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Dynamic Hedge Fund Asset Allocation Under Multiple Regimes

A Dissertation Presented

by

David Cru

to

The Graduate School

in Partial Fulfillment of the

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Portfolio Selection as introduced by Harry Markowitz laid the foundation for Modern Portfolio Theory. However, the assumption that underlying asset returns follow a Normal Distribution and that investors are indifferent to skew and kurtosis is not practically suited for the Hedge Fund environment. Additionally, the Lockup and Notice provisions built into Hedge Fund contracts make portfolio rebalancing difficult and justify the need for dynamic allocation strategies. Market conditions are dynamic, therefore, rebalancing constraints in the face of changing market environments can have a severe impact on return generation. There is a need for sophisticated yet tractable solutions to the multi-period problem of Hedge Fund portfolio construction and rebalancing. In this thesis we Generalize the Hedge Fund asset return distribution to a Multivariate K-mean Gaussian Mixture Distribution; model the multi-period Hedge Fund allocation problem as a Markov Decision Process (MDP); and propose practical rebalancing strategies that represent a convergence of literature on Hedge Fund investing, Regime Switching, and Dynamic Portfolio Optimization.

Dedication

This thesis is dedicated in loving memory of Frederick Geoffrey Schade, whos kindness, brilliance and integrity will always serve as a source of strength and inspiration throughout my life.

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Chapter 1

Introduction

Portfolio Selection as introduced by Harry Markowitz [39] laid the foundation for Modern Portfolio Theory. However, the assumption that underlying asset returns follow a normal distribution and that investors are indifferent to skew and kurtosis are not practically suited for the Hedge Fund environment. Clearly, the non-normal distributions that drive Hedge Fund returns are a well documented phenomenon ([18],[2]), but they are not the only factor that makes Hedge Fund portfolio management difficult. The lockup and notice provisions built into Hedge Fund contracts ([4]) make portfolio rebalancing difficult and justify the need for dynamic strategies.

Neither can the investors ignore the lessons of 2008 nor assume that returns are completely independent. Even prior to the recent systemic breakdown, Hedge Fund performance literature documented significant autocorrelation in Hedge Fund returns attributable to increasing liquidity risk in the underlying holdings ([11],[44],[4]). If the current liquidity crisis can teach us

anything it is that market conditions are dynamic; approaches that cannot account for a changing market environment can leave an investor or portfolio manager overexposed. A growing body of literature documents the need for approaches that account for multiple market regimes particularly in the Hedge Fund space ([24],[45],[55]).

There is a need for sophisticated yet tractable solutions to the problem of Hedge Fund portfolio construction and rebalancing. What we propose is a practical rebalancing strategy that represents a convergence of literature on Hedge Fund investing, Regime Switching, and Dynamic Portfolio Optimization.

1.1 Research Problem

We use the exposition from Bertsekas ([7]) as a suitable starting point. We have an investor that makes a decision in each successive time period with the objective of maximizing his terminal wealth X_T . Let X_0 denote the initial wealth of the investor. X_t is defined to be the wealth of the investor at the beginning of time period t . At each period the investor is given the option of investing in N risky assets with corresponding rates of return r_1, \dots, r_N in excess of a Risk Free asset with constant return r_f . $w_{j,t}$ is defined as the weight allocated to the j^{th} risky asset at time t . The wealth at end of the t^{th} time period is given by:

$$X_{t+1} = (1 + r_f)X_t + \sum_{j=1}^N (r_{j,t} - r_f)w_{j,t}X_t \quad (1.1)$$

The objective is to maximize over $w_{1,1}, \dots, w_{j,t}, \dots, w_{N,T}$

$$J_N(X_0) = E[U(X_T)|X_0 = x_0] \quad (1.2)$$

Where $U(\cdot)$ is a risk sensitive utility function.

We generalize the above problem under the existence of K regimes (states), $S = \{1, \dots, K\}$; the regimes are macro states, such that, each of the N risky assets is in the same regime at any given time t . S_t represents the regime observed at time t . $I_t(i)$ is an indicator function which is defined as follows:

$$I_t(i) = \begin{cases} 1 & \text{if } S_t = i \\ 0 & \text{otherwise} \end{cases}$$

Matrix Q is defined as the transition matrix such that $q_{i,i'}$ corresponds to the one step transition probability from state i to i' .

$$P(S_{t+1} = i' | S_t = i) = q_{i,i'}$$

the wealth dynamics (1.1) becomes as follows:

$$X_{t+1} = (1 + r_f)X_t + \sum_{i=1}^K \sum_{j=1}^N (r_{i,j,t} - r_f)w_{j,t}I_{t+1}(i)X_t \quad (1.3)$$

where $r_{i,j,t}$ is the return of asset j in state i at time t . Please note, that at time t , $I_{t+1}(i)$ is not known with certainty.

1.2 Research Objectives

The dissertation objectives are as follows: 1) develop an asset allocation model for Hedge Fund Returns that follow a Multivariate Gaussian Mixture Distribution; 2) find optimal policies for the multi-period case under a no liquidity restriction assumption; 3) generalize the previous case to account for heterogeneous lockup (no trade regions) and homogeneous notice periods (lookahead windows); 4) provide multiple approximation and simulation approaches to the problem of Portfolio Optimization with heterogeneous lockups; 5) Apply these approaches to the problem of Multi-Period Hedge Fund Asset Allocation.

1.3 Structure of the Thesis

The rest of this thesis is structured as follows: chapter 2 provides background research on the problem with a focus on hedge funds; chapter 3 contains the Asset Allocation Model and a generalization to the Hedge Fund Portfolio Model; chapter 4 contains approaches to solving the Multi-Period Asset Allocation Problem and a number of approximation techniques; chapter 5

contains an out of sample back-test on the model and chapter 6 concludes the discussion.

Chapter 2

Background

2.1 Hedge Funds

Hedge Funds are pooled investment vehicles that can take long and short positions, trade derivatives, use leverage, and invest in almost any opportunity in any market where they anticipate impressive gains at reduced risk ([16],[55]). Hedge Funds as an asset class have seen extraordinary growth. Hedge Funds and Funds of Hedge Funds (FoFs) have seen approximately 20% growth per year over the 15 year period from 1990 to 2005. As of the end of 2005, there were approximately 8,100 funds and 2000 FoFs reporting to one of the 12 major global databases. ([52]).

2.1.1 Investors

High Net Worth investors and Family Offices were the first to actively invest in Hedge Funds and FoFs. However, in the last decade, the investor base has shifted away from High Net Worth individuals to Institutional Investors ([52],[9]).

The largest beneficiary of the growth of Institutional Investor allocation to Hedge Funds have been FoFs. The reason is that most institutions do not have the staff necessary to source, select, perform single manager due diligence, and ultimately monitor the funds post selection ([52]). Institutional Investors have come to rely upon FoFs as extensions of their own management teams for their professional Manager Selection and Due Diligence capabilities ([52]).

2.1.2 Risks

Key structural risks or drawbacks unique to Hedge Fund investing are as follows ([9],[52]):

- **Transparency:** Investors are not given information about the underlying portfolio assets.
- **Complexity:** Managers are able to use diverse asset classes and complex derivative products to produce returns which may not be readily understood by the investor.
- **Fees:** Hedge Fund Fees, comprised of Management and Incentive Fees

are generally significant ([10], [26]).

- **Leverage:** Debt which is invested with the aim of achieving a greater rate of return can affect risk adjusted performance ([6]).
- **Liquidity:** There are two levels of liquidity risk, Lockup Restrictions and illiquidity in the underlying portfolio. Many Hedge Funds have Lockup Provisions. These are two-fold: Lockup Periods which restrict investor divestment and Notice Periods which delay transaction execution (see below). The presence of stale asset prices due to illiquidity can artificially lower estimates of volatility and correlation to traditional indices ([11]).
- **Idiosyncratic:** There have been a number of catastrophic Hedge Fund failures: Long Term Capital Management (1998), Amaranth (2006) and Madoff (2008).

In our research, we focus specifically on the impact of Liquidity Provisions on Portfolio Allocation and Optimization. Fee effect is mitigated by using net as opposed to gross returns. We do not specifically focus on transparency, complexity or leverage in this analysis.

Lockups

Lockup Provisions are classified into "Hard" and "Soft" locks. A "Hard Lock" requires that all initial monies allocated to the fund may not be withdrawn before the end of a pre-specified duration, or, Lockup Period. A "Soft Lock" generally imposes a fee penalty for early redemption. The redemption

Notice Period is the amount of time the investor is required to provide notice before redeeming their share ([4]).

There is a significant body of work documenting the effect of liquidity on Hedge Fund returns. There is evidence that funds with longer lockup periods tend to out-perform those with shorter or no lockups ([35]). This result is verified by Aragon ([4]) who documents a positive concave relationship between a fund's excess return and its redemption Notice Period and minimum investment size. Aragon ([4]) shows that the difference between excess returns on lock versus non-lock funds (lockup premium) are about 4% annually; He further concludes that Hedge Fund out-performance is entirely consistent with the compensation for bearing liquidity risk. The Lockup Premium is consistent with existing literature ([38],[51]) that documents the existence of a Liquidity Premium for Liquidity Risk in illiquid securities.

Illiquidity in Hedge Fund portfolios can have a significant impact on returns. The degree of autocorrelation in a Fund's returns can serve as a proxy for the fund's liquidity exposure ([44]). Chan, Getmansky, Haas and Lo find significant autocorrelation in the following six categories ([44]): Convertible Arbitrage (31.4 %), Fund of Funds (19.6%), Event Driven (18.4%), Emerging Markets (16.5 %), Fixed Income Arbitrage (16.2 %) and Multi-Strategy (14.7%). Getmansky, Lo and Makarov ([41]) further argue that serial correlation is attributable to illiquidity and "performance smoothing"¹. There-

¹For securities that are traded infrequently techniques such as linear extrapolation are used to determine the present price from the most recent transaction price; returns computed in such a manner tend to be smoother, exhibiting lower volatility and higher

fore, due to the relationship between illiquidity and serial correlation, Hedge Fund portfolio illiquidity invalidates the assumption of independent monthly returns.

Much of the research in this area has been on measuring Liquidity Premium². Longstaff ([38]) provides a mathematical calculation for the liquidity premium that is based on the additional return required for utility loss due to liquidity. Browne et al. ([51]) defines liquidity premium as the amount added to an illiquid product to produce the same level of utility as the unrestricted product without the liquidity premium. Derman, Park and Whitt ([14]) propose Discrete Time Markov Chain (DTMC) and Continuous Time Markov Chain (CTMC) models for Hedge Fund Lockup Premiums as a function of the length of the extended lockup periods.

2.1.3 Performance

Liang ([35]) demonstrates that, in a mean-variance context, Hedge Funds typically outperform mutual funds. Agarwal, Mendelson and Naik ([56]) further state that in the 1992 to 1996 period Hedge Funds achieved a monthly return of 1.10% versus 0.85% achieved by Mutual Funds ([56]). In the same period Hedge Funds achieved a monthly standard deviation of 2.40% versus 1.91% for mutual funds. Using a risk adjusted performance measure such as Sharpe Ratio, Liang ([35]) demonstrates that Hedge Funds outperform Mutual Funds on a risk adjusted basis (0.44 vs. 0.26 respectively). These

serial correlation than true economic returns ([44])

²We define Liquidity Premium as the premium demanded by investors for the increased risk in investing in an illiquid security

results justify investment in Hedge Funds as an asset class.

Agarwal and Naik ([2]) expanding on previous work by Fung and Hsieh ([18]) demonstrate that Hedge Fund returns exhibit nonlinear option-like payoffs³. Funds exhibit significant left tail risk that is typically underestimated in the standard mean-variance framework ([2]). Although this claim is supported in the Cremers, Kritzman and Page paper, "Optimal Hedge Fund Allocations: Do Higher Moments Matter?," they find that under log utility mean-variance optimization performs extremely well, even when the distributions of the component Hedge Fund returns are significantly non-normal ([28]).

Persistence

A critical assumption embedded in a Multi-Period Asset Management framework is that Hedge Funds exhibit persistence in returns. Return persistence implies differential manager skill, a necessary prerequisite in allocating to different Hedge Funds within the same strategy group. Studies on Hedge Funds have revealed conflicting, but generally favorable, evidence of persistence.

Brown, Goetzmann and Ibbotson ([54]) use a relative (against the median) annual performance two state (win-win, lose-lose) method to identify differential performance persistence but do not find evidence of persistence.

³Standard performance evaluation models involve regressing a fund's historical return on certain economic benchmarks; funds with option like payoffs cannot be adequately represented this way. Typical approaches include adding nonlinear functions of the benchmark returns as regressors ([18])

However, others find evidence of persistence over shorter periods. Agarwal and Naik ([1]) find a considerable amount of persistence at the quarterly horizon which reduces at the yearly window. Recent work by Jagannathan, Malakhov, and Novikov ([50]) finds that more than 25% of the abnormal performance during a three year interval will spill over into the following three year interval. Derman and Park ([14]) find performance persistence in the following strategies: (i) Convertible Arbitrage, (ii) Dedicated Short Bias, (iii) Fixed Income, (iv) Fund of Funds, and (v) Others.

Biases

Any discussion of Hedge Fund performance requires an overview of listing biases. Literature on this topic addresses four specific biases:

- **Self Selection:** Selection bias occurs because Hedge Funds in a database are not representative of the universe of Hedge Funds ([17]); inclusion of a Hedge Fund in a database is done on a voluntary basis. Aragon ([4]) argues that, though high quality funds have a greater incentive to reveal performance, raising capital is likely the only incentive to report. The impact of self selection bias is mitigated by that disparity.
- **Survivorship:** Survivorship bias occurs if Hedge Fund databases do not contain data on defunct funds. When comparing the HFR and TASS databases Liang ([36]) finds that survivorship bias exceeds 2% per year, 0.6% for the HFR. This is greater than the 3.0% bias found by Fung and Hsieh ([17]). After 1994, both the HFR and TASS database have started maintaining data on dissolved funds.

- **De-listing:** De-listing bias is similar to survivorship. De-listing bias exists if the database does not perfectly observe final return observations of a de-listing fund ([4]). Aragon ([4]) observes that de-listing bias is more severe for funds with share restrictions.
- **Backfilling:** Backfilling bias may arise because databases permit newly-added funds to backfill their performance data. Funds have an incentive to raise capital following above average returns so estimates of performance using backfilled data may be biased upwards ([4]). Using the TASS database, Fung and Hsieh ([17]) estimate the backfilling bias for Hedge Funds at 1.4% per year over the 1994-1998 period. Posthuma and van der Sluis ([46]) estimate backfilling bias at 4% per year. Aragon ([4]) confirms Posthuma and van der Sluis' results finding excess returns are 3-4% lower after controlling for backfilled data.

To mitigate the effect of these biases, in our analysis we compare the results on a cross-sectional basis using HFR index data. In the pilot study in section 5.1, we compare allocations to HFR Indices to the HFR Fund of Fund Composite index; index data limits the affect that any one fund can have on the results. Besides Hedge Funds, the only other asset class in the model is the Risk Free Asset. Lower Hedge Fund returns potentially increase the optimal allocation to the Risk Free asset, however, with the Risk Free Rate at historic lows that effect is diminished.

2.2 Regime Switching Models

Hamilton ([23],[24]) introduces the Regime Switching model with his development of the Autoregressive Regime Switching Process. Since then, similar approaches have been applied to equity return distributions ([25],[22]). Hardy ([25]) uses a Regime Switching Log-Normal Process (RSLN) which implements a Markov process to select among K discrete Log-Normal Distributions; this application is similar in scope to the Regime Switching Model employed here.

Billio, Getmansky and Pelizzon ([42]); Chan, Getmansky, Haas and Lo ([45]); and Tashman and Frey ([55]) address further applications of Regime Switching Models to Hedge Fund returns. Billio, Getmansky and Pelizzon employed a Regime Switching Beta Model; changes in regime correspond to changes in broad market exposure. Chan et. al. ([45]) use a two state model ('Normal' and 'Distressed' regimes) to fit Hedge Fund index returns. Tashman and Frey ([55]) use a Hierarchical Mixture of Experts (HME) model to fit arbitrage Hedge Fund returns; the model features a Multi-Factor Gating Distribution that selects between two Multi-Factor Models.

For the preliminary analysis we fit a Two-State Model (as seen in Chan et. al. [45]) for "Normal" and "Distressed" states to Hedge Fund index returns. The purpose of which is to develop an optimal policy for Hedge Fund Allocation in the presence of multiple regimes.

2.3 Portfolio Optimization

We begin our discussion of Portfolio Optimization with Merton ([40]) and Samuelson ([53]) who examined the problem of Multi-Period Portfolio Optimization in the continuous and discrete-time cases respectively. They examined the problem of Optimal Portfolio Selection under a utility function that satisfies Pratt's ([47]) measure of Constant Relative Risk Aversion (CRRA). Merton specifically addresses Portfolio Selection under a utility function of the form:

$$U(C) = \frac{C^\gamma}{\gamma}$$

The CRRA utility function used by Merton has certain benefits with regards to multi moment optimization and will be discussed further in section 3.2. In the Jondeau and Rockinger paper, "Optimal Portfolio Allocation Under Higher Moments," a Taylor Series expansion is used to approximate the expected utility as a function of the higher moments ([29]). This approach facilitates efficient computation of the optimal portfolio. We use a similar approach in section 3.2.

In absence of forecasts for the returns of the risky assets in a Single Regime Model (see eq. 1.1) the Myopic Policy ⁴ is optimal ([7]). If the forecast for a period i becomes available during the investment process then a Partially Myopic Policy ⁵ is optimal. Multi-Period Solutions become necessary in the

⁴A Myopic Policy assumes that at any point in time the investment opportunity will remain constant thereafter [33]

⁵A Partially Myopic Policy is similar to a Limited Lookahead Policy. The assumption is that after the Lookahead Window the investment opportunity will remain constant

presence of Transaction Costs ([32],[57]) or Liquidity Restrictions ([27],[9]).

Sun, Fan, Chen, Schouwenaars and Albota ([57]) use Certainty Equivalents to develop an Optimal Allocation Policy in the presence of Transaction Costs. Using Certainty Equivalents, they quantify the Risk Neutral Cost of sub-optimality; when this cost exceeds the Transaction Cost of rebalancing, partial or full rebalancing occurs. This approach provides superior results over conventional approaches of periodic and tolerance band rebalancing ([13],[57]). When rebalancing on a periodic basis, the Portfolio Manager adjusts allocations at a predetermined time interval; this approach has the drawback of ignoring market behavior or trading when the portfolio is nearly optimal ([57]). For Tolerance Band rebalancing occurs when an allocation deviates beyond some target limit; exceedences are followed by full rebalancing to the target portfolio. This method reacts to market behavior however the threshold is fixed and any action results in full rebalancing ([57]). Leland ([32]) demonstrates the existence of a "No-Trade Region" around the Optimal Portfolio weights; in the event that the weights exceed the boundaries of the "No-Trade Region" it is optimal to bring them back to the nearest edge of that region. Mulvey and Simsek ([43]) model the problem of rebalancing under Transaction Costs and Market Impact as a generalized network with side conditions.

Some papers deal more specifically with the problem of Hedge Fund Allocation ([27],[9]). Cvitanic et. al. ([27]) develop a model in which a Non-Myopic Investor with incomplete information allocates wealth between a Risk Free Security, a Passive Portfolio and an Actively Managed Portfolio. They find

that that low beta Hedge Funds may serve as natural substitutes for a significant portion of an investor Risk Free Asset holdings; Boyle and Liew support this result ([9]).

There is a growing body of literature that deals specifically with the issue of Portfolio Allocation under Regime Switching. Ang and Bakaert's paper, "International Asset Allocation with Regime Shifts," model the Dynamic Asset Allocation Problem in the presence of regime switches for investors with CRRA preferences ([3]). They examine the effects of asymmetric correlation on the benefit of International Asset Diversification by modeling a US investor with CRRA preference maximizing end of period wealth and dynamically rebalancing in response to regime switches. In the paper, "Optimal Portfolio Choice under Regime Switching, Skew and Kurtosis Preferences," ([21]) Guidolin and Timmerman model a "Buy and Hold" investor's choice of a Simple Stock Portfolio and Risk Free asset over a finite time horizon under a Markov Switching Vector Autoregressive Process and CRRA utility. In, "Asset Allocation under Multivariate Regime Switching," Guidolin and Timmerman expand on their previous work to explore the Asset Allocation in the presence of regimes in the joint distribution of stock and bond returns ([20]). They found that Optimal Asset Allocation varies significantly across regimes and length of the investment horizon.

