Turaev-Viro theory as an extended TQFT

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Abstract of the Dissertation

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In recent years, the application of quantum groups to the study of low-dimensional topology has become an active topic of research. In three-dimensions, these yield the well-known Reshetihkin-Turaev (RT) invariants, which are a mathematical formulation of Chern-Simons theory and Turaev-Viro (TV) theory, which is a convergent form of the Ponzano-Regge state sum formula from Quantum Gravity.

Both RT and TV are more than invariants; they have a far richer structure known as a Topological Quantum Field Theory (TQFT). We describe Turaev-Viro theory as an extended (3-2-1)-TQFT and use this description to prove a theorem relating it to RT theory.

Turaev-Viro theory has several equivalent descriptions. We examine Kitaevs toric code from quantum computation and the Levin-Wen model from condensed matter physics and show that these theories are extended TQFTs coinciding with TV theory.

To Nava

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Chapter 1

Introduction

In the last quarter century, the study of quantum invariants of knots and low-dimensional manifolds has become an immensely active field of research. The subject is multi-disciplinary, combining elements of representation theory, topology, quantum physics, condensed matter theory and quantum computing among others, and is at once highly theoretical and promising to be of practical use in the forseeable future. It has aspects that are simple– the construction of the Jones polynomial via the Kauffmann bracket is easily understandable by an advanced high school student, yet much is the subject of current interest by specialists.

The study of quantum invariants began in 1989 when Witten published the seminal paper 51]. Previously, Jones had defined a polynomial knot invariant which arose from studying represenions of the braid group using subfactors. In his paper, Witten provided a topological interpretation of the Jones Polynomial of a knot $K \subset S^3$: it is the expectation of a Wilson loops associated to K in Chern-Simons theory with gauge group $SU(2)$. This interpretation was revolutionary (and helped earn Witten a Fields Medal), but it was not entirely mathematically rigorous since the formula involved a path integral.

Using the theory of quantum groups, Reshetikhin and Turaev provided a rigorous reformulation of Witten's work. More specifically, they constructed a theory which takes as input a quantum group $\mathfrak{g}$ and a representation V of $\mathfrak{g}$ and produces a knot invariant. These knot invariants include the Jones Polynomial and its colored variants. Their definition is entirely rigorous and they are easily computable. Most importantly, they are unchanged under the Kirby moves (See 5], Section 4.1) and thus are actually 3-manifold invariants. The RT invariant coincides with the Witten invariant only conjecturally; physicists "know" how to compute the path integral using an asymptotic expansion and the stationary phase approximation and these computations agree with RT, but these calculations are not satisfactory to mathematicians.

Another quantum invariant was constructed in 1992 by Turaev and Viro (TV) 49]. This also is a 3-manifold invariant, but it is somewhat different from the Reshetkihin-Turaev invariant. Turaev-Viro also takes as input a quantum group and produces a topological invariant of 3-manifolds. The invariant is defined by taking a state-sum on a triangulation of the manifold and one can check that the result is triangulation-independent. The TV invariant may be viewed as a regularized (convergent) form of the Ponzano-Regge model in 3-dimensional quantum gravity.

Although both theories arose from the study of quantum groups, the exact algebraic requirements for constructing each theory were formalized later using category theory: Reshetikhin-Turaev theory requires a modular tensor category and Turaev-Viro requires a spherical fusion category. A modular category is a type of spherical fusion category with extra structure– a braiding which is non-degenerate in a suitable sense. Given a modular category $\mathcal{A}$ and a spherical fusion category $\mathcal{C}$, we denote the RT invariant corresonding to $\mathcal{A}$ by $Z_{RT,\mathcal{A}}$ and the TV invariant corresponding to $\mathcal{C}$ by $Z_{TV,\mathcal{C}}$.

Since every modular category is spherical fusion, it makes sense to compare RT and TV invariants based on the same modular category.

Theorem 1.0.1 (Turaev)**.** *Let $\mathcal{A}$ be a modular category. Then*

$$Z_{TV,\mathcal{A}}(\mathcal{M}) = |Z_{RT,\mathcal{A}}(\mathcal{M})|^2$$

for any closed 3-manifold $\mathcal{M}$

This theorem is both remarkable and mysterious. The invariants are defined in completely disparate ways, yet for modular categories they yield *almost* the same results. Additionally, it implies that for $\mathcal{A}$ modular, the TV invariant is both real and positivel. This is completely unobvious from the formula.

The RT and TV invariants are more than just invariants of 3-manifolds. They are part of much richer theories known as Topological Quantum Field Theories (TQFTs). Originally, these were studied primarily by physicists as field theories in which there is no metric (e.g., Chern-Simons theory). A complete axiomatic definition was given by Atiyah in 1990:

Definition 1.0.2. An (n+1)-dimensional TQFT is a symmetric monoidal functor

$$\Phi : (n+1)Cob \to Vec_{\mathbb{C}}$$

where (n+1)Cob is the category with objects closed n-manifolds and morphisms cobordisms.

This definition is highly succint and probably too abstruse for the *categorically uninitiated*. It amounts to the following:

Definition 1.0.3. An (n+1)-dimensional TQFT τ is the following collection of data:

- To each closed n-dimensional manifold $\mathcal{N}$, an assignment of a complex vector space $\tau(\mathcal{N})$.
- To each (n+1)-dimensional manifold $\mathcal{M}$ (possibly with boundary), an assignment of a vector $\tau(\mathcal{M})$ in the vector space $\tau(\partial\mathcal{M})$.

- To a homeomorphism $\varphi : \mathcal{N} \to \mathcal{N}'$ of n-manifolds, a natural isomorphism $\tau_* : \tau(\mathcal{N}) \to \tau(\mathcal{N}')$.

- Functorial isomorphisms

  1. $\tau(\overline{\mathcal{N}}) \to \tau(\mathcal{N}^*)$
  2. $\tau(\emptyset) \to \mathbb{C}$
  3. $\tau(\mathcal{N}_1 \coprod \mathcal{N}_2) \to \tau(\mathcal{N}_1) \otimes \tau(\mathcal{N}_2)$

These data are required to satisfying a list of axioms— functoriality, normalization and gluing. For a detailed description of these axioms, see (5], Chapter 4).

Thus, a 3-dimensional TQFT gives numerical invariants of 3-manifolds and vector space invariants of 2-manifolds. It does so in a functorial way and is compatible with cobordisms and gluing of cobordisms.

Reshetikhin-Turaev theory actually is somewhat more than a TQFT. One can define it for 3-manifolds with boundary and embedded ribbon tangles. This gives an example of an extended TQFT, a construction studied by Baez, Walker and greatly popularized by the influential paper of Lurie 35]. Reshetikhin-Turaev theory is a (3-2-1) theory, since it computes values for manifolds of dimensions 3, 2, and 1. In particular, it assigns to the circle S^1 the original category $\mathcal{C}$.

For every monoidal category $\mathcal{C}$ there is an associated category called the Drinfeld center, denoted $Z(\mathcal{C})$, which is braided 39]. The construction of the center is somewhat analogous to taking the center of an algebra, but there are some critical differences. In particular, for a category $\mathcal{C}$, $Z(\mathcal{C})$ is in some sense *larger* than $\mathcal{C}$ and $Z(Z(\mathcal{C}))$ properly contains $Z(\mathcal{C})$ as a subcategory.

Theorem 1.0.4. *(Mueger) If $\mathcal{C}$ is a spherical fusion category, then $Z(\mathcal{C})$ is modular.*

Furthermore it is known that if $\mathcal{C}$ is modular, $Z(\mathcal{C})$ is equivalent as a braided category to $\mathcal{C} \boxtimes \mathcal{C}^{op}$, where $\mathcal{C}^{op}$ obtained from $\mathcal{C}$ by replacing all braiding isomorphisms with their inverses. Using this isomorphism and Theorem 1.0.1, gives

Theorem 1.0.5. *Let $\mathcal{C}$ be a modular tensor category. Then for any closed 3-manifold $\mathcal{M}$,*

$$Z_{TV,\mathcal{C}}(\mathcal{M}) = Z_{RT,Z(\mathcal{C})}(\mathcal{M}) \tag{1.0.1}$$

This result was conjectured to apply for $\mathcal{C}$ any spherical fusion category. It had been proved in several other special cases, such as when $\mathcal{C} = Rep(G)$ for some finite group G, but until recently, the general case was unresolved.

A much stronger connection was conjectured by Turaev:

Conjecture 1.0.6. *Let $\mathcal{C}$ be a spherical fusion category. Then we have an isomorphism of TQFTs*

$$Z_{TV,\mathcal{C}} \cong Z_{RT,Z(\mathcal{C})} \tag{1.0.2}$$

Surprisingly, these TQFTs have a fascinating application to Quantum Computing. One of the main problems with physically building such a device is that quantum systems are very sensitive to interactions with the environment. Although individually small, the effect of such interactions compounds quickly; current models of quantum computers are incapable of performing all but the most trivial calculations. One proposed solution 31], is to find physical systems whose states are *topological* that is, impervious to small perturbations. Thus, the system effectively ignores the interactions with the environment, as long as the interactions are individually small. As is discussed in section 2, this model is equivalent to TV theory. Such physical systems have already been discovered, but they are not sophisticated enough to implement any algorithms. Finding a topological system suitable for universal quantum computation is an active area of research.

1.1 Results

The main result of this thesis is a resolution of Conjecture 1.0.6 in the affirmative. As mentioned above, RT theory is defined as an extended TQFT where cobordisms between surfaces with boundary (equivalently 3-manifolds with boundary and embedded ribbon tangles) are permitted. Together with Alexander Kirillov Jr, we develope a similar description of TV theory:

Theorem 1.1.1. *Turaev-Viro theory has the structure of an extended (3-2-1) TQFT, such that* $Z_{TV,\mathcal{C}}(S^1) = Z(\mathcal{C})$.

This theorem generalizes the construction of TV theory 49] to an extended TQFT. This is important for several reasons. First, this description is necessary to compare TV invariants and RT invariants even at the level of closed 3-manifolds. Further, it provides the first step in describing TV theory as a fully extended (3-2-1-0) theory.

One of the drawbacks of TV theory is that involves triangulating a surface, which is cumbersome even for simple manifolds, such as the torus. To address this, we introduce more general cell decompositions, called piecewise linear cell decompositions 30]. This allows for much more efficient computations of TV invariants.

Using the above description of TV theory, we prove the following:

Theorem 1.1.2. *Let $\mathcal{M}$ be a closed 3-manifold. Then*

$$Z_{TV,\mathcal{C}}(\mathcal{M}) = Z_{RT,Z(\mathcal{C})}(\mathcal{M}) \tag{1.1.1}$$

We next extend the results of Theorem 1.1.2 to show that the two theories are equivalent as extended (3-2-1) TQFTs:

Theorem 1.1.3.

1. *Let Σ be a surface Then there is a natural isomorphism $Z_{TV,\mathcal{C}}(\Sigma) \to Z_{RT,Z(\mathcal{C})}(\Sigma)$.*
2. *The isomorphism from (1) is compatible with cobordisms; if $\mathcal{M} : \Sigma_1 \to \Sigma_2$ is a cobordism, $Z_{TV,\mathcal{C}}(\mathcal{M}) = Z_{RT,Z(\mathcal{C})}(\mathcal{M})$*
3. *These equations also hold for surfaces with boundary and cobordisms between surfaces with boundary (cobordisms with framed embedded tangles), and thus we have an equivalence of extended TQFTs $Z_{TV,\mathcal{C}} \cong Z_{RT,Z(\mathcal{C})}$.*

Another proof of Conjecture 1.0.6 is due to recent work by Turaev and Virelizier 47], but their work is from quite a different perspective.

There are several models in physics that are closely related to Turaev-Viro theory. The first is the toric codel, which Kitaev introduced in 31] as a means of performing fault-tolerant quantum computation. The second is the Levin-Wen model 34] , which was proposed by condensed matter physicists to describe topological phases in certain physical systems.

Both the Kitaev and Levin-Wen models are defined in a similar way. Begin with an oriented surface Σ and a cell decomposition, one constructs a large vector space $\mathcal{H}$, and a Hamiltonian H, which is defined in the following form:

$$H = -\sum_v A_v - \sum_p B_p \tag{1.1.2}$$

Here A_v and B_p are operators depending on the vertices and plaquettes (2-cells) of the cell decomposition. The actual form of these operators as well as the definiton of $\mathcal{H}$ depends on the theory. It is well-known that these operators are commuting projectors and that the lowest eigenspace of H (the *ground state*) depends only on the topology of Σ. It has been conjectured that these models are equivalent to Turaev-Viro theory. Further, the excited states (higher eigenspaces) have been described in the language of extended TQFTs.

Theorem 1.1.4. *Turaev-Viro theory, the Levin-Wen model and the Kitaev model are all equivalent as extended TQFTs.*

1.2 Future Work

One main goal of this project is to describe Turaev-Viro theory as a fully extended TQFT, i.e. defining TV theory on a point, a 1-manifold with boundary, etc. By general theory of extended TQFTs (see 35]), the value $Z_{TV,\mathcal{C}}(pt)$ should be an object in some 3-category that is *fully dualizable*. Futher, such data fully describes the theory; invariants associated to surfaces, 3-manifolds, etc. are determined uniquely by this object.

It had been known that the appropriate 3-category for Turaev-Viro theory should be the 3-category of Bimodule Categories:

Definition 1.2.1. (sketch) The 3-category *BiMod* of bimodule categories has

1. objects: monoidal categories
2. 1-morphisms: $Hom_{BiMod}(\mathcal{A}, \mathcal{B}) = \mathcal{A} - \mathcal{B}$ bimodule categories
3. 2-morphisms: bimodule functors.
4. 3-morphisms natural transformations of bimodule functors.

However, the proof the BiMod is a 3-category and the structure of duals in this category is very recent work, due to Douglas, Schommer-Pries and Snyder 14].

Theorem 1.2.2. .

1. *The 3-category BiMod has the structure of a weak 3-category*
2. *Spherical fusion categories are fully dualizable objects in this category*

Building on this work we would like to describe Turaev-Viro theory as a fully extended theory. This would provide a partial answer to the following long-standing question in physics

Question 1.2.3. *What does Chern-Simons (Reshetikhin-Turaev) theory assign to the point?*

In the case where we start with a modular category $\mathcal{A}$ that is equivalent to $Z(\mathcal{C})$ for some spherical fusion category $\mathcal{C}$, we will have answered this question. The value we assign to the point is the category $\mathcal{C}$. In the general case, a proposed solution is given by so-called conformal nets [**?**], but the full categorical structure as well as a proof that this fully extends RT theory is unknown.

It is important to note that RT theory is more general than TV theory. As shown in Theorem 1.1.3, the TV state-sum with spherical fusion category $\mathcal{C}$ gives at theory equivalent to RT with modular category $Z(\mathcal{C})$, but not every modular category arises as the center of a spherical category. For example, the category $\mathcal{C} = Rep(U_q(\mathfrak{g}))$ is not equivalent to the center of any other category.

Conjecture 1.2.4. *Reshetikhin-Turaev theory is a fully extended theory precisely when the starting modular category is equivalent to the Drinfeld Center of some other category. More generally, TQFTs which arise from triangulations correspond exactly to fully extended theories.*

Our study of TQFTs has been in the oriented setting. It would be interesting to repeat the construction for manifolds equipped with different structures, such as framing or spin structures. As in 35], each of these theories is in correspondence with a different set of object in the 3-category BiMod (for oriented theories Theorem 1.2.2 says these come from spherical fusion categories). Several of these examples are described in 14]. The most interesting (and thus far, unexplored) example is the unoriented setting. The corresponding categorical data should be the same as the oriented case (a spherical fusion category) together with some additional data relating the values for the positively and negatively oriented points.

An interesting problem is to generalize the constructions here to (3+1)-dimensional theories. Crane and Yetter developed such a generalization for

Turaev-Viro theory, but the invariants thus obtained turned out to be uninteresting. More recently, Walker and Wang [5] reframed this model in the langauge of Levin and Wen's string-nets 34] in order to describe topological insulators. The work of Bravyi and Haah 9] has provided interesting examples of (3+1)-dimensional models analogous to Kitaev's toric code, but without the presence of so-called *ribbon* operators.

Chapter 2

Mathematical Preliminaries

In this chapter, we introduce some basic notions from category theory, representation theory and the theory of hopf algebras. We also introduce notation that will be used extensively in subsequent chapters. This chapter is far from comprehensive. We refer the reader to the literature to supplement this chapter:

- **Category Theory**: For general definitions and notions, 36] is the classic reference. 16] contains many important theorems about fusion categories. An excellent reference with many useful results regarding spherical fusion categories and their centers is the two part series by Müger (38],39]).

- **Quantum Topology**: The standard text on 3-D TQFTs is 45]. A more concise alternative is 5]. For a detailed exposition on extended TQFTs, including a proof sketch of the Baez-Dolan cobordism conjecture, see 35]. An alternative approach to extended TQFTs is covered in 50].

- **Hopf Algebras**: 13] is a good reference. For a comprehensive discussion of quantum groups and their applications to topology, 27] is recommended.

- **Quantum Computation**: The standard text is 28]. To learn about topological quantum computation, 52] is the best reference. In particular, Kitaev's model is best covered in the original paper 31].

In this section, we review some prerequisites and collect relevant notation that will be used throughout the paper.

2.1 Monoidal Categories

Definition 2.1.1. A *monoidal* category is the following collection of data:

1. A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

2. For any objects $U, V, W \in \mathcal{C}$, a natural isomorphism

$$\alpha_{UVW} : (U \otimes V) \otimes (W) \cong U \otimes (V \otimes W)$$

3. An object $\mathbf{1} \in \mathcal{C}$ and for each $V \in \mathcal{C}$ isomorphisms

$$\lambda_V : \mathbf{1} \otimes V \to V$$

$$\rho_V : V \otimes \mathbf{1} \to V$$

These data are required to satisfy certain axioms, specifically the *pentagon* and *triangle* axioms (36]).

Example 2.1.2. The following are monoidal categories:

1. $Vec_{\mathbb{K}}$, the category of vector spaces and linear maps over some field $\mathbb{K}$.

2. $Rep_{\mathbb{K}}(R)$, the category of $\mathbb{K}$-representations of a hopf algebra R.

In general, the isomorphisms α, λ, ρ are not identity maps.

Definition 2.1.3. If $\mathcal{C}$ is a monoidal category with α, λ, ρ all identity maps, we say $\mathcal{C}$ is a *strict* monoidal category.

There are virtually no interesting examples of strict monoidal categories. Nevertheless, the following theorem allows us to treat monoidal categories as though they were strict.

Theorem 2.1.4. *Any monoidal category is equivalent to a strict monoidal category*

We will make extensive use of this theorem. In particular, we will suppress the α, λ, ρ isomorphisms throughout.

All monoidal categories we use are are *abelian* over $\mathbb{C}$. That is, the Hom spaces are vector spaces, composition is bilinear, a zero object exists, and there are direct sums, kernels and cokernels. See 5, 36] for a more precise definition.

Definition 2.1.5. An object $X \in \mathcal{C}$ is simple if any injection $U \hookrightarrow X$ is either zero or an isomorphism

Definition 2.1.6. A category $\mathcal{C}$ is *semisimple* if every object X decomposes as a direct sum

$$X = \bigoplus_i X_i$$

where X_i are simple.

Definition 2.1.7. A monoidal category $\mathcal{C}$ is called *rigid* if every object of $\mathcal{C}$ has left and right duals. That is, for each object $X \in \mathcal{C}$, there are $X^*, {}^*X \in \mathcal{C}$ and maps $coev_R : \mathbf{1} \to X \otimes X^*, ev_R : X^* \otimes X \to \mathbf{1}$ and $coev_L : \mathbf{1} \to {}^*X \otimes X, evL : X \otimes {}^*X \to \mathbf{1}$ satisfying

1. $X \xrightarrow{coev_R \otimes id_X} X \otimes X^* \otimes X \xrightarrow{id_X \otimes ev_R} X = id_X$

2. $X^* \xrightarrow{id_{X^*} \otimes coev_R} X^* \otimes X \otimes X^* \xrightarrow{ev_R \otimes id_{X^*}} X^* = id_{X^*}$

3. $X \xrightarrow{id_X \otimes coev_L} X \otimes {}^*X \otimes X \xrightarrow{ev_L \otimes id_X} X = id_X$

4. ${}^*X \xrightarrow{coev_L \otimes id_{^*X}} {}^*X \otimes X \otimes {}^*X \xrightarrow{id_{^*X} \otimes ev_L} {}^*X = id_{^*X}$.

Definition 2.1.8. A *fusion* category is a tensor category $\mathcal{C}$ such that

1. $\mathcal{C}$ is semisimple and linear.

2. $\mathcal{C}$ is rigid.

3. $\mathcal{C}$ has finitely many classes of simple object.

4. Hom spaces are finite-dimensional.

5. $\mathbf{1}$ is simple.

If we exclude the last condition, the category is called *multi-fusion*. We will not need this more general notion in this thesis.

Definition 2.1.9. A rigid monoidal category together with a choice of natural isomorphism

$$\varphi : id \to **$$

is called a *pivotal category*.

Given a pivotal fusion category, one can define the dimension $dim(X)$ of an object X by composing, coevalution, the pivotal map and the evaluation map (see 5] for a precise definition). More generally, such a category contains a trace function on any morphism between an object and itself. A category is called spherical if it is pivotal and $dim(X) = dim(X^*)$ for all objects X. Spherical fusion categories are the main type of category we will use in this thesis. They are discussed in more detail in the next section. Note that in a spherical category the tensor product is not commutative:

$$X \otimes Y \ncong Y \otimes X$$

Definition 2.1.10. A fusion category is *braided* if for each object X, there is a natural isomorphism

$$\beta_X : X \otimes - \to - \otimes X$$

These isomorphisms are required to satisfy certain conditions (the hexagon axiom). See 36] for a detailed description.

Definition 2.1.11. A spherical fusion category is *modular* if it is braided and the braiding satisfies a non-degeneracy condition: The matrix S defined by

$$S_{ij} = tr(\beta_{ji} \circ \beta_{ij})$$

is invertible.

2.2 Spherical Fusion Categories

We fix an algebraically closed field $\mathbf{k}$ of characteristic 0 and denote by $\mathcal{V}ec$ the category of finite-dimensional vector spaces over $\mathbf{k}$.

Throughout this thesis, $\mathcal{C}$ will denote a spherical fusion category over $\mathbf{k}$. We refer the reader to the paper 15] for a detailed description of such categories.

It is important to remember that such categories are *not* braided.

In particular, $\mathcal{C}$ is semisimple with finitely many isomorphism classes of simple objects. We will denote by $\mathrm{Irr}(\mathcal{C})$ the set of isomorphism classes of simple objects. We will also denote by $\mathbf{1}$ the unit object in $\mathcal{C}$ (which is simple).

Two main examples of spherical categories are the category $\mathcal{V}ec^G$ of finite-dimensional G-graded vector spaces (where G is a finite group) and the category $\mathrm{Rep}(U_q\mathfrak{g})$ which is the semisimple part of the category of representations of a quantum group $U_q\mathfrak{g}$ at a root of unity; this last category is actually modular, but we will not be using this.

To simplify the notation, we will assume that $\mathcal{C}$ is a strict pivotal category, i.e. that $V^{**} = V$. As is well-known, this is not really a restriction, since any pivotal category is equivalent to a strict pivotal category.

We will denote, for an object X of $\mathcal{C}$, by

$$d_X = \dim X \in \mathbf{k}$$

its categorical dimension; it is known that for simple X, d_X is non-zero. We will fix, for any simple object $X_i \in \mathcal{C}$, a choice of square root $\sqrt{d_X}$ so that for $X = \mathbf{1}$, $\sqrt{d_{\mathbf{1}}} = 1$ and that for any simple X, $\sqrt{d_X} = \sqrt{d_{X^*}}$.

We will also denote

$$\mathcal{D} = \sqrt{\sum_{x \in \mathrm{Irr}(\mathcal{C})} d_X^2} \tag{2.2.1}$$

(throughout the paper, we fix a choice of the square root). Note that by results of 16], $\mathcal{D} \neq 0$.

We define the functor $\mathcal{C}^{\boxtimes n} \to \mathcal{V}ec$ by

$$\langle V_1, \dots, V_n \rangle = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_1 \otimes \dots \otimes V_n) \tag{2.2.2}$$

for any collection $V_1, \dots, V_n$ of objects of $\mathcal{C}$. Note that pivotal structure gives functorial isomorphisms

$$z \colon \langle V_1, \dots, V_n \rangle \simeq \langle V_n, V_1, \dots, V_{n-1} \rangle \tag{2.2.3}$$

such that $z^n = \mathrm{id}$ (see 5], Section 5.3); thus, up to a canonical isomorphism, the space $\langle V_1, \dots, V_n \rangle$ only depends on the cyclic order of $V_1, \dots, V_n$.

We have a natural composition map

$$\begin{aligned} &\langle V_1, \dots, V_n, X \rangle \otimes \langle X^*, W_1, \dots, W_m \rangle \to \langle V_1, \dots, V_n, W_1, \dots, W_m \rangle \\ &\quad \varphi \otimes \psi \mapsto \varphi \underset{X}{\circ} \psi = \mathrm{ev}_X \circ (\varphi \otimes \psi) \end{aligned} \tag{2.2.4}$$

where $\mathrm{ev}_X \colon X \otimes X^* \to \mathbf{1}$ is the evaluation morphism. It follows from semisimplicity of $\mathcal{C}$ that direct sum of these composition maps gives a functorial isomorphism

$$\bigoplus_{X \in \mathrm{Irr}(\mathcal{C})} \langle V_1, \dots, V_n, X \rangle \otimes \langle X^*, W_1, \dots, W_m \rangle \simeq \langle V_1, \dots, V_n, W_1, \dots, W_m \rangle \tag{2.2.5}$$

Note that for any objects $A, B \in \mathrm{Obj}\, \mathcal{C}$, we have a non-degenerate pairing $\mathrm{Hom}_{\mathcal{C}}(A, B) \otimes \mathrm{Hom}_{\mathcal{C}}(A^*, B^*) \to \mathbf{k}$ defined by

$$(\varphi, \varphi') = (\mathbf{1} \xrightarrow{\mathrm{coev}_A} A \otimes A^* \xrightarrow{\varphi \otimes \varphi'} B \otimes B^* \xrightarrow{\mathrm{ev}_B} \mathbf{1}) \tag{2.2.6}$$

In particular, this gives us a non-degenerate pairing

$$\langle V_1, \dots, V_n \rangle \otimes \langle V_n^*, \dots, V_1^* \rangle \to \mathbf{k}$$

and thus, functorial isomorphisms

$$\langle V_1, \dots, V_n \rangle^* \simeq \langle V_n^*, \dots, V_1^* \rangle \tag{2.2.7}$$

compatible with the cyclic permutations (2.2.3).

We will frequently use graphical representations of morphisms in the category $\mathcal{C}$, using tangle diagrams as in 45] or 5]. However, our convention is that of 5]: a tangle with k strands labeled $V_1, \dots, V_k$ at the bottom and n strands labeled $W_1, \dots, W_n$ at the top is considered as a morphism from $V_1 \otimes \dots \otimes V_k \to W_1 \otimes \dots \otimes W_n$. As usual, by default all strands are oriented going from the bottom to top. Note that since $\mathcal{C}$ is assumed to be a spherical category and not a braided one, no crossings are allowed in the diagrams.

For technical reasons, it is convenient to extend the graphical calculus by allowing, in addition to rectangular coupons, also circular coupons labeled with morphisms $\varphi \in \langle V_1, \dots, V_n \rangle$. This is easily seen to be equivalent to the original formalism: every such circular coupon can be replaced by the usual rectangular one as shown in Figure 2.1.

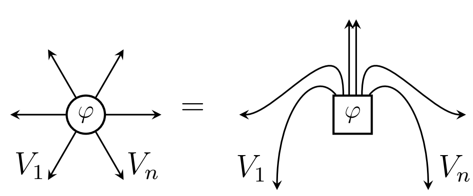
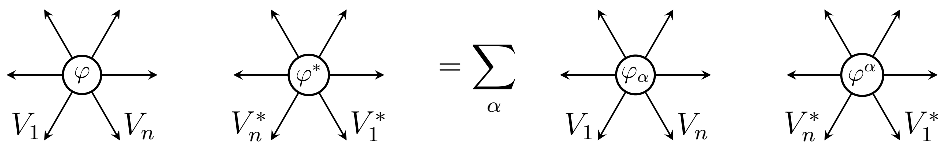


Figure 2.1: Round coupons

We will also use the following convention: if a figure contains a pair of circular coupons, one with outgoing edges labeled $V_1, \dots, V_n$ and the other with edges labeled $V_n^*, \dots, V_1^*$, and the coupons are labeled by pair of letters φ, φ^* (or ψ, ψ^*, or ...) it will stand for summation over the dual bases:



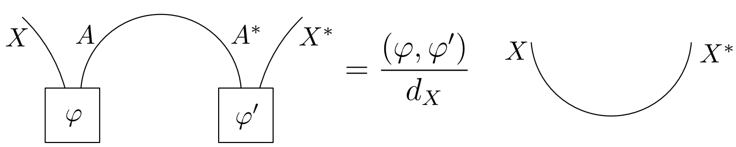
(2.2.8)

where $\varphi_\alpha \in \langle V_1, \dots, V_n \rangle$, $\varphi^\alpha \in \langle V_n^*, \dots, V_1^* \rangle$ are dual bases with respect to pairing (2.4.4).

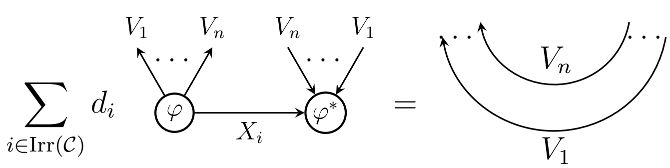
The following lemma, proof of which is left to the reader, lists some properties of this pairing and its relation with the composition maps (2.2.4).

Lemma 2.2.1.

