset B_\delta(x_{n-1})$. By induction we can define $\gamma_{n,j} = g_{n,j}(t_j, F(t_0)) \subset B_\delta(x_{n-j})$, up to $\gamma_{n,n} = g_n(t_n, F(t_0)) \subset B_\delta(x)$.

Like in the case of a repelling orbit we can extract a subsequence of angles converging to an angle s. Landing of g_s at x follows from the same estimates as in 7.1.6 together with the remark that $d(I_{t_m}(g_n) - x) \leq \frac{\delta}{(\eta')^m}$ as $f^m(I_{t_m}(g_n)) = I_{t_0}(g_{n-m}) \subset B_\delta(x_{n-m})$. □

7.2 Accessibility for exponential parameters with bounded postsingular set

Let us define a family of inverse branches for $f(z) = e^z + c$ on $\mathbb{C} - R$, where $R := \{z \in \mathbb{C}, \operatorname{Im} z = \operatorname{Im} c, \operatorname{Re}\, z \le \operatorname{Re}\, c\}$

$$L_n(w) := \log|w - c| + \arg(w - c) + 2\pi i n + c,$$

where $\arg(z)$ takes values in $[0, 2\pi)$.

7.2.1 Statement of theorems and some basic facts

The strategy used for the new proof of Theorem 7.1.1 can be extended to the exponential family to prove the following theorems.

Recall that a point z is accessible if there exists a ray g_s landing at z.

Theorem 7.2.1 (Accessibility for Misiurewicz parameter). *Let $f(z) = f_c(z) = e^z + c$ be postsingularly finite; then any periodic point is accessible, including points in the postsingular orbit.*

Theorem 7.2.2 (Accessibility of periodic orbits for non-recurrent parameters). *Let $f_c(z) = e^z + c$ be non-recurrent with bounded postsingular set ; then any repelling periodic point is accessible.*

Theorem 7.2.3 (Accessibility of the postsingular set). *Let $f_c(z) = e^z + c$, $c \in J$ be non-recurrent with bounded postsingular set; then any point in the postsingular set is accessible.*

Theorem 7.2.1 has been originally proved by Schleicher and Zimmer in [SZ1] using different techniques involving contraction on the space of topological objects called spiders.

As for polynomials, our strategy will be to first prove a uniform bound on the length of fundamental domains $I_T(g_\alpha)$ for some fixed T and some specific family of addresses g_α, then translate this into a uniform shrinking for fundamental domains starting at small t, and eventually study the local dynamics near a repelling periodic orbit.

From now on we will always consider an exponential function f non-recurrent with bounded postsingular set.

The main difficulties with respect to the polynomial case are that Propositions 7.1.3 and 7.1.5 are not valid in such generality, and that a priori the sequence of ray pullbacks near the repelling fixed point might have unbounded addresses and hence no convergent subsequence.

In the following, let α be a fixed point, and $\Omega = \mathbb{C} \setminus (\{c\} \cup \mathcal{P} \cup \{\alpha\})$, where $\mathcal{P}$ denotes the postsingular set $\mathcal{P} = \overline{\underset{n\geq 1}{\cup} f^n(c)}$. As $f(\mathcal{P}) \subset \mathcal{P}$, for any simply connected region $U \subset \Omega$ on which a branch φ of f^{-1} is defined we have $\varphi(U) \subset \Omega$.

Proposition 7.2.4. *The set $\mathbb{C} \setminus \mathcal{P}$, and hence Ω, is connected.*

Proof. As $\mathcal{P}$ is compact, $\mathbb{C} - \mathcal{P}$ has only one unbounded component, and each one of its connected component V_i is open. By density of escaping points, each V_i contains escaping points; as dynamic rays are connected sets, each V_i has to be unbounded, hence there is a unique connected component. □

As Ω is connected, and omits at least three points because c cannot be a fixed point, it admits a well defined hyperbolic metric ρ_Ω.

We will start by proving Theorem 7.2.2 which clearly implies Theorem 7.2.1.

7.2.2 Bounds on fundamental domains for exponentials

The first step is to prove an analog of Proposition 7.1.3 for appropriate families of dynamic rays in the exponential setting.