These papers provide a theoretical basis for our work, however, in Hedge Fund Portfolio Allocation, Transaction Costs are negligible in comparison to the more difficult to quantify Liquidity Restrictions. This project will

supplement existing research in Multi-Period Regime-Based Optimization by providing an allocation and rebalancing approach to Hedge Fund investments with heterogeneous Liquidity Restrictions.

Chapter 3

Hedge Fund Portfolio Model

We aim to provide an Optimal Policy for the problem defined in section 1.1 under the following assumptions:

1. There are K regimes
2. Hedge Fund returns are Normally Distributed in each regime
3. Allocation decisions are made on a periodic (monthly) basis
4. There are no Transaction Costs
5. There is a constant Risk Free Rate of investment
6. The Return Generating Distributions and the Gating Distribution are independent
7. Each fund has a lockup length $L_j = 0, 1, \dots, T$
8. Funds can have a homogenous Notice Period Δ
9. Allocations to fund j cannot be changed during a Lockup

10. Once a fund has exited a Lockup it can no longer reenter a Lockup

3.1 Portfolio Return Distribution

Recall from 1.1 equation 1.3 as follows:

$$X_{t+1} = (1 + r_f)X_t + \sum_{i=1}^K \sum_{j=1}^N (r_{i,j,t} - r_f)w_{j,t}I_{t+1}(i)X_t \quad (3.1)$$

At t , $I_{t+1}(i)$ is not known with certainty, but rather follows a Multinomial Distribution which selects regime i' from regime i with probability $q_{i,i'}$. A key assumption used here is that the Return Generating Distributions are independent of the Switching Mechanism. If the returns of the N risky assets follow a Normal Distribution ($N(\mu_i, \Sigma_i)$) where μ_i is a vector and Σ_i is a matrix in regime i then the return (r_t^P) of a portfolio of N risky assets with allocations $w = (w_1, \dots, w_N)$ at time t follows a Gaussian Mixture Distribution with parameters

$$\Theta_G = (\pi_1, \pi_2, \dots, \pi_K, \mu_1^T w, \mu_2^T w, \dots, \mu_K^T w, w^T \Sigma_1 w, w^T \Sigma_2 w, \dots, w^T \Sigma_K w) \quad (3.2)$$

Let $\mu_i^P = \mu_i^T w$ and $\sigma_i^P = \sqrt{w^T \Sigma_i w}$ and $\pi_1 = q_{i,1}, \pi_2 = q_{i,2}, \dots, \pi_K = q_{i,K}$ respectively then using the Moment Generating Function ($M_r^P(t)$) about r_t^P :

$$M_r^{(P)}(t) = \sum_{i=1}^K \pi_i e^{\mu_i^{(P)} t + \frac{1}{2} \sigma_i^{2(P)} t^2} \quad (3.3)$$

We can solve for the moments and centralized moments of the return distribution r_t^P (see appendix C).

Recall again equation 3.1; we can condition the change in wealth (returns) on the state i at time t :

$$S_t = i \Rightarrow \frac{X_{t+1}}{X_t} = 1 + r_f + \sum_{i'=1}^K \sum_{j=1}^N (r_{i',j} - r_f) I_{i,t+1} w_{j,t} \quad (3.4)$$

The $1 + r_f$ term is a constant as defined above. So let:

$$G_t(I_t) = \sum_{i'=1}^K \sum_{j=1}^N (r_{i',j} - r_f) I_{t+1}(i) w_{j,t}$$

$G_t(I_t)$ represents the excess returns of the portfolio over the Risk Free Rate r_f given weights w_t at time t . Then $G_t(I_t)$ follows a Gaussian Mixture Distribution as follows:

$$G_t(I_t) \sim G(\pi_1, \pi_2, \dots, \pi_K, w_t^T(\mu_1 - r_f), w_t^T(\mu_2 - r_f), \dots, w_t^T(\mu_K - r_f), w_t^T \Sigma_1 w_t, w_t^T \Sigma_2 w_t, \dots, w_t^T \Sigma_K w_t) \quad (3.5)$$

Rewriting equation 3.4 using $G_t(I_t)$ we end up with the following return representation:

$$\frac{X_{t+1}}{X_t} = 1 + r_f + G_t(I_t) \quad (3.6)$$

where $S_t = i$

3.2 Utility Return Function

The objective of this Multi-Period Asset Allocation Problem is to maximize the expected utility ($U(\cdot)$) of the terminal wealth $U(X_T)$. For the purpose of this analysis we use the CRRA Utility similar to Merton ([40]), Ang and Bakaert ([3]), Guidolin and Timmerman ([21],[20]), and Sang and Liew ([9]).

If $G_t(I_t)$ follows a Gaussian Mixture Distribution as described above then we can use a Taylor series approximation (Chen et al. [57], Jondeau and Rockinger [29]) around the mean μ_{G_t} of the excess returns as follows:

$$\begin{aligned}
 U(G_t(I_t)) &= U(\mu_{G_t}) + U'(\mu_{G_t})(G_t(I_t) - \mu_{G_t}) + \frac{U''(\mu_{G_t})}{2!}(G_t(I_t) - \mu_{G_t})^2 + \\
 &+ \frac{U'''(\mu_{G_t})}{3!}(G_t(I_t) - \mu_{G_t})^3 + \frac{U^{iv}(\mu_{G_t})}{4!}(G_t(I_t) - \mu_{G_t})^4 + o(h)
 \end{aligned}
 \tag{3.7}$$

The higher order terms are truncated in the equation above; this facilitates a convenient representation for $U(G_t(I_t))$ in terms of its first four centralized moments as follows:

$$E[U(G_t(I_t))] \approx U(\mu_{G_t}) + \frac{U''(\mu_{G_t})}{2}\sigma_{G_t}^2 + \frac{U'''(\mu_{G_t})}{6}(\mu_{3G_t}) + \frac{U^{iv}(\mu_{G_t})}{24}(\mu_{4G_t})
 \tag{3.8}$$

μ_3 and μ_4 represent the third and fourth centralized moments (see appendix C).

The Taylor Series Approximation of the expected utility provides increased computational efficiency; this justification is similar to that of Jondeau and Rockinger ([29]). As we show in section 4 the number of optimizations we need to run can be quite large $O[(T + 1)^N]$, therefore we need an objective function that balances computational efficiency and higher moment optimization. Loistl ([37]) finds one important caveat, the interval of convergence of the CRRA Utility is:

$$0 < X < 2E(X)$$

This condition is generally non-binding for traditional asset classes, and even in cases where returns may lie outside the interval of convergence the approximation is sufficient for most practical applications ([29],[31]). For applications that deviate significantly outside the interval of convergence the Constant Absolute Risk Aversion (CARA) Utility may be used as in Jondeau and Rockinger ([29]) which converges absolutely for all values of X . Furthermore in section 4.5 we provide an Adaptive Simulation Based Approach that explicitly takes the utility of the terminal wealth at each iteration of the algorithm thereby circumventing the need for the Taylor Approximation.

CRRA Utility

Let the CRRA utility be defined as:

$$U(X) = \frac{X^\alpha}{\alpha}$$

We are tasked with maximizing the utility of the terminal wealth X_T :

$$X_T = \left(\prod_{t=0}^{T-1} (1 + r_f + G_t(I_t)) \right) X_0 \quad (3.9)$$

subject to the initial wealth X_0 . Therefore, the terminal utility is equivalent to the following:

$$\begin{aligned} U(X_T|X_0 = x_0) &= U\left(\left(\prod_{t=0}^{T-1} (1 + r_f + G_t(I_t))\right)X_0\right) \\ E[U(X_T|X_0 = x_0)] &= E\left[\left(\prod_{t=0}^{T-1} (1 + r_f + G_t(I_t))^\alpha\right)U(X_0)\right] \\ \max_{\vec{w}_t} E[U(X_T)|X_0 = x_0] &= \max_{\vec{w}_t} \left(U(X_0) \prod_{t=0}^{T-1} E[(1 + r_f + G_t(I_t))^\alpha] \right) \\ \max_{\vec{w}_t} E[U(X_T)|X_0 = x_0] &= U(X_0) \min_{\vec{w}_t} \left(\prod_{t=0}^{T-1} E[(1 + r_f + G_t(I_t))^\alpha] \right) \\ \max_{\vec{w}_t} \ln E[U(X_T)|X_0 = x_0] &= \ln U(X_0) + \min_{\vec{w}_t} \sum_{t=0}^{T-1} \ln E[(1 + r_f + G_t(I_t))^\alpha] \end{aligned} \quad (3.10)$$

Expanding $(1 + r_f + G_t(I_t))^\alpha$ about the μ_{G_t} term yields:

$$\begin{aligned}
(1 + r_f + G_t(I_t))^\alpha &= (1 + r_f + \mu_{G_t})^\alpha + \alpha(1 + r_f + \mu_{G_t})^{\alpha-1}(G_t(I_t) - \mu_{G_t}) + \\
&+ \frac{\alpha(\alpha - 1)(1 + r_f + \mu_{G_t})^{\alpha-2}}{2}(G_t(I_t) - \mu_{G_t})^2 + \\
&+ \frac{\alpha(\alpha - 1)(\alpha - 2)(1 + r_f + \mu_{G_t})^{\alpha-3}}{6}(G_t(I_t) - \mu_{G_t})^3 + \\
&+ \frac{\alpha(\alpha - 1)(\alpha - 2)(\alpha - 3)(1 + r_f + \mu_{G_t})^{\alpha-4}}{24}(G_t(I_t) - \mu_{G_t})^4
\end{aligned} \tag{3.11}$$

Taking the expectation:

$$\begin{aligned}
E[(1 + r_f + G_t(I_t))^\alpha] &= (1 + r_f + \mu_{G_t})^\alpha + \\
&+ \frac{\alpha(\alpha - 1)(1 + r_f + \mu_{G_t})^{\alpha-2}}{2}\sigma_{G_t}^2 + \\
&+ \frac{\alpha(\alpha - 1)(\alpha - 2)(1 + r_f + \mu_{G_t})^{\alpha-3}}{6}\mu_{3G_t} + \\
&+ \frac{\alpha(\alpha - 1)(\alpha - 2)(\alpha - 3)(1 + r_f + \mu_{G_t})^{\alpha-4}}{24}\mu_{4G_t}
\end{aligned} \tag{3.12}$$

This provides a convenient closed form representation for the expected utility of the return at each time step in terms of the first four portfolio moments. This is a better representation for Hedge Fund Portfolios than Quadratic Utility because of the non-normality of the returns.

3.3 No Liquidity Restriction Model

As before, let S be a set of K distinct macro environmental regimes such that:

$$S = \{1, 2, \dots, K\}$$

S_t represents the state of the environment at time t such that:

$$P(S_t = i' | S_{t-1} = i) = q_{i,i'}$$

In addition, F is the set of available risky assets. The magnitude of set F is N allowing a choice of allocation to a combination of any of the available N assets. let f_j be a fund in set F such that:

$$f_j = (\vec{\mu}_j, \vec{\sigma}_j)$$

In the tuple above $\vec{\mu}_j$ is a column vector of size K , denoting the expected return of fund j in each of the K regimes and $\vec{\sigma}_j$ is a column vector of size K representing the standard deviation of fund j each of the K states.

Under the No Lockup Condition the set of actions available at time t , A_t is the following:

$$A_t = \vec{w}_t$$

where \vec{w}_t is a vector of magnitude N with each entry:

$$w_{j,t} \in [0, 1]$$

Examining equation 1.3 it is apparent that the choice of weights \vec{w}_t does not impact the system dynamics of the model. The dynamics are driven by the static transition probability matrix Q . Therefore, the choice of action at time t only effects the future wealth. In absence of Transaction Costs, when the Liquidity Restrictions are relaxed, the choice of action depends only on the current state at the beginning of time period t .

3.3.1 No Lockup MDP Formulation

Using the Certainty Equivalent approach we start with a special case of a Markov Decision Process (MDP) ([5],[7]) where the regime transition dynamics are independent of the choice of control:

$$D_{NL} = \{S, A, Q, R\}$$

Where S is a set of macro regimes as defined in section 3.3:

$$S = \{1, 2, \dots, K\}$$

A is the set of actions and a is an action in A . The available actions at each state i are given by the weights to the component risky assets which are as follows:

$$A = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_i, \dots, \vec{w}_K\}$$

Such that:

$$\begin{aligned} w_{i,j} &\in [0, 1] \\ \sum_{j=1}^N w_{i,j} &\leq 1 \end{aligned} \tag{3.13}$$

$q(i, i')$ is the transition probability of a transition from state i to i' as follows:

$$q(i, i') = P(S_{t+1} = i' | S_t = i)$$

and $R_t(i, i', a)$ is the immediate reward for transitioning from state i to i' under action $a \in A$:

$$R_t(i, i', a) = (1 + r_f + \sum_{j=1}^N r_{i',j} w_{i,j})^\alpha$$

The value function is shown below

$$V(i) = \min_a \sum_{t=0}^{T-1} \ln E[R_t(S_t, S_{t+1}, a) | S_0 = i] \tag{3.14}$$

3.3.2 No Lockup POMDP Formulation

The state i is not directly observable ([20]) so we generalize this model to a Partially Observable Markov Decision Process (POMDP) ([34]). A POMDP is a tuple $D_{NL}^{\sim} = (S, A, Z, \tilde{Q}, \Omega, \tilde{R})$, where S and A are defined in the same way as in the MDP model. Ω is defined as a finite set of observations of the

process and Z is an Observation Function ([34]) which gives for state at time t a probability distribution over possible observations ([34]). A "belief state" is introduced, where Z_t is a probability distribution over S at time t . The Prior and Predictive Distributions of are given below:

Prior Distribution:

$$P(r_{t-1}^{\vec{r}}|S_t = i) = \phi(r_{t-1}^{\vec{r}}|\vec{\mu}_i, \Sigma_i) \quad (3.15)$$

We can update this distribution as a function of S_{t-1} as follows:

$$\begin{aligned} P(r_{t-1}^{\vec{r}}|S_{t-1} = i) &= \sum_{i'}^K P(r_{t-1}^{\vec{r}}|S_t = i')P(S_t = i'|S_{t-1} = i) \\ &= \sum_{i'=1}^K q_{i,i'}\phi(r_{t-1}^{\vec{r}}|\vec{\mu}_{i'}, \Sigma_{i'}) \end{aligned} \quad (3.16)$$

Which gives us a Prediction Distribution Function as follows:

$$\begin{aligned} P(r_{t-1}^{\vec{r}}|r_{t-2}^{\vec{r}}) &= \sum_{i'=1}^K P(r_{t-1}^{\vec{r}}|S_t = i')P(S_t = i'|r_{t-2}^{\vec{r}}) \\ &= \sum_{i'=1}^K \sum_{i=1}^K P(r_{t-1}^{\vec{r}}|S_t = i')P(S_t = i'|S_{t-1} = i)P(S_{t-1} = i|r_{t-2}^{\vec{r}}) \\ &= \sum_{i=1}^K \sum_{i'=1}^K Z_{t-1}(i)q_{i,i'}\phi(r_{t-1}^{\vec{r}}|\vec{\mu}_{i'}, \Sigma_{i'}) \end{aligned} \quad (3.17)$$

Posterior Distribution

$$\begin{aligned}
Z_t(i') &= P(S_t = i' | r_{t-1}^{\vec{}}) \\
&= \frac{P(r_{t-1}^{\vec{}} | S_t = i') P(S_t = i' | r_{t-2}^{\vec{}})}{P(r_{t-1}^{\vec{}} | r_{t-2}^{\vec{}})} \\
&= \frac{P(r_{t-1}^{\vec{}} | S_t = i') \sum_{i=1}^K P(S_t = i' | S_{t-1} = i) P(S_{t-1} = i | r_{t-2}^{\vec{}})}{\sum_{i=1}^K \sum_{i'=1}^K Z_{t-1}(i) q_{i,i'} \phi(r_{t-1}^{\vec{}} | \mu_{i'}^{\vec{}}, \Sigma_{i'})} \quad (3.18) \\
&= \frac{P(r_{t-1}^{\vec{}} | S_t = i') \sum_{i=1}^K q_{i,i'} Z_{t-1}(i)}{\sum_{i=1}^K \sum_{i'=1}^K Z_{t-1}(i) q_{i,i'} \phi(r_{t-1}^{\vec{}} | \mu_{i'}^{\vec{}}, \Sigma_{i'})}
\end{aligned}$$

Given the Observation Function Z above, we need to update the State Transition and Reward Functions to account for the uncertainty in the states. This gives us new representations for Q and R . We have a new state transition function $\tilde{Q}(r_{t-1}, r_t)$ and reward function $\tilde{R}(r_t, a)$. The state transition function is defined as follows:

$$\tilde{Q}(r_{t-1}^{\vec{}}, r_t^{\vec{}}) = P(r_{t-1}^{\vec{}} | r_{t-2}^{\vec{}}) \quad (3.19)$$

The updated reward function $\tilde{R}(r_{t-1}^{\vec{}}, a)$ is given by

$$\begin{aligned}
\tilde{R}(r_{t-1}, a) &= \sum_{i=1}^K Z_t(i) R_t(i, a) \\
&= \sum_{i=1}^K Z_t(i) E[(1 + r_f + \sum_{i'=1}^K \sum_{j=1}^N (r_{j,t}(i, i') - r_f) w_{j,t} I_{t+1}(i, i'))] \\
&= 1 + r_f + \sum_{i=1}^K \sum_{i'=1}^K Z_t(i) (\sum_{j=1}^N (r_{j,t}(i, i') - r_f) w_{j,t} I_{t+1}(i, i')) \quad (3.20)
\end{aligned}$$

3.4 Lockup Model

Unfortunately many Hedge Funds have Lockup Restrictions that make rebalancing difficult and restrict the set of available actions. For the purpose of this work we introduce three new parameters to the fund specification: a Lockup Length vector (L_j), a Minimum Allocation parameter (M_j) and a Notice Period constant (Δ). We define the Lockup as a "No-Trade" region following an initial allocation to a fund and the Notice Period as the length of time required before an action can effect the system. In the model, once a fund has exited a Lockup it can no longer re-enter the Lockup. Furthermore, we define the Minimum Allocation as the minimum amount required to initiate an investment in a fund. We generalize our definition of each risky asset f_j as follows:

$$f_j = (L_j, M_j, \Delta, \vec{\mu}_j, \vec{\sigma}_j) \quad (3.21)$$

The scalar integer quantity L_j represents the lockup duration in periods for fund j . The addition of a Lockup Parameter expands the scope and complexity of the Multi-Period Asset Allocation Problem and requires State-Space Augmentation techniques to be modeled effectively. For the purpose of the model we make the assumption that Notice Periods are homogeneous across all of the funds which offers a convenient transformation from the K -Regime model to the Aggregate Time Representation, we suspend further discussion to section 4.1.3.

3.4.1 Augmented State Space

Recall, as before each state in S is in one of K regimes at each time $t \leq T$ as follows: $S_t \in \{1, \dots, K\}$, in addition we add three parameters. The first \vec{l}_t a column vector of size N that represents the Lockup Time remaining in fund at time t , each \vec{l}_t is in the set Λ of possible lockup combinations. Each row j of \vec{l}_t is defined as follows:

$$l_{j,t} \in \{0, 1, \dots, L_j\}$$

The second parameter is the aggregate wealth at time t , X_t . Finally, we add the weights \vec{w}_t . The new state variable S^L at time t is given by:

$$S_t^L = \begin{pmatrix} \vec{l}_t \\ S_t \\ \vec{w}_t \\ X_t \end{pmatrix} \quad (3.22)$$

For the purpose of end period liquidity at time T we enforce the condition that $l_{j,T} \in \{0, L_j\}$; an $l_{j,T}$ value of 0 indicates that fund has successfully exited the lockup period prior to time T , similarly, a value of L_j indicates that the fund received no allocation prior to time T . In addition to providing sound economic justification, the condition limits the number of plausible paths thereby limiting the size of the State Space. The choice of Admissible State pair (S_t^L, S_{t+1}^L) constrains the weights \vec{w}_t at time t .

