D-bar Spark Theory and Deligne Cohomology

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Abstract of the Dissertation
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The purpose of this dissertation is to study secondary geometric invariants of smooth manifolds like Cheeger-Simons differential characters, Deligne cohomology and Harvey-Lawson spark characters. Our approach follows Harvey-Lawson spark theory. In particular, we study these secondary geometric invariants via the presentation of smooth hypersparks and give a new description of the ring structure of differential characters. We also study $\bar{d}$-spark theory and the ring functor $\hat{\mathbf{H}}^*(\bullet, p)$ of complex manifolds which is a natural extension of Deligne cohomology. We represent Deligne cohomology classes by $\bar{d}$-sparks and give an explicit product formula for Deligne classes. Massey higher products of secondary geometric invariants are also studied. Moreover, we show a Chern-Weil-type construction of Chern classes in Deligne cohomology for holomorphic vector bundles over complex manifolds. Many applications of our theory are given. Generalized Nadel invariants are defined naturally from our construction of Chern classes and Nadel's conjecture is verified. Studying Chern classes for the normal bundles of holomorphic foliations, we establish an analogue of the Bott vanishing theorem. Applying our representation of analytic Deligne cohomology classes, we give a direct proof of the well known cycle map $\psi : CH^*(X) \to H^{2*}_{\mathcal{D}}(X, \mathbb{Z}(*))$.

To my parents and my wife.

Contents

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Chapter 1

Introduction

The purpose of this thesis is to study secondary geometric invariants of smooth manifolds like Cheeger-Simons differential characters, Deligne cohomology and Harvey-Lawson spark characters.

In this thesis, by secondary geometric invariants associated to a manifold, we mean some kind of invariants refining the primary topological invariants. Not only do they depend on the topological structure of the manifold, but also the smooth, complex or algebraic structure of the manifold. Over the category of smooth manifolds with smooth maps, such invariants were discovered by Chern-Simons [ChS] and Cheeger-Simons [CS], and are called Chern-Simons invariants and differential characters. Invented by Deligne and developed by Beilinson, Deligne cohomology or Deligne-Beilinson cohomology plays the same role over the category of complex analytic manifolds with holomorphic maps. We may also add the Chow ring over the algebraic category to the family of secondary invariants. All these invariants have the following common properties.

- They are equipped with ring structures.

• From them, there are ring homomorphisms to the integral cohomology ring—the primary topological invariants.

• Characteristic classes can be established in these secondary invariants which refine the topological characteristic classes.

The first two of these three invariants, i.e. differential characters and analytic Deligne cohomology, are the main objects that we shall study in this thesis. In particular, we are interested in the following aspects.

• Different representations of these invariants, especially, in terms of sparks.

• Ring structures and Massey higher products.

• Relations between differential characters and Deligne cohomology.

• The theory of characteristic classes.

Our approach employs the spark theory of Harvey-Lawson [HL2] and [HL3].

1.1 Background

The theory of secondary geometric invariants has been discovered and developed over the last four decades. Here we only review a very small part of the development which directly related to this thesis.

Differential Characters. In 1973 Cheeger and Simons [CS] introduced the graded ring of differential characters associated to a smooth manifold X, which is closely related to the famous Chern-Simons invariants in [ChS]. Roughly speaking, a differential character of degree k is a homomorphism from the group of smooth singular k-cycles to $\mathbb{R}/\mathbb{Z}$, whose coboundary is the mod $\mathbb{Z}$ reduction of some degree $k+1$ smooth form. The group of differential characters of degree k is denoted by $\hat{H}^k_{CS}(X)$. A ring structure was also introduced on $\hat{H}^*_{CS}(X)$. From the ring of differential characters, there are two epimorphisms to integral cohomology ring and the ring of smooth closed forms with integral periods respectively. Cheeger and Simons developed a theory of characteristic classes for vector bundles with connections in differential characters, which encompasses both the theory of topological characteristic classes and the Chern-Weil homomorphism. They also showed applications of their theory to conformal geometry, foliation theory and more.

In [HLZ], Harvey, Lawson and Zweck studied the theory of differential characters from a de Rham-Federer viewpoint. A spark was defined to be a current whose exterior differentials can be decomposed into smooth forms and rectifiable currents. An equivalence relation among sparks was introduced and the group of de Rham-Federer spark classes, denoted by $\hat{\mathbf{H}}^*_{spark}(X)$, was established. Using a transversality theorem for currents, they established a ring structure on the group of de Rham-Federer spark classes. Moreover, this ring was shown to be isomorphic to the ring of Cheeger-Simons differential characters.

Spark theory, developed by Harvey and Lawson in [HL2], unifies and expands the approaches to differential characters of Gillet-Soulé [GS], Harris [Har] and Harvey-Lawson-Zweck [HLZ]. Central to their theory are spark complexes, sparks and rings of spark characters which are analogues of cochain complexes, cocycles and cohomology rings in the usual cohomology theory. Roughly speaking, a spark complex is a triple of cochain complexes, two of which are contained in the main one with trivial intersection. A spark is an element in the main complex such that its differential can be represented (uniquely) as the sum of elements from the other two complexes. An equivalence relation among sparks is introduced and the group of spark classes is established. Many examples of spark complexes in geometry, topology and physics were shown in [HL2]. One basic example is the de Rham-Federer spark complex associated to a smooth manifold X, which was introduced in [HLZ]. Another important one is the smooth hyperspark complex, which we shall study in §3.2. We refer to [HL2] for other interesting examples of spark complexes. Furthermore, all these spark complexes are compatible, i.e. connected by quasi-isomorphisms, which implies the groups of spark classes associated to them are all isomorphic. We call these groups the Harvey-Lawson spark characters collectively, denoted by $\hat{\mathbf{H}}^*(X)$. The ring structure on $\hat{\mathbf{H}}^*(X)$ can be defined through the de Rham-Federer spark complex. A striking fact is that the classical secondary invariants, Cheeger-Simons differential characters, can be realized as the groups of spark classes associated to these different spark complexes, i.e. $\hat{\mathbf{H}}^*(X) \cong \hat{H}^k_{CS}(X)$. Therefore, these spark complexes give many different presentations of differential characters just as there are many different presentations of cohomology.

Deligne Cohomology. Let X be a complex manifold and Ω^k denote the sheaf of holomorphic k-forms on X. Deligne cohomology $H^*_{\mathcal{D}}(X, \mathbb{Z}(p))$, introduced by Deligne in 1970's, is defined to be the hypercohomology of Deligne complex

$$\mathbb{Z}_{\mathcal{D}}(p) : 0 \to \mathbb{Z} \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^{p-1} \to 0.$$

From the definition, it is easy to see $H^q_{\mathcal{D}}(X, \mathbb{Z}(0)) = H^q(X, \mathbb{Z})$ and $H^q_{\mathcal{D}}(X, \mathbb{Z}(1)) = H^{q-1}(X, \mathcal{O}^*)$.

In [B], Beilinson studied an analogue of Deligne cohomology, usually called

Deligne-Beilinson cohomology, over open algebraic manifolds. Moreover, he defined a ring structure on Deligne cohomology and Deligne-Beilinson cohomology. Explicitly, Beilinson defined a cup product

$$\cup : \mathbb{Z}_{\mathcal{D}}(p) \otimes \mathbb{Z}_{\mathcal{D}}(p') \to \mathbb{Z}_{\mathcal{D}}(p+p')$$

by

$$x \cup y = \begin{cases} x \cdot y & \text{if } \deg x = 0; \\ x \wedge dy & \text{if } \deg x > 0 \text{ and } \deg y = p'; \\ 0 & \text{otherwise.} \end{cases}$$

The cup product $\cup$ induces a ring structure on

$$\bigoplus_{p,q} H^q_{\mathcal{D}}(X, \mathbb{Z}(p)).$$

Furthermore, Chern classes for algebraic bundles were defined in Deligne cohomology. A good reference on this topic is [EV].

Chow Ring. The Chow ring, invented by Chow [Chow], plays a very important role in algebraic geometry and has been well studied during the last half century. The Chow group $CH^k(X)$ of an algebraic variety X is the abelian group of formal sums of subvarieties of X of codimension k modulo rational equivalence. Grothendieck [Gr] established a theory of Chern classes in Chow ring. We refer to Fulton's famous book [Fu] for more details on the Chow ring.

Relations among Them. Although the definitions of differential characters, Deligne cohomology and Chow ring are very different, they share many common properties as we state earlier. The following facts indicate some of close relations among them.

- Smooth Deligne cohomology $H^*_{\mathcal{D}}(X, \mathbb{Z}(*)^\infty)$, an analogue of Deligne cohomology in the smooth category, is equivalent to Differential characters.

- Deligne cohomology is a subquotient of differential characters.

- There is a well-known cycle map from Chow ring to Deligne cohomology $CH^*(X) \to H^{2*}_{\mathcal{D}}(X, \mathbb{Z}(*))$.

- There is a cycle map from higher Chow groups to Deligne cohomology $CH^p(X, n) \to H^{2p-n}_{\mathcal{D}}(X, \mathbb{Z}(p))$.

- The theories on Characteristic classes are compatible.

These invariants have been studied by many different groups of mathematicians and physicists via many different methods. It is very interesting to compare them and to study one of them from the viewpoint of another.

1.2 What Is New?

Differential characters via smooth hypersparks. The smooth hyperspark complex, defined from the Čech-de Rham double complex, is another important spark complex besides the de Rham-Federer spark complex. It is closely related to n-gerbes with connections in physics and smooth Deligne cohomology. Representing differential characters by smooth hyperspark classes, we get a very nice description of differential characters. For instance, we have

- **Degree 0:** $\hat{\mathbf{H}}^0_{smooth}(X) = \{g : X \to S^1 : g \text{ is smooth }\}$.

- **Degree 1:** $\hat{\mathbf{H}}^1_{smooth}(X)$ = the set of (equivalent classes of) hermitian line bundles with hermitian connections.

- **Degree 2:** $\hat{\mathbf{H}}^2_{smooth}(X)$ =the set of 2-gerbes with connections.
- **Degree** $\mathbf{n} = \dim \mathbf{X}$: $\hat{\mathbf{H}}^n_{smooth}(X) \cong \mathbb{R}/\mathbb{Z}$.

In §3, we focus on this spark complex and give a new description of the ring structure on $\hat{\mathbf{H}}^*(X)$. Explicitly, we introduce a cup product in the Čech-de Rham double complex which determines a ring structure on the associated group of spark classes. As an application of the product formula, we calculate the product for spark characters on the unit circle $\hat{\mathbf{H}}^0_{smooth}(S^1) \otimes \hat{\mathbf{H}}^0_{smooth}(S^1) \to \hat{\mathbf{H}}^1_{smooth}(S^1)$. Moreover, this ring of smooth hyperspark classes is shown to be isomorphic to the rings of de Rham-Federer spark classes, Cheeger-Simons differential characters and smooth Deligne cohomology, which unifies these theories of secondary geometric invariants.

Theorem. $H^*_{\mathcal{D}}(X, \mathbb{Z}(*)^\infty) \cong \hat{\mathbf{H}}^*_{smooth}(X) \cong \hat{\mathbf{H}}^*_{spark}(X) \cong \hat{H}^*_{CS}(X)$.

Deligne cohomology via $\bar{\mathbf{d}}$-sparks. In §4, we study the Harvey-Lawson spark characters of level p, denoted by $\hat{\mathbf{H}}^*(X, p)$, on a complex manifold X. We may consider $\hat{\mathbf{H}}^*(\bullet, p)$ as a contravariant ring functor over the category of complex manifolds with holomorphic maps. In fact, $\hat{\mathbf{H}}^*(X, p)$ can be obtained as the group of spark classes associated to either the Dolbeault-Federer spark complex or the Čech-Dolbeault spark complex, which is the truncated version of the de Rham-Federer spark complex or smooth hyperspark complex. On one hand, $\hat{\mathbf{H}}^*(X, p)$ is a quotient ring of spark characters $\hat{\mathbf{H}}^*(X)$. On the other hand, $\hat{\mathbf{H}}^{k-1}(X, p)$ contains analytic Deligne cohomology $H^k_{\mathcal{D}}(X, \mathbb{Z}(p))$ as a subgroup. Therefore, we can represent a Deligne cohomology class by a spark of level p. Then applying previous work, we give an explicit geometric formulas for the product in Deligne cohomology.

Theorem. *For any Deligne classes* $\alpha \in H^k_{\mathcal{D}}(X, \mathbb{Z}(p))$ *and* $\beta \in H^l_{\mathcal{D}}(X, \mathbb{Z}(q))$*, choose spark classes* $[a] \in \Pi_p^{-1}(\alpha) \subset \hat{\mathbf{H}}^{k-1}(X)$*,* $[b] \in \Pi_q^{-1}(\beta) \subset \hat{\mathbf{H}}^{l-1}(X)$*, where* $\Pi_p : \hat{\mathbf{H}}^*(X) \to \hat{\mathbf{H}}^*(X, p)$ *is the projection. We define*

$$\alpha \cdot \beta \equiv \Pi_{p+q}([a] \cdot [b]) \in H^{k+l}_{\mathcal{D}}(X, \mathbb{Z}(p+q)).$$

Then the product is independent of the choices of spark classes $[a]$ *and* $[b]$ *and coincident with product defined by Beilinson in [B].*

In particular, the product $\alpha \cdot \beta$ *can be represented explicitly in terms of representing sparks* a *and* b *by formulas discussed above.*

Assuming X is a complex manifold, it is transparent to see that a subvariety represents a Deligne cohomology class through the spark presentation of Deligne cohomology. Applying the product formula above, it is easy to see that the intersection of two subvarieties represents the product of their Deligne classes if they intersect properly. In particular, when X is algebraic, we have a direct way to construct the cycle map $CH^*(X) \to H^{2*}_{\mathcal{D}}(X, \mathbb{Z}(*))$.

A theory of characteristic classes. Cheeger and Simons [CS] constructed Chern classes in differential characters for complex vector bundles with connection which refined the usual Chern classes. Two equivalent theories on characteristic classes in secondary invariants were developed by Harvey-Lawson [HL1] and Brylinski-McLaughlin [BrM]. For holomorphic vector bundles over a complex manifold, we show a construction for Chern classes in Deligne cohomology via Cheeger-Simons theory.

Denote by $\mathcal{V}^k(X)$ the set of isomorphism classes of holomorphic vector bundles of rank k on X, and by $\mathcal{V}(X) = \coprod_{k \geq 0} \mathcal{V}^k(X)$ the additive monoid under Whitney sum.

Theorem. *On any complex manifold there is a natural transformation of functors*

$$\hat{d} : \mathcal{V}(X) \to \bigoplus_j H^{2j}_{\mathcal{D}}(X, \mathbb{Z}(j))$$

with the property that

1. $\hat{d} : \mathcal{V}^1(X) \to 1 + H^2_{\mathcal{D}}(X, \mathbb{Z}(1))$ *is an isomorphism,*

2. *For any short exact sequence of holomorphic vector bundles* $0 \to E' \to E \to E'' \to 0$ *one has* $\hat{d}(E) = \hat{d}(E') \cdot \hat{d}(E'')$,

3. *under the natural map* $\kappa : H^{2j}_{\mathcal{D}}(X, \mathbb{Z}(j)) \to H^{2j}(X, \mathbb{Z})$, $\kappa \circ \hat{d} = c$ *(the total integral Chern class).*

The kth Chern character $\widehat{dch}_k(E)$ for a holomorphic vector bundle E can be defined in rational Deligne cohomology in the same way.

In [Z], Zucker indicated that the splitting principle works well in defining Chern classes in Deligne cohomology. In contrast to Zucker's method, our method is constructive since it is possible to explicitly construct representatives of Cheeger-Simons Chern classes via methods of [HL1] or [BrM].

Applications. Two interesting applications follow our construction of Chern classes in Deligne cohomology. The first one is on the Bott vanishing theorem. In 1969, Bott [Bo] constructed a family of connections on the normal bundle of any smooth foliation of a manifold and established the Bott vanishing theorem which says the characteristic classes of the normal bundle are trivial in all sufficiently high degrees. In §7.1, we prove an analogue of the Bott vanishing theorem for Chern classes of the normal bundle of a holomorphic foliation.

Theorem. *Let N be a holomorphic bundle of rank q on a complex manifold X. If N is (isomorphic to) the normal bundle of a holomorphic foliation of X, then for every polynomial P of pure degree $k > 2q$, the associated Chern class in Deligne cohomology satisfies*

$$P(\hat{d}_1(N), ..., \hat{d}_q(N)) \in Im[H^{2k-1}(X, \mathbb{C}^\times) \to H^{2k}_{\mathcal{D}}(X, \mathbb{Z}(k))].$$

The second one is on Nadel invariants. In 1997, Nadel [N] introduced interesting relative invariants for holomorphic vector bundles. Explicitly, for two holomorphic vector bundles E and F over a complex manifold X which are C^∞ isomorphic, Nadel defined invariants $\mathscr{E}^k(E, F) \in H^{2k-1}(X, \mathcal{O})/H^{2k-1}(X, \mathbb{Z})$. He also conjectured that these invariants should coincide with a component of the Abel-Jacobi image of $k!(ch_k(E) - ch_k(F)) \in CH^k_{hom}(X)$ when the setting is algebraic. In §7.2, we construct Nadel-type invariants $\hat{\mathscr{E}}^k(E, F)$ in intermediate Jacobians. Moreover, $\hat{\mathscr{E}}^k(E, F)$ is represented by a smooth $2k-1$-form whose $(0, 2k-1)$ component represents Nadel class $\mathscr{E}^k(E, F) \in H^{2k-1}(X, \mathcal{O})/H^{2k-1}(X, \mathbb{Z})$. In particular, this gives a proof of Nadel's conjecture in a more general context.

Theorem. *E and F are holomorphic vector bundles over complex manifold X. If they are isomorphic as C^∞ complex vector bundles, define*

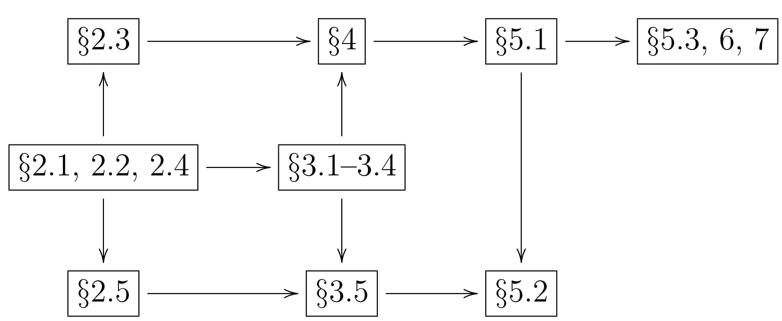
$$\hat{\mathscr{E}}^k(E, F) \equiv k!(\widehat{dch}_k(E) - \widehat{dch}_k(F)) = [\pi_k(k \int_0^1 tr(\eta \wedge (\Omega_t)^{k-1}) dt)] \in \mathcal{J}^k.$$

The kth Nadel invariant $\mathscr{E}^k(E,F)$ *is the image of* $\hat{\mathscr{E}}^k(E,F)$ *under the projection* $\pi : \mathcal{J}^k \to H^{2k-1}(X,\mathcal{O})/H^{2k-1}(X,\mathbb{Z})$. *In particular, Nadel's conjecture is true.*

Massey higher products. Massey products were introduced by Massey [M1] in 1958 as higher order cohomology operations, which generalize the cup product in cohomology theory. For example, the Massey triple product $\mathcal{M}(\alpha,\beta,\gamma)$ of three cohomology classes α, β and γ is defined up to some indeterminacy when both the products $\alpha\beta$ and $\beta\gamma$ vanish. In 1968, Massey gave a first geometric interpretation of higher products. He calculated the triple product in the cohomology of a space associated to the Borromean rings and showed that the Borromean rings can not be unlinked. A graph of this example can be found in [GM]. In their famous paper [DGMS], Deligne, Griffiths, Morgan and Sullivan proved that all Massey higher products are trivial in the de Rham cohomology of Kähler manifolds. In an elegant paper [FG], Fernández and Gray showed that the Iwasawa manifold $I(3)$ has no Kähler structure by calculating a nontrivial Massey triple product in the de Rham cohomology. In §3.5, we establish the Massey higher products in secondary geometric invariants via spark theory. Nontrivial examples are also given. Moreover, in §5.2, we give a new construction of Massey products in analytic Deligne cohomology (c.f. Deninger [De]).

Organization of the thesis. This thesis consists of eight chapters. In §2, we study homological algebra which plays a big role in spark theory. We study secondary geometric invariants over the smooth category in §3. In particular, we focus on the smooth hypersparks and introduce a product formula for smooth hyperspark classes. Then we show an explicit isomorphism between the ring of smooth hyperspark classes and smooth Deligne cohomology. In §4, we study $\bar{d}$-spark theory and the ring functor $\hat{\mathbf{H}}^*(\bullet, p)$ of complex manifolds which is a natural extension of analytic Deligne cohomology. In §5, we represent Deligne cohomology classes by $\bar{d}$-sparks and give an explicit product formula for Deligne classes. In §6, we show a Chern-Weil-type construction of Chern classes in Deligne cohomology for holomorphic vector bundles over complex manifolds. In §7, we give two applications of our theory. First, we establish a version of the Bott vanishing theorem for holomorphic foliations. Second, we define generalized Nadel invariants and give a short proof of Nadel's conjecture. Massey products of secondary geometric invariants are studied in §2.5, §3.5 and §5.2. In the §8, we study the product on hypercohomology, which helps us to prove Theorems 3.4.7 and 5.1.11. You may find a short introduction at the beginning of each chapter.

This thesis has been divided into three papers [H1], [H2] and [H3]. An alternative order to read this thesis may be the following.



The rows 1, 2 and 3 are corresponding to [H2], [H1] and [H3] respectively.

Chapter 2

Basic Algebra

Homological algebra plays a big role in spark theory just as it does in the theory of algebraic topology. In the first part of this chapter, we summarize the homological apparatus invented by Harvey and Lawson to study secondary geometric invariants of manifolds. Explicitly, in §2.1, we introduce the concepts of a spark complex and its associated group of spark classes. The basic 3×3 grid of short exact sequences associated the group of spark classes, as well as other properties, is established. In §2.2, we define quasi-isomorphism between two spark complexes, which induces an isomorphism between the groups of spark classes associated to these spark complexes. It turns out that there are various examples of spark complex appeared naturally in geometry, topology and physics, which will be shown in the next chapter. The generalized spark complex, which will play a key role in §4, is introduced in §2.3. The readers may skip §2.3 at the first time and read it right before reading §4. In the second part of this chapter, we refine our algebraic model and study the spark complex of differential graded algebras. In particular, we focus the ring structure on the group of spark classes. In §2.4, we show explicit product formulas of two spark classes. Moreover, we study Massey higher products in §2.5, which is the main contribution of the author in this chapter.

2.1 Homological Spark Complexes

We introduce the definitions of a homological spark complex and its associated group of homological spark classes. Note that all cochain complexes in this thesis are bounded cochain complexes of abelian groups.

Definition 2.1.1. *A* ***homological spark complex****, or* ***spark complex*** *for short, is a triple of cochain complexes (F^*, E^*, I^*) together with morphisms given by inclusions*

$$I^* \hookrightarrow F^* \hookleftarrow E^*$$

such that

1. $I^k \cap E^k = 0$ *for* $k > 0$*,* $F^k = E^k = I^k = 0$ *for* $k < 0$*,*
2. $H^*(E^*) \cong H^*(F^*)$*.*

Definition 2.1.2. *In a given spark complex (F^*, E^*, I^*), a* ***spark*** *of degree k is an element $a \in F^k$ which satisfies the* ***spark equation***

$$da = e - r$$

for some $e \in E^{k+1}$ and $r \in I^{k+1}$.

Two sparks a, a' of degree k are ***equivalent*** *if*

$$a - a' = db + s$$

for some $b \in F^{k-1}$ and $s \in I^k$.

The set of equivalence classes is called the ***group of spark classes*** *of degree k and denoted by $\hat{\mathbf{H}}^k(F^*, E^*, I^*)$, or $\hat{\mathbf{H}}^k$ for short. Let $[a]$ denote the equivalence class containing the spark a.*

Lemma 2.1.3. *Each spark $a \in F^k$ uniquely determines $e \in E^{k+1}$ and $r \in I^{k+1}$. Moreover, $de = dr = 0$.*

Proof. Uniqueness of e and r is from the fact $I^k \cap E^k = 0$. Taking differential on the spark equation, we get $de - dr = 0$ which implies $de = dr = 0$. □

This is the reason that we represent a spark by only a single $a \in F^k$. We may denote a spark by a triple (a, e, r) or a double (a, r) in the future, especially in the case of generalized sparks in §2.3.

We now show the fundamental exact sequences associated to a spark complex (F^*, E^*, I^*). Let $Z_I^k(E^*)$ denote the space of cycles $e \in E^k$ which are F^*-homologous to some $r \in I^k$, i.e. $e - r$ is exact in F^k.

Lemma 2.1.4. *There exist well-defined surjective homomorphisms*

$$\delta_1 : \hat{\mathbf{H}}^k \to Z_I^{k+1}(E^*) \quad \textit{and} \quad \delta_2 : \hat{\mathbf{H}}^k \to H^{k+1}(I^*)$$

given by

$$\delta_1([a]) = e \quad \textit{and} \quad \delta_2([a]) = [r]$$

where $da = e - r$.

Proof. If a' is equivalent to a, i.e. $a - a' = db + s$, then we have $da' = e - (r + ds)$. So it is easy to see these maps are well-defined.

Consider $e \in Z_I^{k+1}(E^*)$, by definition, there exists $r \in I^{k+1}$ such that $e-r$ is exact in F^{k+1}, i.e. $\exists a \in F^k$ with $da = e-r$. So $\delta_1([a]) = e$. For $[r] \in H^{k+1}(I^*)$, r also represents a class in $H^{k+1}(F^*) \cong H^{k+1}(E^*)$. Choosing a representative $e \in E^{k+1}$ of this class, we have $e - r = da$ for some $a \in F^k$, hence $\delta_2([a]) = [r]$. Both δ_1 and δ_2 are surjective.

□

Let $\hat{\mathbf{H}}_E^k$ denote the space of spark classes that can be represented by a spark $a \in E^k$. Let us also define

$$H_I^k(F^*) \equiv \mathrm{Image}\{H^k(I^*) \to H^k(F^*)\} \equiv \mathrm{Ker}\{H^k(F^*) \to H^k(F^*/I^*)\},$$

$$H^{k+1}(F^*, I^*) \equiv \mathrm{Ker}\{H^{k+1}(I^*) \to H^{k+1}(F^*)\} \equiv \mathrm{Image}\{H^k(F^*/I^*) \to H^{k+1}(I^*)\}.$$

Proposition 2.1.5. *Associated to any spark complex (F^*, E^*, I^*) is the commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^k(F^*)}{H_I^k(F^*)} & \longrightarrow & \hat{\mathbf{H}}_E^k & \longrightarrow & dE^k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^k(F^*/I^*) & \longrightarrow & \hat{\mathbf{H}}^k & \xrightarrow{\delta_1} & Z_I^{k+1}(E^*) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \delta_2} & & \downarrow & & \\
0 & \longrightarrow & H^{k+1}(F^*, I^*) & \longrightarrow & H^{k+1}(I^*) & \longrightarrow & H_I^{k+1}(F^*) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

whose rows and columns are exact.

Proof. To show the exactness of the middle row and column, by Lemma 2.1.4, it suffices to show $\ker\delta_1 = H^k(F^*/I^*)$ and $\ker\delta_2 = \hat{\mathbf{H}}^k_E$. First, by the definition,

$$H^k(F^*/I^*) \equiv \{a \in F^k \mid da \in I^{k+1}\}/\{I^k + dF^{k-1}\}$$

$$= \{[a] \mid da = 0 - r \text{ for some } r \in I^{k+1}\} = \ker\delta_1.$$

Second, it is trivial that $\hat{\mathbf{H}}^k_E \subset \ker\delta_2$. If $[a] \in \ker\delta_2$, then $da = e - r$ with $[r] = 0 \in H^{k+1}(I^*)$. So $\exists s \in I^k$ with $ds = r$, and $d(a+s) = e - 0$, i.e. [a+s]=[a]. By Lemma 2.2.2 in the following section, we can choose $\tilde{a} \in E^k$ such that $[\tilde{a}] = [a+s] = [a] \in \hat{\mathbf{H}}^k_E$. It is straightforward to show the exactness of other rows and columns. □

2.2 Quasi-isomorphism of Spark Complexes

Definition 2.2.1. *Two spark complexes (F^*, E^*, I^*) and $(\bar{F}^*, \bar{E}^*, \bar{I}^*)$ are* ***quasi-isomorphic*** *if there exists a commutative diagram of morphisms*

$$\begin{array}{ccccc} I^* & \xhookrightarrow{i} & F^* & \xhookleftarrow{i} & E^* \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \| \\ \bar{I}^* & \xhookrightarrow{i} & \bar{F}^* & \xhookleftarrow{i} & \bar{E}^* \end{array}$$

inducing an isomorphism

$$i^* : H^*(I^*) \xrightarrow{\cong} H^*(\bar{I}^*).$$

Lemma 2.2.2. *Suppose (F^*, d) is a subcomplex of $(\bar{F}^*, d)$. Then the following conditions are equivalent:*

1. *$H^*(F^*) \cong H^*(\bar{F}^*)$,*

2. *Given* $g \in F^{p+1}$ *and a solution* $\alpha \in \bar{F}^p$ *to the equation* $d\alpha = g$*, there exists* $\gamma \in \bar{F}^{p-1}$ *with* $f = \alpha + d\gamma \in F^p$ *and* $df = g$*,*

3. $H^*(\bar{F}^*/F^*) = 0$.

Proof. (1) $\Rightarrow$ (2). Since $H^*(F^*) \to H^*(\bar{F}^*)$ is injective, g is also exact in F^*, i.e. $\exists h \in F^p$ with $dh = g$. Thus, $\alpha - h$ is closed in $\bar{F}^*$. Since $H^*(F^*) \to H^*(\bar{F}^*)$ is surjective, $\exists \gamma \in \bar{F}^{p-1}$ such that $\alpha - h + d\gamma \in F^p$. Let $f = \alpha + d\gamma \in F^p$, then $df = g$.

(2) $\Rightarrow$ (3). If $[\alpha] \in \bar{F}^p/F^p$ is closed, then $d\alpha = g \in F^{p+1}$. Hence $\exists \gamma \in \bar{F}^{p-1}$ with $f = \alpha + d\gamma \in F^p$, i.e. $[a] = [-d\gamma] \in \bar{F}^p/F^p$ is exact.

(3) $\Rightarrow$ (1) is trivial. □

Proposition 2.2.3. *A quasi-isomorphism of spark complexes* (F^*, E^*, I^*) *and* $(\bar{F}^*, \bar{E}^*, \bar{I}^*)$ *induces an isomorphism*

$$\hat{\mathbf{H}}^k(F^*, E^*, I^*) \cong \hat{\mathbf{H}}^k(\bar{F}^*, \bar{E}^*, \bar{I}^*)$$

of the associated groups of spark classes. Moreover, it induces an isomorphism of the 3×3 *grids associated to the two complexes.*

Proof. Assume $(i, id, i) : (F^*, E^*, I^*) \longrightarrow (\bar{F}^*, \bar{E}^*, \bar{I}^*)$ is a quasi-isomorphism between these two spark complexes, it is plain to see it induces a homomorphism

$$i_* : \hat{\mathbf{H}}^k(F^*, E^*, I^*) \longrightarrow \hat{\mathbf{H}}^k(\bar{F}^*, \bar{E}^*, \bar{I}^*).$$

To see it is surjective, let $\bar{a} \in \bar{F}^k$ is a spark with $d\bar{a} = e - \bar{r}$ where $e \in \bar{E}^{k+1} = E^{k+1}$ and $\bar{r} \in \bar{I}^{k+1}$. Since $d\bar{r} = 0$ and $H^{k+1}(I^*) = H^{k+1}(\bar{I}^*)$, there exists $\bar{s} \in \bar{I}^k$ such that $r = \bar{r} + d\bar{s} \in I^{k+1}$. Therefore, $d(\bar{a} - \bar{s}) = e - r$. Now $e - r \in F^{k+1}$ and $H^*(F^*) \cong H^*(\bar{F}^*)$, the lemma above implies there exists $\bar{b} \in F^{k-1}$ with $a = \bar{a} - \bar{s} + d\bar{b} \in F^k$ and hence $da = e - r$.

Suppose that $a \in F^k$ is an (F^*, E^*, I^*)-spark which is equivalent to zero as an $(\bar{F}^*, \bar{E}^*, \bar{I}^*)$-spark, i.e. $da = e - r$ with $e \in E^{k+1}$, $r \in I^{k+1}$ and $a = d\bar{b} + \bar{s}$ with $\bar{b} \in \bar{F}^{k-1}$, $\bar{s} \in \bar{I}^k$. Then $e - r = da = d\bar{s}$ which implies $e = 0$ and $d\bar{s} = -r$. Applying the lemma to $I^* \subset \bar{I}^*$, there exist $\bar{t} \in \bar{I}^{k-1}$, $s \in I^k$ with $s = \bar{s} + d\bar{t}$. Therefore, $a - s = d(\bar{b} - \bar{t})$. Since $H^k(F^*) \cong H^k(\bar{F}^*)$, there exists $b \in F^{k-1}$ with $a - s = db$. That is, a is equivalent to zero as an (F^*, E^*, I^*)-spark also, and the map is injective.

It is routine to verify that the associated 3×3 grids are isomorphic. $\square$

2.3 Generalized Spark Complexes

We define a **generalized** spark complex and its associated group of spark classes, which are generalizations of a spark complex and its associated group of spark classes. When we mention a spark complex in this section, Chapter 4 and after, we mean this generalized spark complex defined below.

Definition 2.3.1. *A* ***generalized homological spark complex****, or* ***spark complex*** *for short, is a triple of cochain complexes (F^*, E^*, I^*) together with morphisms*

$$I^* \xrightarrow{\Psi} F^* \hookleftarrow E^*$$

such that

1. *$I^k \cap E^k = 0$ for $k > 0$, $F^k = E^k = I^k = 0$ for $k < 0$,*

2. $H^*(E^*) \cong H^*(F^*)$,

3. $\Psi|_{I^0} : I^0 \to F^0$ *is injective.*

Definition 2.3.2. *In a given spark complex* (F^*, E^*, I^*)*, a* ***spark*** *of degree* k *is a pair* $(a, r) \in F^k \oplus I^{k+1}$ *which satisfies the* ***spark equations***

1. $da = e - \Psi(r)$ *for some* $e \in E^{k+1}$,

2. $dr = 0$.

Two sparks $(a, r), (a', r')$ *of degree* k *are* ***equivalent*** *if there exists a pair* $(b, s) \in F^{k-1} \oplus I^k$ *such that*

1. $a - a' = db + \Psi(s)$,

2. $r - r' = -ds$.

The set of equivalence classes is called the ***group of spark classes*** *of degree* k *and denoted by* $\hat{\mathbf{H}}^k(F^*, E^*, I^*)$*, or* $\hat{\mathbf{H}}^k$ *for short. Let* $[(a, r)]$ *denote the equivalence class containing the spark* (a, r).

In the case of spark complexes in §2.1, as Harvey and Lawson introduced in [HL2], we require $I^* \to F^*$ to be injective. Therefore, e and r are uniquely determined by a. We usually denote a spark by a single element a for short. The generalized spark complex was introduced in [HL3], where $\Psi : I^* \to F^*$ was not required to be injective. Hence, r is not determined uniquely by a and we have to remember r for a spark and denote a spark by (a, r).

We now derive the fundamental exact sequences associated to a spark complex (F^*, E^*, I^*). Let $Z^k(E^*) = \{e \in E^k : de = 0\}$ and set

$$Z^k_I(E^*) \equiv \{e \in Z^k(E^*) : [e] = \Psi_*([r]) \text{ for some } [r] \in H^k(I^*)\}$$

where $[e]$ denotes the class of e in $H^k(E^*) \cong H^k(F^*)$.

Lemma 2.3.3. *There exist well-defined surjective homomorphisms*

$$\delta_1 : \hat{\mathbf{H}}^k \to Z_I^{k+1}(E^*) \quad \text{and} \quad \delta_2 : \hat{\mathbf{H}}^k \to H^{k+1}(I^*)$$

given by

$$\delta_1([(a,r)]) = e \quad \text{and} \quad \delta_2([(a,r)]) = [r]$$

where $da = e - \Psi(r)$.

Proof. If (a', r') is equivalent to (a, r), i.e. $a - a' = db + \Psi(s)$ and $r - r' = -ds$, then we have $da' = e - \Psi(r + ds)$. So it is easy to see these maps are well-defined.

Consider $e \in Z_I^{k+1}(E^*)$, by definition, there exists $r \in I^{k+1}$ such that $e - \Psi(r)$ is exact in F^{k+1}, i.e. $\exists a \in F^k$ with $da = e - \Psi(r)$. So $\delta_1([(a,r)]) = e$. For $[r] \in H^{k+1}(I^*)$, $\Psi(r)$ also represents a class in $H^{k+1}(F^*) \cong H^{k+1}(E^*)$. Choosing a representative $e \in E^{k+1}$ of this class, we have $e - \Psi(r) = da$ for some $a \in F^k$, hence $\delta_2([(a,r)]) = [r]$. Both δ_1 and δ_2 are surjective.