1. *If X is simple and $\varphi \in \langle X, A \rangle$, $\varphi' \in \langle A^*, X^* \rangle$ then*



2.



(*we use here convention* (6.0.1)).

3. *If X is simple, $\varphi \in \langle A, X \rangle$, $\varphi' \in \langle X^*, A^* \rangle$, $\psi \in \langle X^*, B \rangle$, $\psi' \in \langle B^*, X \rangle$,*

then

$$(\varphi \underset{X}{\circ} \psi, \psi' \underset{X^*}{\circ} \varphi') = \frac{1}{d_X}(\varphi, \varphi')(\psi', \psi) \tag{2.2.9}$$

(*see Figure* 2.2).

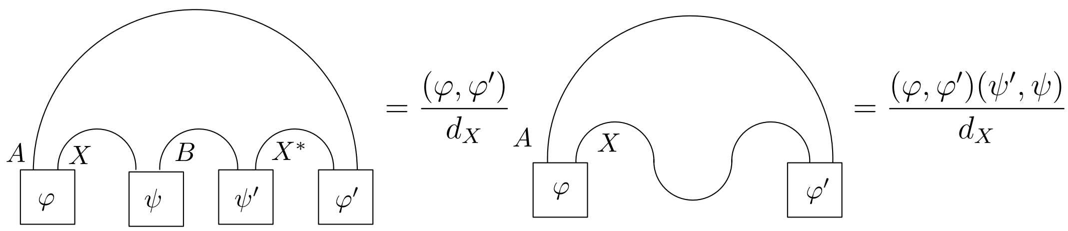


Figure 2.2: Compatibility of pairing with composition.

Corollary 2.2.2. *Let X be a simple object. Define the rescaled composition map*

$$\langle V_1, \dots, V_n, X\rangle \otimes \langle X^*, W_1, \dots, W_m\rangle \to \langle V_1, \dots, V_n, W_1, \dots, W_m\rangle$$
$$\varphi \otimes \psi \mapsto \varphi \underset{X}{\bullet} \psi = \sqrt{d_X}\ \mathrm{ev}_X \circ (\varphi \otimes \psi) \tag{2.2.10}$$

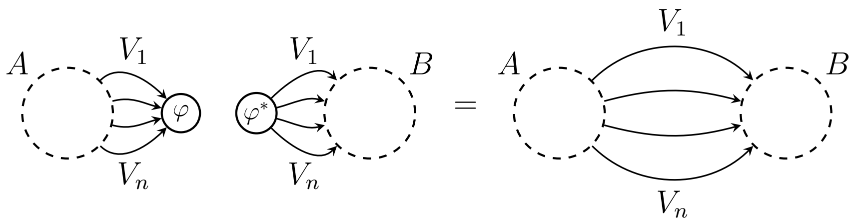
Then the rescaled composition map agrees with the pairing:

$$(\varphi \underset{X}{\bullet} \psi, \psi' \underset{X^*}{\bullet} \varphi') = (\varphi, \varphi')(\psi', \psi)$$

(*same notation as in* (2.2.9)).

The following result, which easily follows from Lemma 2.2.1, will also be very useful.

Lemma 2.2.3. *If the subgraphs A, B are not connected, then*



Finally, we will need the following result, which is the motivation for the name "spherical category".

Let Γ be an oriented graph embedded in the sphere S^2, where each edge e is colored by an object $V(e) \in \mathcal{C}$, and each vertex v is colored by a morphism $\varphi_v \in \langle V(e_1)^{\pm}, \dots V(e_n)^{\pm} \rangle$, where $e_1, \dots, e_n$ are the edges adjacent to vertex v, taken in clockwise order, and $V(e_i)^{\pm} = V(e_i)$ if e_i is outgoing edge, and $V^*(e_i)$ if e_i is the incoming edge.

By removing a point from S^2 and identifying $S^2 \setminus pt \simeq \mathbb{R}^2$, we can consider Γ as a planar graph. Replacing each vertex v by a circular coupon labeled by morphism φ_v as shown in Figure 2.3, we get a graph of the type discussed above and which therefore defines a number $Z_{RT}(\Gamma) \in \mathbf{k}$ (see, e.g., 5] or 45]).

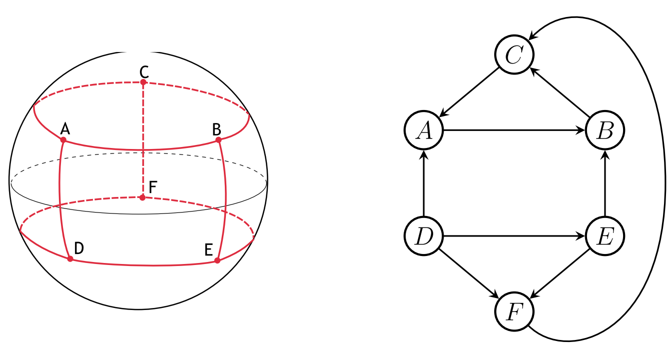


Figure 2.3: Graph on a sphere and its "flattening" to the plane

Theorem 2.2.4. *7] The number $Z_{RT}(\Gamma) \in \mathbf{k}$ does not depend on the choice of a point to remove from S^2 or on the choice of order of edges at vertices compatible with the given cyclic order and thus defines an invariant of colored graphs on the sphere.*

2.3 Drinfeld Center

We will also need the notion of Drinfeld center of a spherical fusion category. Recall that the Drinfeld center $Z(\mathcal{C})$ of a fusion category $\mathcal{C}$ is defined as the category whose objects are pairs (Y, φ_Y), where Y is an object of $\mathcal{C}$ and φ_Y – a functorial isomorphism $Y \otimes - \to - \otimes Y$ satisfying certain compatibility conditions (see 38]).

As before, we will frequently use graphical presentation of morphisms which involve objects both of $\mathcal{C}$ and $Z(\mathcal{C})$. In these diagrams, we will show objects of $Z(C)$ by double green lines and the half-braiding isomorphism $\varphi_Y : Y \otimes V \to V \otimes Y$ by crossing as in Figure 2.4.

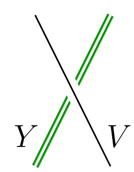


Figure 2.4: The half-braiding

We list here main properties of $Z(\mathcal{C})$, all under the assumption that $\mathcal{C}$ is a spherical fusion category over an algebraically closed field of characteristic zero.

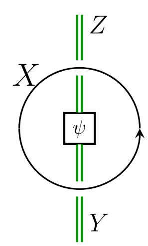
Theorem 2.3.1. *39] $Z(\mathcal{C})$ is a modular category; in particular, it is semisimple with finitely many simple objects, it is braided and has a pivotal structure which coincides with the pivotal structure on $\mathcal{C}$.*

We have an obvious forgetful functor $F\colon Z(\mathcal{C}) \to \mathcal{C}$. To simplify the notation, we will frequently omit it in the formulas, writing for example $\mathrm{Hom}_{\mathcal{C}}(Y, V)$ instead of $\mathrm{Hom}_{\mathcal{C}}(F(Y), V)$, for $Y \in \mathrm{Obj}\, Z(\mathcal{C})$, $V \in \mathrm{Obj}\, \mathcal{C}$. Note, however, that if $Y, Z \in \mathrm{Obj}\, Z(\mathcal{C})$, then $\mathrm{Hom}_{Z(\mathcal{C})}(Y, Z)$ is different from $\mathrm{Hom}_{\mathcal{C}}(Y, Z)$: namely, $\mathrm{Hom}_{Z(\mathcal{C})}(Y, Z)$ is a subspace in $\mathrm{Hom}_{\mathcal{C}}(Y, Z)$ consisting of those morphisms that commute the with the half-braiding. The following lemma is extremely useful.

Lemma 2.3.2. *Let $Y, Z \in \mathrm{Obj}\, Z(\mathcal{C})$. Define the operator*

$P\colon \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ by the following formula:

$$P\psi = \frac{1}{\mathcal{D}^2} \sum_{X \in \mathrm{Irr}(\mathcal{C})} d_X$$

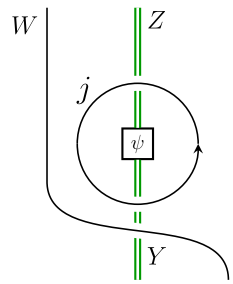


Then P is a projector onto the subspace $\mathrm{Hom}_{Z(\mathcal{C})}(Y, Z) \subset \mathrm{Hom}_{\mathcal{C}}(Y, Z)$.

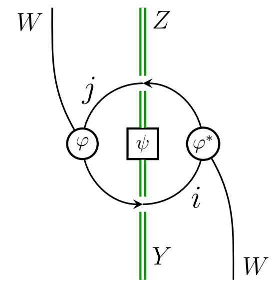
Proof. It is immediate from the definition that if $\psi \in \mathrm{Hom}_{Z(\mathcal{C})}(Y, Z)$, then $P\psi = \psi$. On the other hand, using Lemma 2.2.1, we get that for any $\psi \in$

$\mathrm{Hom}_{\mathcal{C}}(Y, Z)$, one has $(P\psi)\varphi_Y = \varphi_Z(P\psi)$:

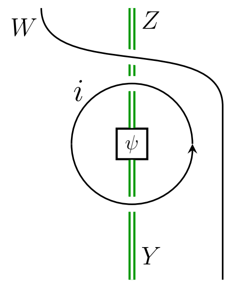
$$(P\psi)\varphi_Y = \frac{1}{\mathcal{D}^2}\sum_j d_j$$



$$= \frac{1}{\mathcal{D}^2}\sum_{i,j} d_i d_j$$



$$= \frac{1}{\mathcal{D}^2}\sum_i d_i$$



$$= \varphi_Z(P\psi),$$

(as before, we are using convention (6.0.1)). □

The following theorem is a refinement of 16], Proposition 5.4.

Theorem 2.3.3. *Let* $F\colon Z(\mathcal{C}) \to \mathcal{C}$ *be the forgetful functor and* $I\colon \mathcal{C} \to Z(\mathcal{C})$ *the (left) adjoint of* F*:* $\mathrm{Hom}_{Z(\mathcal{C})}(I(V), X) = \mathrm{Hom}_{\mathcal{C}}(V, F(X))$*. Then for* $V \in \mathrm{Obj}\,\mathcal{C}$*, one has*

$$I(V) = \bigoplus_{i\in\mathrm{Irr}(\mathcal{C})} X_i \otimes V \otimes X_i^* \tag{2.3.1}$$

with the half braiding given by

Note that instead of normalizing factor $\sqrt{d_i}\sqrt{d_j}$ we could have used d_i or d_j — each of this would give an equivalent definition.

Proof. Denote $Y = \bigoplus_{i\in\mathrm{Irr}(\mathcal{C})} X_i \otimes V \otimes X_i^*$. It follows from Lemma 2.2.1 that the morphisms $Y \otimes W \to W \otimes Y$ defined by Figure 2.5 satisfy the compatibility

$$\bigoplus_{i,j \in \mathrm{Irr}(\mathcal{C})} \sqrt{d_i}\sqrt{d_j}$$

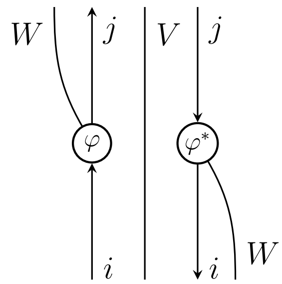
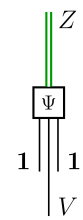


Figure 2.5: Half-braiding $I(V) \otimes W \to W \otimes I(V)$.

relations required of half braiding and thus define on Y a structure of an object of $Z(\mathcal{C})$. Now, define for any $Z \in \mathrm{Obj}\, Z(\mathcal{C})$, maps

$$\mathrm{Hom}_{Z(\mathcal{C})}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(V, Z)$$

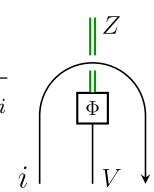
$$\Psi \mapsto \Psi \circ P_0 =$$



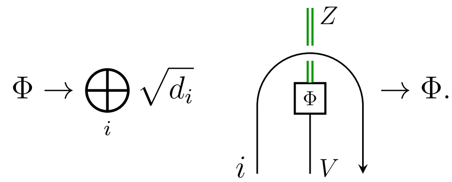
where P_0 is the embedding $V = \mathbf{1} \otimes V \otimes \mathbf{1} \to Y = \bigoplus X_i \otimes V \otimes X_i^*$ and

$$\mathrm{Hom}_{\mathcal{C}}(V, Z) \to \mathrm{Hom}_{Z(\mathcal{C})}(Y, Z)$$

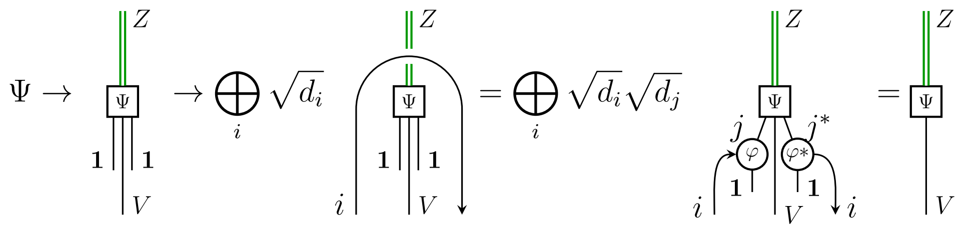
$$\Phi \mapsto \bigoplus_{i \in \mathrm{Irr}(\mathcal{C})} \sqrt{d_i}$$



It follows from Lemma 2.3.2 that these two maps are inverse to each other. Composition in one direction is easy. First suppose $\Phi \in \mathrm{Hom}_{\mathcal{C}}(V, Z)$. The computation is shown below.



The composition in opposite order is as follows:



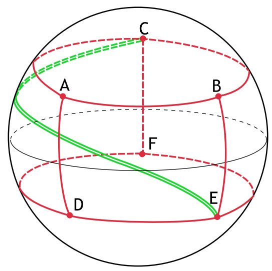
The first equality holds by functoriality of the half-braiding and Figure 2.5. The second equality is obvious. Therefore, the two maps are inverses to one another and we have $\mathrm{Hom}_{Z(\mathcal{C})}(Y, Z) = \mathrm{Hom}_{\mathcal{C}}(V, Z)$; thus, $Y = I(V)$. □

An easy generalization of Theorem 2.2.4 allows us to consider graphs in which some of the edges are labeled by objects of $Z(\mathcal{C})$.

Let $\hat{\Gamma}$ be a graph which consists of a usual graph Γ embedded in S^2 as in Theorem 2.2.4 and a finite collection of non-intersecting oriented arcs γ_i such that endpoints of each arc γ are vertices of graph Γ, and each vertex has a neighborhood in which arcs γ_i do not intersect edges of Γ; however, arcs γ_i are allowed to intersect edges of Γ away from vertices. Note that this implies that for each vertex v, we have a natural cyclic order on the set of all edges of $\hat{\Gamma}$ (including arcs γ_i) adjacent to v.

Let us color such diagram, labeling each edge of Γ by an object of $\mathcal{C}$, each arc

γ by an object of $Z(\mathcal{C})$, and each vertex v by a vector $\varphi_v \in \langle V^{\pm}(e_1), \dots, V^{\pm}(e_n)\rangle$ where $e_1, \dots, e_n$ are edges of $\hat{\Gamma}$ adjacent to v (including the arcs γ_i), and the signs are chosen as in Theorem 2.2.4.



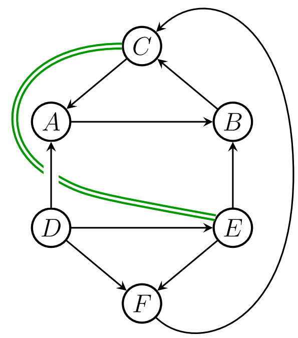


Figure 2.6: The graph $\hat{\Gamma}$ on the sphere and its flattening to the plane. The arc γ is shown by a double line.

As before, by removing a point from S^2 and choosing a linear order of edges (including the arcs) at every vertex, we get a diagram in the plane; however, now the projections of arcs γ_i can intersect edges of Γ as shown in Figure 2.6. Let us turn this into a tangle diagram by replacing each intersection by a picture where the arch γ_i goes under the edges of Γ, as shown in Figure 2.6.

Such a diagram defines a number $Z_{RT}(\hat{\Gamma})$ defined in the usual way, with the extra convention shown in Figure 2.4.

Theorem 2.3.4. *The number $Z_{RT}(\hat{\Gamma}) \in \mathbf{k}$ does not depend on the choice of a point to remove from S^2 orand thus defines an invariant of colored graphs on the sphere. Moreover, this number is invariant under homotopy of arcs γ_i.*

Proof. The fact that it is independent of the choice of point to remove and thus is an invariant of a graph on the sphere immediately follows from Theorem 2.2.4: replacing every crossing by a coupon colored by half-braiding φ_Y gives a graph as in Theorem 2.2.4. Invariance under homotopy of arcs γ follows from compatibility conditions on half-braiding shown in Figure 2.7.

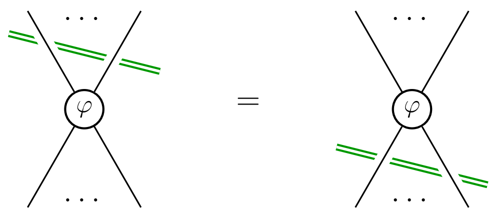


Figure 2.7: Invariance under homotopy

□

Finally, we will need a construction generalizing the cyclic isomorphism from (2.2.3).

For any $Y \in \operatorname{Obj} Z(\mathcal{C})$, we define a functor functor $\mathcal{C}^{\boxtimes n} \to \mathcal{V}ec$ by

$$\langle V_1, \ldots, V_n \rangle_Y = \operatorname{Hom}_{\mathcal{C}}(\mathbf{1}, Y \otimes V_1 \otimes \cdots \otimes V_n) \tag{2.3.2}$$

for any collection $V_1, \ldots, V_n$ of objects of $\mathcal{C}$. As before, we have functorial isomorphisms

$$z_Y : \langle V_1, \ldots, V_n \rangle_Y \simeq \langle V_n, V_1, \ldots, V_{n-1} \rangle_Y \tag{2.3.3}$$

obtained as composition

$$\langle Y, V_1, \ldots, V_n \rangle \to \langle V_n, Y, V_1, \ldots, V_{n-1} \rangle \to \langle Y, V_n, V_1, \ldots, V_{n-1} \rangle$$

(the first isomorphism is the cyclic isomorphism (2.2.3), the second one is the inverse of half-braiding φ_Y). Note however that in general we do not have $z_Y^n = \mathrm{id}$.

2.4 Hopf Algebras

Basic definitions

Throughout the paper, we denote by R a finite dimensional Hopf algebra over $\mathbb{C}$ with

- multiplication $\mu_R\colon R \otimes R \to R$,
- unit $\eta_R\colon \mathbb{C} \to R$
- comultiplication $\Delta_R\colon R \to R \otimes R$
- counit $\epsilon_R\colon R \to \mathbb{C}$
- antipode $S_R\colon R \to R$

We will drop the subscript R when there is no ambiguity.

We will use the Sweedler notation, writing $\Delta(x) = x' \otimes x''$, $\Delta^2(x) = x' \otimes x'' \otimes x'''$, etc.; summation will be implicit in these formulas. If the number of factors is large, we will also use the alternative notation writing $\Delta^{(n-1)}(x) = x^{(1)} \otimes x^{(2)} \otimes \cdots \otimes x^{(n)}$.

We will denote by R^* the dual Hopf algebra. We will use Greek letters $\alpha, \beta, \ldots$ for elements of R^*. We will also use the Sweedler notation for comultiplication in R^*, writing $\Delta_{R^*}(\alpha) = \alpha' \otimes \alpha''$; thus,

$$\langle \alpha' \otimes \alpha'', x_1 \otimes x_2 \rangle = \langle \alpha, x_1 x_2 \rangle \tag{2.4.1}$$

where $\langle\,\rangle$ stands for the canonical pairing $R^* \otimes R \to \mathbb{C}$.

From now on, we will also assume that R is semisimple. The following theorem shows that in fact this condition can be replaced by one of several equivalent conditions.

Theorem 2.4.1. *[3] Let R be a finite-dimensional Hopf algebra over a field of characteristic zero. Then the following are equivalent:*

1. *R is semisimple.*

2. *R^* is semisimple.*

3. *$S_R^2 = \mathrm{id}$.*

Haar integral

Let R be as described above. Then we have a distinguished element in R called the Haar integral which is defined by the following conditions:

1. $hx = xh = \epsilon_R(x)h \quad$ for all $x \in R$.

2. $h^2 = h$.

The following theorem lists important properties of the Haar integral.

Theorem 2.4.2. *Let R be a semisimple, finite-dimensional Hopf algebra. Then*

1. *h exists and is unique.*

2. *$S(h) = h$.*

3. For any n, the element

$$h_n = \Delta^{(n-1)}(h) \in R^{\otimes n}$$

is cyclically invariant. In particular, for $n = 2$, we have $\Delta^{op}(h) = \Delta(h)$.

Lemma 2.4.3. *The Haar integral acts by 1 in the trivial representation of R and by 0 in any irreducible non-trivial representation.*

Proof. This is an immediate consequence of the definition of Haar integral. □

Corollary 2.4.4. *Let $W_1, \dots, W_n$ be finite-dimensional representations of R. Then $h_n = \Delta^{n-1}(h)$ acts in $W_1 \otimes \cdots \otimes W_n$ by projection onto the invariant subspace*

$$\{w \in W_1 \otimes \cdots \otimes W_n \mid \Delta^{n-1}(x)w = \varepsilon(x)w\} \simeq \mathrm{Hom}_R(\mathbf{1}, W_1 \otimes \cdots \otimes W_n)$$

Representations

Since R is semisimple, any representation of R is completely reducible. We will denote by V_i, $i \in I$, a set of representatives of isomorphism classes of irreducible representations of R; we will also use the notation $d_i = \dim V_i$. In particular, the trivial one-dimensional representation of R will be denoted

$$\mathbf{1} = V_0.$$

We will frequently use the notation

$$\langle W_1, \dots, W_n \rangle = \mathrm{Hom}_R(\mathbf{1}, W_1 \otimes \cdots \otimes W_n) \simeq \{w \in W_1 \otimes \cdots \otimes W_n \mid \Delta^{n-1}(x)w = \varepsilon(x)w\}$$

(compare with Corollary 2.4.4).

Given a representation V of R, we can define the dual representation V^*. As a vector space V^* is the ordinary dual space to V. The action of R is defined by

$$\langle x\alpha, v\rangle = \langle \alpha, S(x)v\rangle \tag{2.4.2}$$

where $\alpha \in V^*, v \in V$, and $x \in R$. Note that since $S^2 = \mathrm{id}$, the usual vector space isomorphism $V^{**} \simeq V$ is an isomorphism of representations; thus, V^{**} is canonically isomorphic to V as a representation of R.

Since a dual of an irreducible representation is irreducible, we have an involution $\vee\colon I \to I$ such that $V_i^* \simeq V_{i^\vee}$. This isomorphism is not canonical; however, one does have a canonical isomorphism

$$V_i \otimes V_i^* \simeq V_{i^\vee}^* \otimes V_{i^\vee}. \tag{2.4.3}$$

For any two representations V, W of R, we denote by $\mathrm{Hom}_R(V,W)$ the space of R-morphisms from V to W. We have a non-degenerate pairing

$$\mathrm{Hom}_R(V,W) \otimes \mathrm{Hom}_R(V^*,W^*) \to \mathbb{C} \tag{2.4.4}$$

given by

$$\langle \varphi, \psi\rangle = (\mathbf{1} \to (V \otimes V^*) \xrightarrow{\varphi\otimes\psi} W \otimes W^* \to \mathbf{1})$$

By semisimplicity, we have a canonical isomorphism

$$R \cong \bigoplus_{i\in I} \mathrm{End}_{\mathbb{C}}(V_i) = \bigoplus_i V_i \otimes V_i^*. \tag{2.4.5}$$

It is convenient to write the Hopf algebra structure of R in terms of this isomorphism.

Lemma 2.4.5. *Under isomorphism* (4.4.1)*, multiplication, comultiplication, unit, counit, and antipode of R are given by*

- ***Multiplication**: $\mu_R \colon V_i \otimes V_i^* \otimes V_j \otimes V_j^* \xrightarrow{1\otimes ev \otimes 1} \delta_{i,j} V_i \otimes V_i^*$, where $ev \colon V_i^* \otimes V_i \to \mathbb{C}$ is the evaluation map.*

- ***Comultiplication**: Let φ_α be a basis for $\mathrm{Hom}_R(V_i, V_j \otimes V_k)$ and $\varphi^\alpha \in \mathrm{Hom}_R(V_i^*, V_k^* \otimes V_j^*)$ the dual basis with respect to the pairing given by* (2.4.4)*. Then*

$$\Delta_R \colon \bigoplus_i V_i \otimes V_i^* \xrightarrow{\sum d_i \varphi_\alpha \otimes \varphi^\alpha} \bigoplus_{j,k} V_j \otimes V_k \otimes V_k^* \otimes V_j^*$$
$$\xrightarrow{\tau_{23} \circ \tau_{34}} \bigoplus_{j,k} V_j \otimes V_j^* \otimes V_k \otimes V_k^*$$

- ***Unit**: $\eta_R = \bigoplus_i \sum_{j=1}^{dim V_i} v_j \otimes v_j^* = \sum_i \mathrm{id}_{V_i}$, where $\{v_j\}$ is a basis for V_i and $\{v_j^*\}$ is the dual basis.*

- ***Counit**: For $a \otimes b \in V_i \otimes V_i^*$, $\epsilon_R(a \otimes b) = \delta_{i,0} b(a)$*

- ***Antipode**: If $a \otimes b \in V_i \otimes V_i^*$, then*

$$S(a \otimes b) = b \otimes a \in V_i^* \otimes V_i \simeq V_{i^\vee} \otimes V_{i^\vee}^*$$

  (*see* (2.4.3)).

In this language, the Haar integral is given by the canonical element $1 \in V_0 \otimes V_0^*$.

Graphical calculus

We will frequently use graphical presentation of morphisms between representations of R. We will use slightly different conventions from those in Section 2.1 , representing a morphism $\varphi\colon W_1 \otimes \dots W_k \to W_1' \otimes \dots \otimes W_l'$ by a tangle with k strands labeled $W_1, \dots, W_k$ at the top and l strands labeled $W_1', \dots, W_l'$ at the bottom (here, morphisms are graphically represented top to bottom; we are using the aptly-named *pessimistic* convention). We will also use the usual cap and cup tangles to represent evaluation and coevaluation morphisms.

Dual Hopf algebra

Given a semisimple Hopf algebra R, we will define the following version of the dual Hopf algebra

$$\overline{R} = (R^{op})^* \tag{2.4.6}$$

where R^{op} denotes the algebra R with opposite multiplication.

Since comultiplication in $\overline{R}$ is opposite to comultiplication in R^*, notation α', α'' is ambiguous. We adopt the following convention: notation α', α'' (or, equivalently, $\alpha^{(1)}$, $\alpha^{(2)}$,...) always refers to comultiplication in R^*. Thus, comultiplication in $\overline{R}$ is given by

$$\Delta_{\overline{R}}(\alpha) = \alpha'' \otimes \alpha'.$$

Note that $\overline{\overline{R}}$ is canonically isomorphic to R as a vector space but as a Hopf algebra, has opposite multiplication and comultiplication. Thus, the map

$$S\colon R \to \overline{\overline{R}}$$

is an isomorphism of Hopf algebras.

Note that by Theorem 2.4.1, $\overline{R}$ is also semisimple and thus has a unique Haar integral. We will denote by

$$\bar{h} \in \overline{R} \tag{2.4.7}$$

the Haar integral of $\overline{R}$.

Lemma 2.4.6. *Let R be a semisimple Hopf algebra. Then the Haar integral of $\overline{R}$ is given by*

$$\langle \bar{h}, x\rangle = \frac{1}{\dim R} \operatorname{tr}_R(x)$$

where $\operatorname{tr}_R(x) = \sum d_i \operatorname{tr}_{V_i}(x)$ is the trace of action of x in the (left or right) regular representation.

We can also rewrite the Haar integral of $\overline{R}$ in terms of the isomorphism $R \simeq \bigoplus V_i \otimes V_i^*$ (see (4.4.1)).

Lemma 2.4.7. *Let $x_k = v_k \otimes w_k \in V_{i_k} \otimes V_{i_k}^*$; using isomorphism (4.4.1), each x_k can be considered as element of R. Then*

$$\langle \bar{h}, x_1 \dots x_n\rangle = \begin{cases} \frac{d_i}{\dim R}\langle v_1, w_n\rangle\langle w_1, v_2\rangle \dots \langle w_{n-1}, v_n\rangle, & i_1 = i_2 = \dots = i_n = i, \\ 0 & \textit{otherwise} \end{cases}$$

(*see Figure* 2.8).

$$\langle \bar{h}, (v_1 \otimes w_1) \cdots (v_n \otimes w_n) \rangle = \frac{d_i}{\dim R}$$

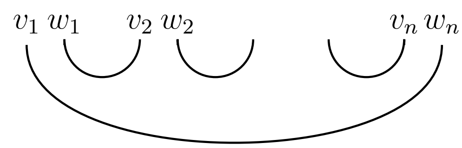


Figure 2.8: The Haar integral of $\overline{R}$

Regular action

With have two obvious actions of R on itself, called left and right regular actions:

- $L_x \colon y \mapsto xy$
- $R_x \colon y \mapsto= yS(x)$

Note that these actions commute.

In a similar way, the dual Hopf algebra $\overline{R}$ can also be endowed with two commuting actions of $\overline{R}$.