Theorem 7.2.5 (Bounded fundamental domains for exponentials). *Let g_s be a periodic dynamic ray ,*

$$A := \{s \in \mathcal{S} \textit{ such that } \sigma^n(s) = s_0 \textit{ for some } n \geq 0\}$$

and $\{g_\alpha\}_{\alpha \in A}$ be the collection of pullbacks of g_{s_0}. Then there exists T such that:

- *If $\alpha = a_m ... a_2 a_1 s_0$, then $g_\alpha(t) = L_{a_m} \circ \cdots \circ L_{a_1} g_{s_0}(F^m(t))$ for all $t > T, \alpha \in A$. (*)*
- *For $t \geq T$, Re $(g_\alpha(t)) > C$ for some constant C depending only on g_{s_0} (**)*
- *For $t \geq T$, $\ell(I_t(g_\alpha)) \leq B$ where B depends on g_{s_0} and on t, but not on α. (***)*

Theorem 7.2.5 is a consequence of the following proposition.

Proposition 7.2.6 (Branches of the logarithm). *Let $f(z) = e^z + c$; there is $M > 0$ such that for any z with Re $f^j(z) > M$ for all $j > 0$, and for all $m \in \mathbb{N}$,*

1. *Re $(L_{a_m} \circ \ldots L_{a_1})(f^m(z)) \geq$ Re $(L_0^m f^m(z))$*
2. *$|(L_{a_m} \circ \ldots L_{a_1})(f^m(z)) - c| \geq |L_0^m f^m(z) - c|$*

Proof. The proof works by induction on m.

Let $M >$ Re c. For $m = 1$, equality holds in 1. because all preimages of a point are $2\pi i$ translate of each other. Also, 2. holds because

$$\begin{aligned}|(L_{a_1})(f(z)) - c| &= |\log|f(z) - c| + i\arg(z - c) + 2\pi i n| \\ &> |\log|f(z) - c| + i\arg(z - c)| = |L_0 f(z)| \text{ if Re } z > M\end{aligned}$$

(recall we defined the argument as a function taking values from $-\pi$ to π). For $m > 1$, let us assume 1.,2. hold for m and let us show they also hold for $m+1$.

To show 2. it is enough to show 1., because

$$|\operatorname{Im}(L_{a_m} \circ \dots L_{a_1})(f^m(z)) - c)| > |\operatorname{Im} L_0^m f^m(z) - c|.$$

To show 1. observe that

$$\begin{aligned} &\operatorname{Re}\,(L_{a_{m+1}} \circ L_{a_m} \circ \dots \circ L_{a_1}) f^{m+1}(z) = \log|(L_{a_m} \circ \dots \circ L_{a_1})(f^{m+1}(z)) - c| > \\ &> \log|L_0^m(f^{m+1}(z)) - c| = \operatorname{Re}\, L_0^{m+1}(f^{m+1}(z)) \end{aligned}$$

where the inequality follows by property 2. in the induction hypothesis and monotonicity of the real logarithm (we are using the induction hypothesis for $w = f(z)$, using $\operatorname{Re} f^j(z) > M$ for all j). □

Proof of Theorem 7.2.5. Property (*) is equivalent to Remark 2.

Let T be such that (*) holds and such that the dynamic ray g_{s_0} and all its pullbacks $L_0^m(g_{s_0})$ are approximately straight, i.e. estimates in Theorem 3.1.3 hold up to some small ε for each $L_0^m(g_{s_0})$, $m \geq 0$. Property (**) follows from property 1 in Proposition 7.2.6 and from the fact that $\operatorname{Re} L_0^m(g_{s_0})(t) \geq T - \varepsilon + \operatorname{Re} c$ for all $t > T$ by asymptotic estimates in Theorem 3.1.3.

Property (***) follows from the fact that $\ell(I_t(L_0^m(g_{s_0}))) \leq C_t \sim e^{-t}$, and $\ell(I_t(g_\alpha)) \leq \ell(I_t(L_0^m(g_{s_0})))$ for all α by estimate 2 in Proposition 7.2.6 and the fact that $|L'_a|(z) = \frac{1}{|z-c|}$, hence contraction of length is minimal along branches of L_0. □

Theorem 7.2.5 gives us the equivalent of Lemma 7.1.3 for polynomials; now we will state and prove an equivalent to Proposition 7.1.5.

The next proposition is a general fact following from normality of inverse branches; its equivalent for rational functions can be found in [L0], Proposition 3.