□

Lemma 2.3.4. *Define $\hat{\mathbf{H}}_E^k \equiv \ker \delta_2$, then $\hat{\mathbf{H}}_E^k \cong E^k / Z_I^k(E^*)$.*

Proof. Let $\alpha \in \hat{\mathbf{H}}_E^k$ be represented by (a, r) with spark equations $da = e - \Psi(r)$ and $dr = 0$. Then we have $[r] = \delta_2(\alpha) = 0$, i.e. $r = -ds$ for some $s \in I^k$. So $d(a - \Psi(s)) = e$, by Lemma 2.2.2 and the fact $H^*(F^*) \cong H^*(E^*)$, there exists $b \in F^{k-1}$ such that $a' \equiv a - \Psi(s) + db \in E^k$. Hence α can be represented by spark $(a', 0)$ with $a' \in E^k$. If $(a', 0)$ is equivalent to 0, then $a' = db' + \Psi(s')$ for some $b' \in F^{k-1}$ and $s' \in I^k$ with $ds' = 0$, i.e. $a' \in Z_I^k(E^*)$. □

Remark 2.3.5. *From last proof, it is easy to see that $\hat{\mathbf{H}}^k_E$ is the space of spark classes that can be represented by sparks of type $(a, 0)$ with $a \in E^k$.*

Definition 2.3.6. *Associated to any spark complex (F^*, E^*, I^*) is the cone complex (G^*, D) defined by setting*

$$G^k \equiv F^k \oplus I^{k+1} \text{ with differential } D(a, r) = (da + \Psi(r), -dr).$$

Consider the homomorphism $\Psi_* : H^k(I^*) \to H^k(F^*) = H^k(E^*)$, and define

$$H^k_I(E^*) \equiv \text{Image}\{\Psi_*\} \quad \text{and} \quad Ker^k(I^*) \equiv \ker\{\Psi_*\}.$$

Proposition 2.3.7. *[HL3] There are two fundamental short exact sequences*

1. $0 \longrightarrow H^k(G^*) \longrightarrow \hat{\mathbf{H}}^k \xrightarrow{\delta_1} Z^{k+1}_I(E^*) \longrightarrow 0$;
2. $0 \longrightarrow \hat{\mathbf{H}}^k_E \longrightarrow \hat{\mathbf{H}}^k \xrightarrow{\delta_2} H^{k+1}(I^*) \longrightarrow 0.$

Moreover, associated to any spark complex (F^*, E^*, I^*) is the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^k(E^*)}{H^k_I(E^*)} & \longrightarrow & \hat{\mathbf{H}}^k_E & \longrightarrow & dE^k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^k(G^*) & \longrightarrow & \hat{\mathbf{H}}^k & \xrightarrow{\delta_1} & Z^{k+1}_I(E^*) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \delta_2} & & \downarrow & & \\
0 & \longrightarrow & Ker^{k+1}(I^*) & \longrightarrow & H^{k+1}(I^*) & \longrightarrow & H^{k+1}_I(E^*) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

whose rows and columns are exact.

Proof. The proof is similar to the proof of Proposition 2.1.5. See [HL3] for details. $\square$

We can also talk about quasi-isomorphism between two generalized spark complexes.

Definition 2.3.8. *Two spark complexes* (F^*, E^*, I^*) *and* $(\bar{F}^*, \bar{E}^*, \bar{I}^*)$ *are quasi-isomorphic if there exists a commutative diagram of morphisms*

$$\begin{array}{ccccc} I^* & \xrightarrow{\Psi} & F^* & \overset{i}{\hookleftarrow} & E^* \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle i} & & \| \\ \bar{I}^* & \xrightarrow{\bar{\Psi}} & \bar{F}^* & \overset{i}{\hookleftarrow} & \bar{E}^* \end{array}$$

inducing an isomorphism

$$\psi^* : H^*(I^*) \xrightarrow{\cong} H^*(\bar{I}^*).$$

Proposition 2.3.9. *[HL3] A quasi-isomorphism of spark complexes* (F^*, E^*, I^*) *and* $(\bar{F}^*, \bar{E}^*, \bar{I}^*)$ *induces an isomorphism*

$$\hat{\mathbf{H}}^*(F^*, E^*, I^*) \cong \hat{\mathbf{H}}^*(\bar{F}^*, \bar{E}^*, \bar{I}^*)$$

of the associated groups of spark classes. Moreover, it induces an isomorphism of the 3×3 *grids associated to these two complexes.*

Proof. We omit the proof which is similar to the proof of Proposition 2.2.3. $\square$

2.4 Ring Structure

In this section, we define the ring structure on the group of spark classes $\hat{\mathbf{H}}^*$. For simplicity, we first assume all cochain complexes in a spark complex are differential graded algebras.

Definition 2.4.1. *By a* ***differential graded algebra****, or* ***DGA*** *for short, we mean a cochain complex F^* with a graded commutative product which is compatible with the differential (the Leibniz rule), i.e.*

1. $a \cdot b = (-1)^{kl} b \cdot a$,

2. $d(a \cdot b) = da \cdot b + (-1)^k a \cdot db$.

where $a \in F^k$ and $b \in F^l$.

If a cochain complex F^* is a DGA, then there is an induced ring structure on the cohomology group $H^*(F^*)$ and the product $[a] \cdot [b]$ can be represented by the element $a \cdot b$. (We may omit $\cdot$ and write ab for the product in the future.)

A well-known example for DGA is the de Rham complex associated to a smooth manifold with wedge product.

Definition 2.4.2. *We say a spark complex (F^*, E^*, I^*) is a* ***spark complex of differential graded algebras****, if F^* is a differential graded algebra, and E^*, I^* are differential graded subalgebras of F^*.*

We show in the following theorem that the product in F^* induces a product in $\hat{\mathbf{H}}^*$.

Theorem 2.4.3. *If (F^*, E^*, I^*) is a spark complex of differential graded algebras, then there is an induced ring structure on $\hat{\mathbf{H}}^*$ with the formula*

$$[a] \cdot [b] = [af + (-1)^{k+1} rb] = [as + (-1)^{k+1} eb] \in \hat{\mathbf{H}}^{k+l+1}$$

where $[a] \in \hat{\mathbf{H}}^k$ with $da = e - r$ and $[b] \in \hat{\mathbf{H}}^l$ with $db = f - s$.

Proof. It is easy to verify

$$d(af + (-1)^{k+1} rb) = d(as + (-1)^{k+1} eb) = ef - rs,$$

$$(af + (-1)^{k+1} rb) - (as + (-1)^{k+1} eb) = (-1)^k d(ab).$$

So $af + (-1)^{k+1} rb$ and $as + (-1)^{k+1} eb$ are sparks and represent the same spark class.

To show that the product is independent of the choices of representatives, we assume that the spark $a' \in F^k$ represents the same spark class with a and $da' = e' - r'$. Then $\exists c \in F^{k-1}$ and $t \in I^k$ with $a - a' = dc + t$. We have

$$(af + (-1)^{k+1} rb) - (a'f + (-1)^{k+1} r'b) = d(cf + (-1)^k (tb)) + ts.$$

By the same calculation we can show the product is also independent of the choices of representatives of the second factor. □

Corollary 2.4.4. *$\hat{\mathbf{H}}^*$ is a graded commutative ring, for $\alpha \in \hat{\mathbf{H}}^k$ and $\beta \in \hat{\mathbf{H}}^l$, we have*

$$\alpha \cdot \beta = (-1)^{(k+1)(l+1)} \beta \cdot \alpha.$$

Proof.

$$[a] \cdot [b] = [af + (-1)^{k+1} rb] = [(-1)^{k(l+1)} fa + (-1)^{k+1+(k+1)l} br]$$

$$= (-1)^{(k+1)(l+1)}[br + (-1)^{l+1} fa] = [b] \cdot [a].$$

□

Corollary 2.4.5. *From the formula*

$$d(af + (-1)^{k+1} rb) = ef - rs,$$

it is easy to see the group homomorphisms

$$\delta_1 : \hat{\mathbf{H}}^* \to Z_I^{*+1}(E^*) \quad \textit{and} \quad \delta_2 : \hat{\mathbf{H}}^* \to H^{*+1}(I^*)$$

are ring homomorphisms.

It is reasonable to shift the index of our notation $\hat{\mathbf{H}}^*$ from the last two corollaries. However, we keep our notation for the consistency with historical papers [CS] [HL2].

Corollary 2.4.6. *A quasi-isomorphism of differential graded spark complexes* (F^*, E^*, I^*) *and* $(\bar{F}^*, \bar{E}^*, \bar{I}^*)$ *induces an ring isomorphism*

$$\hat{\mathbf{H}}^*(F^*, E^*, I^*) \cong \hat{\mathbf{H}}^*(\bar{F}^*, \bar{E}^*, \bar{I}^*).$$

Remark 2.4.7. *In our future examples, the product on* F^* *may not be well-defined in general. However, if for any two spark classes in* $\hat{\mathbf{H}}^*$*, there always* *exist good representatives such that their products are well-defined, then we can define the ring structure on* $\hat{\mathbf{H}}^*$ *as well. The case we will meet is similar to the case when cap product of singular homology is defined: the intersection of two cycles may not be well-defined, but we can always deform one a little bit such that they meet transversally and then the intersection is well-defined.*

2.5 Massey Products

If F^* is a DGA, then $H^*(F^*)$ is a graded ring with the induced product. Moreover, we can define higher operations on $H^*(F^*)$.

Definition 2.5.1. *Assume* $\alpha \in H^i(F^*)$, $\beta \in H^j(F^*)$ *and* $\gamma \in H^k(F^*)$ *with* $\alpha\beta = 0 \in H^{i+j}(F^*)$ *and* $\beta\gamma = 0 \in H^{j+k}(F^*)$. *Choose representatives* $a \in \alpha$, $b \in \beta$ *and* $c \in \gamma$, *then there exist* $A \in F^{i+j-1}$ *and* $B \in F^{j+k-1}$ *such that* $ab = dA$ *and* $bc = dB$. *We define the* ***Massey triple product***

$$\mathcal{M}(\alpha, \beta, \gamma) \triangleq [aB + (-1)^{i+1}Ac] \in H^{i+j+k-1}/(\alpha H^{j+k-1} + H^{i+j-1}\gamma).$$

Proposition 2.5.2. *The Massey triple product is well-defined.*

Proof. First, $d(aB + (-1)^{i+1}Ac) = (-1)^i adB + (-1)^{i+1}dAc = (-1)^i abc + (-1)^{i+1}abc = 0$, so $aB + (-1)^{i+1}Ac \in F^{i+j+k-1}$ is a cocycle and represents a class in $H^{i+j+k-1}(F^*)$.

It is easy to verify that the class $[aB+(-1)^{i+1}Ac] \in H^{i+j+k-1}(F^*)$ is independent of choices of representatives a, b and c. Considering different choices of A and B, the Massey triple product is well-defined in $H^{i+j+k-1}/(\alpha H^{j+k-1} + H^{i+j-1}\gamma)$. □

The Massey triple product in the cohomology of a DGA is a special case of Massey higher products in the cohomology of twisted complexes which we refer to [K] [De] for interested readers.

Example 2.5.3. *Let X be a smooth manifold and $\mathcal{E}^*(X)$ denote the de Rham complex on X. $\mathcal{E}^*(X)$ is a graded differential algebra. Hence we define the Massey triple product in de Rham cohomology $H^*_{DR}(X)$.*

A famous result in [DGMS] says $\mathcal{E}^*(X)$ is formal when X is Kähler. In particular, all Massey higher products in $H^*_{DR}(X)$ are trivial.

Let (F^*, E^*, I^*) be a spark complex of differential graded algebras. We define the Massey triple product in the ring of spark classes $\hat{\mathbf{H}}^*$ as follows.

Let $\alpha \in \hat{\mathbf{H}}^i$, $\beta \in \hat{\mathbf{H}}^j$ and $\gamma \in \hat{\mathbf{H}}^k$ be three spark classes. Choose representatives $a \in \alpha$, $b \in \beta$ and $c \in \gamma$ with the spark equations

$$da = e - r, \quad db = f - s, \quad dc = g - t.$$

Assume $\alpha\beta \in \hat{\mathbf{H}}^{i+j+1}_E \subset \hat{\mathbf{H}}^{i+j+1}$ and $\beta\gamma \in \hat{\mathbf{H}}^{j+k+1}_E \subset \hat{\mathbf{H}}^{j+k+1}$, i.e.

$$\alpha\beta = [a][b] = [af + (-1)^{i+1} rb] = [\phi], \quad \beta\gamma = [b][c] = [bg + (-1)^{j+1} sc] = [\psi]$$

for some $\phi \in E^{i+j+1}$ and $\psi \in E^{j+k+1}$.

Then there exist $A \in F^{i+j}$, $B \in F^{j+k}$, $X \in I^{i+j+1}$ and $Y \in I^{j+k+1}$ such that

$$af + (-1)^{i+1} rb = dA + X + \phi, \quad bg + (-1)^{j+1} sc = dB + Y + \psi.$$

At the same time,

$$\begin{aligned} d(af+(-1)^{i+1}rb) &= d(dA+X+\phi) \\ \Rightarrow \quad ef-rs &= dX+d\phi, \end{aligned}$$

hence

$$ef = d\phi, \quad -rs = dX.$$

Similarly, we have

$$fg = d\psi, \quad -st = dY.$$

Consider the element $a\psi + (-1)^i rB + (-1)^{i+1}Ag + (-1)^j Xc \in F^{i+j+k+1}$.

$$\begin{aligned} d(a\psi) &= da\psi + (-1)^i ad\psi \\ &= e\psi - r\psi + (-1)^i afg \end{aligned}$$

$$\begin{aligned} d((-1)^i rB) &= (-1)^{i+i+1} rdB \\ &= -r(bg + (-1)^{j+1}sc - Y - \psi) \\ &= -rbg + (-1)^j rsc + rY + r\psi \end{aligned}$$

$$\begin{aligned} d((-1)^{i+1}Ag) &= (-1)^{i+1}dAg \\ &= (-1)^{i+1}(af + (-1)^{i+1}rb - X - \phi)g \\ &= (-1)^{i+1}afg + rbg + (-1)^i Xg + (-1)^i \phi g \end{aligned}$$

$$\begin{aligned} d((-1)^j Xc) &= (-1)^j(dXc + (-1)^{i+j+1}Xdc) \\ &= (-1)^j(-rsc + (-1)^{i+j+1}X(g-t)) \\ &= (-1)^{j+1}rsc + (-1)^{i+1}Xg + (-1)^i Xt \end{aligned}$$

Finally, we have

$$d(a\psi + (-1)^i rB + (-1)^{i+1}Ag + (-1)^j Xc) = e\psi + (-1)^i \phi g + rY + (-1)^i Xt,$$

where $e\psi + (-1)^i\phi g \in E^{i+j+k+2}$ and $rY + (-1)^i Xt \in I^{i+j+k+2}$.

So $a\psi+(-1)^i rB+(-1)^{i+1}Ag+(-1)^j Xc \in F^{i+j+k+1}$ represents a spark class. We define the Massey triple product of α, β and γ, denoted by $\mathcal{M}(\alpha,\beta,\gamma)$, by this spark class.

In fact, the Massey triple product is well-defined in $\hat{\mathbf{H}}^{i+j+k+1}$ up to the subgroup $\alpha\hat{\mathbf{H}}^{j+k} + \hat{\mathbf{H}}^{i+j}\gamma$.

First, it is routine to verify that the triple product is independent of the choices of representatives a, b and c. Moreover, we may consider the different choices of A, B, X, Y, ϕ and ψ. Explicitly, if we have $af + (-1)^{i+1}rb = dA + X + \phi = dA' + X' + \phi'$, then

$$a\psi+(-1)^i rB+(-1)^{i+1}Ag+(-1)^j Xc-(a\psi+(-1)^i rB+(-1)^{i+1}A'g+(-1)^j X'c)$$

$$= (-1)^j(X-X')c + (-1)^{i+1}(A-A')g.$$

Notice that $d(A-A') = -(\phi-\phi') - (X-X')$, hence $A-A'$ represents a spark class. Moreover

$$[A-A'][c] = [(A-A')g + (-1)^{i+j+1}(X-X')c],$$

hence the difference $(-1)^j(X-X')c+(-1)^{i+1}(A-A')g \in (-1)^{i+1}[A-A'][c] \in \hat{\mathbf{H}}^{i+j}\gamma$.

Similarly, if $bg + (-1)^{j+1}sc = dB + Y + \psi = dB' + Y' + \psi'$, then

$$a\psi+(-1)^i rB+(-1)^{i+1}Ag+(-1)^j Xc-(a\psi'+(-1)^i rB'+(-1)^{i+1}Ag+(-1)^j Xc)$$

$$= a(\psi-\psi') + (-1)^i r(B-B').$$

$B-B'$ is a spark satisfying the spark equation $d(B-B') = -(\psi-\psi')-(Y-Y')$. The product

$$[a][B-B'] = [-a(\psi-\psi')+(-1)^{i+1}r(B-B')] = [-(a(\psi-\psi')+(-1)^i r(B-B'))],$$

Therefore,

$$a(\psi-\psi') + (-1)^i r(B-B') \in -[a][B-B'] \in \alpha\hat{\mathbf{H}}^{j+k}.$$

Theorem 2.5.4. *Let (F^*, E^*, I^*) be a spark complex of differential graded algebras and $\hat{\mathbf{H}}^*$ be the ring of associated spark classes. If three classes $\alpha \in \hat{\mathbf{H}}^i$, $\beta \in \hat{\mathbf{H}}^j$ and $\gamma \in \hat{\mathbf{H}}^k$ satisfy that $\alpha\beta \in \hat{\mathbf{H}}_E^{i+j+1} \subset \hat{\mathbf{H}}^{i+j+1}$ and $\beta\gamma \in \hat{\mathbf{H}}_E^{j+k+1} \subset \hat{\mathbf{H}}^{j+k+1}$, then the Massey triple product of α, β and γ, denoted by $\mathcal{M}(\alpha,\beta,\gamma)$, is well-defined in $\hat{\mathbf{H}}^{i+j+k+1}/(\alpha\hat{\mathbf{H}}^{j+k} + \hat{\mathbf{H}}^{i+j}\gamma)$.*

The Massey triple product is well-defined in $H^*(I^*)$. Also, it is easy to see that we can define the Massey triple product in $Z_I^*(E^*)$ formally. Explicitly, let $B^*(E^*) \subset Z_I^*(E^*)$ be groups of boundaries, then

$$\mathcal{M}(e, f, g) \equiv e \cdot d^{-1}(fg) + (-1)^{i+1} d^{-1}(ef) \cdot g \in Z_I^{i+j+k-1}(E^*)/(eZ_I^{j+k-1} + Z_I^{i+j-1} g)$$

is well-defined for $e \in Z_I^i(E^*)$, $f \in Z_I^j(E^*)$ and $g \in Z_I^k(E^*)$ provided that $ef \in B^{i+j}(E^*)$ and $fg \in B^{j+k}(E^*)$. By $d^{-1}(fg)$ we mean any element whose boundary is fg.

Corollary 2.5.5. *The Massey triple product is compatible with the ring homomorphisms*

$$\delta_1 : \hat{\mathbf{H}}^* \to Z_I^{*+1}(E^*) \quad \textit{and} \quad \delta_2 : \hat{\mathbf{H}}^* \to H^{*+1}(I^*).$$

Proof. In the proof of the last theorem, we see $\alpha\beta \in \hat{\mathbf{H}}_E^{i+j+1}$ and $\beta\gamma \in \hat{\mathbf{H}}_E^{j+k+1}$ imply

$$ef = d\phi, \quad fg = d\psi \quad \text{and} \quad -rs = dX, \quad -rt = dY.$$

So $\mathcal{M}(e, f, g)$ and $\mathcal{M}([r], [s], [t])$ are well-defined, and

$$\mathcal{M}(e, f, g) = e\psi + (-1)^i \phi g,$$

$$\mathcal{M}([r], [s], [t]) = [-(rY + (-1)^i Xt)].$$

From the formula

$$d(a\psi + (-1)^i rB + (-1)^{i+1} Ag + (-1)^j Xc) = e\psi + (-1)^i \phi g + rY + (-1)^i Xt,$$

we have

$$\delta_1([a\psi + (-1)^i rB + (-1)^{i+1} Ag + (-1)^j Xc]) = e\psi + (-1)^i \phi g,$$

$$\delta_2([a\psi + (-1)^i rB + (-1)^{i+1} Ag + (-1)^j Xc]) = [-(rY + (-1)^i Xt)].$$

This shows the compatibility of the Massey triple product with ring homomorphisms δ_1 and δ_2. □

We can construct the **Massey quadruple product** or even higher products following the same idea. Here we sketch a construction of the quadruple product.

First recall the Massey quadruple product in the cohomology ring $H^*(F^*)$ of a differential graded algebra F^*. Assume $\alpha_i \in H^{k_i}(F^*)$ $(i = 1, 2, 3, 4)$ satisfy that the cup products $\alpha_i \alpha_{i+1} = 0$ for $i = 1, 2, 3$ and that the Massey triple products $\mathcal{M}(\alpha_i, \alpha_{i+1}, \alpha_{i+2})$ for $i = 1, 2$ vanish simultaneously (see [O]). Then we can define the Massey quadruple product $\mathcal{M}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. It follows the vanishing condition above that $\exists$ representatives $a_i \in F^{k_i}$ for α_i and a_{12}, a_{23}, a_{34}, a_{13}, $a_{24} \in F^*$ such that

$$da_{12} = a_1 a_2, \qquad da_{23} = a_2 a_3, \qquad da_{34} = a_3 a_4$$

and

$$da_{13} = a_1 a_{23} + (-1)^{k_1+1} a_{12} a_3, \qquad da_{24} = a_2 a_{34} + (-1)^{k_2+1} a_{23} a_4.$$

Then we define $\mathcal{M}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in H^{k-2}(F^*)$ as the class represented by

$$a_1 a_{24} + (-1)^{k_1+1} a_{12} a_{34} + (-1)^{k_1+k_2} a_{13} a_4 \in F^{k-2}$$

up to some indeterminacy, where $k = k_1 + k_2 + k_3 + k_4$.

Now we show a construction of the Massey quadruple product in $\hat{\mathbf{H}}^*$ associated to a spark complex (F^*, E^*, I^*). Let $\alpha_i \in \hat{\mathbf{H}}^{k_i}$ $(i = 1, 2, 3, 4)$ be four spark classes satisfying that the products $\alpha_i \alpha_{i+1} \in \hat{\mathbf{H}}^*_E$ for $i = 1, 2, 3$ and that $\mathcal{M}(\alpha_i, \alpha_{i+1}, \alpha_{i+2})$ for $i = 1, 2$ are well-defined and in $\hat{\mathbf{H}}^*_E$. Then there exist $a_i \in F^{k_i}$ representing α_i with spark equation $da_i = e_i - r_i$ and $A_{i,j} \in F^*$, $\phi_{i,j} \in E^*$, $X_{i,j} \in I^*$ $(i, j) \in \{(1, 2), (2, 3), (3, 4), (1, 3), (2, 4)\}$ satisfying the following equations

$$a_i e_{i+1} + (-1)^{k_i+1} r_i a_{i+1} = \phi_{i,i+1} + dA_{i,i+1} + X_{i,i+1}, \quad i = 1, 2, 3,$$

and

$$a_i \phi_{i+1,i+2} + (-1)^{k_i} r_i A_{i+1,i+2} + (-1)^{k_i+1} A_{i,i+1} e_{i+2} + (-1)^{k_{i+1}} X_{i,i+1} a_{i+2}$$

$$= \phi_{i,i+2} + dA_{i,i+2} + X_{i,i+2}, \quad i = 1, 2.$$

We define the Massey quadruple product $\mathcal{M}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in $\hat{\mathbf{H}}^{k-2}$ (up to some indeterminacy) represented by the spark

$$a_1\phi_{24} + (-1)^{k_1} r_1 A_{24} + (-1)^{k_1+1} A_{12}\phi_{34} + (-1)^{k_2+1} X_{12} A_{34} + (-1)^{k_1+k_2+1} A_{13} e_4 + (-1)^{k_3} X_{13} a_4.$$

By direct calculation, its differential equals

$$e_1\phi_{24}+(-1)^{k_1}\phi_{12}\phi_{34}+(-1)^{k_1+k_2}\phi_{13}e_4+r_1X_{24}+(-1)^{k_1+1}X_{12}X_{34}+(-1)^{k_1+k_2}X_{13}r_4.$$

It follows that the quadruple product is compatible with δ_1 and δ_2.

In §3.5, we shall study Massey higher products in secondary geometric invariants. Moreover, a theory on the Massey products in Deligne cohomology will be developed in §5.2.

Chapter 3

Secondary Geometric Invariants: From the Viewpoint of Spark Theory

In this chapter, we study secondary geometric invariants of smooth manifolds from the viewpoint of Harvey-Lawson spark theory. In §3.1, we study several examples of spark complex associated to a smooth manifold X and define Harvey-Lawson spark characters $\hat{\mathbf{H}}^*(X)$. In §3.2, we focus on the smooth hyperspark complex and define the ring structure on $\hat{\mathbf{H}}^*(X)$ via the smooth hyperspark complex. This ring structure is shown to be equivalent to the one introduced in [HLZ] via the de Rham-Federer spark complex. We study explicit examples for low dimensional manifolds in §3.3. In particular, we calculate the product of two spark characters of degree 0 on the unit circle. In §3.4, we study smooth Deligne cohomology and show an explicit construction of the isomorphism between groups of spark classes and the (p, p) part of smooth Deligne cohomology groups associated to a smooth manifold. Moreover, we show that this is an isomorphism of ring structures. Since Harvey and Lawson [HL2] showed the ring isomorphism between $\hat{\mathbf{H}}^*(X)$ and Cheeger-

Simons differential characters, we conclude that spark characters, differential characters and smooth Deligne cohomology ((p,p) part) are equivalent ring functors over the category of smooth manifolds. In the last section, we study Massey higher products in these secondary geometric invariants. We mainly discuss $(\mathbb{R}/\mathbb{Z})$-characters in this chapter, but all theorems are still true for $(\mathbb{C}/\mathbb{Z})$-characters. All spark complexes in this chapter are spark complex in the sense of Definition 2.1.1.

3.1 Spark Characters

We show our main examples of homological spark complexes and define the Harvey-Lawson spark characters associated to a smooth manifold.

Let X be a smooth manifold of dimension n. Let $\mathcal{E}^k$ denote the sheaf of smooth differential k-forms on X, $\mathcal{D}^k$ the sheaf of currents of degree k on X. Let $\mathcal{R}^k$ and $\mathcal{IF}^k$ denote the sheaf of rectifiable currents of degree k and the sheaf of integrally flat currents of degree k on X respectively. Note that

$$\mathcal{IF}^k(U) = \{r + ds : r \in \mathcal{R}^k(U) \text{ and } s \in \mathcal{R}^{k-1}(U)\}$$

3.1.1 de Rham-Federer Sparks

Definition 3.1.1. *The* ***de Rham-Federer spark complex*** *associated to a smooth manifold X is obtained by taking*

$$F^k = \mathcal{D}'^k(X), \quad E^k = \mathcal{E}^k(X), \quad I^k = \mathcal{IF}^k(X).$$

Remark 3.1.2. *The condition* $H^k(\mathcal{D}'^*(X)) = H^k(\mathcal{E}^*(X)) = H^k(X, \mathbb{R})$ *is standard. For a proof of the fact* $\mathcal{E}^k(X) \cap \mathcal{IF}^k(X) = \{0\}$ *for* $k > 0$*, we refer to [HLZ, Lemma 1.3].*

Definition 3.1.3. *A* ***de Rham-Federer spark*** *of degree* k *is a current* $a \in \mathcal{D}'^k(X)$ *with the spark equation*

$$da = e - r$$

where $e \in \mathcal{E}^{k+1}(X)$ *is smooth and* $r \in \mathcal{IF}^{k+1}(X)$ *is integrally flat.*

Two sparks a *and* a' *are* ***equivalent*** *if there exist* $b \in \mathcal{D}'^{k-1}(X)$ *and* $s \in \mathcal{IF}^k(X)$ *with*

$$a - a' = db + s.$$

The equivalence class determined by a spark a *will be denoted by* $[a]$ *and the group of de Rham-Federer spark classes will be denoted by* $\hat{\mathbf{H}}^k_{spark}(X)$*.*

Let $\mathcal{Z}^k_0(X)$ denote the group of closed degree k forms on X with integral periods. Note that $H^k(\mathcal{IF}^*(X)) = H^k(X, \mathbb{Z})$. By Lemma 2.1.4 and Proposition 2.1.5, we have

Proposition 3.1.4. *[HLZ] There exist well-defined surjective homomorphisms*

$$\delta_1 : \hat{\mathbf{H}}^k(X) \to \mathcal{Z}^{k+1}_0(X) \quad \text{and} \quad \delta_2 : \hat{\mathbf{H}}^k(X) \to H^{k+1}(X, \mathbb{Z})$$

given by

$$\delta_1([a]) = e \quad \text{and} \quad \delta_2([a]) = [r]$$

where $da = e - r$*.*

Associated to the de Rham-Federer spark complex is the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^k(X,\mathbb{R})}{H^k_{free}(X,\mathbb{Z})} & \longrightarrow & \hat{\mathbf{H}}^k_\infty(X) & \longrightarrow & d\mathcal{E}^k(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^k(X,\mathbb{R}/\mathbb{Z}) & \longrightarrow & \hat{\mathbf{H}}^k(X) & \xrightarrow{\delta_1} & \mathcal{Z}^{k+1}_0(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \delta_2} & & \downarrow & & \\
0 & \longrightarrow & H^{k+1}_{tor}(X,\mathbb{Z}) & \longrightarrow & H^{k+1}(X,\mathbb{Z}) & \longrightarrow & H^{k+1}_{free}(X,\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where $\hat{\mathbf{H}}^k_\infty(X) \cong \mathcal{E}^k(X)/\mathcal{Z}^k_0(X)$ *denote the group of spark classes of degree* k *which can be represented by smooth forms.*

3.1.2 Hypersparks and Smooth Hypersparks

Suppose $\mathcal{U} = \{U_i\}$ is a good cover of X (with each intersection U_I contractible). We have the Čech-Current bicomplex $\bigoplus_{p,q\geq 0} C^p(\mathcal{U}, \mathcal{D}'^q)$. Now we are concerned with the total complex of Čech-Current bicomplex $\bigoplus_{p+q=*} C^p(\mathcal{U}, \mathcal{D}'^q)$ with total differential $D = \delta + (-1)^p d$.

Definition 3.1.5. *By the* ***hyperspark complex*** *we mean the spark complex defined as*

$$(F^*, E^*, I^*) = (\bigoplus_{p+q=*} C^p(\mathcal{U}, \mathcal{D}'^q), \mathcal{E}^*(X), \bigoplus_{p+q=*} C^p(\mathcal{U}, \mathcal{IF}^q)).$$

Remark 3.1.6. *We should verify the triple of complexes above is a spark* *complex. There is a natural inclusion* $\mathcal{E}^*(X) \hookrightarrow \bigoplus_{p+q=*} C^p(\mathcal{U}, \mathcal{D}'^q)$*, given by*

$$\mathcal{E}^*(X) \hookrightarrow \mathcal{D}'^*(X) \hookrightarrow C^0(\mathcal{U}, \mathcal{D}'^*) \hookrightarrow \bigoplus_{p+q=*} C^p(\mathcal{U}, \mathcal{D}'^q).$$

For $k > 0$*,* $\mathcal{E}^k(X) \cap \bigoplus_{p+q=k} C^p(\mathcal{U}, \mathcal{IF}^q)) = \mathcal{E}^k(X) \cap C^0(\mathcal{U}, \mathcal{IF}^k)) = \mathcal{E}^k(X) \cap \mathcal{IF}^k(X) = \{0\}$*.*

And it is easy to see $H^*(F^*) = H^*(\mathcal{D}'^*(X)) = H^*(X, \mathbb{R}) = H^*(E^*)$*, and also* $H^*(I^*) = H^*(C^*(\mathcal{U}, \mathbb{Z})) = H^*(X, \mathbb{Z})$*.*

Definition 3.1.7. *A* ***hyperspark*** *of degree* k *is an element*

$$a \in \bigoplus_{p+q=k} C^p(\mathcal{U}, \mathcal{D}'^q)$$

with the spark equation

$$Da = e - r$$

where $e \in \mathcal{E}^{k+1}(X) \subset C^0(\mathcal{U}, \mathcal{D}'^{k+1})$ *and* $r \in \bigoplus_{p+q=k+1} C^p(\mathcal{U}, \mathcal{IF}^q)$*.*

Two hypersparks a *and* a' *are said to be* ***equivalent*** *if there exist* $b \in \bigoplus_{p+q=k-1} C^p(\mathcal{U}, \mathcal{D}'^q)$ *and* $s \in \bigoplus_{p+q=k} C^p(\mathcal{U}, \mathcal{IF}^q))$ *satisfying*

$$a - a' = Db + s.$$

The equivalence class determined by a hyperspark a *will be denoted by* $[a]$*, and the group of hyperspark classes will be denoted by* $\hat{\mathbf{H}}^k_{hyper}(X)$*.*

Proposition 3.1.8.

$$\hat{\mathbf{H}}^k_{spark}(X) \cong \hat{\mathbf{H}}^k_{hyper}(X).$$

Proof. It is easy to see that there is a natural inclusion from the de Rham-Federer spark complex to the hyperspark complex which is a quasi-isomorphism.

□

We may consider the de Rham-Federer spark complex as a spark subcomplex of the hyperspark complex, now we introduce another spark subcomplex of the hyperspark complex, which is called the smooth hyperspark complex.

Definition 3.1.9. *By the* ***smooth hyperspark complex*** *we mean the spark complex*

$$(F^*, E^*, I^*) = (\bigoplus_{p+q=*} C^p(\mathcal{U}, \mathcal{E}^q), \mathcal{E}^*(X), C^*(\mathcal{U}, \mathbb{Z})).$$

Definition 3.1.10. *A* ***smooth hyperspark*** *of degree k is an element*

$$a \in \bigoplus_{p+q=k} C^p(\mathcal{U}, \mathcal{E}^q)$$

with the spark equation

$$Da = e - r$$

where $e \in \mathcal{E}^{k+1}(X) \subset C^0(\mathcal{U}, \mathcal{E}^{k+1})$ and $r \in C^{k+1}(\mathcal{U}, \mathbb{Z})$.

Two smooth hypersparks a and a' are ***equivalent*** *if there exist*

$$b \in \bigoplus_{p+q=k-1} C^p(\mathcal{U}, \mathcal{E}^q) \quad \text{and} \quad s \in C^k(\mathcal{U}, \mathbb{Z})$$

satisfying $a - a' = Db + s$.

The equivalence class determined by a smooth hyperspark a will be denoted by $[a]$, and the group of smooth hyperspark classes will be denoted by $\hat{\mathbf{H}}^k_{smooth}(X)$.

One can easily verify that the smooth hyperspark complex is quasi-isomorphic to the hyperspark complex. Hence, we have

Proposition 3.1.11.

$$\hat{\mathbf{H}}^k_{smooth}(X) \cong \hat{\mathbf{H}}^k_{hyper}(X).$$

Corollary 3.1.12.

$$\hat{\mathbf{H}}^k_{spark}(X) \cong \hat{\mathbf{H}}^k_{smooth}(X).$$

We can consider the hyperspark complex as a bridge which connects the de Rham-Federer spark complex and the smooth hyperspark complex.

3.1.3 Harvey-Lawson Spark Characters

We defined three homological spark complexes associated to a smooth manifold X, and showed the natural isomorphisms between the groups of spark classes associated to them. We refer to [HL2] for more very interesting spark complexes whose groups of spark classes are all isomorphic to each other. We denote the groups of spark classes by $\hat{\mathbf{H}}^*(X)$ collectively, and call them the **Harvey-Lawson spark characters** associated to X.

An important fact is that $\hat{\mathbf{H}}^*(X)$ has a ring structure which is functorial with respect to smooth maps between manifolds. This ring structure on $\hat{\mathbf{H}}^*(X)$ was defined in [HLZ] via the de Rham-Federer spark complex. The main technical difficulty is that the wedge product of two currents may not be well-defined. However, we can always choose good representatives in the following sense:

Proposition 3.1.13. *[HLZ, Proposition 3.1] Given classes* $\alpha \in \hat{\mathbf{H}}^k_{spark}(X)$ *and* $\beta \in \hat{\mathbf{H}}^l_{spark}(X)$ *there exist representatives* $a \in \alpha$ *and* $b \in \beta$ *with* $da = e - r$ *and* $db = f - s$ *so that* $a \wedge s$, $r \wedge b$ *and* $r \wedge s$ *are well-defined flat currents on* X *and* $r \wedge s$ *is rectifiable.*

Theorem 3.1.14. *[HLZ, Theorem 3.5] Setting*

$$\alpha * \beta \equiv [a \wedge f + (-1)^{k+1} r \wedge b] = [a \wedge s + (-1)^{k+1} e \wedge b] \in \hat{\mathbf{H}}^{k+l+1}_{spark}(X)$$

gives $\hat{\mathbf{H}}^*_{spark}(X)$ *the structure of a graded commutative ring such that* $\delta_1 : \hat{\mathbf{H}}^*_{spark}(X) \to \mathcal{Z}^{*+1}_0(X)$ *and* $\delta_2 : \hat{\mathbf{H}}^*_{spark}(X) \to H^{*+1}(X, \mathbb{Z})$ *are ring homomorphisms.*

Proof. It is easy to verify

$$d(a \wedge f + (-1)^{k+1} r \wedge b) = d(a \wedge s + (-1)^{k+1} e \wedge b) = e \wedge f - r \wedge s,$$

$$(a \wedge f + (-1)^{k+1} r \wedge b) - (a \wedge s + (-1)^{k+1} e \wedge b) = (-1)^k d(a \wedge b).$$

So $a \wedge f + (-1)^{k+1} r \wedge b$ and $a \wedge s + (-1)^{k+1} e \wedge b$ are sparks and represent the same spark class.