Using (2.4.2), we can also define two actions of R on $\overline{R}$:

- $L^*_x \colon \lambda \mapsto x.\lambda$, where $\langle x.\lambda, y \rangle = \langle \lambda, S(x)y \rangle$
- $R^*_x \colon \lambda \mapsto \lambda.S(x)$, where $\langle \lambda.x, y \rangle = \langle \lambda, yS(x) \rangle$

and two actions of $\overline{R}$ on on R:

- $L^*_\alpha \colon x \mapsto \alpha.x := \langle \alpha, S(x') \rangle x''$, so that $\langle \lambda, \alpha.x \rangle = \langle S(\alpha)\lambda, x \rangle$

- $R^*_\alpha\colon x \mapsto x.S(\alpha)\rangle = x'\langle \alpha, x''\rangle$, so that $\langle \lambda, x.\alpha\rangle = \langle \lambda S(\alpha), x\rangle$

Note that notation $x.\alpha$ is ambiguous, as it can mean $L^*_x(\alpha)$ or $R^*_{S(\alpha)}(x)$. In most cases the meaning will be clear from the context.

Note that we can also define left and right action of the Hopf algebra $\overline{\overline{R}}$ on $\overline{R}$. It is easy to see that these actions are given by

- $L_t\colon \lambda \mapsto \lambda.t$, where $t \in \overline{\overline{R}}$ is considered as an element of R via trivial vector space isomorphism.

- $R_t\colon \lambda \mapsto S(t).\lambda$, where $t \in \overline{\overline{R}}$ is considered as an element of R via trivial vector space isomorphism.

We will use these actions (together with left and right regular actions of R on itself) repeatedly throughout the rest of the paper. All the operators we will discuss can be defined in terms of them.

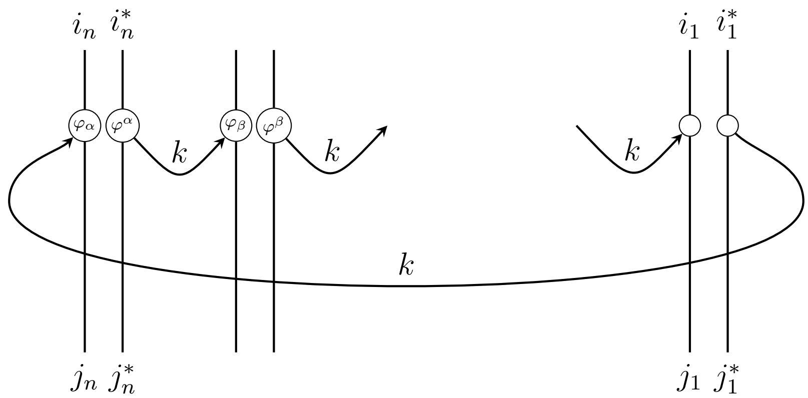
Lemma 2.4.8. *Let $\bar{h}$ be the Haar integral of $\overline{R}$. Consider the left regular action of $\bar{h}$ on $R^{\otimes n}$:*

$$\begin{aligned} L^*_{\bar{h}}\colon R^{\otimes n} &\to R^{\otimes n} \\ x_n \otimes \cdots \otimes x_1 &\mapsto \bar{h}^{(n)}.x_n \otimes \cdots \otimes \bar{h}^{(1)}x_1 \\ &= \langle \bar{h}, S(x'_n \ldots x'_1)\rangle x''_n \otimes \cdots \otimes x''_1 \quad = \langle \bar{h}, x'_n \ldots x'_1\rangle x''_n \otimes \cdots \otimes x''_1 \end{aligned} \tag{2.4.8}$$

(recall that comultiplication in $\overline{R}$ is given by $\Delta^{(n-1)}\alpha = \alpha^{(n)} \otimes \cdots \otimes \alpha^{(1)}$).

Then, after identifying each copy of R with $\bigoplus V_i \otimes V_i^*$ (see (4.4.1)), $L^*_{\bar{h}}$ is *given by the following picture:*

$$\sum_{i_1,\ldots,i_n,j_1,\ldots,j_n,k} \frac{d_{i_1}\ldots d_{i_n} d_k}{\dim R} \sum_{\alpha,\beta,\ldots}$$



where $\varphi_\alpha, \varphi^\alpha$ are as in Lemma 2.4.5.

Note that the crossings in the picture are just permutation of factors (there is no braiding in the category Rep R*) and the whole map is not a morphism of representations.*

Proof. Follows by combining the formula for multiplication and comultiplication in R (Lemma 2.4.5) and Lemma 2.4.7. □

2.5 Drinfeld Double

In this section, we discuss the *Drinfeld Double* of a Hopf algebra. This construction takes a Hopf algebra R and constructs a quasitriangular Hopf algebra $D(R)$. This is done by taking a twisted tensor product of R and $\overline{R}$. We will be concerned both with $D(R)$ and its dual. This can prove very confusing when writing expressions, so we stick to the following conventions:

1. We use letters from the Latin alphabet $(x, y, z, ...)$ to refer to elements from R.

2. Letters from the Greek alphabet $(\alpha, \beta, \gamma, ...)$ to refer to elements from R^*

3. We use Sweedler notation when taking coproducts, but always indexing by coproducts in R and R^*. For example, in $\overline{R}$, comultiplication is given be $\Delta(\alpha) = \alpha'' \otimes \alpha'$.

Theorem 2.5.1. *The following operations define on the vector space $R \otimes \overline{R}$ a structure of a Hopf algebra. This Hopf algebra will be denoted $D(R)$ and called the Drinfeld double of R.*

1. *Multiplication:*

$$(x \otimes \alpha) \cdot (y \otimes \beta) = xy'' \otimes \alpha^y \beta,$$

where $\alpha^y \in \overline{R}$ is defined by

$$\langle \alpha^y, z \rangle = \langle \alpha, y''' z S^{-1}(y') \rangle \tag{2.5.1}$$

2. *Unit:* $1_{D(R)} = 1_R \otimes 1_{\overline{R}}$

3. *Comultiplication:* $x \otimes \alpha \mapsto (x' \otimes \alpha'') \otimes (x'' \otimes \alpha')$

4. *Counit:* $x \otimes \alpha \mapsto \epsilon_R(x) \epsilon_{\overline{R}}(\alpha)$

5. *Antipode:* $S(x \otimes \alpha) = S(\alpha) S(x) = S(x'') S(\alpha)^{S(x)}$

The Hopf algebra $D(R)$ is actually quasitriangular, with R-matrix given by

$$R = \bigoplus_{\alpha} x_\alpha \otimes x^\alpha \tag{2.5.2}$$

The algebraic structure of $D(R)$ is obtained from a bicrossed product of R and $\overline{R}$ (27]).This leads to some messy formulas. Luckily, the haar integral is easy to define:

Lemma 2.5.2. *Let R be a semisimple, finite-dimensional Hopf algebra. Then $D(R)$ is semisimple with Haar integral given by*

$$h_{D(R)} = h_R \otimes h_{\overline{R}}. \tag{2.5.3}$$

Moreover, both $h, \bar{h}$ are central in $D(R)$.

We now define an action of $D(R)$ on R. This action is very important and will be used in Chapter 7.

Lemma 2.5.3. *For $a \in R$, $\alpha \in \overline{R}$, define the operators $p_a \colon R \otimes R \to R \otimes R$ and $q_\alpha \colon R \otimes R \to R \otimes R$ by*

$$p_a(u \otimes v) = a'u \otimes vS(a'')$$
$$q_\alpha(u \otimes v) = \langle \alpha, S(u'v') \rangle u'' \otimes v'' = \alpha''.u \otimes \alpha'.v.$$

Then these operators satisfy the commutation relations of $D(R)$: the map

$$D(R) \to \operatorname{End}(R \otimes R)$$
$$a \otimes \alpha \mapsto p_a q_\alpha$$

is a morphism of algebras.

Proof. This follows by explicit computation ([1]), using the following formulas:

$$\alpha.(xy) = x''(\alpha_x.y) = ({}_y\alpha.x)y'', \qquad \text{where}$$
$$\langle \alpha_x, z\rangle = \langle \alpha, zS(x')\rangle$$
$$\langle {}_y\alpha, z\rangle = \langle \alpha, S(y')z\rangle$$

□

Chapter 3

Turaev-Viro Theory as an Extended TQFT

3.1 Polytope decompositions

It will be convenient to rewrite the definition of Turaev–Viro (TV) invariants using not just triangulations, but more general cellular decompositions. In this section we give precise definitions of these decompositions.

In what follows, the word "manifold" denotes a compact, oriented, piecewise-linear (PL) manifold; unless otherwise specified, we assume that it has no boundary. Note that in dimensions 2 and 3, the category of PL manifolds is equivalent to the category of topological manifolds. For an oriented manifold M, we will denote by $\overline{M}$ the same manifold with opposite orientation, and by ∂M, the boundary of M with induced orientation.

Instead of triangulated manifolds as in 7], we prefer to consider more general cellular decompositions, allowing individual cells to be arbitrary polytopes (rather than just simplices); moreover, we will allow the attaching maps to identify some of the boundary points, for example gluing polytopes so that some of the vertices coincide. On the other hand, we do not want to consider arbitrary cell decompositions (as is done, say, in 40]), since it would make describing the elementary moves between two such decompositions more complicated. The following definition is the compromise; for lack of a better word, we will call such decompositions *polytope decompositions*.

Recall that a cellular decomposition of a manifold M is a collection of inclusion maps $B^d \to M$, where B^d is the (open) d-dimensional ball, satisfying certain conditions. Equivalently, we can replace d-dimensional balls with d-dimensional cubes $I^d = (0,1)^d$. For a PL manifold, we will call such a cellular decomposition a PL decomposition if each inclusion map $(0,1)^d \to M$ is a

PL map. In particular, every triangulation of a PL manifold gives such a cellular decomposition (each d-dimensional simplex is PL homeomorphic to a d-dimensional cube).

We will call a cell *regular* if the corresponding map $(0,1)^d \to M$ extends to a map of the closed cube $[0,1]^d \to M$ which is a homeomorphism onto its image.

Definition 3.1.1. A polytope decomposition of a 2- or 3-dimensional PL manifold M (possibly with boundary) is a cellular decomposition which can be obtained from a triangulation by a sequence of moves M1—M3 below (for $\dim M = 2$, only moves M1, M2).

M1: removing a vertex Let v be a vertex which has a neighborhood whose intersection with the 2-skeleton is homeomorphic to the "open book" shown below with $k \geq 1$ leaves; moreover, assume that all leaves in the figure are distinct 2-cells and the two 1-cells are also distinct (i.e., not two ends of the same edge). Then move M1 removes vertex v and replaces two 1-cells adjacent to it with a single 1-cell.

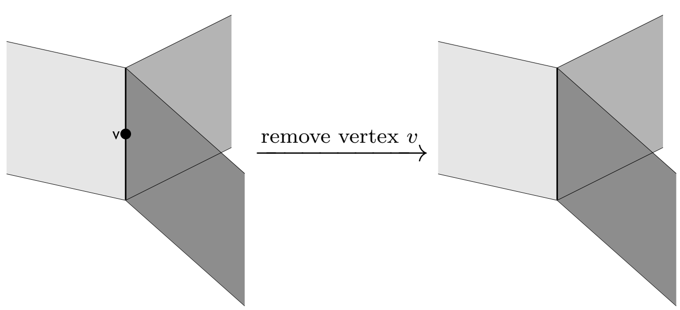


Figure 3.1: Move M1

M2: removing an edge Let e be a 1-cell which is regular and which is adjacent to exactly two distinct 2-cells c_1, c_2 as shown in the figure below. Then the move M2 removes the edge e and replaces the cells c_1, c_2 with a single cell c.

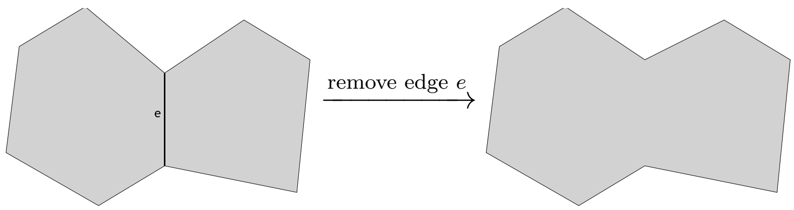


Figure 3.2: Move M2

M3: removing a 2-cell Let c be a 2-cell which is regular and which is adjacent to exactly two distinct 3-cells F_1, F_2 as shown in the figure below. Then the move M2 removes the 2-cell c and replaces the cells F_1, F_2 with a single cell F.

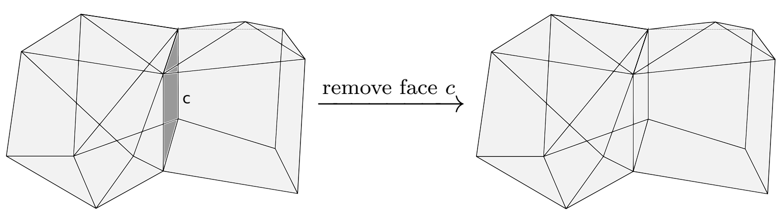


Figure 3.3: Move M3

A 2 or 3-dimensional PL manifold M with boundary together with a choice of polytope decomposition will be called a *combinatorial manifold*; for $\dim M = 2$, we will also use the term "combinatorial surface". We will use script letters to denote combinatorial manifolds and Roman letters for underlying PL manifolds.

Note that the extension of the inclusion maps $(0,1)^d \to M$ to the boundary does not have to be injective.

If F is an oriented d-dimensional cell of a combinatorial manifold $\mathcal{M}$ (i.e., a pair consisting of a cell and its orientation), we can define its boundary ∂F in the obvious way, as a formal union of oriented $(d-1)$-dimensional cells. Note that ∂F can contain the same (unoriented) cell C more than once: for example, one could have $\partial F = \dots \cup C \cup \overline{C} \dots$.

Lemma 3.1.2. *If $\mathcal{M}$ is a combinatorial manifold of dimension d with boundary, then*

$$\bigcup_F \partial F = \Big(\bigcup_{C\in\partial M} C\Big) \cup \Big(\bigcup_{c_{in}} c'_{in} \cup c''_{in}\Big)$$

where F runs over the set of d-cells of M (each taken with induced orientation), C runs over the set of $(d-1)$-cells of ∂M (each taken with induced orientation), and c_{in} runs over the set of (unoriented) $(d-1)$-cells in the interior of M, with c', c'' denoting two possible orientations of c (so that $\overline{c'} = c''$).

The main result of this section is the following theorem.

Theorem 3.1.3. *Let M be a PL 2- or 3-manifold without boundary. Then any two polytope decompositions of M can be obtained from each other by a finite sequence of moves M1–M3 and their inverses (if $\dim M = 2$, only moves M1, M2 and their inverses).*

Proof. It is immediate from the definition that it suffices to prove that any two triangulations can be obtained one from another by a sequence of moves M1–M3 and their inverses. On the other hand, since it is known that any two triangulations are related by a sequence of Pachner bistellar moves 42], it suffices to show that each Pachner bistellar move can be presented as a sequence of moves M1–M3 and their inverses. For $\dim M = 2$, this is left as an easy exercise to the reader; for $\dim M = 3$, this is shown in Figure 3.4, Figure 3.5.

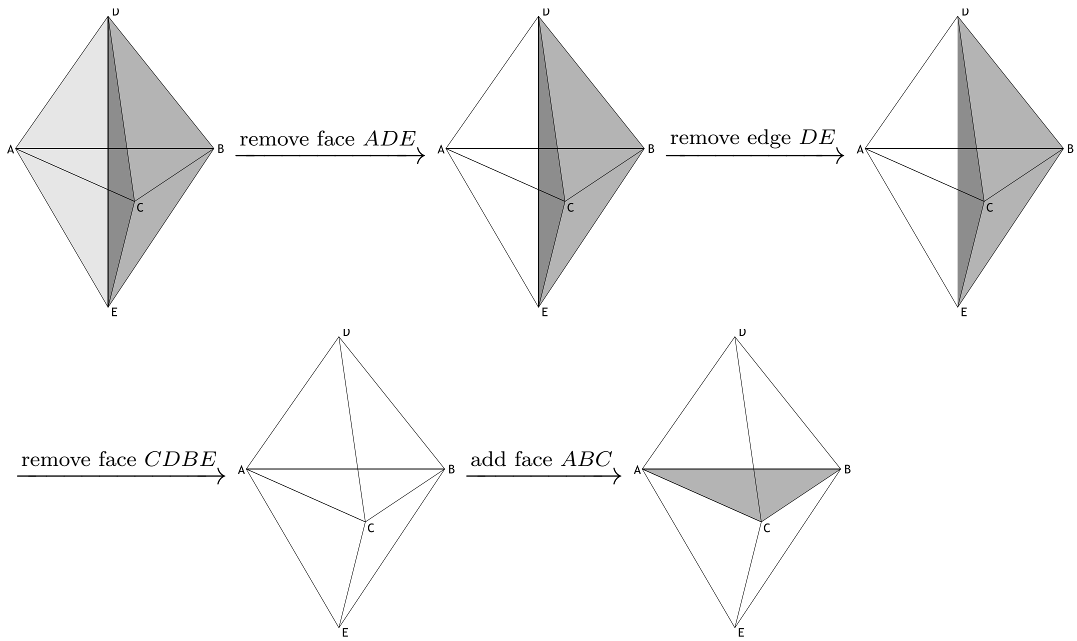


Figure 3.4: Pachner 3-2 move as composition of elementary moves

□

This can be generalized to manifolds with boundary.

Theorem 3.1.4. *Let M be a PL 2- or 3-manifold with boundary and let $\mathcal{N}$ be a polytope decomposition of ∂M. Then*

1. *$\mathcal{N}$ can be extended to a polytope decomposition $\mathcal{M}$ of M.*
2. *Any two polytope decompositions $\mathcal{M}_1, \mathcal{M}_2$ of M which coincide with $\mathcal{N}$ on ∂M can be obtained from each other by a finite sequence of moves M1–*

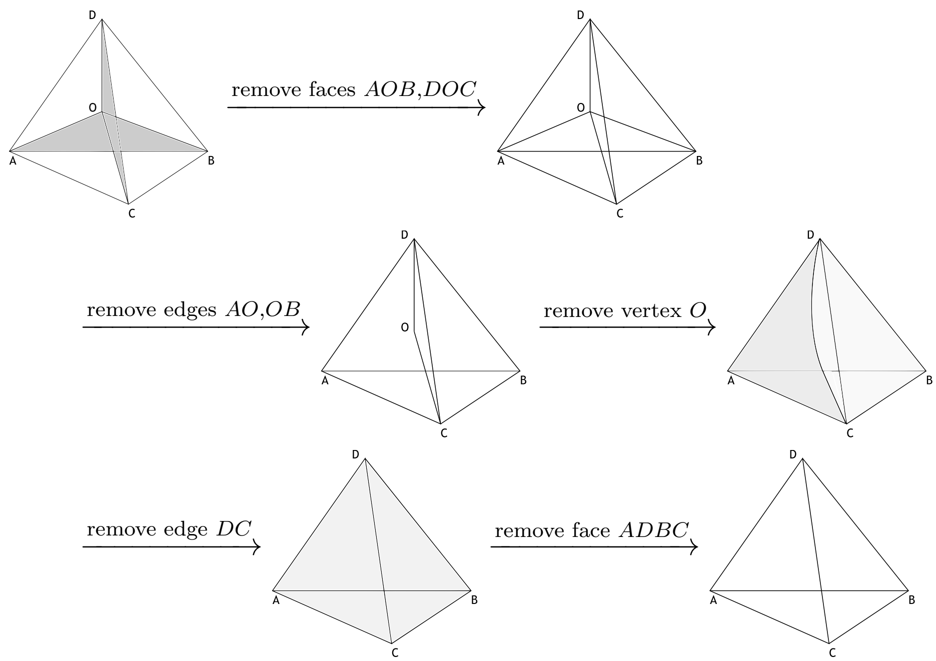


Figure 3.5: Pachner 4-1 move as composition of elementary moves

M3 and their inverses which do not change the polytope decomposition of ∂M.

Proof. The theorem immediately follows from the following two lemmas.

Lemma 3.1.5. *If $\mathcal{N}$ is a triangulation, then the statement of the theorem holds.*

Lemma 3.1.6. *If $\mathcal{N}$ is obtained from another polytope decomposition $\mathcal{N}'$ of ∂M by a move M1, M2 (only M1 if $\dim M = 2$), and the statement of the theorem holds for $\mathcal{N}'$, then the statement of the theorem holds for $\mathcal{N}$.*

Proof of Lemma 3.1.5. Follows from the relative version of Pachner moves 12].

□

Proof of Lemma 3.1.6. We will do the proof in the case when $\dim M = 3$ and $\mathcal{N}$ is obtained from $\mathcal{N}'$ by erasing an edge e separating two cells c_1, c_2. The proof in other cases is similar and left to the reader.

Let $\mathcal{M}'$ be a polytope decomposition of the M which agrees with $\mathcal{N}'$ on ∂M; by assumption such a decomposition exists. Denote $c = c_1 \cup e \cup c_2$. Let us glue to $\mathcal{M}'$ another copy of 2-cell c along the boundary of $c_1 \cup e \cup c_2$ and a 3-cell F filling the space between $c_1 \cup e \cup c_2$ and c as shown in Figure 3.6

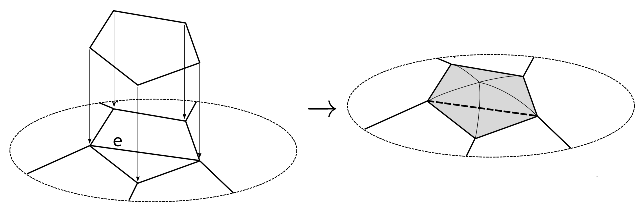


Figure 3.6: Proof of Lemma 3.1.6

This gives a new manifold $\tilde{M}$ which is obviously homeomorphic to M, together with a polytope decomposition $\tilde{M}$ such that its restriction to the boundary is $\mathcal{N}$. This proves existence of extension. Moreover, it is immediate from the assumption on $\mathcal{N}'$ that any two polytope decompositions $\tilde{M}_1$, $\tilde{M}_2$ obtained in this way from polytope decomposition $\mathcal{M}'_1, \mathcal{M}'_2$ extending $\mathcal{N}'$ can be obtained from each other by a sequence of moves M1, M2 and their inverses which do not change decomposition of ∂M.

To prove the second part, let $\mathcal{M}_1$, $\mathcal{M}_2$ be two polytope decompositions which coincide with $\mathcal{N}$ on ∂M. Let us add 2-cells c_1, c_2 and an edge e to to each of these decomposition as shown in Figure 3.7; this gives new decompositions $\tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2$ which are of of the form discussed above and thus can be obtained from each other by a sequence of moves M1, M2 and their inverses which do not change decomposition of ∂M.

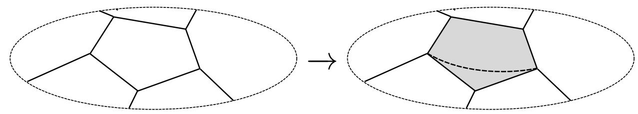


Figure 3.7: Proof of Lemma 3.1.6

□

□

Finally, we will need a slight generalization of this result.

Theorem 3.1.7. *Let M be a 3-manifold with boundary and let $X \subset \partial M$ be a subset homeomorphic to a 2-manifold with boundary. Let $\mathcal{N}$ be a polytope decomposition of a X. Then*

1. *$\mathcal{N}$ can be extended to a polytope decomposition $\mathcal{M}$ of M*

2. *Any two polytope decompositions $\mathcal{M}_1, \mathcal{M}_2$ of M which coincide with $\mathcal{N}$ on X can be obtained from each other by a finite sequence of moves M1–M3 and their inverses which do not change the polytope decomposition of X.*

A proof is similar to the proof of the previous theorem; details are left to the reader.

3.2 TV invariants from polytope decompositions

In this section, we recall the definition of Turaev–Viro (TV) invariants of 3-manifolds. Our exposition essentially follows the approach of Barrett and Westbury 7]; however, instead of triangulations we use more general polytope decompositions as defined in the previous section.

Let $\mathcal{C}$ be a spherical fusion category, and $\mathcal{M}$ — a combinatorial 3-manifold. We denote by E the set of oriented edges (1-cells) of $\mathcal{M}$. Note that each 1-cell of $\mathcal{M}$ gives rise to two oriented edges, with opposite orientations.

Definition 3.2.1. An *labeling* of $\mathcal{M}$ is a map $l\colon E \to \operatorname{Obj}\mathcal{C}$ which assigns to every oriented edge e of $\mathcal{M}$ an object $l(e) \in \operatorname{Obj}\mathcal{C}$ such that $l(\bar{e}) = l(e)^*$. A labeling is called simple if for every edge, $l(e)$ is simple.

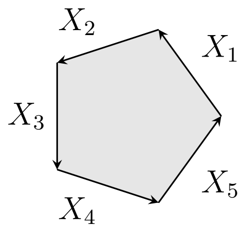
Two labelings are called equivalent if $l_1(e) \simeq l_2(e)$ for every e.

Given a combinatorial 3-manifold $\mathcal{M}$ and a labeling l, we define, for every oriented 2-cell C, the state space

$$H(C, l) = \langle l(e_1), l(e_2), \dots, l(e_n) \rangle, \qquad \partial C = e_1 \cup e_2 \dots \cup e_n \tag{3.2.1}$$

where the edges $e_1, \dots, e_n$ are taken in the counterclockwise order on ∂C as shown in Figure 3.8.

Note that by (2.2.3), up to a canonical isomorphism, the state space only depends on the cyclic order of $e_1, \dots, e_n$ (which is defined by C) and does not depend on the choice of the starting point.



$$H(C,l) = \langle X_1, X_2, \dots, X_5 \rangle$$

Figure 3.8: Defining the state space for a 2-cell

If $\mathcal{N}$ is an oriented 2-dimensional combinatorial manifold, we define the state space

$$H(\mathcal{N}, l) = \bigotimes_C H(C, l)$$

where the product is over all 2-cells C, each taken with orientation induced from orientation of $\mathcal{N}$.

Finally, we define

$$H(\mathcal{N}) = \bigoplus_l H(\mathcal{N}, l), \tag{3.2.2}$$

where the sum is over all simple labelings up to equivalence.

In the case when $\mathcal{N}$ is a triangulated surface, this definition coincides with the one in 7].

Note that it is immediate from (2.2.7) that we have canonical isomorphism

$$H(\overline{\mathcal{N}}) = H(\mathcal{N})^*. \tag{3.2.3}$$

Next, we define the TV invariant of 3-manifolds. Let $\mathcal{M}$ be a combinatorial 3-manifold with boundary. Fix a labeling l of edges of $\mathcal{M}$. Then every 3-cell F defines a vector

$$Z(F, l) \in H(\partial F, l)$$

defined as follows. Recall that F is an inclusion $F\colon (0,1)^3 \to M$. The pull-back of the polytope decomposition of $\mathcal{M}$ gives a polytope decomposition of $\partial(0,1)^3 \simeq S^2$. Consider the dual graph Γ of this decomposition and choose an orientation for every edge of this dual graph (arbitrarily) as shown in Figure 3.9.

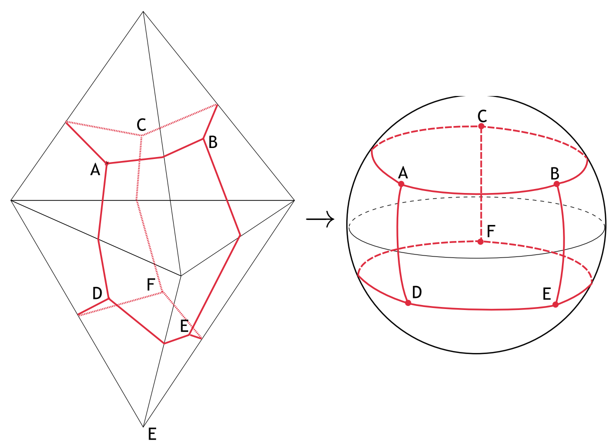


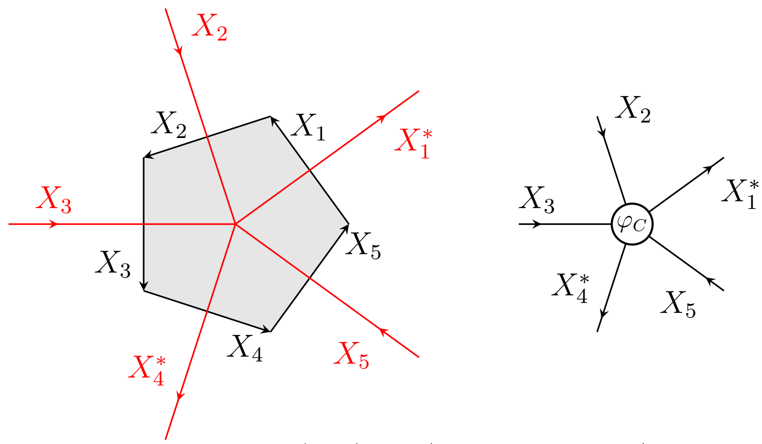
Figure 3.9: The dual graph on the boundary of a 3-cell

Note that a labeling l of $\mathcal{M}$ defines a labeling of edges of this dual graph as shown in Figure 3.10. Moreover, choose, for every face $C \in \partial F$, an element $\varphi_C \in H(C,l)^* = \langle l(e_n)^*, \ldots, l(e_1)^* \rangle$. Then this collection of morphisms defines a coloring of vertices of Γ.

By Theorem 2.2.4, we get an invariant $Z_{RT}(\Gamma) \in \mathbf{k}$, which depends on the choice of labeling of edges l and on the choice of morphisms φ_C. We define $Z(F,l) \in \otimes_C H(C,l)$ by

$$(Z(F,l), \otimes\varphi_C) = Z_{RT}(\Gamma, l, \{\varphi_C\}). \qquad (3.2.4)$$

Again, if F is a tetrahedron, then this coincides with the definition in 7];



$$\varphi_C \in H(C,l)^* = \langle X_5^*, X_4^*, \dots, X_1^* \rangle$$

Figure 3.10: Coloring of the dual graph

if $\mathcal{C}$ is the category of representations of quantum $\mathfrak{sl}_2$, these numbers are the $6j$-symbols.

We can now give a definition of the TV invariants of combinatorial 3-manifolds.