3.4.2 Lockup Dynamics

The System Dynamics are modified by the addition of the Lockup vector into the State Space specification. Unlike the Regime Transition mechanics the choice of action A_t effects the Lockup State Transitions \vec{l}_t . If the Lockup entry corresponding to fund j at time $t - 1$ is equal to L_j and there is no allocation to fund j , then the j th entry of the Lockup vector at time t remains equal to L_j . If there is an allocation that meets the Minimum Balance Constraint at time t then the process will move sequentially from L_j to 0 from t to $t + L_j$; during this period the Admissible Action Set is constrained so that no adjustment can be made to fund j . Once the fund passes through the Lockup Process then the allocations can be modified freely.

$$l_{j,t} = \begin{cases} L_j & \text{if } l_{j,t-1} = L_j, w_{j,t-1} = 0 \\ L_j - 1 & \text{if } l_{j,t-1} = L_j, M_j < w_{j,t-1}X_{t-1} \\ l_{j,t-1} - 1 & \text{if } 0 < l_{j,t-1} < L_j, 0 < w_{j,t-1} \\ 0 & \text{if } l_{j,t-1} = 0 \end{cases} \quad (3.23)$$

During the Lockup Period for fund j no action can be taken that directly changes the allocation to fund j , however, the allocation can be changed indirectly due to the Wealth Dynamics of the system. If the fund is in Lockup then the allocation to fund j at time $t + 1$ will be equal to the proportion of its total wealth allocation at time $t + 1$ as follows:

$$w_{j,t+1} = \frac{(1 + r_f + r_{j,t})w_{j,t}X_t}{X_{t+1}} \text{ if } 0 < l_{j,t} < L_j \quad (3.24)$$

As we see the relationship above forces a constraint on the action set. In order to accommodate this constraint and model this problem as a standard MDP we can separate the weights into Actionable and Inactionable components.

3.4.3 Splitting the Weights

We split the weights into two components. The first component (\vec{w}_t^l) is the Inactionable Component that represents the drift in allocations given that no action is taken. The second component (\vec{w}_t^a) represents the Actionable Component that can be directly manipulated.

$$\vec{w}_t = [\vec{w}_t^l, \vec{w}_t^a] \quad (3.25)$$

In order to determine the action A_t at time t , we need a method to select between the components of \vec{w}_t . To facilitate this purpose, we introduce a new parameter ($l_{j,t}^a$) that takes a value of 1 if fund j is in its Lock Period and 0 otherwise. Adding this parameter to the wealth dynamics allows us to switch between the Actionable and Inactionable weights as follows:

$$w_{j,t} = l_{j,t}^a w_{j,t}^l + (1 - l_{j,t}^a) w_{j,t}^a$$

$$l_{j,t}^a = \begin{cases} 1 & \text{if } 0 < l_{j,t} < L_j \\ 0 & \text{otherwise} \end{cases} \quad (3.26)$$

We combine each of these components to form the new problem formulation.

3.4.4 Lockup Formulation

We extend the POMDP model in section 3.3.1, the new model is $\tilde{D}_L = (S^L, A^L, Z, \tilde{Q}^L, \Omega, \tilde{R})$. Where the new state set S^L is defined:

$$S^L = \begin{cases} \vec{l} & : \vec{l} \in \Lambda, l_j = \{0, 1, \dots, L_j\} \\ S & : S \in \{1, 2, \dots, K\} \\ \vec{w}^l & : w_j \in [0, 1] \\ X & : X \in \mathfrak{R} \end{cases} \quad (3.27)$$

A^L is the new action set such that $A^L = \{\vec{w}_0^a, \vec{w}_1^a, \dots, \vec{w}_{T-1}^a\}$. Z is defined similarly to \tilde{D}_{NL} , $Z_t(i) = P(S_t = i | r_{t-1})$. Ω , likewise is defined as a finite set of observations of the process. $\tilde{Q}^L = (\tilde{Q}, \lambda, \varpi, \chi)$ corresponds to the new system dynamics, where Q is defined as in section 3.3.1, namely:

$$P(S_t = i' | S_{t-1} = i) = q_{i,i'}$$

and \tilde{Q} is given as in section 3.3.2 by

$$\tilde{Q} = P(r_t | r_{t-1})$$

λ defines the Lockup State \vec{l}_t transitions such that:

$$\begin{aligned} \vec{l}_t &:= \lambda(S_{t-1}, A_{t-1}) \\ \lambda(S_{t-1})_j &= \begin{cases} L_j & \text{if } l_{j,t-1} = L_j, w_{j,t-1} = 0 \\ L_j - 1 & \text{if } l_{j,t-1} = L_j, M_j < w_{j,t-1} X_{t-1} \\ l_{j,t-1} - 1 & \text{if } 0 < l_{j,t-1} < L_j, 0 < w_{j,t-1} \\ 0 & \text{if } l_{j,t-1} = 0 \end{cases} \end{aligned} \quad (3.28)$$

χ controls the change in wealth X_t :

$$\begin{aligned} X_t &:= \chi(S_t^L, S_{t-1}^L, A_{t-1}) \\ \chi(S_t^L, S_{t-1}^L) &= (1 + r_f + \sum_{j=1}^N r_{j,t-1} w_{j,t-1}) X_{t-1} \end{aligned} \quad (3.29)$$

ϖ effects the dynamics of the current weight vector \vec{w}_t as follows:

$$\begin{aligned} \vec{w}_t &:= \varpi(S_t^L, S_{t-1}^L, A_t) \\ \varpi(S_t^L, S_{t-1}^L, A_t)_j &= l_{j,t}^a w_{j,t}^l + (1 - l_{j,t}^a) w_{j,t}^a \\ w_{j,t}^l &= \frac{(1 + r_f + r_{j,t-1}) w_{j,t-1} X_{t-1}}{X_t} \\ l_{j,t}^a &= \begin{cases} 1 & \text{if } 0 < l_{j,t} < L_j \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (3.30)$$

The expected reward function is defined as in section 3.3.2 as

$$\tilde{R}(r_{t-1}, a) = \sum_{i=1}^K Z_t(i) R_t(i, a)$$

3.4.5 System Dynamics

We provide here a synopsis of the system in time order to help clarify state - action precedence in the system:

1. At the beginning of time t we have, $(S_{t-1}^L, A_{t-1}, X_t, r_{t-1}^{\vec{}})$ the components of S_t^L
2. from $r_{t-1}^{\vec{}}$ we calculate Z_t
3. from \vec{l}_t we calculate \vec{l}_t^a
4. from r_t and S_t we determine $A_t = w_t^a$
5. we calculate $\vec{w}_t := \varpi(S_t^L, S_{t-1}^L, A_t)$
6. we observe \vec{r}_t
7. from \vec{r}_t, \vec{w}_t we calculate r_t^P
8. r_t^P and X_t gives X_{t+1}
9. At the end of time t we have, $(S_t^L, A_t, X_{t+1}, \vec{r}_t, w_{t+1}^l)$ the components of S_{t+1}^L
10. increment t

System Dynamics

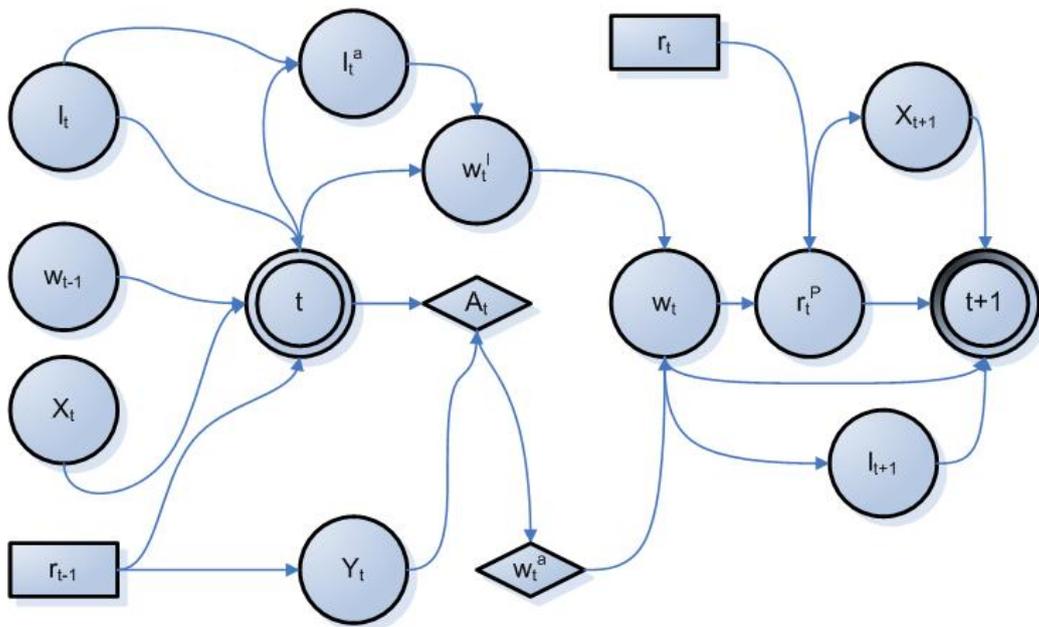


Figure 3.1: System Dynamics of the Lockup POMDP

Chapter 4

Algorithm

We begin with the following quantities:

1. Ω : A $T_\Omega \times N$ matrix of historical fund returns
2. \vec{L} : A Lockup Length vector of size N
3. \vec{M} : An Initial Allocation vector of size N
4. r_f : Risk Free Rate
5. X_0 : Initial wealth scalar quantity
6. α : Risk Aversion Constant
7. T : Max length of the finite time algorithm in discrete time periods.

4.1 Model Fitting

There are two distinct components in modeling the State Space. The first is the Regime Dynamics and the second is the Lockup Dynamics. The two

components can be modeled independently of one another because the choice of action A_t has no effect on the regime transitions.

4.1.1 Regime Dynamics

We fit a K-means Gaussian Mixture distribution using the Expectation Maximization (EM) algorithm (see appendix D) to the historical matrix Ω of returns in excess of the Risk Free Rate r_f . Assuming the number of historical datapoints is T_Ω , the size of Ω is $N \times T_\Omega$. The output of the EM Algorithm is a parameter vector:

$$\Theta_G = (\pi_1, \pi_2, \dots, \pi_K, \mu_1, \mu_2, \dots, \mu_K, \Sigma_1, \Sigma_2, \dots, \Sigma_K)$$

Where π_i represents the probability of being in fund i at any given t_Ω . We also obtain a $K \times T_\Omega$ matrix Y where $Y_{i,t_\Omega} = P(S_{t_\Omega} = i | \Theta_G)$ which represents the probability that the return at time t_Ω came from regime i . We use the approach outlined in section 3.3.1 for the calculation of $Z_t(i)$ to calculate Y_{i,t_Ω} , i.e $Y(i, t_\Omega) = P(S_{t_\Omega} = i | r_{t_\Omega}^{\rightarrow}, Y_{t_\Omega-1}^{\rightarrow}, \Theta_G)$

We can use the proportions data from Y to obtain the transition matrix Q where $q_{i,i'}$ represents the one step probability of transitioning from state i to i' ([30]) in a Time Homogeneous Markov Chain. Given the matrix Y we can minimize the squared errors under the assumption that for two given periods t_Ω and $t_\Omega + 1$ the probability of being in state i' at time $t_\Omega + 1$ is equal to the probability of being in state i at time t_Ω and transitioning to state i' . We can formalize this as follows:

$$\min_Q \sum_{t_\Omega=0}^{T_\Omega} (\vec{Y}_{t_\Omega} - Y_{t_\Omega-1}^T Q)^T (\vec{Y}_{t_\Omega} - Y_{t_\Omega-1}^T Q) \quad (4.1)$$

$$\sum_{i=1}^K q_{i,i'} = 1 \quad \forall i$$

We end up with a parameter set Θ_G and a Transition Matrix Q . We can use the values above to initialize the Baum-Welch (Forward Backward) algorithm as described in appendix D. Additionally, we can use a penalized likelihood function as in the Bayesian Information Criterion (BIC) to rank different multi-regime models as in McLachlan ([19]) and Frey ([15]).

4.1.2 Lockup Dynamics

The most significant issue in modeling this problem is the size of the Lockup Component in the State Space. the number of possible states \vec{l}_j is:

$$\prod_{\{j \in \vec{L} | 0 < L_j < T\}} (L_j + 1) \leq (T + 1)^N$$

Fortunately, as shown above, we benefit from the fact that only funds with a lockup increase the size of the lockup components in the state space. We can add any number of non-lock funds to the model without increasing the size of this component in the model. In most practical applications of this problem we are concerned with a relatively small number of funds. In a larger scale model we can perform this optimization at the asset allocation level.

Let P be the set of all admissible paths for N funds from time 0 to T , and $p \in P$ be one admissible path in the set such that $p = \{\vec{l}_0, \vec{l}_1, \dots, \vec{l}_T\}$. We

can create a Multinomial Decision Tree or Multinomial Path Tree (MPT) (see appendix E) and a matrix PM which enumerates each of the admissible path sequences. The number of possible paths is:

$$\bar{P} = \prod_{\{j \in \bar{L} | 0 < L_j < T\}} (T - L_j + 2) \leq (T + 1)^N$$

The size of matrix PM is:

$$\prod_{\{j \in \bar{L} | 0 < L_j < T\}} (T - L_j + 2)(T + 1) \leq (T + 1)^{N+1}$$

We see that PM is a $\bar{P} \times (T + 1)$ matrix where each cell represents an index to a Lockup State \vec{l}_j . This representation allows us to approximate the solution by conditioning on a given path.

4.1.3 Notice Periods and Time Aggregation

In the previous section we see that the Lockup State Model increases exponentially in the length of time T and number of funds N . We set a hard upper bound on the number of paths $(T + 1)^N$ however for large enough values of T and N the problem becomes intractable. If we reduce the rebalancing frequency we can significantly reduce the number of paths and make the problem more manageable; this transformation is similar to modeling the problem with a Notice Period Δ . The transformation is accomplished by aggregating Δ number of time periods together thereby reducing the upper bound on the number of paths:

$$\bar{P}_\Delta = \prod_{\{j \in \bar{L} | 0 < \frac{L_j}{\Delta} < \frac{T}{\Delta}\}} \left(\frac{T}{\Delta} - \lceil \frac{L_j}{\Delta} \rceil + 2 \right) \leq (\lfloor \frac{T}{\Delta} \rfloor + 1)^N$$

The transformation is accomplished by enumerating all possible regime sequences that can occur in the Δ time periods and expanding the K -Mean Gaussian Mixture Distribution to a κ Distribution where $\kappa = K^\Delta$. Let

$$S_\tau^\Delta = \{S_t, S_{t+1}, \dots, S_{t+\Delta}\}$$

represents the new regime state S^Δ at time $\tau = \Delta t$. We can expand the Augmented State Space to accommodate the new formulation as follows:

$$S_\tau^L = \begin{pmatrix} \vec{l}_\tau \\ S_\tau^\Delta \\ X_\tau \end{pmatrix} \quad (4.2)$$

The Updated Return Distribution

The Return Distribution $r_\tau^{\vec{\Delta}}$ can be modeled by a Multivariate Gaussian Mixture of Mixtures as follows:

$$f(r_\tau^{\vec{\Delta}}) = \sum_{\iota=1}^{\kappa} P(S_\tau^\Delta = \iota) \left[\sum_{t=\Delta\tau}^{\Delta\tau+\Delta-1} f(r_\tau^{\vec{\Delta}} | S_\tau^\Delta = \iota) \right] \quad (4.3)$$

where the component probability $P(S_\tau^\Delta = \iota)$ is given by:

$$\begin{aligned}
P(S_\tau^\Delta = \iota) &= \sum_{\{i_1, i_2, \dots, i_{\Delta-1}\}} P(S_t = i_1, S_{t+1} = i_2, \dots, S_{t+\Delta-1} = i_{\Delta-1}) I_{\iota, \tau}^\Delta(i_1, i_2, \dots, i_{\Delta-1}) = \\
&= \sum_{\{i_1, i_2, \dots, i_{\Delta-1}\}} P(S_t = i_1) P(S_{t+1} = i_2 | S_t = i_1) \dots \\
&\dots P(S_{t+\Delta-1} = i_{\Delta-1} | S_{t+\Delta-2} = i_{\Delta-2}) I_{\iota, \tau}^\Delta(i_1, i_2, \dots, i_{\Delta-1}) = \\
&= \sum_{\{i_1, i_2, \dots, i_{\Delta-1}\}} \pi_{i_1} q_{i_1, i_2} \dots q_{i_{\Delta-2}, i_{\Delta-1}} i_{\Delta-1} I_{\iota, \tau}^\Delta(i_1, i_2, \dots, i_{\Delta-1})
\end{aligned} \tag{4.4}$$

and the component densities $\sum_{t=\Delta\tau}^{\Delta\tau+\Delta-1} f(\vec{r}_t | S_t^\Delta = \iota)$ are given by:

$$\sum_{t=\Delta\tau}^{\Delta\tau+\Delta-1} f(\vec{r}_t | S_t^\Delta = \iota) = \sum_{t=\Delta\tau}^{\Delta\tau+\Delta-1} \sum_{i \in \{i_1, i_2, \dots, i_{\Delta-1}\}} f(\vec{r}_t | S_t = i) I_{\iota, \tau}^\Delta(i_1, i_2, \dots, i_{\Delta-1}) \tag{4.5}$$

where $I_{\iota, \tau}^\Delta(i_1, i_2, \dots, i_{\Delta-1})$ is an indicator function as follows:

$$I_{\iota, \tau}^\Delta(i_1, i_2, \dots, i_{\Delta-1}) = \begin{pmatrix} 1 & \text{if } S_t = i_1, S_{t+1} = i_2, \dots, S_{t+\Delta-1} = i_{\Delta-1} \\ 0 & \text{otherwise} \end{pmatrix} \tag{4.6}$$

Transformation from \mathbf{t} to τ

After reducing the total time $T_\Delta = \lfloor \frac{T}{\Delta} \rfloor$ and individual lockup $L_j^\Delta = \lceil \frac{L_j}{\Delta} \rceil$ we can apply the same techniques as in the original (non aggregate) case. In the case that we consider Δ to be the result of a Notice Period Restriction on the

problem then we need one minor adjustment. A Notice Period Restriction lags the action at time τ of the model; the decision determining A_{t_Δ} is made at time $\tau - 1$. Thus, we can only use the returns up to time $t - \Delta$ in the original model to determine appropriate action at τ .

Reducing the rebalancing frequency penalizes the optimality by further restricting the action space such that $A_\tau \subseteq A_t$ therefore $Z_\Delta \geq Z^*$.