To show that the product is independent of the choices of representatives, assume the spark $a' \in \mathcal{D}'^k(X)$ represent the same spark class with a and $da' = e' - r'$. Then $\exists c \in \mathcal{D}'^{k-1}(X)$ and $t \in \mathcal{IF}^k(X)$ with $a - a' = dc + t$. We have

$$(a \wedge f + (-1)^{k+1} r \wedge b) - (a' \wedge f + (-1)^{k+1} r' \wedge b) = d(c \wedge f + (-1)^k (t \wedge b)) + t \wedge s.$$

By the same calculation we can show the product is also independent of the choices of representatives of the second factor.

We can calculate

$$\beta*\alpha = [b\wedge e+(-1)^{l+1}s\wedge a] = (-1)^{(k+1)(l+1)}[a\wedge s+(-1)^{k+1}e\wedge b] = (-1)^{(k+1)(l+1)}\alpha*\beta,$$

i.e. the product is graded commutative.

It is easy to show the product is associative. □

The following theorems on the functoriality were shown in [HLZ].

Theorem 3.1.15. *[HLZ] Any smooth map $f : X \to Y$ between two smooth manifolds induces a graded ring homomorphism*

$$f^* : \hat{\mathbf{H}}^*(Y) \to \hat{\mathbf{H}}^*(X)$$

compatible with δ_1 and δ_2. Moreover, if $g : Y \to Z$ is smooth, then $(g \circ f)^* = f^* \circ g^*$.

Therefore, we can consider $\hat{\mathbf{H}}^*(\bullet)$ as a graded ring functor on the category of smooth manifolds and smooth maps.

Theorem 3.1.16. *[HLZ](Gysin map) Any smooth proper submersion $f : X \to Y$ between two smooth manifolds induces a Gysin homomorphism*

$$f_* : \hat{\mathbf{H}}^*(X) \to \hat{\mathbf{H}}^{*-d}(Y)$$

compatible with δ_1 and δ_2, where $d = \dim X - \dim Y$.

3.1.4 Cheeger-Simons Differential Characters

Cheeger and Simons introduced differential characters in their remarkable paper [CS].

Let X be a smooth manifold. And let $C_k(X) \supset Z_k(X) \supset B_k(X)$ denote the groups of smooth singular k-chains, cycles and boundaries.

Definition 3.1.17. *The group of* ***differential characters*** *of degree k is defined by*

$$\hat{H}^k_{CS}(X, \mathbb{R}/\mathbb{Z}) = \{h \in \hom(Z_k(X), \mathbb{R}/\mathbb{Z}) : dh \equiv \omega \mod \mathbb{Z}, \text{ for some } \omega \in \mathcal{E}^{k+1}(X)\}.$$

Similarly, we can define $\hat{H}^k_{CS}(X, \mathbb{C}/\mathbb{Z})$. We write $\hat{H}^k_{CS}(X)$ when the coefficient is clear in the context.

Remark 3.1.18. *For any $\sigma \in C_{k+1}(X)$, $(dh)(\sigma) = h \circ \partial(\sigma)$. In the definition above, $dh \equiv \omega \mod \mathbb{Z}$ means $h \circ \partial(\sigma) \equiv \int_{\Delta_{k+1}} \sigma^*(\omega) \mod \mathbb{Z}$, $\forall \sigma \in C_{k+1}(X)$.*

Cheeger and Simons also defined the ring structure on $\hat{H}^*_{CS}(X)$ and showed the functoriality of $\hat{H}^*_{CS}(X)$. Harvey, Lawson and Zweck [HL2][HLZ] established the equivalency of differential characters and spark characters.

Theorem 3.1.19. *[HL2] [HLZ]*

$$\hat{\mathbf{H}}^*(X) \cong \hat{H}^*_{CS}(X).$$

Therefore, we may consider that de Rham-Federer spark classes, hyperspark classes and smooth hyperspark classes are different representations of Cheeger-Simons differential characters.

3.2 Ring Structure via the Smooth Hyperspark Complex

We introduced the Harvey-Lawson spark characters and established the ring structure. In this section, we give a new description of the ring structure via the smooth hyperspark complex.

Consider the smooth hyperspark complex

$$(F^*, E^*, I^*) = (\bigoplus_{p+q=*} C^p(\mathcal{U}, \mathcal{E}^q), \mathcal{E}^*(X), C^*(\mathcal{U}, \mathbb{Z})).$$

Recall there is a cup product on the cochain complex $C^*(\mathcal{U}, \mathbb{Z})$ which induces the ring structure on $H^*(X, \mathbb{Z})$.

Proposition 3.2.1. *For $a \in C^r(\mathcal{U}, \mathbb{Z})$ and $b \in C^s(\mathcal{U}, \mathbb{Z})$, we define cup product*

$$(a \cup b)_{i_0,\ldots,i_{r+s}} \equiv a_{i_0,\ldots,i_r} \cdot b_{i_r,\ldots,i_{r+s}}.$$

This product induces an associative, graded commutative product on $\check{H}^*(\mathcal{U}, \mathbb{Z}) \cong H^*(X, \mathbb{Z})$.

Proof. It is easy to verify that $\delta(a \cup b) = \delta a \cup b + (-1)^r a \cup \delta b$ (the Leibniz rule), so the product descends to cohomology. The associativity is trivial. However, a direct proof of graded commutativity is quite complicated, see [Br1, Proposition 1.3.7] and [GH]. □

Now we want to define a cup product on the cochain complex

$$(\bigoplus_{p+q=*} C^p(\mathcal{U}, \mathcal{E}^q), D = \delta + (-1)^p d)$$

which is compatible with products on $\mathcal{E}^*(X)$ and $C^*(\mathcal{U}, \mathbb{Z})$ and descends to its cohomology. A first try is to define

$$(a \cup b)_{i_0,\ldots,i_{r+s}} \equiv a_{i_0,\ldots,i_r} \wedge b_{i_r,\ldots,i_{r+s}} \text{ for } a \in C^r(\mathcal{U}, \mathcal{E}^p) \text{ and } b \in C^s(\mathcal{U}, \mathcal{E}^q).$$

But it turns out that this cup product does not satisfy the Leibniz rule. We modify the product and define

$$(a \cup b)_{i_0,\ldots,i_{r+s}} \equiv (-1)^{js} a_{i_0,\ldots,i_r} \wedge b_{i_r,\ldots,i_{r+s}} \text{ for } a \in C^r(\mathcal{U}, \mathcal{E}^j) \text{ and } b \in C^s(\mathcal{U}, \mathcal{E}^k).$$

Proposition 3.2.2. *We define a **cup product** on the complex* $\bigoplus_{p+q=*} C^p(\mathcal{U}, \mathcal{E}^q)$ *as*

$$(a \cup b)_{i_0,\ldots,i_{r+s}} \equiv (-1)^{js} a_{i_0,\ldots,i_r} \wedge b_{i_r,\ldots,i_{r+s}} \in C^{r+s}(\mathcal{U}, \mathcal{E}^{j+k}),$$

for $a \in C^r(\mathcal{U}, \mathcal{E}^j)$ *and* $b \in C^s(\mathcal{U}, \mathcal{E}^k)$. *This product is associative and satisfies the Leibniz rule, hence it induces a product on its cohomology.*

Proof. Associativity: for $a \in C^r(\mathcal{U}, \mathcal{E}^j)$, $b \in C^s(\mathcal{U}, \mathcal{E}^k)$ and $c \in C^t(\mathcal{U}, \mathcal{E}^l)$, we have

$$((a \cup b) \cup c)_{i_0,\ldots,i_{r+s+t}} = (-1)^{(j+k)t} (a \cup b)_{i_0,\ldots,i_{r+s}} \wedge c_{i_{r+s},\ldots,i_{r+s+t}}$$

$$= (-1)^{jt+kt+js} a_{i_0,\ldots,i_r} \wedge b_{i_r,\ldots,i_{r+s}} \wedge c_{i_{r+s},\ldots,i_{r+s+t}},$$

and

$$(a \cup (b \cup c))_{i_0,\ldots,i_{r+s+t}} = (-1)^{j(s+t)} a_{i_0,\ldots,i_r} \wedge (b \cup c)_{i_r,\ldots,i_{r+s+t}}$$

$$= (-1)^{js+jt+kt} a_{i_0,\ldots,i_r} \wedge b_{i_r,\ldots,i_{r+s}} \wedge c_{i_{r+s},\ldots,i_{r+s+t}}.$$

Hence, the associativity follows.

The Leibniz rule: We want to check the Leibniz rule $D(a \cup b) = Da \cup b + (-1)^{r+j} a \cup Db$ for $a \in C^r(\mathcal{U}, \mathcal{E}^j)$ and $b \in C^s(\mathcal{U}, \mathcal{E}^k)$. We **fix the notation**

$$(a \wedge b)_{i_0,\ldots,i_{r+s}} \equiv a_{i_0,\ldots,i_r} \wedge b_{i_r,\ldots,i_{r+s}},$$

i.e. $a \cup b = (-1)^{js} a \wedge b$.

It is easy to check that

$$d(a \wedge b) = da \wedge b + (-1)^j a \wedge db$$

and

$$\delta(a \wedge b) = \delta a \wedge b + (-1)^r a \wedge \delta b.$$

Furthermore, we verify that $D(a \cup b) = Da \cup b + (-1)^{r+j} a \cup Db$.

$$\begin{aligned}
& D(a \cup b) \\
= \; & (\delta + (-1)^{r+s} d)(a \cup b) \\
= \; & (\delta + (-1)^{r+s} d)(-1)^{js}(a \wedge b) \\
= \; & (-1)^{js} \delta(a \wedge b) + (-1)^{r+s+js} d(a \wedge b) \\
= \; & (-1)^{js}(\delta a \wedge b + (-1)^r a \wedge \delta b) + (-1)^{r+s+js}(da \wedge b + (-1)^j a \wedge db) \\
= \; & (-1)^{js} \delta a \wedge b + (-1)^{js+r} a \wedge \delta b + (-1)^{r+s+js} da \wedge b + (-1)^{r+s+j+js} a \wedge db,
\end{aligned}$$

$$
\begin{aligned}
& Da \cup b + (-1)^{r+j} a \cup Db \\
= \; & (\delta + (-1)^r d) a \cup b + (-1)^{r+j} a \cup (\delta + (-1)^s d) b \\
= \; & \delta a \cup b + (-1)^r da \cup b + (-1)^{r+j} a \cup \delta b + (-1)^{r+j+s} a \cup db \\
= \; & (-1)^{js} \delta a \wedge b + (-1)^{r+(j+1)s} da \wedge b + (-1)^{r+j+j(s+1)} a \wedge \delta b + (-1)^{r+j+s+js} a \wedge db \\
= \; & (-1)^{js} \delta a \wedge b + (-1)^{r+s+js} da \wedge b + (-1)^{js+r} a \wedge \delta b + (-1)^{r+s+j+js} a \wedge db.
\end{aligned}
$$

So the Leibniz rule is verified. □

Remark 3.2.3. *It is easy to see this product is compatible with products on subcomplexes $\mathcal{E}^*(X)$ and $C^*(\mathcal{U}, \mathbb{Z})$.*

The main result of this section is the following theorem.

Theorem 3.2.4. *For two smooth hyperspark classes $\alpha \in \hat{\mathbf{H}}^k_{smooth}(X)$ and $\beta \in \hat{\mathbf{H}}^l_{smooth}(X)$, choose representatives $a \in \alpha$ and $b \in \beta$ with spark equations $Da = e - r$ and $Db = f - s$, where*

$$
a \in \bigoplus_{p+q=k} C^p(\mathcal{U}, \mathcal{E}^q), \quad e \in \mathcal{E}^{k+1}(X) \subset C^0(\mathcal{U}, \mathcal{E}^{k+1}), \quad r \in C^{k+1}(\mathcal{U}, \mathbb{Z}) \subset C^{k+1}(\mathcal{U}, \mathcal{E}^0),
$$

$$
b \in \bigoplus_{p+q=l} C^p(\mathcal{U}, \mathcal{E}^q), \quad f \in \mathcal{E}^{l+1}(X) \subset C^0(\mathcal{U}, \mathcal{E}^{l+1}), \quad s \in C^{l+1}(\mathcal{U}, \mathbb{Z}) \subset C^{l+1}(\mathcal{U}, \mathcal{E}^0).
$$

The product

$$
\alpha * \beta \equiv [a \cup f + (-1)^{k+1} r \cup b] = [a \cup s + (-1)^{k+1} e \cup b] \in \hat{\mathbf{H}}^{k+l+1}_{spark}(X)
$$

is well-defined and gives $\hat{\mathbf{H}}^*_{smooth}(X)$ the structure of a graded commutative *ring such that* $\delta_1 : \hat{\mathbf{H}}^*_{smooth}(X) \to \mathcal{Z}^{*+1}_0(X)$ *and* $\delta_2 : \hat{\mathbf{H}}^*_{smooth}(X) \to H^{*+1}(X, \mathbb{Z})$ *are ring homomorphisms.*

Proof. Since the cup product satisfies the Leibniz rule:

$$D(a \cup b) = Da \cup b + (-1)^{\deg a} a \cup Db,$$

we have

$$\begin{aligned} & D(a \cup f + (-1)^{k+1} r \cup b) \\ = \; & Da \cup f + (-1)^k a \cup Df + (-1)^{k+1} Dr \cup b + (-1)^{k+1+k+1} r \cup Db \\ = \; & (e - r) \cup f + r \cup (f - s) \\ = \; & e \cup f - r \cup f + r \cup a - r \cup s \\ = \; & e \wedge f - r \cup s. \end{aligned}$$

Similarly, we can check

$$D(a \cup s + (-1)^{k+1} e \cup b) = e \wedge f - r \cup s,$$

and

$$(a \cup f + (-1)^{k+1} r \cup b) - (a \cup s + (-1)^{k+1} e \cup b) = (-1)^k d(a \cup b).$$

Therefore, $a \cup f + (-1)^{k+1} r \cup b$ and $a \cup s + (-1)^{k+1} e \cup b$ are sparks and represent the same spark class.

Assume the spark a' represent the same spark class with a and $da' = e' - r'$.

Then $\exists c \in \bigoplus_{p+q=k+1} C^p(\mathcal{U}, \mathcal{E}^q)$ and $t \in C^k(\mathcal{U}, \mathbb{Z})$ with $a - a' = Dc + t$. We have

$$(a \cup f + (-1)^{k+1} r \cup b) - (a' \cup f + (-1)^{k+1} r' \cup b) = D(c \cup f + (-1)^k (t \cup b)) + t \cup s.$$

By the same calculation we can show the product is also independent of the choices of representatives of the second factor. It is easy to check the associativity. It is not easy to give a direct proof of graded commutativity. However, we can see the graded commutativity as a corollary of next theorem. □

Remark 3.2.5. *Since* $f \in C^0(\mathcal{U}, \mathcal{E}^{l+1})$ *and* $r \in C^{k+1}(\mathcal{U}, \mathcal{E}^0)$, *we have*

$$\alpha * \beta \equiv [a \cup f + (-1)^{k+1} r \cup b] = [a \wedge f + (-1)^{k+1} r \wedge b].$$

Theorem 3.2.6. *The products for de Rham-Federer spark classes in Theorem 3.1.14 and for smooth hyperspark classes in Theorem 3.2.4 give the same ring structure. Hence, the isomorphism* $\hat{\mathbf{H}}^*_{spark}(X) \cong \hat{\mathbf{H}}^*_{smooth}(X)$ *is a ring isomorphism.*

Proof. We can define a cup product on the cochain complex $\bigoplus_{p+q=*} C^p(\mathcal{U}, \mathcal{D}'^q)$ as

$$(a \cup b)_{i_0,\dots,i_{r+s}} \equiv (-1)^{js} a_{i_0,\dots,i_r} \wedge b_{i_r,\dots,i_{r+s}},$$

for $a \in C^r(\mathcal{U}, \mathcal{D}'^j)$ and $b \in C^s(\mathcal{U}, \mathcal{D}'^k)$ whenever all $a_{i_0,\dots,i_r} \wedge b_{i_r,\dots,i_{r+s}}$ make sense.

For two spark classes

$$\alpha \in \hat{\mathbf{H}}^k_{hyper}(X) \cong \hat{\mathbf{H}}^k_{spark}(X) \cong \hat{\mathbf{H}}^k_{smooth}(X)$$

and

$$\beta \in \hat{\mathbf{H}}^l_{hyper}(X) \cong \hat{\mathbf{H}}^l_{spark}(X) \cong \hat{\mathbf{H}}^l_{smooth}(X).$$

We can choose representatives of α and β by two ways. First, we choose a smooth hyperspark $a \in \bigoplus_{p+q=k} C^p(\mathcal{U}, \mathcal{E}^q) \subset \bigoplus_{p+q=k} C^p(\mathcal{U}, \mathcal{D}'^q)$ representing α; Second, we choose a de Rham-Federer spark $a' \in \mathcal{D}'^k(X) \subset \bigoplus_{p+q=l} C^p(\mathcal{U}, \mathcal{D}'^q)$ representing α. We may choose b and b' correspondingly as well. Moreover we can choose a' and b' to be "good" representatives in the sense of Proposition 3.1.13.

Assume the spark equations for a, b, a' and b' are

$$Da = e - r, \qquad Db = f - s,$$

and

$$Da' = e - r', \qquad Db' = f - s'$$

where $e \in \mathcal{E}^{k+1}(X)$, $r \in C^{k+1}(\mathcal{U}, \mathbb{Z})$, $f \in \mathcal{E}^{l+1}(X)$, $s \in C^{l+1}(\mathcal{U}, \mathbb{Z})$, $r' \in \mathcal{IF}^{k+1}(X)$ and $s' \in \mathcal{IF}^{l+1}(X)$. Note that all cup products $a \cup b$, $a \cup f$, $r \cup b$, $a' \cup b'$, $a' \cup f$, $r' \cup b'$, etc. are well-defined in $\bigoplus_{p+q=*} C^p(\mathcal{U}, \mathcal{D}'^q)$. Then we can define product via hyperspark complex by choosing all representatives in either the smooth hyperspark complex or the de Rham-Federer complex.

Moreover, the product does not depend on the choices of representatives. In fact, since a and a', b and b' represent the same spark classes, there exist $c \in \bigoplus_{p+q=k-1} C^p(\mathcal{U}, \mathcal{D}'^q)$, $t \in \bigoplus_{p+q=k} C^p(\mathcal{U}, \mathcal{IF}^q)$, $c' \in \bigoplus_{p+q=l-1} C^p(\mathcal{U}, \mathcal{D}'^q)$ and $t' \in \bigoplus_{p+q=l} C^p(\mathcal{U}, \mathcal{IF}^q)$ such that $a - a' = Dc + t$ and $b - b' = Dc' + t'$.

Then

$$
\begin{aligned}
& (a \cup f + (-1)^{k+1} r \cup b) - (a' \cup f + (-1)^{k+1} r' \cup b') \\
= \ & (a \cup f + (-1)^{k+1} r \cup b) - (a \cup f + (-1)^{k+1} r \cup b') \\
& \quad + (a \cup f + (-1)^{k+1} r \cup b') - (a' \cup f + (-1)^{k+1} r' \cup b') \\
= \ & (-1)^{k+1} r \cup (b - b') + (a - a') \cup f + (-1)^{k+1} (r - r') \cup b' \\
= \ & (-1)^{k+1} r \cup (Dc' + t') + (Dc + t) \cup f + (-1)^{k+1} (-Dt) \cup b' \\
= \ & (-1)^{k+1} r \cup t' + D(r \cup c') + D(c \cup f) + t \cup f + (-1)^k D(t \cup b') - t \cup (f - s') \\
= \ & D(r \cup c' + c \cup f + (-1)^k t \cup b') + (-1)^{k+1} r \cup t' + t \cup s'
\end{aligned}
$$

The calculation above shows $(a \cup f + (-1)^{k+1} r \cup b)$ and $(a' \cup f + (-1)^{k+1} r' \cup b')$ represent the same spark class whenever the cup products in the sums $r \cup c' + c \cup f + (-1)^k t \cup b' \in \bigoplus_{p+q=k+l} C^p(\mathcal{U}, \mathcal{D}'^q)$ and $(-1)^{k+1} r \cup t' + t \cup s' \in \bigoplus_{p+q=k+l+1} C^p(\mathcal{U}, \mathcal{IF}^q)$ are well-defined. On one hand, it is trivial to see $r \cup c'$, $c \cup f$ and $(-1)^{k+1} r \cup t'$ are well-defined. On the other hand, because t is only related to a and a', we can change b' and s' if necessary, so that $(-1)^k t \cup b'$ and $t \cup s'$ are well-defined. $\square$

3.3 Examples

Using the representation of secondary invariants by the smooth hyperspark classes, we give a concrete description of secondary invariants of low degrees. Moreover, applying the product formula in the last section, we calculate the product of smooth hyperspark classes explicitly when the manifold is simple.

Degree 0: A smooth hyperspark of degree 0 is an element $a \in C^0(\mathcal{U}, \mathcal{E}^0)$ satisfying the spark equation

$$Da = e - r \text{ with } e \in \mathcal{E}^1(X) \text{ and } r \in C^1(\mathcal{U}, \mathbb{Z}).$$

Moreover,

$$Da = e - r \qquad \Leftrightarrow \quad \delta a = -r \in C^1(\mathcal{U}, \mathbb{Z}) \quad \text{and} \quad da = e \in \mathcal{E}^1(X)$$

$$\Leftrightarrow \quad \delta a \in C^1(\mathcal{U}, \mathbb{Z}).$$

Two smooth hypersparks a and a' are equivalent if and only if $a - a' \in C^0(\mathcal{U}, \mathbb{Z})$. Consider the exponential of a smooth hyperspark $g \equiv e^{2\pi i a}$. $\delta a \in C^1(\mathcal{U}, \mathbb{Z})$ implies g is a global circle valued function, and $a - a' \in C^0(\mathcal{U}, \mathbb{Z}) \Leftrightarrow e^{2\pi i a} = e^{2\pi i a'}$. Therefore, we have

$$\hat{\mathbf{H}}^0_{smooth}(X) = \{g : X \to S^1 : g \text{ is smooth }\}.$$

Degree 1: A smooth hyperspark of degree 1 is an element

$$a = a^{0,1} + a^{1,0} \in C^0(\mathcal{U}, \mathcal{E}^1) \oplus C^1(\mathcal{U}, \mathcal{E}^0)$$

satisfying the spark equation

$$Da = e - r \text{ with } e \in \mathcal{E}^2(X) \text{ and } r \in C^2(\mathcal{U}, \mathbb{Z})$$

which is equivalent to equations $\begin{cases} \delta a^{1,0} = -r \in C^2(\mathcal{U}, \mathbb{Z}) \\ \delta a^{0,1} - d a^{1,0} = 0 \\ d a^{0,1} = e \in \mathcal{E}^2(X) \end{cases}$.

If $g = e^{2\pi i a^{1,0}}$ then the spark equation is equivalent to $\begin{cases} \delta g = 0 \\ \delta a^{0,1} - \frac{1}{2\pi i} d \log g = 0 \\ d a^{0,1} = e \in \mathcal{E}^2(X) \end{cases}$.

Note that we can write $a^{0,1} = \{a_i^{0,1}\}$ where $a_i^{0,1} \in \mathcal{E}^1(U_i)$, and $g = \{g_{ij}\}$ where each g_{ij} is a circle valued function on U_{ij}. Then

$$\delta g = 0 \quad \Leftrightarrow \quad g_{jk} g_{ki} g_{ij} = 1$$

i.e. g_{ij} are transition functions of a hermitian line bundle, and

$$\delta a^{0,1} - \frac{1}{2\pi i} d \log g = 0 \quad \Leftrightarrow \quad a_j^{0,1} - a_i^{0,1} = \frac{1}{2\pi i} \frac{d g_{ij}}{g_{ij}}$$

which means $a_i^{0,1}$ is the connection 1-form on U_i.

Therefore, it is easy to see $\hat{\mathbf{H}}^1_{smooth}(X) =$ the set of (equivalent classes of) hermitian line bundles with hermitian connections.

Degree 2: Roughly speaking, $\hat{\mathbf{H}}^2_{smooth}(X)$ is the set of 2-gerbes with connections. The descriptions of spark classes of degree 2 or more are complicated. We refer to [Br1] for interested readers.

Degree n $= \dim X$: From Proposition 3.1.4, we have $\hat{\mathbf{H}}^n_{smooth}(X) \cong H^n(X, \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$. Furthermore, every spark class of top degree can be represented by a global top form. And integrating this form over X (modulo $\mathbb{Z}$) gives the isomorphism $\hat{\mathbf{H}}^n_{smooth}(X) \cong \mathbb{R}/\mathbb{Z}$.

Ring structure on $\hat{\mathbf{H}}^*_{smooth}(S^1)$.

Now we calculate the product $\hat{\mathbf{H}}^0_{smooth}(S^1) \otimes \hat{\mathbf{H}}^0_{smooth}(S^1) \to \hat{\mathbf{H}}^1_{smooth}(S^1)$.

Let X be the unit circle S^1. Fix a small number $\varepsilon > 0$ and an open cover $\mathcal{U} = \{U_1, U_2, U_3\}$ where

$$U_1 = \{e^{2\pi it} : t \in (-\varepsilon, \frac{1}{3})\}, \quad U_2 = \{e^{2\pi it} : t \in (\frac{1}{3}-\varepsilon, \frac{2}{3})\}, \quad U_3 = \{e^{2\pi it} : t \in (\frac{2}{3}-\varepsilon, 1)\}.$$

Let $a = (a_1, a_2, a_3) \in C^0(\mathcal{U}, \mathcal{E}^0) = \mathcal{E}^0(U_1) \oplus \mathcal{E}^0(U_2) \oplus \mathcal{E}^0(U_3)$ be a smooth hyperspark representing a spark class $\alpha \in \hat{\mathbf{H}}^0_{smooth}(S^1)$. Since $\delta a \in C^1(\mathcal{U}, \mathbb{Z})$, we have

$$(a_2 - a_1) \mid_{U_{12}} \in \mathbb{Z}, \quad (a_3 - a_1) \mid_{U_{13}} \in \mathbb{Z}, \quad (a_3 - a_2) \mid_{U_{23}} \in \mathbb{Z}.$$

Moreover, two smooth hypersparks represent the same spark class if and only if the difference of them is in $C^0(\mathcal{U}, \mathbb{Z})$, so we can choose the representative a to be of form:

$$a_1 = a_2 \mid_{U_{12}}, \quad a_2 = a_3 \mid_{U_{23}}, \quad a_1 + N = a_3 \mid_{U_{13}}, \quad a_1(x_0) \in [0, 1)$$

where N is an integer and $x_0 = e^{2\pi i \cdot 0} \in U_1$. It is easy to see the representative of this form is unique for any class. Assume the spark equation for a is $Da = e - r$ for $e \in \mathcal{E}^1(S^1)$, $r = (r_{12}, r_{23}, r_{13}) \in C^1(\mathcal{U}, \mathbb{Z})$. Then $da = e$ is a global 1-form and $\delta a = (0, 0, N) = -r$. If we have another smooth hyperspark b of this form representing spark class β with $db = f$, $\delta b = (0, 0, N')$, then by the product formula in the last section, the product $\alpha\beta$ can be represented by $a \cup f - r \cup b$. In the case $r = 0$, i.e. $N = 0$, a is a global function and the product is represented by the global 1-form af. Evaluating the integral $\int_{S^1} af$ mod $\mathbb{Z}$, we get a number in $\mathbb{R}/\mathbb{Z}$ which representing the product under the isomorphism $\hat{\mathbf{H}}^1_{smooth}(S^1) \cong \mathbb{R}/\mathbb{Z}$. To calculate the general product, we need the following lemma.

Let $\tilde{\mathcal{S}}$ be the set $\{f \in C^\infty(\mathbb{R}) : f(x+1) - f(x) \in \mathbb{Z}\}$. In fact, $\tilde{\mathcal{S}}$ is a group. We say $f \sim g$ if and only if $f(x) - g(x) \equiv N \in \mathbb{Z}$. Define the quotient group $\mathcal{S} = \tilde{\mathcal{S}}/\sim$. Note that we can identify $\mathcal{S}$ with the set $\{f \in C^\infty(\mathbb{R}) : 0 \leq f(0) < 1, f(x+1) - f(x) \in \mathbb{Z}\}$.

Lemma 3.3.1. *There exists a group isomorphism $\hat{\mathbf{H}}^0_{smooth}(S^1) \cong \mathcal{S}$. Moreover, for any $f(x) \in \mathcal{S}$, we have the decomposition*

$$f(x) = Nx + C + \sum_{k=1}^{\infty}(A_k \sin(2\pi kx) + B_k \cos(2\pi kx)).$$

Hence, we have the corresponding decomposition of a spark class.

Proof. For any spark class $\alpha \in \hat{\mathbf{H}}^0_{smooth}(S^1)$, there exists a unique representative $a = (a_1, a_2, a_3) \in C^0(\mathcal{U}, \mathcal{E}^0)$ with

$$a_1 = a_2 \mid_{U_{12}}, \quad a_2 = a_3 \mid_{U_{23}}, \quad a_1 + N = a_3 \mid_{U_{13}}, \quad a_1(x_0) \in [0, 1).$$

We can lift a to a smooth function $\tilde{a} \in \mathcal{S}$ uniquely, and establish a $1-1$ correspondence between $\hat{\mathbf{H}}^0_{smooth}(S^1)$ and the set $\mathcal{S}$.

For any smooth function $f \in \mathcal{S}$ with $f(x+1) - f(x) = N$ for some $N \in \mathbb{Z}$, $f(x) - Nx$ is periodic. Hence we have the Fourier expansion

$$f(x) - Nx = C + \sum_{k=1}^{\infty}(A_k \sin(2\pi kx) + B_k \cos(2\pi kx)).$$

On the other hand, under the $1-1$ correspondence, every component of the Fourier expansion is still in $\mathcal{S}$, and hence represents a spark class. $\square$

Now let us calculate the product $\hat{\mathbf{H}}^0_{smooth}(S^1)\otimes\hat{\mathbf{H}}^0_{smooth}(S^1) \to \hat{\mathbf{H}}^1_{smooth}(S^1)$. We use identification $\hat{\mathbf{H}}^0_{smooth}(S^1) \cong \mathcal{S}$ and $\hat{\mathbf{H}}^1_{smooth}(S^1) \cong \mathbb{R}/\mathbb{Z}$, and represent the product as $\mathcal{S}\otimes\mathcal{S} \to \mathbb{R}/\mathbb{Z}$.

Theorem 3.3.2. *For $a, b \in \mathcal{S}$ with decompositions*

$$a = Nx + C + \sum_{k=1}^{\infty}(A_k \sin(2\pi kx) + B_k \cos(2\pi kx))$$

and

$$b = N'x + C' + \sum_{k=1}^{\infty}(A'_k \sin(2\pi kx) + B'_k \cos(2\pi kx)),$$

the product

$$a * b = \frac{NN'}{2} + CN' - C'N + \sum_{k=1}^{\infty}(A'_k B_k - A_k B'_k)\pi k \mod \mathbb{Z}.$$

Proof. First, we calculate the product $[\sin 2\pi kx] * [\cos 2\pi k'x]$. Since $\sin 2\pi kx$ corresponds to a smooth hyperspark

$$a_k = (\sin 2\pi kx\mid_{U_1}, \sin 2\pi kx\mid_{U_2}, \sin 2\pi kx\mid_{U_3})$$

with spark equations

$$da_k = d\sin 2\pi kx = 2\pi k \cos 2\pi kx dx \quad \text{and} \quad \delta a_k = (0,0,0) = 0.$$

Similarly, $\cos 2\pi k'x$ corresponds a smooth hyperspark $b_{k'}$ with

$$db_{k'} = -2\pi k' \sin 2\pi k' x dx \quad \text{and} \quad \delta b_{k'} = 0.$$

Then by the product formula we have

$$\begin{aligned}
& [\sin 2\pi kx] * [\cos 2\pi k'x] \\
= & \int_0^1 \sin 2\pi kx d \cos 2\pi k'x \\
= & -2\pi k' \int_0^1 \sin 2\pi kx \sin 2\pi k'x dx \\
= & -2\pi k' \int_0^1 \frac{1}{2}(\cos 2\pi(k-k')x - \cos 2\pi(k+k')x) dx \\
= & \begin{cases} -\pi k, & k = k' \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Similarly, we can calculate

$$\begin{aligned}
& [C] * [Nx] = \int_0^1 C N dx = CN \\
& [C] * [\sin 2\pi kx] = \int_0^1 C d \sin 2\pi kx = 0 \\
& [C] * [\cos 2\pi kx] = \int_0^1 C d \cos 2\pi kx = 0 \\
& [C] * [C'] = 0 \\
& [\sin 2\pi kx] * [\sin 2\pi k'x] = \int_0^1 \sin 2\pi kx d \sin 2\pi k'x = 0 \\
& [\cos 2\pi kx] * [\cos 2\pi k'x] = \int_0^1 \cos 2\pi kx d \cos 2\pi k'x = 0
\end{aligned}$$

$$[\sin 2\pi kx] * [Nx] = \int_0^1 \sin 2\pi kx N dx = 0$$
$$[\cos 2\pi kx] * [Nx] = \int_0^1 \cos 2\pi kx N dx = 0$$

Nx corresponds to a smooth hyperspark $z_n = (Nx \mid_{U_1}, Nx \mid_{U_2}, Nx \mid_{U_3})$ with spark equations

$$dz_n = dNx = Ndx \text{ and } \delta z_n = (0,0,N).$$

So the product of Nx and $N'x$ can be represented by smooth hyperspark

$$NxdN'x + (-1)^1(-(0,0,N))N'x = NN'xdx + (0,0,NN'x) \in C^0(\mathcal{U},\mathcal{E}^1) \oplus C^1(\mathcal{U},\mathcal{E}^0).$$

Let $\frac{1}{2}NN'x^2$ denote the element

$$(\frac{1}{2}NN'x^2 \mid_{U_1}, \frac{1}{2}NN'x^2 \mid_{U_2}, \frac{1}{2}NN'x^2 \mid_{U_3}) \in \mathcal{E}^0(U_1) \oplus \mathcal{E}^0(U_2) \oplus \mathcal{E}^0(U_3) = C^0(\mathcal{U},\mathcal{E}^0).$$

Then

$$D(\frac{1}{2}NN'x^2) = d(\frac{1}{2}NN'x^2) + \delta(\frac{1}{2}NN'x^2) = NN'xdx + (0,0,NN'(x+\frac{1}{2})).$$

Hence, $NN'xdx + (0,0,NN'x)$ is equivalent to

$$NN'xdx + (0,0,NN'x) - D(\frac{1}{2}NN'x^2) = 0 - (0,0,\frac{1}{2}NN').$$

And $-(0,0,\frac{1}{2}NN') \in C^1(\mathcal{U},\mathbb{R})$ equals $-\frac{1}{2}NN' \equiv \frac{1}{2}NN' \mod \mathbb{Z}$ under the isomorphism

$$H^1(S^1,\mathbb{R})/H^1(S^1,\mathbb{Z}) \cong H^1(S^1,\mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}.$$

So we have

$$[Nx] * [N'x] = \frac{NN'}{2}$$

Finally, by distributivity and graded commutativity of the product, we have

$$[Nx+C+\sum_{k=1}^{\infty}(A_k \sin 2\pi kx+B_k \cos 2\pi kx)][N'x+C'+\sum_{k=1}^{\infty}(A'_k \sin 2\pi kx+B'_k \cos 2\pi kx)]$$

$$= \frac{NN'}{2} + CN' - C'N + \sum_{k=1}^{\infty}(A'_k B_k - A_k B'_k)\pi k \mod \mathbb{Z}$$

□

The next example that we shall discuss is the product of two smooth hypersparks of degree 1 on a 3-dimensional manifold X. Since $\hat{\mathbf{H}}^1_{smooth}(X)$ is the set of hermitian line bundles with hermitian connections and $\hat{\mathbf{H}}^3_{smooth}(X) \cong \mathbb{R}/\mathbb{Z}$, the product associates a number modulo $\mathbb{Z}$ to two hermitian line bundles with hermitian connections.

For two smooth hyperspark classes $\alpha, \beta \in \hat{\mathbf{H}}^1_{smooth}(X)$, assume

$$a = a^{0,1}+a^{1,0} \in C^0(\mathcal{U},\mathcal{E}^1)\oplus C^1(\mathcal{U},\mathcal{E}^0) \text{ and } b = b^{0,1}+b^{1,0} \in C^0(\mathcal{U},\mathcal{E}^1)\oplus C^1(\mathcal{U},\mathcal{E}^0)$$

are representatives of α and β respectively with spark equations

$$Da = e - r \text{ and } Db = f - s.$$

Then we have $\begin{cases} \delta a^{1,0} = -r \in C^2(\mathcal{U}, \mathbb{Z}) \\ \delta a^{0,1} - da^{1,0} = 0 \\ da^{0,1} = e \in \mathcal{E}^2(X) \end{cases}$ and $\begin{cases} \delta b^{1,0} = -s \in C^2(\mathcal{U}, \mathbb{Z}) \\ \delta b^{0,1} - db^{1,0} = 0 \\ db^{0,1} = f \in \mathcal{E}^2(X) \end{cases}$.

By the product formula, we have $\alpha\beta = [a \cup f + r \cup b]$ where

$$a \cup f + r \cup b = a^{0,1} \wedge f + a^{1,0} \wedge f + r \wedge b^{0,1} + r \wedge b^{1,0}$$

$$\in C^0(\mathcal{U}, \mathcal{E}^3) \oplus C^1(\mathcal{U}, \mathcal{E}^2) \oplus C^2(\mathcal{U}, \mathcal{E}^1) \oplus C^3(\mathcal{U}, \mathcal{E}^0).$$

$a \cup f + r \cup b$ is a cycle in $\bigoplus_{i+j=3} C^i(\mathcal{U}, \mathcal{E}^j)$ representing a class in $H^3(X, \mathbb{R}) \cong \mathbb{R}$. In general, it is hard to identify this class under the isomorphism $\hat{\mathbf{H}}^3_{smooth}(X) \cong \mathbb{R}/\mathbb{Z}$. However, when one of α and β represents a flat bundle, it is easier to calculate the product.