Definition 3.2.2. Let $\mathcal{M}$ be a combinatorial 3-manifold with boundary and $\mathcal{C}$ – a spherical category. Then for any coloring l, define a vector

$$Z_{TV}(\mathcal{M}, l) \in H(\partial\mathcal{M}, l)$$

by

$$Z_{TV}(\mathcal{M}, l) = \mathrm{ev}\Big(\bigotimes_F Z(F, l)\Big)$$

where

- F runs over all 3-cells in M, each taken with the induced orientation, so that

$$\bigotimes_F Z(F,l) \in \bigotimes_F H(\partial F, l) = H(\partial\mathcal{M}, l) \otimes \bigotimes_c H(c', l) \otimes H(c'', l)$$

(compare with Lemma 3.1.2)

- c runs over all unoriented 2-cells in the interior of M, c', c'' are the two orientations of such a cell, so that $c' = \overline{c''}$.

- ev is the tensor product over all c of evaluation maps $H(c', l) \otimes H(c'', l) = H(c', l) \otimes H(c', l)^* \to \mathbf{k}$

Finally, we define

$$Z_{TV}(\mathcal{M}) = \mathcal{D}^{-2v(\mathcal{M})} \sum_l \Big(Z_{TV}(\mathcal{M}, l) \prod_e d_{l(e)}^{n_e} \Big)$$

where

- the sum is taken over all equivalence classes of simple labelings of $\mathcal{M}$,

- e runs over the set of all (unoriented) edges of $\mathcal{M}$

- $\mathcal{D}$ is the dimension of the category $\mathcal{C}$ (see (2.2.1)), and

  $$v(\mathcal{M}) = \text{number of internal vertices of } \mathcal{M} + \frac{1}{2}(\text{number of vertices on } \partial\mathcal{M})$$

- $d_{l(e)}$ is the categorical dimension of $l(e)$ and

$$n_e = \begin{cases} 1, & e \text{ is an internal edge} \\ \frac{1}{2}, & e \in \partial\mathcal{M} \end{cases}$$

It is easy to see that in the special case of triangulated manifold, this coincides with the construction in 7].

Theorem 3.2.3. *If M is a PL manifold without boundary, then the number $Z_{TV}(M) \in \mathbf{k}$ defined in Definition 3.2.2 does not depend on the choice of polytope decomposition of M: for any two choices of polytope decomposition, the resulting invariants are equal.*

The proof of this theorem will be given in Section 3.3.

These invariants can be extended to a TQFT. Namely, let $\mathcal{M}$ be a combinatorial 3-cobordism between two 2-dimensional combinatorial manifolds $\mathcal{N}_1, \mathcal{N}_2$, i.e a combinatorial manifold $\mathcal{M}$ with boundary such that $\partial\mathcal{M} = \overline{\mathcal{N}_1} \sqcup \mathcal{N}_2$ (note that the combinatorial structure on M automatically defines a combinatorial structure on ∂M). Then $H(\partial\mathcal{M}) = H(\mathcal{N}_1)^* \otimes H(\mathcal{N}_2) = \mathrm{Hom}_{\mathbf{k}}(H(\mathcal{N}_1), H(\mathcal{N}_2))$, so Definition 3.2.2 defines an element $Z(\mathcal{M}) \in \mathrm{Hom}_{\mathbf{k}}(H(\mathcal{N}_1), H(\mathcal{N}_2))$, i.e. a linear operator

$$Z(\mathcal{M})\colon H(\mathcal{N}_1) \to H(\mathcal{N}_2).$$

Theorem 3.2.4.

1. *So defined invariant satisfies the gluing axiom: if $\mathcal{M}$ is a combinatorial 3-manifold with boundary $\partial\mathcal{M} = \mathcal{N}_0 \cup \mathcal{N} \cup \overline{\mathcal{N}}$, and $\mathcal{M}'$ is the manifold obtained by identifying boundary components $\mathcal{N}, \overline{\mathcal{N}}$ of $\partial\mathcal{M}$ with the*

obvious cell decomposition, then we have

$$Z_{TV}(\mathcal{M}') = \mathrm{ev}_{H(\mathcal{N})}\, Z_{TV}(\mathcal{M}) = \sum_{\alpha} (Z_{TV}(\mathcal{M}), \varphi_\alpha \otimes \varphi^\alpha),$$

where ev *is the evaluation map* $H(\mathcal{N}) \otimes H(\overline{\mathcal{N}}) \to \mathbf{k}$*, and* $\varphi_\alpha \in H(\mathcal{N})$*,* $\varphi^\alpha \in H(\overline{\mathcal{N}})$ *are dual bases.*

2. *If a* M *is a 3-manifold with boundary, and* $\mathcal{M}', \mathcal{M}''$ *are two polytope decompositions of* M *which agree on the boundary, then* $Z(\mathcal{M}') = Z(\mathcal{M}'') \in H(\partial\mathcal{M}') = H(\partial\mathcal{M}'')$*.*

3. *For a combinatorial 2-manifold* $\mathcal{N}$*, define* $A_{\mathcal{N}} \colon H(\mathcal{N}) \to H(\mathcal{N})$ *by*

$$A_{\mathcal{N}} = Z_{TV}(\mathcal{N} \times I) \tag{3.2.5}$$

Then $A_{\mathcal{N}}$ *is a projector:* $A_{\mathcal{N}}^2 = A_{\mathcal{N}}$*.*

4. *For a combinatorial 2-manifold* $\mathcal{N}$*, define the vector space*

$$Z_{TV}(\mathcal{N}) = \mathrm{Im}(A_{\mathcal{N}} \colon H(\mathcal{N}) \to H(\mathcal{N})) \tag{3.2.6}$$

where A *is the projector* (3.2.5)*. Then the space* $Z_{RT}(N)$ *is an invariant of PL manifolds: if* $\mathcal{N}', \mathcal{N}''$ *are two different polytope decompositions of the same PL manifold* N*, then one has a canonical isomorphism* $Z(\mathcal{N}') \simeq Z(\mathcal{N}'')$*.*

5. *The assignments* $N \mapsto Z_{TV}(N)$*,* $M \mapsto Z_{TV}(M)$ *give a functor from the category of PL 3-cobordisms to the category of finite-dimensional vector*

spaces and thus define a $2+1$*-dimensional TQFT.*

Proof. Part (1) is immediate from the definition.

Part (2) will be proved in Section 3.3.

To prove part (3), note that gluing of two cylinders again gives a cylinder, so (3) follows from (1) and (2).

To prove (4), let $\mathcal{N}', \mathcal{N}''$ be two different polytope decompositions of N. Consider the cylinder $C = N \times I$ and choose a polytope decomposition of C which agrees with $\mathcal{N}'$ on $N \times \{0\}$ and agrees with $\mathcal{N}''$ on $N \times \{1\}$ (existence of such a decomposition follows from Theorem 3.1.4). Consider the corresponding operator $F_1 = Z(C)\colon H(\mathcal{N}') \to H(\mathcal{N}'')$. In a similar way, define an operator $F_2\colon H(\mathcal{N}'') \to H(\mathcal{N}')$. Then it follows from (2) that $F_1F_2 = A_{\mathcal{N}''}$, and $F_2F_1 = A_{\mathcal{N}'}$. Thus, F_1, F_2 give rise to mutually inverse isomorphisms $Z_{TV}(\mathcal{N}') \to Z_{TV}(\mathcal{N}'')$.

Part (5) follows immediately from (1)–(4).

□

Note that in the PL category, gluing along a boundary component is well defined: gluing together PL manifolds results canonically in a PL manifodl (unlike the smooth category).

Example 3.2.5. Let G be a finite group and $\mathcal{C} = \mathcal{V}ec^G$ — the category of G-graded vector spaces, with obvious tensor structure. Then a simple labeling is just labeling of edges of $\mathcal{M}$ with elements of the group G, and for a 2-cell C, we have

$$H(C,l) = \begin{cases} \mathbf{k}, & \prod_{\partial C} l(e) = 1 \\ 0, & \text{otherwise} \end{cases}$$

Thus, we see that in this case the state space $H(\mathcal{N})$ is the space of flat G—connections (which depends on the choice of polytope decomposition!). It is well-known that in this case the projector $A = Z_{TV}(\Sigma \times I)$ is the operator of averaging over the action of the gauge group $G^{v(\mathcal{N})}$, where $v(\mathcal{N})$ is the set of vertices of $\mathcal{N}$. Thus the space $Z(N)$ is the space of gauge equivalence classes of G–connections.

Example 3.2.6. We verify $Z_{TV}(S^2) = \mathbf{k}$ as is required by the definition of a TQFT. We pick the polytope decomposition of S^2 consisting of one vertex, one edge and two faces as shown in Figure 3.11. Using the fact that for X_i, X_j

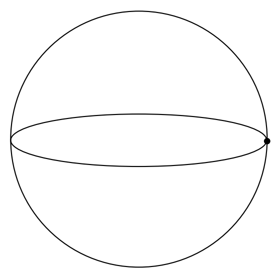


Figure 3.11: The polytope decomposition of S^2

simple $\mathrm{Hom}(X_i, X_j) = \delta_{ij}\mathbf{k}$, it is easy to see that $H(S^2) = \bigoplus_i \langle X_i \rangle \otimes \langle X_i^* \rangle = \mathbf{k}$. It remains to show that $A\colon H(S^2) \to H(S^2)$ is the identity map or equivalently, the induced map $H(S^2) \otimes H(S^2)^* \to \mathbf{k}$ equals the canonical pairing defined in Section **??**. Consider the cylinder $S^2 \times I$ with cell decomposition as in Figure 3.12. Note that both boundary edges must be labeled by $\mathbf{1}$. The

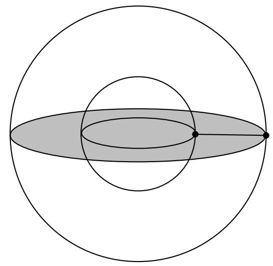
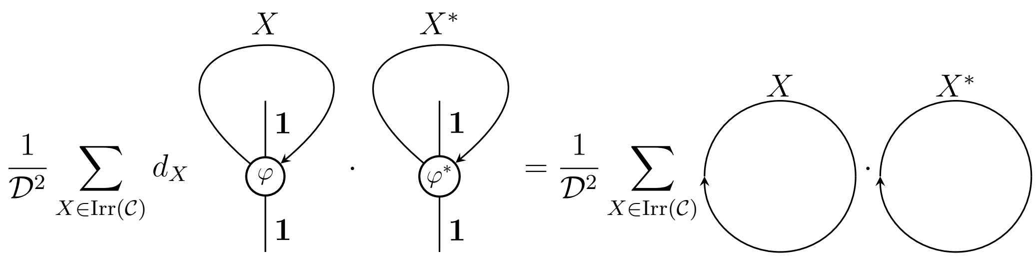


Figure 3.12: The cylinder over S^2

computation is then straightforward:



$$= \frac{1}{\mathcal{D}^2} \sum_{X \in \mathrm{Irr}(\mathcal{C})} d_X^2 = 1.$$

The first equality follows from the normalization of the pairing. The other two equalities are obvious.

3.3 Proof of independence of polytope decomposition

In this section, we give proofs of Theorem 7.2.1, Theorem 7.4.1, i.e. prove that TV invariants are independent of the choice of polytope decomposition. The proof is based on Theorem 3.1.3, Theorem 3.1.4, which state that any two decompositions can be obtained from one another by a sequence of moves

M1–M3 and their inverses.

First, we fix some notation. Unless otherwise stated, we denote simple objects in $\mathcal{C}$ by $X_i, X_j \dots$ and arbitrary objects by $A, B, \dots$. We let $N_1^{i_1 \dots i_k} = \dim(\langle X_{i_1}, \dots, X_{i_k} \rangle)$.

We will now show that the TV state sum is invariant under M1–M3.

Invariance under M1

First we consider move M1. Note that by applying M2 and M3, we can transform an open book with any number of pages to one with only one page (see Figure 3.13). Thus, it suffices to prove invariance under M1 in this special

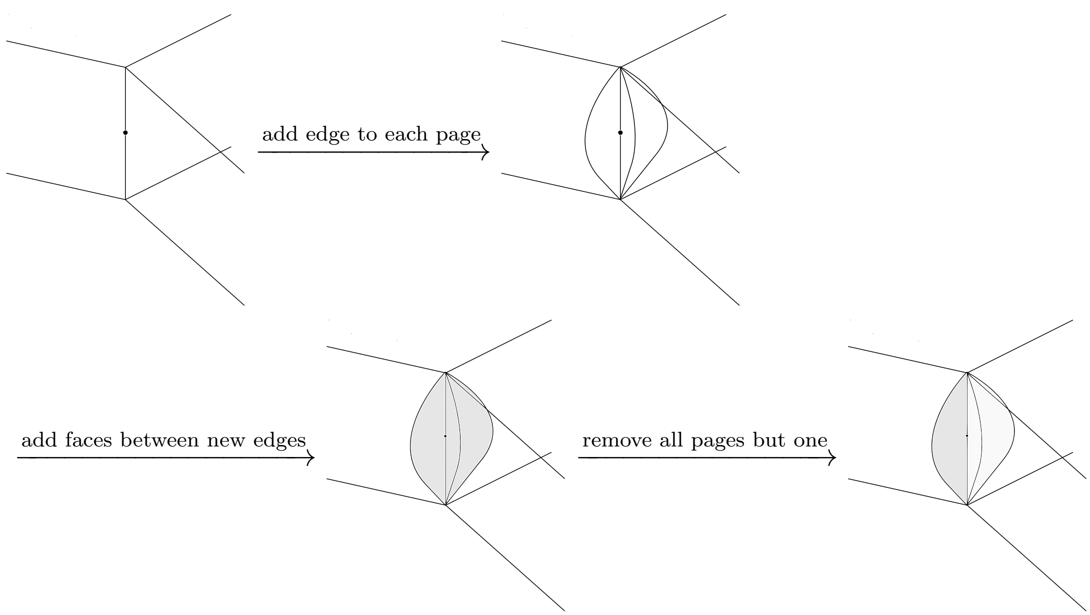
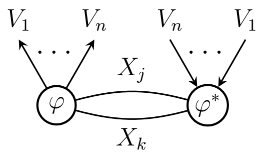
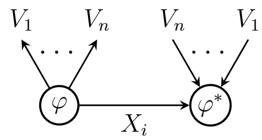


Figure 3.13: Decomposing an open book into a single page book

case. Drawing the dual graph in the vicinity of the vertex, invariance under

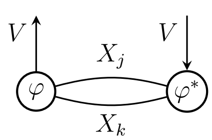
M1 is equivalent to the following equality:

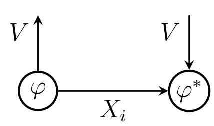
$$\frac{1}{\mathcal{D}^2}\sum_{j,k\in\mathrm{Irr}(\mathcal{C})} d_j d_k$$ 

$$= \sum_{i\in\mathrm{Irr}(\mathcal{C})} d_i$$ 

Note the normalizing factor $\frac{1}{\mathcal{D}^2}$ which comes from the fact that we are removing a vertex.

Using semisimplicity of $\mathcal{C}$, it is easy to see that it suffices to show this equality in the special case when $V = V_1 \otimes \cdots \otimes V_n$ is simple:

$$\frac{1}{\mathcal{D}^2}\sum_{j,k\in\mathrm{Irr}(\mathcal{C})} d_j d_k$$ 

$$= \sum_{i\in\mathrm{Irr}(\mathcal{C})} d_i$$ 

By Lemma 2.2.1, the right-hand side is equal to $\mathrm{coev}_V : \mathbf{1} \to V \otimes V^*$. Since $\mathrm{Hom}(\mathbf{1}, V \otimes V^*)$ is one-dimensional, the left-hand side is also a multiple of coev_V. Composing it with the evaluation morphism ev_V, we get

$$\begin{aligned}\frac{1}{\mathcal{D}^2}\sum_{j,k} d_j d_k N_1^{Vjk} &= \frac{1}{\mathcal{D}^2}\sum_{j,k} N_{k^*}^{Vj} d_k d_j \\ &= \frac{1}{\mathcal{D}^2}\sum_{j}\Big(\sum_{k} N_{k^*}^{Vj} d_k\Big) d_j = \frac{1}{\mathcal{D}^2}\sum_{j}(d_V d_j)d_j = d_V,\end{aligned}$$

which proves that the left-hand side is equal to coev_V.

Invariance under M2

The invariance under M2 is seen as follows. By definition, the edge being removed is incident to exactly two faces c_1, c_2. Each face bounds the same two 3-cells F_1, F_2. In Figure 3.14, we draw the dual graphs. In each of the summands we have two graphs corresponding to cells F_1, F_2, separated by a dot. The equality follows immediately from the fact that if $\varphi_\alpha, \varphi^\alpha$ and ψ_β, ψ^β are dual bases, then so are $\varphi_\alpha \underset{X_i}{\bullet} \psi_\beta$, $\psi^\beta \underset{X_i^*}{\bullet} \varphi^\alpha$ (cf. Corollary 2.2.2).

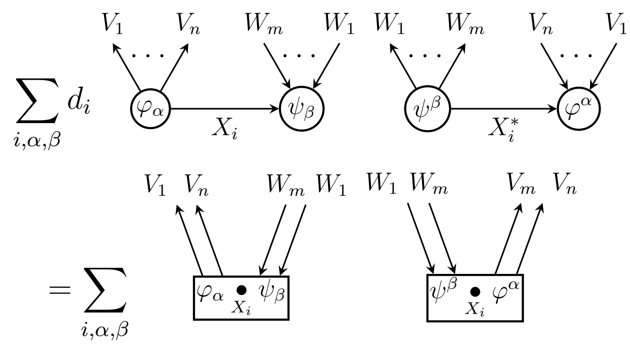


Figure 3.14:

Invariance under M3

Finally, we consider M3. In this case the invaraince immediately follows from Lemma 2.2.3, whith two subgraphs corresponding to two 3-cells separated by the 2-cell being removed.

3.4 Surfaces with boundary

In this section we extend the definition of TV TQFT to surfaces with boundary (and 3-manifolds with corners). Recall that according to general ideas of extended field theory (see 35]), an extended 3d TQFT should assign to a closed 1-manifold a 2-vector space, or an abelian category, and to a 2-cobordism between two 1-manifolds, a functor between corresponding categories (which in the special case of cobordism between two empty 1-manifolds gives a functor $\mathcal{V}ec \to \mathcal{V}ec$, i.e. a vector space). In this section we show that the extension of the TV TQFT to 1-manifolds assigns to a circle S^1 the category $Z(\mathcal{C})$—the Drinfeld center of the original spherical category $\mathcal{C}$. This result was proved by Turaev in the special case when the original category $\mathcal{C}$ is ribbon (see 45]); the general case has remained a conjecture.

For technical reasons, it is more convenient to replace surfaces with boundaries by surfaces with embedded disks. These two notions give equivalent theories: given a surface with boundary, we can glue a disk to every boundary circle and get a surface with embedded disks; conversely, given a surface with embedded disks, one can remove the disks to get a surface with boundary. Moreover, in order to accommodate real-life examples, we need to consider framing. This leads to the following definition.

We denote

$$D^2 = [0,1] \times [0,1]$$

and will call it *the standard disk* (it is, of course, a square, but this is what a disk looks like in PL setting). We will also the marked point P_0 on the boundary of D^2

$$P_0 = (0,1) \in \partial D^2$$

Definition 3.4.1. A framed embedded disk D in a PL surface N is the image of a PL map

$$\varphi \colon D^2 \to N$$

which is a homeomorphism with the image, together with the point $P = \varphi(P_0) \subset \partial D$.

An *extended surface* is a PL surface N together with a finite collection of disjoint framed embedded disks (see Figure 3.15). We will denote the set of embedded disks by $D(N)$.

A *coloring* of an extended surface is a choice of an object $Y_\alpha \in \operatorname{Obj} Z(\mathcal{C})$ for every embedded disk D_α.

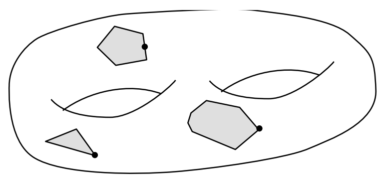


Figure 3.15: Extended surface

Next, we can define cobordisms between such surfaces. As usual, such a cobordism will be a 3-manifold with boundary together with some "tubes" inside which connect the embedded disks on the boundary of M. The following gives a precise definition in the PL category.

Definition 3.4.2. Let M be a PL 3-manifold with boundary.

An open *embedded tube* $T \subset M$ is the image of a PL map

$$\varphi\colon [0,1] \times D^2 \to M$$

which is satisfies the conditions below, together with the oriented arc $\gamma = \varphi([0,1] \times \{P_0\})$ (which we will call the *longitude*).

The map φ should satisfy:

1. φ is a homeomorphism onto its image

2. $T \cap \partial M = \varphi(\{0\} \times D^2) \cup \varphi(\{1\} \times D^2)$

We will call the disks $B_0 = \varphi(\{0\} \times D^2)$ and $B_1 = \varphi(\{1\} \times D^2)$ the *bottom* and *top* disks of the tube.

A closed embedded tube $T \subset M$ is the image of a PL map

$$\varphi\colon S^1 \times D^2 \to M$$

which is satisfies the conditions below, together with the oriented arc $\gamma = \varphi([0,1] \times \{P_0\})$ (the *longitude*) and the disk $B = \varphi(\{0\} \times D^2) \subset T$.

The map φ should satisfy:

1. φ is a homeomorphism onto its image

2. $T \cap \partial M = \varnothing$

The longitude γ determines the framing of the tube; the disk B is convenient for technical reasons; later we will get rid of it.

Definition 3.4.3. An extended 3-manifold M is an oriented PL 3-manifold with boundary together with a finite collection of disjoint framed tubes $T_i \subset M$. We denote the set of tubes of M by $T(M)$.

A coloring of an extended 3-manifold M is a choice of an object $Y_\alpha \in \operatorname{Obj} Z(\mathcal{C})$ for every tube T_α.

Note that if M is an extended 3-manifold, then its boundary ∂M has a natural structure of an extended surface: the embedded disks are the bottom and top disks of the open tubes, and the marked points on the boundary of

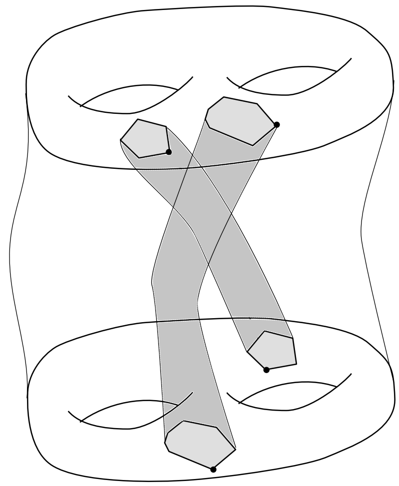


Figure 3.16: Extended 3-manifold

embedded disks are the endpoints of the longitude arcs γ_α, where α runs over the set of all open tubes in M. Moreover, a coloring of M defines a coloring of ∂M: if an open tube T_α is colored with $Y_\alpha \in \operatorname{Obj} Z(\mathcal{C})$, we color the embedded disk $\varphi_\alpha(\{1\} \times D^2)$ with Y_α and the embedded disk $\varphi_\alpha(\{0\} \times D^2)$ with Y_α^*.

Our main goal will be extending the TV invariants to such extended surfaces and cobordisms. Namely, we will

1. Define, for every colored extended surface N, the space $Z_{TV}(N, \{Y_\alpha\})$ which

   - functorially depends on colors Y_α
   - is functorial under homeomorphisms of extended surfaces
   - has natural isomorphisms $Z_{TV}(\overline{N}, \{Y_\alpha^*\}) = Z_{TV}(N, \{Y_\alpha\})^*$
   - satisfies the gluing axiom for surfaces

2. Define, for every colored extended 3-manifold M, a vector $Z_{TV}(M) \in Z_{TV}(\partial M)$ (or, equivalently, for any colored extended 3-cobordism M be-

tween colored extended surfaces N_1, N_2, a linear map $Z_{TV}(M)\colon Z_{TV}(N_1) \to Z_{TV}(N_2)$) so that this satisfies the gluing axiom for extended 3-manifolds.

In the next chapter, we will show that this extended theory actually coincides with the Reshetikhin–Turaev theory for the modular category $Z(\mathcal{C})$:

$$Z_{RT,Z(\mathcal{C})} = Z_{TV,\mathcal{C}}.$$

The construction of the theory proceeds similar to the construction of TV invariants. Namely, we will first define $Z_{TV}(N)$, $Z_{TV}(M)$ for manifolds with a polytope decomposition and then show that the so-defined objects are independent of the choice of a polytope decomposition and thus define an invariant of extended manifolds.

3.5 Extended combinatorial surfaces

We begin by generalizing the definition of a polytope decomposition to extended surfaces.

Definition 3.5.1. A combinatorial extended surface $\mathcal{N}$ is a an extended surface N together with a polytope decomposition such that

1. The interior of each embedded disk is one of the 2-cells of the polytope decomposition.

2. Each marked point P_α on the boundary of an embedded disk is a vertex (0-cell) of the polytope decomposition.

We can now define the state space for such a surface. Let $\mathcal{N}$ be a combinatorial extended surface, and $Y_\alpha, \alpha \in D(N)$, — a coloring of $\mathcal{N}$. Let l be a labeling of edges of $\mathcal{N}$. Then we define the state space

$$H(\mathcal{N}, \{Y_\alpha\}, l) = \bigotimes_C H(C, l)$$

where the product is over all 2-cells of $\mathcal{N}$ (including the embedded disks) and

$$H(C, l) = \begin{cases} \langle Y_\alpha, l(e_1), l(e_2), \dots, l(e_n)\rangle & C = D_\alpha \text{ – an embedded disk} \\ \langle l(e_1), l(e_2), \dots, l(e_n)\rangle & C \text{ – an ordinary 2-cell of } \mathcal{N} \end{cases}$$

where $e_1, e_2, \dots$ are edges of C traveled counterclockwise; for the embedded disks, we also require that we start with the marked point P_α; for ordinary 2-cells of $\mathcal{N}$ the choice of starting point is not important.

As usual, we now define

$$H(\mathcal{N}, \{Y_\alpha\}) = \bigoplus_l H(\mathcal{N}, \{Y_\alpha\}, l) \tag{3.5.1}$$

where the sum is taken over all equivalence classes of simple labellings l of edges of $\mathcal{N}$.

Note that so defined state space is functorial in Y_α and functorial under homeomorphism of extended surfaces; it is also immediate from the definition that one has a canonical isomorphism

$$H(\overline{\mathcal{N}}, Y_\alpha^*) = H(\mathcal{N}, Y_\alpha)^*.$$

Example 3.5.2. Let $\mathcal{N}$ be the sphere with n embedded disks and the cell decomposition shown in Figure 3.17. Then

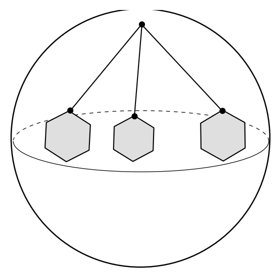


Figure 3.17: n-punctured sphere

$$
\begin{aligned}
&H(\mathcal{N}, Y_1, \dots, Y_n) \\
&\quad = \bigoplus_{X_1,\dots,X_n,U_1,\dots,U_n \in \mathrm{Irr}(\mathcal{C})} \langle X_1, U_1, X_1^*, \dots, X_n, U_n, X_n^* \rangle \otimes \langle U_1^*, Y_1 \rangle \otimes \cdots \otimes \langle U_n^*, Y_n \rangle \\
&\quad \simeq \bigoplus_{X_1,\dots,X_n \in \mathrm{Irr}(\mathcal{C})} \langle X_1, Y_1, X_1^*, \dots, X_n, Y_n, X_n^* \rangle.
\end{aligned}
$$

where the last isomorphism is given by direct sum of rescaled compositions (2.2.10).

The first main result of this chapter is the gluing axiom for the so defined state space.

Theorem 3.5.3. *Let $\mathcal{N}$ be a combinatorial extended surface and D_a, D_b — two distinct embedded disks. Let $\mathcal{N}'$ be the extended surface obtained by removing the disks D_a, D_b and connecting the resulting boundary circles with a cylinder with the polytope decomposition consisting of a single 2-cell and a single 1-cell as shown below:*

Thus, the set D' of embedded disks of $\mathcal{N}'$ is $D' = D(\mathcal{N}) \setminus \{a, b\}$

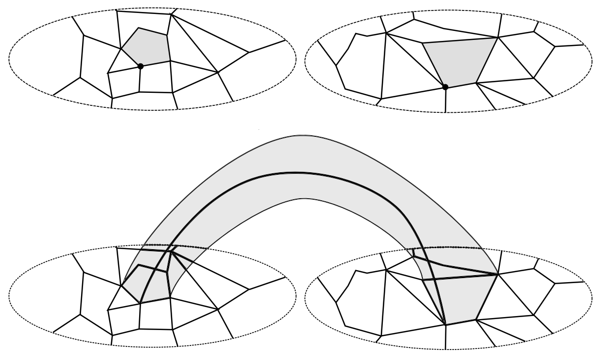


Figure 3.18: Gluing of extended surfaces. To help visualize the cylinder, it is colored light gray.

Then one has a natural isomorphism

$$H(\mathcal{N}', \{Y_\alpha\}_{\alpha\in D'}) = \bigoplus_{Z\in \mathrm{Irr}(Z(\mathcal{C}))} H(\mathcal{N}, \{Y_\alpha\}_{\alpha\in D'}, Z, Z^*)$$

where objects Z, Z^* are assigned to embedded disks D_a, D_b.

Proof. For a given labeling l of edges of $\mathcal{N}$, let

$$H_0(l) = \bigotimes_C H(C, l)$$

where the product is taken over all 2-cells of $\mathcal{N}$ (including the embedded disks) except D_a, D_b. Then

$$H(\mathcal{N}, \{Y_\alpha\}, Z, Z^*, l) = H_0(l) \otimes \langle Z, A\rangle \otimes \langle Z^*, B\rangle$$

where $A = l(e_1) \otimes l(e_2) \cdots \otimes l(e_n)$, where $e_1, e_2, \ldots$ are edges of D_a traveled counterclockwise starting with the marked point P_a, and similarly for B.