Transformation from τ to t

The transformation indicated above has the effect of removing admissible paths in the neighborhood of path $p = (\Delta l_0^\Delta, \Delta l_1^\Delta, \dots, \Delta l_{T_\Delta}^\Delta)$. Let ϵ_Δ be defined as follows:

$$\epsilon_\Delta = \max_j \left(\lceil \frac{L_j}{\Delta} \rceil - \frac{L_j}{\Delta} \right) \quad (4.7)$$

We need to search over all feasible paths in the neighborhood of path p . Let $l_{j,t}^p$ be a $l_{j,t} \in p$, a path would only contain lockup states that have the following property:

$$\lfloor l_{j,t}^p - \Delta(1 - \epsilon_\Delta) \rfloor < l_{j,t} < \lceil l_{j,t}^p + \Delta(1 + \epsilon_\Delta) \rceil \quad (4.8)$$

4.2 No Lockup Lower Bound

The Lockup Formulation is a generalization of the No Lockup case. It is thus possible to provide a lower bound (denoted by Z_L) on the optimal solution

denoted Z^* by solving the No Lockup Problem (see 3.3.1). We can use this result to assess the quality of our approximations in the following sections.

$$\begin{aligned}
Z_L &:= \min_w \sum_{t=0}^{T-1} \ln E[(1 + r_f + \mu_{A_t})^\alpha] \\
&\quad \text{s.t.} \\
&\quad 0 \leq w_{j,t} \forall j, t \\
&\quad \sum_{j=1}^N w_{j,t} \leq 1 \forall j, t
\end{aligned} \tag{4.9}$$

The effect of the Lockups in the model restrict the action space A . For any optimal solution to Z^* , w^* would be feasible to the No Lockup case because $A^* \subseteq A$ therefore:

$$Z^* \geq Z_L$$

4.3 Path Enumeration Approximation

For a given MPT or PM matrix it is possible to iterate over all admissible paths in the set and establish an approximation to the Optimal Policy, we call this the Path Enumeration Approximation (PEA) and it is identical to the Certainty Equivalent approach. For a given path $p \in P$ the Lockup State transitions from 0 to T are deterministic and therefore \vec{l}_t and \vec{l}_t^a are constants over each p . Additionally, we can define an $K \times T$ matrix IA where its j, t -th component is given by:

$$IA_{j,t} = \begin{pmatrix} 1 & \text{if an initial allocation to fund } j \text{ was made prior or at time } t \\ 0 & \text{otherwise} \end{pmatrix} \quad (4.10)$$

We can expand on the problem specified in section 3 in order to account for the constraints imposed by the fund Lockup Parameters as follows:

$$\begin{aligned} Z_A(p) &:= \min_w \sum_{t=0}^{T-1} \ln E[(1 + r_f + \mu_{A_t})^\alpha] \\ &\quad s.t. \\ E[X_{t+1}] &= 1 + r_f + \sum_{i=1}^K \sum_{i'=1}^K \sum_{j=1}^N (Y_{i,0} Q_{i,i'}^{t+1} \mu_{j,i'} w_{j,t}) E[X_t] \\ l_{j,t}^a w_{j,t+1} &= \frac{w_{j,t}(1 + r_f + \mu_{A_{j,t}})}{E[X_t]} l_{j,t}^a \quad \forall j, t < T \\ w_{j,t} &\leq IA_{j,t} \quad \forall j, t < T \end{aligned} \quad (4.11)$$

$$w_{j,t}(IA_{j,t} - IA_{j,t-1})E[X_t] \geq M_j(IA_{j,t} - IA_{j,t-1}) \quad \forall j, 1 \leq t < T$$

$$w_{j,t}IA_{j,t}E[X_t] \geq M_jIA_{j,t} \quad \forall j, t = 0$$

$$0 \leq w_{j,t} \quad \forall j, t$$

$$\sum_{j=1}^N w_{j,t} \leq 1 \quad \forall t$$

Let $Z_A(p)$ be the solution to the above problem for all $p \in P$, the best approximation to Z^* is given by:

$$Z_A := \min_{p \in P} Z_A(p)$$

The disadvantage to this approach is that we solve the stochastic problem as if it were deterministic by forcing the selection of p at time 0 and not allowing the information that becomes available at time t to determine choice of the next $l_{t+1}^{\vec{}}$. The limitation is that information about the current state S_t for $t > 0$ becomes available at the end of $t-1$. Therefore the best approximation for \vec{Y}_t is given by:

$$\widehat{\vec{Y}}_t \approx \vec{Y}_0^T Q^t \approx \lim_{t \rightarrow \infty} Q^t = \pi$$

We see that the estimate $\widehat{\vec{Y}}_t$ converges to the Steady State Mixing Probabilities for that Gaussian Mixture Distribution as time horizon approaches ∞ . For example, if the Regime Transition Dynamics would favor another path at time $t > 0$ then the choice of \vec{w}_0 may not leave enough cash available in the Risk Free Rate to take advantage of that opportunity. Therefore we now have that:

$$Z_A \geq Z^* \geq Z_L$$

Where Z_L represents a global lower bound on the lockup problem which is infeasible to the lockup case. Z_A represents the optimal expected utility if we restrict our choice of feasible path to time 0. We are able to refine that bound further.

4.4 Bounded Approximation

In the previous case the expectation operator gives us a forecast of the regimes probabilities Y given an initial forecast on \vec{Y}_0 . We can, however, refine our lower bound on Z^* by including Y in the objective function for each given p in the example above as follows:

$$\begin{aligned}
Z_B(p) &= \min_{w,Y} \sum_{t=0}^{T-1} \ln E[(1 + r_f + \mu_{A_t})^\alpha] \\
&\quad s.t. \\
E[X_{t+1}] &= 1 + r_f + \sum_{i=1}^K \sum_{j=1}^N (Y_{i,t} \mu_{j,i} w_{j,t}) E[X_t] \\
l_{j,t}^a w_{j,t+1} &= \frac{w_{j,t} (1 + r_f + \mu_{A_{j,t}})}{E[X_t]} l_{j,t}^a \quad \forall j, t < T \\
w_{j,t} &\leq IA_{j,t} \quad \forall j, t < T \\
w_{j,t} (IA_{j,t} - IA_{j,t-1}) E[X_t] &\geq M_j (IA_{j,t} - IA_{j,t-1}) \quad \forall j, 1 \leq t < T \\
w_{j,t} IA_{j,t} E[X_t] &\geq M_j IA_{j,t} \quad \forall j, t = 0 \\
0 &\leq w_{j,t} \quad \forall j, t \\
\sum_{j=1}^N w_{j,t} &\leq 1 \quad \forall t \\
\sum_{i=1}^K Y_{i,t} &= 1 \quad \forall t
\end{aligned} \tag{4.12}$$

Solving the above problem creates the "best case" combination of regime transitions and actions for a given path. In effect this transfers control of the Regime Switching Process to the set of admissible actions removing the

randomness in the regime transitions. For any given path p the expected case $Z_A(p)$ regime transition sequence will always yield a higher expected utility than the "best case" $Z_B(p)$ transition sequence. We can update the approximation as shown below:

$$Z_A \geq Z^* \geq Z_B := \min_p Z_B(p)$$

We can use this approximation to create a candidate set $P^* \subseteq P$ of magnitude \bar{P}^* potential paths.

4.4.1 Path Elimination

We can use the results from the Bounded Path Approximation to filter P . Under Z_A we make the decision at time 0 about which path we are going to choose and do not deviate from that choice regardless of the information available at time t . Under Z_B we assume that all information about $\{S_0, S_1, \dots, S_T\}$ is available at time 0 such that:

$$\begin{aligned} \min_{p \in P} Z_A(p) &\geq Z^* \\ \min_{p \in P} Z_B(p) &\leq Z^* \end{aligned} \tag{4.13}$$

The second inequality implies that the solution to Z_B is infeasible, this is because Z_B requires that the regime transition $\{S_0, S_1, \dots, S_T\}$ be fully specified at time 0.

For any two paths $\{p_1, p_2\} \in P$, if $Z_A(1) \leq Z_B(2)$ then $U(p_1) \leq U(p_2)$.

We can generalize this as follows:

$$P^* = \{ \forall p \mid Z_B(p) \leq Z_A \}$$

We can use the candidate set P^* and adaptive decision making to arrive at an Optimal Discretized Policy for Z^*

4.5 Adaptive Simulation

We can use simulation and an Adaptive Learning Algorithm called the Pursuit Algorithm [49] to arrive at an Optimal Policy for the problem. We assume the reader is familiar with the algorithm however, for more information on the Pursuit Algorithm see Appendix F. The automaton is defined by (A, Q, R, T) [49] and the environment is defined by (A, R, D) , these quantities are specified in Appendix F. In order to cast the problem in a format for the learning automaton we need a finite discrete representation of the action space.

4.5.1 Discretizing the Action Space

A fund can be in one of three distinct states with respect to its lockup:

- Unallocated: Prior to an initial allocation the weight with respect to that fund $w_{j,t}$ must be equal to 0.
- Locked: After the initial allocation is made the weight can only change with respect to the wealth dynamics no direct intervention is possible.

- Unlocked: After the end of the Lockup Period all changes made to the weight of that fund are admissible subject to $w_{j,t} \in [0, 1]$.

Given that the action space is unrestricted with respect to fund j in the unlocked fund state and that this state is persistent with regards to the lockup dynamics then any decision made during this state does not need to be made prior to entering this state. Therefore, the key questions are: when to enter fund j and what the initial allocation to fund j should be. We see that if we allocated all of the wealth at time 0 then a situation can arise at time t where an allocation to a new fund may have greater utility, if we have not left enough fluid capital then we would be unable to take advantage of that opportunity at that time. However, leaving too much in the Risk Free Asset may yield a lower utility so the choice of initial allocation time and amount is very important.

We can create an Initial Weight vector as follows:

$$w_j^{IA} \in [0, 1] \forall j$$

The combination of $(w^{\vec{IA}}, p)$ at time 0 allows us to select a weight vector \vec{w}_0 and lockup vector \vec{l}_1 . Though there are a finite number of paths $p \in P^*$, the weight vector $w^{\vec{IA}}$ is continuous so in order to represent the action space in a finite, discrete way in which the space is uniformly discretized into \bar{A} distinct actions such that each row can take on $\frac{\bar{A}}{N}$ distinct actions. We can create an \bar{A} by \bar{P}^* matrix of actions A^ϕ .

4.5.2 Sampling the Discrete Action Space

The first step of the Pursuit Algorithm is to seed the probability vector:

$$p^\phi(w^{\vec{I}A}, p) = \frac{1}{\hat{r}} \forall i \in A^\phi$$

where \hat{r} is the finite number of admissible actions and vector \hat{d}^ϕ containing the estimates of feedback from the system. We can use a modification to the (PEA) technique (4.3) which we will call the Simulated Path Enumeration Algorithm (SPEA) with respect to all $(w^{\vec{I}A}, p)$ for each $p \in P^*$ by solving the non-linear optimization below:

$$\begin{aligned} \hat{d}^\phi(w^{\vec{I}A}, p) &:= \min_w \sum_{t=0}^{T-1} \ln E[(1 + r_f + \mu_{A_t})^\alpha] \\ &\quad s.t. \\ l_{j,t}^a w_{j,t+1} &= \frac{w_{j,t}(1 + \mu_{A_{j,t}})}{w_t'(1 + \mu_{A_t})} l_{j,t}^a \forall j, t < T \\ w_{j,t} &\leq I a_{j,t} \forall j, t < T \\ w_{j,t}(I A_{j,t} - I A_{j,t-1}) E[X_t] &\geq M_j(I A_{j,t} - I A_{j,t-1}) \forall j, t \\ w_{j,t}(I A_{j,t} - I A_{j,t-1}) &= w_j^{IA} \forall j, t \\ 0 &\leq w_{j,t} \forall j, t \\ \sum_{j=1}^N w_{j,t} &\leq 1 \forall t \end{aligned} \tag{4.14}$$

Certain combinations of $w^{\vec{I}A}$ and p will be infeasible; in these cases we use the heuristic that the estimate corresponding to these values is equal to 0. We can then use the estimates of \hat{d}^ϕ to create a more robust probability

matrix p^ϕ as follows:

$$p_{\hat{j}, \hat{p}}^\phi = \frac{|\hat{d}_{\hat{j}, \hat{p}}^\phi|}{\sum_{m=1}^{\bar{A}} \sum_{n=1}^{\bar{P}^*} |\hat{d}_{m,n}^\phi|} \quad (4.15)$$

$$\sum_{\hat{j}=1}^{\bar{A}} \sum_{\hat{p}=1}^{\bar{P}^*} p_{\hat{j}, \hat{p}}^\phi = 1$$

It is apparent that the second equality follows logically from the first. We can now sample the probability matrix by selecting each action \hat{j}, \hat{p} with probability $p_{\hat{j}, \hat{p}}^\phi$.

4.5.3 Simulating the System Reaction

The Pursuit Algorithm works through sampling the actions and getting feedback from the system. We can use a similar approach to simulate a $N \times T$ multivariate time series matrix r^G from a Gaussian Mixture Model using parameters Θ_G and the Transition Matrix Q . We introduce two new quantities

- $\Theta_G(\pi^G)$: The parameter vector of the Gaussian Mixture with the Steady State Proportions π replaced by a product of the current Y_t^G vector and the one step transition probabilities.
- Y^G : a $K \times T$ matrix of probabilities that the return at time t came from regime i

Algorithm 4.5.1: SIMULATEMIXTURETIMESERIES($\Theta_G, \vec{Y}_0, Q, T, N$)

```

local  $r^G, Y^G, \pi^G$  for  $t \leftarrow 1$  to  $T$ 
  do
     $\left\{ \begin{array}{l} \vec{Y}_t^G \leftarrow [Y_{t-1}^G]' * Q \end{array} \right.$ 
for  $t \leftarrow 0$  to  $T - 1$ 
  do
     $\left\{ \begin{array}{l} \pi^G \leftarrow [y_t^G]' * Q \\ r_t^G \leftarrow \text{SIMULATEMULTIVARIATEMIXTURERETURNS}(\pi^G, \Theta_G) \\ Y_{t+1}^G \leftarrow \text{CALCULATEREGIMEPROBABILITIES}(r_t^G, \Theta_G) \end{array} \right.$ 
return  $(r^G, Y^G)$ 

```

In the pseudocode above SimulateMultivariateMixtureReturns draws an vector of returns of size N from a the Mixture Distributions using the parameter vector $\Theta_G(\pi^G)$ and a function CalculateRegimeProbabilities which returns a vector of size K where:

$$Y_{i,t} = P(S_t = i | r_t^G, \Theta_G(\pi_t^G))$$

Once we have the proportion Y^G and return r^G matrices we can use them in conjunction with the action A^{IA} and initial wealth X_0 to produce an $N \times T$ matrix of weights w^G , wealth vector X^G of size $T + 1$ and utility value Z^G as follows:

Algorithm 4.5.2: GETSIMULATEDWEIGHTS($Y^G, r^G, w^{\vec{IA}}, p, X_0, Q, \Theta_G, \vec{M}, r_f, T, \alpha, p^{IA}, A^{IA}, M$)

local $w^G, w^{SPEA}, X^G, IA, l^a, p^{SPEA}, Y^{SPEA}, p^{resample} X_0^G \leftarrow X_0$

for $t \leftarrow 0$ **to** $T - 1$

do

$$\left\{ \begin{array}{l} p^{SPEA} \leftarrow \{p_t, p_{t+1}, \dots, p_T\} \\ Y^{SPEA} \leftarrow \{\vec{Y}_t, \vec{Y}_{t+1}, \dots, \vec{Y}_T\} \\ (IA, l^a, w^{\vec{IA}}, \vec{M}) \leftarrow \text{GETPATHVARIABLES}(p^{SPEA}) \\ \textbf{while not} \text{ PATHISFEASIBLE}(p^{SPEA}, \vec{M}, w^{\vec{IA}}, IA, l^a, X) \\ \quad \textbf{do} \\ \quad \left\{ \begin{array}{l} p \leftarrow \text{SAMPLEACTIONSPACE}(p^{IA}, A^{IA}, p, t, MPT) \\ p^{SPEA} \leftarrow \{p_t, p_{t+1}, \dots, p_T\} \\ (IA, l^a, w^{\vec{IA}}, \vec{M}) \leftarrow \text{GETPATHVARIABLES}(p^{SPEA}) \\ w^{SPEA} \leftarrow \text{SPEA}(Y^{SPEA}, \Theta_G, X^G, IA, l^a, vecw^{IA}, \vec{M}, T - t) \\ w_t^{\vec{G}} \leftarrow w_0^{SPEA} \\ X_{t+1}^G \leftarrow X_t^G * (1 + r_f + w_t^{\vec{G}'} * r_t^{\vec{G}}) \end{array} \right. \\ Z^G \leftarrow \frac{(X_T^G)^\alpha}{\alpha} \end{array} \right.$$

return (w^G, X^G, Z^G)

In the algorithm above we introduce three new functions:

- **GetPathVariables:** this function updates the IA, l^a matrices and initial allocation vector $w^{\vec{IA}}$ function of the lockup states remaining in path p . We assume that funds allocated to prior to time t are now initially allocated to at time $T - t$ and adjust the minimum vector M and initial allocation vector w^{IA} accordingly.

- `IsPathFeasible`: this function checks to see whether the allocation made at time t is still feasible. If so it returns true, otherwise it returns false.
- `SampleActionSpace`: this function resamples the potential paths remaining at time t given that choices of lockup state transitions from 0 to $t - 1$.

Once these adjustments are made, we can implement the Pursuit Algorithm ([49]) to give an ϵ -optimal choice of action A^{IA} to implement at time 0. At times $t > 0$ we can reimplement the algorithm assuming the choice of allocations and lockup states prior to time t , wealth at time t , regime probability \vec{Y}_t and distribution parameters Θ_G, Q .

Chapter 5

Out-of-Sample Back-Test

Results

We run two tests on the model: the first is a small example that consists of two Hedge Fund strategy indices and a broad market index ($N = 3$); the second is a larger scale model consisting of multiple Hedge Fund strategy indices ($N = 6$) and the S&P 500 Broad Market Index. In the first example we fit a ($K = 3$) Gaussian Mixture Distribution and examine the results over the following four time periods ($T = 4$). In the second example we fit a ($K = 2$) mixture and examine the results over the following two years of monthly periodic data ($T = 24$). We test the results against a number of Fund of Hedge Fund and Broad Market Indices as well as two Markowitz optimizations. The Markowitz optimizations proxy two strategies, one Risk Seeking and the other Risk Adverse. We assume the following:

- We have one million dollars to initially invest $X_0 = 1000000$

Table 5.1: HFRX Index Statistics (Jan 98 - Dec 07), ESA

Index	Mean	Std	Skew	Kurtosis	JB Stat	P-Value
EH	0.0093	0.0222	0.5837	5.3593	34.6451	< 0.001
EMN	0.0022	0.0107	-0.1468	3.2321	0.500	0.7003
SP	0.0038	0.0429	-0.5056	3.6575	7.2737	0.0299

- Each fund requires 250K minimum investment $M_j = 250000$
- The risk free rate is 1% annualized $r_f = 0.0842\%$

5.1 Equity Strategy Allocation (ESA)

In order to demonstrate the approach we use a three regime model, labeling one 'Regime 1', the second 'Regime 2' and the last 'Regime 3'. A multivariate Gaussian Mixture Model is fit using the Expectation Maximization (see appendix D) to the monthly excess returns of two Investible Hedge Fund Research (HFRX) Strategy Indices (see appendix B): HFRX Equity Hedge Index (EH), HFRX Equity Market Neutral Index (EMN) and the S&P 500 Index (SP). The returns are fit from HFRX index inception to the end of 2007 (Jan 1998 - Dec 2007).

Table 5.2: Estimated ESA Correlation Matrix

Index	EH	EMN	SP
EH	1.0000	0.1837	0.0018
EMN	0.1837	1.0000	0.0318
SP	0.0018	0.0318	1.0000

Table 5.3: Mixture Distribution Parameters

	Regime 1, $\hat{\pi}_1 = 0.3670$		Regime 2, $\hat{\pi}_2 = 0.2025$		Regime 3, $\hat{\pi}_3 = 0.4305$	
Index	$\hat{\mu}_1$	$\hat{\sigma}_1$	$\hat{\mu}_2$	$\hat{\sigma}_2$	$\hat{\mu}_3$	$\hat{\sigma}_3$
EH	0.0063	0.0244	0.0109	0.0209	0.0098	0.0216
EMN	0.0000	0.0117	0.0033	0.0099	0.0026	0.0104
SP	0.0142	0.0348	-0.0145	0.0500	0.0117	0.0356

5.1.1 Fitting the Multivariate Gaussian Mixture

A Jarque-Bera Test ([12]) run on the returns of the HFRX Indices indicates that the index returns deviate significantly from normal (see table above). Jarque-Bera Test is a two-sided goodness of fit test.