Lemma 3.3.3. *If $\beta \in H^1(X, \mathbb{R}/\mathbb{Z}) \subset \hat{\mathbf{H}}^1_{smooth}(X)$ represents a flat bundle on X, then their exists a smooth hyperspark $b = b^{0,1} + b^{1,0}$ representing β with $b^{0,1} = 0$ and $b^{1,0} \in C^1(\mathcal{U}, \mathbb{R})$.*

Proof. For any flat line bundle, there exists a trivialization with constant transition functions and zero connection forms (with respect to a local basis). □

By the lemma, if β is flat, we have $\alpha\beta = [a \cup f + r \cup b] = [r \wedge b^{1,0}]$ where $r \wedge b^{1,0} \in C^3(\mathcal{U}, \mathbb{R}) \subset C^3(\mathcal{U}, \mathcal{E}^0)$ is a Čech cycle representing a cohomology class in $H^3(X, \mathbb{R})$. Hence, we proved the following proposition.

Proposition 3.3.4. *X is a 3-dimensional manifold. Let $\alpha \in \hat{\mathbf{H}}^1_{smooth}(X)$ and $\beta \in H^1(X, \mathbb{R}/\mathbb{Z}) \subset \hat{\mathbf{H}}^1_{smooth}(X)$. Choosing representatives as above, we have $\alpha\beta = [r \wedge b^{1,0}] \in H^3(X, \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$.*

Remark 3.3.5. *It is easy to generalize this proposition to the product*

$$\hat{\mathbf{H}}^{n-2}_{smooth}(X) \otimes \hat{\mathbf{H}}^1_{smooth}(X) \to \hat{\mathbf{H}}^n_{smooth}(X)$$

for an n-dimensional manifold X when the second factor $\beta \in H^1(X, \mathbb{R}/\mathbb{Z}) \subset \hat{\mathbf{H}}^1_{smooth}(X)$.

Remark 3.3.6. *From this proposition, we see the product $\alpha\beta$ only depends the first Chern class $[r]$ of α. We can also see this fact from the next lemma.*

Moreover, the product coincides with the natural product

$$H^2(X, \mathbb{Z}) \otimes H^1(X, \mathbb{R}/\mathbb{Z}) \to H^3(X, \mathbb{R}/\mathbb{Z}).$$

Lemma 3.3.7. *X is a smooth manifold. If $\alpha \in \hat{\mathbf{H}}^k_\infty(X) \subset \hat{\mathbf{H}}^k_{smooth}(X)$ and $\beta \in H^l(X, \mathbb{R}/\mathbb{Z}) \subset \hat{\mathbf{H}}^l_{smooth}(X)$, then $\alpha\beta = 0$.*

Proof. Note that $H^l(X, \mathbb{R}/\mathbb{Z}) = \ker \delta_1$ and $\hat{\mathbf{H}}^k_\infty(X) = \ker \delta_2$. So we can choose representatives a and b with spark equations $Da = e - 0$ and $Db = 0 - s$. By the product formula we have $\alpha\beta = 0$. □

3.4 Smooth Deligne Cohomology

Deligne cohomology, which was invented by Deligne in 1970's, is closely related to spark characters and differential characters. In this section, we introduce "smooth Deligne cohomology" [Br1], a smooth analog of Deligne cohomology and establish its relation with spark characters.

Definition 3.4.1. *Let X be a smooth manifold. For $p \geq 0$, the* ***smooth Deligne complex*** $\mathbb{Z}_{\mathcal{D}}(p)^{\infty}$ *is the complex of sheaves:*

$$0 \to \mathbb{Z} \xrightarrow{i} \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{p-1} \to 0$$

where $\mathcal{E}^k$ denotes the sheaf of real-valued differential k-forms on X. The hypercohomology groups $\mathbb{H}^q(X, \mathbb{Z}_{\mathcal{D}}(p)^{\infty})$ are called the ***smooth Deligne cohomology*** *groups of X, and are denoted by $H^q_{\mathcal{D}}(X, \mathbb{Z}(p)^{\infty})$.*

Example 3.4.2. *It is easy to see $H^q_{\mathcal{D}}(X, \mathbb{Z}(0)^{\infty}) = H^q(X, \mathbb{Z})$ and $H^q_{\mathcal{D}}(X, \mathbb{Z}(1)^{\infty}) = H^{q-1}(X, \mathbb{R}/\mathbb{Z})$.*

There is a cup product [Br1] [EV]

$$\cup : \mathbb{Z}_{\mathcal{D}}(p)^{\infty} \otimes \mathbb{Z}_{\mathcal{D}}(p')^{\infty} \to \mathbb{Z}_{\mathcal{D}}(p+p')^{\infty}$$

by

$$x \cup y = \begin{cases} x \cdot y & \text{if } \deg x = 0; \\ x \wedge dy & \text{if } \deg x > 0 \text{ and } \deg y = p'; \\ 0 & \text{otherwise.} \end{cases}$$

The cup product $\cup$ is a morphism of complexes and associative, hence induces a ring structure on

$$\bigoplus_{p,q} H^q_{\mathcal{D}}(X, \mathbb{Z}(p)^{\infty}).$$

We may calculate the smooth Deligne cohomology groups of a manifold X

with dimension n by the following two short exact sequences of complexes of sheaves:

1. $0 \to \mathcal{E}^{*<p}[-1] \to \mathbb{Z}_{\mathcal{D}}(p)^{\infty} \to \mathbb{Z} \to 0$,

2. $0 \to \mathcal{E}^{*\geq p}[-p-1] \to \mathbb{Z}_{\mathcal{D}}(n+1)^{\infty} \to \mathbb{Z}_{\mathcal{D}}(p)^{\infty} \to 0$

where $\mathcal{E}^{*<p}[-1]$ denotes the complex of sheaves $\mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{p-1}$ shifted by 1 position to the right, and $\mathcal{E}^{*\geq p}[-p-1]$ denotes the complex of sheaves $\mathcal{E}^p \xrightarrow{d} \mathcal{E}^{p+1} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^n$ shifted by $p+1$ positions to the right.

It turns out $H^p_{\mathcal{D}}(X, \mathbb{Z}(p)^{\infty})$ is the most interesting part among all the smooth Deligne cohomology groups.

Theorem 3.4.3. *We can put $H^p_{\mathcal{D}}(X, \mathbb{Z}(p)^{\infty})$ into the following two short exact sequences:*

1. $0 \longrightarrow \mathcal{E}^{p-1}(X)/\mathcal{Z}^{p-1}_0(X) \longrightarrow H^p_{\mathcal{D}}(X, \mathbb{Z}(p)^{\infty}) \longrightarrow H^p(X, \mathbb{Z}) \longrightarrow 0$

2. $0 \longrightarrow H^{p-1}(X, \mathbb{R}/\mathbb{Z}) \longrightarrow H^p_{\mathcal{D}}(X, \mathbb{Z}(p)^{\infty}) \longrightarrow \mathcal{Z}^p_0(X) \longrightarrow 0$

Proof. (1) From the short exact sequence $0 \to \mathcal{E}^{*<p}[-1] \to \mathbb{Z}_{\mathcal{D}}(p)^{\infty} \to \mathbb{Z} \to 0$, we get the long exact sequence of hypercohomology:

$$\cdots \to \mathbb{H}^{p-1}(\mathbb{Z}) \to \mathbb{H}^p(\mathcal{E}^{*<p}[-1]) \to \mathbb{H}^p(\mathbb{Z}_{\mathcal{D}}(p)^{\infty}) \to \mathbb{H}^p(\mathbb{Z}) \to \mathbb{H}^{p+1}(\mathcal{E}^{*<p}[-1]) \to \cdots .$$

First, we have $\mathbb{H}^p(\mathbb{Z}) = H^p(X, \mathbb{Z})$. Since the sheaf $\mathcal{E}^k$ is soft for every k, it is easy to see $\mathbb{H}^p(\mathcal{E}^{*<p}[-1]) = \mathbb{H}^{p-1}(\mathcal{E}^{*<p}) = \mathcal{E}^{p-1}(X)/d\mathcal{E}^{p-2}(X)$ and $\mathbb{H}^{p+1}(\mathcal{E}^{*<p}[-1]) = \mathbb{H}^p(\mathcal{E}^{*<p}) = 0$. And $\mathbb{H}^p(\mathbb{Z}_{\mathcal{D}}(p)^{\infty}) = H^p_{\mathcal{D}}(X, \mathbb{Z}(p)^{\infty})$ by notation. So we have

$$\cdots \to H^{p-1}(X, \mathbb{Z}) \to \mathcal{E}^{p-1}(X)/d\mathcal{E}^{p-2}(X) \to H^p_{\mathcal{D}}(X, \mathbb{Z}(p)^{\infty}) \to H^p(X, \mathbb{Z}) \to 0.$$

Note that the map $H^{p-1}(X,\mathbb{Z}) \to \mathcal{E}^{p-1}(X)/d\mathcal{E}^{p-2}(X)$ is induced by morphism of complexes of sheaves $i : \mathbb{Z} \to \mathcal{E}^{*<p}$ which is composition of $i : \mathbb{Z} \to \mathcal{E}^*$ and projection $p : \mathcal{E}^* \to \mathcal{E}^{*<p}$. Hence $H^{p-1}(X,\mathbb{Z}) \to \mathcal{E}^{p-1}(X)/d\mathcal{E}^{p-2}(X)$ factors through $\mathbb{H}^{p-1}(\mathcal{E}^*) = H^{p-1}(X,\mathbb{R})$, and the image is $\mathcal{Z}_0^{p-1}(X)/d\mathcal{E}^{p-2}(X)$. Finally, we get the short exact sequence

$$0 \longrightarrow \mathcal{E}^{p-1}(X)/\mathcal{Z}_0^{p-1}(X) \longrightarrow H^p_{\mathcal{D}}(X,\mathbb{Z}(p)^\infty) \longrightarrow H^p(X,\mathbb{Z}) \longrightarrow 0.$$

(2) From the short exact sequence $0 \to \mathcal{E}^{*\geq p}[-p-1] \to \mathbb{Z}_{\mathcal{D}}(n+1)^\infty \to \mathbb{Z}_{\mathcal{D}}(p)^\infty \to 0$, we get the long exact sequence of hypercohomology:

$$\cdots \to \mathbb{H}^p(\mathcal{E}^{*\geq p}[-p-1]) \to \mathbb{H}^p(\mathbb{Z}_{\mathcal{D}}(n+1)^\infty) \to \mathbb{H}^p(\mathbb{Z}_{\mathcal{D}}(p)^\infty) \to$$

$$\mathbb{H}^{p+1}(\mathcal{E}^{*\geq p}[-p-1]) \to \mathbb{H}^{p+1}(\mathbb{Z}_{\mathcal{D}}(n+1)^\infty) \to \cdots.$$

The complex of sheaves $\mathbb{Z}_{\mathcal{D}}(n+1)^\infty$ is quasi-isomorphic to $\mathbb{R}/\mathbb{Z}[-1]$, so $\mathbb{H}^p(\mathbb{Z}_{\mathcal{D}}(n+1)^\infty) = H^{p-1}(X,\mathbb{R}/\mathbb{Z})$. Also, it is easy to see $\mathbb{H}^{p+1}(\mathcal{E}^{*\geq p}[-p-1]) = \mathbb{H}^0(\mathcal{E}^{*\geq p}) = \mathcal{Z}^p(X)$, and $\mathbb{H}^p(\mathcal{E}^{*\geq p}[-p-1]) = 0$. Thus, we get

$$0 \to H^{p-1}(X,\mathbb{R}/\mathbb{Z}) \to H^p_{\mathcal{D}}(X,\mathbb{Z}(p)^\infty) \to \mathcal{Z}^p(X) \to H^p(X,\mathbb{R}/\mathbb{Z}) \to \cdots.$$

To complete our proof, we have to determine the kernel of the map $\mathcal{Z}^p(X) \to H^p(X,\mathbb{R}/\mathbb{Z})$. Note this map is induced by $i : \mathcal{E}^{*\geq p}[-p-1] \to \mathbb{Z}_{\mathcal{D}}(n+1)^\infty$, which is composition of $i : \mathcal{E}^{*\geq p}[-p-1] \to \mathcal{E}^*[-1]$ and $i : \mathcal{E}^*[-1] \to \mathbb{Z}_{\mathcal{D}}(n+1)^\infty$. So the map $\mathcal{Z}^p(X) \to H^p(X,\mathbb{R}/\mathbb{Z})$ is composition of $\mathcal{Z}^p(X) \to H^p(X,\mathbb{R})$ and $H^p(X,\mathbb{R}) \to H^p(X,\mathbb{R}/\mathbb{Z})$, and it is easy to see the kernel is $\mathcal{Z}_0^p(X)$.

□

Remark 3.4.4. *By similar calculations, it is easy to determine other part of the smooth Deligne cohomology groups:*

$$H^q_{\mathcal{D}}(X, \mathbb{Z}(p)^\infty) = \begin{cases} H^{q-1}(X, \mathbb{R}/\mathbb{Z}), & \textit{when } (q < p); \\ H^q(X, \mathbb{Z}), & \textit{when } (q > p). \end{cases}$$

In the last theorem, we saw the (p, p)-part of smooth Deligne cohomology satisfies the same short exact sequences with spark characters in Proposition 3.1.4. It is not surprising we have the isomorphism:

Theorem 3.4.5.

$$H^p_{\mathcal{D}}(X, \mathbb{Z}(p)^\infty) \cong \hat{\mathbf{H}}^{p-1}(X).$$

Proof. It suffices to show the isomorphism $H^p_{\mathcal{D}}(X, \mathbb{Z}(p)^\infty) \cong \hat{\mathbf{H}}^{p-1}_{smooth}(X)$.

Step 1: Choose a good cover $\{\mathcal{U}\}$ of X and take Čech resolution for the complex of sheaves $\mathbb{Z}_{\mathcal{D}}(p)^\infty \longrightarrow \mathcal{C}^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p)^\infty)$.

Then

$$H^q_{\mathcal{D}}(X, \mathbb{Z}(p)^\infty) \equiv \mathbb{H}^q(\mathbb{Z}_{\mathcal{D}}(p)^\infty) \cong \mathbb{H}^q(Tot(\mathcal{C}^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p)^\infty)))$$

$$\cong H^q(Tot(C^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p)^\infty)))$$

where $C^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p)^\infty)$ are the groups of global sections of sheaves $\mathcal{C}^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p)^\infty)$ and look like the following double complex.

$$
\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & & & \vdots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & & & \uparrow{\scriptstyle\delta} \\
C^p(\mathcal{U},\mathbb{Z}) & \xrightarrow{(-1)^p i} & C^p(\mathcal{U},\mathcal{E}^0) & \xrightarrow{(-1)^p d} & C^p(\mathcal{U},\mathcal{E}^1) & \xrightarrow{(-1)^p d} & C^p(\mathcal{U},\mathcal{E}^2) & \xrightarrow{(-1)^p d} & \cdots & \xrightarrow{(-1)^p d} & C^p(\mathcal{U},\mathcal{E}^{p-1}) \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & & & \uparrow{\scriptstyle\delta} \\
\vdots & & \vdots & & \vdots & & \vdots & & & & \vdots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & & & \uparrow{\scriptstyle\delta} \\
C^2(\mathcal{U},\mathbb{Z}) & \xrightarrow{i} & C^2(\mathcal{U},\mathcal{E}^0) & \xrightarrow{d} & C^2(\mathcal{U},\mathcal{E}^1) & \xrightarrow{d} & C^2(\mathcal{U},\mathcal{E}^2) & \xrightarrow{d} & \cdots & \xrightarrow{d} & C^2(\mathcal{U},\mathcal{E}^{p-1}) \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & & & \uparrow{\scriptstyle\delta} \\
C^1(\mathcal{U},\mathbb{Z}) & \xrightarrow{-i} & C^1(\mathcal{U},\mathcal{E}^0) & \xrightarrow{-d} & C^1(\mathcal{U},\mathcal{E}^1) & \xrightarrow{-d} & C^1(\mathcal{U},\mathcal{E}^2) & \xrightarrow{-d} & \cdots & \xrightarrow{-d} & C^1(\mathcal{U},\mathcal{E}^{p-1}) \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & & & \uparrow{\scriptstyle\delta} \\
C^0(\mathcal{U},\mathbb{Z}) & \xrightarrow{i} & C^0(\mathcal{U},\mathcal{E}^0) & \xrightarrow{d} & C^0(\mathcal{U},\mathcal{E}^1) & \xrightarrow{d} & C^0(\mathcal{U},\mathcal{E}^2) & \xrightarrow{d} & \cdots & \xrightarrow{d} & C^0(\mathcal{U},\mathcal{E}^{p-1})
\end{array}
$$

Step 2:

Let $M_p^* \equiv Tot(C^*(\mathcal{U},\mathbb{Z}_{\mathcal{D}}(p)^\infty))$ denote the total complexes of the double complex $C^*(\mathcal{U},\mathbb{Z}_{\mathcal{D}}(p)^\infty)$ with differential

$$
D_p(a) = \begin{cases} (\delta + (-1)^r i)(a), & \text{when} \quad a \in C^r(\mathcal{U},\mathbb{Z}); \\ (\delta + (-1)^r d)(a), & \text{when} \quad a \in C^r(\mathcal{U},\mathcal{E}^j), j < p-1; \\ \delta a, & \text{when} \quad a \in C^r(\mathcal{U},\mathcal{E}^{p-1}). \end{cases}
$$

Now, we show $H^p(M_p^*) \cong \hat{\mathbf{H}}^{p-1}_{smooth}(X)$.

Let $\tilde{a} = r + a = r + \Sigma_{i=0}^{p-1} a^{i,p-1-i} \in M_p^p$ where $r \in C^p(\mathcal{U},\mathbb{Z})$ and $a^{i,p-1-i} \in C^i(\mathcal{U},\mathcal{E}^{p-1-i})$. We define a map $H^p(M_p^*) \longrightarrow \hat{\mathbf{H}}^{p-1}_{smooth}(X)$ which maps $[\tilde{a}] \mapsto [a]$ for $\tilde{a} \in \ker D_p$.

$$
\tilde{a} \in \ker D_p \Leftrightarrow D_p\tilde{a} = 0 \Leftrightarrow D_p a + (-1)^p i(r) = 0 \text{ and } \delta r = 0
$$

$$\Leftrightarrow Da = D_p a + da^{0,p-1} = da^{0,p-1} - (-1)^p r.$$

Note that

$$\delta a^{0,p-1} - da^{1,p-2} = 0 \quad \Rightarrow \quad \delta da^{0,p-1} = d\delta a^{0,p-1} = dda^{1,p-2} = 0$$

$$\Rightarrow \quad da^{0,p-1} \in \mathcal{E}^p(X) = \ker \delta : C^0(\mathcal{U}, \mathcal{E}^p) \to C^1(\mathcal{U}, \mathcal{E}^p).$$

Therefore, $\tilde{a} \in \ker D_p$ implies a is a smooth hyperspark of degree $p-1$. On the other hand, if a is a smooth hyperspark with spark equation $Da = e - r$ with $e \in \mathcal{E}^p(X)$ and $r \in C^p(\mathcal{U}, \mathbb{Z})$, it is clear to see $\tilde{a} \equiv (-1)^p r + a \in \ker D_p$.

Moreover, it is easy to see $\tilde{a}' = r' + a' \in \ker D_p$ with $\tilde{a} - \tilde{a}' \in ImD_p$ if and only if a and a' represent the same spark class.

Hence, the map $[\tilde{a}] \to [a]$ gives an isomorphism $H^p(M_p^*) \cong \hat{\mathbf{H}}^{p-1}_{smooth}(X)$.

□

It is shown in [HLZ] [HL2] that there is a natural isomorphism $\hat{\mathbf{H}}^{p-1}(X) \cong \hat{H}^{p-1}_{CS}(X)$, so we get [Br1, Proposition 1.5.7.] as a corollary.

Corollary 3.4.6.

$$H^p_{\mathcal{D}}(X, \mathbb{Z}(p)^\infty) \cong \hat{H}^{p-1}_{CS}(X).$$

In fact, $\bigoplus_p H^p_{\mathcal{D}}(X, \mathbb{Z}(p)^\infty) \subset \bigoplus_{p,q} H^q_{\mathcal{D}}(X, \mathbb{Z}(p)^\infty)$ is a subring, where the product coincides with the products on spark characters and differential character, i.e. we have the following **ring isomorphism**:

Theorem 3.4.7.

$$H^*_{\mathcal{D}}(X,\mathbb{Z}(*)^\infty) \cong \hat{\mathbf{H}}^*(X) \cong \hat{H}^*_{CS}(X).$$

Proof. It is shown in [HLZ] that $\hat{\mathbf{H}}^*(X)$ and $\hat{H}^*_{CS}(X)$ are isomorphic as rings. So we only need to verify that the product on $H^*_{\mathcal{D}}(X,\mathbb{Z}(*)^\infty)$ agrees with the product on $\hat{\mathbf{H}}^*(X)$.

We can make use of the isomorphism:

$$H^p_{\mathcal{D}}(X,\mathbb{Z}(p)^\infty) \cong H^p(M^*_p) \cong \hat{\mathbf{H}}^{p-1}_{smooth}(X).$$

First, fix two smooth Deligne cohomology classes

$$\alpha \in H^p_{\mathcal{D}}(X,\mathbb{Z}(p)^\infty) \text{ and } \beta \in H^q_{\mathcal{D}}(X,\mathbb{Z}(q)^\infty),$$

and let

$$\tilde{a} = r + a = r + \sum_{i=0}^{p-1} a^{i,p-1-i} \in M^p_p \text{ be a representative of } \alpha$$

and

$$\tilde{b} = s + b = s + \sum_{i=0}^{q-1} b^{i,q-1-i} \in M^q_q \text{ be a representative of } \beta$$

where

$$r \in C^p(\mathcal{U},\mathbb{Z}), a^{i,p-1-i} \in C^i(\mathcal{U},\mathcal{E}^{p-1-i}),$$

and

$$s \in C^q(\mathcal{U},\mathbb{Z}), b^{i,q-1-i} \in C^i(\mathcal{U},\mathcal{E}^{q-1-i}).$$

On one hand, we calculate $\alpha \cup \beta$ by original product formula (See appendix §8):

$$\alpha \cup \beta = [r \cup \tilde{b} + a \cup db^{0,q-1}].$$

On the other hand, let $[a]$ and $[b]$ be the image of α and β under the isomorphism $H^k(M_k^*) \cong \hat{\mathbf{H}}^{k-1}_{smooth}(X)$, $k = p, q$ with spark equations

$Da = e - (-1)^p r$ and $Db = f - (-1)^q s$ where $e = da^{0,p-1}, f = db^{0,q-1}$ are global forms.

We apply product formula on $\hat{\mathbf{H}}^*_{smooth}(X)$, and get

$$[a][b] = [a \cup f + (-1)^p (-1)^p r \cup b] = [a \cup db^{0,q-1} + r \cup b]$$

which is the image of $[r \cup \tilde{b} + a \cup db^{0,q-1}] = [a \cup db^{0,q-1} + r \cup b + r \cup s]$ under the isomorphism of $H^{p+q}(M_{p+q}^*) \cong \hat{\mathbf{H}}^{p+q-1}_{smooth}(X)$.

We get the products are the same.

□

3.5 Massey Products in Secondary Geometric Invariants

In this section, we apply the theory in §2.5 to the de Rham-Federer spark complex and the smooth hyperspark complex associated to a smooth manifold X, and define the Massey triple product on spark characters $\hat{\mathbf{H}}^*(X)$.

In §2.5, we defined the Massey triple product in the ring of spark classes $\hat{\mathbf{H}}^*$ associated to a spark complex of differential graded algebras. Strictly speaking, neither the de Rham-Federer spark complex nor the smooth hyperspark complex is spark complex of differential graded algebras. However, all arguments in §2.5 are still valid for the de Rham-Federer spark complex and the smooth hyperspark complex, since the product is on the class level.

Now we rewrite §2.5 in the context of the de Rham-Federer spark complex. Thanks to Proposition 3.1.4, for any spark classes, we can always choose good representatives such that the wedge product makes sense. Therefore, we just assume all representatives chosen are good in the sense of Proposition 3.1.4. We can also apply the theory in §2.5 to the smooth hyperspark complex. In that case, we do not worry about the choice of representatives, since products are always well-defined. In the following theorem, $\hat{\mathbf{H}}^*(X)$ denotes the group of spark classes associated to the de Rham-Federer spark complex or the smooth hyperspark complex, and $\hat{\mathbf{H}}^*_\infty(X)$ is the subgroup whose elements can be represented by global smooth forms, i.e. $\ker \delta_2$.

Theorem 3.5.1. *If three spark classes $\alpha \in \hat{\mathbf{H}}^i(X)$, $\beta \in \hat{\mathbf{H}}^j(X)$ and $\gamma \in \hat{\mathbf{H}}^k(X)$ satisfy that $\alpha\beta \in \hat{\mathbf{H}}^{i+j+1}_\infty(X) \subset \hat{\mathbf{H}}^{i+j+1}(X)$ and $\beta\gamma \in \hat{\mathbf{H}}^{j+k+1}_\infty(X) \subset \hat{\mathbf{H}}^{j+k+1}(X)$, then the Massey triple product of α, β and γ, denoted by $\mathcal{M}(\alpha,\beta,\gamma)$, is well-defined in $\hat{\mathbf{H}}^{i+j+k+1}(X)/(\alpha\hat{\mathbf{H}}^{j+k}(X) + \hat{\mathbf{H}}^{i+j}(X)\gamma)$.*

Proof. Choose representatives $a \in \alpha$, $b \in \beta$ and $c \in \gamma$ with the spark equations

$$da = e - r, \quad db = f - s, \quad dc = g - t.$$

If $\alpha\beta \in \hat{\mathbf{H}}^{i+j+1}_\infty(X) \subset \hat{\mathbf{H}}^{i+j+1}(X)$ and $\beta\gamma \in \hat{\mathbf{H}}^{j+k+1}_\infty(X) \subset \hat{\mathbf{H}}^{j+k+1}(X)$, then

$$\alpha\beta = [a][b] = [af + (-1)^{i+1} rb] = [\phi], \quad \beta\gamma = [b][c] = [bg + (-1)^{j+1} sc] = [\psi]$$

for some $\phi \in \mathcal{E}^{i+j+1}(X)$ and $\psi \in \mathcal{E}^{j+k+1}(X)$. Hence there exist $A \in \mathcal{D}'^{i+j}(X)$, $B \in \mathcal{D}'^{j+k}(X)$, $X \in \mathcal{IF}^{i+j+1}(X)$ and $Y \in \mathcal{IF}^{j+k+1}(X)$ such that

$$af + (-1)^{i+1} rb = dA + X + \phi, \quad bg + (-1)^{j+1} sc = dB + Y + \psi.$$

Note that

$$d(a\psi + (-1)^i rB + (-1)^{i+1} Ag + (-1)^j Xc) = e\psi + (-1)^i \phi g + rY + (-1)^i Xt.$$

So $a\psi + (-1)^i rB + (-1)^{i+1} Ag + (-1)^j Xc \in \mathcal{D}'^{i+j+k+1}(X)$ represents a spark class.

We define

$$\mathcal{M}(\alpha, \beta, \gamma) \equiv [a\psi + (-1)^i rB + (-1)^{i+1} Ag + (-1)^j Xc]$$

$$\in \hat{\mathbf{H}}^{i+j+k+1}(X)/(\alpha \hat{\mathbf{H}}^{j+k}(X) + \hat{\mathbf{H}}^{i+j}(X)\gamma).$$

It is easy to verify that $\mathcal{M}(\alpha, \beta, \gamma)$ is well-defined. (See §2.5.) □

Similar to Corollary 2.5.5, we have

Corollary 3.5.2. *The Massey triple product is compatible with the ring homomorphisms*

$$\delta_1 : \hat{\mathbf{H}}^*(X) \to \mathcal{Z}_0^{*+1}(X) \quad \textit{and} \quad \delta_2 : \hat{\mathbf{H}}^*(X) \to H^{*+1}(X, \mathbb{Z}).$$

Proof. Note that the equations

$$af + (-1)^{i+1} rb = dA + X + \phi, \quad bg + (-1)^{j+1} sc = dB + Y + \psi$$

imply $ef = d\phi$, $fg = d\psi$, $-rs = dX$, $-st = dY$. So we can define the Massey triple product for e, f, g and $[r]$, $[s]$, $[t]$.

From the formula

$$d(a\psi + (-1)^i rB + (-1)^{i+1} Ag + (-1)^j Xc) = e\psi + (-1)^i \phi g + rY + (-1)^i Xt,$$

it is easy to see the compatibility of the Massey triple product with ring homomorphisms δ_1 and δ_2. □

Because of the surjectivity of δ_1 and δ_2 and their compatibility with the Massey triple product, the examples mentioned in the introduction on the Borromean rings and Iwasawa manifold naturally show the nontriviality of the Massey triple product in spark characters. Moreover, for three spark classes whose the Massey triple product is well-defined, even the Massey triple products of their images under δ_1 and δ_2 are trivial, their Massey triple product may not be trivial. The following example may be the simplest case.

Example 3.5.3. *Let $X = S^1 \times S^1 \times S^1 = \mathbb{R}^3/\mathbb{Z}^3$ be three dimensional torus and x, y, z be coordinates of $\mathbb{R}^3$. Define $\alpha, \gamma \in \hat{\mathbf{H}}^1(X)$ and $\beta \in \hat{\mathbf{H}}^0(X)$ as follows. Let α be the spark class $[a = \lambda dx] \in \ker \delta_1 \cap \ker \delta_2$ and γ be the spark class $[c = \lambda' dz] \in \ker \delta_1 \cap \ker \delta_2$. The spark equations of α and γ can be written as $da = e - r = 0 - 0$ and $dc = g - t = 0 - 0$. Let β be a spark class represented by any current b with the spark equation $db = f - s$ such that $f = dy$.*

Calculate the products $\alpha\beta$ *and* $\beta\gamma$*,*

$$\alpha\beta = [a \wedge f + (-1)^2 r \wedge b] = [\lambda dx \wedge dy] \in \hat{\mathbf{H}}^2_\infty(X),$$

$$\beta\gamma = [b\wedge g + (-1)^1 s \wedge c] = [b \wedge t + (-1)^1 f \wedge c + d(b\wedge c)] = [-\lambda' dy \wedge dz + d(b \wedge dz)] \in \hat{\mathbf{H}}^2_\infty(X).$$

So the Massey triple product is well-defined and represented by $-\lambda\lambda' dx\wedge dy \wedge dz$*.*

From the 3×3 *diagram in Proposition 3.1.4, we see the class*

$$[-\lambda\lambda' dx \wedge dy \wedge dz] \in \ker \delta_1 \cap \ker \delta_2 = H^3(X,\mathbb{R})/H^3(X,\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}.$$

It is easy to see the subgroup of $\hat{\mathbf{H}}^3(X)$ *generated by* α *and* γ *is* $\{[(m\lambda + n\lambda')dx \wedge dy \wedge dz] \mid m, n \in \mathbb{Z}\}$*. Therefore, for general* λ*,* λ'*, the Massey triple product is nontrivial.*

Question. *It is not clear what is geometric meaning behind the nontrivial Massey products from the example above. We are still looking for more examples and trying to explain this phenomena.*

Question. *We study secondary geometric invariants over smooth manifolds via spark theory. One may ask whether there exist similar secondary invariants over orbifolds. There are several papers on this topic, e.g. [LU] [LM]. In fact, we can develop an orbifold spark theory which may be the correct secondary invariant theory for orbifolds. It is interesting to compare our theory with [LU] [LM].*

Chapter 4

$\bar{d}$-sparks of Level p

In this chapter, we study secondary geometric invariants on complex manifolds. We introduce the groups of $\bar{d}$-spark classes of level p which are generalizations of the group of $\bar{\partial}$-spark classes in [HL3]. These groups are obtained from a family of generalized spark complexes in the sense of Definition 2.3.1. The main examples are the Dolbeault-Federer spark complex, the Čech-Dolbeault spark complex and the Čech-Dolbeault hyperspark complex which are presented in §4.1, §4.2 and §4.3 respectively. The associated groups of $\bar{d}$-spark classes are shown to be quotient groups of differential characters and spark characters. A ring structure is induced from the one on spark characters. Moreover, in Proposition 4.1.4, we see that the Deligne cohomology groups are contained in the groups of $\bar{d}$-spark classes. So we may represent a Deligne cohomology class by a $\bar{d}$-spark and give a product formula, which will be shown in the next Chapter. Note that all characters in this chapter is of coefficient $\mathbb{C}/\mathbb{Z}$.

4.1 Dolbeault-Federer Sparks of Level p

We studied the de Rham-Federer spark complex associated a smooth manifold X in §3.1.1. Recall the de Rham-Federer spark complex is a triple

$$(\mathcal{D}'^*(X), \mathcal{E}^*(X), \mathcal{IF}^*(X)).$$

Now let X be a **complex manifold** and $\mathcal{E}^*$ and $\mathcal{D}'^*$ denote the sheaves **complex-valued** smooth forms and currents respectively. We still denote by $\hat{\mathbf{H}}^*_{spark}(X)$ the associated group of spark classes.

Now we introduce a new spark complex, the Dolbeault-Federer spark complex of level p, which is closely related to the de Rham-Federer spark complex.

For a complex manifold X, we can decompose the space of smooth k-forms by types:

$$\mathcal{E}^k(X) \equiv \bigoplus_{r+s=k} \mathcal{E}^{r,s}(X).$$

And similarly,

$$\mathcal{D}'^k(X) \equiv \bigoplus_{r+s=k} \mathcal{D}'^{r,s}(X).$$

Fix an integer $p > 0$ and consider the truncated complex $(\mathcal{D}'^*(X,p), d_p)$ with

$$\mathcal{D}'^k(X,p) \equiv \bigoplus_{r+s=k, r<p} \mathcal{D}'^{r,s}(X) \quad \text{and} \quad d_p \equiv \pi_p \circ d$$

where $\pi_p : \mathcal{D}'^k(X) \to \mathcal{D}'^k(X,p)$ is the natural projection

$$\pi_p(a) = a^{0,k} + ... + a^{p-1,k-p+1}.$$

Similarly, we can define $(\mathcal{E}^*(X,p), d_p)$.

Definition 4.1.1. *By the* ***Dolbeault-Federer spark complex of level*** $\mathbf{p}$*, or more simply, the* $\bar{\mathbf{d}}$***-spark complex of level*** $\mathbf{p}$ *we mean the triple* (F_p^*, E_p^*, I_p^*)

$$F_p^* \equiv \mathcal{D}'^*(X,p), \quad E_p^* \equiv \mathcal{E}^*(X,p), \quad I_p^* \equiv \mathcal{IF}^*(X)$$

with maps

$$E_p^* \hookrightarrow F_p^* \quad \textit{and} \quad \Psi_p : I_p^* \to F_p^*$$

where $\Psi_p = \pi_p \circ i$.

Remark 4.1.2. *The triple* $(F_p^*, E_p^*, I_p^*) \equiv (\mathcal{D}'^*(X,p), \mathcal{E}^*(X,p), \mathcal{IF}^*(X))$ *is a spark complex.*

Proof. First,

$$H^*(F_p^*) \cong H^*(E_p^*) \cong \mathbb{H}^*(\Omega^{*<p}) \equiv H^*(X,p).$$

where $\mathbb{H}^*(\Omega^{*<p}) \equiv H^*(X,p)$ denotes the hypercohomology of complex of sheaves

$$0 \to \Omega^0 \to \Omega^1 \to \Omega^2 \to \cdots \to \Omega^{p-1} \to 0$$

and Ω^k is the sheaf of holomorphic k-forms on X.

For the proof of $\Psi_p(I_p^k) \cap E_p^k = \{0\}$ for $k > 0$, we refer to [HL3, Appendix B]. □

Definition 4.1.3. *A* ***Dolbeault-Federer spark of level*** $\mathbf{p}$ *of degree* k*, or* *a* $\bar{\mathbf{d}}$-***spark of level*** $\mathbf{p}$ *is a pair*

$$(a, r) \in \mathcal{D}'^k(X, p) \oplus \mathcal{IF}^{k+1}(X)$$

satisfying the spark equations

$$d_p a = e - \Psi_p(r) \quad \text{and} \quad dr = 0$$

for some $e \in \mathcal{E}^{k+1}(X, p)$.