Proposition 7.2.7 (Shrinking under inverse iterates)**.** *Let L be a compact set such that $L \cap J \neq \emptyset$ and $L \cap \mathcal{P} = \emptyset$. Then*

$$\operatorname{diam}(f_\lambda^{-m}(L)) \to 0$$

uniformly in f_λ^{-m} for any branch f_λ^{-m} of f^{-m} such that $f_\lambda^{-m}(x_m) \cap K \neq \emptyset$ for some compact set K and some sequence $x_m \in L \cap J$.

Proof. As $L \cap \mathcal{P} = \emptyset$, inverse branches are well defined and univalent. Suppose by contradiction that there is $\varepsilon > 0$, $m_k \to \infty$, and λ_k branches of f^{-m_k} such that $\operatorname{diam}(f_{\lambda_k}^{-m_k})(L) > \varepsilon$ for any m_k, λ_k.

By normality of inverse branches, we can extract a subsequence converging uniformly to a function φ . As $f_{\lambda_k}^{-m_k}(x_k) \in K$ for some $x_k \in L \cap J$, there is a finite accumulation point $y \in J$ for the sequence $\{x_k \in L \cap J\}$. By Hurwitz theorem and by convergence to φ, if φ is not constant, there is a neighborhood V of $\varphi(y)$ such that $(f_{\lambda_k}^{-m_k})(L) \supset V$ for all k. This implies that $f^{\tilde{m}_k}(V) \subset L$ for some sequence $\tilde{m}_k \to \infty$, contradicting the fact that $y \in J$.

So ϕ must be equal to a constant and $\operatorname{diam}(f_{\lambda_k}^{-m_k})(L) \to 0$. □

Theorem 7.2.8 (Fundamental domains shrinking for exponentials)**.** *Let g_0 be a periodic dynamic ray, $\{g_\alpha\}_A$ its family of pullbacks as defined in Theorem 7.2.5. Given an $\varepsilon > 0$ and a compact set K there exists $t_\varepsilon = t_\varepsilon(K)$ such that $\ell(I_t(g_\alpha)) < \varepsilon$ whenever $t < t_\varepsilon$ and $g_\alpha(t) \in K$.*

Lemma 7.2.9. *(Bound on bounded pullbacks) Let $\{g_\alpha\}$ be the collection of pullbacks of some periodic ray g_0, $\alpha \in A$ as defined in Theorem 7.2.5.*

Given any $\varepsilon > 0$ there exists M, if $|\alpha| > M$ then $\ell(I_T(g_\alpha)) < \varepsilon \leq \frac{B}{M}$ where T, B are given by Proposition 7.2.5. Moreover for all M there exists a compact set K_M such that $\{I_t(g_\alpha)\} \subset K_M$ for all α such that $|\alpha| < M$.

The proof of Lemma 7.2.9 follows from estimates similar to the estimates in Proposition 7.2.6.

Lemma 7.2.10. *Let Ω be as in Proposition 7.2.4, and*

$$\mathbb{H}_C = \{z \in \mathbb{C}, Re\ z > C\}$$

where C is given by Theorem 7.2.5. Then there exists a disk B_R centered at c such that $\rho_\Omega < 1$ on $\mathbb{H}_C - B_R$.

Proof. As the postsingular set is bounded, we have that $\Omega \supset \mathbb{C} - B_r$ for some disk B_r of radius r, hence $\rho_\Omega < \rho_{\mathbb{C}-B_r}$. On the other side, $\rho_{\mathbb{C}-B_r} < 1$ on $\mathbb{C} - B_R$ for some $R > r$, as $\rho_{\mathbb{C}-B_r}(z) \sim \frac{-1}{|z|\log|z|}$ for $|z| \to \infty$. □

Proof of Theorem 7.2.8. Fix a compact set K and $\varepsilon > 0$. As for any compact set K there is a constant C_K such that $1 = \rho_{eucl} < C_K \rho_{hyp}$ on K, we can find ε' such that a curve γ intersecting K has euclidean length smaller than ε whenever it has hyperbolic length smaller than ε'.

By Lemma 7.2.9 we can find M such that $\ell(I_T(g_\alpha)) < \varepsilon'$ if $|\alpha| > M$, and $K_M \subset \mathbb{H}_C$ such that $\{I_T(g_\alpha) \subset K_M\}$ for each α with $|\alpha| < M$.