A Gaussian Mixture Distribution of ($K = 3$) components is fit to the returns using the Expectation Maximization (EM) algorithm (see appendix D). The distribution parameters are as follows:

Given the Fitted Parameters $\Theta_G = (\pi_1, \pi_2, \pi_3, \mu_1, \mu_2, \mu_3, \Sigma_1, \Sigma_2, \Sigma_3)$ and the regime probability matrix Y , $P(S_{t_\Omega} = i | \Theta_G, r_{t_\Omega}^{\vec{r}})$ we can use it to produce the following transition matrix Q :

Table 5.4: Estimated ESA Transition Probabilities

	Regime 1	Regime 2	Regime 3
Regime 1	0.4045	0.1556	0.4399
Regime 2	0.2967	0.3515	0.3518
Regime 3	0.1058	0.4027	0.4914

We can verify that $(\lim_{t \rightarrow \infty} Q^t)_{i,i'} = \pi_{i'}$

5.1.2 Lockup State Modeling

The next step is to model the Lockup State Space given the vector \vec{L} and build the PM matrix and MPT. We see that given the Lockup vector \vec{L} we are able to model 12 distinct paths each corresponding to a different sequence of Lockup State transitions \vec{l}_t . The Figure 5.1 below depicts MPT for the three fund, four period model with L_1 equal to three periods, L_2 equal to two periods and L_3 equal to 0.

5.1.3 Single Period Mixture Portfolio Modeling

We run Single Period Optimizations (shown below), the CRRA utility results have a parameter α that adjusts the Risk Aversion level; the more negative the α parameter the larger the degree of Risk Aversion [9]. We can fix the regime i and run the Single Period Optimization over different values of alpha to examine the sensitivity to alpha and the mixing component π :

For small magnitude α the results are largely risk seeking however, the

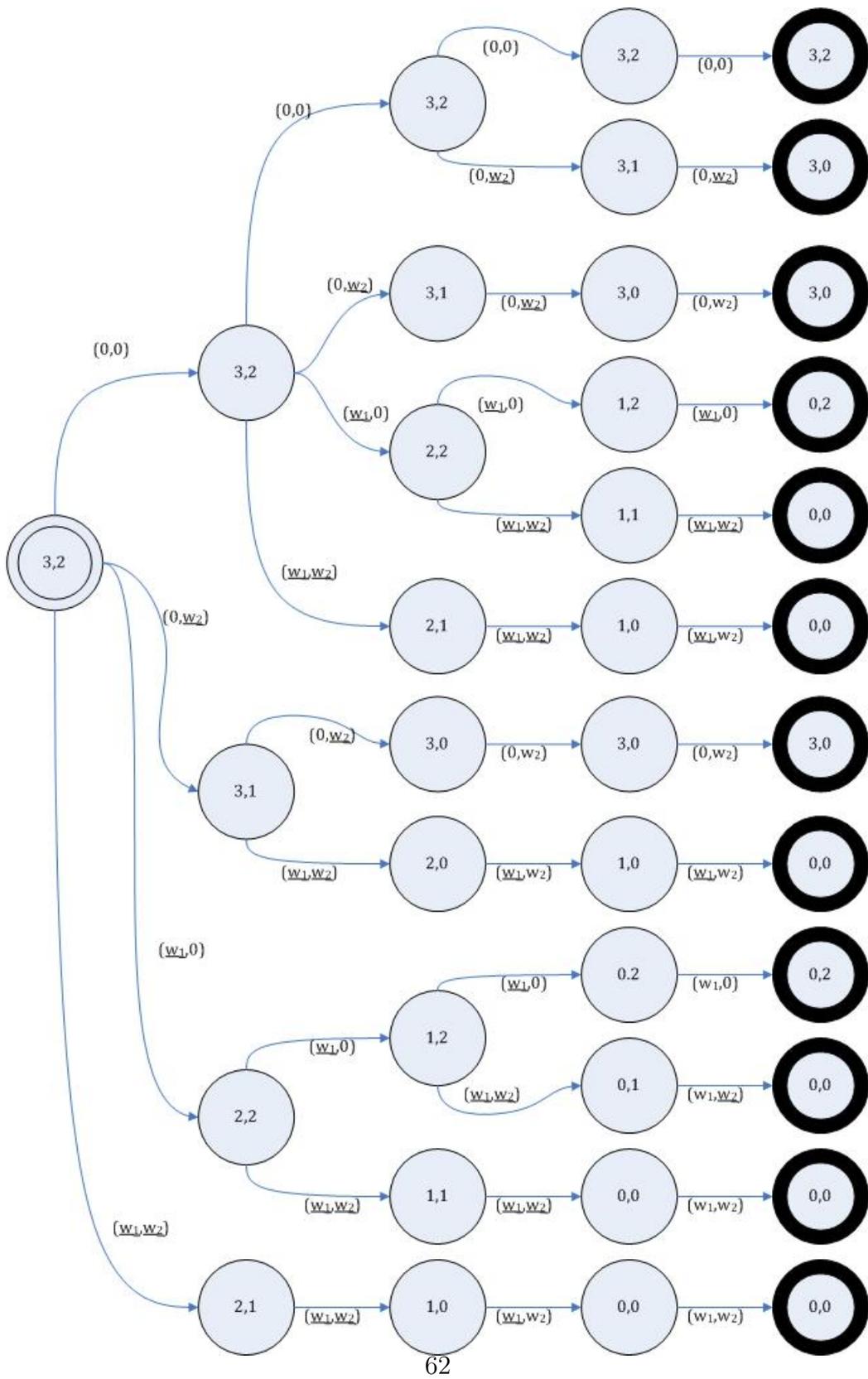


Figure 5.1: Decision Tree for ESA Model

Table 5.5: Optimal ESA CRRA Utility Portfolio Parameters

Regime	α	EH	EMN	SP	r_f	Mean	Std	Skew	Kurtosis
Regime 1	-1	0.5213	0.0000	0.4787	0.0000	0.0086	0.0219	-0.1552	3.29567
	-10	0.7337	0.0000	0.2663	0.0000	0.0086	0.0194	-0.0290	3.0311
	-20	0.7271	0.0000	0.2512	0.0217	0.0084	0.0190	-0.0256	3.0256
	-50	0.2992	0.0000	0.1032	0.5976	0.0039	0.0078	-0.0255	3.0255
Regime 2	-1	1.0000	0.0000	0.0000	0.0000	0.0091	0.0223	-0.0352	3.0588
	-10	1.0000	0.0000	0.0000	0.0000	0.0091	0.0223	-0.0352	3.0588
	-20	0.7755	0.1579	0.0666	0.0000	0.0076	0.0179	-0.0228	3.0471
	-50	0.3203	0.0718	0.0280	0.5799	0.0036	0.0095	-0.0232	3.0481
Regime 3	-1	1.0000	0.0000	0.0000	0.0000	0.0099	0.0217	-0.0186	3.0334
	-10	1.0000	0.0000	0.0000	0.0000	0.0099	0.0217	-0.0186	3.0334
	-20	0.8274	0.1726	0.0000	0.0000	0.0086	0.0184	-0.0253	3.0486
	-50	0.3629	0.2013	0.0001	0.4357	0.0084	0.0094	-0.0363	3.0756

model is sensitive to larger magnitude α values which have the effect of diversifying the portfolio across multiple assets in different regimes. For the remainder of the discussion we will use an $\alpha = -20$ heuristic approach to run the Multi-Period Optimizations.

5.1.4 Multiperiod No Lockup Optimization

We fit the model to index returns from Jan 1996 to Dec 2007 and use the 2008-2009 index returns to test the strategy against two Markowitz Optimized portfolios. We use the assumption that the previous month's returns are available immediately at the start of each month. The transition probabilities at each time step are calculated based on the prior month returns and transition matrix Q such that $P(S_{t+1}|r_{t-1}^{\rightarrow}, \Theta_G) = P(S_t|r_{t-1}^{\rightarrow}, \Theta_G)^T Q$. CRRA Utility ($\alpha = -20$) Optimal Portfolios are constructed at each step through

EH	EMN	SP	ESA	MRS	MRA
-0.1566	-0.0666	-0.2210	-0.1398	-0.1573	-0.1465

Myopic Single Period Methods and the portfolio returns are computed ¹.

We see that the back tested results are superior to most of the component indices, specifically the HFRX Equity Hedge Index and the S&P 500 Broad Market Index. It also is superior to the two Markowitz optimizations targeting the expected return of the Risk Seeking $\alpha = -1$ and Risk Adverse $\alpha = -20$ portfolios. we also are able to extract the No-Lockup lower bound on Z^* , $Z_L = -0.5125$ that we can use to analyze the results of the path based approximation techniques.

5.1.5 Multiperiod Lockup Optimization

We run the (PEA) and (BPEA) Algorithms across the 12 paths and calculate the expected utility $Z_A(p)$ and upper bound utility $Z_B(p)$ for each path p. In all cases we see that:

$$Z_A(p) \geq Z_B(p) \geq Z_L$$

¹Source: Hedge Fund Research, Inc., 2010, <http://www.hedgefundresearch.com>

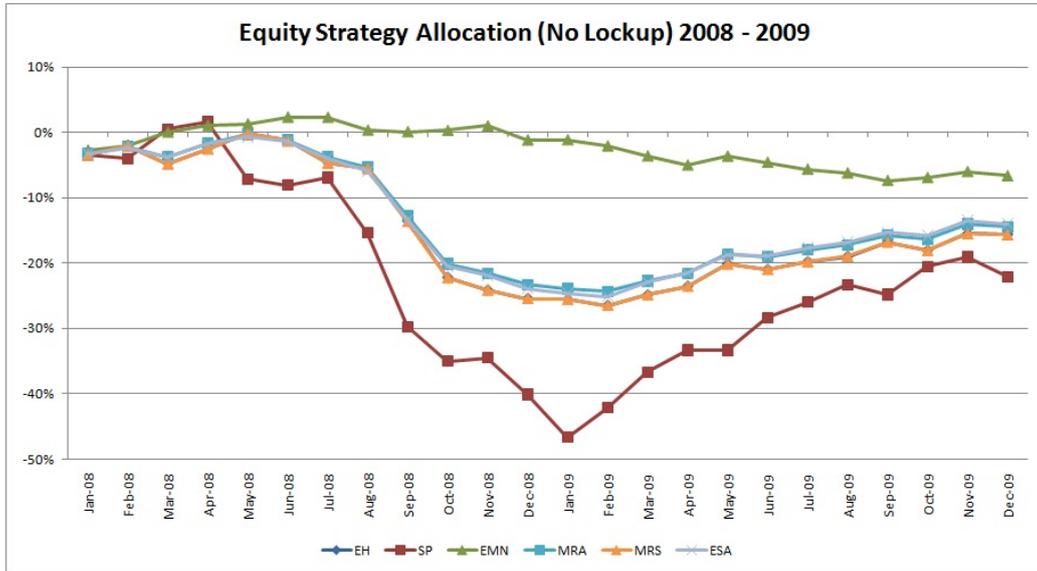


Figure 5.2: Backtested Returns of the ESA Model

Using the results above we are able to create an upper bound Z_U on Z^* as follows:

$$Z_U = \min_{p \in P} Z_A(p) = -0.3164$$

Using the results above we see that the optimal $Z^* \in [-0.5126, -0.3164]$. The path based results are shown below:

We are further able to use Z_L and Z_U to filter the path set P and create a candidate path set P^* by including any path p for which:

$$Z_B(p) \leq Z_U$$

The results (shown below) demonstrate that the eligible paths in P^* are

Table 5.7: ESA (BPEA) Results

Path (p)	Z_L	Z_A	Z_B	Z_U
1	-0.3164	-0.3156	-0.4833	-0.5125
2	-0.3164	-0.3132	-0.4701	-0.5125
3	-0.3164	-0.3140	-0.4829	-0.5125
4	-0.3164	-0.3164	-0.4893	-0.5125
5	-0.3164	-0.2528	-0.3980	-0.5125
6	-0.3164	-0.0929	-0.2021	-0.5125
7	-0.3164	-0.2475	-0.3960	-0.5125
8	-0.3164	-0.2482	-0.3846	-0.5125
9	-0.3164	-0.2495	-0.4094	-0.5125
10	-0.3164	-0.0860	-0.2294	-0.5125
11	-0.3164	-0.0834	-0.2695	-0.5125
12	-0.3164	-0.0741	-0.2921	-0.5125

Table 5.8: ESA Selected Path Lockup State Transitions

Index	l_0	l_1	l_2	l_3	l_4
EH	3	2	1	0	0
EMN	2	2	2	2	2
SP	0	0	0	0	0

$\{p_1, p_2, p_3, p_4, p_5, p_7, p_8, p_9\}$ which results in a 33.33% reduction in the number of paths. We use the new candidate path set P^* as an input to the (SPEA) algorithm.

We can also see that the approximation algorithm yields p_4 as the highest magnitude path in terms of its utility $Z_A(p_4)$, path p_4 is equivalent to the following Lockup State transitions:

We see that the choice of path p_4 implies an allocation to the Equity Hedge index (EH) at time 0 and no allocation to Equity Market Neutral

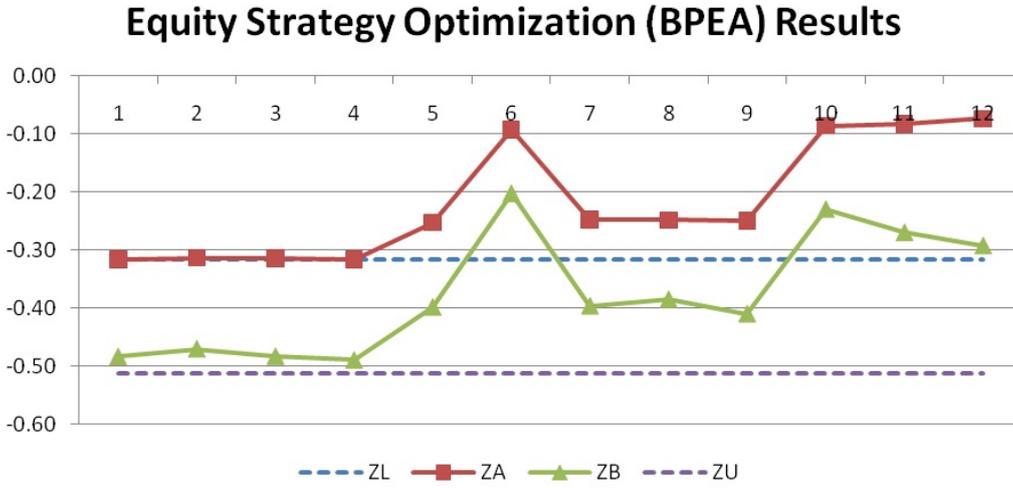


Figure 5.3: (BPEA) Results for the ESA Model

(EMN) for the duration of the model.

5.1.6 (SPEA) Results

The first step is to discretize the initial allocation vector. Given that we have three funds and eight eligible paths if we use a Discretization Factor $\delta = 20$ we get a new Discrete Action Space $\bar{A}^\phi = 20^3$ by $\bar{P}^* = 8$ number of distinct actions:

$$\bar{A}^\phi \bar{P}^* = 64000$$

However, not all initial allocation, path combinations are admissible. In many cases the initial allocation do not satisfy the constraints imposed by the path. If we filter the initial allocation, path combination and impose a probability 0 of selection of an inadmissible action we reduce the size of the Pursuit

Table 5.9: (SPEA) Algorithm ESA Path Breakdown Evolution

Iteration	p_1	p_2	p_3	p_4	p_5	p_7	p_8	p_9
1	0.1511	0.1462	0.1454	0.0714	0.1384	0.1431	0.1375	0.0670
50	0.0603	0.0000	0.0000	0.0313	0.9090	0.0000*	0.0000	0.0000
100	0.0670	0.0000	0.0000	0.0340	0.9061	0.0005	0.0000	0.0000
500	0.0532	0.0000	0.0000	0.0345	0.9013	0.0063	0.0000	0.0000
1000	0.0454	0.0000	0.0000	0.0910	0.8636	0.0000	0.0000	0.0000
5000	0.0000	0.0000	0.0000	0.3069	0.6931	0.0000	0.0000	0.0000
10000	0.0000	0.0000	0.0000	0.4633	0.5367	0.0000	0.0000	0.0000
25000	0.0000	0.0000	0.0000	0.7537	0.2463	0.0000	0.0000	0.0000
40792	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000

algorithm action set. We can do this by eliminating combinations that with probability 1 don't satisfy Minimum Allocation Requirements, Lockup State Constraints or that would yield inadmissible weight vectors ($\sum_{j=1}^N w_{j,t} > 1$ $0 \leq t < T$).

Filtering by admissible action combinations yields 1939 different initial allocation, path combinations that we can use to run the algorithm. This decreases the number of iterations required for convergence. We see that though initially the algorithm favors path p_5 the algorithm converges on path p_4 at iteration 40792 which has a higher $Z_A(p_4)$.

* value less than 0.0001

The corresponding initial allocation choice ($w^{\vec{I}A}$) is shown in the table below:

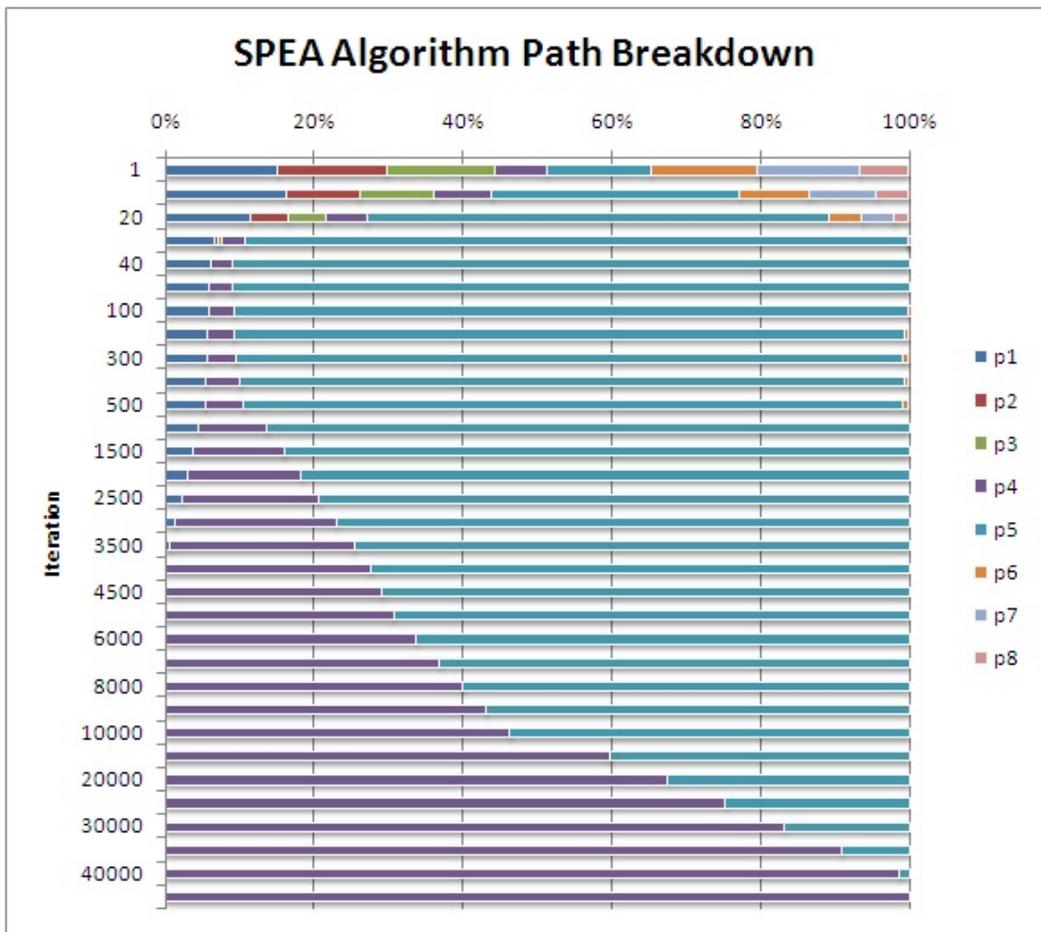


Figure 5.4: SPEA Path Breakdown Evolution

Table 5.10: ESA Initial Allocation

	EH	SP	EMN
w^{IA}	0.85	0	0.05

(SPEA) Additional Passes

We can increase the precision of the initial allocation vector by rerunning the (SPEA) Algorithm over allocation points in the neighborhood of w^{IA} for path p_4 . This is accomplished by searching in the feasible region of $w_j^{IA} \pm \delta$. The benefit here is that the algorithm will need to sample fewer points than if we had operated at that precision from the start, however we may miss global optimum if it is in the neighborhood of another (a^ϕ, p) combination.