Two Dolbeault-Federer sparks of level p, (a, r) *and* (a', r') *are* ***equivalent*** *if there exist* $b \in \mathcal{D}'^{k-1}(X, p)$ *and* $s \in \mathcal{IF}^k(X)$ *such that*

$$a - a' = d_p b + \Psi_p(s) \quad \text{and} \quad r - r' = -ds.$$

The equivalence class determined by a spark (a, r) *will be denoted by* $[(a, r)]$, *and the group of Dolbeault-Federer spark classes of level* p *of degree* k *will be denoted by* $\hat{\mathbf{H}}^k_{spark}(X, p)$ *or* $\hat{\mathbf{H}}^k(X, p)$ *for short.*

Applying Proposition 2.3.7, we have

Proposition 4.1.4. *Let* $H^{k+1}_{\mathbb{Z}}(X, p)$ *denote the image of map* $\Psi_{p*} : H^{k+1}(X, \mathbb{Z}) \to H^{k+1}(X, p)$, *and* $\mathcal{Z}^{k+1}_{\mathbb{Z}}(X, p)$ *denote the set of* d_p*-closed forms in* $\mathcal{E}^{k+1}(X, p)$ *which represent classes in* $H^{k+1}_{\mathbb{Z}}(X, p)$. *Let* $\hat{\mathbf{H}}^k_\infty(X, p)$ *denote the spark classes representable by smooth forms, and* $H^{k+1}_{\mathcal{D}}(X, \mathbb{Z}(p))$ *denote the Deligne cohomology group.*

The 3×3 *diagram for* $\hat{\mathbf{H}}^k(X, p)$ *can be written as*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^k(X,p)}{H^k_{\mathbb{Z}}(X,p)} & \longrightarrow & \hat{\mathbf{H}}^k_\infty(X,p) & \longrightarrow & d_p\mathcal{E}^k(X,p) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{k+1}_{\mathcal{D}}(X,\mathbb{Z}(p)) & \longrightarrow & \hat{\mathbf{H}}^k(X,p) & \xrightarrow{\delta_1} & \mathcal{Z}^{k+1}_{\mathbb{Z}}(X,p) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \delta_2} & & \downarrow & & \\
0 & \longrightarrow & \ker \Psi_* & \longrightarrow & H^{k+1}(X,\mathbb{Z}) & \xrightarrow{\Psi_{p*}} & H^{k+1}_{\mathbb{Z}}(X,p) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

A special and the most interesting case is when X is Kähler and $k = 2p-1$,

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J}^p(X) & \longrightarrow & \hat{\mathbf{H}}^{2p-1}_\infty(X,p) & \longrightarrow & d_p\mathcal{E}^{2p-1}(X,p) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p)) & \longrightarrow & \hat{\mathbf{H}}^{2p-1}(X,p) & \xrightarrow{\delta_1} & \mathcal{Z}^{2p}_{\mathbb{Z}}(X,p) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \delta_2} & & \downarrow & & \\
0 & \longrightarrow & Hdg^{p,p}(X) & \longrightarrow & H^{2p}(X,\mathbb{Z}) & \xrightarrow{\Psi_{p*}} & H^{2p}_{\mathbb{Z}}(X,p) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where $\mathcal{J}^p(X)$ denotes the pth intermediate Jacobian and $Hdg^{p,p}(X)$ is the set of the Hodge classes.

Proof. The proof follows Proposition 2.3.7 directly. The only nontrivial part is to show $H^{k+1}_{\mathcal{D}}(X,\mathbb{Z}(p)) \cong \ker \delta_1$. We postpone our proof to §5 where we study Deligne cohomology in detail. □

Remark 4.1.5. *The $\bar{d}$-spark complex is a generalization of $\bar{\partial}$-spark complex in [HL3] which corresponds the special case $p = 1$.*

4.1.1 Ring Structure

We can establish the ring structure on $\hat{\mathbf{H}}^*(X,p)$ by identifying it as a quotient ring of $\hat{\mathbf{H}}^*(X)$.

Consider the following commutative diagram

$$\begin{array}{ccccc} \mathcal{IF}^*(X) & \xrightarrow{i} & \mathcal{D}'^*(X) & \xleftarrow{i} & \mathcal{E}^*(X) \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \pi_p} & & \downarrow{\scriptstyle \pi_p} \\ \mathcal{IF}^*(X) & \xrightarrow{\pi_p \circ i} & \mathcal{D}'^*(X,p) & \xleftarrow{i} & \mathcal{E}^*(X,p) \end{array}$$

which induces a group homomorphism $\Pi_p : \hat{\mathbf{H}}^*(X) \to \hat{\mathbf{H}}^*(X,p)$. Furthermore, we have

Theorem 4.1.6. *The morphism of spark complexes* $(\pi_p, \pi_p, id) : (F^*, E^*, I^*) \to (F_p^*, E_p^*, I_p^*)$ *induces a surjective group homomorphism*

$$\Pi_p : \hat{\mathbf{H}}^*(X) \to \hat{\mathbf{H}}^*(X,p)$$

whose kernel is an ideal. Hence, $\hat{\mathbf{H}}^*(X,p)$ *carries a ring structure.*

Proof. It's straightforward to see that the diagram above commutes and π_p commutes with differentials. Consequently, the induced map $(a,r) \mapsto (\pi_p(a), r)$ on sparks descends to a well defined homomorphism $\Pi_p : \hat{\mathbf{H}}^k(X) \to \hat{\mathbf{H}}^k(X,p)$ as claimed.

To prove the surjectivity, consider a spark $(A,r) \in F_p^k \oplus I^{k+1}$ with $dr = 0$ and $d_pA = e - \Psi_p(r)$ for some $e \in E_p^{k+1}$. We can choose some smooth form which represents same cohomology class with r in $H^{k+1}(F^*) \cong H^{k+1}(E^*)$, so there exist $a_0 \in F^k$, $e_0 \in E^{k+1}$ such that $da_0 = e_0 - r$. We have $\pi_p(da_0) = \pi_p(e_0 - r) \Rightarrow d_p(\pi_p a_0) = \pi_p e_0 - \Psi_p r$. Hence, $d_p(A - \pi_p a_0) = e - \pi_p e_0$ is a smooth form. It follows by Lemma 2.2.2 that there exist $b \in F_p^{k-1}$ and

$f \in E_p^k$ with $A - \pi_p a_0 = f + d_p b$. Set $a = a_0 + f + db$ and note that $da = da_0 + df + ddb = e_0 - r + df = (e_0 + df) - r$. Hence, (a, r) is a spark of degree k and $\pi_p a = \pi_p(a_0 + f + db) = \pi_p a_0 + f + d_p b = A$. So Π_p is surjective.

We need the following lemma to show the kernel is an ideal.

Lemma 4.1.7. *On* $\hat{\mathbf{H}}^k(X)$, *one has that* $\ker(\Pi_p) = \{\alpha \in \hat{\mathbf{H}}^k(X) : \exists (a, 0) \in \alpha$ *where* a *is smooth and* $\pi_p(a) = 0\}$. *In particular,* $\ker(\Pi_p) \subset \hat{\mathbf{H}}^k_\infty(X)$.

Proof. One direction is clear. Suppose $\alpha \in \ker(\Pi_p)$ and choose any spark $(a, r) \in \alpha$. $\Pi_p(\alpha) = 0$ means that there exist $b \in F_p^{k-1}$ and $s \in I^k$ with

$$\begin{cases} \pi_p(a) = d_p b + \Psi_p(s) = \pi_p(db + s) \\ r = -ds \end{cases}$$

Replace (a, r) by $(\tilde{a}, 0) = (a - db - s, r + ds)$, note that $\pi_p(\tilde{a}) = \pi_p(a - db - s) = 0$.

In fact, we can choose $\tilde{a}$ to be smooth. $d\tilde{a} = da - ds = e - r - ds = e$ is a smooth form, it follows by Lemma 2.2.2 and the fact $H^*(F^p\mathcal{D}'^*(X)) = H^*(F^p\mathcal{E}^*(X))$ that we can choose $\tilde{a}$ to be smooth. Note that $F^0\mathcal{D}'^* \supset F^1\mathcal{D}'^* \supset \cdots \supset F^p\mathcal{D}'^* \supset \cdots$ is the naive filtration. □

By the product formula of $\hat{\mathbf{H}}^*(X)$, it is easy to see the kernel is an ideal. In fact, if α and β are two spark classes, and $\alpha \in \ker(\Pi_p)$, then we can choose representatives $(a, 0)$ and (b, s) for α and β respectively, with spark equations $da = e - 0$ and $db = f - s$, where a, e, f are smooth. By the product formula, $\alpha\beta$ can be represented by $(a \wedge f + (-1)^{\deg a + 1} 0 \wedge b, 0 \wedge s) = (a \wedge f, 0)$ which is in $\ker(\Pi_p)$.

Hence, $\hat{\mathbf{H}}^*(X, p)$ carries a ring structure induced from $\hat{\mathbf{H}}^*(X)$.

□

4.1.2 Functoriality

Proposition 4.1.8. *There are commutative diagrams*

$$\begin{array}{ccc} \hat{\mathbf{H}}^k(X) & \xrightarrow{\delta_1} & \mathcal{Z}^{k+1}_{\mathbb{Z}}(X) \\ \downarrow{\scriptstyle \Pi_p} & & \downarrow{\scriptstyle \pi_p} \\ \hat{\mathbf{H}}^k(X,p) & \xrightarrow{\delta_1} & \mathcal{Z}^{k+1}_{\mathbb{Z}}(X,p) \end{array} \qquad \begin{array}{ccc} \hat{\mathbf{H}}^k(X) & \xrightarrow{\delta_2} & H^{k+1}(X,\mathbb{Z}) \\ \downarrow{\scriptstyle \Pi_p} & & \downarrow{\scriptstyle =} \\ \hat{\mathbf{H}}^k(X,p) & \xrightarrow{\delta_2} & H^{k+1}(X,\mathbb{Z}) \end{array}$$

Proof. Let $\alpha \in \hat{\mathbf{H}}^k(X)$. Choose a representative $(a, r) \in \alpha$ with spark equation $da = e - r$. Then $\pi_p \circ \delta_1(\alpha) = \pi_p(e)$, and $\delta_1 \circ \Pi_p(\alpha) = \delta_1 \circ \Pi_p([(a, r)]) = \delta_1([(\pi_p(a), r)]) = \pi_p(e)$ since $(\pi_p(a), r)$ is a $\bar{d}$-spark of level p with spark equation $d_p(\pi_p a) = \pi_p(e) - \Psi_p r$. Hence, the first diagram is commutative. We can verify the second one by the same way.

□

Moreover, we have the following theorem

Theorem 4.1.9. *Any holomorphic map $f : X \to Y$ between complex manifolds induces a graded ring homomorphism*

$$f^* : \hat{\mathbf{H}}^*(Y, p) \to \hat{\mathbf{H}}^*(X, p)$$

with the property that if $g : Y \to Z$ is holomorphic, then $(g \circ f)^* = f^* \circ g^*$.

Proof. By Theorem 3.1.15, it suffices to show $f^*(\ker \Pi_p) \subset (\ker \Pi_p)$ which is directly from Lemma 4.1.7. □

Corollary 4.1.10. $\hat{\mathbf{H}}^*(\bullet, p)$ *is a graded ring functor on the category of complex manifolds and holomorphic maps.*

Theorem 4.1.11. *(Gysin map) Any proper holomorphic submersion map $f : X^{m+r} \to Y^m$ between complex manifolds induces a Gysin homomorphism*

$$f_* : \hat{\mathbf{H}}^*(X,p) \to \hat{\mathbf{H}}^{*-2r}(Y,p-r).$$

Proof. It was shown in Theorem 3.1.16 that f induces a Gysin map $f_* : \hat{\mathbf{H}}^*(X) \to \hat{\mathbf{H}}^{*-r}(Y)$. Moreover, it is clear that $f^*(\ker \Pi_p) \subset (\ker \Pi_{p-r})$ from Lemma 4.1.7. □

4.2 Čech-Dolbeault Sparks of Level p

We now consider other presentations of the $\bar{d}$-spark classes. We introduce the Čech-Dolbeault spark complex of level p which is a generalization of the Čech-Dolbeault spark complex in [HL3].

Recall that, for a complex manifold X, we can decompose the space of smooth k-forms over an open set $U \subset X$ by types:

$$\mathcal{E}^k(U) \equiv \bigoplus_{r+s=k} \mathcal{E}^{r,s}(U).$$

Let $\mathcal{E}^k(U,p) = \bigoplus_{r+s=k, r<p} \mathcal{E}^{r,s}(U)$, and $\mathcal{E}^k_p$ denote the subsheaf of $\mathcal{E}^k$ with $\mathcal{E}^k_p(U) = \mathcal{E}^k(U,p)$. And similarly, we can define the sheaf $\mathcal{D}'^k_p$ with $\mathcal{D}'^k_p(U) = \mathcal{D}'^k(U,p)$.

Suppose $\mathcal{U}$ is a good cover of X and consider the total complex of the following double complex with total differential $D_p = \delta + (-1)^r d_p$:

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots \\
\uparrow{\scriptstyle d_p} & & \uparrow{\scriptstyle -d_p} & & \uparrow{\scriptstyle d_p} & & & & \uparrow{\scriptstyle (-1)^r d_p} \\
C^0(\mathcal{U},\mathcal{E}_p^2) & \xrightarrow{\delta} & C^1(\mathcal{U},\mathcal{E}_p^2) & \xrightarrow{\delta} & C^2(\mathcal{U},\mathcal{E}_p^2) & \xrightarrow{\delta} \cdots & \cdots & \xrightarrow{\delta} & C^r(\mathcal{U},\mathcal{E}_p^2) \xrightarrow{\delta} \cdots \\
\uparrow{\scriptstyle d_p} & & \uparrow{\scriptstyle -d_p} & & \uparrow{\scriptstyle d_p} & & & & \uparrow{\scriptstyle (-1)^r d_p} \\
C^0(\mathcal{U},\mathcal{E}_p^1) & \xrightarrow{\delta} & C^1(\mathcal{U},\mathcal{E}_p^1) & \xrightarrow{\delta} & C^2(\mathcal{U},\mathcal{E}_p^1) & \xrightarrow{\delta} \cdots & \cdots & \xrightarrow{\delta} & C^r(\mathcal{U},\mathcal{E}_p^1) \xrightarrow{\delta} \cdots \\
\uparrow{\scriptstyle d_p} & & \uparrow{\scriptstyle -d_p} & & \uparrow{\scriptstyle d_p} & & & & \uparrow{\scriptstyle (-1)^r d_p} \\
C^0(\mathcal{U},\mathcal{E}_p^0) & \xrightarrow{\delta} & C^1(\mathcal{U},\mathcal{E}_p^0) & \xrightarrow{\delta} & C^2(\mathcal{U},\mathcal{E}_p^0) & \xrightarrow{\delta} \cdots & \cdots & \xrightarrow{\delta} & C^r(\mathcal{U},\mathcal{E}_p^0) \xrightarrow{\delta} \cdots
\end{array}
$$

It is easy to see the row complexes are exact everywhere except in the first column on the left, and

$$\{\ker(\delta) \text{ on the left column}\} \cong \{\text{global sections of sheaves } \mathcal{E}_p^*\} = \mathcal{E}^*(X,p).$$

Hence,

$$H^*(\bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{E}_p^s)) \cong H^*(\mathcal{E}^*(X,p)) \cong H^*(X,p).$$

Note that every column complex is exact everywhere except at the bottom and the level of p from the bottom.

Now we consider the triple of complexes

$$(F_p^*, E_p^*, I_p^*) \equiv (\bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{E}_p^s), \mathcal{E}^*(X,p), C^*(\mathcal{U},\mathbb{Z})).$$

And we have

Proposition 4.2.1. *The triple (F_p^*, E_p^*, I_p^*) defined above is a spark complex (even in the sense of Definition 2.1.1), which is called the **Čech-Dolbeault spark complex of level p**, or the **smooth hyperspark complex of level p**.*

Proof. We have shown that $E_p^* \hookrightarrow F_p^*$ induces an isomorphism $H^*(E_p^*) \cong H^*(F_p^*)$. Also there is an injective cochain map $I_p^* \equiv C^*(\mathcal{U}, \mathbb{Z}) \hookrightarrow C^*(\mathcal{U}, \mathcal{E}_p^0) \hookrightarrow \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{E}_p^s) \equiv F_p^*$.

$E_p^k \cap I_p^k = \{0\}$ for $k > 0$ is trivial. □

Definition 4.2.2. *A* ***Čech-Dolbeault spark of level* p** *of degree k, or a* ***smooth hyperspark of level* p** *is an element*

$$a \in \bigoplus_{r+s=k} C^r(\mathcal{U}, \mathcal{E}_p^s)$$

with the spark equation

$$D_p a = e - r$$

where $e \in \mathcal{E}_p^{k+1}(X) \subset C^0(\mathcal{U}, \mathcal{E}_p^{k+1})$ is of bidegree $(0, k+1)$ and $r \in C^{k+1}(\mathcal{U}, \mathbb{Z})$.

Two Čech-Dolbeault sparks of level p, a and a' are ***equivalent*** *if there exist $b \in \bigoplus_{r+s=k-1} C^r(\mathcal{U}, \mathcal{E}_p^s)$ and $s \in C^k(\mathcal{U}, \mathbb{Z})$ satisfying*

$$a - a' = D_p b + s.$$

The equivalence class determined by a Čech-Dolbeault spark a will be denoted by $[a]$, and the group of Čech-Dolbeault spark classes of level p will be denoted by $\hat{\mathbf{H}}^k_{smooth}(X, p)$.

Recall that the **smooth hyperspark complex** in the last chapter is defined by

$$(F^*, E^*, I^*) = (\bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{E}^s), \mathcal{E}^*(X), C^*(\mathcal{U}, \mathbb{Z})).$$

The associated group of smooth hyperspark classes is denoted by $\hat{\mathbf{H}}^*_{smooth}(X)$.

The relation between the smooth hyperspark complex and the Čech-Dolbeault spark complex of level p is the same as the relation between the de Rham-Federer spark complex and the Dolbeault-Federer spark complex of level p. We have the natural morphism $(\pi_p, \pi_p, id) : (F^*, E^*, I^*) \longrightarrow (F^*_p, E^*_p, I^*_p)$. Explicitly, we have the following commutative diagram

$$\begin{array}{ccccc} C^*(\mathcal{U}, \mathbb{Z}) & \xrightarrow{i} & \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{E}^s) & \xleftarrow{i} & \mathcal{E}^*(X) \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \pi_p} & & \downarrow{\scriptstyle \pi_p} \\ C^*(\mathcal{U}, \mathbb{Z}) & \xrightarrow{i} & \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{E}^s_p) & \xleftarrow{i} & \mathcal{E}^*(X, p) \end{array}$$

Theorem 4.2.3. *The morphism of spark complexes* $(\pi_p, \pi_p, id) : (F^*, E^*, I^*) \to (F^*_p, E^*_p, I^*_p)$ *induces a surjective group homomorphism*

$$\Pi_p : \hat{\mathbf{H}}^*_{smooth}(X) \to \hat{\mathbf{H}}^*_{smooth}(X, p)$$

whose kernel is an ideal. Hence, $\hat{\mathbf{H}}^*_{smooth}(X, p)$ *carries a ring structure.*

Proof. The proof is similar to Theorem 4.1.6. It's plain to see that the diagram above commutes and π_p commutes with differentials. Hence, the induced map $a \mapsto \pi_p(a)$ on sparks descends to a group homomorphism $\Pi_p : \hat{\mathbf{H}}^k_{smooth}(X) \to \hat{\mathbf{H}}^k_{smooth}(X, p)$.

To prove the surjectivity, consider a spark $a \in F^k_p$ with $D_p a = e - r$ for some $e \in E^{k+1}_p$ and $r \in I^{k+1}_p = I^{k+1}$. We can choose some smooth form which represents same cohomology class with r in $H^{k+1}(F^*) \cong H^{k+1}(E^*)$, so there exist $a_0 \in F^k$, $e_0 \in E^{k+1}$ such that $Da_0 = e_0 - r$. We have

$$\pi_p(Da_0) = \pi_p(e_0) - r \Rightarrow D_p(\pi_p a_0) = \pi_p e_0 - r.$$

Hence, $D_p(a-\pi_p a_0) = e-\pi_p e_0$ is a smooth form. It follows by Lemma 2.2.2 that there exist $b \in F_p^{k-1}$ and $f \in E_p^k$ with $a - \pi_p a_0 = f + D_p b$. Set $\tilde{a} = a_0 + f + Db$ and note that $D\tilde{a} = Da_0 + Df + DDb = e_0 - r + df = (e_0 + df) - r$. Hence, $\tilde{a}$ is a spark of degree k and $\pi_p \tilde{a} = \pi_p(a_0 + f + Db) = \pi_p a_0 + f + D_p b = a$. So Π_p is surjective.

We need the following lemma to show the kernel is an ideal.

Lemma 4.2.4. *On* $\hat{\mathbf{H}}^k_{smooth}(X)$*, one has that* $\ker(\Pi_p) = \{\alpha \in \hat{\mathbf{H}}^k_{smooth}(X) : \exists a \in \alpha$ *where* $a \in \mathcal{E}^k(X) \subset C^0(\mathcal{U}, \mathcal{E}^k)$ *and* $\pi_p(a) = 0\}$*. In particular,* $\ker \Pi_p \subset \hat{\mathbf{H}}^k_\infty(X)$*.*

Proof. One direction is clear. Suppose $\alpha \in \ker(\Pi_p)$ and choose any spark $a \in \alpha$ with $Da = e - r$. $\Pi_p(\alpha) = 0$ means that there exist $b \in F_p^{k-1}$ and $s \in I_p^k = I^k$ with $\pi_p(a) = D_p b + s = \pi_p(Db) + s$ which implies $D_p(\pi_p a) = \delta s$. On the other hand, $D_p(\pi_p a) = \pi_p(Da) = \pi_p e - r$. So we have $\pi_p e = 0$ and $-r = \delta s$. Replace a by $\bar{a} = a - Db - s$, then $\bar{a}$ represents the same class as a and $\pi_p(\bar{a}) = \pi_p(a - Db) - s = 0$.

In fact, we can choose $\bar{a}$ in $\mathcal{E}^k(X) \subset C^0(\mathcal{U}, \mathcal{E}^k)$. Since

$$D\bar{a} = Da - \delta s = e - (r + \delta s) = e$$

is a global smooth form, it follows by Lemma 2.2.2 and the fact

$$H^*(\bigoplus_{r+s=*} C^r(\mathcal{U}, F^p\mathcal{E}^s)) \cong H^*(F^p\mathcal{E}^*(X))$$

that we can choose $\bar{a}$ to be smooth. Note that $F^0\mathcal{E}^* \supset F^1\mathcal{E}^* \supset \cdots \supset F^p\mathcal{E}^* \supset \cdots$ is the naive filtration. $\square$

By the product formula of $\hat{\mathbf{H}}^*_{smooth}(X)$, it is easy to see the kernel is an ideal. Hence, $\hat{\mathbf{H}}^*_{smooth}(X,p)$ carries a ring structure induced from $\hat{\mathbf{H}}^*_{smooth}(X)$.

$\square$

4.3 Čech-Dolbeault Hypersparks of Level p

Now we introduce the Čech-Dolbeault hyperspark complex of level p which set up a bridge connecting the Čech-Dolbeault spark complex of level p and the Dolbeault-Federer spark complex of level p.

Fix a good cover $\mathcal{U}$ of X and consider total complex of the following double complex with total differential $D_p = \delta + (-1)^r d_p$:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots \\
\uparrow{\scriptstyle d_p} & & \uparrow{\scriptstyle -d_p} & & \uparrow{\scriptstyle d_p} & & & & \uparrow{\scriptstyle (-1)^r d_p} \\
C^0(\mathcal{U},\mathcal{D}'^2_p) & \xrightarrow{\delta} & C^1(\mathcal{U},\mathcal{D}'^2_p) & \xrightarrow{\delta} & C^2(\mathcal{U},\mathcal{D}'^2_p) & \xrightarrow{\delta} \cdots & & \cdots \xrightarrow{\delta} & C^r(\mathcal{U},\mathcal{D}'^2_p) \xrightarrow{\delta} \cdots \\
\uparrow{\scriptstyle d_p} & & \uparrow{\scriptstyle -d_p} & & \uparrow{\scriptstyle d_p} & & & & \uparrow{\scriptstyle (-1)^r d_p} \\
C^0(\mathcal{U},\mathcal{D}'^1_p) & \xrightarrow{\delta} & C^1(\mathcal{U},\mathcal{D}'^1_p) & \xrightarrow{\delta} & C^2(\mathcal{U},\mathcal{D}'^1_p) & \xrightarrow{\delta} \cdots & & \cdots \xrightarrow{\delta} & C^r(\mathcal{U},\mathcal{D}'^1_p) \xrightarrow{\delta} \cdots \\
\uparrow{\scriptstyle d_p} & & \uparrow{\scriptstyle -d_p} & & \uparrow{\scriptstyle d_p} & & & & \uparrow{\scriptstyle (-1)^r d_p} \\
C^0(\mathcal{U},\mathcal{D}'^0_p) & \xrightarrow{\delta} & C^1(\mathcal{U},\mathcal{D}'^0_p) & \xrightarrow{\delta} & C^2(\mathcal{U},\mathcal{D}'^0_p) & \xrightarrow{\delta} \cdots & & \cdots \xrightarrow{\delta} & C^r(\mathcal{U},\mathcal{D}'^0_p) \xrightarrow{\delta} \cdots
\end{array}$$

It is easy to see the row complexes are exact everywhere except the first column on the left, and

$$\{\ker(\delta) \text{ on the left column}\} \cong \{\text{global sections of sheaves } \mathcal{D}'^*_p\} = \mathcal{D}'^*(X,p).$$

Hence,

$$H^*\Big(\bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{D}'^s_p)\Big) \cong H^*(\mathcal{D}'^*(X,p)) \cong H^*(\mathcal{E}^*(X,p)) \cong H^*(X,p).$$

Note that every column complex is exact everywhere except at the bottom and the level of p from the bottom.

Now we consider the triple of complexes

$$(F_p^*, E_p^*, I_p^*) \equiv (\bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{D}_p'^s), \mathcal{E}^*(X,p), \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{IF}^s)).$$

And we have

Proposition 4.3.1. *The triple of complexes* (F_p^*, E_p^*, I_p^*) *as defined above is a spark complex, which is called the* ***Čech-Dolbeault hyperspark complex of level*** $\mathbf{p}$*, or more simply, the* ***hyperspark complex of level*** $\mathbf{p}$*.*

Proof. We have shown that $E_p^* \hookrightarrow F_p^*$ induces an isomorphism $H^*(E_p^*) \cong H^*(F_p^*)$. Also there is a map

$$\Psi_p : I_p^* \equiv \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{IF}^s) \hookrightarrow \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{D}'^s) \xrightarrow{\pi_p} \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{D}_p'^s) \equiv F_p^*.$$

And $E_p^k \cap I_p^k = \{0\}$ for $k > 0$ follows [HL3, Appendix B]. □

Definition 4.3.2. *A* ***Čech-Dolbeault hyperspark of level*** $\mathbf{p}$ *of degree* k*, or* ***hyperspark of level*** $\mathbf{p}$ *is a pair*

$$(a, r) \in \bigoplus_{r+s=k} C^r(\mathcal{U}, \mathcal{D}_p'^s) \oplus \bigoplus_{r+s=k+1} C^r(\mathcal{U}, \mathcal{IF}^s)$$

with the spark equations

$$D_p a = e - \Psi_p r \quad \textit{and} \quad Dr = 0$$

where $e \in \mathcal{E}_p^{k+1}(X) \subset C^0(\mathcal{U}, \mathcal{D}'^{k+1}_p)$ *is of bidegree* $(0, k+1)$.

Two Čech-Dolbeault sparks of level p, (a, r) *and* (a', r') *are* ***equivalent*** *if there exist*

$$b \in \bigoplus_{r+s=k-1} C^r(\mathcal{U}, \mathcal{D}'^s_p) \quad \textit{and} \quad s \in \bigoplus_{r+s=k} C^r(\mathcal{U}, \mathcal{IF}^s)$$

satisfying

$$a - a' = D_p b + s \quad \textit{and} \quad r = -Ds.$$

The equivalence class determined by a Čech-Dolbeault hyperspark (a, r) *will be denoted by* $[(a, r)]$, *and the group of Čech-Dolbeault hyperspark classes of level* p *will be denoted by* $\hat{\mathbf{H}}^k_{hyper}(X, p)$.

Recall the **hyperspark complex**

$$(F^*, E^*, I^*) = (\bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{D}'^s), \mathcal{E}^*(X), \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{IF}^s))$$

defined in §2.1. The hyperspark complex and the Čech-Dolbeault hyperspark complex of level p is related by the natural morphism (π_p, π_p, id) : $(F^*, E^*, I^*) \longrightarrow (F^*_p, E^*_p, I^*_p)$. Explicitly, we have the following commutative diagram

$$\begin{array}{ccccc} \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{IF}^s) & \xrightarrow{i} & \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{D}'^s) & \xleftarrow{i} & \mathcal{E}^*(X) \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \pi_p} & & \downarrow{\scriptstyle \pi_p} \\ \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{IF}^s) & \xrightarrow{i} & \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{D}'^s_p) & \xleftarrow{i} & \mathcal{E}^*(X, p) \end{array}$$

Similar to last two sections, we have the following lemma and theorem

Lemma 4.3.3. *On* $\hat{\mathbf{H}}^k_{hyper}(X)$, *one has that* $\ker(\Pi_p) = \{\alpha \in \hat{\mathbf{H}}^k_{hyper}(X) : \exists a \in \alpha$ *where* $a \in \mathcal{E}^k(X) \subset C^0(\mathcal{U}, \mathcal{D}'^k)$ *and* $\pi_p(a) = 0\}$. *In particular,* $\ker(\Pi_p) \subset$

$\hat{\mathbf{H}}^k_\infty(X)$.

Theorem 4.3.4. *The morphism of spark complexes* $(\pi_p, \pi_p, id) : (F^*, E^*, I^*) \to (F^*_p, E^*_p, I^*_p)$ *induces a surjective group homomorphism*

$$\Pi_p : \hat{\mathbf{H}}^*_{hyper}(X) \to \hat{\mathbf{H}}^*_{hyper}(X, p)$$

whose kernel is an ideal. Hence, $\hat{\mathbf{H}}^*_{hyper}(X, p)$ *carries a ring structure.*

In §3.1, we showed that both the de Rham-Federer spark complex and the smooth hyperspark complex are quasi-isomorphic to the hyperspark complex. Hence,

$$\hat{\mathbf{H}}^*_{spark}(X) \cong \hat{\mathbf{H}}^*_{hyper}(X) \cong \hat{\mathbf{H}}^*_{smooth}(X).$$

Similarly, we establish relations among the Dolbeault-Federer spark complex, the Čech-Dolbeault spark complex and the Čech-Dolbeault hyperspark complex of level p.

Theorem 4.3.5. *We have morphisms of spark complexes*

$$\begin{array}{ccccc}
\boxed{\begin{array}{c}\textit{the de Rham-Federer}\\ \textit{spark complex}\end{array}} & \xrightarrow{i} & \boxed{\textit{the hyperspark complex}} & \xleftarrow{i} & \boxed{\begin{array}{c}\textit{the smooth}\\ \textit{hyperspark complex}\end{array}} \\
\downarrow \pi_p & & \downarrow \pi_p & & \downarrow \pi_p \\
\boxed{\begin{array}{c}\textit{the Dolbeault-Federer}\\ \textit{spark complex of level p}\end{array}} & \xrightarrow{i} & \boxed{\begin{array}{c}\textit{the Čech-Dolbeault hyperspark}\\ \textit{complex of level p}\end{array}} & \xleftarrow{i} & \boxed{\begin{array}{c}\textit{the Čech-Dolbeault spark}\\ \textit{complex of level p}\end{array}}
\end{array}$$

where horizontal morphisms are quasi-isomorphisms.

Hence we get induced homomorphisms

$$\begin{array}{ccccc}
\hat{\mathbf{H}}^*_{spark}(X) & \xrightarrow{=} & \hat{\mathbf{H}}^*_{hyper}(X) & \xleftarrow{=} & \hat{\mathbf{H}}^*_{smooth}(X) \\
\downarrow \Pi_p & & \downarrow \Pi_p & & \downarrow \Pi_p \\
\hat{\mathbf{H}}^*_{spark}(X, p) & \xrightarrow{=} & \hat{\mathbf{H}}^*_{hyper}(X, p) & \xleftarrow{=} & \hat{\mathbf{H}}^*_{smooth}(X, p)
\end{array}$$

where the horizontal ones are isomorphism.

Proof. It is easy to see we have the following two commutative diagrams

$$\begin{array}{ccccc}
\mathcal{IF}^*(X) & \xrightarrow{\Psi_p} & \mathcal{D}'^*(X,p) & \xleftarrow{i} & \mathcal{E}^*(X,p) \\
\downarrow i & & \downarrow i & & \| \\
\bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{IF}^s) & \xrightarrow{\Psi_p} & \bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{D}'^s_p) & \xleftarrow{i} & \mathcal{E}^*(X,p)
\end{array}$$

and

$$\begin{array}{ccccc}
C^*(\mathcal{U},\mathbb{Z}) & \xrightarrow{i} & \bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{E}^s_p) & \xleftarrow{i} & \mathcal{E}^*(X,p) \\
\downarrow i & & \downarrow i & & \| \\
\bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{IF}^s) & \xrightarrow{\Psi_p} & \bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{D}'^s_p) & \xleftarrow{i} & \mathcal{E}^*(X,p)
\end{array}$$

where

$$i:\mathcal{IF}^*(X)\longrightarrow \bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{IF}^s) \quad \text{and} \quad i:C^*(\mathcal{U},\mathbb{Z})\longrightarrow \bigoplus_{r+s=*} C^r(\mathcal{U},\mathcal{IF}^s)$$

are quasi-isomorphisms of cochain complexes.

□

So far, we have introduced three families of spark complexes associated to a complex manifold X, and showed the natural isomorphisms between the groups of spark classes associated to them. We denote the groups of spark classes by $\hat{\mathbf{H}}^*(X,p)$ collectively, and call them the **Harvey-Lawson spark characters of level p associated to** X. The ring structure of $\hat{\mathbf{H}}^*(X,p)$ is induced from the ring structure of $\hat{\mathbf{H}}^*(X)$. We may define the product in $\hat{\mathbf{H}}^*(X)$ via the de Rham-Federer spark complex or the smooth hyperspark complex. In the next chapter, we shall study analytic Deligne cohomology and define the product for Deligne cohomology.

Chapter 5

Analytic Deligne Cohomology and Product Formula via Spark Presentation

In this chapter, we study analytic Deligne cohomology and its ring structure. In §5.1, we represent Deligne cohomology classes by $\bar{d}$-spark classes and show an explicit product formula for Deligne cohomology classes. The main results are Theorems 5.1.6, 5.1.9 and 5.1.11. In §5.2, we define Massey higher products in Deligne cohomology. An explicit formula for the Massey triple product is also shown. As an application of our theory, we show that every algebraic cycle represents a Deligne cohomology class and derive the ring homomorphism $\psi : CH^*(X) \to H^{2*}_{\mathcal{D}}(X, \mathbb{Z}(*))$ in §5.3.

5.1 Ring Structure on Deligne Cohomology

Definition 5.1.1. *Let X be a complex manifold. For $p \geq 0$, the **Deligne complex** $\mathbb{Z}_{\mathcal{D}}(p)$ is the complex of sheaves:*

$$0 \to \mathbb{Z} \xrightarrow{i} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1} \to 0$$

where Ω^k denotes the sheaf of holomorphic k-forms on X. The hypercohomology groups $\mathbb{H}^q(X, \mathbb{Z}_{\mathcal{D}}(p))$ are called the ***Deligne cohomology*** *groups of X, and are denoted by $H^q_{\mathcal{D}}(X, \mathbb{Z}(p))$.*

Remark 5.1.2. *In Deligne complex $\mathbb{Z}_{\mathcal{D}}(p)$, we always consider that $\mathbb{Z}$ is of degree 0, and Ω^k is of degree $k+1$.*

Example 5.1.3. *It is easy to see $H^q_{\mathcal{D}}(X, \mathbb{Z}(0)) = H^q(X, \mathbb{Z})$ and $H^q_{\mathcal{D}}(X, \mathbb{Z}(1)) = H^{q-1}(X, \mathcal{O}^*)$.*

In [B], Beilinson defined a cup product

$$\cup : \mathbb{Z}_{\mathcal{D}}(p) \otimes \mathbb{Z}_{\mathcal{D}}(p') \to \mathbb{Z}_{\mathcal{D}}(p + p')$$

by

$$x \cup y = \begin{cases} x \cdot y & \text{if } \deg x = 0; \\ x \wedge dy & \text{if } \deg x > 0 \text{ and } \deg y = p'; \\ 0 & \text{otherwise.} \end{cases}$$

The cup product $\cup$ is a morphism of complexes and associative [EV] [Br1], hence induces a ring structure on

$$\bigoplus_{p,q} H^q_{\mathcal{D}}(X, \mathbb{Z}(p)).$$

We are identifying the Deligne cohomology groups with subgroups of the groups of $\bar{d}$-spark classes. Then we give a product formula for Deligne cohomology.

Lemma 5.1.4. *We have the short exact sequence*

$$0 \to H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) \to \hat{\mathbf{H}}^{k-1}_{spark}(X, p) \to \mathcal{Z}^k_{\mathbb{Z}}(X, p) \to 0$$

which is the middle row of 3×3 diagram for the group of $\bar{d}$-spark classes of level p. Hence, for any Deligne class $\alpha \in H^k_{\mathcal{D}}(X, \mathbb{Z}(p))$, there exists a spark representative

$$(a, r) \in \mathcal{D}'^{k-1}(X, p) \oplus \mathcal{IF}^k(X) \quad \textit{with} \quad d_p a = -\Psi_p(r), \quad dr = 0.$$

Proof. By Proposition 2.3.7 it suffices to show

$$H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) \cong H^{k-1}(\mathrm{Cone}(\Psi : \mathcal{IF}^*(X) \to \mathcal{D}'^*(X, p))).$$

By definition, $H^*_{\mathcal{D}}(X, \mathbb{Z}(p))$ is the hypercohomology of the complex of sheaves

$$0 \to \mathbb{Z} \to \Omega^0 \to \Omega^1 \to \ldots \to \Omega^{p-1} \to 0.$$

In other words, it is the hypercohomology of the Cone $(\mathbb{Z} \to \Omega^{*<p})[-1]$.