For pullbacks of rays whose fundamental domains are in $K_M \cup B_R$, the claim follows from Proposition 7.2.7. For pullbacks of rays g_α with $|\alpha| > M$, the claim follows by the fact that $\ell_\Omega(I_T(g_\alpha)) < \varepsilon'$, hence $\ell_\Omega(\psi^n I_T(g_\alpha)) < \varepsilon'$ for all $n \in \mathbb{N}$ and $\ell(\psi^n I_T(g_\alpha)) < \varepsilon$ whenever $\psi^n I_T(g_\alpha)$ intersects K. □

7.2.3 Proof of Accessibility Theorems

Proof of Theorem 7.2.2. The same inductive construction from the proof of Theorem 7.1.6 can be used to obtain a ray landing at the fixed point α, to show that it lands and that it is periodic, once we show that the sequence of external addresses s_n obtained by the construction is bounded. However this is now easy to show: by construction $g_n(t_0) \in U$ for large n and $F^N(t_0) > T$ for some N. As $f^N(U)$ is a compact set, $\operatorname{Im} f^N(g_n(t_0))$ is bounded, hence $|s_n|$ is bounded by claim (*) in Proposition 7.2.5. □

A theorem by Rempe relates accessibility of repelling periodic orbits for a map f_c to the fact that c belongs to a parabolic wake ([R0], Lemma 5.14.1).

Lemma 7.2.11. *Suppose that an exponential map f_c with nonescaping singular value has some periodic orbit of period n which is not accessible. Then there exists a period n hyperbolic component W such that c belongs to the wake of W, but does not belong to any of the parabolic wakes attached to parabolic parameters on ∂W.*

Together with Theorem 7.2.2, Lemma 7.2.11 implies that non-recurrent parameters with bounded post-singular set belong to parabolic wakes attached to the period one component.

Corollary 7.2.12 (Corollary of Theorem 7.2.2)**.** *If c is a non-recurrent parameter with bounded postsingular set, it is contained in a parabolic wake attached to the boundary of the period one hyperbolic component W_0.*

Proof. We can find a curve γ joining c to W_0 such that all fixed points can be continued analytically. Call α is the analytic continuation of the attracting fixed point. If c does not belong to any parabolic wake attached to W_0, α would be the landing point of only one periodic ray, which would then need to be fixed. But all fixed rays already have landing points, because all dynamic rays land at different points for parameters in W_0. □

To prove Theorem 7.2.3, let us recall that the postsingular set is hyperbolic by Theorem 6.1.7.

Proof of Theorem 7.2.3. The construction is the same as in the proof of Theorem 7.1.9 using the fact that $\mathcal{P}$ is bounded together with Proposition 7.2.7 to show shrinking of fundamental domains. □

Chapter 8

Hyperbolic metric

Here we will recollect some notions about the hyperbolic metric, with a special attention in including those standard facts often used in dynamics but rarely written in expository accounts about the hyperbolic metric. For references see [Al], [McM],[L].

We will say that a Riemann Surface S is *hyperbolic* if its universal covering is the unit disk $\mathbb{D}$.

By the Uniformization Theorem (see e.g. [Mi]), any open subset of $\mathbb{C}$ whose complement contains at least two points has $\mathbb{D}$ as universal covering.

On the unit disk D there is a unique metric of constant curvature -1 called the *hyperbolic metric* on $\mathbb{D}$, which can be written explicitly as

$$ds = \frac{2|dz|}{1-|z|^2} =: \rho_{\mathbb{D}}|dz|,$$

where dz is the euclidean metric and $\rho_{\mathbb{D}}$ is the *hyperbolic density* in $\mathbb{D}$.

It can be checked by direct computation that the hyperbolic metric is invariant under Möbius transformations, i.e. automorphisms of the disk.

If Ω is a hyperbolic Riemann surface, which means if we have a covering map $\pi : \mathbb{D} \to \Omega$, the hyperbolic metric on Ω is defined as the pushforward under π of the hyperbolic metric on $\mathbb{D}$ (recall that π is open and locally invertible).

If we denote by $\rho_\Omega, |dw|$ the hyperbolic density and the euclidean metric on Ω respectively, this means that the hyperbolic metric ds on Ω is given by

$$ds = \rho_\Omega(w)|dw| = \frac{2|dw|}{|\pi'(z)|(1-|z|^2)}, \tag{8.0.1}$$

where $z \in \{\pi^{-1}(w)\}$.