5.2 Hedge Fund Strategy Allocation (HFSA)

We perform a larger scale Strategy Based Optimization on a two regime model, labeling one 'Regime 1', the second 'Regime 2' by expanding the (ESA) model to include three additional Hedge Fund indices: the HFRX Convertible Arbitrage Index (CV); the HFRX Merger Arbitrage Index (MA) and an actively traded Global Macro portfolio proxied by the HFRX Macro Index (M). We call this the Hedge Fund Strategy Allocation (HFSA) problem. We use a two regime model as in Chan et. al. ([45]), labeling one 'Regime 1' and the other 'Regime 2' and fit the excess returns from index

Table 5.11: HFSA HFRX Index Stats (Jan 98 - Dec 07)

Index	Mean	Std	Skew	Kurtosis	JB Stat	P-Value
EH	0.0093	0.0222	0.5837	5.3593	34.6451	< 0.001
SP	0.0038	0.0429	-0.5056	3.6575	7.2737	0.0299
EMN	0.0022	0.0107	-0.1468	3.2321	0.500	0.7003
MA	0.0055	0.0111	-1.2937	6.8038	105.8160	< 0.001
M	0.0085	0.0266	0.0466	3.7549	2.8928	0.1630
CV	0.0049	0.0121	-0.8424	4.1471	20.7734	0.0029

Table 5.12: HFSA Mixture Distribution Parameters

Index	Regime 1, $\hat{\pi}_1 = 0.7573$		Regime 2, $\hat{\pi}_2 = 0.2427$	
	$\hat{\mu}_1$	$\hat{\sigma}_1$	$\hat{\mu}_2$	$\hat{\sigma}_2$
EH	0.0077	0.0220	0.0144	0.0218
SP	0.0020	0.0436	0.0094	0.0392
EMN	0.0025	0.0104	0.0011	0.0113
MA	0.0045	0.0119	0.0088	0.0071
M	0.0072	0.0260	0.0129	0.0275
CV	0.0042	0.0124	0.0074	0.0104

inception to the end of 2007 (Jan 1998 - Dec 2007).

5.2.1 (HFSA) Mixture Distribution

A Jarque-Bera Test ([12]) run on the returns of the HFRX Indices indicates that the index returns deviate significantly from normal (see table above).

The distribution parameters are as follows:

We see that in the first regime the Macro strategy outperforms and the

Table 5.13: HFSA Estimated Transition Probabilities

	Regime 1	Regime 2
Regime 1	0.8343	0.1657
Regime 2	0.5199	0.4801

Table 5.14: HFSA Lockup Lengths

	EH	SP	EMN	MA	M	CV
Lockup	6	0	6	12	0	12

Equity Benchmark index (SP) significantly underperforms. In Regime 2 the Equity Benchmark outperforms the hedge fund indices in the model. This yields the following Transition Matrix Q :

5.2.2 Lockup State Modeling

We model the lockup space using a sample lockup vector \vec{L} (shown below) by dividing the strategies into: Liquid (SP,M), Moderately Liquid (EH,EMN) and Illiquid (MA,CV). We use the assumption here that Liquid strategies have ($L_j = 0$) period liquidity, Moderately Liquid strategies have semi-annual liquidity ($L_j = 6$) and Illiquid strategies have annual liquidity ($L_j = 12$). We assume that there are available *S&P* and Macro investible indices that we can allocate to on a monthly basis.

We wish to analyze the portfolios produced over the following two years

Table 5.15: Optimal HFSA No Lockup Weights

Regime	EH	SP	EMN	MA	M	CV	r_f
Regime 1	0.3473	0.0280	0.0000	0.4177	0.1607	0.0463	0.0000
Regime 2	0.4453	0.0306	0.0000	0.4088	0.1152	0.0000	0.0000
Long Run	0.3714	0.0289	0.0000	0.4402	0.1548	0.0047	0.0000

(2008 - 2009) so we use $T = 24$ time periods. However, given the lockup vector \vec{L} and time T we obtain $\bar{P} = 78400$ distinct admissible paths. Given the state space reduction technique discussed in section 4.1.3 we can use $\Delta = 3$ quarterly data and remodel the problem with $\bar{P}^\Delta = 2304$ by increasing the number of regimes $K = 2$ to $\kappa = 8$.

5.2.3 HFSA Single Period Results

As in the (ESA) model we can run a Single Period Optimization to examine the portfolios produced. Here we see that the model tends to prefer the Equity Hedge and Merger Arbitrage strategies, though it consistently allocates to macro and the S&P. In all portfolios the Equity Market Neutral and Convertible Arbitrage strategies receive no allocation.

As expected in 'Regime 2' we see superior equity returns and position ourselves more aggressively in the Equity Hedge and S&P assets. Although Macro performs better in 'Regime 2' the transient nature of that regime favors assets that perform relatively well in either regime.

Table 5.16: Optimal HFSA No Lockup Portfolio Parameters

Regime	Z	Mean	Std	Skewness	Kurtosis
Regime 1	-0.1146	0.0068	0.0136	-0.0259	2.9987
Regime 2	-0.1511	0.0089	0.0142	-0.0564	3.0221
Long Run	-0.1233	0.0074	0.0138	-0.0362	3.0023

We can use the target returns for Risk Seeking $\alpha = -1$ and Risk Adverse $\alpha = -20$ to build run Markowitz Optimizations and analyze the multi-period results produced by the algorithm with respect to the optimization and target Fund of Hedge Fund Benchmarks.

5.2.4 Multiperiod No Lockup Results

We fit the model to index returns from Jan 1996 to Dec 2007 and use the 2008-2009 index returns to perform an out of sample backtest against a number of Hedge Fund Benchmarks: the HFRX Absolute Return Index (AR); the HFRX Global Hedge Fund Index (GHF); the HFRX Equal Weighted Strategy Index (EWS); the Market Directional Index (MD) and the two Markowitz Portfolios. As before, we use the assumption that the previous month's returns are available immediately at the start of each month ².

We see the the HFSA algorithm outperformed all of the benchmark indices and the two Markowitz portfolios. It was however inferior to the HFRX

²Source: Hedge Fund Research, Inc., 2010, <http://www.hedgefundresearch.com>

Table 5.17: Multi-Period Strategy Allocation (No Lockup) Results 2008 - 2009

EH	SP	EMN	MA	M	CV	ESA	HFSA
-0.1566	-0.2210	-0.0667	0.1216	-0.0365	-0.4070	-0.1398	-0.0299
AR	GHF	EWS	MD	MRS	MRA	ESA	$HFSA_L$
-0.1697	-0.12971	-0.12963	-0.09073	-0.1430	-0.710	-0.1398	-0.0216

Merger Arbitrage index which was the only index that produced a positive rate of return over that period. It was able to avoid allocating to Convertible Arbitrage and the S&P Index which were the two worst performing indices over the period. It was also superior to the ESA portfolio because of its ability to incorporate arbitrage strategies.

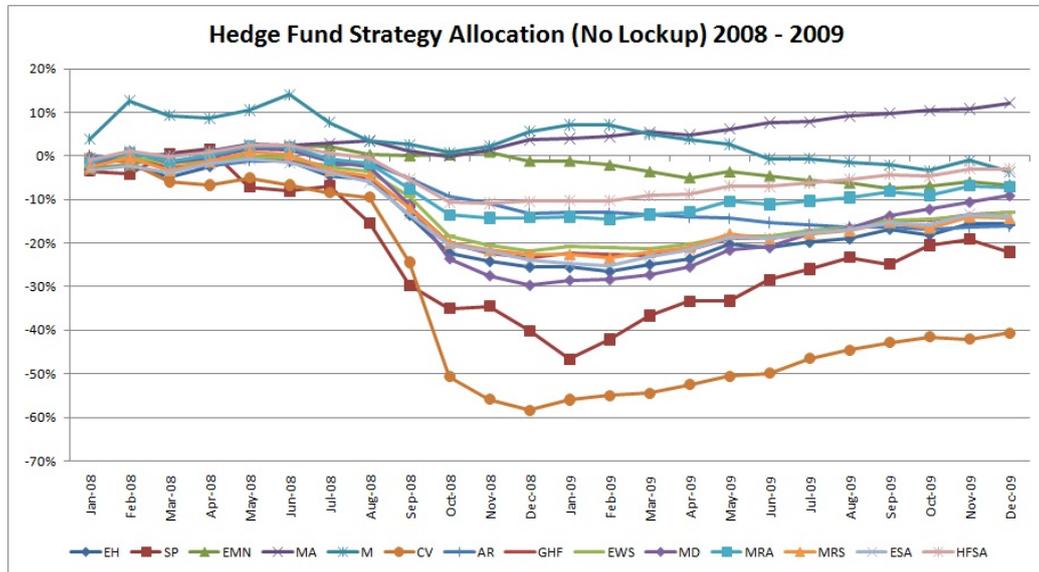


Figure 5.5: Backtested Returns of the HFSA Model

As before we are able to extract the No-Lockup lower bound on Z^* ,

Table 5.18: HFSA Aggregate Lockup Lengths

	EH	SP	EMN	MA	M	CV
Lockup	2	0	2	4	0	4

$Z_L = -5.2193$ that we can use to analyze the results of the path based approximation techniques.

5.2.5 HFSA Aggregate Time (BPEA)

As mentioned in section 5.2.2 we can use Time Aggregation to reduce the number of admissible paths. This is done by reducing the rebalancing frequency but incurring a penalty to optimality. In the HFSA model we can reduce the allocation frequency from monthly to quarterly data by setting $\Delta = 3$ and $T = 8$ and the number of regimes to $\kappa = K^\Delta = 8$. This transforms the lockup vector \vec{L} as shown below:

For each path p_Δ there is a corresponding path p such that:

$$\begin{aligned} Z_A(p_\Delta) &\geq Z_A(p) \geq Z_L \\ Z_B(p_\Delta) &\geq Z_B(p) \end{aligned} \tag{5.1}$$

This is due to the fact that the strategy described by p_Δ can be produced in the Non Aggregate Model by only admitting actions on a Δ period basis. Running the (BPEA) Algorithm over $P_\Delta = 2304$ paths produces the results shown in figure 5.6. We can also obtain a lower bound $Z_L^\Delta = -3.0242$ on

the Aggregate State Model using a similar approach to the one outlined in section 5.2.4 run on quarterly time intervals.

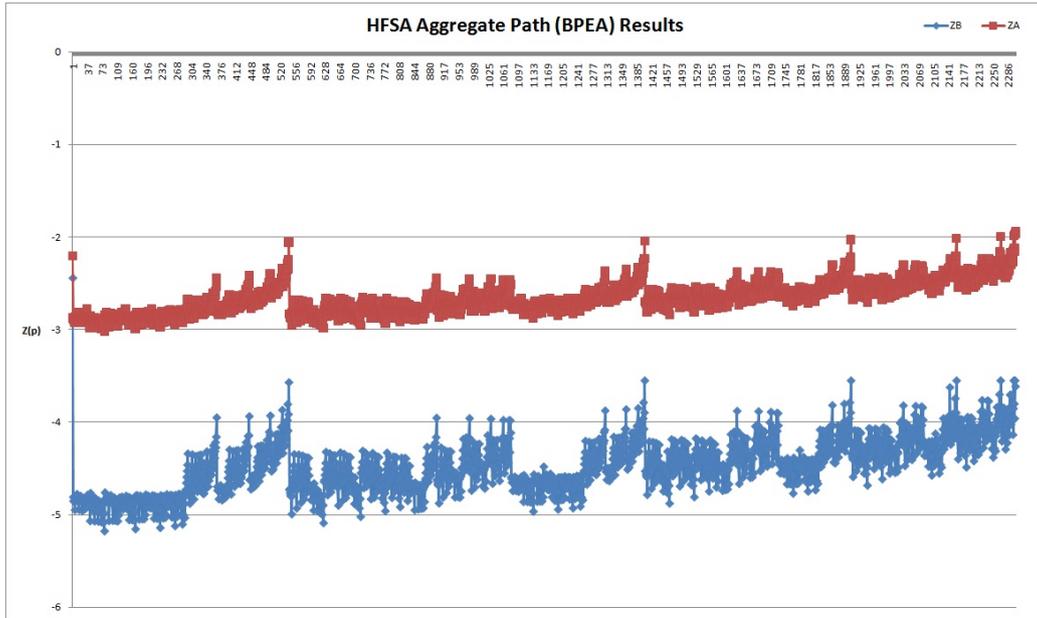


Figure 5.6: HFSA Aggregate Path (BPEA) Results

Searching over the aggregate paths P_Δ we see that the path with the best (PEA) utility is p_{78}^Δ which has a utility $Z_A^\Delta(78) = -3.0241$. The Aggregate Lockup State transitions corresponding to p_{78}^Δ are shown in table 5.19. We see that the model allocates to the Equity Hedge Index (EH) and the Merger Arbitrage Index (MA) at time t_0^Δ .

Table 5.19: HFSA Selected Path Lockup State Transitions

Index	l_0^Δ	l_1^Δ	l_2^Δ	l_3^Δ	l_4^Δ	l_5^Δ	l_6^Δ	l_7^Δ	l_8^Δ
EH	2	1	0	0	0	0	0	0	0
SP	0	0	0	0	0	0	0	0	0
EMN	2	2	2	2	2	2	2	2	2
MA	4	3	2	1	0	0	0	0	0
M	0	0	0	0	0	0	0	0	0
CV	4	4	4	4	4	4	4	4	4

5.2.6 HFSA (BPEA) State Translation

Performing Lockup State Aggregation on the HFSA Model yields an aggregate path p^Δ , however the time aggregation eliminates admissible paths in P with initial allocation w^{IA} within Δ periods of p^Δ . In order to find a solution to the (PEA) approximation for P we need to search over all admissible paths in the neighborhood of p^Δ . The path p_{78}^Δ initially allocates to the Equity Hedge (EH) and Merger Arbitrage (MA) indices at time 0 and does not allocate to Equity Market Neutral (EMN) or Convertible Arbitrage (CV). The paths in the neighborhood of p^Δ are paths that allocate to EH or MA between time t_0 and t_Δ exclusive. We see that there are only 9 paths that fit the criteria for eligibility in P^* ; the (BPEA) utility of these paths are shown in figure 5.7.

Searching over P^* yields path p_{462} as the best (PEA) utility path. We see that this path initially allocates to EH and MA at time t_0 . The Lockup State transitions are shown in table 5.20 below.

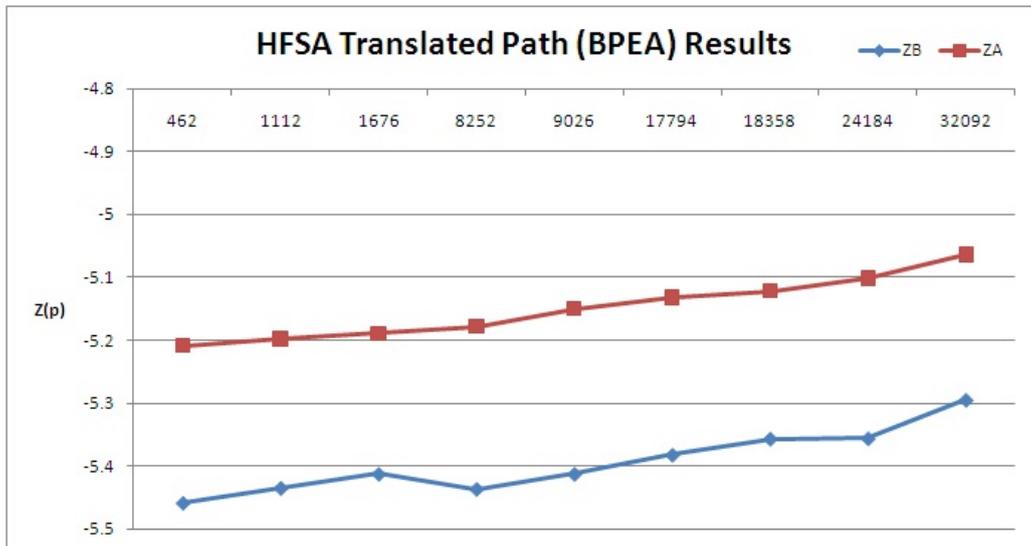


Figure 5.7: HFSA Translated Path (BPEA) Results

Running the algorithm against the 2008-2009 Returns yields a return of -0.0216 which is in line with the results produced in the No-Lockup model.

Table 5.20: HFSA Selected Path Lockup State Transitions

Index	\vec{l}_0	\vec{l}_1	\vec{l}_2	\vec{l}_3	\vec{l}_4	\vec{l}_5	\vec{l}_6	\vec{l}_7	\vec{l}_8	\vec{l}_9	\vec{l}_{10}	\vec{l}_{11}	\vec{l}_{12}
EH	6	5	4	3	2	1	0	0	0	0	0	0	0
SP	0	0	0	0	0	0	0	0	0	0	0	0	0
EMN	6	6	6	6	6	6	6	6	6	6	6	6	6
MA	12	11	10	9	8	7	6	5	4	3	2	1	0
M	0	0	0	0	0	0	0	0	0	0	0	0	0
CV	12	12	12	12	12	12	12	12	12	12	12	12	12
Index	\vec{l}_{13}	\vec{l}_{14}	\vec{l}_{15}	\vec{l}_{16}	\vec{l}_{17}	\vec{l}_{18}	\vec{l}_{19}	\vec{l}_{20}	\vec{l}_{21}	\vec{l}_{22}	\vec{l}_{23}	\vec{l}_{24}	
EH	0	0	0	0	0	0	0	0	0	0	0	0	
SP	0	0	0	0	0	0	0	0	0	0	0	0	
EMN	6	6	6	6	6	6	6	6	6	6	6	6	
MA	0	0	0	0	0	0	0	0	0	0	0	0	
M	0	0	0	0	0	0	0	0	0	0	0	0	
CV	12	12	12	12	12	12	12	12	12	12	12	12	

Chapter 6

Conclusion

Clearly the Hedge Fund Portfolio Allocation problem is non-trivial. Many of the assumptions that underlie Mean-Variance Optimization are violated by the Non-Normal Return Distributions and Portfolio Rebalancing Constraints. The Lockup Constraints transform the Quadratic Optimization Problem into a Combinatorial Problem with "curse of dimensionality" issues when it comes to the time horizon T and number of assets N in the problem. Additionally, the dynamic nature of financial markets motivates the need for approaches that can effectively address Lockup Constraints in both favorable and unfavorable market regimes.

We proposed a model for the Asset Return Dynamics under multiple regimes and a risk-averse utility function that is sensitive to higher moment distributions. Using the Taylor Approximation, we were able to approximate the CRRA utility function and maintain the property of additivity over the discrete time periods in the model. We then cast the Hedge Fund Portfolio

Problem as a Markov Decision Process (MDP) and a Partially Observable Markov Decision Process (POMDP) by modeling the separate Lockup and Regime Dynamics. We decomposed the allocation vector to contain both Actionable and Inactionable Components, the latter which was then modeled into the State Space.

We developed a Single Period Optimization Problem to find a solution to the Hedge Fund Portfolio Allocation problem under the No-Lockup Condition; generalizing this model to the multi-period case we were able to find a lower bound Z_L on the Optimal Solution. By further generalizing the model to account for Lockups, we developed multiple approaches to approximate the Optimal Policy for various sizes of the State Space. Initially, we developed the Path Enumeration Approximation (PEA) algorithm which enumerates over eligible paths in the model and approximates the optimal utility Z^* by solving a Non-Linear Constrained Non-Linear Optimization Problem. By moving the State Transition Dynamics into the action set; we were able to create a path dependent bounded approximation technique (BPEA) that facilitated a finer approximation on Z^* . Using (PEA) and (BPEA) results we were able to filter the path set P to create a Candidate Path Set P^* .