Consider the acyclic resolutions:

$$\mathbb{Z} \to \mathcal{IF}^* \quad \text{and} \quad \Omega^k \to \mathcal{D}'^{k,*}.$$

And we have quasi-isomorphism of complexes of sheaves

$$\mathrm{Cone}(\mathbb{Z} \to \Omega^{*<p}) \simeq \mathrm{Cone}(\Psi : \mathcal{IF}^* \to \bigoplus_{s+t=*, s<p} \mathcal{D}'^{s,t}),$$

and hence

$$H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) \cong \mathbb{H}^{k-1}(\mathrm{Cone}(\mathbb{Z} \to \Omega^{*<p})) \cong \mathbb{H}^{k-1}(\mathrm{Cone}(\Psi : \mathcal{IF}^* \to \bigoplus_{s+t=*, s<p} \mathcal{D}'^{s,t}))$$

$$\cong H^{k-1}(\mathrm{Cone}(\Psi : \mathcal{IF}^*(X) \to \mathcal{D}'^*(X, p))).$$

Then for any Deligne class $\alpha \in H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) \subset \hat{\mathbf{H}}^{k-1}(X, p)$, we can find a representative $(a, r) \in \mathcal{D}'^{k-1}(X, p) \oplus \mathcal{I}^k(X)$ with $d_p a = e - \Psi_p(r)$, $dr = 0$. And we have $e = 0$ since $\alpha \in \ker \delta_1$.

□

Applying the representation of Deligne cohomology classes in terms of currents above, we define a product in Deligne cohomology

$$H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) \otimes H^l_{\mathcal{D}}(X, \mathbb{Z}(q)) \longrightarrow H^{k+l}_{\mathcal{D}}(X, \mathbb{Z}(p+q)).$$

First, for any Deligne class $\alpha \in H^k_{\mathcal{D}}(X, \mathbb{Z}(p))$, we choose a spark representative

$$(a, r) \in \mathcal{D}'^{k-1}(X, p) \oplus \mathcal{IF}^k(X) \quad \text{with} \quad d_p a = -\Psi_p(r), \quad dr = 0.$$

Similarly, for any $\beta \in H^l_{\mathcal{D}}(X, \mathbb{Z}(q))$, we choose a spark representative

$$(b, s) \in \mathcal{D}'^{l-1}(X, q) \oplus \mathcal{IF}^l(X) \quad \text{with} \quad d_q b = -\Psi_q(s), \quad ds = 0.$$

Since $\Pi_p : \hat{\mathbf{H}}^*(X) \to \hat{\mathbf{H}}^*(X,p)$ is surjective, there exist

$$(\tilde{a}, r) \in \mathcal{D}'^{k-1}(X) \oplus \mathcal{IF}^k(X) \quad \text{with} \quad \Pi_p[(\tilde{a}, r)] = [(\pi_p(\tilde{a}), r)] = [(a, r)] = \alpha,$$

and

$$(\tilde{b}, s) \in \mathcal{D}'^{l-1}(X) \oplus \mathcal{IF}^l(X) \quad \text{with} \quad \Pi_q[(\tilde{b}, s)] = [(\pi_q(\tilde{b}), s)] = [(b, s)] = \beta.$$

$\tilde{a}$ and $\tilde{b}$ represent spark classes in $\hat{\mathbf{H}}^*(X)$. Write the spark equations for $\tilde{a}$ and $\tilde{b}$ as

$$d\tilde{a} = e - r \quad \text{and} \quad d\tilde{b} = f - s,$$

where $\pi_p \tilde{a} = a$, $\pi_p e = 0$ and $\pi_q \tilde{b} = b$, $\pi_q f = 0$.

By the product formula in Theorem 3.1.14

$$[(\tilde{a}, r)] * [(\tilde{b}, s)] = [(\tilde{a} \wedge f + (-1)^k r \wedge \tilde{b}, r \wedge s)] = [(\tilde{a} \wedge s + (-1)^k e \wedge \tilde{b}, r \wedge s)].$$

Since

$$d(\tilde{a} \wedge f + (-1)^k r \wedge \tilde{b}) = e \wedge f - r \wedge s \quad \text{and} \quad \pi_{p+q}(e \wedge f) = 0,$$

we get $\Pi_{p+q}[(\tilde{a} \wedge f + (-1)^k r \wedge \tilde{b}, r \wedge s)] \in H_{\mathcal{D}}^{k+l}(X, \mathbb{Z}(p+q))$, and define it as $\alpha * \beta$.

Now we show the product is well-defined, i.e. it is independent of the choices of representatives α and β. If we have another representative $(a', r') \in \alpha$ and a lift $(\tilde{a}', r')$ with $\Pi_p[(\tilde{a}', r')] = [(a', r')] = \alpha$, then $[(\tilde{a}, r) - (\tilde{a}', r')] \in \ker \Pi_p$. By Lemma 4.1.7, there exists a representative of spark class $[(\tilde{a}, r) -$

$(\tilde{a}', r')]$, which is of form $(c, 0)$ where c is smooth and $\pi_p(c) = 0$. Then we have

$$\begin{aligned}
& \Pi_{p+q}([(\tilde{a}, r)] * [(\tilde{b}, s)] - [(\tilde{a}', r')] * [(\tilde{b}, s)]) \\
= \ & \Pi_{p+q}([(\tilde{a}, r) - (\tilde{a}', r')] * [(\tilde{b}, s)]) \\
= \ & \Pi_{p+q}([(c, 0)] * [(\tilde{b}, s)]) \\
= \ & \Pi_{p+q}([(c \wedge f + (-1)^k 0 \wedge \tilde{b}, 0)]) \\
= \ & 0
\end{aligned}$$

Similarly, we can show the product does not depend on representatives of β either.

Remark 5.1.5. *In the process above, we can always choose good representatives a, $\tilde{a}$, b, $\tilde{b}$, r and s in sense of Proposition 3.1.13 such that all wedge products are well defined.*

Theorem 5.1.6. Product formula of Deligne cohomology I

For any Deligne classes $\alpha \in H^k_{\mathcal{D}}(X, \mathbb{Z}(p))$ and $\beta \in H^l_{\mathcal{D}}(X, \mathbb{Z}(q))$, there exist spark representations (a, r) for α and (b, s) for β as above. Let $(\tilde{a}, r)$ and $(\tilde{b}, s)$ be de Rham-Federer sparks which are lifts of (a, r) and (b, s). Then we define the product $\alpha * \beta$ in $H^{k+l}_{\mathcal{D}}(X, \mathbb{Z}(p+q))$ by

$$\alpha * \beta = \Pi_{p+q}[(\tilde{a} \wedge f + (-1)^k r \wedge \tilde{b}, r \wedge s)] = [(\pi_{p+q}(\tilde{a} \wedge f + (-1)^k r \wedge \tilde{b}), r \wedge s)].$$

Proof. We have shown the product is well defined. In Theorem 5.1.11, we shall verify that this product is equivalent to Beilinson's definition. □

Remark 5.1.7. *Suppose X is a algebraic manifold and $CH^*(X)$ is the Chow ring of X. Considering every nonsingular subvariety as a integrally flat current, we can define the group homomorphism*

$$\psi : CH^p(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)).$$

By our product formula, it is quite easy to see this map induces a ring homomorphism, i.e. the ring structure of Deligne cohomology is compatible with the ring structure of the Chow ring. We shall explain this in the next section.

In Theorem 5.1.6, we represent Deligne cohomology classes by currents and show an explicit product formula of Deligne classes. Now we derive another product formula by representing Deligne classes in terms of the Čech-Dolbeault sparks. Then we show this product is equivalent to the product defined by Beilinson in Theorem 5.1.11.

Lemma 5.1.8. *We have the short exact sequence*

$$0 \to H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) \to \hat{\mathbf{H}}^{k-1}_{smooth}(X, p) \to \mathcal{Z}^k_{\mathbb{Z}}(X, p) \to 0.$$

For any Deligne class $\alpha \in H^k_{\mathcal{D}}(X, \mathbb{Z}(p))$, there exists a representative

$$(a, r) \in \bigoplus_{r+s=k-1} C^r(\mathcal{U}, \mathcal{E}^s_p) \oplus C^k(\mathcal{U}, \mathbb{Z}) \quad \textit{with} \quad D_p a = -r, \quad \delta r = 0.$$

Proof. Note that we use (a, r) to represent a Čech-Dolbeault spark here although we can omit r. The reason is that we can make the proof of Theorem 5.1.11 clearer with this representation.

Applying the following quasi-isomorphisms of complexes of sheaves:

$$\mathbb{Z} \simeq \mathcal{C}^*(\mathcal{U}, \mathbb{Z}) \quad \text{and} \quad \Omega^{*<p} \simeq \mathcal{E}_p^* \simeq \bigoplus_{r+s=*} \mathcal{C}^r(\mathcal{U}, \mathcal{E}_p^s),$$

we get

$$H_{\mathcal{D}}^k(X, \mathbb{Z}(p)) \cong \mathbb{H}^{k-1}(\text{Cone}(\mathbb{Z} \to \Omega^{*<p}) \cong \mathbb{H}^{k-1}(\text{Cone}(\mathcal{C}^*(\mathcal{U}, \mathbb{Z}) \to \bigoplus_{r+s=*} \mathcal{C}^r(\mathcal{U}, \mathcal{E}_p^s)))$$

$$\cong H^{k-1}(\text{Cone}(C^*(\mathcal{U}, \mathbb{Z}) \to \bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{E}_p^s))).$$

By Proposition 2.3.7 and definition of $\hat{\mathbf{H}}^*_{smooth}(X,p)$, we have the short exact sequence:

$$0 \to H_{\mathcal{D}}^k(X, \mathbb{Z}(p)) \to \hat{\mathbf{H}}^{k-1}_{smooth}(X,p) \to \mathcal{Z}_{\mathbb{Z}}^k(X,p) \to 0.$$

Hence, for any Deligne class $\alpha \in H_{\mathcal{D}}^k(X, \mathbb{Z}(p)) \subset \hat{\mathbf{H}}^{k-1}_{smooth}(X,p)$, we can find a representative $(a,r) \in \bigoplus_{r+s=k-1} C^r(\mathcal{U}, \mathcal{E}_p^s) \oplus C^k(\mathcal{U}, \mathbb{Z})$ with $D_p a = e - r$, $\delta r = 0$. And we have $e = 0$ since $\alpha \in \ker \delta_1$. □

Via the Čech-Dolbeault spark complex, we establish another product formula for Deligne cohomology. The construction is similar to the former one, but we still show the construction in detail to help the readers to understand Theorem 5.1.11.

Our goal is to define the product in Deligne cohomology

$$H_{\mathcal{D}}^k(X, \mathbb{Z}(p)) \otimes H_{\mathcal{D}}^l(X, \mathbb{Z}(q)) \longrightarrow H_{\mathcal{D}}^{k+l}(X, \mathbb{Z}(p+q)).$$

First, we choose a Čech-Dolbeault spark representative

$$(a,r)\in\bigoplus_{r+s=k-1}C^r(\mathcal{U},\mathcal{E}^s_p)\oplus C^k(\mathcal{U},\mathbb{Z})\quad\text{with}\quad D_pa=-r,\quad\delta r=0$$

for Deligne class $\alpha\in H^k_{\mathcal{D}}(X,\mathbb{Z}(p))$, and a Čech-Dolbeault spark representative

$$(b,s)\in\bigoplus_{r+s=l-1}C^r(\mathcal{U},\mathcal{E}^s_p)\oplus C^l(\mathcal{U},\mathbb{Z})\quad\text{with}\quad D_qb=-s,\quad\delta s=0$$

for $\beta\in H^l_{\mathcal{D}}(X,\mathbb{Z}(q))$.

Since $\Pi_p:\hat{\mathbf{H}}^*_{smooth}(X)\to\hat{\mathbf{H}}^*_{smooth}(X,p)$ is surjective, there exist smooth hypersparks

$$(\tilde{a},r)\in\bigoplus_{r+s=k-1}C^r(\mathcal{U},\mathcal{E}^s)\oplus C^k(\mathcal{U},\mathbb{Z})\quad\text{with}\quad\Pi_p[(\tilde{a},r)]=[(a,r)]=\alpha,$$

and

$$(\tilde{b},s)\in\bigoplus_{r+s=l-1}C^r(\mathcal{U},\mathcal{E}^s)\oplus C^l(\mathcal{U},\mathbb{Z})\quad\text{with}\quad\Pi_q[(\tilde{b},s)]=[(b,s)]=\beta.$$

Write the spark equations of $\tilde{a}$ and $\tilde{b}$ as

$$D\tilde{a}=e-r\quad\text{and}\quad D\tilde{b}=f-s,$$

where $\pi_p\tilde{a}=a$, $\pi_pe=0$ and $\pi_q\tilde{b}=b$, $\pi_qf=0$.

By the product formula of $\hat{\mathbf{H}}^*_{smooth}(X)$ constructed in §3.2,

$$[(\tilde{a},r)]*[(\tilde{b},s)]=[\tilde{a}\cup f+(-1)^kr\cup\tilde{b},r\cup s]=[\tilde{a}\cup s+(-1)^ke\cup\tilde{b},r\cup s].$$

We have

$$D(\tilde{a} \cup f + (-1)^k r \cup \tilde{b}) = e \wedge f - r \cup s \text{ and } \pi_{p+q}(e \wedge f) = 0,$$

so we get $\Pi_{p+q}[(\tilde{a} \cup f + (-1)^k r \cup \tilde{b}, r \cup s)] \in H_{\mathcal{D}}^{k+l}(X, \mathbb{Z}(p+q))$, which is defined to be $\alpha * \beta$.

The product is only dependent on the spark classes α and β. If we have another representative $(a', r') \in \alpha$ and a lift $(\tilde{a}', r')$ with $\Pi_p[(\tilde{a}', r')] = [(a', r')] = \alpha$, then $[(\tilde{a}, r) - (\tilde{a}', r')] \in \ker \Pi_p$. By Lemma 4.2.4, we can choose a representative of spark class $[(\tilde{a}, r) - (\tilde{a}', r')]$, which is of form $(c, 0)$ where c is smooth and $\pi_p(c) = 0$. Then we have

$$\begin{aligned}
& \Pi_{p+q}([(\tilde{a}, r)] * [(\tilde{b}, s)] - [(\tilde{a}', r')] * [(\tilde{b}, s)]) \\
= \; & \Pi_{p+q}([(\tilde{a}, r) - (\tilde{a}', r')] * [(\tilde{b}, s)]) \\
= \; & \Pi_{p+q}([(c, 0)] * [(\tilde{b}, s)]) \\
= \; & \Pi_{p+q}([(c \cup f + (-1)^k 0 \cup \tilde{b}, 0)]) \\
= \; & 0
\end{aligned}$$

Similarly, we can show the product does not depend on representatives of β either.

Theorem 5.1.9. Product formula of Deligne cohomology II

For any Deligne classes $\alpha \in H_{\mathcal{D}}^k(X, \mathbb{Z}(p))$ and $\beta \in H_{\mathcal{D}}^l(X, \mathbb{Z}(q))$, there exist Čech-Dolbeault spark representations (a, r) for α and (b, s) for β as above. Let $(\tilde{a}, r)$ and $(\tilde{b}, s)$ be smooth hypersparks which are lifts of (a, r) and (b, s). Then *we define the product $\alpha * \beta$ in $H_{\mathcal{D}}^{k+l}(X, \mathbb{Z}(p+q))$ by*

$$\alpha * \beta = \Pi_{p+q}[(\tilde{a} \cup f + (-1)^k r \cup \tilde{b}, r \cup s)] = [(\pi_{p+q}(\tilde{a} \cup f + (-1)^k r \cup \tilde{b}), r \cup s)].$$

Theorem 5.1.10. *Two product formulas in Theorem 5.1.6 and Theorem 5.1.9 are equivalent.*

Proof. The product formula in Theorem 5.1.6 and Theorem 5.1.9 are based on product formulas of $\hat{\mathbf{H}}^*_{spark}(X, p)$ and $\hat{\mathbf{H}}^*_{smooth}(X, p)$ established in Theorem 3.1.14 and 3.2.4 respectively. Note that $\hat{\mathbf{H}}^*_{spark}(X, p) \cong \hat{\mathbf{H}}^*_{smooth}(X, p)$ and the ring structures on them are compatible. Hence, product formulas in Theorems 5.1.6 and 5.1.9 are equivalent as well. □

Our product formula is quite explicit compared with the product in [B] which is defined on the sheaf level. Now we verify that these products are equivalent.

Theorem 5.1.11. *The products Theorem 5.1.6 and 5.1.9 are equivalent to Beilinson's product.*

Proof. It is sufficient to show that the product formula in Theorem 5.1.9 is the same as Beilinson's product which is induced from the cup product on the sheaf level.

The outline of the proof is following: First, we construct an explicit isomorphism between $H_{\mathcal{D}}^k(X, \mathbb{Z}(p))$ and $\ker \delta_1 : \hat{\mathbf{H}}^{k-1}_{smooth}(X, p) \to \mathcal{Z}^k_{\mathbb{Z}}(X, p)$; Then, we calculate the product induced by $\cup : \mathbb{Z}_{\mathcal{D}}(p) \otimes \mathbb{Z}_{\mathcal{D}}(q) \to \mathbb{Z}_{\mathcal{D}}(p+q)$ using Čech resolution; Finally, we calculate the product via smooth hypersparks defined earlier in this section, and compare these two products.

Step 1: Fix a good cover $\{\mathcal{U}\}$ of X and take Čech resolution for the complex of sheaves $\mathbb{Z}_{\mathcal{D}}(p) \longrightarrow \mathcal{C}^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p))$.

Then

$$H^q_{\mathcal{D}}(X, \mathbb{Z}(p)) \equiv \mathbb{H}^q(\mathbb{Z}_{\mathcal{D}}(p)) \cong \mathbb{H}^q(Tot(\mathcal{C}^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p)))) \cong H^q(Tot(C^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p))))$$

where $C^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p))$ are the groups of global sections of sheaves $\mathcal{C}^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p))$ and look like the following double complex.

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & & & \vdots \\
\uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & & & \uparrow\delta \\
C^k(\mathcal{U},\mathbb{Z}) & \xrightarrow{(-1)^k i} & C^k(\mathcal{U},\Omega^0) & \xrightarrow{(-1)^k\partial} & C^k(\mathcal{U},\Omega^1) & \xrightarrow{(-1)^k\partial} & C^k(\mathcal{U},\Omega^2) & \xrightarrow{(-1)^k\partial} & \cdots & \xrightarrow{(-1)^k\partial} & C^k(\mathcal{U},\Omega^{p-1}) \\
\uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & & & \uparrow\delta \\
\vdots & & \vdots & & \vdots & & \vdots & & & & \vdots \\
\uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & & & \uparrow\delta \\
C^2(\mathcal{U},\mathbb{Z}) & \xrightarrow{i} & C^2(\mathcal{U},\Omega^0) & \xrightarrow{\partial} & C^2(\mathcal{U},\Omega^1) & \xrightarrow{\partial} & C^2(\mathcal{U},\Omega^2) & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & C^2(\mathcal{U},\Omega^{p-1}) \\
\uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & & & \uparrow\delta \\
C^1(\mathcal{U},\mathbb{Z}) & \xrightarrow{-i} & C^1(\mathcal{U},\Omega^0) & \xrightarrow{-\partial} & C^1(\mathcal{U},\Omega^1) & \xrightarrow{-\partial} & C^1(\mathcal{U},\Omega^2) & \xrightarrow{-\partial} & \cdots & \xrightarrow{-\partial} & C^1(\mathcal{U},\Omega^{p-1}) \\
\uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & & & \uparrow\delta \\
C^0(\mathcal{U},\mathbb{Z}) & \xrightarrow{i} & C^0(\mathcal{U},\Omega^0) & \xrightarrow{\partial} & C^0(\mathcal{U},\Omega^1) & \xrightarrow{\partial} & C^0(\mathcal{U},\Omega^2) & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & C^0(\mathcal{U},\Omega^{p-1})
\end{array}$$

Let $M_p^* \equiv Tot(C^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p)))$ denote the total complex of the double complex $C^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p))$ with differential

$$\Delta_p(a) = \begin{cases} (\delta + (-1)^r i)(a), & \text{when} \quad a \in C^r(\mathcal{U}, \mathbb{Z}); \\ (\delta + (-1)^r \partial)(a), & \text{when} \quad a \in C^r(\mathcal{U}, \Omega^j), \quad j < p-1; \\ \delta a, & \text{when} \quad a \in C^r(\mathcal{U}, \Omega^{p-1}). \end{cases}$$

Now we construct a map

$$\varphi_p : H^*(M_p^*) \cong H^*_{\mathcal{D}}(X, \mathbb{Z}(p)) \longrightarrow \ker \delta_1 \subset \hat{\mathbf{H}}^{*-1}_{smooth}(X, p).$$

Assume that a cycle $\tilde{a} \in M_p^k$ represents a Deligne class in $H^k_{\mathcal{D}}(X, \mathbb{Z}(p))$, and

$$\tilde{a} = r + a = r + \sum_{i+j=k-1, j<p} a^{i,j}$$

where $r \in C^k(\mathcal{U}, \mathbb{Z})$ and $a^{i,j} \in C^i(\mathcal{U}, \Omega^j)$.

Note that $a^{i,j} \in C^i(\mathcal{U}, \Omega^j) \subset C^i(\mathcal{U}, \mathcal{E}^{j,0}) \subset C^i(\mathcal{U}, \mathcal{E}^j_p)$, so $a \in \bigoplus_{i+j=k-1} C^i(\mathcal{U}, \mathcal{E}^j_p)$. And it is easy to see

$$\Delta_p \tilde{a} = 0 \quad \Leftrightarrow \quad D_p a + (-1)^k r = 0 \text{ and } \delta r = 0,$$

where D_p is the differential of the total complex of double complex $\bigoplus_{r+s=*} C^r(\mathcal{U}, \mathcal{E}^s_p)$ defined in §4.2. Hence, $\varphi_p : \tilde{a} \to (a, (-1)^k r)$ gives a map from cycles to smooth hypersparks. Moreover, assume $\tilde{a}$ and $\tilde{a}'$ represent the same Deligne class, i.e. $\tilde{a} - \tilde{a}' = \Delta_p \tilde{b}$ is a boundary, where $\tilde{b} = s + \Sigma_{i+j=k-2, j<p} b^{i,j}$ for $s \in C^{k-1}(\mathcal{U}, \mathbb{Z})$ and $b^{i,j} \in C^i(\mathcal{U}, \Omega^j)$. Then

$$a - a' + r - r' = \tilde{a} - \tilde{a}' = \Delta_p \tilde{b} = \delta s + (-1)^{k-1} i(s) + D_p b$$

implies

$$a - a' = (-1)^{k-1} s + D_p b \quad \text{and} \quad (-1)^k r - (-1)^k r' = -(-1)^{k-1} \delta s,$$

i.e. $(a, (-1)^k r)$ and $(a', (-1)^k r')$ represent the same spark class. So the map (also denoted by φ_p)

$$\varphi_p : H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) \to \hat{\mathbf{H}}^{k-1}_{smooth}(X, p)$$

which maps a Deligne class $[\tilde{a}]$ to a smooth hyperspark class $[(a, (-1)^k r)]$ is well-defined. Since $\varphi_p([\tilde{a}]) = [(a, (-1)^k r)]$ satisfies the spark equation $D_p a = 0 - (-1)^k r$, so we have $Im\varphi_p \subset \ker \delta_1$. Therefore, $\varphi_p : H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) \to \ker \delta_1$ give an explicit isomorphism between $H^k_{\mathcal{D}}(X, \mathbb{Z}(p))$ and $\ker \delta_1$.

Step 2: The product formula for Deligne classes is induced by the cup product

$$\cup : \mathbb{Z}_{\mathcal{D}}(p) \otimes \mathbb{Z}_{\mathcal{D}}(q) \to \mathbb{Z}_{\mathcal{D}}(p+q)$$

with the formula

$$x \cup y = \begin{cases} x \cdot y & \text{if } \deg x = 0; \\ x \wedge dy & \text{if } \deg x > 0 \text{ and } \deg y = q; \\ 0 & \text{otherwise.} \end{cases}$$

In the §8, we showed the explicit product formula on Čech cycles.

Assume

$$\alpha \in H^k_{\mathcal{D}}(X, \mathbb{Z}(p)) \quad \text{and} \quad \beta \in H^l_{\mathcal{D}}(X, \mathbb{Z}(q)),$$

and let

$$\tilde{a} = r + a = r + \sum_{i+j=k-1, j<p} a^{i,j} \in M^k_p \text{ be a representative of } \alpha$$

and

$$\tilde{b} = s + b = s + \sum_{i+j=l-1, i<q} b^{i,j} \in M_q^l \text{ be a representative of } \beta$$

where

$$r \in C^k(\mathcal{U}, \mathbb{Z}), \quad a^{i,j} \in C^i(\mathcal{U}, \Omega^j),$$

and

$$s \in C^l(\mathcal{U}, \mathbb{Z}), \quad b^{i,j} \in C^i(\mathcal{U}, \Omega^j).$$

By the §8, we calculate

$$\begin{aligned} & \alpha \cup \beta \\ = \ & [\tilde{a} \cup \tilde{b}] \\ = \ & [rs + \Sigma_{i+j=l-1, j<q} (-1)^{0 \cdot i} r \cdot b^{i,j} + \Sigma_{i+j=k-1, j<p} (-1)^{(j+1)\cdot(l-q)} a^{i,j} \wedge \partial b^{l-q, q-1}] \\ = \ & [r \cup \tilde{b} + (-1)^{l-q} a \cup \partial b^{l-q, q-1}] \end{aligned}$$

Note that $a^{i,j}$ is of bidegree $(i, j+1)$ in the Čech-Deligne double complex $C^*(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p))$. This explains the sign $(-1)^{(j+1)\cdot(l-q)}$ in front of the last summand in the second line to the bottom. On the other hand, considering $a^{i,j} \in C^i(\mathcal{U}, \Omega^j) \subset C^i(\mathcal{U}, \mathcal{E}^j)$ as an element in the Čech-de Rham double complex, it is of degree (i, j). By the definition of $\cup$ in Proposition 3.2.2, we get the last equality.

Step 3: Let us calculate the product of α and β by the formula in Theorem 6.9.

$\varphi_p(\tilde{a}) = (a, (-1)^k r)$ and $\varphi_q(\tilde{b}) = (b, (-1)^l s)$ are two smooth hypersparks which represent Deligne classes α and β respectively. The spark equations associated to them are:

$$D_p a = 0 - (-1)^k r, \quad \delta(-1)^k r = 0$$

and

$$D_p b = 0 - (-1)^l s, \quad \delta(-1)^l s = 0.$$

Because of the surjectivity of the map $\Pi_p : \hat{\mathbf{H}}^*_{smooth}(X) \to \hat{\mathbf{H}}^*_{smooth}(X,p)$, there exist

$$(A, (-1)^k r) \in \bigoplus_{i+j=k-1} C^i(\mathcal{U}, \mathcal{E}^j) \oplus C^k(\mathcal{U}, \mathbb{Z}) \text{ with } \Pi_p[(A, (-1)^k r)] = [(a, (-1)^k r)] = \alpha,$$

and

$$(B, (-1)^l s) \in \bigoplus_{i+j=l-1} C^i(\mathcal{U}, \mathcal{E}^j) \oplus C^l(\mathcal{U}, \mathbb{Z}) \text{ with } \Pi_q[(B, (-1)^l s)] = [(b, (-1)^l s)] = \beta.$$

Assume the spark equations for A and B are $DA = e - (-1)^k r$, $DB = f - (-1)^l s$, then $\pi_p A = a$, $\pi_p e = 0$ and $\pi_q B = b$, $\pi_q f = 0$.

By the product formula in Theorem 3.2.4,

$$\begin{aligned} & [(A, (-1)^k r)] * [(B, (-1)^l s)] \\ = \; & [A \cup f + (-1)^k (-1)^k r \cup B, (-1)^k r \cup (-1)^l s] \\ = \; & [A \cup f + r \cup B, (-1)^{k+l} r \cup s]. \end{aligned}$$

We have

$$D(A \cup f + r \cup B) = e \wedge f - (-1)^{k+l} r \cup s \text{ and } \pi_{p+q}(e \wedge f) = 0,$$

so we get $\Pi_{p+q}[(A \cup f + r \cup B, (-1)^{k+l} r \cup s)] \in H_{\mathcal{D}}^{k+l}(X, \mathbb{Z}(p+q))$, and define it as $\alpha * \beta$.

We compare two results under isomorphism

$$\varphi_{p+q} : H_{\mathcal{D}}^{k}(X, \mathbb{Z}(p+q)) \to \ker \delta_1.$$

The following lemma shows that $\varphi_{p+q}(r \cup \tilde{b} + (-1)^{l-q} a \cup \partial b^{l-q,q-1}) = (r \cup b + (-1)^{l-q} a \cup \partial b^{l-q,q-1}, (-1)^{k+l} r \cup s)$ and $(\pi_{p+q}(A \cup f + r \cup B), (-1)^{k+l} r \cup s)$ represent the same class. □

Lemma 5.1.12.

$$r \cup b + (-1)^{l-q} a \cup \partial b^{l-q,q-1} = \pi_{p+q}(A \cup f + r \cup B) + (-1)^k D_{p+q}(a \cup (B - b)).$$

Proof. Compare

$$Db = (-1)^{l-q} \partial b^{l-q,q-1} + D_p b = (-1)^{l-q} \partial b^{l-q,q-1} - (-1)^l s$$

and

$$DB = f - (-1)^l s,$$

we have

$$(-1)^{l-q} \partial b^{l-q,q-1} = f - D(B - b). \tag{5.1}$$

$$
\begin{aligned}
& (-1)^k D_{p+q}(a \cup (B-b)) \\
= \;& (-1)^k \pi_{p+q} D(a \cup (B-b)) \\
= \;& (-1)^k \pi_{p+q}(Da \cup (B-b) + (-1)^{k-1} a \cup D(B-b)) \\
\overset{*}{=} \;& (-1)^k \pi_{p+q}(D_p a \cup (B-b)) - \pi_{p+q}(a \cup D(B-b)) \\
= \;& (-1)^k \pi_{p+q}(-(-1)^k r \cup (B-b)) - \pi_{p+q}(a \cup D(B-b)) \\
= \;& -\pi_{p+q}(r(B-b)) - \pi_{p+q}(a \cup D(B-b))
\end{aligned} \tag{5.2}
$$

The equality $*$ follows that $\pi_p(Da - D_p a) = 0$ and $\pi_q(B-b) = 0$.

By 5.1 and 5.2, we have

$$
\begin{aligned}
& \text{Right hand side} \\
= \;& \pi_{p+q}(A \cup f + r \cup B) + (-1)^k D_{p+q}(a \cup (B-b)) \\
= \;& \pi_{p+q}(a \cup f) + \pi_{p+q}(r \cup B) - \pi_{p+q}(r(B-b)) - \pi_{p+q}(a \cup D(B-b)) \\
= \;& \pi_{p+q}(a \cup f - a \cup D(B-b)) + \pi_{p+q}(r \cup b) \\
= \;& \pi_{p+q}((-1)^{l-q} a \cup \partial b^{l-q,q-1}) + \pi_{p+q}(r \cup b) \\
= \;& (-1)^{l-q} a \cup \partial b^{l-q,q-1} + r \cup b \\
= \;& \text{Left hand side.}
\end{aligned}
$$

□

Question. *It would be interesting to find and study other spark complexes in geometry, topology and physics. In particular, for smooth quasi-projective varieties, we may consider forms with logarithmic poles and develop a spark* *theory closely related to Deligne-Beilinson cohomology [B].*

5.2 Massey Products in Deligne Cohomology

We defined the Massey triple product in spark characters $\hat{\mathbf{H}}^*(X)$ in §3.5. In fact, it induces the Massey triple product in Deligne cohomology.

Assume that we have three Deligne cohomology classes

$$\alpha \in H^i_{\mathcal{D}}(X, \mathbb{Z}(p_1)) \subset \hat{\mathbf{H}}^{i-1}(X, p_1),$$

$$\beta \in H^j_{\mathcal{D}}(X, \mathbb{Z}(p_2)) \subset \hat{\mathbf{H}}^{j-1}(X, p_2),$$

$$\gamma \in H^k_{\mathcal{D}}(X, \mathbb{Z}(p_3)) \subset \hat{\mathbf{H}}^{k-1}(X, p_3)$$

with their spark presentations:

$$\alpha : (a, r) \in \mathcal{D}'^{i-1}(X, p_1) \oplus \mathcal{IF}^i(X) \quad \text{with} \quad d_{p_1} a = -\Psi_{p_1}(r), \quad dr = 0;$$

$$\beta : (b, s) \in \mathcal{D}'^{j-1}(X, p_2) \oplus \mathcal{IF}^j(X) \quad \text{with} \quad d_{p_2} b = -\Psi_{p_2}(s), \quad ds = 0;$$

$$\gamma : (c, t) \in \mathcal{D}'^{k-1}(X, p_3) \oplus \mathcal{IF}^k(X) \quad \text{with} \quad d_{p_3} c = -\Psi_{p_3}(t), \quad dt = 0.$$

We choose lifts of α, β, γ in $\hat{\mathbf{H}}^*(X)$ with representatives

$$(\tilde{a}, r) \in \mathcal{D}'^{i-1}(X) \oplus \mathcal{IF}^i(X) \text{ with } d\tilde{a} = e - r \text{ where } e \in \mathcal{E}^i(X) \text{ and } \pi_{p_1} e = 0;$$

$$(\tilde{b}, s) \in \mathcal{D}'^{j-1}(X) \oplus \mathcal{IF}^j(X) \text{ with } d\tilde{b} = f - s \text{ where } f \in \mathcal{E}^j(X) \text{ and } \pi_{p_2} f = 0;$$

$$(\tilde{c}, t) \in \mathcal{D}'^{k-1}(X) \oplus \mathcal{IF}^k(X) \text{ with } d\tilde{c} = g - t \text{ where } g \in \mathcal{E}^k(X) \text{ and } \pi_{p_3} g = 0.$$

When $\alpha\beta = \beta\gamma = 0$, we define the Massey triple product for the triple (α, β, γ), denoted by $\mathcal{M}(\alpha, \beta, \gamma)$, as follows.

In the last section, we define

$$\alpha\beta = \Pi_{p_1+p_2}([\tilde{a}][\tilde{b}]) = \Pi_{p_1+p_2}([\tilde{a} \wedge f + (-1)^i r \wedge \tilde{b}]).$$

And $\alpha\beta = 0 \in H_{\mathcal{D}}^{i+j}(X, \mathbb{Z}(p_1 + p_2)) \subset \hat{\mathbf{H}}^{i+j-1}(X, p_1 + p_2)$ (together with Lemma 4.1.7) implies

$$[\tilde{a}][\tilde{b}] = [\tilde{a} \wedge f + (-1)^i r \wedge \tilde{b}] \in \ker \Pi_{p_1+p_2} \subset \hat{\mathbf{H}}_{\infty}^{i+j-1}(X) \subset \hat{\mathbf{H}}^{i+j-1}(X).$$

Moreover, Lemma 4.1.7 tells us that any element in $\ker \Pi_{p_1+p_2}$ can be represented by a smooth form ϕ with the property $\pi_{p_1+p_2}(\phi) = 0$. So we can assume that $[\tilde{a}][\tilde{b}] = [\tilde{a} \wedge f + (-1)^i r \wedge \tilde{b}] = [\phi]$ for $\phi \in \mathcal{E}^{i+j-1}(X)$ with $\pi_{p_1+p_2}(\phi) = 0$, i.e. $\exists A \in \mathcal{D}'^{i+j-2}(X)$ and $X \in \mathcal{IF}^{i+j-1}(X)$ such that

$$\tilde{a} \wedge f + (-1)^i r \wedge \tilde{b} = dA + X + \phi.$$

Similarly, $\beta\gamma = 0 \in H_{\mathcal{D}}^{j+k}(X, \mathbb{Z}(p_2 + p_3)) \subset \hat{\mathbf{H}}^{j+k-1}(X, p_2 + p_3)$ implies

$$[\tilde{b}][\tilde{c}] = [\tilde{b} \wedge g + (-1)^j s \wedge \tilde{c}] \in \ker \Pi_{p_2+p_3} \subset \hat{\mathbf{H}}_{\infty}^{j+k-1}(X) \subset \hat{\mathbf{H}}^{j+k-1}(X),$$

and $\exists B \in \mathcal{D}'^{j+k-2}(X)$, $Y \in \mathcal{IF}^{j+k-1}(X)$ and $\psi \in \mathcal{E}^{j+k-1}(X)$ with $\pi_{p_2+p_3}(\psi) = 0$ such that

$$\tilde{b} \wedge g + (-1)^j s \wedge \tilde{c} = dB + Y + \psi.$$

As in §3.5, the Massey triple product of $[\tilde{a}]$,$[\tilde{b}]$ and $[\tilde{c}]$ in spark characters is defined by

$$\mathcal{M}([\tilde{a}],[\tilde{b}],[\tilde{c}]) = [\tilde{a}\psi + (-1)^{i-1}rB + (-1)^{i}Ag + (-1)^{j+1}X\tilde{c}]$$

$$\in \hat{\mathbf{H}}^{i+j+k-2}/([\tilde{a}]\hat{\mathbf{H}}^{j+k-2} + \hat{\mathbf{H}}^{i+j-2}[\tilde{c}]).$$

Note that

$$d(\tilde{a}\psi+(-1)^{i-1}rB+(-1)^{i}Ag+(-1)^{j+1}X\tilde{c}) = e\psi+(-1)^{i-1}\phi g+rY+(-1)^{i-1}Xt,$$

and

$$\pi_{p_1+p_2+p_3}(e\psi + (-1)^{i-1}\phi g) = 0.$$

Therefore, we define the Massey triple product of α, β and γ as

$$\Pi_{p_1+p_2+p_3}([\tilde{a}\psi + (-1)^{i-1}rB + (-1)^{i}Ag + (-1)^{j+1}X\tilde{c}]) =$$

$$[(\pi_{p_1+p_2+p_3}(\tilde{a}\psi + (-1)^{i-1}rB + (-1)^{i}Ag + (-1)^{j+1}X\tilde{c}), rY + (-1)^{i-1}Xt)]$$

$$\in H_{\mathcal{D}}^{i+j+k-1}(X,\mathbb{Z}(p_1+p_2+p_3)) \subset \hat{\mathbf{H}}^{i+j+k-2}(X,p_1+p_2+p_3).$$

The triple product is well-defined up to some indeterminacy due to different choices of A, B, X, Y, ϕ and ψ. If we have $\tilde{a}f + (-1)^{i}r\tilde{b} = dA + X + \phi = dA' + X' + \phi'$, then

$$\tilde{a}\psi+(-1)^{i-1}rB+(-1)^{i}Ag+(-1)^{j-1}X\tilde{c}-(\tilde{a}\psi+(-1)^{i-1}rB+(-1)^{i}A'g+(-1)^{j-1}X'\tilde{c})$$

$$= (-1)^{j-1}(X-X')\tilde{c} + (-1)^{i}(A-A')g.$$

Notice that $d(A-A') = -(\phi-\phi') - (X-X')$ and $\pi_{p_1+p_2}(\phi) = \pi_{p_1+p_2}(\phi') = 0$, hence $A-A'$ represents a spark class which is located in $H_{\mathcal{D}}^{i+j-1}(X, \mathbb{Z}(p_1+p_2))$. Moreover,

$$[A-A'][\tilde{c}] = [(A-A')g + (-1)^{i+j-1}(X-X')\tilde{c}],$$

Hence the difference

$$[(-1)^{j-1}(X-X')c + (-1)^i(A-A')g] = (-1)^i[A-A'][\tilde{c}]$$

$$\in H_{\mathcal{D}}^{i+j-1}(X, \mathbb{Z}(p_1+p_2))\gamma \subset \hat{\mathbf{H}}^{i+j-2}(X, p_1+p_2)\gamma.$$

Similarly, if $\tilde{b}g + (-1)^j s\tilde{c} = dB + Y + \psi = dB' + Y' + \psi'$, then

$$\tilde{a}\psi + (-1)^{i-1}rB + (-1)^i Ag + (-1)^{j-1}X\tilde{c} - (\tilde{a}\psi' + (-1)^{i-1}rB' + (-1)^i Ag + (-1)^{j-1}X\tilde{c})$$

$$= \tilde{a}(\psi-\psi') + (-1)^{i-1}r(B-B').$$

$B-B'$ is a spark satisfying the spark equation $d(B-B') = -(\psi-\psi') - (Y-Y')$ with $\pi_{p_2+p_3}(\psi-\psi') = 0$, so $[B-B'] \in H_{\mathcal{D}}^{j+k-1}(X, \mathbb{Z}(p_2+p_3))$.