On the other hand, for a given labeling l' of edges of $\mathcal{N}'$, we have

$$H(\mathcal{N}', \{Y_\alpha\}, l') = H_0(l) \otimes \langle A \otimes l(e) \otimes B \otimes l(e)^* \rangle$$

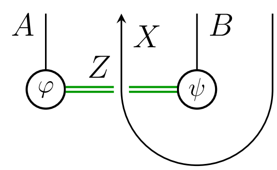
where l is the restriction of labeling l' to edges of $\mathcal{N}$, and e is the added edge connecting marked points P_a, P_b.

Thus, the theorem immediately follows from the following lemma.

Lemma 3.5.4. *For any $A, B \in \operatorname{Obj}\mathcal{C}$, the map*

$$\bigoplus_{Z \in \operatorname{Irr}(Z(\mathcal{C}))} \langle Z, A \rangle \otimes \langle Z^*, B \rangle \to \bigoplus_{X \in \operatorname{Irr}(\mathcal{C})} \langle A, X, B, X^* \rangle$$

$$\varphi \otimes \psi \mapsto \bigoplus_{X \in \operatorname{Irr}(\mathcal{C})} \frac{\sqrt{d_X}\sqrt{d_Z}}{\mathcal{D}}$$



(3.5.2)

is an an isomorphism.

(The factor $\sqrt{d_X}\sqrt{d_Z}/\mathcal{D}$ is introduced to make this isomorphism agree with pairing (2.4.4).)

Proof. By Theorem 2.3.3, we have

$$\bigoplus_{X \in \operatorname{Irr}(\mathcal{C})} \langle A \otimes X \otimes B \otimes X^* \rangle = \bigoplus_X \operatorname{Hom}_{\mathcal{C}}(A^*, X \otimes B \otimes X^*)$$

$$= \operatorname{Hom}_{\mathcal{C}}(A^*, FI(B)) = \operatorname{Hom}_{Z(\mathcal{C})}(I(A^*), I(B))$$

On the other hand,

$$\bigoplus_{Z \in \text{Irr}(Z(\mathcal{C}))} \langle Z, A \rangle \otimes \langle Z^*, B \rangle = \bigoplus_Z \text{Hom}_{\mathcal{C}}(Z^*, A) \otimes \text{Hom}_{\mathcal{C}}(Z, B)$$
$$= \bigoplus_Z \text{Hom}_{Z(\mathcal{C})}(Z^*, I(A)) \otimes \text{Hom}_{Z(\mathcal{C})}(Z, I(B))$$
$$= \bigoplus_Z \text{Hom}_{Z(\mathcal{C})}(I(A)^*, Z) \otimes \text{Hom}_{Z(\mathcal{C})}(Z, I(B))$$
$$= \text{Hom}_{Z(\mathcal{C})}(I(A)^*, I(B))$$

(using semisimplicity of $Z(\mathcal{C})$). □

This completes the proof of the lemma and thus the theorem. □

3.6 Invariants of extended 3-manifolds

We begin by generalizing the definition of a polytope decomposition to extended 3-manifolds as defined in Definition 3.4.3.

Definition 3.6.1. A combinatorial extended 3-manifold $\mathcal{M}$ is an extended PL 3-manifold with a polytope decomposition such that

- For an open tube T_α, its interior is a single 3-cell of the decomposition. Moreover, the interior of the "bottom disk" $B_0 = \varphi_\alpha(\{0\} \times D^2)$ is a single 2-cell of the decomposition, and the marked point P on the boundary of the bottom disk is a vertex of the decomposition, and similarly for the top disk $B_1 = \varphi_\alpha(\{1\} \times D^2)$.
- For a closed tube T_α, the interior of the disk $B_\alpha = \varphi_\alpha(\{0\} \times D^2)$ is a single 2-cell of the decomposition, the marked point $P_\alpha \in \partial B_\alpha$ is a vertex

of the decomposition, and the complement $\mathrm{Int}(T_\alpha) - B_\alpha$ is a single 3-cell of the decomposition.

Note that this implies that the restriction of such a polytope decomposition to the boundary of ∂M satisfies the conditions of Definition 3.5.1 and thus defines on ∂M the structure of a combinatorial extended surface. It also this implies that $\mathcal{M}$ contains two kinds of 3-cells: usual cells (which are not contained in any tube) and "tube cells", i.e. cells contained in one of the tubes. The boundary of a usual 3-cell is a union of usual 2-cells; the boundary of a 3-cell corresponding to an open tube contains usual 2-cells and two embedded disks; the boundary of a 3-cell corresponding to a closed tube contains usual 2-cells and two copies of the disk B_α with opposite orientation.

Finally, note that we have imposed no restriction on the longitude of the tube: it is allowed (and usually will) intersect the edges of the decomposition of the boundary tubes.

The following theorem is an analog of Theorem 3.1.4.

Theorem 3.6.2. *Let M be an extended 3-manifold. Then any two polytope decompositions $\mathcal{M}', \mathcal{M}''$ of $\mathcal{M}$ which satisfy the conditions of Definition* 3.6.1 *and agree on ∂M can be obtained from each other by a sequence of moves M1—M3 and their inverses such that all intermediate decompositions also satisfy the conditions of Definition* 3.6.1 *and agree with $\mathcal{M}', \mathcal{M}''$ on ∂M.*

Proof. Let us consider the manifold $\tilde{M}$ obtained by removing from M the interior of every tube and also the interior of the embedded disks on the boundary of M. Then $\tilde{M}$ is a manifold with boundary

$$\partial\tilde{M} = (\partial M - \cup\,\mathrm{Int}(D_\alpha)) \cup \partial\tilde{M}_{free}$$

where the "free boundary" $\partial\tilde{M}_{free}$ is the union of side surfaces $I \times \partial D^2$ of the tubes (for closed tubes, $S^1 \times \partial D^2$).

Obviously, polytope decompositions $\mathcal{M}', \mathcal{M}''$ satisfying the conditions of the theorem determine decomposition of $\tilde{M}$ which agree on the subset $X = (\partial M - \cup\,\mathrm{Int}(D_\alpha)) \subset \partial\tilde{M}$. Now the result follows from Theorem 3.1.7. □

Recall that for usual oriented 3-cell F and a choice of edge labeling l, we have defined the vector $Z_{TV}(F, l) \in H(\partial F, l)$ defined by (3.2.4). We can now generalize it to tube cells. Namely, let l be an edge coloring of an extended combinatorial 3-manifold $\mathcal{M}$ and let $T_\alpha \subset M$ be an open tube, with the longitude γ_α and color $Y_\alpha \in Z(\mathcal{C})$. Since T is homeomorphic to $[0, 1] \times D^2 \simeq D^3$— a 3-ball, the boundary ∂T is homeomorphic to S^2; thus, the polytope decomposition of T defines a polytope decomposition of S^2.

Let Γ be the dual graph of this cell decomposition. We can connect the marked points on the top and bottom disks to the vertex of the dual graph corresponding to these disks; together with the longitude γ, this gives an oriented arc on the surface of the sphere whose endpoints are two distinct vertices of Γ. For every 2-cell $C \in \partial F$ (including the embedded disks), choose a vector $v_C \in H(C, l)^*$. Thus, we get a graph $\hat{\Gamma}$ of the type considered in Section 2.3, i.e. colored graph Γ on the surface of the sphere together with a colored framed arc inside as shown in Figure 3.19. By Theorem 2.3.4 this

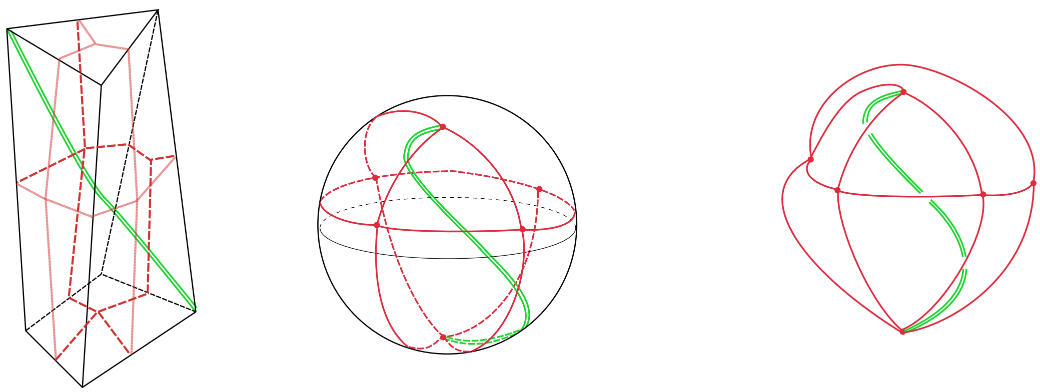


Figure 3.19: Dual graph for a tube cell. The longitude is shown by double green line.

defines a number $Z_{RT}(\hat{\Gamma})$; as before, we let

$$(Z(F,l), \otimes v_C) = Z_{RT}(\hat{\Gamma}). \tag{3.6.1}$$

In a similar way we define the invariant for closed tubes.

We can now generalize the constructions of Section 3.2 to extended 3-manifolds.

Definition 3.6.3. Let $\mathcal{M}$ be an extended combinatorial 3-manifold with boundary and $\mathcal{C}$ – a spherical category. Then for any edge coloring l and a coloring Y_α of the tubes $T_\alpha \subset \mathcal{M}$, define the vector

$$Z_{TV}(\mathcal{M}, \{Y_\alpha\}, l) \in H(\partial\mathcal{M}, \{Y_\alpha\}, l)$$

by

$$Z_{TV}(\mathcal{M}, l) = \mathrm{ev}\Big(\bigotimes_F Z(F,l)\Big)$$

where

- F runs over all 3-cells in M (including the tube cells), each taken with the induced orientation, so that

$$\bigotimes_F Z(F,l) \in \bigotimes_F H(\partial F, l) = H(\partial\mathcal{M}, l) \otimes \bigotimes_c H(c', l) \otimes H(c'', l)$$

(compare with Lemma 3.1.2)

- c runs over all unoriented 2-cells in the interior of M, including the disks B_α inside the closed tubes, and c', c'' are the two orientations of such a cell, so that $c' = \overline{c''}$.

- ev is the tensor product over all c of evaluation maps $H(c', l) \otimes H(c'', l) = H(c', l) \otimes H(c', l)^* \to \mathbf{k}$

Finally, we define

$$Z_{TV}(\mathcal{M}, \{Y_\alpha\}) = \mathcal{D}^{-2v(\mathcal{M})} \sum_l \Big(Z_{TV}(\mathcal{M}, \{Y_\alpha\}, l) \prod_e d_{l(e)}^{n_e} \Big) \tag{3.6.2}$$

where

- the sum is taken over all equivalence classes of simple labellings of $\mathcal{M}$,

- e runs over the set of all (unoriented) edges of $\mathcal{M}$

- $\mathcal{D}$ is the dimension of the category $\mathcal{C}$ (see (2.2.1)), and

$$v(\mathcal{M}) = \text{number of internal vertices of } \mathcal{M} + \frac{1}{2}(\text{number of vertices on } \partial\mathcal{M})$$

- $d_{l(e)}$ is the categorical dimension of $l(e)$ and

$$n_e = \begin{cases} 1, & e \text{ is an internal edge} \\ \frac{1}{2}, & e \in \partial\mathcal{M} \end{cases}$$

Note that in this definition, edges and vertices on the boundary of the tubes are considered internal unless they are also on $\partial\mathcal{M}$.

Theorem 3.6.4.

1. *$Z_{TV}(\mathcal{M})$ satisfies the gluing axiom: if $\mathcal{M}$ is an extended combinatorial 3-manifold with boundary $\partial\mathcal{M} = \mathcal{N}_0 \cup \mathcal{N} \cup \overline{\mathcal{N}}$, and $\mathcal{M}'$ is the manifold obtained by identifying boundary components $\mathcal{N}, \overline{\mathcal{N}}$ of $\partial\mathcal{M}$ with the obvious cell decomposition (if $\mathcal{N}$ contains embedded disks, then we may need to erase them so that the interior of resulting tubes have exactly one 3-cell), then we have*

$$Z_{TV}(\mathcal{M}') = \mathrm{ev}_{H(\mathcal{N})}\, Z_{TV}(\mathcal{M}) = \sum_{\alpha} (Z_{TV}(\mathcal{M}), \varphi_\alpha \otimes \varphi^\alpha),$$

   where ev *is the evaluation map $H(\mathcal{N}) \otimes H(\overline{\mathcal{N}}) \to \mathbf{k}$, and $\varphi_\alpha \in H(\mathcal{N})$, $\varphi^\alpha \in H(\overline{\mathcal{N}})$ are dual bases.*

2. *If a M is an extended PL 3-manifold, and $\mathcal{M}', \mathcal{M}''$ are two polytope decompositions of M which agree on the boundary, then $Z(\mathcal{M}', \{Y_\alpha\}) = Z(\mathcal{M}'', \{Y_\alpha\})$.*

3. For a combinatorial 2-manifold $\mathcal{N}$, define $A\colon H(\mathcal{N}) \to H(\mathcal{N})$ by

$$A = Z_{TV}(\mathcal{N} \times I) \tag{3.6.3}$$

Then A is a projector: $A^2 = A$.

4. For a combinatorial extended 2-manifold $\mathcal{N}$, define the vector space

$$Z_{TV}(\mathcal{N}) = \operatorname{Im}(A\colon H(\mathcal{N}) \to H(\mathcal{N})) \tag{3.6.4}$$

where A is the projector (3.6.3)*. Then the space $Z_{RT}(\mathcal{N})$ is an invariant of PL manifolds: if $\mathcal{N}', \mathcal{N}''$ are two different polytope decompositions of the same extended PL manifold N, then one has a canonical isomorphism $Z(\mathcal{N}') \simeq Z(\mathcal{N}'')$.*

Proof. The proof is parallel to the proof of Theorem 7.4.1. The only new ingredient is in the proof of part (1), i.e. the gluing axiom for 3-manifolds: if the component of boundary along which we are gluing contains embedded disks, we need to erase them so that in the resulting manifold, interior of each tube is exactly one 3-cell. Thus, we need to check that our that $Z(\mathcal{M})$ is unchanged under this operation. The proof of this is similar to invariance under M3 move proved in Section 3.3. Details are left to the reader.

□

Finally, we also note that our extended theory satisfies the gluing axiom for extended surfaces.

Theorem 3.6.5. *Under the assumptions of Theorem* 3.5.3*, one has a natural isomorphism*

$$Z(\mathcal{N}', \{Y_\alpha\}_{\alpha\in D'}) = \bigoplus_{Z\in \mathrm{Irr}(Z(\mathcal{C}))} Z(\mathcal{N}, \{Y_\alpha\}_{\alpha\in D'}, Z, Z^*)$$

where objects Z, Z^* are assigned to embedded disks a, b.

Proof. Recall that by Theorem 3.5.3, one has an isomorphism

$$G\colon \bigoplus_{Z\in \mathrm{Irr}(Z(\mathcal{C}))} H(\mathcal{N}, \{Y_\alpha\}_{\alpha\in D'}, Z, Z^*) \xrightarrow{\sim} H(\mathcal{N}', \{Y_\alpha\}_{\alpha\in D'})$$

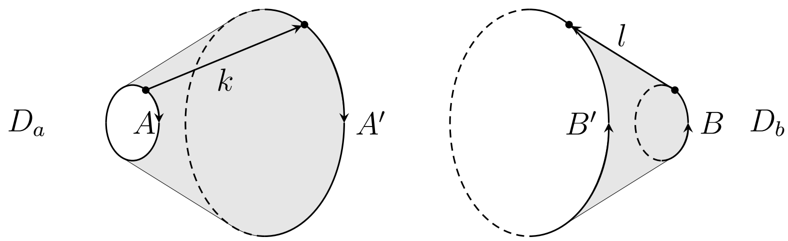
Since $Z(\mathcal{N})$ is defined as the image of the projector $A\colon H(\mathcal{N}) \to H(\mathcal{N})$, and similarly for $Z(\mathcal{N}')$, it suffices to prove that the following diagram is commutative:

$$\begin{array}{ccc} H(\mathcal{N}, Z, Z^*) & \xrightarrow{G} & H(\mathcal{N}') \\ A\downarrow & & \downarrow A' \\ H(\mathcal{N}, Z, Z^*) & \xrightarrow{G} & H(\mathcal{N}') \end{array}$$

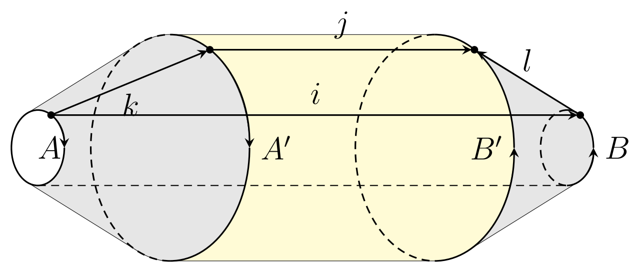
or equivalently, that for any $\varphi \in H(\mathcal{N}, Z, Z^*)$, $\varphi' \in H(\overline{\mathcal{N}}, Z, Z^*)$, we have

$$(Z(\mathcal{N} \times I, Z, Z^*), \varphi \otimes \varphi') = (Z(\mathcal{N}' \times I), G(\varphi) \otimes G(\varphi')).$$

Comparing both sides, we see that the only difference is that $\mathcal{N} \times I$ contains a pair of cylinders $D_a \times I, D_b \times I$:



whereas $\mathcal{N}' \times I$ contains instead a single cell $C \times I$, where C is the cylinder connecting boundary circles $\partial D_a, \partial D_b$:



Thus, it suffices to prove that for any collection

$$\varphi_a \in H(D_a) = \langle A, Z \rangle, \qquad \varphi_b \in H(D_b) = \langle B, Z^* \rangle,$$

$$\varphi'_a \in H(D_a)^* = \langle (A')^*, Z^* \rangle, \qquad \varphi_b \in H(D_b)^* = \langle (B')^*, Z \rangle,$$

$$\psi_a \in H(\partial D_a \times I) = \langle X_k, A', X_k^*, A^* \rangle, \qquad \psi_b \in H(\partial D_b \times I) = \langle X_l, B', X_l^*, B^* \rangle$$

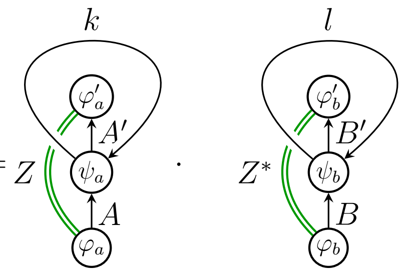
we have

$$\begin{aligned} &Z(D_a \times I, \varphi_a \otimes \varphi'_a \otimes \psi_a) \cdot Z(D_b \times I, \varphi_b \otimes \varphi'_b \otimes \psi_b) \\ &\quad = \sum_{i,j} \sqrt{d_i}\sqrt{d_j}\ Z(C \times I, \psi_a, \psi_b, G(\varphi_a \otimes \varphi_b), G(\varphi'_a \otimes \varphi'_b)) \end{aligned}$$

(the factors $\sqrt{d_i}$, $\sqrt{d_j}$ appear because $\mathcal{N}' \times I$ contains two extra edges on the boundary, labeled i, j.)

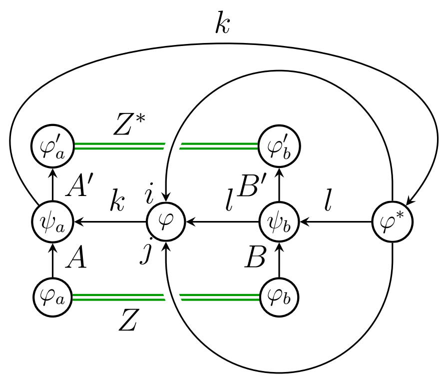
The left hand side is given by

$LHS =$

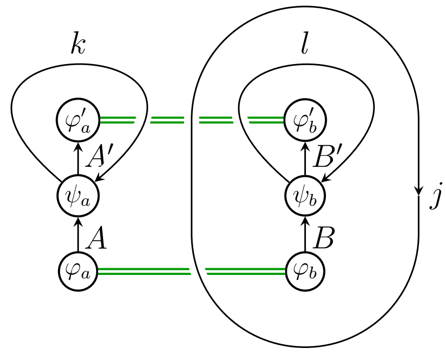


Combining explicit computation given in Section 3.7 with the formula for gluing in Lemma 3.5.4, we see that the right hand side is given by

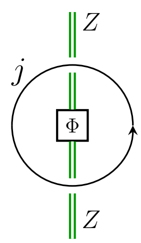
$RHS = \sum_{i,j} \frac{d_i d_j d_Z}{\mathcal{D}^2}$



Using Lemma 2.2.1, we can rewrite it as

$$RHS = \sum_j \frac{d_j d_Z}{\mathcal{D}^2}$$ 

Since it follows from Lemma 2.3.2 that for any simple $Z \in \operatorname{Obj} Z(\mathcal{C})$ and a morphism $\Phi \in \operatorname{Hom}_{\mathcal{C}}(Z, Z)$, we have

$$\frac{1}{\mathcal{D}^2} \sum_j d_j$$  $$= \frac{1}{d_Z} \operatorname{tr}(\Phi) \operatorname{id}_Z$$

this easily implies that the LHS is equal to RHS.

□

Example 3.6.6. Let $\mathcal{N}$ be the sphere with n embedded disks, colored by objects $Y_1, \ldots, Y_n \in \operatorname{Obj} Z(\mathcal{C})$ (see Example 3.5.2). Then

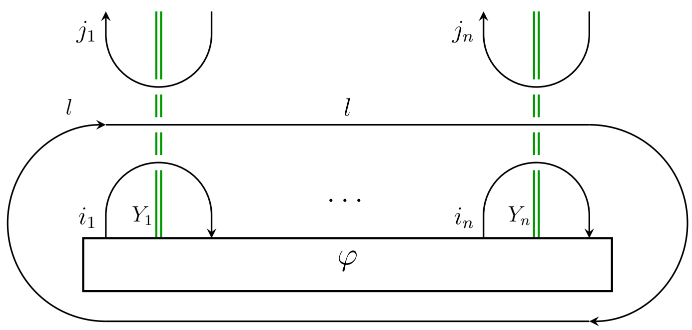
$$Z(\mathcal{N}, Y_1, \ldots, Y_n) = \operatorname{Hom}_{Z(\mathcal{C})}(\mathbf{1}, Y_1 \otimes Y_2 \otimes \cdots \otimes Y_n).$$

Indeed, by Example 3.5.2, we have

$$H(\mathcal{N}, Y_1, \ldots, Y_n) = \bigoplus_{i_1,\ldots,i_n \in \mathrm{Irr}(\mathcal{C})} \langle X_{i_1}, Y_1, X_{i_1}^*, \ldots, X_{i_n}, Y_n, X_{i_n}^* \rangle.$$

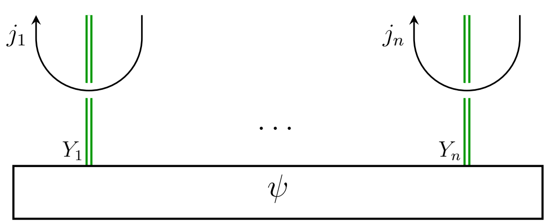
By a direct computation done in Section 3.7, we see that the operator $A = Z(\mathcal{N} \times I) \colon H(\mathcal{N}) \to H(\mathcal{N})$ is given by

$$\varphi \mapsto \frac{1}{\mathcal{D}^{2(n+1)}} \sum_{l, j_1,\ldots,j_n \in \mathrm{Irr}(\mathcal{C})} d_l \prod_{a=1}^{n} \sqrt{d_{i_a}} \sqrt{d_{j_a}}$$



Consider now the subspace $W \subset H(\mathcal{N}, Y_1, \ldots, Y_n)$ spanned by elements of the form

$$\bigoplus_{j_1,\ldots,j_n} \prod_{a=1}^{n} \sqrt{d_{j_a}}$$



$\psi \in \mathrm{Hom}_{Z(\mathcal{C})}(\mathbf{1}, Y_1 \otimes \cdots \otimes Y_n)$

Clearly, $W \simeq \mathrm{Hom}_{Z(\mathcal{C})}(\mathbf{1}, Y_1 \otimes \cdots \otimes Y_n)$.

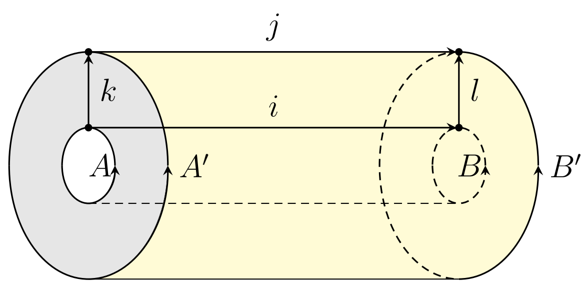
Now, it follows from the previous computation and Lemma 2.3.2 that for any $\varphi \in H(\mathcal{N}, Y_1, \ldots, Y_n)$, we have $A\varphi \in W$; on the other hand, it is immediate that if $\psi \in W$, then $A\psi = \psi$. Therefore, A is the projector onto $W \simeq \mathrm{Hom}_{Z(\mathcal{C})}(\mathbf{1}, Y_1 \otimes \cdots \otimes Y_n)$.

3.7 Some computations

In this section we give some explicit computations of the TV invariants.

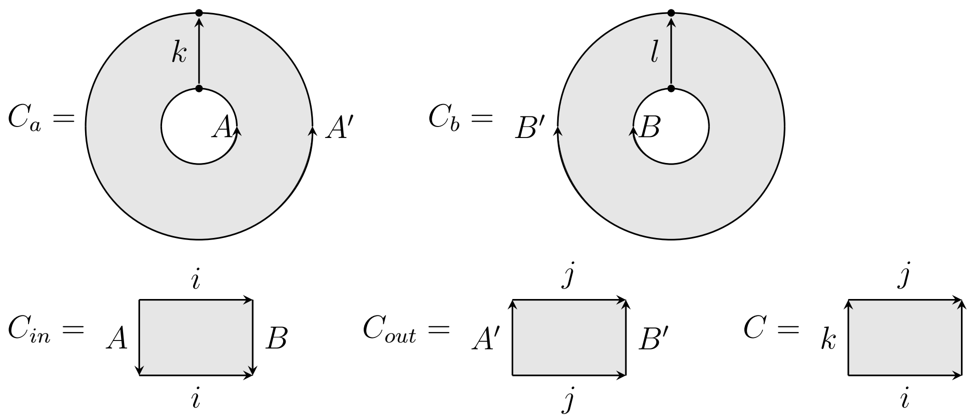
Cylinder over an annulus

Let $F = S^1 \times I \times I$ be the cylinder over an annulus, as shown below. Then

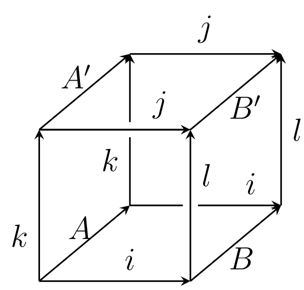


$$\partial F = C_a \cup C_b \cup C_{in} \cup C_{out} \cup C \cup \overline{C}$$

where C_a, C_b are the left and right annuli, C_{in} and C_{out} are the inner and outer cylinders, and C is the internal cell:



The pullback of the cell decomposition of $\partial F \simeq S^2$ to the sphere is homeomorphic to the cube shown below:



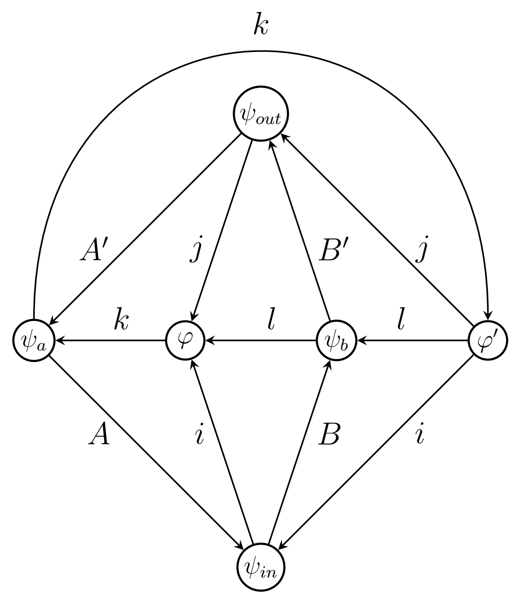
Drawing the dual graph, we see that given a collection

$$\psi_L \in H(\overline{C_a}) = \langle A, X_k, (A')^*, X_k^* \rangle, \qquad \psi_R \in H(\overline{C_b}) = \langle B_1^*, X_l, B', X_l^* \rangle$$

$$\psi_{in} \in H(\overline{C_{in}}) = \langle A^*, X_i, B, X_i^* \rangle, \qquad \psi_{out} \in H(\overline{C_{out}}) = \langle A', X_j, (B')^*, X_j^* \rangle$$

$$\varphi \in H(C) = \langle X_l, X_j^*, X_k^*, X_j \rangle, \qquad \varphi' \in H(\overline{C}) = \langle X_j^*, X_k, X_j, X_l^* \rangle,$$

the value $(Z(F), \psi_L \otimes \psi_R \otimes \psi_{in} \otimes \psi_{out} \otimes \varphi \otimes \varphi')$ is given by the following graph:



Sphere with n holes

Let $\mathcal{N}$ be the sphere with n embedded disks, colored by objects $Y_1, \dots, Y_n \in \operatorname{Obj} Z(\mathcal{C})$ (see Example 3.5.2). Choose the cell decomposition of N as in Example 3.5.2; then

$$\begin{aligned} &H(\mathcal{N}, Y_1, \dots, Y_n) \\ &\quad= \bigoplus_{X_1,\dots,X_n,U_1,\dots,U_n \in \operatorname{Irr}(\mathcal{C})} \langle X_1, U_1, X_1^*, \dots, X_n, U_n, X_n^* \rangle \otimes \langle U_1^*, Y_1 \rangle \otimes \dots \otimes \langle U_n^*, Y_n \rangle \\ &\quad\simeq \bigoplus_{X_1,\dots,X_n \in \operatorname{Irr}(\mathcal{C})} \langle X_1, Y_1, X_1^*, \dots, X_n, Y_n, X_n^* \rangle. \end{aligned}$$

Consider now the cylinder $\mathcal{N} \times I$ with the cell decomposition shown in Figure 3.20.