For smaller scale models, we discretized the action space A to form A^ϕ and combined this with the candidate paths P^* under the Discretization Factor δ . We developed a version of the Adaptive Learning Pursuit algorithm (SPEA) to find the optimal action choice a^ϕ at time t . For larger scale problems, we developed an approximation technique that aggregates the time period

increments Δ and a technique to translate the results back to the original problem; this was similar to a proposed technique for handling homogeneous notice periods.

We performed out of sample tests on a small (ESA) and large scale (HFSA) problem over the 2008-2009 Hedge Fund Market. We compared the results against the HFRX Hedge Fund Indices and two Markowitz Portfolios, one Risk Seeking (MRS) and one Risk Adverse (MRA). We saw that over the 2008-2009 period, the ESA and HFSA algorithms compared favorably against their peers significantly limiting downside risk on the portfolios during these turbulent markets.

Our main contribution is that we developed an approach to address the problem of Hedge Fund Asset allocation for a finite time investment horizon. We addressed Minimum Allocation Constraints, Lockups and homogeneous Notice Period issues against an environment that transitioned through multiple economic regimes. Finally casting the problem as a Markov Decision Process (MDP), we developed and combined multiple techniques to approximate the Optimal Policy for different scales of the problem.

6.1 Further Work

Key areas of further research in the Hedge Fund Portfolio Asset Allocation problem we identify in this thesis are in the modeling and algorithmic efficiency aspects of the problem. We make certain assumptions on the model

that can be further generalized. The first assumption made is that asset returns follow a K-Means Gaussian Mixture Distribution with respect to the multiple regimes in the model. We also assume that the Risk Free Asset produces a constant rate of growth r_f and we ignore Hedge Fund fee structures.

Other assumptions are made in the Lockup Dynamics. Firstly we assume hard rather than soft Lockups and that once a fund exits a Lockup there are no further Minimum Allocation Requirements to the fund. We also assume that complete asset redemption doesn't reset the Lockup. We assume homogeneous Notice Periods, however, in practice Hedge Funds can exhibit Notice Period restrictions of various size.

The algorithm performance is influenced significantly by the number of assets and the length of the time horizon. While Path Enumeration is fairly effective in handling problems of this scale, it is impractical for very large scale models. Moreover, discretization of the initial allocation vector further exacerbates this issue. In the fitting of the distribution, we are limited by the amount of data typically available with Hedge Funds. Mixture Modeling requires the fitting of a large number of parameters. It is possible that using Hierarchical Factor Based Modeling can increase the Maximum Likelihood Fit of the Asset Return Distribution.

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Appendix A

Notation Reference

A.1 Problem Space

A.1.1 Fund Notation

- F : Set of admissible funds
- A : Set of admissible actions
- a_t : Admissible action set at time t
- f_j : Individual fund in set F
- L_j : Lockup Length of fund j
- M_j : Minimum Allocation to fund j
- Δ : Notice Period length
- r_f : Risk Free Rate

- N : Number of assets in the model
- L_j^Δ : Aggregate regime lockup for fund j

A.1.2 State Space

- T : Termination period
- X_t : wealth at time t
- S_t : Regime State at time t
- \vec{l}_t : lockup state at time t
- S_t^L : Augmented lockup State at time t
- τ : aggregate time point Δt
- S_τ^Δ : aggregate regime state at time τ
- κ : total number of aggregate states
- S_τ^L : Augmented State at time τ
- ι : Aggregate regime indicator
- T_Δ : Aggregate regime termination period

A.2 System Dynamics

A.2.1 Return Distribution

- S : Set of regimes

- K : Number of regimes in set S
- π_i : Mixing Component for regime i
- $\vec{\mu}_i$: Mean vector for regime i
- Σ_i : Covariance matrix for regime i
- Θ_G : Distribution parameter vector
- Q : Regime transition matrix
- $q_{i,i'}$: one step transition probability from i to i'
- \vec{r}_t : Return of the N assets at time t
- \vec{r}_τ^Δ : Aggregate regime return of the N assets at time τ
- r_t^P : Portfolio return at time t

A.2.2 Wealth Dynamics

- α : Risk Aversion Coefficient
- $r_{i,j,t}$: Return of fund j in regime i at time t
- a_t : Action taken at time t
- $w_{j,t}$: Weight to fund j at time t
- $w_{j,t}^a$: actionable weight to fund j at time t
- $w_{j,t}^l$: Inactionable weight to fund j at time t

A.2.3 Lockup Dynamics

- $l_{j,t}^a$: Lockup indicator at time t for fund j
- $IA_{j,t}$: Initial allocation indicator at time t for fund j

A.2.4 MDP and POMDP Notation

- D_{NL} : No Lock MDP Model
- R : Reward Function
- \tilde{D}_{NL} : No Lock POMDP Model
- \tilde{Q} : POMDP Regime Transition Function
- \tilde{R} : POMDP Reward Function
- Ω : POMDP Observations
- Z : Observation Regime Probability Distribution
- \tilde{D}_L : Lockup POMDP Model
- S^L : Lockup POMDP State Set
- A^L : Lockup POMDP Action Set
- \tilde{Q}^L : Lockup POMDP System Dynamics
- λ : Lockup POMDP Lockup State Transition Dynamics
- ϖ : Lockup POMDP Weight Dynamics
- χ : Lockup POMDP wealth Dynamics

A.3 Algorithm Notations

A.3.1 Algorithms

- (PEA): Path Enumeration Approximation
- (BPEA): Bounded Path Enumeration Approximation
- (SPEA): Simulated Path Enumeration Algorithm

A.3.2 Path Parameters

- P : Admissible path set
- p : Individual path in set P
- \bar{P} : Number of paths in P
- P_Δ : Aggregate admissible path set
- p_Δ : Individual path in P_Δ
- \bar{P}_Δ : Magnitude of the set P_Δ

A.3.3 Utility Values

- Z^* : Expected utility of the Optimal Policy
- Z_A : Minimum utility of (PEA)
- $Z_A(p)$: Utility of (PEA) for path p
- Z_B : Minimum utility of (BPEA)

- $Z_B(p)$: Utility of (BPEA) for path p
- Z_U : Upper bound on Z^*
- Z_L : Lower bound on Z^*

A.3.4 Model Fitting

- Ω : Matrix of historical fund returns
- $Y_{i,t}$: Probability that the return at time t came from regime i
- T_Ω : Number of historical datapoints
- t_Ω : Current evaluated historical datapoint

A.3.5 Pursuit Parameters

- A^ϕ : Discretized action space
- \bar{A}^ϕ : Magnitude of the discretized action space
- \hat{d}^ϕ : Utility estimation matrix
- p^ϕ : Sampling matrix
- \hat{r} : Number of admissible actions
- δ : Discretization Factor
- $w^{\vec{I}A}$: Initial Allocation Vector

Appendix B

HFRX Data

¹ The HFRX Indices (“HFRX”) are a series of benchmarks of hedge fund industry performance which are engineered to achieve representative performance of a larger universe of hedge fund strategies. Hedge Fund Research, Inc. (“HFR, Inc.”) employs the HFRX Methodology (UCITSIII compliant), a proprietary and highly quantitative process by which hedge funds are selected as constituents for the HFRX Indices. This methodology includes robust classification, cluster analysis, correlation analysis, advanced optimization and Monte Carlo simulations. More specifically, the HFRX Methodology defines certain qualitative characteristics, such as: whether the fund is open to transparent fund investment and the satisfaction of the index manager’s due diligence requirements. Production of the HFRX Methodology results in a model output which selects funds that, when aggregated and weighted, have the highest statistical likelihood of producing a return series that is most representative of the reference universe of strategies.

¹Source: Hedge Fund Research, Inc., 2010, <http://www.hedgefundresearch.com>

Constituents of HFRX Indices are selected and weighted by the complex and robust process described above. The model output constitutes a subset of strategies which are representative of a larger universe of hedge fund strategies, geographic constituencies or groupings of funds maintaining certain specific characteristics.

In order to be considered for inclusion in the HFRX Indices, a hedge fund must be currently open to new transparent investment, maintain a minimum asset size (typically \$50 Million) and meet the duration requirement (generally, a 24 month track record). These criteria may vary slightly by index.

B.1 Strategy Definitions

B.1.1 HFRX Absolute Return Index

The HFRX Absolute Return Index is designed to be representative of the overall composition of the hedge fund universe. It is comprised of all eligible hedge fund strategies; including but not limited to convertible arbitrage, distressed securities, equity hedge, equity market neutral, event driven, macro, merger arbitrage, and relative value arbitrage. As a component of the optimization process, the index selects constituents which characteristically exhibit lower volatilities and lower correlations to standard directional benchmarks of equity market and hedge fund industry performance.

B.1.2 HFRX Convertible Arbitrage Index

Convertible Arbitrage includes strategies in which the investment thesis is predicated on realization of a spread between related instruments in which one or multiple components of the spread is a convertible fixed income instrument. Strategies employ an investment process designed to isolate attractive opportunities between the price of a convertible security and the price of a non-convertible security, typically of the same issuer. Convertible arbitrage positions maintain characteristic sensitivities to credit quality the issuer, implied and realized volatility of the underlying instruments, levels of interest rates and the valuation of the issuers equity, among other more general market and idiosyncratic sensitivities.

B.1.3 HFRX Equal Weighted Strategies Index

The HFRX Equal Weighted Strategies Index is designed to be representative of the overall composition of the hedge fund universe. It is comprised of all eligible hedge fund strategies; including but not limited to convertible arbitrage, distressed securities, equity hedge, equity market neutral, event driven, macro, merger arbitrage, and relative value arbitrage. The HFRX Equal Weighted Strategies Index applies an equal weight to all constituent strategy indices.

B.1.4 HFRX Equity Hedge Index

Equity Hedge strategies maintain positions both long and short in primarily equity and equity derivative securities. A wide variety of investment pro-

cesses can be employed to arrive at an investment decision, including both quantitative and fundamental techniques; strategies can be broadly diversified or narrowly focused on specific sectors and can range broadly in terms of levels of net exposure, leverage employed, holding period, concentrations of market capitalizations and valuation ranges of typical portfolios. Equity Hedgemanagers would typically maintain at least 50%, and may in some cases be substantially entirely invested in equities, both long and short.

B.1.5 HFRX Equity Hedge Index

Equity Market Neutral strategies employ sophisticated quantitative techniques of analyzing price data to ascertain information about future price movement and relationships between securities, select securities for purchase and sale. These can include both Factor-based and Statistical Arbitrage/Trading strategies. Factor-based investment strategies include strategies in which the investment thesis is predicated on the systematic analysis of common relationships between securities. In many but not all cases, portfolios are constructed to be neutral to one or multiple variables, such as broader equity markets in dollar or beta terms, and leverage is frequently employed to enhance the return profile of the positions identified. Statistical Arbitrage/Trading strategies consist of strategies in which the investment thesis is predicated on exploiting pricing anomalies which may occur as a function of expected mean reversion inherent in security prices; high frequency techniques may be employed and trading strategies may also be employed on the basis on technical analysis or opportunistically to exploit new information

the investment manager believes has not been fully, completely or accurately discounted into current security prices. Equity Market Neutral Strategies typically maintain characteristic net equity market exposure no greater than 10% long or short.

B.1.6 HFRX Global Hedge Fund Index

The HFRX Global Hedge Fund Index is designed to be representative of the overall composition of the hedge fund universe. It is comprised of all eligible hedge fund strategies; including but not limited to convertible arbitrage, distressed securities, equity hedge, equity market neutral, event driven, macro, merger arbitrage, and relative value arbitrage. The strategies are asset weighted based on the distribution of assets in the hedge fund industry.

B.1.7 HFRX Macro Index

Macro strategy managers which trade a broad range of strategies in which the investment process is predicated on movements in underlying economic variables and the impact these have on equity, fixed income, hard currency and commodity markets. Managers employ a variety of techniques, both discretionary and systematic analysis, combinations of top down and bottom up theses, quantitative and fundamental approaches and long and short term holding periods. Although some strategies employ RV techniques, Macro strategies are distinct from RV strategies in that the primary investment thesis is predicated on predicted or future movements in the underlying instruments, rather than realization of a valuation discrepancy between secu-

rities. In a similar way, while both Macro and equity hedge managers may hold equity securities, the overriding investment thesis is predicated on the impact movements in underlying macroeconomic variables may have on security prices, as opposes to EH, in which the fundamental characteristics on the company are the most significant and integral to investment thesis.

B.1.8 HFRX Market Directional Index

The HFRX Market Directional Index is designed to be representative of the overall composition of the hedge fund universe. It is comprised of all eligible hedge fund strategies; including but not limited to convertible arbitrage, distressed securities, equity hedge, equity market neutral, event driven, macro, merger arbitrage, and relative value arbitrage. As a component of the optimization process, the index selects constituents which characteristically exhibit higher volatilities and higher correlations to standard directional benchmarks of equity market and hedge fund industry performance.

B.1.9 HFRX Merger Arbitrage Index

Merger Arbitrage strategies which employ an investment process primarily focused on opportunities in equity and equity related instruments of companies which are currently engaged in a corporate transaction. Merger Arbitrage involves primarily announced transactions, typically with limited or no exposure to situations which pre-, post-date or situations in which no formal announcement is expected to occur. Opportunities are frequently presented in cross border, collared and international transactions which incorporate

multiple geographic regulatory institutions, with typically involve minimal exposure to corporate credits. Merger Arbitrage strategies typically have over 75% of positions in announced transactions over a given market cycle.

Appendix C

Portfolio Moments

$$M_x^{(P)}(t) = \sum_{i=1}^K \pi_i e^{\mu_i^{(P)}t + \frac{1}{2}\sigma_i^{2(P)}t^2} \quad (\text{C.1})$$

let $A_i = \mu_i^{(P)}t + \frac{1}{2}\sigma_i^{2(P)}t^2$

C.1 First Moment

$$M_x^{(P)'}(t) = \sum_{i=1}^K \pi_i (\mu_i^{(P)} + \sigma_i^{2(P)}t) e^{A_i} \quad (\text{C.2})$$

$$M_x^{(P)'}(0) = \sum_{i=1}^K \pi_i \mu_i^{(P)} \quad (\text{C.3})$$

C.1.1 Portfolio Mean

$$\mu_p = \sum_{i=1}^K \pi_i \mu_i^{(P)} \quad (\text{C.4})$$

C.2 Second Moment

$$\begin{aligned} M_x^{(P)''} &= \sum_{i=1}^K [\pi_i (\mu_i^{(P)} + \sigma_i^{2(P)} t) e^{A_i}]' = \\ &= \sum_{i=1}^K [\pi_i \sigma_i^{2(P)} e^{A_i} + \pi_i (\mu_i^{(P)} + \sigma_i^{2(P)} t)^2 e^{A_i}] \end{aligned} \quad (\text{C.5})$$

$$M_x^{(P)''}(0) = \sum_{i=1}^K [\pi_i \sigma_i^{2(P)} + \pi_i \mu_i^{2(P)}] \quad (\text{C.6})$$

C.2.1 Portfolio Variance

$$\sigma_p^2 = \sum_{i=1}^K [\pi_i \sigma_i^{2(P)} + \pi_i \mu_i^{2(P)}] - \left(\sum_{i=1}^K \pi_i \mu_i^{(P)} \right)^2 \quad (\text{C.7})$$

C.3 Third Moment

$$\begin{aligned} M_x^{(P)'''}(t) &= \sum_{i=1}^K [\pi_i \sigma_i^{2(P)} (\mu_i^{(P)} + \sigma_i^{2(P)} t) e^{A_i} + \\ &\quad + 2\pi_i \sigma_i^{2(P)} (\mu_i^{(P)} + \sigma_i^{2(P)} t) e^{A_i} + \\ &\quad + \pi_i (\mu_i^{(P)} + \sigma_i^{2(P)} t)^3 e^{A_i}] \end{aligned} \quad (\text{C.8})$$

$$\begin{aligned}
M_x^{(P)'''(0)} &= \sum_{i=1}^K [\pi_i \mu_i^{(P)} \sigma_i^{2(P)} + 2\pi_i \mu_i^{(P)} \sigma_i^{2(P)} + \pi_i \mu_i^{3(P)}] = \\
&= \sum_{i=1}^K [3\pi_i \mu_i^{(P)} \sigma_i^{2(P)} + \pi_i \mu_i^{3(P)}]
\end{aligned} \tag{C.9}$$

C.3.1 Portfolio Skew

$$\begin{aligned}
\mu_3(X) &= E[(X - \mu)^3] = E[X^3 - 3X^2\mu + 3X\mu^2 - \mu^3] = \\
&= E[X^3] - 3E[X^2]\mu + 3\mu^3 - \mu^3 = E[X^3] - 3E[X^2]\mu + 2\mu^3
\end{aligned} \tag{C.10}$$

$$\begin{aligned}
\mu_{3_p} &= \sum_{i=1}^K [3\pi_i \mu_i^{(P)} \sigma_i^{2(P)} + \pi_i \mu_i^{3(P)}] - \\
&\quad - 3 \left(\sum_{i=1}^K [\pi_i \sigma_i^{2(P)} + \pi_i \mu_i^{2(P)}] \right) \left(\sum_{i=1}^K \pi_i \mu_i^{(P)} \right) + \\
&\quad + 2 \left(\sum_{i=1}^K \pi_i \mu_i^{(P)} \right)^3
\end{aligned} \tag{C.11}$$

$$skew_p = \frac{\mu_{3_p}}{\sigma_p^3} \tag{C.12}$$

C.4 Fourth Moment

$$\begin{aligned}
M_x^{(P)''''}(t) &= \sum_{i=1}^K [3\pi_i(\sigma_i^{2(P)})^2 e^{A_i} + \\
&\quad + 3\pi_i\sigma_i^{2(P)}(\mu_i^{(P)} + \sigma_i^{2(P)}t)^2 e^{A_i} + \\
&\quad + 3\pi_i\sigma_i^{2(P)}(\mu_i^{(P)} + \sigma_i^{2(P)}t)^2 e^{A_i} + \\
&\quad + \pi_i(\mu_i^{(P)} + \sigma_i^{2(P)}t)^4 e^{A_i}] = \tag{C.13} \\
&= \sum_{i=1}^K [3\pi_i\sigma_i^{4(P)} e^{A_i} + \\
&\quad + 6\pi_i\sigma_i^{2(P)}(\mu_i^{(P)} + \sigma_i^{2(P)}t)^2 e^{A_i} + \\
&\quad + \pi_i(\mu_i^{(P)} + \sigma_i^{2(P)}t)^4 e^{A_i}]
\end{aligned}$$

$$\begin{aligned}
M_x^{(P)''''}(0) &= \sum_{i=1}^K [3\pi_i\sigma_i^{4(P)} + \\
&\quad + 6\pi_i\sigma_i^{2(P)}\mu_i^{2(P)} + \\
&\quad + \pi_i\mu_i^{4(P)}] \tag{C.14}
\end{aligned}$$

C.4.1 Portfolio Kurtosis

$$\begin{aligned}
\mu_4(X) &= E[(X - \mu)^4] = E[X^4] - 4E[X^3]\mu + 6E[X^2]\mu^2 - 4E[X]\mu^3 + \mu^4 = \\
&= E[X^4] - 4E[X^3]\mu + 6E[X^2]\mu^2 - 3\mu^4 \tag{C.15}
\end{aligned}$$

$$\begin{aligned}
\mu_{4_p} = & \sum_{i=1}^K [3\pi_i \sigma_i^{4(P)} + 6\pi_i \sigma_i^{2(P)} \mu_i^{2(P)} + \pi_i \mu_i^{4(P)}] - \\
& - 4 \left(\sum_{i=1}^K [3\pi_i \mu_i^{(P)} \sigma_i^{2(P)} + \pi_i \mu_i^{3(P)}] \right) \left(\sum_{i=1}^K \pi_i \mu_i^{(P)} \right) + \\
& + 6 \left(\sum_{i=1}^K [\pi_i \sigma_i^{2(P)} + \pi_i \mu_i^{2(P)}] \right) \left(\sum_{i=1}^K \pi_i \mu_i^{(P)} \right)^2 - \\
& - 3 \left(\sum_{i=1}^K \pi_i \mu_i^{(P)} \right)^4
\end{aligned} \tag{C.16}$$

$$kurt_p = \frac{\mu_{4_p}}{\sigma_p^4} \tag{C.17}$$

Appendix D

EM Algorithm

The Expectation Maximization Algorithm ([19],[8]) is a method for finding the Maximum Likelihood Estimate of parameters of an underlying distribution from a given data set when the data is incomplete or has missing values ([8]).