The product

$$[\tilde{a}][B-B'] = [-\tilde{a}(\psi-\psi') + (-1)^i r(B-B')] = -[\tilde{a}(\psi-\psi') + (-1)^{i-1}r(B-B')].$$

Therefore,

$$[\tilde{a}(\psi-\psi') + (-1)^{i-1}r(B-B')] = -[\tilde{a}][B-B']$$

$$\in \alpha H_{\mathcal{D}}^{j+k-1}(X, \mathbb{Z}(p_2+p_3)) \subset \alpha\hat{\mathbf{H}}^{j+k-2}(X, p_2+p_3).$$

Finally, we have

Theorem 5.2.1. *If three Deligne classes* $\alpha \in H^i_{\mathcal{D}}(X, \mathbb{Z}(p_1))$, $\beta \in H^j_{\mathcal{D}}(X, \mathbb{Z}(p_2))$ *and* $\gamma \in H^k_{\mathcal{D}}(X, \mathbb{Z}(p_3))$ *satisfy that* $\alpha\beta = 0$ *and* $\beta\gamma = 0$, *then the Massey triple product of* α, β *and* γ, *denoted by* $\mathcal{M}(\alpha, \beta, \gamma)$, *is well-defined in*

$$H^{i+j+k-1}_{\mathcal{D}}(X, \mathbb{Z}(p_1+p_2+p_3))/(\alpha H^{j+k-1}_{\mathcal{D}}(X, \mathbb{Z}(p_2+p_3)) + H^{i+j-1}_{\mathcal{D}}(X, \mathbb{Z}(p_1+p_2))\gamma).$$

In his thesis [Sc], Schwarzhaupt showed an example where the Massey triple product of three Deligne classes is not zero but a torsion. So far, we do not have a clear answer whether the triple product must be a torsion.

Question. *Are Massey triple products of (integer-valued) Deligne classes always torsion?*

5.3 Application to Algebraic Cycles

We begin this section by observing that, from the viewpoint of spark theory, it is trivial that every analytic subvariety of a complex manifold represents a Deligne cohomology class. Furthermore, when two cycles intersect properly, their intersection represents the product of the Deligne classes they represent. We shall then give a proof of the rational invariance of these Deligne classes in the algebraic setting, thereby giving the well known ring homomorphism

$$\psi : CH^*(X) \to H^{2*}_{\mathcal{D}}(X, \mathbb{Z}(*)).$$

Let V be a subvariety of complex manifold X with codimension p. Then integration over the regular part of V

$$[V](\alpha) \equiv \int_{\text{Reg } V} \alpha, \quad \forall \text{ smooth form } \alpha \text{ with compact support}$$

defines a degree (p,p) current $[V]$ on X. Moreover, $[V]$ is rectifiable [Har], hence $[V] \in \mathcal{IF}^{2p}(X)$. It is easy to see V represents a Deligne class.

Proposition 5.3.1. $(0,[V])$ *represents a spark class in* $\hat{\mathbf{H}}^{2p-1}(X,p)$. *Moreover, this class belongs to* $H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p)) = \ker \delta_1 \subset \hat{\mathbf{H}}^{2p-1}(X,p)$.

Proof. Since $[V]$ is of type (p,p), we have $\Psi_p([V]) = 0$ and $(0,[V])$ satisfies the spark equation $d0 = 0 - \Psi_p([V])$. □

Proposition 5.3.2. *Let* V*,* W *be two subvarieties which intersect properly. Then*

$$[(0,[V])] * [(0,[W])] = [(0,[V \cap W])].$$

Proof. Let V, W be two subvarieties in X with codimension p and q respectively. Let r and s denote currents $[V]$ and $[W]$, then $r \wedge s = [V \cap W]$. Now we calculate the product of two Deligne classes $[(0,r)]$ and $[(0,s)]$. First, fix a lift of $(0,r)$, say (a,r) with spark equation $da = e - r$ and a lift of $(0,s)$, (b,s) with $db = f - s$. Note that $\pi_p(a) = 0$, $\pi_p(e) = 0$ and $\pi_q(b) = 0$, $\pi_q(f) = 0$. By product formula, $[(a,r)][(b,s)] = [(a \wedge f + r \wedge b, r \wedge s)]$. Since $\pi_{p+q}(a \wedge f + r \wedge b) = 0$, we have

$$[(0,[V])] * [(0,[W])] = \Pi_{p+q}([(a \wedge f + r \wedge b, r \wedge s)]) = [(0, r \wedge s)] = [(0,[V \cap W])].$$

□

Proposition 5.3.3. *If X is an algebraic manifold of dimension n and V is an algebraic cycle which is rationally equivalent to zero, then V represents zero Deligne class.*

Proof. Assume V is an algebraic cycle with dimension k and codimension p. If V is rationally equivalent to zero, in particular, V represents zero homology class, then $V = dS$ for some rectifiable current with degree $2p-1$ (and real dimension $2k+1$). Hence $(0, V)$ is equivalent to $(\pi_p(S), 0)$ as sparks of level p. $(\pi_p(S), 0)$ represents zero class if and only of

$\pi_p(S) = d_pA + \Psi_pR$ for $A \in \mathcal{D}'^{2p-2}(X, p)$ and closed current $R \in \mathcal{IF}^{2p-1}(X)$,

i.e.

$$[\pi_p(S)] = 0 \in H^{2p-1}(X, \mathbb{C})/F^pH^{2p-1}(X, \mathbb{C}) \oplus H^{2p-1}(X, \mathbb{Z}) \equiv \mathcal{J}^p,$$

which means the Abel-Jacobi invariant of V is zero. It is well known that the Abel-Jacobi invariant is trivial for a cycle rationally equivalent to zero. So we are done. We give a short and direct proof of this fact now.

If V is rationally equivalent to zero, then there is a cycle $W \subset \mathbb{P}^1 \times X$ of codimension p, such that $V = \pi^{-1}(1) - \pi^{-1}(0)$ where $\pi : W \to \mathbb{P}^1$, the restriction of the projection $pr_1 : \mathbb{P}^1 \times X \to \mathbb{P}^1$, is equidimensional over $\mathbb{P}^1$. Define $V_z = \pi^{-1}(z) - \pi^{-1}(0)$, then we have a map $\mu : \mathbb{P}^1 \to \mathcal{J}^p$ which assigns z the Abel-Jacobi invariant of V_z. We shall show that μ is holomorphic, hence a constant map to zero.

Let us recall the construction of the Abel-Jacobi map briefly. If V is a cycle homologous to zero, then $V = dS$. Integrating over S, $\int_S$ defines a class in

$$H^{2p-1}(X,\mathbb{C})/F^pH^{2p-1}(X,\mathbb{C}) = F^{n-p+1}H^{2n-2p+1}(X,\mathbb{C})^*.$$

If $dS' = V$, then the difference $\int_S - \int_{S'}$ lies in the image of map

$$H_{2n-2p+1}(X,\mathbb{Z}) \to F^{n-p+1}H^{2n-2p+1}(X,\mathbb{C})^*.$$

Therefore, we get the Abel-Jacobi invariant of V defined in

$$\mathcal{J}^p \equiv H^{2p-1}(X,\mathbb{C})/F^pH^{2p-1}(X,\mathbb{C}) \oplus H^{2p-1}(X,\mathbb{Z}).$$

Now we focus on the map μ. Let γ_z be a curve on $\mathbb{P}^1$ connecting 0 and z and $S_z = \pi^{-1}(\gamma_z)$ with $dS_z = V_z$. We want to show that $\mu : z \mapsto \int_{S_z}$ is holomorphic. Note that

$$F^{n-p+1}H^{2n-2p+1}(X,\mathbb{C}) \cong \bigoplus_{\substack{r+s=2n-2p+1\\ r\geq n-p+1}} \mathcal{H}^{r,s}(X)$$

where $\mathcal{H}^{r,s}(X)$ is the group of harmonic (r,s) forms. So it suffices to show $\mu_\alpha : z \mapsto \int_{S_z}\alpha$ is holomorphic for every $\alpha \in \mathcal{H}^{r,s}(X)$, $r+s = 2n-2p+1$, $r \geq n-p+1$.

Let ν be a vector field in a small neighborhood U of z in $\mathbb{P}^1$, and $\tilde{\nu}$ be a lift of ν in $U \times X$. If ν is of type $(0,1)$, we have

$$\nu \int_{S_z} \alpha = \int_{\pi^{-1}(z)} \tilde{\nu} \lrcorner \alpha = 0$$

for any $\alpha \in \mathcal{H}^{r,s}(X)$, $r+s=2n-2p+1$, $r \geq n-p+1$. The last equality follows from the fact $\tilde{\nu} \lrcorner \alpha$ has no component of type $(n-p, n-p)$.

$\square$

By the last propositions and Chow's moving lemma, it is easy to see

Theorem 5.3.4. *The map* $V \mapsto [(0,[V])]$ *induces a ring homomorphism*

$$\psi : CH^*(X) \to H^{2*}_{\mathcal{D}}(X, \mathbb{Z}(*)).$$

Question. *There is also a cycle map for higher Chow groups* $CH^p(X,n) \to H^{2p-n}_{\mathcal{D}}(X, \mathbb{Z}(p))$ *(c.f. [KLM]). It is interesting if we can define this map from our viewpoint.*

Question. *Inspired by Massey products in Deligne cohomology, we may define Massey products in higher Chow groups. However, it is not easy to find nontrivial examples or show the triviality.*

Chapter 6

Characteristic Classes in Secondary Geometric Invariants

The main purpose of this chapter is to give a Chern-Weil-type construction of Chern classes in Deligne cohomology for holomorphic vector bundles. In [Z], Zucker indicated that it is possible to define Chern classes in analytic Deligne cohomology by splitting principle method. When the setting is algebraic, theses classes are the image of Grothendieck-Chern classes under the ring homomorphism $\psi : CH^*(X) \to H^{2*}_{\mathcal{D}}(X, \mathbb{Z}(*))$. Our approach is different and constructive. In [CS], [HL1] and [BrM], characteristic classes for a vector bundle with a connection were constructed in differential characters or spark characters. For a holomorphic vector bundle with a compatible connection, we define Chern classes in Deligne cohomology by projecting corresponding classes in differential characters, which is independent of the choice of the connection. The usual properties of Chern classes are verified. Two important applications are shown in the next chapter.

6.1 Characteristic Classes in Differential Characters

In §3, we studied secondary geometric invariants systematically from the viewpoint of spark theory and showed the ring isomorphism among Cheeger-Simons differential characters, Harvey-Lawson spark characters and smooth Deligne cohomology. In a recent paper [SS], Simons and Sullivan showed the ring of differential characters are uniquely characterized by its properties on functoriality and compatibility with the ring functors $H^*(\bullet, \mathbb{Z})$ and $\mathcal{Z}_0^*(\bullet)$. The theory of characteristic classes in differential characters were developed in [CS]. These characteristic classes are defined for **vector bundles with connections**. Similar to the characteristic classes in primary topological invariants, the characteristic classes in secondary geometric invariants are uniquely determined by their properties, i.e. functoriality and compatibility with topological characteristic classes and Chern-Weil homomorphism.

Because of the different descriptions of differential characters, there exist different ways to construct the characteristic classes in secondary invariants. In [CS], Cheeger and Simons defined characteristic classes in differential characters which refines both topological characteristic classes and Chern-Weil homomorphism. Harvey and Lawson constructed characteristic classes in the group of de Rham-Federer spark classes in [HL1]. Brylinski and McLaughlin [BrM] showed a construction of characteristic classes in smooth Deligne cohomology. All these three constructions satisfy the following properties and are essentially equivalent.

We write this theorem in terms of Chern classes. Let E be a complex vector bundle with connection ∇ over smooth manifold X. Let $c_k(E)$ and $c_k(\Omega^\nabla)$ denote the kth integral Chern class and Chern-Weil form representing $c_k(E) \otimes \mathbb{R}$ in the curvature of ∇.

Theorem 6.1.1. *[CS] There exist Chern classes $\hat{c}_k(E, \nabla) \in \hat{\mathbf{H}}^{2k-1}(X)$ satisfying*

1. $\delta_1(\hat{c}_k(E, \nabla)) = c_k(\Omega^\nabla)$,

2. $\delta_2(\hat{c}_k(E, \nabla)) = c_k(E)$,

3. *If $f : Y \to X$ is smooth, then $f^*(\hat{c}_k(E, \nabla)) = \hat{c}_k(f^*(E), f^*(\nabla))$,*

4. *If E' is another complex vector bundle with connection ∇', then*

$$\hat{c}(E \oplus E', \nabla \oplus \nabla') = \hat{c}(E, \nabla) * \hat{c}(E', \nabla'),$$

   where $\hat{c} = 1 + \hat{c}_1 + \hat{c}_2 + ...$ denotes the total Chern class.

We call $\hat{c}_k(E, \nabla)$ Cheeger-Simons Chern classes.

6.2 Chern Classes for Holomorphic Bundles in Deligne Cohomology

In this section we construct Chern classes in Deligne cohomology for holomorphic bundles E over a complex manifold X.

Let $E \to X$ be a smooth complex vector bundle over complex manifold X. ∇ is a unitary connection. We take the projections of the Cheeger-Simons

Chern classes and define

$$\hat{d}_k(E, \nabla) \equiv \Pi_k(\hat{c}_k(E, \nabla)) \in \hat{\mathbf{H}}^{2k-1}(X, k).$$

By Theorem 6.1.1 and Proposition 4.1.8, we have

$$\delta_1(\hat{d}_k(E, \nabla)) = \pi_k(c_k(\Omega^\nabla)) \quad \text{and} \quad \delta_2(\hat{d}_k(E, \nabla)) = c_k(E).$$

Now suppose that E is **holomorphic** and is provided with a hermitian metric h. Let ∇ be the associated **canonical hermitian connection**. Then $c_k(\Omega^\nabla)$ is of type k, k and we have

$$\delta_1(\hat{d}_k(E, \nabla)) = \pi_k(c_k(\Omega^\nabla)) = 0 \quad \Longrightarrow \quad \hat{d}_k(E, \nabla) \in \ker(\delta_1) = H^{2k}_{\mathcal{D}}(X, \mathbb{Z}(k)).$$

Proposition 6.2.1. *The class $\hat{d}_k(E, \nabla) \in H^{2k}_{\mathcal{D}}(X, \mathbb{Z}(k))$ defined above is independent of the choice of hermitian metric.*

Proof. Let h_0, h_1 be hermitian metrics on E with canonical connections ∇_0, ∇_1 respectively. Then

$$\hat{c}_k(E, \nabla_1) - \hat{c}_k(E, \nabla_0) = [T]$$

where $[T]$ is the differential character represented by the smooth transgression form

$$T = T(\nabla_1, \nabla_0) \equiv k \int_0^1 C_k(\nabla_1 - \nabla_0, \Omega_t, ..., \Omega_t) dt$$

where $C_k(X_1, ..., X_k)$ is the polarization of the kth elementary symmetric function and where Ω_t is the curvature of the connection $\nabla_t \equiv t\nabla_1 + (1 - t)\nabla_0$. Fix a local holomorphic frame field for E and let H_i be the hermitian matrix representing the metric h_i with respect to this trivialization. Then

$$\nabla_1 - \nabla_0 = \theta_1 - \theta_0 \text{ where } \theta_j \equiv \partial H_j \cdot H_j^{-1}.$$

In this framing, $\nabla_t = d + \theta_t$ where $\theta_t = t\theta_1 + (1-t)\theta_0$ and so its curvature $\Omega_t = d\theta_t - \theta_t \wedge \theta_t$ only has Hodge components of type $1,1$ and $2,0$. It follows that the Hodge components

$$T^{p,q} = 0 \text{ for } p < q.$$

So we have $\hat{d}_k(E, \nabla_1) - \hat{d}_k(E, \nabla_0) = \Pi_k([T]) = 0$.

□

Remark 6.2.2. *In the proof of last proposition, it is easy to see that we can choose any connection compatible to the complex structure ($\nabla^{0,1} = \bar{\partial}$) to define the Chern classes in Deligne cohomology.*

By the proposition above, each holomorphic vector bundle of rank r has a well defined total Chern class in Deligne cohomology

$$\hat{d}(E) = 1 + \hat{d}_1(E) + ... + \hat{d}_r(E) \in \bigoplus_{j=0}^{r} H_{\mathcal{D}}^{2j}(X, \mathbb{Z}(j)).$$

Denote by $\mathcal{V}^k(X)$ the set of isomorphism classes of holomorphic vector bundles of rank k on X, and by $\mathcal{V}(X) = \coprod_{k\geq 0} \mathcal{V}^k(X)$ the additive monoid under Whitney sum.

Theorem 6.2.3. *On any complex manifold there is a natural transformation* *of functors*

$$\hat{d} : \mathcal{V}(X) \to \bigoplus_j H^{2j}_{\mathcal{D}}(X, \mathbb{Z}(j))$$

with the property that:

1. $\hat{d}(E \oplus F) = \hat{d}(E) * \hat{d}(F)$,
2. $\hat{d} : \mathcal{V}^1(X) \to 1 + H^2_{\mathcal{D}}(X, 1)$ *is an isomorphism,*
3. *under the natural map* $\kappa : H^{2j}_{\mathcal{D}}(X, \mathbb{Z}(j)) \to H^{2j}(X, \mathbb{Z})$, $\kappa \circ \hat{d} = c$ *(the total integral Chern class).*

Proof. (1)Suppose E and F are holomorphic bundles with hermitian connections ∇ and ∇', then we have

$$\hat{d}(E) = 1 + \hat{d}_1(E) + \hat{d}_2(E) + \cdots = 1 + \Pi_1(\hat{c}_1(E, \nabla)) + \Pi_2(\hat{c}_2(E, \nabla)) + \cdots$$

and similarly $\hat{d}(F) = 1 + \Pi_1(\hat{c}_1(F, \nabla')) + \Pi_2(\hat{c}_2(F, \nabla')) + \cdots$.

Since

$$\hat{c}(E \oplus F, \nabla \oplus \nabla') = \hat{c}(E, \nabla) * \hat{c}(F, \nabla'),$$

we have

$$\begin{aligned}
\hat{d}_k(E \oplus F) &= \Pi_k(\hat{c}_k(E \oplus F, \nabla \oplus \nabla')) \\
&= \Pi_k(\sum_{i=0}^{k} \hat{c}_i(E, \nabla) \cdot \hat{c}_{k-i}(F, \nabla')) \\
&= \sum_{i=0}^{k} \Pi_i(\hat{c}_i(E, \nabla)) \cdot \Pi_{k-i}(\hat{c}_{k-i}(F, \nabla')) \\
&= \sum_{i=0}^{k} \hat{d}_i(E) \cdot \hat{d}_{k-i}(F).
\end{aligned}$$

It is easy to see the second to last equality from our definition of product of Deligne cohomology classes. Recall when we defined the product of two Deligne classes, we first lifted them to two sparks, then did multiplication and projected the product back.

(2) is true because $H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong H^1(X, \mathcal{O}^*)$.

(3) follows $\delta_2(\hat{d}_k(E, \nabla)) = c_k(E)$. □

Following Grothendieck we define the holomorphic K-theory of X to be the quotient

$$K_{\text{hol}}(X) \equiv \mathcal{V}(X)^+ / \sim$$

where $\sim$ is the equivalence relation generated by setting $[E] \sim [E' \oplus E'']$ when there exists a short exact sequence of holomorphic bundles $0 \to E' \to E \to E'' \to 0$. The next theorem tells us the natural transformation $\hat{d}$ defined above descends to a natural transformation

$$\hat{d} : K_{\text{hol}}(X) \to \bigoplus_j H^{2j}_{\mathcal{D}}(X, \mathbb{Z}(j)).$$

Theorem 6.2.4. *For any short exact sequence of holomorphic vector bundles on X*

$$0 \to E' \to E \to E'' \to 0$$

one has $\hat{d}(E) = \hat{d}(E') * \hat{d}(E'')$.

Proof. We have $E' \oplus E'' \cong E$ as smooth bundles, so we consider them as the same bundle with different holomorphic structures. The purpose is to show these two holomorphic bundles have the same total Chern class valued in Deligne cohomology. The idea of the proof is the following. We fix a hermitian metric on this smooth bundle, choose local holomorphic bases for those two holomorphic structures respectively, and calculate the hermitian connections with respect to them. Then we calculate the smooth transgression form which represents the difference of Cheeger-Simons Chern classes of these two holomorphic bundles, and show that under the projection Π_k, this transgression form represents a zero spark class in $\hat{\mathbf{H}}^{2k-1}(X,k)$. Hence $\hat{d}(E) = \hat{d}(E' \oplus E'') = \hat{d}(E') * \hat{d}(E'')$.

Choose a C^∞-splitting

$$0 \longrightarrow E' \xrightarrow{i} E \underset{\sigma}{\overset{\pi}{\rightleftarrows}} E'' \longrightarrow 0.$$

Fix hermitian metrics h_1 and h_2 for E' and E'' respectively, and define a hermitian metric $h = h_1 \oplus h_2$ on E via the smooth isomorphism (i, σ) : $E' \oplus E'' \to E$.

Over a small open set $U \subset X$, we choose a local holomorphic basis $\{e_1, e_2, ..., e_m\}$ for E' and a local holomorphic basis $\{e_{m+1}, e_{m+2}, ..., e_{m+n}\}$ for E''. Then choose a local holomorphic basis $\{\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_m, \tilde{e}_{m+1}, \tilde{e}_{m+2}, ..., \tilde{e}_{m+n}\}$ for E such that $\tilde{e}_i = e_i$ for $1 \le i \le m$ and $\tilde{e}_{m+j}$ is a holomorphic lift of e_{m+j} for $1 \le j \le n$. Assume $g = (g_{ij})$ is the transition matrix for these two bases, i.e. $\tilde{e}_i = \sum_{j=1}^{m+n} g_{ij} e_j$. Then it is easy to know g has the form

$$g = \begin{pmatrix} I_m & 0 \\ A & I_n \end{pmatrix} \quad \text{and} \quad g^{-1} = \begin{pmatrix} I_m & 0 \\ -A & I_n \end{pmatrix}$$

where I_m and I_n is the identity matrices of rank m and n, and A is the nontrivial part of g.

Let H_1 and H_2 be the hermitian matrices representing the metrics h_1 and

h_2 with respect to the bases $\{e_1, e_2, ..., e_m\}$ and $\{e_{m+1}, e_{m+2}, ..., e_{m+n}\}$. Let H and $\tilde{H}$ be the hermitian matrices representing the metric h with respect to bases $\{e_i\}_{i=1}^{m+n}$ and $\{\tilde{e}_i\}_{i=1}^{m+n}$. Then we have

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \quad \text{and} \quad \tilde{H} = gHg^*$$

where $g^* = \bar{g}^t$ is the transpose conjugate of g.

Fix the hermitian metric h, we calculate the canonical hermitian connections with respect to two holomorphic structures. For $E' \oplus E''$, the hermitian connection ∇_0 can be written locally as the matrix (w.r.t. the basis $\{e_i\}$)

$$\theta_0 = \partial H \cdot H^{-1}.$$

For E, the hermitian connection ∇_1 can be written locally as the matrix (w.r.t. the basis $\{\tilde{e}_i\}$)

$$\tilde{\theta}_1 = \partial\tilde{H}\cdot\tilde{H}^{-1} = \partial(gHg^*)(gHg^*)^{-1} = \partial g\cdot g^{-1} + g\partial H\cdot H^{-1}g^{-1} + gH\partial g^*(g^*)^{-1}H^{-1}g^{-1}.$$

We change the basis and write ∇_1 as the matrix with respect to the basis $\{e_i\}$

$$\begin{aligned} \theta_1 &= d(g^{-1}) \cdot g + g^{-1}\tilde{\theta}_1 g \\ &= -g^{-1}dg + g^{-1}(\partial g \cdot g^{-1} + g\partial H \cdot H^{-1}g^{-1} + gH\partial g^*(g^*)^{-1}H^{-1}g^{-1})g \\ &= -g^{-1}dg + g^{-1}\partial g + \partial H \cdot H^{-1} + H\partial g^*(g^*)^{-1}H^{-1} \\ &= -g^{-1}\bar{\partial} g + \theta_0 + H\partial g^*(g^*)^{-1}H^{-1} \\ &= \theta_0 + H\partial g^*(g^*)^{-1}H^{-1} - g^{-1}\bar{\partial} g \end{aligned}$$

Let $\eta \equiv \theta_1 - \theta_0 = H\partial g^*(g^*)^{-1}H^{-1} - g^{-1}\bar{\partial}g$ and $\eta^{1,0} = H\partial g^*(g^*)^{-1}H^{-1}$, $\eta^{0,1} = -g^{-1}\bar{\partial}g$ be the $(1,0)$ and $(0,1)$ components of η respectively. Then we have

$$\eta^{1,0} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \begin{pmatrix} 0 & \partial A^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_m & -A^* \\ 0 & I_n \end{pmatrix} \begin{pmatrix} H_1^{-1} & 0 \\ 0 & H_2^{-1} \end{pmatrix} = \begin{pmatrix} 0 & H_1\partial A^* H_2^{-1} \\ 0 & 0 \end{pmatrix}$$

and

$$\eta^{0,1} = -\begin{pmatrix} I_m & 0 \\ -A & I_n \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \bar{\partial}A & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ \bar{\partial}A & 0 \end{pmatrix}.$$

Define a family of connections ∇_t with connection matrices $\theta_t = \theta_0 + t\eta$ for $0 \le t \le 1$. Let $\Omega_t = d\theta_t - \theta_t \wedge \theta_t$ be the curvature of the connection θ_t. It is easy to see

$$\Omega_t^{0,2} = t\bar{\partial}\eta^{0,1} - t^2(\eta^{0,1} \wedge \eta^{0,1}) = 0$$

and

$$\Omega_t^{1,1} = \bar{\partial}\theta_0 + t(\bar{\partial}\eta^{1,0} + \partial\eta^{0,1} - \theta_0 \wedge \eta^{0,1} - \eta^{0,1} \wedge \theta_0) - t^2(\eta^{1,0} \wedge \eta^{0,1} + \eta^{0,1} \wedge \eta^{1,0}).$$

Note that

$$\eta^{0,1} \wedge \eta^{0,1} = \begin{pmatrix} 0 & 0 \\ -\bar{\partial}A & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 \\ -\bar{\partial}A & 0 \end{pmatrix} = 0.$$

We will use this trick again in the later calculation.

Suppose that Φ is an symmetric invariant k-multilinear function on the Lie algebra $\mathfrak{gl}_{m+n}(\mathbb{C})$. Then the two connections ∇_0 and ∇_1 on E give rise to two

Cheeger-Simons differential characters $\hat{\Phi}_0$ and $\hat{\Phi}_1$, and the difference

$$\hat{\Phi}_0 - \hat{\Phi}_1 = [T_\Phi]$$

where $[T_\Phi]$ is the character associated to the smooth form

$$T_\Phi = k\int_0^1 \Phi(\eta, \Omega_t, \Omega_t, ..., \Omega_t)dt.$$

Our goal is to show that $\Pi_k([T_\Phi]) = [\pi_k(T_\Phi)]$ represents a zero spark class. So it suffices to show $\pi_k(T_\Phi)$ is a d_k-exact form. In fact, we shall show $\pi_k(T_\Phi)$ is a form of pure type $(k-1,k)$ and equals $\bar{\partial}S = d_kS$ for some $(k-1,k-1)$ form S.

Lemma 6.2.5. *$T_\Phi^{i,2k-1-i} = 0$ for $i < k-1$, i.e. $\pi_k(T_\Phi) = T_\Phi^{k-1,k}$, where $T_\Phi^{i,2k-1-i}$ is the $(i, 2k-1-i)$ Hodge component of T_Φ.*

Proof. Note that we have $\Omega_t^{0,2} = 0$, i.e. Ω_t is of type $(1,1)$ and $(2,0)$. Hence it is easy to see $T_\Phi^{i,2k-1-i} = 0$ for $i < k-1$ from the expression $T_\Phi = k\int_0^1 \Phi(\eta, \Omega_t, \Omega_t, ..., \Omega_t)dt$. □

In order to show T_Φ is $\bar{\partial}$-exact for general Φ, we first show T_{Ψ_k} is $\bar{\partial}$-exact for $\Psi_k(A_1, A_2, ..., A_k) = tr(A_1 \cdot A_2 \cdot ... \cdot A_k)$.

Lemma 6.2.6. *Let $\Psi_k(A_1, A_2, ..., A_k) = tr(A_1 \cdot A_2 \cdot ... \cdot A_k)$ and $T = T_{\Psi_k} = k\int_0^1 tr(\eta \wedge (\Omega_t)^{k-1})dt$. Then $T^{k-1,k}$ is $\bar{\partial}$-exact. Explicitly, $T^{0,1} = 0$ when $k = 1$, and for $k \geq 2$,*

$$T^{k-1,k} = k\int_0^1 tr(\eta^{0,1} \wedge (\Omega_t^{1,1})^{k-1})dt = k\bar{\partial}\int_0^1 -tr(\eta^{0,1} \wedge \eta^{1,0} \wedge (\Omega_t^{1,1})^{k-2}) \cdot tdt.$$

Proof. When $k = 1$, $T^{0,1} = \int_0^1 tr(\eta^{0,1})dt = 0$ since $\eta^{0,1}$ has the form

$$\begin{pmatrix} 0 & 0 \\ -\bar{\partial}A & 0 \end{pmatrix}.$$

When $k \geq 2$, it is easy to see

$$T^{k-1,k} = k\int_0^1 tr(\eta^{0,1} \wedge (\Omega_t^{1,1})^{k-1})dt$$

by comparing Hodge components on both sides. So it suffices to show the identity

$$tr(\eta^{0,1} \wedge (\Omega_t^{1,1})^{k-1}) = -\bar{\partial}tr(\eta^{0,1} \wedge \eta^{1,0} \wedge (\Omega_t^{1,1})^{k-2}) \cdot t.$$

First, we introduce some basic identities. We know that in our theory, Chern classes in Deligne cohomology are independent of the choice of hermitian metric, and the question above is local. So we fix local bases and choose hermitian metrics h_1 and h_2 such that $H_1 = I_m$ and $H_2 = I_n$ locally. Now we have

$$\eta^{1,0} = \begin{pmatrix} 0 & H_1\partial A^* H_2^{-1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \partial A^* \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \eta^{0,1} = \begin{pmatrix} 0 & 0 \\ -\bar{\partial}A & 0 \end{pmatrix}$$

$$\Omega_t^{1,1} = t(\bar{\partial}\eta^{1,0} + \partial\eta^{0,1}) - t^2(\eta^{1,0} \wedge \eta^{0,1} + \eta^{0,1} \wedge \eta^{1,0}) = td\eta - t^2\eta \wedge \eta.$$

Note in the equation above, we use the fact $\eta^{1,0}$ is ∂-exact, $\eta^{0,1}$ is $\bar{\partial}$-exact, and $\eta^{1,0} \wedge \eta^{1,0} = 0$, $\eta^{0,1} \wedge \eta^{0,1} = 0$ by matrix multiplication.

By calculation, we have

$$[\Omega_t^{1,1}, \eta] \equiv \Omega_t^{1,1} \wedge \eta - \eta \wedge \Omega_t^{1,1} = td(\eta \wedge \eta).$$

Take $(2,1)$ and $(1,2)$ components respectively, we have

$$[\Omega_t^{1,1}, \eta^{1,0}] = t\partial(\eta \wedge \eta) \quad \text{and} \quad [\Omega_t^{1,1}, \eta^{0,1}] = t\bar{\partial}(\eta \wedge \eta).$$

The next observation is

$$\bar{\partial}\Omega_t^{1,1} = -t^2\bar{\partial}(\eta \wedge \eta) = -t[\Omega_t^{1,1}, \eta^{0,1}].$$

Using identities above, it is easy to conclude

$$\bar{\partial}((\Omega_t^{1,1})^n) = -t[(\Omega_t^{1,1})^n, \eta^{0,1}].$$

Now we are ready to calculate.

$$\begin{aligned}
& -\bar{\partial} tr(\eta^{0,1} \wedge \eta^{1,0} \wedge (\Omega_t^{1,1})^n) \cdot t \\
= \;& -t \cdot tr(\bar{\partial}(\eta^{0,1} \wedge \eta^{1,0} \wedge (\Omega_t^{1,1})^n)) \\
= \;& -t \cdot tr(-\eta^{0,1} \wedge \bar{\partial}\eta^{1,0} \wedge (\Omega_t^{1,1})^n + \eta^{0,1} \wedge \eta^{1,0} \wedge \bar{\partial}(\Omega_t^{1,1})^n) \\
= \;& -t \cdot tr(-\eta^{0,1} \wedge \bar{\partial}\eta^{1,0} \wedge (\Omega_t^{1,1})^n + \eta^{0,1} \wedge \eta^{1,0} \wedge (-t)[(\Omega_t^{1,1})^n, \eta^{0,1}]) \\
= \;& t \cdot tr(\eta^{0,1} \wedge \bar{\partial}\eta^{1,0} \wedge (\Omega_t^{1,1})^n + t\eta^{0,1} \wedge \eta^{1,0} \wedge (\Omega_t^{1,1})^n \wedge \eta^{0,1} \\
& \qquad\qquad -t\eta^{0,1} \wedge \eta^{1,0} \wedge \eta^{0,1} \wedge (\Omega_t^{1,1})^n)
\end{aligned}$$

$$
\begin{aligned}
&= t \cdot tr(\eta^{0,1} \wedge \bar{\partial}\eta^{1,0} \wedge (\Omega_t^{1,1})^n - t\eta^{0,1} \wedge \eta^{1,0} \wedge \eta^{0,1} \wedge (\Omega_t^{1,1})^n) \\
&\qquad\qquad + t^2 \cdot tr(\eta^{0,1} \wedge \eta^{1,0} \wedge (\Omega_t^{1,1})^n \wedge \eta^{0,1}) \\
&= tr(\eta^{0,1} \wedge (t\bar{\partial}\eta^{1,0} - t^2\eta^{1,0} \wedge \eta^{0,1}) \wedge (\Omega_t^{1,1})^n) + t^2 \cdot tr(\eta^{0,1} \wedge \eta^{0,1} \wedge \eta^{1,0} \wedge (\Omega_t^{1,1})^n) \\
&\overset{*}{=} tr(\eta^{0,1} \wedge \Omega_t^{1,1} \wedge (\Omega_t^{1,1})^n) + 0 \\
&= tr(\eta^{0,1} \wedge (\Omega_t^{1,1})^{n+1})
\end{aligned}
$$

Put $n = k-2$, we are done.

Note that in the second to last equality, we use the trick $\eta^{0,1} \wedge \eta^{0,1} = 0$ several times.