This cell decomposition contains $(n+1)$ 3-cells: n open tubes and one large

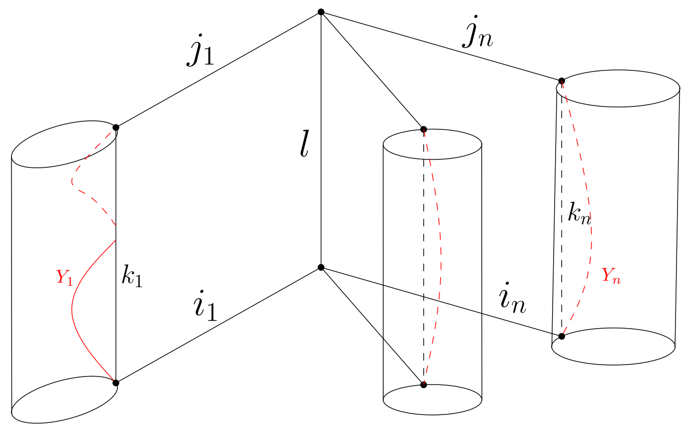


Figure 3.20: Cylinder over sphere with n embedded disks

3-cell. Thus, the invariant $Z(\mathcal{N} \times I)$ is given by

$$(Z(\mathcal{N} \times I), \varphi \otimes \varphi') = \sum_{l,k_1,\dots,k_n \in \mathrm{Irr}(\mathcal{C})} D \cdot Z_0 \cdot \dots \cdot Z_n$$

where

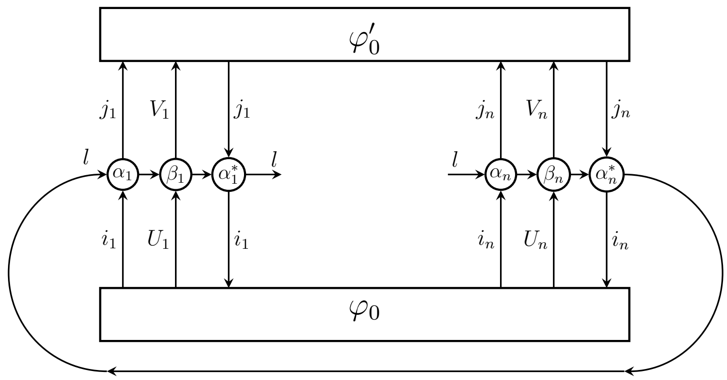
$$\begin{aligned}
\varphi &= \varphi_0 \otimes \varphi_1 \otimes \cdots \otimes \varphi_n \in H(\mathcal{N}, i_1, \dots, i_n) \\
&= \langle X_{i_1}, U_1, X_{i_1}^*, \dots, X_{i_n}, U_n, X_{i_n}^* \rangle \otimes \langle U_1^*, Y_1 \rangle \otimes \cdots \otimes \langle U_n^*, Y_n \rangle \\
\varphi' &= \varphi'_0 \otimes \varphi'_1 \otimes \cdots \otimes \varphi'_n \in H(\mathcal{N}, j_1, \dots, j_n)^* \\
&= \langle X_{j_n}, V_n^*, X_{j_n}^*, \dots, X_{j_1}, V_1^*, X_{j_1}^* \rangle \otimes \langle V_1, Y_1^* \rangle \otimes \cdots \otimes \langle V_n, Y_n^* \rangle
\end{aligned}$$

D is the normalization factor:

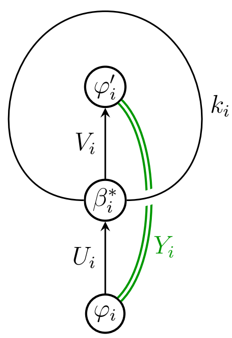
$$D = \frac{1}{\mathcal{D}^{2(n+1)}} d_l \prod_{a=1}^{n} d_{k_a} \sqrt{d_{i_a}} \sqrt{d_{j_a}} \sqrt{d_{U_a}} \sqrt{d_{V_a}}$$

and $Z_0, \dots, Z_n$ are the factors corresponding to the $(n+1)$ 3-cells of the decomposition:

$Z_0 =$



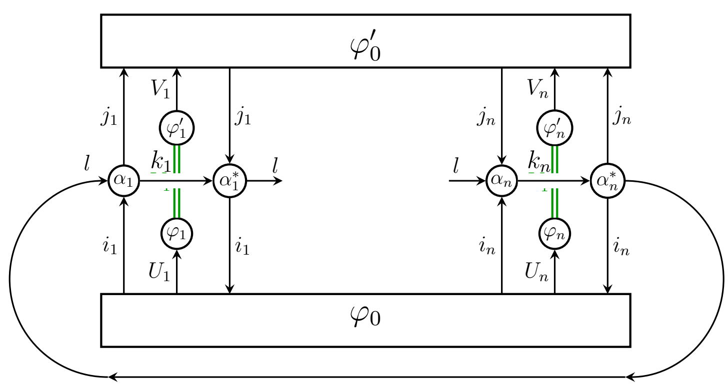
$Z_i =$



Using Lemma 2.2.3, we see that it can be rewritten as follows:

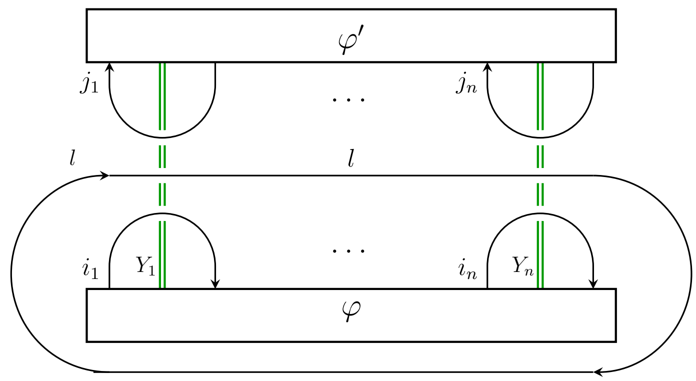
$(Z(\mathcal{N} \times I), \varphi \otimes \varphi') =$

$$\sum_{l,k_1,\ldots,k_n \in \mathrm{Irr}(\mathcal{C})} D$$



Thus, identifying $H(\mathcal{N}, i_1, \ldots, i_n) \simeq \langle X_{i_1}, Y_1, X_{i_1}^*, \ldots)$ as in Example 3.5.2 and using Lemma 2.2.1, we see that

$$(Z(\mathcal{N} \times I), \varphi \otimes \varphi') = \sum_{l \in \mathrm{Irr}(\mathcal{C})} \frac{1}{\mathcal{D}^{2(n+1)}} d_l \prod_{a=1}^{n} \sqrt{d_{i_a}} \sqrt{d_{j_a}}$$



Chapter 4

Relating RT and TV invariants

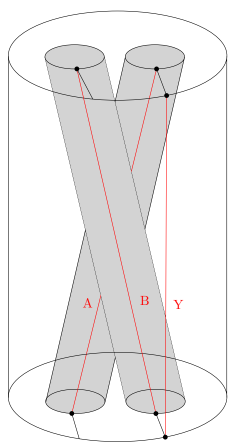
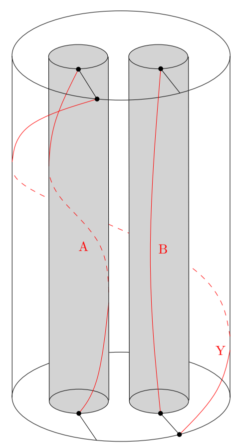
In this chapter, we finish proving that for a closed 3-manifold $\mathcal{M}$ (possibly with an embedded link inside), $Z_{TV,\mathcal{C}}(\mathcal{M}) = Z_{RT,Z(\mathcal{C})}(\mathcal{M})$. Turaev and Virelizier recently posted a proof of this formula [4], but the methods are different, and in particular involves state sums on skeletons of 3-manifolds and the theory of Hopf monads in monoidal category.

In contrast, we prove the theorem via the results of the previous chapter. Namely, we use the *extended* TV theory. First we prove that the theories coincide for S^3 with a link inside. We do this by decomposing S^3 into a finite collection of *building blocks*, demonstrating that the theories coincide on these blocks, and then using the gluing axiom. Next, we describe graphically the vector space assigned to the torus and examine the action of the mapping class group. In particular, we show that the generators T and S of the mapping class group act by multiplication by the twist and s-matrices respectively, just as they do in Z_{RT}. Finally, we prove the surgery formula for our theory, which implies the main theorem as a corollary.

4.1 Sphere

In this section we compute the TV state sum for several extended 3-manifolds, which we call *generators*. If S^3_L denotes S^3 with an embedded link L inside, we can decompose S^3_L into a finite union of these generators. Using the computations in this section and the gluing axiom for our TQFT, we conclude that the theories give the same answer for S^3_L. In what follows, all links are framed and oriented.

Consider the following extended 3-manifold structure on $\mathcal{N}$ where $\mathcal{N}$ is the cobordism between 3-punctured spheres that interchanges two of the embedded disks with longitudes labeled as pictured [1]. Clearly, the two picture are homeomorphic If we take a modular category as input data, Reshetikhin-

$\mathcal{N} =$  $\cong$ 

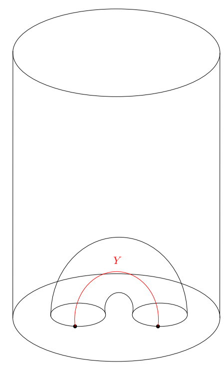
Turaev theory gives $Z_{RT}(\mathcal{N}) = Id_Y \otimes \sigma_{AB}$, where σ is the braiding 5]. We now show that Turaev-Viro theory gives the same answer.

Lemma 4.1.1. *Let $\mathcal{C}$ be a spherical category. Then there is a canonical isomorphism $Z_{TV,\mathcal{C}}(\mathcal{N}) \cong Z_{RT,Z(\mathcal{C})}(\mathcal{N})$*

[1]We have removed a solid cylinder from the figure to make the diagram more manageable

Proof. See Section 5.3 □

Now let $\mathcal{M} = S^2 \times I$ with a single open embedded tube colored by $Y \in Z(\mathcal{C})$ as shown. As in the previous lemma, we have removed a solid cylinder from $\mathcal{M}$. By definition, $Z_{RT,Z(\mathcal{C})}(\mathcal{M}) = ev_Y : Y^* \otimes Y \to \mathbf{1}$, the evaluation map.



Lemma 4.1.2. $Z_{TV,\mathcal{C}}(\mathcal{M}) \cong Z_{RT,Z(\mathcal{C})}(\mathcal{M})$.

Proof. See Section 5.3 □

It follows by an identical calculation that $Z_{TV}(\mathcal{M}')$ gives the coevaluation map, where $\mathcal{M}'$ is similar to $\mathcal{M}$ but inverted.

We can now use the above two lemmas to state the following result.

Theorem 4.1.3. *Let $M = S^3_L$ be the 3-sphere with an embedded link L inside with components colored by $Y \in Irr(Z(\mathcal{C}))$. Then $Z_{TV,\mathcal{C}}(M) = Z_{RT,Z(\mathcal{C})}(M)$.*

Proof. After isotoping L appropriately, we can cut M into regions, each of which is isomorphic to $\mathcal{N}, \mathcal{M}$ or $\mathcal{M}'$ from the above lemmas or to B^3. The theorem then follows from the lemmas and the gluing axiom 2]. □

Note that it is known that for RT, $Z_{RT,\mathcal{A}}(S^3_L) = \frac{1}{\mathrm{Dim}(\mathcal{A}}F(L)$, so the theorem gives

$$Z_{TV,\mathcal{C}}(S^3_L) = Z_{RT,Z(\mathcal{C})}(S^3_L) = \frac{1}{\mathrm{Dim}(Z(\mathcal{C}))}F(L) = \frac{1}{\mathcal{D}^2}F(L) \qquad (4.1.1)$$

which can also be verified by direct computation. Here we use the fact that Dim(Z($\mathcal{C}$))= Dim($\mathcal{C}$)^2 39].

4.2 Computations with the Torus

In the previous section, we established that $Z_{RT}(S^3_L) = Z_{TV}(S^3_L)$, where S^3_L is the 3-sphere with an embedded link L inside. It is a classical result that any connected, closed 3-manifold may be obtained from S^3 via surgery along a framed link L, or, more precisely, along a tubular neighborhood of L. A tubular neighborhood of a link is simply a disjoint union of solid tori. In this section, we study the vector space $Z_{TV}(\mathbb{T}^2)$ and describe graphically an inner product, an orthonormal basis and action of the mapping class group (MCG) of the torus. We denote the standard 2-torus $S^1 \times S^1$ by $\mathbb{T}^2$.

The following result is well-known (see e.g. 39]):

Lemma 4.2.1. $Z_{TV,\mathcal{C}}(\mathbb{T}^2)$ *has basis indexed by isomorphism classes of irreducible objects of* $\mathcal{Z}(\mathcal{C})$*.*

Proof. The torus may be obtained from the 2-punctured sphere, S^2_2 by gluing together its two boundary circles. It follows directly from the gluing axiom for surfaces (Theorems 8.4, 8.5 in 2]) that

$$Z_{TV,\mathcal{C}}(\mathbb{T}^2) = \bigoplus_{Z \in Irr(Z(\mathcal{C}))} \langle Z, Z^* \rangle_{Z(\mathcal{C})}. \tag{4.2.1}$$

□

We can also see this explicitly by computing $Z_{TV}(\mathbb{T}^2 \times I)$, as is done in Section 5.3.

We denote by $\mathbf{T}^2_Z$ the solid torus with a closed embedded tube inside with (untwisted) longitude labeled by $Z \in Irr(Z(\mathcal{C}))$ as shown in Figure 4.1. Then $Z_{TV,\mathcal{C}}(\mathbf{T}^2_Z)$ is a vector in $Z_{TV,\mathcal{C}}(\mathbb{T}^2)$. We denote this vector by $[Z]$.

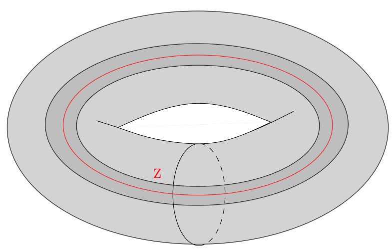


Figure 4.1: The solid torus with a closed embedded tube labelled by $Z \in Z(\mathcal{C})$

Lemma 4.2.2. $\{[Z]\}_{Z \in Irr(Z(\mathcal{C}))}$ *form a basis in* $Z(\mathbb{T}^2)$.

A simple computation shows that $[Z] = \frac{1}{\sqrt{d_Z}} coev_Z$, under the identification 4.2.1 . Note that $coev_{\mathcal{C}}|_{Z(\mathcal{C})} = coev_{Z(\mathcal{C})}$.

4.3 Surgery

If M, N are manifolds with boundary and $\varphi : \partial M \longrightarrow \partial \overline{N}$ is a homeomorphism, we may glue M and N along their boundaries to obtain a closed

3-manifold, denoted $M \sqcup_\varphi N$. Such a map φ induces a linear map between the corresponding vector spaces, and we denote this map $\varphi_*\colon Z(\partial M) \to Z(\partial \overline{N})$. By the gluing axiom, $Z(M \sqcup_\varphi N) = (\varphi_* Z(M), Z(N))$.

Lemma 4.3.1. *$M \sqcup_\varphi N$ only depends on the isotopy class of φ*

For a proof of this lemma, see 5].

Definition 4.3.2. For a closed manifold M, the *mapping class group* $\Gamma(M)$ is the group of isotopy classes of homeomorphisms $M \longrightarrow M$.

Since any 3-manifold may be obtained from S^3 by surgery along an embedded link, or equivalently along a collection of solid tori, we will be primarily concerned with the following example:

Example 4.3.3. $\Gamma(\mathbb{T}^2) = SL_2(\mathbb{Z})$, which is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. If we pick generators α, β for $H_1(\partial \mathbf{T}^2, \mathbb{Z})$ as in Figure 4.2, then S acts by interchanging [2] α and β and T is a Dehn twist (See Figure 4.2). Further, the action of T extends to a homeomorphism of the solid torus.

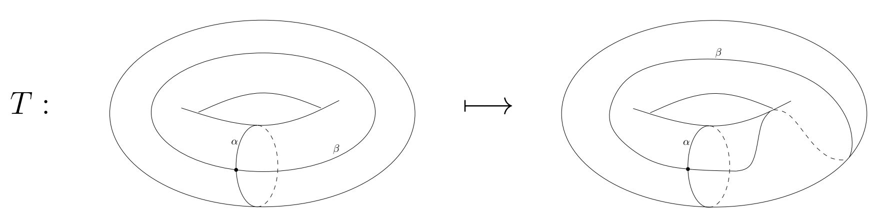


Figure 4.2: The action of T on the Torus

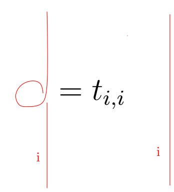
We now describe an inner product on $Z(\mathbb{T}^2)$.

[2]More precisely S(α) = β, S(β) = -α

Lemma 4.3.4. *Define a bilinear form on* $Z(\mathbb{T}^2)$ *by* $\langle [Z],[W]\rangle = Z(\boldsymbol{T}^2_Z \sqcup_U \boldsymbol{T}^2_W)$, *where* $U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. *Then* $\{[Z]\}_{Z\in Irr(Z(\mathcal{C}))}$ *is orthonormal with respect to this form.*

Proof. See Section 5.3. □

Since $Z(\mathcal{C})$ is a modular category, it has an s-matrix $\tilde{s}$ (5]) and a twist matrix t where $t_{i,j}$ is zero for $i \neq j$ and is defined for $i = j$ by



Let S and T be as above. The following theorem shows that S and T act on $Z(\mathbb{T}^2)$ by the s-matrix and the twist matrix respectively.

Theorem 4.3.5. *1.* $S_*[Z] = \sum_{W\in Irr(Z(\mathcal{C}))} \frac{\tilde{s}_{ZW}}{\mathcal{D}^2}[W]$.

2. $T_*[Z] = t_{Z,Z}[Z]$

Proof. 1. By the gluing axiom, $(S_*[Z],[W]) = Z_{TV,\mathcal{C}}(\mathbf{T}^2_Z \sqcup_S \mathbf{T}^2_W) = Z_{TV,\mathcal{C}}(S^3_L)$, where L is the Hopf link with components labelled by Z and W. By 4.1.3, this equals $Z_{RT,Z(\mathcal{C})}(S^3_L) = \tilde{s}_{ZW}$.

2. By the gluing axiom $(T_*[Z],[W]) = Z_{TV,\mathcal{C}}(\mathbf{T}^2_Z \sqcup_T \mathbf{T}^2_W)$. This manifold is homeomorphic to $S^2 \times S^1$ with two unlinked embedded closed tubes labelled Z and W respectively, the Z tube with a single positive twist. This follows directly from the fact that $\mathbf{T}^2 \sqcup_U \mathbf{T}^2 = S^2 \times S^1$ combined with the fact that the map T extends to a homeomorphism of solid tori.

Replacing the twist with a multiplicative factor $t_{Z,Z}$, we get $Z_{TV,\mathcal{C}}(\mathbf{T}^2_Z \sqcup_T \mathbf{T}^2_W) = t_{Z,Z} Z_{TV,\mathcal{C}}(\mathbf{T}^2_Z \sqcup_U \mathbf{T}^2_W) = t_{Z,Z}\delta_{Z,W}$, where the final equality holds by Lemma 4.3.4.

$\square$

The following computations will be particularly useful:

1. $S_*[\mathbf{1}] = \sum_{W \in Irr(Z(\mathcal{C}))} \frac{d_W}{\mathcal{D}^2}[W]$. This follows immediately from Theorem 4.5(1) and the fact that $\tilde{s}_{\mathbf{1},Z} = d_Z$.

2. $S_* \sum_Z \frac{d_Z}{\mathcal{D}^2}[Z] = [\mathbf{1}]$. This follows from the equation $\sum_Z d_Z \tilde{s}_{Z,W} = \delta_{W,\mathbf{1}}\mathcal{D}^2$ (See 5], Chapter 3).

Let us briefly outline what we have accomplished so far.

1. In Section 4.1, we demonstrated that $Z_{TV,\mathcal{C}}(S^3_L) = Z_{RT,Z(\mathcal{C})}(S^3_L)$. This was done by decomposing S^3_L as a union of several *types* of 3-manifolds with boundary, showing the result holds for each type, and using the gluing axiom.

2. In Section 4.2 we showed that $Z_{TV,\mathcal{C}}(\mathbb{T}^2) \cong Z_{RT,Z(\mathcal{C})}(\mathbb{T}^2)$ and that this isomorphism agrees with the action of the mapping class group $\Gamma(\mathbb{T}^2)$

We will now connect these results. Let K be a framed knot with framing $n \in \mathbb{Z}$ and $\mathcal{M}_K$ a 3-manifold with K embedded. Let $\mathcal{M}'$ be the result of performing surgery in $\mathcal{M}$ along K. We can write the gluing map $\varphi : \mathbb{T}^2 \to \partial(\mathcal{M} - \mathbb{T}^2)$ as $\varphi = T^n \circ S$.

Lemma 4.3.6. *$Z_{TV,\mathcal{C}}(\mathcal{M}') = \frac{1}{\mathcal{D}^2} Z_{TV,\mathcal{C}}(\mathcal{M}_K)$.*

Proof. Recall that $Z_{TV,\mathcal{C}}(\mathcal{M}_K) \equiv \sum_i d_i Z_{TV,\mathcal{C}}(\mathcal{M}_{K_i})$, where K_i is the knot K colored by $i \in Irr(\mathcal{C})$ and $\mathcal{M}_K = (\mathcal{M} - \mathbf{T}^2) \sqcup_{Id} \mathbf{T}^2$, where $\mathbf{T}^2$ is a tubular neighborhood of K. Then

$$Z_{TV,\mathcal{C}}(\mathcal{M}_K) = (\sum d_i[i], Z(\mathcal{M}_K - \mathbf{T}^2)) = (\sum d_i t^n_{i,i}[i], Z(\mathcal{M}_K - \mathbf{T}^2))$$

$$= \mathcal{D}^2(T^n_* S_*[\mathbf{1}], Z(\mathcal{M}_K - \mathbf{T}^2) = \mathcal{D}^2 Z_{TV,\mathcal{C}}((\mathcal{M} - \mathbf{T}^2) \sqcup_{T^n \circ S} \mathbf{T}^2) = \mathcal{D}^2 Z_{TV,\mathcal{C}}(\mathcal{M}').$$

□

We can slightly generalize Lemma 4.3.6. The proof is similar.

Lemma 4.3.7. *Let $\mathcal{M}_L$ be a 3-manifold with a framed link L inside. Let $\mathcal{M}'_{L'}$ be the result of performing surgery on $\mathcal{M}_L$ along a single component of L. (Note that $|L'| = |L| - 1$). Then $Z_{TV,\mathcal{C}}(\mathcal{M}'_{L'}) = \frac{1}{\mathcal{D}^2} Z_{TV,\mathcal{C}}(\mathcal{M}_L)$.*

Note that if L is a knot, the lemma reduces to Lemma 4.3.6. Now we state the main theorem of this section, which relates the Reshetikhin-Turaev and Turaev-Viro invariants. The proof makes repeated use of Lemma 4.3.7 and is very simple. Since we will be working with two categories, $\mathcal{C}$ and its Drinfeld Center $Z(\mathcal{C})$, we will replace all potentially ambiguous shorthand in what follows. For example, we will write $\mathrm{Dim}(\mathcal{C})$ instead of $\mathcal{D}$.

Theorem 4.3.8. *Let $\mathcal{M}$ be a closed, oriented 3-manifold with a colored link inside. Then $Z_{TV,\mathcal{C}}(\mathcal{M}) = Z_{RT,Z(\mathcal{C})}(\mathcal{M})$.*

Proof. For simplicity, we can assume $\mathcal{M}$ has no embedded link. The general case is proved in the exact same fashion. We can obtain $\mathcal{M}$ from S^3_L via surgery along a tubular neighborhood of L. Evidently, we can perform surgery along each component of L individually. Using Lemma 4.3.7 repeatedly, we obtain

$$Z_{TV,\mathcal{C}}(\mathcal{M}) = \frac{1}{\text{Dim}(\mathcal{C})^{2|L|}} Z_{TV,\mathcal{C}}(S^3_L) = \sum \frac{1}{\text{Dim}(\mathcal{C})^{2|L|}} \prod d_{Y_i} Z_{TV,\mathcal{C}}(S^3; L_i; Y_i).$$

By Theorem 2.3, this equals

$$\sum \frac{1}{\text{Dim}(\mathcal{C})^{(2|L|+2)}} \prod d_{Y_i} F(L_i; Y_i) = \sum \frac{1}{\text{Dim}(Z(\mathcal{C}))^{(|L|+1)}} \prod d_{Y_i} F(L_i; Y_i)$$

$$\equiv Z_{RT,Z(\mathcal{C})}(\mathcal{M}).$$

□

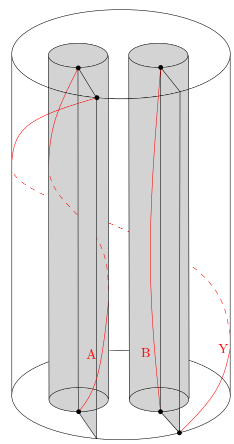
We have shown that $Z_{TV,\mathcal{C}}$ and $Z_{RT,Z(\mathcal{C})}$ give the same 3-manifold invariants. Using the gluing axiom and the fact that the theories agree on the n-puntured sphere, one can easily show that for Σ_g a closed genus g surface, $\text{Dim}(Z_{TV,\mathcal{C}}(\Sigma_g)) = \text{Dim}(Z_{RT,Z(\mathcal{C})}(\Sigma_g))$ [3]. Thus, the vector spaces associated to 2-manifolds are isomorphic. This is not enough, however, to show the TQFTs are isomorphic. We will have to construct a canonical isomorphism between the spaces and show these satisfy certain compatibility conditions. This is the subject of the next chapter.

[3]In fact it is not hard to explicitly compute this this common dimension. See 45]

4.4 Some Proofs

In this section we include some of the computations described in the chapter.

Proof of Lemma 4.1.1



Decompose $\mathcal{N}$ into a combinatorial 3-manifold. The tubes labeled A and B are shaded gray and the tube labeled Y lies on the outside of the 3-cell. The decomposition has 4 vertices and 16 edges, of which 4 are internal and 12 lie on the boundary. The decomposition has four 3-cells, of which 3 are labeled tubes. Orienting and coloring edges, we get 4 graphs, one for each 3-cell. Note that in the following diagrams we often replace coupons with vertices and omit labeling vertices by the appropriate vectors. As always, we label vertices corresponding to dual Hom-spaces with dual basis vectors and sum over all these 'paired' bases (Recall that these correspond to internal 2-cells). We also label the bottom and top vertices by φ and φ' respectively.