D.1 Mixture Distribution EM Algorithm

In the context of Multivariate Gaussian Mixture modeling the missing variable y is an assignment variable that assigns each return x_i to one of K regimes. There are two steps, an expectation (E) step and a maximization (M) step.

E-Step The algorithm begins with an estimate for the underlying distribution parameters $\Theta_G^{(i-1)}$. The first step is to find the expected value of the complete-data log-likelihood with respect to the unknown y given the

observed x and the current parameter estimates $\Theta_G^{(i-1)}$ ([8]).

$$Q(\Theta_G, \Theta_G^{(i-1)}) = E[\log p(x, y | \Theta_G) | x, \Theta_G^{(i-1)}] \quad (\text{D.1})$$

M-step The (M) step maximizes the expectation computed in the (E) step as follows:

$$\Theta_G^i = \arg \max_{\Theta_G} Q(\Theta_G, \Theta_G^{(i-1)}) \quad (\text{D.2})$$

Given N total returns x and a current parameter estimate Θ_G new parameters are estimated as follows ([8]):

$$\alpha = \frac{1}{N} \sum_{i=1}^N p(y | x_i, \Theta_G) \quad (\text{D.3})$$

$$\mu^{new} = \frac{\sum_{i=1}^N x_i p(y | x_i, \Theta_G)}{\sum_{i=1}^N p(y | x_i, \Theta_G)} \quad (\text{D.4})$$

$$\Sigma^{new} = \frac{\sum_{i=1}^N p(y | x_i, \Theta_G) (x_i - \mu^{new})(x_i - \mu^{new})}{\sum_{i=1}^N p(y | x_i, \Theta_G)} \quad (\text{D.5})$$

D.2 Baum-Welch Algorithm

The Baum-Welch or Forward Backward Algorithm is a version of the Expectation Maximization Algorithm that is used to fit Hidden Markov Models or Partially Observable Markov Decision Processes ([48], [15]). An HMM is characterized by the following:

- N : The number of states in the model, with an individual state $S = \{S_1, S_2, \dots, S_N\}$
- Q : Observation state $Q = \{q_1, \dots, q_t, \dots, q_T\}$
- O : Observation vector with components $O = \{x_0, x_1, \dots, x_{K-1}\}$
- B : Observation Density:

$$b_j(O) = \sum_{m=1}^N c_{j,m} \phi(O | \mu_{j,m}, \Sigma_{j,m})$$

- π : The initial state distribution $\pi = \{\pi_i\}$ where:

$$\pi_i = P(q_1 = S_i)$$

- A : The state transition probability distribution $A = \{a_{i,j}\}$ where:

$$a_{i,j} = P(q_{t+1} = S_j | q_t = S_i)$$

We simplify the notation for an HMM as follows:

$$\lambda = (A, B, \pi)$$

and we will use the symbol θ to refer to the collection of parameters in the system:

$$\theta = \{\pi, a, \mu, \Sigma\}$$

D.2.1 Forward Recursion

Let $b_t(i)$ be defined as follows:

$$b_t(i) = P(x_t | q_t = s_i, \theta)$$

Then α follows:

$$\begin{aligned} \alpha_t(i) &= P(x_1, \dots, x_t, q_t = s_i | \theta) \\ &= P(x_t | q_t = s_i) \sum_{k=1}^N P(x_1, \dots, x_{t-1}, q_{t-1} = s(k) | \theta) P(q_t = i | q_{t-1} = s_k) \\ &= b_t(i) \sum_{k=1}^N \alpha_{t-1}(k) a(k, i) \\ \alpha_1(i) &= \pi_i b_1(i) \end{aligned} \tag{D.6}$$

D.2.2 Backward Recursion

Let β be defined as follows:

$$\begin{aligned} \beta_t(i) &= P(x_{t+1}, \dots, x_T | q_t = s_i, \theta) \\ &= \sum_{k=1}^N P(q_{t+1} = s_k | q_t = s_i) P(x_{t+1} | q_{t+1} = s_k, \theta) P(x_{t+2}, \dots, x_T | q_{t+1} = s_k, \theta) \\ &= \sum_{k=1}^M a(i, k) b_{t+1}(k) \beta_{t+1}(k) \\ \beta_T(i) &= 1 \end{aligned} \tag{D.7}$$

D.2.3 Transition Expectation

Let γ be defined as follows:

$$\gamma_t(i) = P(q_t = S_i | O, \theta) = \frac{\alpha_t(i)\beta_t(i)}{\sum_{k=1}^N \alpha_t(k)\beta_t(k)} \quad (\text{D.8})$$

and ϵ be defined as:

$$\epsilon_t(i, j) = \frac{\alpha_t(i)a(i, j)b_{t+1}(j)\beta_{t+1}(j)}{\sum_{k=1}^N \sum_{l=1}^N \alpha_t(k)a(k, l)\beta_{t+1}(l)} \quad (\text{D.9})$$

D.2.4 Parameter Updating

We update the parameters as follows:

$$\begin{aligned} \pi_i &= \gamma_1(i) \\ a(i, j) &= \frac{\sum_{t=1}^{T-1} \epsilon_t(i, j)}{\sum_{t=1}^{T-1} \gamma_t(i)} \\ \mu_i &= \frac{\sum_{t=1}^T \gamma_t(i)x_t}{\sum_{t=1}^T \gamma_t(i)} \\ \Sigma_i &= \frac{\sum_{t=1}^T \gamma_t(i)[(x_t - \mu_i)'(x_t - \mu_i)]}{\sum_{t=1}^T \gamma_t(i)} \end{aligned} \quad (\text{D.10})$$

Appendix E

Multinomial Decision Tree

Pseudocode

We present the psuedocode for building the Multinomial Decision Tree or Multinomial Path Tree (MPT) that models the lockup dynamics and admissible paths.

E.1 Structure

A node in the tree contains the following properties:

t: A scalar quantity representing the current time period

l: An $N \times 1$ vector representing the current state

index: A scalar quantity representing an index to the most recent calculated state

Next: A vector list containing the next admissible states

E.2 Functions

E.2.1 IsNodeAdmissible

Algorithm E.2.1: ISNODEADMISSIBLE(*Node* : *PN*, *N*, *L*, *T*)

```
for  $j \leftarrow 1$  to  $N$ 
  do
    if not (( $PN.l[j] = L[j]$ ) or ( $PN.l[j] = 0$ ))
      then
        if ( $PN.l[j] > T - PN.t$ )
          then
            return (FALSE)
            exit
return (TRUE)
```

E.2.2 GetNextNode

Algorithm E.2.2: GETNEXTNODE(*Node* : *PN*)

```
local CNindex  $\leftarrow$  PN.Counter
bindex  $\leftarrow$  CONVERTTOBINARYARRAY(index)
sindex  $\leftarrow$  CONVERTTOINTEGERARRAY(PN.l)
CN.l  $\leftarrow$  sindex - bindex
CN.t  $\leftarrow$  PN.t + 1
CN.index  $\leftarrow$  0
CN.Next  $\leftarrow$  NEWLIST()
return (CN)
```

E.2.3 InsertNextNode

Algorithm E.2.3: INSERTNEXTNODE(*Node* : *PN*, *N*, *L*, *T*)

```
for index  $\leftarrow$  1 to  $2^N$ 
  do
  {
    PN.index = index
    CN  $\leftarrow$  GETNEXTNODE(PN)
    if ISNODEADMISSIBLE(CN, N, L, T)
      then
      {
        PN.Next  $\leftarrow$  INSERTNEXTNODE(CN, N, L, T)
      }
  }
return (PN)
```

E.3 Build Path Tree Algorithm

Algorithm E.3.1: BUILDPATHTREE(N, L, T)

main

$root.l \leftarrow L$

$root.index \leftarrow 0$

$root.t \leftarrow 0$

$root.Next \leftarrow \text{GETNEXTNODE}(root, N, L, T)$

Appendix F

Pursuit Algorithm

The Pursuit Algorithm ¹ is a special type of estimator algorithm [49]. It is simple and it converges rapidly in simulations.

F.1 Learning Automata Problem Formulation

A learning automata is a stochastic automaton in a feedback connection with a random environment. The output of the automaton (called the action) is input to the environment and the output of the environment (called the reaction) is input to the automaton [49].

Automaton is defined by (A, Q, R, T) and the environment by (A, R, D) where [49]:

- $A = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ set of actions to the automaton. $A(k)$ action of the automaton at instant k . A is the set of outputs of the automaton and

¹This section is taken from K. Rajarman and P.S. Sastry's paper "Finite Time Analysis of the Pursuit Algorithm for Learning Automata"

inputs to the environment.

- R is the set of reactions from the environment. $\beta(k)$ is the reaction received by the automaton at instant k . It is assumed that $\beta(k) \in [0, M]$
- $D = \{d_1, d_2, \dots, d_r\}$ is the set of average reward values where

$$d_i(k) = E[\beta(k) | \alpha(k) = \alpha_i]$$

- Q is the state of the automaton defined by

$$Q(k) = (p(k), \hat{d}(k))$$

where

$$\begin{aligned} p(k) &= [p_1(k), \dots, p_r(k), 0 \leq p_i \leq 1 \\ \sum_{i=1}^r p_i(k) &= 1, \forall k \end{aligned} \tag{F.1}$$

and

$$\hat{d}(k) = [\hat{d}_1(k), \dots, \hat{d}_r(k)]$$

is the vector of estimates of the average reward values at the k -th instant

- T is the learning algorithm that is used by the automaton to update

its state we have:

$$Q(k+1) = T(Q(k), \alpha(k), \beta(k))$$

F.2 Pursuit Algorithm Pseudocode: Discrete Case

We introduce the following objects:

- e_i r dimensional vector with the i th component unity and all others 0
- $I\{A\}$ Indicator function of event A ,

$$I\{A\} = \begin{pmatrix} 1 & \text{if the event } A \text{ occurs} \\ 0 & \text{otherwise} \end{pmatrix} \quad (\text{F.2})$$

- $X_i(k)$ Total reward obtained for the i -th action till the k -th instant:

$$X_i(k) = \sum_{j=1}^{k-1} \beta(j) I\{\alpha(j) = \alpha_i\}$$

- $Y_i(k)$ Number of times the i -th action is chosen till the k -th instant:

$$Y_i(k) = \sum_{j=1}^{k-1} I\{\alpha(j) = \alpha_i\}$$

- $M(k)$ The index of the average reward value estimate defined as:

$$\hat{d}_{M(k)}(k) = \max_j \{\hat{d}_j(k)\}$$

Algorithm F.2.1: DISCRETEPURSUITALGORITHM($\tau, r, maxiter, A$)

main

local $X, Y, \hat{d}, M, p, k, \alpha, \beta$, **ifor** $i \leftarrow 1$ **to** r

do $p[i] = \frac{1}{r}$

$\hat{d} \leftarrow \text{INITIALIZE}(\hat{d})$

$k \leftarrow 0$

$M \leftarrow \{i | \hat{d}[i] = \max_{\{1 \leq i \leq r\}} \hat{d}\}$

while ($(p[M] \leq 1 - \tau)$ **and** ($k \leq maxiter$))

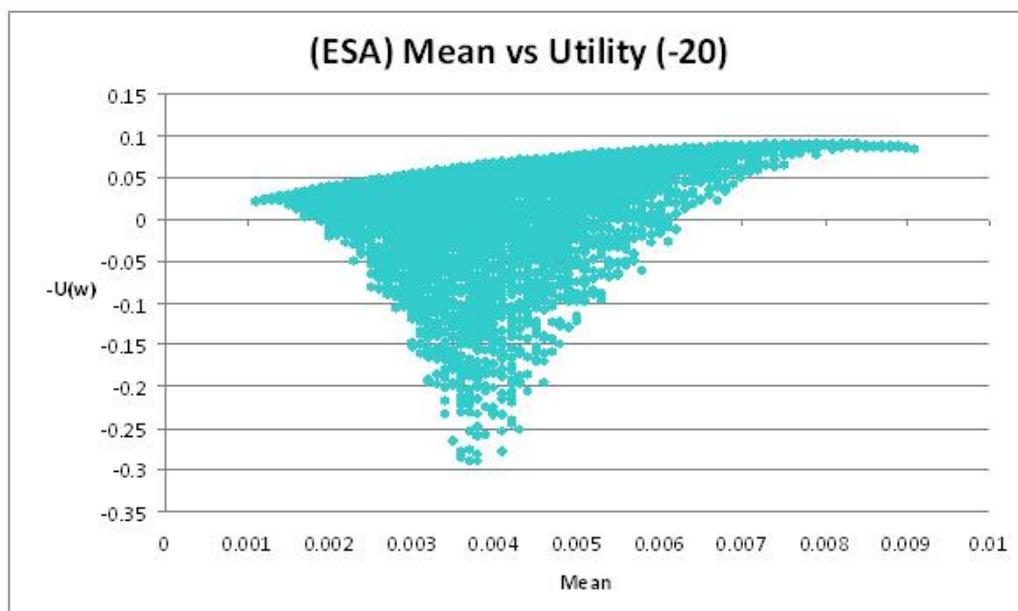
$$\left\{ \begin{array}{l} \{\alpha, i, \beta\} \leftarrow \text{SAMPLEACTION}(p, A) \\ X[i] \leftarrow X[i] + \beta \\ Y[i] \leftarrow Y[i] + 1 \\ X[j] \leftarrow X[j] \forall j \neq i \\ Y[j] \leftarrow Y[j] \forall j \neq i \\ \hat{d}[i] \leftarrow \frac{X[i]}{Y[i]} \forall i \\ M \leftarrow \{i | \hat{d}[i] = \max_{\{1 \leq i \leq r\}} \hat{d}\} \\ p[j] \leftarrow \max\{p[j] - \tau, 0\}, \forall j \neq M \\ p[M] \leftarrow 1 - \sum_{j \neq M} p[j] \\ k \leftarrow k + 1 \end{array} \right.$$

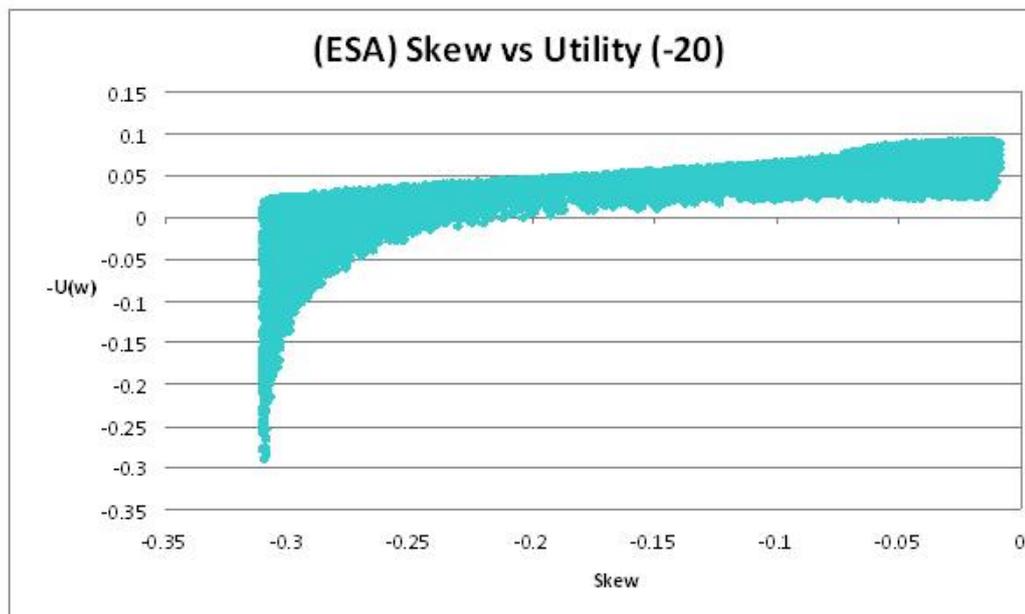
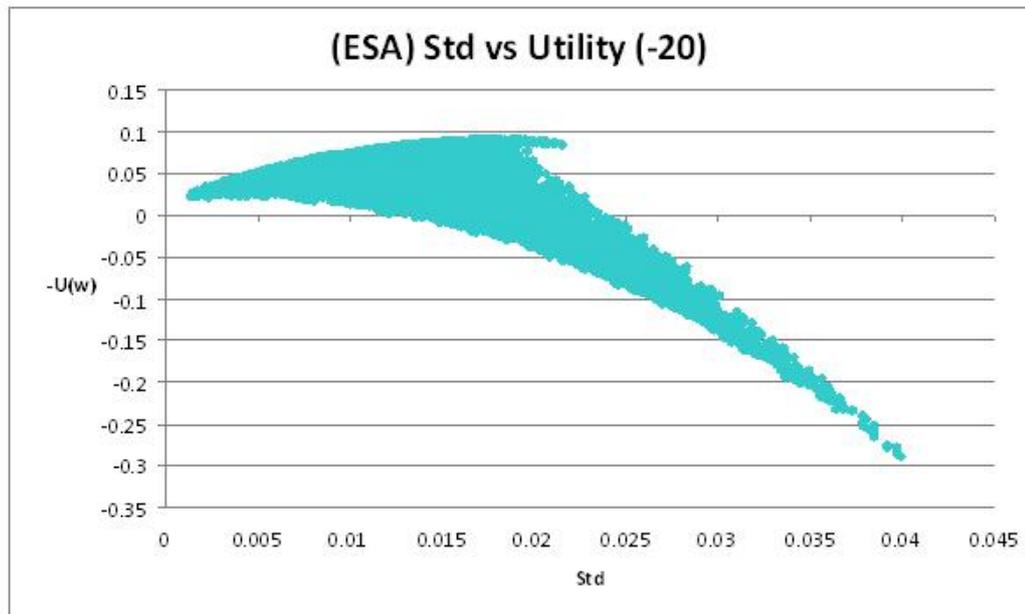
return (\hat{d})

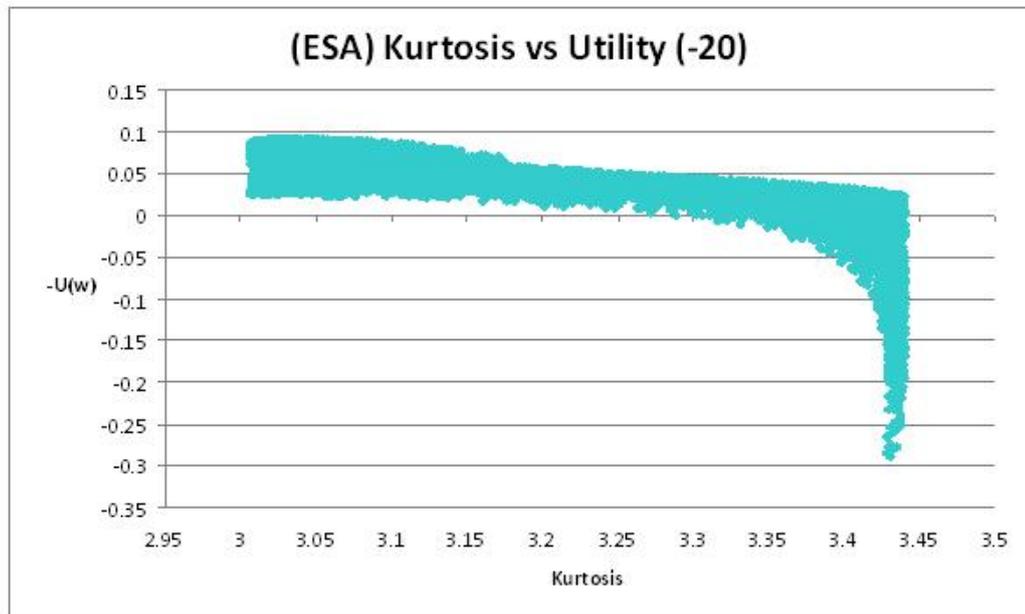
Appendix G

Utility Charts

G.1 ESA Model







G.2 HFSA Model