Apart from $\mathbb{D}$, the two most important cases for which the hyperbolic density can be written up explicitly are given by the upper half plane $\mathbb{H}$ and the punctured disk $\mathbb{D}^*$:

$$\rho_{\mathbb{H}}(w) = \frac{1}{\operatorname{Im} w}$$
$$\rho_{\mathbb{D}^*}(w) = \frac{1}{|w||\log|w||}$$

Given a curve γ contained in a hyperbolic Riemann surface Ω, we can define its hyperbolic length

$$l_{\text{hyp}}\gamma := \int_\gamma \rho_\Omega.$$

The two elementary main results needed about the hyperbolic metric are the following ([L], Scharz Lemma):

Theorem 8.0.13. Schwarz-Pick Lemma *If $f : S \to S'$ is a holomorphic map between Riemann surfaces, either f is a local isometry for the hyperbolic metric, or f strictly decreases all nonzero distances in the hyperbolic metrics of S, S' respectively.*

Proposition 8.0.14. Inverse monotonicity *If $\Omega \subset \Omega' \subset \mathbb{C}$ are regions admitting a hyperbolic metric, then $\rho_{\Omega'} \leq \rho_\Omega$. If equality holds at some $z \in \Omega$, then $\Omega = \Omega'$.*

Proof. Apply the Schwarz Lemma to the identity map from Ω to Ω'. □

Corollary 8.0.15. *If $f : S \to S'$ is a conformal map between Riemann surfaces, and it is surjective, then the hyperbolic length of a curve $\gamma \subset S$ equals the hyperbolic length of its image $f(\gamma) \subset S'$.*

8.0.4 Boundary behavior of the hyperbolic metric

The hyperbolic density ρ_Ω at some point z tends to infinity when z approaches the boundary of Ω. However, it is easy to obtain an elementary upper bound in term of the euclidean distance of z from the boundary, $d(z, \partial\Omega) =: d(z)$([L], Lemma 1.86)

Theorem 8.0.16. ***Estimates on simply connected domains*** *If $\Omega \subset \mathbb{C}$ is a simply connected region omitting at least two points, and $d(z)$ is the distance of a point z from $\partial\Omega$, $\frac{1}{2d(z)} \leq \rho_\Omega(z) \leq 2/d(z)$.*

Proof. The second inequality follows from the fact that the disk $D := B_{d(z)}(z) \subset \Omega$, hence by 8.0.14 $\rho_\Omega(z) \leq \rho_D \leq 2/d(z)$.

For the first inequality consider the Riemann map Φ from $\mathbb{D}$ to Ω mapping 0 to z, and noting that $|\Phi'(0)| \geq 4d(z)$ by Koebe 1/4-Theorem. The claim follows from 8.0.1 and the expression of $\rho_{\mathbb{D}}$. □

The most general theorem that can be obtained for an upper bound is the following (keep in mind the expression of the hyperbolic metric on $\mathbb{D}^*$), ([L], Theorem 1.67):

Theorem 8.0.17. *For any hyperbolic domain $\Omega \subset \mathbb{C}$, there exists some constant k such that*

$$\rho_\Omega(z \geq) \frac{k}{d(z)|\log d(z)|},$$

where $d(z)$ is the distance of z from $\partial\Omega$.

Remark: As this is a local property, the theorem above also holds if we let $d(z)$ be the spherical distance between z and $\partial\Omega$.

It requires a little bit more work to compare euclidean and hyperbolic length of curves approaching the boundary of a hyperbolic domain. An example of a theorem in this spirit is the following:

Theorem 8.0.18. ***Curves going to the boundary*** *Let $\Omega \subset \mathbb{C}$ be an open region admitting a hyperbolic metric.*

Let $\{\gamma_n\}$ be a family of curves $\gamma_n : [0,1] \to \Omega$ which have uniformly bounded hyperbolic length and such that $\gamma_n(0) \to \partial\Omega$. Then the euclidean length $\ell(\gamma_n) \to 0$, in fact $\ell\gamma \leq C\,\mathrm{dist}(\gamma, \partial\Omega)$.