The state sum is therefore:

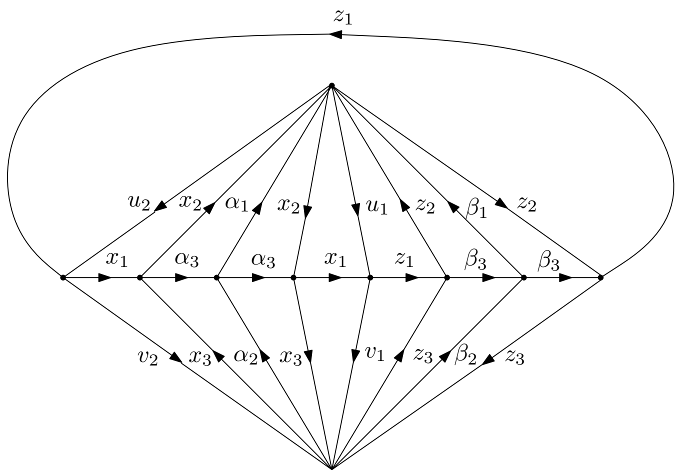


Figure 4.3: Big 3-cell

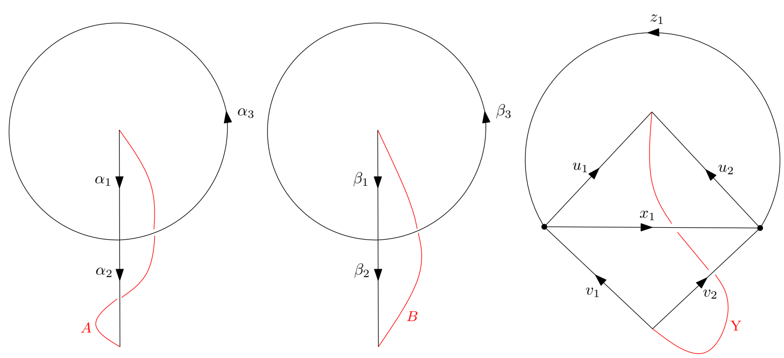


Figure 4.4: Dual graphs of the 3 tube cells

$$Z_{TV}(\mathcal{N}) = \sum DK_3K_YK_AK_B$$

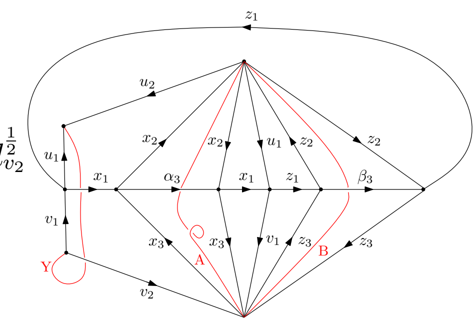
where K_3, K_Y, K_A, K_B denote the evaluations of each of the labeled graphs picture above and D is the following unsightly term:

$$D = \mathcal{D}^{-8} \sum d_{x_1} d_{z_1} d_{\alpha_3} d_{\beta_3} d_{\alpha_1}^{\frac{1}{2}} d_{\alpha_2}^{\frac{1}{2}} d_{\beta_1}^{\frac{1}{2}} d_{\beta_2}^{\frac{1}{2}} d_{x_2}^{\frac{1}{2}} d_{x_3}^{\frac{1}{2}} d_{z_2}^{\frac{1}{2}} d_{z_3}^{\frac{1}{2}} d_{u_1}^{\frac{1}{2}} d_{u_2}^{\frac{1}{2}} d_{v_1}^{\frac{1}{2}} d_{v_2}^{\frac{1}{2}}$$

Recall that each vertex is labeled by a morphism in the corresponding Hom-space. Pairing dual morphisms, we can glue in the 3 tubes.

$$Z_{TV}(\mathcal{N}) =$$

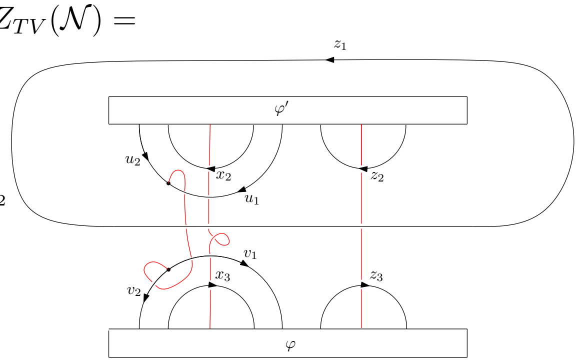
$$\sum \mathcal{D}^{-8} d_{x_1} d_{z_1} d_{\alpha_3} d_{\beta_3} d_{x_2}^{\frac{1}{2}} d_{x_3}^{\frac{1}{2}} d_{z_2}^{\frac{1}{2}} d_{z_3}^{\frac{1}{2}} d_{u_1}^{\frac{1}{2}} d_{u_2}^{\frac{1}{2}} d_{v_1}^{\frac{1}{2}} d_{v_2}^{\frac{1}{2}}$$



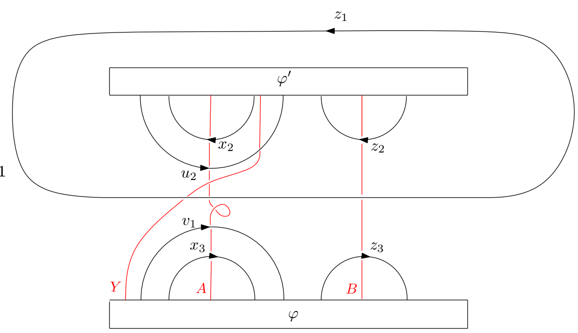
Here we have used the pairing of dual graphs and semisimplicity (Lemma 2.2.1). Using Lemma 2.2.1 three more times, we get

$$Z_{TV}(\mathcal{N}) =$$

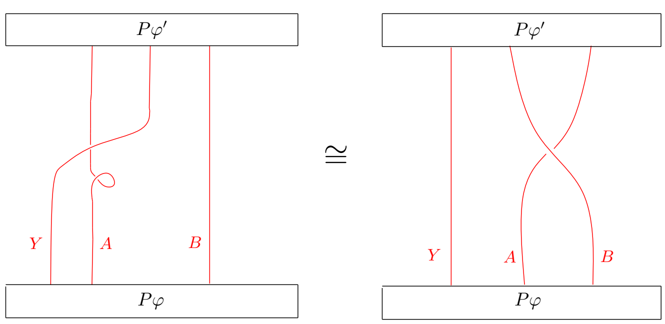
$$\sum \mathcal{D}^{-8} d_{z_1} d_{x_2}^{\frac{1}{2}} d_{x_3}^{\frac{1}{2}} d_{z_2}^{\frac{1}{2}} d_{z_3}^{\frac{1}{2}} d_{u_1}^{\frac{1}{2}} d_{u_2}^{\frac{1}{2}} d_{v_1}^{\frac{1}{2}} d_{v_2}^{\frac{1}{2}}$$



If we pair some more and cancel opposite twists, this equals

$\sum \mathcal{D}^{-8} d_{z_1} d_{x_2}^{\frac{1}{2}} d_{x_3}^{\frac{1}{2}} d_{z_2}^{\frac{1}{2}} d_{z_3}^{\frac{1}{2}} d_{u_2}^{\frac{1}{2}} d_{v_1}^{\frac{1}{2}}$ 

It follows from Lemma 2.2 and Example 8.6 of 2] that the loop labeled z_1 projects the top morphism onto $Hom_{Z(\mathcal{C})}(1, Y \otimes A \otimes B)$. After projection, the diagram may be depicted as below where P is the projector from Lemma 2.3.2 and the diagram on the left follows from the fact that for a projector, $(P\varphi, \varphi')$ $= (P\varphi, P\varphi')$. The final isomorphism may be easily verified by the reader.



$= \big(P\varphi', (1_Y \otimes \sigma_{AB})P\varphi\big)$

Proof of Lemma 4.1.2

We choose the following combinatorial structure on $\mathcal{M}$:

- Three 3-cells, one of which is an open embedded tube, with longitude labeled by Y$\in Irr(Z(\mathcal{C}))$.
- 8 vertices, all of which lie on the boundary
- 15 edges, three of which are internal.

For each 3-cell, we get a graph. Gluing these graphs together using Lemma 2.2.1

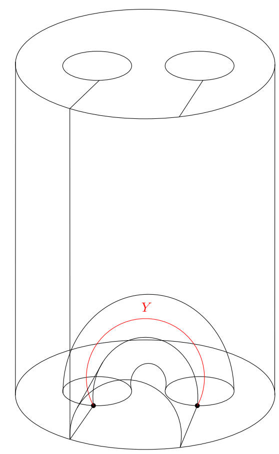
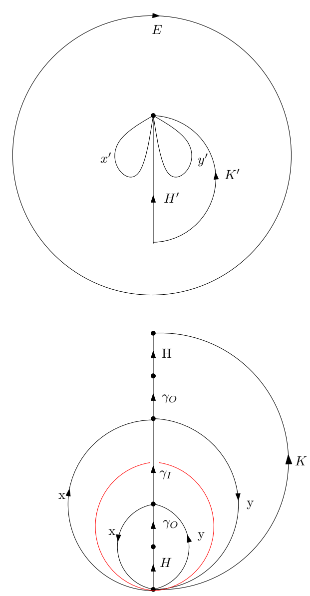


Figure 4.5: Decomposition of $\mathcal{M}$

yields the following graph:



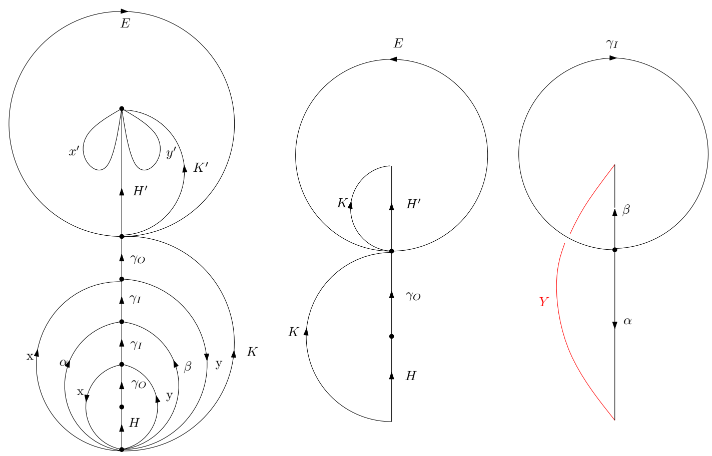
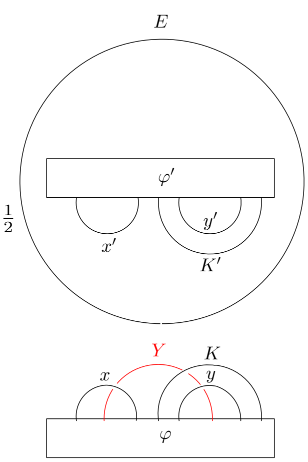


Figure 4.6: The graphs corresponding to the three 3-cells from Figure 6. From left to right, these correspond to the main 3-cell, the outer 3-cell and the embedded tube cell

Simplifying the graph using Lemma 2.2.1, the state-sum formula becomes

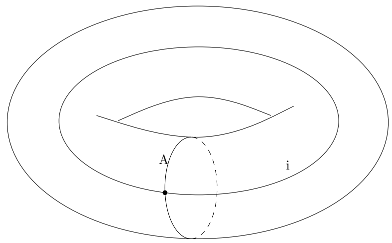
$$Z_{TV,\mathcal{C}}(M) = \sum \mathcal{D}^{-8} d_E (d_x d_{x'} d_y d_{y'} d_K d_{K'})^{\frac{1}{2}}$$



Using the same arguments as the previous calculation, one can see that this gives $(P\varphi', ev_Y(P\varphi))$, where $ev_Y : Y^* \otimes Y \to \mathbf{1}$ is the evaluation map in $\mathrm{Z}(\mathcal{C})$.

Computation of $Z_{TV}(\mathbb{T}^2 \times I)$ from Lemma 4.2.1

Let us take the following polytope decomposition of $\mathbb{T}^2$ consisting of one vertex, two edges and one face. Then



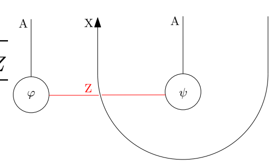
$$H(\mathbb{T}^2) = \bigoplus_{A,X \in Irr\mathcal{C}} \langle A, X, A^*, X^* \rangle.$$

. We will often make use of the following isomorphisms:

$$H(\mathbb{T}^2) \cong \bigoplus_{Z,A} \langle Z, A \rangle \otimes \langle Z^*, A^* \rangle \cong \bigoplus_{Z} \langle Z, Z^* \rangle_{\mathcal{C}} \tag{4.4.1}$$

where the first isomorphism is given by the map G from Lemma 3.5.4, and the second is given by a direct sum of composition maps ((2.2.5)). Choosing bases $\{\varphi_{Z,A,i}\}$ in $\langle Z, A \rangle$ and $\{\psi_{Z,A,j}\}$ in $\langle Z^*, A^* \rangle$, we can write a basis in $H(\mathbb{T}^2)$ as

$$\eta_{Z,A,i,j} = \sum_{X \in Irr(\mathcal{C})} \frac{\sqrt{d_X}\sqrt{d_Z}}{\mathcal{D}}$$



Composing the map G with the direct sum of composition maps, we see that $H(\mathbb{T}^2) \cong \bigoplus_{Z} \langle Z, Z^* \rangle_{\mathcal{C}}$. We now show that $Z_{TV,\mathcal{C}}(\mathbb{T}^2 \times I)$ computes the projection onto $\bigoplus_{Z \in Irr(Z(\mathcal{C}))} \langle Z, Z^* \rangle_{Z(\mathcal{C})}$.

Consider the decomposition of $\mathbb{T}^2 \times I$ shown below. . The top and bottom

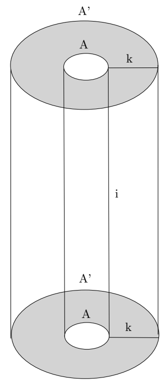
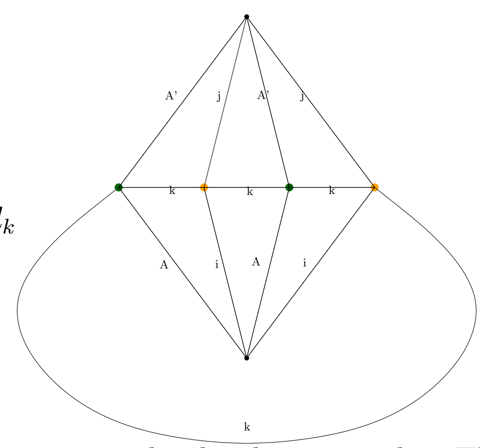


Figure 4.7: Decomposition of $\mathbb{T}^2 \times I$

of the figure are shaded to emphasize that they are to be identified. This decomposition consists of 2 vertices both on the boundary, 5 edges 4 of which lie on the boundary, and a single 3-cell.

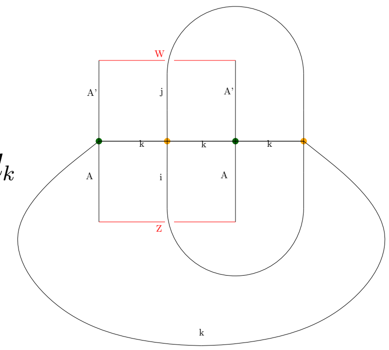
$$Z_{TV}(\mathbb{T}^2 \times I) = \sum \frac{1}{\mathcal{D}^2}(d_A d_{A'} d_i d_j)^{\frac{1}{2}} d_k$$



where we have labelled vertices dual to one another by the same color. This gives a map $\Phi : \bigoplus_{A,i} \langle A, X_i, A^*, X_i^* \rangle \longrightarrow \bigoplus_{A',j} \langle A', X_j, A'^*, X_j^* \rangle$. Composing on both sides by G yields a map $G^{-1}\Phi G : \bigoplus_{Z,A} \langle Z, A \rangle \otimes \langle Z^*, A^* \rangle \longrightarrow \bigoplus_{W,A'} \langle W, A' \rangle \otimes$

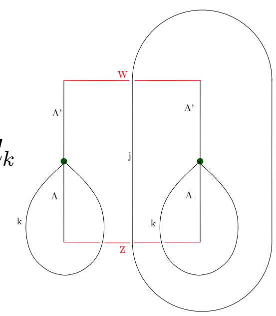
$\langle W^*, A'^* \rangle$.

$$\sum \frac{1}{\mathcal{D}^4}(d_A d_{A'})^{\frac{1}{2}} d_i d_j d_k$$



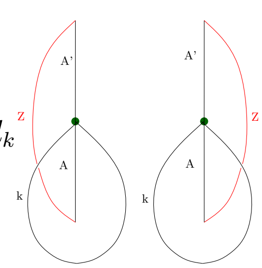
Pairing the two orange vertices yields

$$\bigoplus_{Z,W} \sum \frac{1}{\mathcal{D}^4}(d_A d_{A'})^{\frac{1}{2}} d_i d_k$$

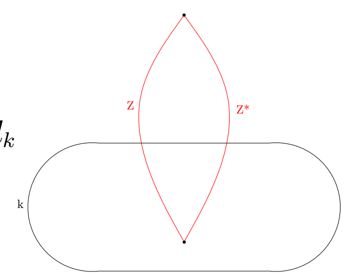


By Schur's Lemma, this diagram evaluates to 0 unless $W = Z^*$. If $W = Z^*$, this equals (Lemma 2.3.2)

$$\bigoplus_{Z} \sum \frac{1}{\mathcal{D}^2}(d_A d_{A'})^{\frac{1}{2}} d_k$$



$$= \bigoplus_{Z} \sum \frac{1}{\mathcal{D}^2} d_k$$



where the equality follows from pairing the dual vertices and using the rescaled composition map twice. By Lemma 2.3.2 this is a projector onto

$$\bigoplus_{Z \in Irr(Z(\mathcal{C}))} Hom_{Z(\mathcal{C})}(\mathbf{1}, Z \otimes Z^*).$$

This computation also shows that the projection is compatible with 4.4.1.

Proof of Lemma 4.3.4

$\mathbf{T}^2_Z \sqcup_U \mathbf{T}^2_W = S^2 \times S^1$ with two unlinked embedded tubes labelled by Z and W. Choose a decomposition of $S^2 \times S^1$ as pictured in Figure 9. This decomposition

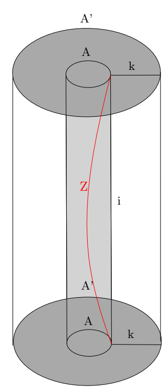
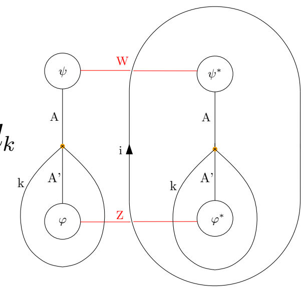


Figure 4.8: Decomposition of $S^2 \times S^1$ with two closed embedded tubes. The tube labelled W is to be glued on the outside of the pictured cylinder and the top and bottom of the picture are identified.

has two vertices, five edges and three 3-cells, two of which are embedded tubes.

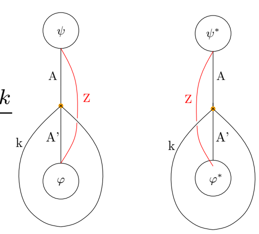
The computation of the state sum is similar to that in Lemma 4.2.1. We get

$$Z(\mathbf{T}^2_Z \sqcup_U \mathbf{T}^2_W) = \sum \frac{1}{\mathcal{D}^4} d_A d_{A'} d_i d_k$$



where the orange vertices are labelled by dual vectors. By Lemma 2.3.2 this equals

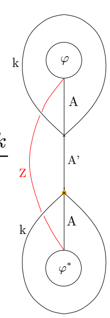
$$\delta_{Z,W} \sum_{A,A',k} \frac{1}{\mathcal{D}^2} \frac{d_A d_{A'} d_k}{d_Z}$$



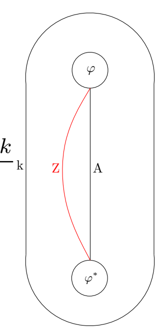
.

Finally, pairing dual vertices using Lemma 2.2.1, we get

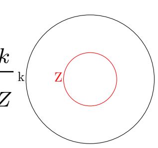
$$\delta_{Z,W} \sum_{A,A',k} \frac{1}{\mathcal{D}^2} \frac{d_A d_{A'} d_k}{d_Z}$$



$$= \delta_{Z,W} \sum_{A,k} \frac{1}{\mathcal{D}^2} \frac{d_A d_k}{d_Z}$$



$$= \delta_{Z,W} \sum_{k} \frac{1}{\mathcal{D}^2} \frac{d_k}{d_Z}$$



$$= \delta_{Z,W}$$

Chapter 5

Comparison of the Theories

In this chapter, we show that the Reshetikhin-Turaev and Turaev-Viro TQFTs are isomorphic at the level of surfaces. Namely, if Σ is a closed surface, we show that there is a nautural isomorphism $Z_{TV,\mathcal{C}}(\Sigma) \cong Z_{RT,Z(\mathcal{C})}(\Sigma)$ of vector spaces. We also note that we actually get an equivalence of extended (3-2-1) theories if we impose mild restrictions on the allowed types of manifolds with corners.

It is easy to compute the dimensions of the above spaces:

$$\operatorname{Dim} Z_{TV}(\Sigma_g) = \operatorname{Dim} Z_{RT}(\Sigma_g) = \mathcal{D}^{2g-2} \sum_{i \in \operatorname{Irr}(\mathcal{C})} d_i^{2-2g} \tag{5.0.1}$$

where $\mathcal{D}$ is the dimension of $\mathcal{C}$ and d_i is the dimension of simple object X_i. The vector spaces are therefore isomorphic, but this is not enough. We need to exhibit a natural isomorphism between the spaces.

The same issue occurs in general when attempting to define any 2D modular functor. For example, in RT theory, one decomposes the surface Σ into a union of punctured spheres[1], evaluates Z_{RT} for each of them, and uses the gluing axiom to obtain $Z_{RT}(\Sigma)$. A priori, this appears to depend on the choice of decomposition of Σ. Refining earlier work by Moore and Seiberg, Bakalov and Kirillov 6] proposed a set of *moves* (The "Lego-Teichmüller Game"). relating any two such decompositions. One can show that each of these moves corresponds to a certain natural isomorphism of vector spaces and any two "paths" between two chosen decompositions yield the same map. The space $Z(\Sigma)$ is therefore well defined.

We will apply the results described above to TV theory. In Section 3.7 , we constructed an isomorphism

$$Z_{TV}(\Sigma) \cong \mathrm{Hom}_{Z(\mathcal{C})}(\mathbf{1}, Y_1 \otimes \cdots \otimes Y_n) \tag{5.0.2}$$

where Σ is an n-punctured sphere with boundary components labeled by $Y_1, \ldots Y_n \in Irr(Z(\mathcal{C}))$. Notice that the space on the right of this equation is by definition $Z_{RT,Z(\mathcal{C})}(\Sigma; Y_1, \ldots Y_n)$.

It is important to note that RT is defined using "pairs-of-pants" decompositions of surfaces, while TV is defined via cell decompositions. Since the latter is a local construction and the former is inherently nonlocal, comparing the two requires a natural way of passing between them. The solution is simple and is provided immediately by the surface parametrizations defined in 6]. Using these, we can compute maps between TV state spaces that correspond to each of the moves between cut systems and check that such maps are compatible with the projector $H_{TV}(\Sigma) \longrightarrow Z_{TV}(\Sigma)$. Thus, we get a natural

[1] Following 6], we call this a *cut sytem*.

identification $Z_{RT}(\Sigma) \cong Z_{TV}(\Sigma)$.

In both RT and TV theory, once we know the value of the TQFT on a punctured sphere, we can use the gluing axiom to define $Z(\Sigma)$ for any surface. Thus, we get a well-defined vector space, up to natural isomorphism that depends only on the topology of Σ. We do this in each case by defining "intermediate" vector spaces which do depend on some choices[2] and demonstrating that we can identify all such spaces naturally. The key result of this chapter is that we can pass between the theories in a natural way, so that $Z_{TV,\mathcal{C}}(\Sigma) \cong Z_{RT,Z(\mathcal{C})}(\Sigma)$ independent of any choices.

This chapter is organized as follows. First, we briefly review the theory of parametrized surfaces from 6]. Next, we examine the effect of passing between parametrizations on the associated TV state spaces. In particular, we show that each of the moves yields a natural map between state spaces, which under projection gives the same identification between vector spaces as that in RT. This establishes an equivalence of theories at the level of surfaces. Finally, we consider extended 3-manifolds with boundary and show that both theories give the same answer. Along the way, we explain how Turaev-Viro theory works for 3-manifolds with embedded ribbon graphs.

[2]The parametrization in RT and the cell decomposition in TV.

5.1 Surface Decompositions

In this section we briefly review the notion of a parametrized surfaces. For a complete exposition, see 6]. Informally, a parametrization is a way of writing a surface Σ as the union on punctured spheres, together with a fixed identification of each punctured sphere with a *standard* sphere. The *standard sphere* with n punctures is defined formally as

$$S_{0,n} = \mathbb{CP}^1 \backslash \{D_1, \ldots, D_n\}; D_j = \{z||z - z_j| < \epsilon\}, z_1 < \cdots < z_n \tag{5.1.1}$$

where ϵ is sufficiently small so the boundary circles do not intersect. We also fix a point $p_i \in \partial D_i$. Note that we have fixed an ordering of the boundary circles, so we can refer to the set of boundary components by $\{\mathbf{1}, \ldots, \mathbf{n}\}$.

Definition 5.1.1. An *extended surface* is a compacted oriented surface Σ , possibly with boundary, together with a fixed point p_α on each boundary component $(\partial\Sigma)_\alpha$.

Note that there are several other equivalent ways of defining an extended surface (See 5]).

Definition 5.1.2. A *colored* extended surface is an extended surface together which a choice of label $Z_\alpha \in Z(\mathcal{C})$ for each marked point p_α.

We now give the main definition of this section. Let Σ be a colored extended surface.

Definition 5.1.3. A *parametrization* of Σ consists of

1. A finite set C of non-intersecting simple closed curves on Σ such that $\Sigma\backslash C$ is of genus zero. We call C the set of *cuts* and fix a point on each cut.

2. For each component Σ_a of $\Sigma\backslash C$, a homeomorphism $\psi : \Sigma_a \to S_{0,n_a}$

Two parametrizations are considered equivalent if they are isotopic [3]. There is a nice graphical way of describing parametrized surfaces. Namely, take the standard sphere with the graph as shown in Figure 5.1. This graph connects

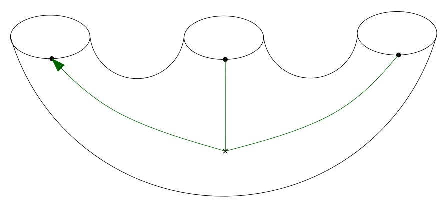


Figure 5.1: The graph on $S_{0,3}$

a single internal vertex to each of the points p_α fixed on the boundary and labels the edge connected to circle **1** by an arrow.

To depict a parametrization of any surface Σ, we draw the cuts on Σ. Then for each connected component Σ_α, we pull back the graph on the standard sphere by ψ_α to obtain a graph M_α on Σ_α. Clearly, such data are equivalent (up to isotopy) to specifying a parametrization and henceforth we will refer to a parametrization as a pair (C, M) where C is a set of cuts on Σ and $M = \cup_\alpha M_\alpha$. When possible, we will often draw the graphs M_α in the plane, ignoring the sufaces into which they are embedded. The reader should have no difficulty passing between such a graph and the surface it represents.

[3]Both the set of cuts and the homemorphisms of boundary components are considered up to isotopy

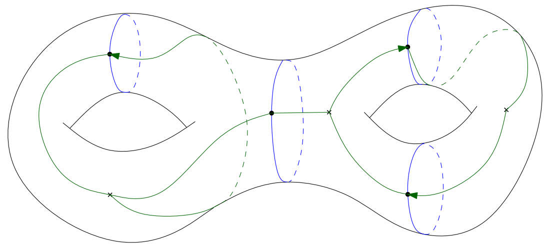


Figure 5.2: A parametrization of a genus two surface $\Sigma = S_{0,3} \sqcup S_{0,3} \sqcup S_{0,2}$. The blue lines are cuts and the green lines are graphs M_α

Definition 5.1.4. Let Σ be a parametrized sphere with n boundary components colored by $Z_1, \ldots, Z_n \in \mathrm{Irr}(Z(\mathcal{C}))$. We define the Reshetikhin-Turaev invariant of Σ to be

$$Z_{RT,Z(\mathcal{C})}(\Sigma; Z_1, \ldots, Z_n) = \mathrm{Hom}_{Z(\mathcal{C})}(\mathbf{1}, Z_1 \otimes \cdots \otimes Z_n) \tag{5.1.2}$$

More generally, we can defined the Reshetikhin-Turaev invariant for any colored, parametrized surface as follows. $\Sigma \backslash C$ is a union of genus zero surfaces with boundary, each equipped with parametrization inherited from Σ. Let $\Sigma_\alpha, \Sigma_\beta$ be two such components separated by a cut c. Then c corresponds to two boundary circles, one on Σ_α and the other on Σ_β. We may color these components by assigning $Z \in \mathrm{Irr}(Z(\mathcal{C}))$ to one component and Z^* to the other.

Definition 5.1.5. Let (Σ, P) be a colored parametrized surface.

$$Z_{RT,Z(\mathcal{C})}(\Sigma, P) = \bigoplus_{Y_{1_\alpha},\ldots Y_{n_\alpha}} \bigotimes_{\alpha} Z_{RT,Z(\mathcal{C})}(\Sigma_\alpha; Y_{1_\alpha}, \ldots Y_{n_\alpha}) \tag{5.1.3}$$

where the product is over all connected components of $\Sigma \backslash C$, and we color all *newly created* boundary components as described above, summing over all possible colorings.

We will typically denote a simple object $Y_i \in Z(\mathcal{C})$ by its index i. In all that follows, i^* represents the dual object Y_i^*, which is also simple. To simplify formulas, many authors attempt to pick a function $f : Irr(\mathcal{C}) \to Irr(\mathcal{C})$ so that $Y_i^* = Y_{f(i)}$, but one should avoid doing this at all costs since it is often impossible to do so in a consistent manner (See 5], Remark 2.4.2).

Example 5.1.6. Let Σ be the torus with one puncture and parametrization as shown on the left hand side of Figure 5.1 and boundary disk labeled by Y. Then $Z_{RT,Z(\mathcal{C})}(\Sigma) = \bigoplus_{i \in Irr(Z(\mathcal{C}))} Hom_{Z(\mathcal{C})}(\mathbf{1}, Y \otimes i \otimes i^*)$.

Now we describe a set of *moves* between parametrizations of a surface. As we'll see below, we can relate any two decompositions by a finite composition of these moves:

1. The Z-move cyclically permutes the boundary components.

2. The B-move *braids* one boundary component about an adjacent one.

3. The F-move removes a cut. If a cut separates $S_{0,n}$ and $S_{0,m}$, deleting the cut gives a component homeomorphic to $S_{0,m+n-2}$ together with a graph inherited from the original components. Notice that we connect circle **1** from the one sphere to circle **m** of the other, thus resulting in a graph which inherits a natural ordering of boundary circles.