□

Recall in the Chern-Weil theory, the kth Chern character of a vector bundle is represented by the form $\frac{1}{k!}\Psi_k(\Omega, \Omega, ..., \Omega) = \frac{1}{k!}tr(\Omega^k)$ where Ω is the curvature of any connection. Any symmetric invariant k-multilinear function Φ on the Lie algebra $\mathfrak{gl}_{m+n}(\mathbb{C})$ is generated by Ψ_1, Ψ_2, ... Ψ_k, i.e. we have

$$
\Phi = \sum_{n=1}^{k} \sum_{\substack{i_1+\cdots+i_n=k \\ i_1>0,\cdots,i_n>0}} a_{i_1 i_2 \cdots i_n} \Psi_{i_1} \otimes \Psi_{i_2} \otimes ... \otimes \Psi_{i_n}.
$$

Hence, $T_\Phi = k\int_0^1 \Phi(\eta, \Omega_t, \Omega_t, ..., \Omega_t)dt$ where $\Phi(\eta, \Omega_t, \Omega_t, ..., \Omega_t)$ is a sum with summands like $\Psi_{i_1}(\eta, \Omega_t, \Omega_t, ..., \Omega_t)\Psi_{i_2}(\Omega_t, \Omega_t, ..., \Omega_t)...\Psi_{i_n}(\Omega_t, \Omega_t, ..., \Omega_t)$. For $j > 1$, $\Psi_{i_j}(\Omega_t, \Omega_t, ..., \Omega_t)$ is a closed (i_j, i_j) form representing $i_j!$ times the i_jth Chern character. And from last lemma, we know $\Psi_{i_1}(\eta, \Omega_t, \Omega_t, ..., \Omega_t)$ has types (i_1-1, i_1) and higher, and its (i_1-1, i_1) component is $\bar{\partial}$-exact. Therefore, T_Φ is of types $(k-1, k)$ and higher, and $\pi_k(T) = T^{k-1,k} = \bar{\partial}S^{k-1,k-1}$ for some $(k-1, k-1)$ form $S^{k-1,k-1}$. □

Remark 6.2.7. *By the theorems on uniqueness of Chern classes in Deligne cohomology in [B], [EV] and [Br2], our theory on Chern classes is equivalent to all others. In particular, when the setting is algebraic, Chern classes defined above are compatible with Grothendieck-Chern classes under the cycle map* $\psi : CH^*(X) \to H^{2*}_{\mathcal{D}}(X, \mathbb{Z}(*))$.

Cheeger and Simons also defined Chern characters for vector bundles with connections, which are located in rational differential characters $\hat{H}^*(X, \mathbb{R}/\mathbb{Q})$. For holomorphic vector bundles, we can project Chern characters in differential characters to get Chern characters in rational Deligne cohomology $H^{2*}_{\mathcal{D}}(X, \mathbb{Q}(*))$. Define

$$\widehat{dch}_k(E) \equiv \Pi_k(\hat{ch}_k(E, \nabla)) \in H^{2k}_{\mathcal{D}}(X, \mathbb{Q}(k)),$$

where ∇ is the hermitian connection associated to a hermitian metric.

Since $\hat{ch}(E \oplus E', \nabla \oplus \nabla') = \hat{ch}(E, \nabla) + \hat{ch}(E', \nabla')$, we have

Theorem 6.2.8. *If E and F be two holomorphic vector bundles on complex manifold X, then*

$$\widehat{dch}(E \oplus F) = \widehat{dch}(E) + \widehat{dch}(F).$$

Moreover,

Theorem 6.2.9. *For any short exact sequence of holomorphic vector bundles on X*

$$0 \to E' \to E \to E'' \to 0$$

one has $\widehat{dch}(E) = \widehat{dch}(E') + \widehat{dch}(E'')$.

Recently, Grivaux [G] defined Chern classes in rational Deligne cohomology for analytic coherent sheaves over complex manifolds. By a result of Voisin

[V], not every analytic coherent sheaf has a locally free resolution. However, applying Green's result on simplicial resolution for coherent sheaves (see [TT]), we may define Chern classes in Deligne cohomology for coherent sheaves, if we can define Chern classes for simplical holomorphic vector bundles by our method.

Question. *Is it possible to define Chern classes in Deligne cohomology for analytic coherent sheaves through simplicial resolution?*

Chapter 7

Holomorphic Foliations and Nadel Invariants

We show two important applications of our theory on Chern classes. The first one is on the Bott vanishing theorem for holomorphic foliations. Bott showed an obstruction of a vector bundle to be (isomorphic to) the normal bundle of some smooth foliation. We give an analogue of the Bott vanishing theorem on holomorphic foliations in §7.1. The second application is on Nadel's invariants. Nadel introduced interesting relative invariants for a pair of holomorphic vector bundles. In §7.2, we construct Nadel-type invariants in the intermediate Jacobians and give a direct proof of Nadel's conjecture.

7.1 Bott Vanishing for Holomorphic Foliations

In [Bo], Bott constructed a family of connections on the normal bundle of any smooth foliation of a manifold and established the Bott vanishing theorem. Roughly speaking, it says the characteristic classes of the normal bundle are trivial in all sufficiently high degrees. In particular, for the holomorphic case, we have the following theorem.

Theorem 7.1.1. *[BB]**Bott vanishing for holomorphic foliations.** Let X be a complex manifold of dimension n and F an integrable holomorphic subbundle of TX with rank k. Let $\varphi \in \mathbb{C}[X_1, X_2, ..., X_n]$ be symmetric and homogeneous polynomial of degree l, where $n - k < l \leq n$. Let ∇ be a Bott connection for $N \cong T/F$. Then $\varphi(\Omega^{\nabla}) = 0$.*

Note that the tangent bundle of a foliation is integrable and this theorem tells us that the characteristic class of the normal bundle N corresponding to φ vanishes if $\deg \varphi$ is high. Furthermore, if a Bott connection is chosen, the Chern-Weil form vanishes!

Now we show a version of the Bott vanishing theorem for Chern classes in Deligne cohomology.

Suppose that N is the normal bundle to a holomorphic foliation of codimension p on a complex manifold X. Then there are two natural families of connections to consider on N, the family of Bott connections and the family of canonical hermitian connections.

Proposition 7.1.2. *N is a holomorphic vector bundle on X. Let $P(c_1, ..., c_q)$ be a polynomial in Chern classes which is of pure degree $2k$ with $k > 2q$. Then the projection image of Cheeger-Simons Chern class $\Pi_k(P(\hat{c}_1, ..., \hat{c}_q)) \in \hat{H}^{2k-1}(X, k)$ for Bott connections agrees with the Chern class in Deligne cohomology $P(\hat{d}_1, ..., \hat{d}_q)$ for the canonical hermitian connections.*

Proof. In fact, this is a direct corollary of Remark 6.2.2 since Bott connections are compatible with the holomorphic structure of N [BB, Remark 3.26].

Let ∇ be a Bott connection, $\tilde{\nabla}$ be the canonical hermitian connection for some hermitian metric and θ, $\tilde{\theta}$ be their connection forms. Notice that both

θ, $\tilde{\theta}$ are of type $(1, 0)$.

Let $\Phi(X_1, ..., X_k)$ be the symmetric invariant k-multilinear function on the Lie algebra $\mathfrak{gl}_q(\mathbb{C})$ such that $P(\sigma_1(X), ..., \sigma_q(X)) = \Phi(X, ..., X)$ where σ_j is the j^{th} elementary symmetric function of the eigenvalues of X. Then the difference between the Cheeger-Simons Chern class associated to P for the two connections ∇ and $\tilde{\nabla}$ is the character associated to the smooth form

$$T = k \int_0^1 \Phi(\theta - \tilde{\theta}, \Omega_t, ..., \Omega_t) dt$$

where $\Omega_t = d\theta_t - \theta_t \wedge \theta_t$. Since θ_t is of type $(1, 0)$ and Ω_t is of type $(1, 1)$ and $(2, 0)$, we have $T^{p,q} = 0$ for all $p < q$.

Therefore,

$$\begin{aligned}
& P(\hat{d}_1, ..., \hat{d}_q) - \Pi_k(P(\hat{c}_1(N, \nabla), ..., \hat{c}_q(N, \nabla))) \\
= \; & \Pi_k(P(\hat{c}_1(N, \tilde{\nabla}), ..., \hat{c}_q(N, \tilde{\nabla}))) - \Pi_k(P(\hat{c}_1(N, \nabla), ..., \hat{c}_q(N, \nabla))) \\
= \; & \Pi_k(T) = 0
\end{aligned}$$

□

Theorem 7.1.3. *Let N be a holomorphic bundle of rank q on a complex manifold X. If N is (isomorphic to) the normal bundle of a holomorphic foliation of X, then for every polynomial P of pure degree $k > 2q$, the associated refined Chern class satisfies*

$$P(\hat{d}_1(N), ..., \hat{d}_q(N)) \in Im[H^{2k-1}(X, \mathbb{C}^\times) \to H^{2k}_{\mathcal{D}}(X, \mathbb{Z}(k))].$$

Proof. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{2k-1}(X,\mathbb{C}^\times) & \longrightarrow & \hat{H}^{2k-1}(X) & \longrightarrow & \mathcal{Z}^{2k}_{\mathbb{Z}}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^{2k}_{\mathcal{D}}(X,\mathbb{Z}(k)) & \longrightarrow & \hat{H}^{2k-1}(X,k) & \longrightarrow & \mathcal{Z}^{2k}_{\mathbb{Z}}(X,k) & \longrightarrow & 0 \end{array}$$

We know $P(\hat{d}_1,...,\hat{d}_q) \in H^{2k}_{\mathcal{D}}(X,\mathbb{Z}(k))$ and $P(\hat{c}_1,...,\hat{c}_q) \in \hat{H}^{2k-1}(X)$. By last proposition, we know they have the same images in $\hat{H}^{2k-1}(X,k)$. And by the Bott vanishing theorem, we have $P(\hat{c}_1,...,\hat{c}_q) \in H^{2k-1}(X,\mathbb{C}^\times)$. Then we get the conclusion. □

7.2 Nadel Invariants for Holomorphic Vector Bundles

In his beautiful paper [N], Nadel introduced interesting relative invariants for holomorphic vector bundles. Explicitly, for two holomorphic vector bundles E and F over a complex manifold X and a C^∞ isomorphism $f : E \to F$, Nadel defined invariants

$$\mathscr{E}^k(E,F,f) \in H^{2k-1}(X,\mathcal{O}) \quad \text{and} \quad \mathscr{E}^k(E,F) \in H^{2k-1}(X,\mathcal{O})/H^{2k-1}(X,\mathbb{Z}).$$

He also conjectured that these invariants should coincide with a component of the Abel-Jacobi image of $k!(ch_k(E) - ch_k(F)) \in CH^k(X)$ when the setting is algebraic. In [Be], Berthomieu developed a relative K-theory and gave an affirmative answer to Nadel's conjecture. In his proof of Nadel's conjecture, the hard part is to show the compatibility of two theories on Characteristic classes, i.e. Cheeger-Simons Chern classes and Deligne-Beilinson Chern classes.

In this section, we generalize Nadel theory and give a short proof of his conjecture. From our point of view, if E and F are two holomorphic vector bundles whose underlying C^∞ vector bundles are isomorphic, then their usual Chern classes coincide, and the difference of their kth Chern classes in Deligne cohomology $\hat{d}_k(E) - \hat{d}_k(F)$ is located in the intermediate Jacobians $\mathcal{J}^k(X)$. Hence we can define relative invariants for a pair (E, F) directly in $\mathcal{J}^k(X)$. In particular, we shall express the difference of kth Chern character $\widehat{dch}_k(E) - \widehat{dch}_k(F)$ by transgression forms whose $(0, 2k-1)$ components are exactly the Nadel invariants. This will prove Nadel's conjecture in more general context (not necessarily algebraic). Note that the Nadel-type invariants are naturally defined from our viewpoint, and the Nadel's conjecture is clear after a computation of $(0, 2k-1)$ component of the transgression form.

Let E and F be two holomorphic vector bundles over complex manifold X, and $g : E \to F$ be a C^∞ bundle isomorphism. Fix a hermitian metric h for E and F. Over a small open set $U \subset X$, choose local holomorphic bases $\{e_i\}_{i=1}^r$ and $\{f_i\}_{i=1}^r$ for E and F respectively, and denote also by g the transition matrix of the C^∞ bundle isomorphism with respect to bases $\{e_i\}_{i=1}^r$ and $\{f_i\}_{i=1}^r$. Let H and $\tilde{H}$ be the hermitian matrices representing the metric h with respect to bases $\{e_i\}_{i=1}^r$ and $\{f_i\}_{i=1}^r$. Then we have $\tilde{H} = gHg^*$.

Now we calculate the canonical hermitian connections with respect to two holomorphic structures. For E, the hermitian connection ∇_0 can be written locally as the matrix (w.r.t. the basis $\{e_i\}$)

$$\theta_0 = \partial H \cdot H^{-1}.$$

And for F, the hermitian connection ∇_1 can be written locally as the matrix (w.r.t. the basis $\{f_i\}$)

$$\tilde{\theta}_1 = \partial\tilde{H}\cdot\tilde{H}^{-1} = \partial(gHg^*)(gHg^*)^{-1} = \partial g\cdot g^{-1} + g\partial H\cdot H^{-1}g^{-1} + gH\partial g^*(g^*)^{-1}H^{-1}g^{-1}.$$

We change the basis and write ∇_1 as the matrix with respect to the basis $\{e_i\}$

$$\begin{aligned}
\theta_1 &= d(g^{-1})\cdot g + g^{-1}\tilde{\theta}_1 g \\
&= -g^{-1}dg + g^{-1}(\partial g\cdot g^{-1} + g\partial H\cdot H^{-1}g^{-1} + gH\partial g^*(g^*)^{-1}H^{-1}g^{-1})g \\
&= -g^{-1}dg + g^{-1}\partial g + \partial H\cdot H^{-1} + H\partial g^*(g^*)^{-1}H^{-1} \\
&= \theta_0 + H\partial g^*(g^*)^{-1}H^{-1} - g^{-1}\bar{\partial}g
\end{aligned}$$

Let $\eta = \theta_1 - \theta_0 = H\partial g^*(g^*)^{-1}H^{-1} - g^{-1}\bar{\partial}g$ and $\eta^{1,0} = H\partial g^*(g^*)^{-1}H^{-1}$, $\eta^{0,1} = -g^{-1}\bar{\partial}g$ be the $(1,0)$ and $(0,1)$ components of η respectively. Define a family of connections ∇_t with connection matrices $\theta_t = \theta_0 + t\eta$ for $0 \le t \le 1$. Let $\Omega_t = d\theta_t - \theta_t\wedge\theta_t$ be the curvature of the connection θ_t.

$$\begin{aligned}
\Omega_t^{0,2} &= t\bar{\partial}\eta^{0,1} - t^2(\eta^{0,1}\wedge\eta^{0,1}) \\
&= t\bar{\partial}(-g^{-1}\bar{\partial}g) - t^2(-g^{-1}\bar{\partial}g)\wedge(-g^{-1}\bar{\partial}g) \\
&= (t-t^2)g^{-1}\bar{\partial}g\wedge g^{-1}\bar{\partial}g.
\end{aligned}$$

Suppose that Φ is an symmetric invariant k-multilinear function on the Lie algebra $\mathfrak{gl}_{m+n}(\mathbb{C})$. Then the two connections ∇_0 and ∇_1 on E give rise to two

Cheeger-Simons differential characters $\hat{\Phi}_0$ and $\hat{\Phi}_1$, and the difference

$$\hat{\Phi}_0 - \hat{\Phi}_1 = [T_\Phi]$$

where $[T_\Phi]$ is the character associated to the smooth form

$$T_\Phi = k\int_0^1 \Phi(\eta, \Omega_t, \Omega_t, ..., \Omega_t)dt.$$

The difference of Chern classes $\Pi_k([T_\Phi]) = \Pi_k(\hat{\Phi}_0 - \hat{\Phi}_1) \in \mathscr{I}^k$ is a spark class which is represented by the smooth form $\pi_k(T_\Phi)$. In particular, when Φ is the form $\Phi(A_1, A_2, ..., A_k) = tr(A_1 \cdot A_2 \cdot ... \cdot A_k)$ (representing $k!$ times the kth Chern character), we have

$$T_\Phi = k\int_0^1 tr(\eta \wedge (\Omega_t)^{k-1})dt \quad \text{and} \quad \pi_k(T_\Phi) = \pi_k(k\int_0^1 tr(\eta \wedge (\Omega_t)^{k-1})dt).$$

The $(0, 2k-1)$ component

$$\begin{aligned}
\pi_k(T_\Phi)^{0,2k-1} &= T_\Phi^{0,2k-1} \\
&= k\int_0^1 tr(\eta^{0,1} \wedge (\Omega_t^{0,2})^{k-1})dt \\
&= k\int_0^1 tr(-g^{-1}\bar{\partial}g \wedge ((t-t^2)g^{-1}\bar{\partial}g \wedge g^{-1}\bar{\partial}g)^{k-1})dt \\
&= k\int_0^1 -t^{k-1}(1-t)^{k-1}tr((g^{-1}\bar{\partial}g)^{2k-1})dt \\
&= -k\int_0^1 t^{k-1}(1-t)^{k-1}dt \cdot tr((g^{-1}\bar{\partial}g)^{2k-1}) \\
&= \pm\mathscr{E}^k(E, F, g)
\end{aligned}$$

Definition 7.2.1. *X is a complex manifold, E and F are two holomorphic* *vector bundles over X whose underlying C^∞ vector bundle are isomorphic. Define Nadel-type invariants*

$$\hat{\mathscr{E}}^k(E,F) \equiv k!(\widehat{dch}_k(E) - \widehat{dch}_k(F)) = [\pi_k(k\int_0^1 tr(\eta \wedge (\Omega_t)^{k-1})dt)] \in \mathcal{J}^k.$$

Note that we have a natural projection

$$\pi : \mathcal{J}^k \equiv H^{2k-1}(X,\mathbb{C})/F^kH^{2k-1}(X,\mathbb{C}) \oplus H^{2k-1}(X,\mathbb{Z}) \to H^{2k-1}(X,\mathcal{O})/H^{2k-1}(X,\mathbb{Z}).$$

By the calculations above, we have

Theorem 7.2.2. *The kth Nadel invariant $\mathscr{E}^k(E,F)$ is the image of $\hat{\mathscr{E}}^k(E,F)$ under the map $\pi : \mathcal{J}^k \to H^{2k-1}(X,\mathcal{O})/H^{2k-1}(X,\mathbb{Z})$. That is, Nadel's conjecture is true.*

Chapter 8

Products on Hypercohomology

The purpose of the chapter is to demonstrate the abstract nonsense behind the following well known fact. It will help us to understand the proofs of Theorems 3.4.7 and 5.1.11.

Theorem. *If there are three complexes of sheaves of abelian groups* $\mathcal{F}^*, \mathcal{G}^*, \mathcal{H}^*$ *over a manifold* X *and a cup product*

$$\cup : \mathcal{F}^* \otimes \mathcal{G}^* \longrightarrow \mathcal{H}^*$$

which commutes with differentials, then $\cup$ *induces an product on their hypercohomology*

$$\cup : \mathbb{H}^*(X, \mathcal{F}^*) \otimes \mathbb{H}^*(X, \mathcal{G}^*) \longrightarrow \mathbb{H}^*(X, \mathcal{H}^*).$$

Although the above fact is well known, it is hard to find reference on how to realize the product on the cocycle level. We shall show an explicit formula of the induced product on Čech cocycles.

8.1 An Easy Case

Let us start from an easy case which can be found in standard textbooks on homological algebra.

Theorem 8.1.1. *Suppose we have three sheaves $\mathcal{F}, \mathcal{G}, \mathcal{H}$ over X and a cup product*

$$\cup : \mathcal{F} \otimes \mathcal{G} \longrightarrow \mathcal{H}.$$

Then $\cup$ induces an product on their cohomology

$$\cup : H^*(X, \mathcal{F}) \otimes H^*(X, \mathcal{G}) \longrightarrow H^*(X, \mathcal{H}).$$

We shall show a proof using Čech resolution. In particular, we are interested in the product formula in terms of Čech cocycles representing the cohomology classes.

First, let us recall some facts on Čech cohomology. Fix an open covering $\mathcal{U}$ of X, we have Čech resolution of $\mathcal{F}$:

$$\mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \to \cdots$$

Čech cohomology of sheaf $\mathcal{F}$ with respect to $\mathcal{U}$ is defined as

$$\check{H}^*(\mathcal{U}, \mathcal{F}) \equiv H^*(C^*(\mathcal{U}, \mathcal{F}))$$

where $C^k(\mathcal{U}, \mathcal{F})$ is group of global sections of sheaf $\mathcal{C}^k(\mathcal{U}, \mathcal{F})$. When the open covering $\mathcal{U}$ is acyclic with respect to $\mathcal{F}$, we have the canonical isomorphism

$$H^*(X,\mathcal{F}) \cong \check{H}^*(\mathcal{U},\mathcal{F}) \equiv H^*(C^*(\mathcal{U},\mathcal{F})).$$

The idea of the proof is to show this chain of morphisms

$$\check{H}^*(\mathcal{U},\mathcal{F}) \otimes \check{H}^*(\mathcal{U},\mathcal{G}) \xrightarrow{\tau_*} H^*(Tot(C^*(\mathcal{U},\mathcal{F}) \otimes C^*(\mathcal{U},\mathcal{G})))$$

$$\xrightarrow{\phi_*} \check{H}^*(\mathcal{U},\mathcal{F}\otimes\mathcal{G}) \xrightarrow{\cup_*} \check{H}^*(\mathcal{U},\mathcal{H}).$$

Then we get the desired cup product by choosing a good covering $\mathcal{U}$ and composing these three morphisms.

Proof. Fix an open covering $\mathcal{U}$ of X.

We construct a morphism of complexes

$$\phi : Tot(C^*(\mathcal{U},\mathcal{F}) \otimes C^*(\mathcal{U},\mathcal{G})) \longrightarrow C^*(\mathcal{U},\mathcal{F}\otimes\mathcal{G}).$$

For $a \in C^r(\mathcal{U},\mathcal{F})$, $b \in C^s(\mathcal{U},\mathcal{G})$ we put

$$\phi(a\otimes b)_{i_0,\cdots,i_{r+s}} = a_{i_0,\cdots,i_r} \otimes b_{i_r,\cdots,i_{r+s}}$$

We fix the differential $D = \delta_{\mathcal{F}} \otimes id + id \otimes (-1)^r \delta_{\mathcal{G}}$ on the total complex of double complex

$$\bigoplus_{r,s} C^r(\mathcal{U},\mathcal{F}) \otimes C^s(\mathcal{U},\mathcal{G}),$$

where $\delta_{\mathcal{F}}$, $\delta_{\mathcal{G}}$ are Čech differentials on $C^*(\mathcal{U},\mathcal{F})$ and $C^*(\mathcal{U},\mathcal{G})$ respectively. It is easy to verify that ϕ is a chain map, i.e. commutative with differentials.

Therefore, ϕ induce a map

$$\phi_* : H^*(Tot(C^*(\mathcal{U},\mathcal{F}) \otimes C^*(\mathcal{U},\mathcal{G}))) \longrightarrow H^*(C^*(\mathcal{U},\mathcal{F}\otimes\mathcal{G})) \equiv \check{H}^*(\mathcal{U},\mathcal{F}\otimes\mathcal{G}).$$

In addition, $\cup : \mathcal{F}\otimes\mathcal{G} \longrightarrow \mathcal{H}$ induces a map on Čech cohomology

$$\cup_* : \check{H}^*(\mathcal{U},\mathcal{F}\otimes\mathcal{G}) \longrightarrow \check{H}^*(\mathcal{U},\mathcal{H}).$$

Furthermore, there is a natural map

$$\check{H}^*(\mathcal{U},\mathcal{F}) \otimes \check{H}^*(\mathcal{U},\mathcal{G}) \xrightarrow{\tau_*} H^*(Tot(C^*(\mathcal{U},\mathcal{F}) \otimes C^*(\mathcal{U},\mathcal{G})))$$

induced by

$$C^*(\mathcal{U},\mathcal{F}) \otimes C^*(\mathcal{U},\mathcal{G}) \xrightarrow{\tau} Tot(C^*(\mathcal{U},\mathcal{F}) \otimes C^*(\mathcal{U},\mathcal{G}))).$$

Finally, we get a map

$$\check{H}^*(\mathcal{U},\mathcal{F}) \otimes \check{H}^*(\mathcal{U},\mathcal{G}) \xrightarrow{\tau_*} H^*(Tot(C^*(\mathcal{U},\mathcal{F}) \otimes C^*(\mathcal{U},\mathcal{G})))$$

$$\xrightarrow{\phi_*} \check{H}^*(\mathcal{U},\mathcal{F}\otimes\mathcal{G}) \xrightarrow{\cup_*} \check{H}^*(\mathcal{U},\mathcal{H}).$$

And it is easy to see, for two Čech cocycles $a \in C^r(\mathcal{U},\mathcal{F})$ and $b \in C^s(\mathcal{U},\mathcal{G})$, the cup product of $[a]$ and $[b]$ can be represented by $a \cup b$ which is defined by

$$(a \cup b)_{i_0,\cdots,i_{r+s}} = a_{i_0,\cdots,i_r} \cup b_{i_r,\cdots,i_{r+s}} \tag{8.1}$$

When the covering $\mathcal{U}$ is acyclic with respect to $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$, we define

$$\cup : H^*(X, \mathcal{F}) \otimes H^*(X, \mathcal{G}) \longrightarrow H^*(X, \mathcal{H})$$

by above construction. □

8.2 The General Case

Now we consider the general case, i.e. cup product on hypercohomology of complexes of sheaves. In fact, it is only nationally more difficult than the easy case. From now on, we assume the covering $\mathcal{U}$ is acyclic with respect to all sheaves we deal with and identify Čech (hyper)cohomology with sheaf (hyper)cohomology.

Theorem 8.2.1. *Let $(\mathcal{F}^*, d_{\mathcal{F}}), (\mathcal{G}^*, d_{\mathcal{G}}), (\mathcal{H}^*, d_{\mathcal{H}})$ be complexes of sheaves of abelian groups over a manifold X. If there is a cup product*

$$\cup : \mathcal{F}^* \otimes \mathcal{G}^* \longrightarrow \mathcal{H}^*$$

which commutes with differentials, then $\cup$ induces an product on their hypercohomology:

$$\cup : \mathbb{H}^*(X, \mathcal{F}^*) \otimes \mathbb{H}^*(X, \mathcal{G}^*) \longrightarrow \mathbb{H}^*(X, \mathcal{H}^*).$$

Proof. Fix an open covering $\mathcal{U}$ of X, for a complex of sheaves $\mathcal{A}^*$ ($\mathcal{A}^* = \mathcal{F}^*$, $\mathcal{G}^*$ or $\mathcal{H}^*$) we have Čech resolution $\mathcal{A}^* \to \mathcal{C}^*(\mathcal{U}, \mathcal{A}^*)$. Then

$$\mathbb{H}^*(X, \mathcal{A}^*) \equiv \mathbb{H}^*(Tot(\mathcal{C}^*(\mathcal{U}, \mathcal{A}^*))) \equiv H^*(Tot(C^*(\mathcal{U}, \mathcal{A}^*)))$$

where $C^*(\mathcal{U}, \mathcal{A}^*)$ denotes the group of global sections of sheaves $\mathcal{C}^*(\mathcal{U}, \mathcal{A}^*)$, and $Tot(C^*(\mathcal{U}, \mathcal{A}^*))$ denotes the total complex of double complex $\bigoplus_{r,p} C^r(\mathcal{U}, \mathcal{A}^p)$ with total differential $D_{\mathcal{A}} = \delta + (-1)^r d_{\mathcal{A}}$.

Similar to the case when $\mathcal{F}^*$, $\mathcal{G}^*$ are $\mathcal{H}^*$ are single sheaves, the cup product

$$\cup : \mathbb{H}^*(X, \mathcal{F}^*) \otimes \mathbb{H}^*(X, \mathcal{G}^*) \longrightarrow \mathbb{H}^*(X, \mathcal{H}^*)$$

is defined as the composition of three maps

$$\mathbb{H}^*(X, \mathcal{F}^*) \otimes \mathbb{H}^*(X, \mathcal{G}^*) \xrightarrow{\tau_*} H^*(Tot(C^*(\mathcal{U}, \mathcal{F}^*) \otimes C^*(\mathcal{U}, \mathcal{G}^*)))$$

$$\xrightarrow{\phi_*} \mathbb{H}^*(X, \mathcal{F}^* \otimes \mathcal{G}^*) \xrightarrow{\cup_*} \mathbb{H}^*(X, \mathcal{H}^*).$$

1) The first map is induced by

$$Tot(C^*(\mathcal{U}, \mathcal{F}^*)) \otimes Tot(C^*(\mathcal{U}, \mathcal{G}^*)) \xrightarrow{\tau_*} Tot(C^*(\mathcal{U}, \mathcal{F}^*) \otimes C^*(\mathcal{U}, \mathcal{G}^*)))$$

where $Tot(C^*(\mathcal{U}, \mathcal{F}^*) \otimes C^*(\mathcal{U}, \mathcal{G}^*))$ is the total complex of

$$\bigoplus_{r,s,p,q} (C^r(\mathcal{U}, \mathcal{F}^p)) \otimes (C^s(\mathcal{U}, \mathcal{G}^q))$$

with differential $D = D_{\mathcal{F}} \otimes id + id \otimes (-1)^{r+p} D_{\mathcal{G}}$.

2) Now we construct ϕ which induces the second map ϕ_*:

$$\phi : Tot(C^*(\mathcal{U}, \mathcal{F}^*) \otimes C^*(\mathcal{U}, \mathcal{G}^*)) \longrightarrow Tot(C^*(\mathcal{U}, \mathcal{F}^* \otimes \mathcal{G}^*)).$$

For $a \in C^r(\mathcal{U}, \mathcal{F}^p)$, $b \in C^s(\mathcal{U}, \mathcal{G}^q)$ we define $\phi(a \otimes b) \in C^{r+s}(\mathcal{U}, \mathcal{F}^p \otimes \mathcal{G}^q))$ by

$$\phi(a \otimes b)_{i_0,\cdots,i_{r+s}} = (-1)^{ps} a_{i_0,\cdots,i_r} \otimes b_{i_r,\cdots,i_{r+s}}.$$

Note that $\mathcal{F}^* \otimes \mathcal{G}^*$ is the total complex of double complex $\bigoplus_{p,q} \mathcal{F}^p \otimes \mathcal{G}^q$ with differential $d_{\mathcal{F}\otimes\mathcal{G}} = d_{\mathcal{F}} \otimes id + id \otimes (-1)^p d_{\mathcal{G}}$. And $Tot(C^*(\mathcal{U}, \mathcal{F}^* \otimes \mathcal{G}^*))$ is the total complex of $\bigoplus_{r,p,q} C^r(\mathcal{U}, \mathcal{F}^p \otimes \mathcal{G}^q)$ with differential $D_{\mathcal{F}\otimes\mathcal{G}} = \delta + (-1)^r d_{\mathcal{F}\otimes\mathcal{G}}$.

We have to verify that ϕ is a chain map, i.e. commutative with differentials. In fact, the purpose that we put a sign $(-1)^{ps}$ in the definition of ϕ is to make ϕ to be a chain map.

For an element $a \otimes b \in C^r(\mathcal{U}, \mathcal{F}^p) \otimes C^s(\mathcal{U}, \mathcal{G}^q)$, we calculate $\phi(D(a \otimes b))$ and $D_{\mathcal{F}\otimes\mathcal{G}}(\phi(a \otimes b))$ respectively.

$$\begin{aligned}
& \phi(D(a \otimes b)) \\
= \; & \phi(D_{\mathcal{F}} a \otimes b + (-1)^{r+p} a \otimes D_{\mathcal{G}} b) \\
= \; & \phi((\delta a + (-1)^r d_{\mathcal{F}} a) \otimes b + (-1)^{r+p} a \otimes (\delta b + (-1)^s d_{\mathcal{G}} b)) \\
= \; & \phi(\delta a \otimes b + (-1)^r d_{\mathcal{F}} a \otimes b + (-1)^{r+p} a \otimes \delta b + (-1)^{r+p+s} a \otimes d_{\mathcal{G}} b) \\
= \; & (-1)^{ps} \delta a \otimes b + (-1)^{(p+1)s+r} d_{\mathcal{F}} a \otimes b \\
& \qquad + (-1)^{p(s+1)+r+p} a \otimes \delta b + (-1)^{ps+r+p+s} a \otimes d_{\mathcal{G}} b \\
= \; & (-1)^{ps} \delta a \otimes b + (-1)^{ps+s+r} d_{\mathcal{F}} a \otimes b \\
& \qquad + (-1)^{ps+r} a \otimes \delta b + (-1)^{ps+r+p+s} a \otimes d_{\mathcal{G}} b
\end{aligned}$$

$$
\begin{aligned}
& D_{\mathcal{F}\otimes\mathcal{G}}(\phi(a\otimes b)) \\
= \;& D_{\mathcal{F}\otimes\mathcal{G}}((-1)^{ps}(a\otimes b)) \\
= \;& (\delta+(-1)^{r+s}d_{\mathcal{F}\otimes\mathcal{G}})((-1)^{ps}(a\otimes b)) \\
= \;& (-1)^{ps}\delta(a\otimes b)+(-1)^{ps+r+s}d_{\mathcal{F}\otimes\mathcal{G}}(a\otimes b) \\
= \;& (-1)^{ps}(\delta a\otimes b+(-1)^{r}a\otimes\delta b)+(-1)^{ps+r+s}(d_{\mathcal{F}}a\otimes b+(-1)^{p}a\otimes d_{\mathcal{G}}b) \\
= \;& (-1)^{ps}\delta a\otimes b+(-1)^{ps+r}a\otimes\delta b+(-1)^{ps+s+r}d_{\mathcal{F}}a\otimes b+(-1)^{ps+r+p+s}a\otimes d_{\mathcal{G}}b \\
= \;& \phi(D(a\otimes b))
\end{aligned}
$$

Therefore, ϕ is a chain map and induces a map

$$\phi_*: H^*(Tot(C^*(\mathcal{U},\mathcal{F}^*)\otimes C^*(\mathcal{U},\mathcal{G}^*))) \longrightarrow H^*(C^*(\mathcal{U},\mathcal{F}^*\otimes\mathcal{G}^*)) \equiv \mathbb{H}^*(X,\mathcal{F}^*\otimes\mathcal{G}^*).$$

3) Because $\cup$ commutes with differentials, it is easy to see $\cup$ induces a chain map:

$$Tot(C^*(\mathcal{U},\mathcal{F}^*\otimes\mathcal{G}^*)) \longrightarrow Tot(C^*(\mathcal{U},\mathcal{H}^*)).$$

Therefore, we have an induced map on hypercohomology

$$\cup_*: \mathbb{H}^*(X,\mathcal{F}^*\otimes\mathcal{G}^*) \longrightarrow \mathbb{H}^*(X,\mathcal{H}^*).$$

From above process, we can realize the cup product on Čech cocycles. Assume $\alpha\in\mathbb{H}^k(X,\mathcal{F}^*)$ and $\beta\in\mathbb{H}^l(X,\mathcal{G}^*)$. Let $a=\sum_{r+p=k}a^{r,p}\in\bigoplus_{r+p=k}C^r(\mathcal{U},\mathcal{F}^p)$ be a Čech cocycle representing α and $b=\sum_{s+q=l}b^{s,q}\in\bigoplus_{s+q=l}C^s(\mathcal{U},\mathcal{G}^q)$ be a Čech cocycle representing β.

We define

$$a \cup b \equiv \sum_{r+p=k, s+q=l} (-1)^{ps} a^{r,p} \cup b^{s,q},$$

where the cup product $\cup$ in the right hand side is defined as formula 8.1. Then $\alpha \cup \beta$ is represented by $a \cup b$.

□

Let us take a look at the case of smooth Deligne cohomology.

The cup product

$$\cup : \mathbb{Z}_{\mathcal{D}}(p)^{\infty} \otimes \mathbb{Z}_{\mathcal{D}}(q)^{\infty} \to \mathbb{Z}_{\mathcal{D}}(p+q)^{\infty}$$

is defined by

$$x \cup y = \begin{cases} x \cdot y & \text{if } \deg x = 0; \\ x \wedge dy & \text{if } \deg x > 0 \text{ and } \deg y = q; \\ 0 & \text{otherwise.} \end{cases}$$

Assume

$$\alpha \in H^p_{\mathcal{D}}(X, \mathbb{Z}(p)^{\infty}) \text{ and } \beta \in H^q_{\mathcal{D}}(X, \mathbb{Z}(q)^{\infty}),$$

and let

$$\tilde{a} = r + a = r + \sum_{i=0}^{p-1} a^{i,p-1-i} \in M^p_p \text{ be a representative of } \alpha$$

and

$$\tilde{b} = s + b = s + \sum_{i=0}^{q-1} b^{i,q-1-i} \in M_q^q \text{ be a representative of } \beta$$

where

$$r \in C^p(\mathcal{U}, \mathbb{Z}), a^{i,p-1-i} \in C^i(\mathcal{U}, \mathcal{E}^{p-1-i}),$$

and

$$s \in C^q(\mathcal{U}, \mathbb{Z}), b^{i,q-1-i} \in C^i(\mathcal{U}, \mathcal{E}^{q-1-i}).$$

Then we calculate

$$\begin{aligned}
\alpha \cup \beta &= [\tilde{a} \cup \tilde{b}] \\
&= [rs + \Sigma_i (-1)^{0 \cdot i} r \cdot b^{i,q-1-i} + \Sigma_j (-1)^{(p-j)0} a^{j,p-1-j} \wedge db^{0,q-1}] \\
&= [r \cdot \tilde{b} + a \wedge db^{0,q-1}] \\
&= [r \cup \tilde{b} + a \cup db^{0,q-1}]
\end{aligned}$$

Note that in the last line above, we use $\cup$ in the sense of Proposition 3.2.2.

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