Sketch of the proof. By Proposition 8.0.14, we can assume $\Omega = \mathbb{D}$ and $\gamma_n(0) \to 0$, or $\Omega = \mathbb{H}$ and $\mathrm{Im}\,\gamma_n(0) \to 0$. In this cases the proof follows by straightforward calculations. □

Bibliography

[Al] Lars V. Ahlfors, *Conformal invariants: topics in geometric function theory*, McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Dsseldorf-Johannesburg, 1973.

[Ba1] Noel Baker, *Repulsive fixed points of entire functions*, Math. Z. 104 (1968), 252-256.

[Bo] Heinrich Bock, *Uber das Iterationsverhalten meromorpher Funktionen auf der Juliamenge*, Aach- ener Beiträge zur Mathematik 23, RWTH Aachen, 1998.

[BR] Noel Baker, Phil J. Rippon, *Iteration of exponential functions* Ann. Acad. Sci. Fenn. Ser. A I Math.9 (1984), 4–77.

[BD1] Clara Bodelon, Robert Devaney, Lisa Goldberg, Michael Hayes, John Hubbard and Gareth Roberts, *Hairs for the complex exponential family*, Internat. J. Bifurc. Chaos 9 (1999).

[BD2] Clara Bodelon, Robert Devaney, Lisa Goldberg, Michael Hayes, John H. Hubbard, Gareth Roberts, *Dynamical convergence of polynomials to the exponential*, J. Differ. Equations Appl. 6 (2000), no. 3, 275–307.

[De] Robert L. Devaney, *Structural instability of exp(z)*, Proc. Amer. Math. Soc. 94 (1985), no. 3, 545548.

[DK] Robert L. Devaney, Michael Krych, *Dynamics of Exp(z)*, Ergodic Theory Dynam. Systems 4 (1984), no. 1, 35–52.

[Do] , Adrien Douady, *Algorithms for computing angles in the Mandelbrot set*, Chaotic Dynamics and Fractals, ed. Barnsley and Demko, Acad. Press, 1986, pp. 155-168.

[DH] Adrien Douady, John H. Hubbard, *Etude dynamique des polynomes complexes*, Prepublications mathematiques d'Orsay (1984/1985), no. 2/4.

[EL] Alexandre Eremenko, Mikhail Lyubich, *Dynamical properties of some classes of entire functions* Ann. Inst. Fourier (Grenoble) 42 (1992), no. 4, 989–1020.

[EL2] Alexandre Eremenko, Mikhail Lyubich, *Dynamics of analytic transformations*, Leningrad Math. J. 1 (1990), no. 3, 563634.

[FRS] Markus Förster, Lasse Rempe, Dierk Schleicher, *Classification of escaping exponential maps*, Proc. Amer. Math. Soc. 136 (2008), no. 2, 651–663.

[FS] Markus Förster, Dierk Schleicher, *Parameter rays in the space of exponential maps*, Ergodic Theory Dynam. Systems 29 (2009), no. 2, 515–544.

[GM] Lisa Goldberg, John Milnor, *Fixed points of polynomial maps. Part II. Fixed point portraits*, Ann. Sci. cole Norm. Sup. (4) 26 (1993), no. 1, 51–98.

[GS] Jacek Graczyk and Grzegorz Swiatek, *Generic hyperbolicity in the logistic family*, Ann. of Math. (2) 146 (1997), no. 1.

[H] John Hubbard, *Local connectivity of Julia sets and bifurcation loci: three theorems of J. C. Yoccoz*, 467-511 of Topological Methods in Modern Mathematics, ed. Goldberg and Phillips, Publish or Perish (1993).

[HSS] John H. Hubbard, Dierk Schleicher, Mitsuhiro Shishikura, *Exponential Thurston maps and limits of quadratic differentials*, J. Amer. Math. Soc. 22 (2009), no. 1, 77–117.

[LSV] Bastian Laubner, Dierk Schleicher, Vlad Vicol, *A combinatorial classification of postsingularly finite complex exponential maps*, Discrete Contin. Dyn. Syst. 22 (2008), no. 3, 663-682.

[MS] *The Mandelbrot set, Theme and Variations*, Ed. Tan Lei, London Math. Soc. Lect. Note Ser. 274, Cambridge Univ. Press, 2000.

[LS] Mitsuhiro Shishikura, Tan Lei, *An alternative proof of Mane's theorem on non-expanding Julia sets*, in: Tan Lei et al., *The Mandelbrot Set, Theme and Variations*, Cambridge Univ. Press (2000).

[L0] Mikhail Lyubich, *Entropy properties of rational endomorphisms of the Riemann sphere*, Ergodic Theory Dynam. Systems 3 (1983), no.2, 351–385.

[L1] Mikhail Lyubich, *Dynamics of quadratic polynomials I-II*, Acta Math. 178 (1997), 185-297.

[L] Mikhail Lyubich, *Conformal geometry and dynamics of quadratic polynomials*, to appear.

[MS] Peter Makienko, G. Serra, *Poincare series and instability of exponential maps*, arXiv:math/0506143v1 [math.DS].

[MSS] R. Mañe, P. Sad, and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. Ecole Norm. Sup. (4) 16 (1983), no. 2, 193217.

[McM] Curtis McMullen, *Complex Dynamics and Renormalization*, Annals of Mathematics Studies (1994), Princeton University Press.

[Mi] John Milnor, *Dynamics in one complex variable*, Princeton University Press (2006).

[Mi1] John Milnor, *Periodic orbits, external rays and the Mandelbrot set: an expository account*, arXiv:math/9905169 [math.DS].

[R0] Lasse Rempe, *Dynamics of exponential maps*, Doctoral Thesis, Christian-Albrechts-Universitat Kiel (2003), http://e-diss.uni-kiel.de/diss781/.

[R1] Lasse Rempe, *A landing theorem for periodic rays of exponential maps*, Proc. Amer. Math. Soc. 134 (2006), no. 9, 2639–2648 (electronic).

[R2] Lasse Rempe, *Topological dynamics of exponential maps*, Ergodic Theory Dynam. Systems 26 (2006), no. 6, 1939–1975.

[R3] Lasse Rempe, *Rigidity of escaping dynamics for transcendental entire functions*, Acta Math., 203 (2009), 235267.

[RS1] Lasse Rempe, Dierk Schleicher, *Combinatorics of bifurcations in exponential parameter space*, in: Phil Rippon, Gwyneth Stallard, *Transcendental Dynamics and Complex Analysis*, London Mathematical Society Lecture Note Series 348 Cambridge University Press (2008),317370; arXiv:math/0408011 [math.DS].

[RS2] Lasse Rempe, Dierk Schleicher, *Bifurcation loci of exponential maps and quadratic polynomials: local connectivity, triviality of fibers and density of hyperbolicity*, in: Mikhail Lyubich et al, Fields Institute Communications Volume 53: *Holomorphic dynamics and renormalization.*

A volume in honour of John Milnor's 75th birthday (2008), 177-196; arXiv:0805.1658 [math.DS].

[RvS] Lasse Rempe, Sebastian Van Strien, *Absence of line fields and Mane's theorem for non-recurrent transcendental functions*, arXiv:0802.0666 [math.DS].

[RvS2] Lasse Rempe, Sebastian Van Strien, *Density of hyperbolicity for classes of real transcendental entire functions and circle maps*, manuscript.

[S0] Dierk Schleicher, *On the dynamics of iterated exponential maps*, Habilitation thesis, TU Munchen (1999).

[S1] Dierk Schleicher, *On fibers and local connectivity of Mandelbrot and Multibrot sets*, Fractal geometry and applications: a jubilee of Benoit Mandelbrot. Part 1, 477–517, Proc. Sympos. Pure Math., 72, Part 1, Amer. Math. Soc., Providence, RI, (2004).

[S2] Dierk Schleicher, *Attracting dynamics of exponential maps*, Ann. Acc. Sc. Fenn. Vol 28, 2003, 3-34.

[SZ1] Dierk Schleicher, Johannes Zimmer, *Escaping points of exponential maps*, J. London Math. Soc. (2) 67 (2003), no. 2, 380–400.

[SZ2] Dierk Schleicher, Johannes Zimmer, *Periodic Points and dynamic rays of exponential maps*, Ann. Acad. Sci. Fenn. Math. 28 (2003), no. 2, 327–354.

[Sl] Zbigniew Slodkowski, *Holomorphic motions and polynomial hulls*, Proc. Amer. Math. Soc. 111 (1991), 347-355.

[Ye] Zhuan Ye, *Structural instability of exponential functions*, Trans. Amer. Math. Soc. 344 (1994), no. 1, 379-389.