4. The S-move interchanges meridians and longitudes of the punctured torus.

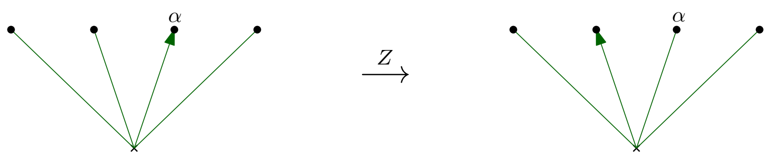


Figure 5.3: Z-move

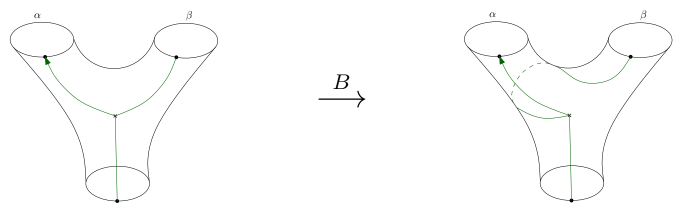


Figure 5.4: B-move

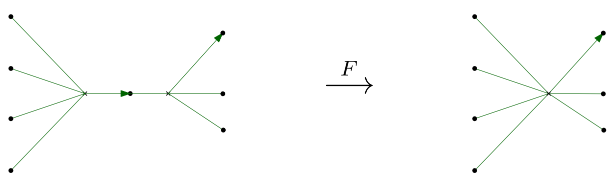


Figure 5.5: F-move

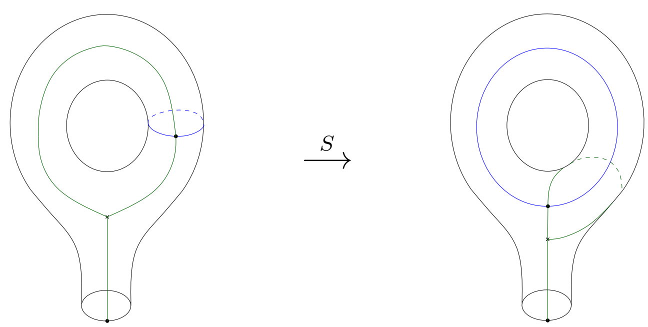


Figure 5.6: S-move

Theorem 5.1.7. *Let $A = (\Sigma, C, M)$ and $A' = (\Sigma, C', M')$ be two parametrizations of a surface. Then A and A' are related by a finite sequence of Z, B, F and S moves described above.*

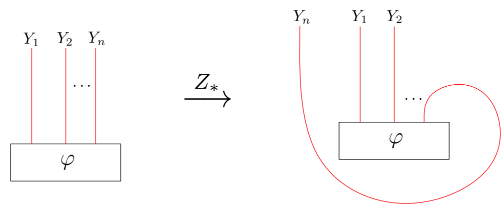
This result has its origins in conformal field theory. It was conjectured by Moore and Seiberg and rigorously proved in 6]. It is a generalization of a result by Hatcher and Thurston (24]), which describes moves between surfaces decomposed into spheres, cylinders and pairs-of-pants, but doesn't take into account the full data of a parametrization. The theorem in 6] does a lot more in fact: it provides a complete set of relations between the above moves, but we will not need this part explicitely.

These moves are important for defining a 2-dimensional (extended) modular functor $\mathcal{F}$. Given the vector space associated to the punctured sphere, one should be able to use the gluing axiom to describe $\mathcal{F}(\Sigma)$ for a surface of any genus. Different parametrizations should give naturally isomorphic vector spaces; one can check that this is so by verifying that it is true for each of the *simple* moves between parametrizations. If P, P' are two parametrizations of a surface Σ related by a single Z, B, F or S move, we can explicitely describe the correspondence between associated vector spaces in RT theory:

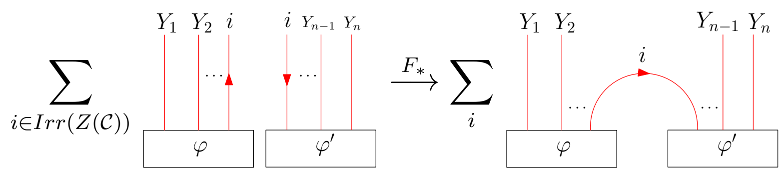
Lemma 5.1.8. *Let P, P' be two parametrizations of a surface Σ and let $X : (\Sigma, P) \longrightarrow (\Sigma, P')$ be any composition of Z, B, F and S moves connecting P and P'. Then X induces an isomorphism $X_* : Z_{RT,Z(\mathcal{C})}(\Sigma, P) \xrightarrow{\cong} Z_{RT,Z(\mathcal{C})}(\Sigma, P')$. This isomorphism is independent of the choice of X. In terms of the generators,*

1. *The Z-move corresponds to the rotation isomorphism:*

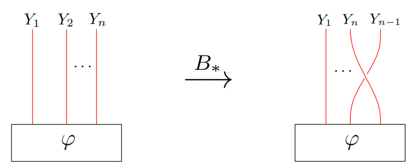
$\langle Y_1, \ldots, Y_n \rangle \to \langle Y_n, Y_1, \ldots, Y_{n-1} \rangle$



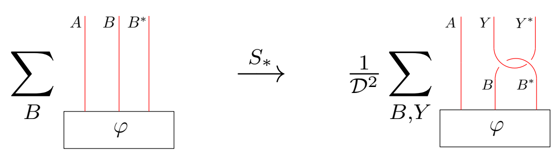
2. *The F-move gives the composition isomorphism. That this is an isomorphism follows directly from semisimplicity.*



3. *The B-move gives the braiding isomorphism*



4. *The S-move gives multiplication by the S-matrix*



5.2 Parametrized surfaces and cell decompositions

In this section, we state and prove the main result of the chapter: TV and RT theories assign the same vector space (up to natural isomorphism) surface Σ, which may have boundary. If $\partial\Sigma \neq \emptyset$, we fix a coloring of each boundary component $\partial\Sigma_i$ by $Z_i \in \mathrm{Irr}(Z(\mathcal{C}))$.

Given two cell decompositions Δ, Δ' of a surface Σ, there is a natural map

$$\Psi_{\Delta',\Delta} : H(\Sigma, \Delta) \longrightarrow H(\Sigma, \Delta') \tag{5.2.1}$$

obtained by computing a state sum on the cylinder $\Sigma \times I$, with a decomposition chosen to agree with Δ on $\Sigma \times 0$ and Δ' on $\Sigma \times 1$. Note, that this map does not depend on the choice of the internal decomposition. We will refer to this map as the *cylinder* map.

The cylinder map is not an isomorphism in general since the dimension of $H_{TV,\mathcal{C}}(\Sigma, \Delta)$ depends on the number of edges of Δ, but it is almost an isomorphism. More precisely, define the space $Z_{TV,\mathcal{C}}(\Sigma, \Delta) = Im(\Psi_{\Delta,\Delta})$. Then

$$\Psi_{\Delta,\Delta'} : Z_{TV,\mathcal{C}}(\Sigma, \Delta) \longrightarrow Z_{TV,\mathcal{C}}(\Sigma, \Delta') \tag{5.2.2}$$

is a natural isomorphism. We can refer to this space as $Z_{TV,\mathcal{C}}(\Sigma)$, since up to natural isomorphism it doesn't depend on the cell decomposition.

Given a parametrized surface Σ, there is a natural way to obtain a cell decomposition of Σ . We have a fixed collection of closed curves dividing Σ into the union of punctured spheres. These cuts become 1-cells in the cell decomposition. Further, for each punctured sphere thus obtained, we have a graph from our parametrization terminating at fixed points on the boundary circles. Each edge of this graph becomes a 1-cell and the points at which the 1-cells terminate become vertices. It is easy to see that these choices define a cell decomposition in the sense of Section 3.1. We call the cell decomposition obtained in this way, the *associated* cell decomposition to parametrization P.

Recall that for a punctured sphere with standard cell decomposition (Figure 5.1), we have a projection $H_{TV,\mathcal{C}}(S^2) \xrightarrow{\pi} Z_{TV,\mathcal{C}}(S^2) \cong Z_{RT,Z(\mathcal{C})}(S^2)$. The associated inclusion map i can be described graphically: The normalization

$$\varphi \xrightarrow{i} \bigoplus_{x_1,\ldots,x_n} \prod_{j=1}^{n} \sqrt{d_j}$$

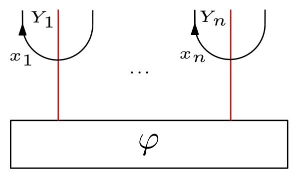


Figure 5.7: $i : Z_{RT,Z(\mathcal{C})}(S^2) \hookrightarrow H_{TV,\mathcal{C}}(S^2)$

factors are chosen to agree with that in 2], so that $\pi \circ i = Id$.

We have two parallel notions in TV and RT theory. On the RT side, we have surface parametrizations and passing between any two parametrizations gives an isomorphism as described earlier in Lemma 5.1.8. On the TV, side, we have cell decompositions; passing between any two cell decompositions gives a natural isomorphism obtained from a cylinder as described earlier. The following theorem, which implies the main result in this chapter, shows that these two notions are the same, up to projection.

Theorem 5.2.1. *Let P, P' be two parametrizations of a surface Σ with associated cell decompositions Δ, Δ' respectively. Then the diagram in Figure* 5.8 *commutes.*

Here, X_* is the map described in Lemma 5.1.8*, j is the map described in Figure* 5.7 *followed by projection to $Z_{TV,\mathcal{C}}(\Sigma)$and Ψ is the isomorphism described in* (5.2.2)*.*

Proof. To show the diagram commutes, we will verify that it does for each of the generators Z, B, F and S. The Z and B moves are essentially immediate, while the F and S moves require some work. Throughout the proof,

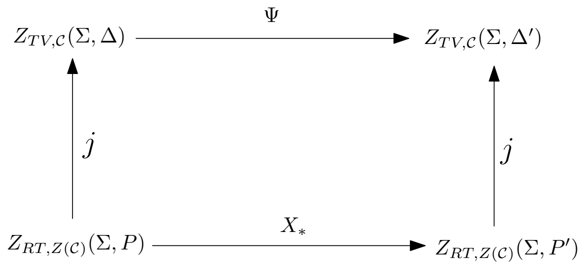


Figure 5.8:

our convention will be that diagrams of surfaces represent the vector spaces associated to them. In particular, a parametrized surface (Σ, P) represents $Z_{RT,Z(\mathcal{C})}(\Sigma, P)$ and a cell-decomposed surface (Σ, Δ) represents $Z_{TV,\mathcal{C}}(\Sigma, \Delta)$. In the diagrams below, we have written Ψ from Figure 5.8 as the composition of several elementary steps for the reader's edification. We have moved several of the large diagrams to Section 5.3.

The Z-move

This follows directly from the natural isomorphism from Lemma 5.1.8(1).

The B-move

A proof of this fact may be found in 2] (lemma 2.1), where we provide an explicit computation.

The F-move

We will show that the diagram in Figure 5.16 commutes . The arrow labeled F is the isomorphism described in Lemma 5.1.8, those labeled i are inclusion maps (Figure 5.7), and G is the gluing isomorphism at the level of state-spaces

(Theorem 7.3, 2]) . The other maps are all *cylinder* maps 5.2.1. Notice that this can be done in fewer steps, but the cylinder maps will be more difficult to realize. Figure 5.16 by contrast contains cylinder maps that are all easy to compute.

To check that this diagram commutes we begin with a vector in $Z_{RT,Z(\mathcal{C})}$ and proceed about the diagram in two ways. In Figure 5.17 we give the answer. The explicit computation at each stage left to the reader.

The S-move

We will show that the diagram in Figure 5.18 commutes. We have omitted some intermediate steps on the right side of the diagram as they are much the same as those on the left. Notice that the diagrams connected by the horizontal arrow labeled S are parametrized surface while the others are of cell-decomposed surfaces. We have chosen a convenient cell decomposition as the terminating point of the diagram which is easy to work with since there are simple maps α, β to this space which can be though of as contractions along edges u_1 and u_2 respectively (Figure 5.9). If we start on the bottom left

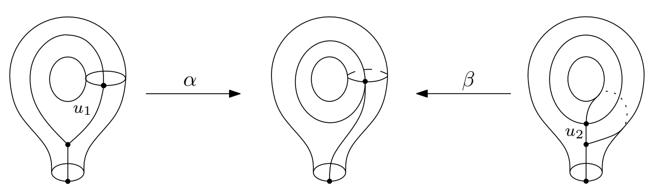


Figure 5.9: To identify the spaces on the left and the right, we use cylinder maps α, β to the space in the center and compare the images of these maps.

of figure Figure 5.18 and proceed around in two different ways, we get two vectors, φ_1, φ_2 in the same space as shown in Figure 5.10

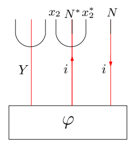
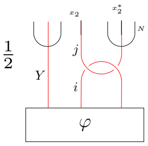
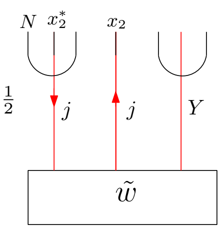
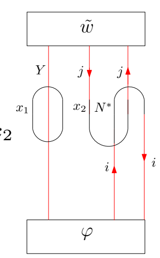
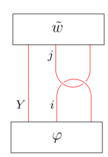
$\varphi_1 = \frac{1}{\mathcal{D}} \sum_{x_1, x_2, N} (d_{x_1} d_{x_2} d_N)^{\frac{1}{2}}$  $; \varphi_2 = \frac{1}{\mathcal{D}^3} \sum_{x_1, x_2, N} (d_{x_1} d_{x_2} d_N)^{\frac{1}{2}}$ 

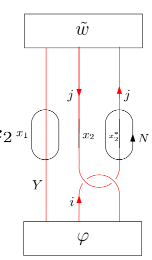
Figure 5.10:

We can easily verify that these vectors are the same by picking some vector w in the dual space and comparing the pairings $\langle \varphi_1, w \rangle$ and $\langle \varphi_1, w \rangle$. Let w be given by

$\frac{1}{\mathcal{D}} \sum_{i, N, x_1, x_2} (d_N d_{x_1} d_{x_2})^{\frac{1}{2}}$ 

where $\tilde{w}$ is some vector in $\bigoplus_j Hom_{Z(\mathcal{C})}(\mathbf{1}, j^* \otimes j \otimes Y)$. Then

$\langle w, \varphi_1 \rangle = \frac{1}{\mathcal{D}^2} \sum d_N d_{x_1} d_{x_2}$  $=$  $= \langle \tilde{w}, S_* \varphi \rangle$

$= \frac{1}{\mathcal{D}^4} \sum d_N d_{x_1} d_{x_2}$  $= \langle w, \varphi_2 \rangle$ $\square$

As an immediate consequence, we get

Theorem 5.2.2. *For any surface (possibly with boundary), we have a natural isomorphism*

$$Z_{RT,Z(\mathcal{C})}(\Sigma) \cong Z_{TV,\mathcal{C}}(\Sigma)$$

We conclude the chapter by combining results in 2], 3] and this paper to prove the following theorem.

Theorem 5.2.3. *Let $\mathcal{C}$ be a spherical fusion category. Then $Z_{TV,\mathcal{C}} \cong Z_{RT,Z(\mathcal{C})}$ as (3-2-1) TQFTs.*

We have already shown that the two TQFTs give the same answer for a closed 3-manifold with an embedded link 2] (Theorem 4.8), and for a surface, possibly with boundary, as shown in Theorem 7.4.1. It remains to show that the theories give the same answer on any 3-manifold with corners.

Let us very briefly review the RT construction for 3-manifolds with corners. For more details, see 5], 45]. Fix a spherical fusion category $\mathcal{C}$.

Definition 5.2.4. An extended 3-manifold $\mathcal{M}$ is an oriented PL 3-manifold with boundary, together with a finite collection of disjoint framed tubes $T_i \subset \mathcal{M}$.

An extended 3-manifold as described above is equivalent to a 3-manifold with an embedded framed tangle in the obvious way. We will use both descriptions interchangeably.

Notice that a tube T_i may terminate on $\partial\mathcal{M}$ in which case we call it an *open* tube, or it may close on itself, forming a solid torus, in which case we call it a *closed* tube.

Definition 5.2.5. A coloring of an extended 3-manifold $\mathcal{M}$ is a choice of color of simple object $Y \in Z(\mathcal{C})$ for each open tube T_i.

We wish now to generalize the famous theorem which states that any closed 3-manifold may be obtained from S^3 via surgery along a framed link.

Definition 5.2.6. A framed link with *coupons* is a framed link where components are allowed to coincide at multivalent vertices, called *coupons*. We often draw coupons as rectangles instead of vertices (Figure 5.11).

Given $L \subset \mathbb{R}^3$, an oriented, framed link with coupons, we can color L as follows. As before, assign to each edge, a simple object $Y \in Z(\mathcal{C})$. To each coupon C_i assign a morphism $\varphi_i \in W_i \equiv \mathrm{Hom}_{Z(\mathcal{C})}(\mathbf{1}, Z_1^{\epsilon_1} \otimes \cdots \otimes Z_n^{\epsilon_n})$, where $Z_1 \dots Z_n$ are the colors of the edges incident the coupon in clockwise cylic order, and $\epsilon_i = 1$ if the strand labeled by Z_i is oriented away from the coupon and -1 otherwise.

As shown in 45], we can evaluate such a link $L \subset \mathbb{R}^3$ to get a number $Z_{RT}(L) \in \mathbb{C}$ in a way that is invariant under isotopy of L. Further, since $\mathcal{C}$ (and hence $Z(\mathcal{C})$) is a spherical category, we can actually view L as lying in S^3.

Equivalently, if we leave the coupons of L uncolored, this construction gives a vector $v \in \bigotimes_i W_i^*$, where the tensor product is over all coupons in L and W_i is the Hom-space associated to coupon C_i. Thus, such a link with uncolored coupons gives a vector space V and a vector $v_L \in V$. Both can be seen to be invariant under isotopy of $L \in S^3$.

We are most interested in a particular type of oriented, framed link with coupons:

Definition 5.2.7. A *special* link X is a framed link with coupons such that some of the link components and coupons are colored by objects and morphisms, respectively, such that the following conditions hold:

- Any uncolored link component is either an annulus, or has both ends on the same uncolored coupon, in which case they are required to be adjacent to one another.

- Uncolored coupons are all of the form shown in Figure 5.11

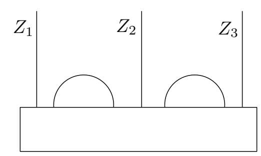


Figure 5.11: A uncolored coupon C in a special link X. Colored strands have a single end terminating on C. Uncolored strands have both ends terminating on C. Further, the ends are adjacent to one another.

Theorem 5.2.8. *Let $\mathcal{M}$ be an extended 3-manifold as in 2]. Then $\mathcal{C}$ may be obtained from S^3 via surgery along some special link $X \subset S^3$, where we define surgery along X by*

$$M_X = M_L \backslash \bigcup_i T(C_i) \tag{5.2.3}$$

Here, we do ordinary surgery along all annular link components L, giving M_L, and remove handlebodies $T(C_i)$ which are tubular neighborhoods of uncolored coupons C_i, as shown in Figure 5.12*.*

Using the surgery description of an extended 3-manifold $\mathcal{M}$, we can easily define the RT invariant for such a manifold. Namely, we express $\mathcal{M}$ as the result of surgery along a *special* link $X \subset S^3$, and define

$$Z_{RT}(\mathcal{M}) \equiv Z_{RT}(X) \tag{5.2.4}$$

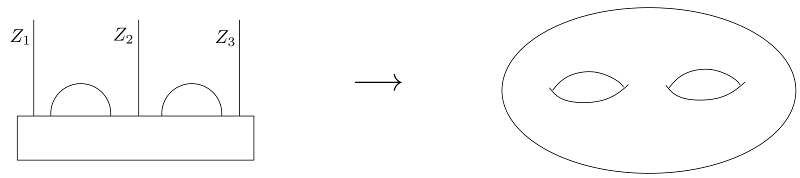


Figure 5.12: A coupon C determines a handlebody $T(C)$ by taking a tubular neighborhood of C and its uncolored strands. The genus of $T(C)$ equals the number of uncolored strands incident to C. The colored strands determine extra data attached to this handlebody, namely marked points and tangent vectors on $\partial(T(C))$ (not pictured), and are important in definining Reshetikhin-Turaev theory. For more details, see 5].

where $Z_{RT}(X)$ is obtained by evaluating the special link X, summing over all possible colorings of unlabeled link components, and using the convention that whenever we color an unlabeled component by simply object $Y \in Z(\mathcal{C})$, we multiply by d_Y, its categorical dimension. As noted above, if X has any uncolored coupons, then $Z_{RT}(X)$ is a vector, not a number.

We can also define Turaev-Viro theory on manifolds with embedded special links.

Definition 5.2.9. Let $\mathcal{M} = S^3_X$ be the 3-sphere with a special link X inside. Then

$$Z_{TV,\mathcal{C}}(\mathcal{M}) \equiv Z_{TV,\mathcal{C}}(\mathcal{M}') \tag{5.2.5}$$

where $\mathcal{M}'$ denotes the manifold with boundary obtained by removing tubular neighborhoods of each coupon.

Lemma 5.2.10. *Let $\mathcal{N}$ be a handlebody of genus g. Then $Z_{TV,\mathcal{C}}(\mathcal{N}) = Z_{RT,Z(\mathcal{C})}(\mathcal{N})$.*

This equality is to be interpreted as follows: Under the canonical isomorphism $Z_{TV,\mathcal{C})}(\Sigma) \cong Z_{RT,Z(\mathcal{C})}(\Sigma)$, where $\Sigma = \partial\mathcal{N}$, the two sides of the equation are identified.

Proof. As computed in 5] (Example 4.5.3),

$$Z_{RT,Z(\mathcal{C})}(\mathcal{N}) = (id : \mathbf{1} \to (\mathbf{1} \otimes \mathbf{1})^{\otimes g}) \in \bigoplus \mathrm{Hom}(1, Z_1 \otimes Z_1^* \otimes \cdots \otimes Z_g \otimes Z_g^*). \tag{5.2.6}$$

. One shows that this equals $Z_{TV,\mathcal{C}}(\mathcal{N})$ by explicit state-sum computation. For $g = 1$, this can be deduced from 3], (Lemma 3.2). The general case is left to the reader as an exercise. □

Our goal is to show that $Z_{TV,\mathcal{C}}(\mathcal{M}) \cong Z_{RT,Z(\mathcal{C})}(\mathcal{M})$ for any extended 3-manifold $\mathcal{M}$. The idea is to convert our extended 3-manifold to a closed 3-manifold $\mathcal{N}$ (possibly with a link inside) by gluing handlebodies of appropriate genus to each component of $\partial\mathcal{M}$.

For simplicity, assume $\partial\mathcal{M}$ has a single component of genus g. Given a vector $\psi \in Z_{TV,\mathcal{C}}(\partial\overline{M})$, we can try to find a handlebody $\mathcal{H}_g$ with an embedded colored tangle, such that $Z_{TV,\mathcal{C}}(\mathcal{H}_g) = \psi$. By the gluing axiom, we get

$$\langle Z_{TV,\mathcal{C}}(\mathcal{M}), \psi\rangle = \langle Z_{RT,Z(\mathcal{C})}(\mathcal{M}), \psi\rangle$$

If we can do this for any ψ, then we are done.

Unfortunately, it is almost never possible to produce such a handlebody, even when $g = 0$. For example, if Σ is the 3-punctured sphere with boundary components labeled by Z_1, Z_2, Z_3, the space $Z_{TV,\mathcal{C}}(\Sigma)$ may be quite large, but there are no extended 3-manifolds with boundary Σ. Indeed Σ is cobordant to $\emptyset$ if and only if it has an even number of punctures.

One approach is to redefine $Z_{TV,\mathcal{C}}(\Sigma)$ as the vector space generated by $\{Z_{TV,\mathcal{C}}(\mathcal{M})\}$, where $\mathcal{M}$ ranges over all extended 3-manifolds with $\partial\mathcal{M} = \Sigma$. The theorem then follows from the above argument and some minor details, which we omit. This approach certainly works, but it is undesirable, e.g. it assigns a zero-dimensional vector space to any surface with an odd number of boundary circles.

A better approach is to use coupons.

Definition 5.2.11. Let $\mathcal{K}$ be the 3-ball with an special link inside as shown in Figure 5.13. We define

$$Z_{TV,\mathcal{C}}(\mathcal{K}) = \psi \in Hom_{Z(\mathcal{C})}(\mathbf{1}, Z_1 \otimes Z_2 \otimes Z_3 \otimes \cdots \otimes Z_N) \tag{5.2.7}$$

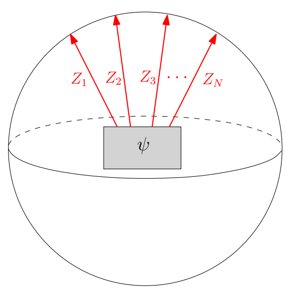


Figure 5.13: The manifold $\mathcal{K}$ is the 3-ball B^3 with a special link consisting of a single coupon and N strands connecting the coupon to ∂K. The coupon is labeled by some morphism $\psi \in \mathrm{Hom}_{Z(\mathcal{C})}(\mathbf{1}, Z_1 \otimes \cdots \otimes Z_N)$.

Notice that if we removed from $\mathcal{K}$ a tubular neighborhood of the coupon, we would be left with a cylinder over the N punctured sphere. We think of the coupon as a handlebody $\mathcal{H}$ which satisfies $Z_{TV,\mathcal{C}}(\mathcal{H}) = \psi$.

Also note that we have defined the value of $Z_{TV,\mathcal{C}}(\mathcal{K})$. It is not a result we can deduce from standard Turaev-Viro theory. However, it is consistent with the rest of our theory. In particular, we can treat $\mathcal{K}$ as an ordinary extended 3-manifold and use the gluing axiom, the graphical calculus descibed in 2]. In particular, if we view the braiding isomorphism, the cup and the cap (3], Section 2) as special examples of coupons, we get the same result as in Definition 5.2.11.

Theorem 5.2.12. *Let $\mathcal{M} = S^3_X$ be the 3-sphere with a special link X inside. Then*

$$Z_{TV,\mathcal{C}}(\mathcal{M}) = Z_{RT,Z(\mathcal{C})}(\mathcal{M}) \tag{5.2.8}$$

Proof. A special case of this theorem is proved in 3] (Theorem 2.3), where the result is proved if X is a colored link (with no coupons). If every coupon of X has the same form as that in Figure 5.13 (every strand that is incident to a coupon is colored and touches the coupon exactly once), the theorem follows immediately from Theorem 2.3 in 3], Definition 5.2.11 and the gluing axiom.

The situation is slightly more complicated if there are coupons of X that have uncolored strands (see Figure 5.11). We could try to come up with an analogous defintion to Definition 5.2.11 for the more complicated coupons, but in this case, the result follows from the state sum formula and Definition 5.2.11.

Let $\mathcal{H}$ be the extended 3-manifold shown in Figure 5.14. $\mathcal{H}$ is a cobordism, so by standard theory, it gives a linear map

$$Z_{TV,\mathcal{C}}(\mathcal{H}) : Z_{TV,\mathcal{C}}(S^2, Z_1, Z_1^*, Z_2, Z_2^*) \longrightarrow Z_{TV,\mathcal{C}}(\Sigma_2) \tag{5.2.9}$$

Notice that the left hand side of (5.2.9) is naturally a subspace of $Z_{TV,\mathcal{C}}(\Sigma_2)$.

Lemma 5.2.13. *The map defined above is given by*

$$Z_{TV,\mathcal{C}}(\mathcal{H}) = Id : Z_{TV,\mathcal{C}}(S^2, Z_1, Z_1^*, Z_2, Z_2^*) \longrightarrow Z_{TV,\mathcal{C}}(\Sigma_2) \tag{5.2.10}$$

where by Id, we mean the identification of the domain with its image under the identity map. If we sum up over all possible colorings of the strands, the two sides of the equation are naturally isomorphic, and we get the identity map.

Proof. We decompose $\mathcal{H}$ as shown in Figure 5.15.

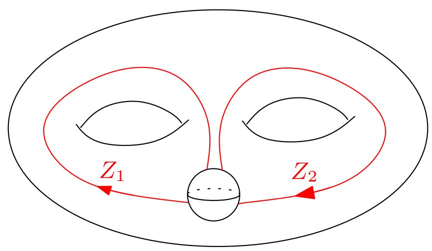


Figure 5.14: The extended manifold $\mathcal{H}$ is obtained by taking a handlebody of genus 2, removing a 3-ball and embedding a colored tangle inside as shown. Thus, $\mathcal{H}$ is a cobordism between the sphere with 4 holes and Σ_2, a suface of genus 2.

The result follows immediately from (2], Example 9.2), where the computation is done in detail. □

An analogous result holds for a handlebody of any genus.

Now we can use Lemma 5.2.13 to finish proving Theorem 5.2.12. Suppose X contains a coupon C with uncolored strands beginning and ending on C (see Figure 5.11 for an example.) Let T(C) be a tubular neighborhood of C, as described earlier. It is a handlebody of some genus g. By definition,

$$Z_{TV,\mathcal{C}}(S^3_X) = Z_{TV,\mathcal{C}}(S^3_X \backslash T(C)).$$

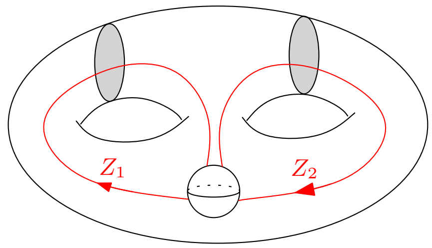


Figure 5.15: The decomposition of $\mathcal{H}$ is given by cutting along the grey disks. The complement of these disks is a cylinder over the sphere with 4 holes. We choose the same cell decomposition (not pictures) for this cylinder as in 2], Example 9.2.

Using Lemma 5.2.13, we can glue an extended manifold $\mathcal{H}$ to $S^3_X \backslash T(C)$. (Here, we sum up over all colorings of strands inside $\mathcal{H}$.) Up to natural isomorphism $Z_{TV,\mathcal{C}}(\mathcal{H})$ is the identity map, so gluing it to $S^3_X \backslash T(C)$ does not change the value of $Z_{TV,\mathcal{C}}(S^3_X \backslash T(C))$.

Our new manifold may be described as $S^3_{X'}$, where X' is the same as X except the coupon C is replaced by a coupon C with no uncolored strands incident to it (so $T(C')$ has genus zero). Repeating this, we reduce X to a special link all of whose coupons have no uncolored strands incident to them. But we already know the theorem to be true in this case! □

We know from before that any extended 3-manifold may be obtained from S^3_X by doing surgery along the annular components of X, and removing tubular neighborhoods of coupons of X. Combining Theorem 5.2.12 with the surgery formula from 3] (Lemma 4.7) gives a proof of Theorem 5.2.3.

5.3 Diagrams

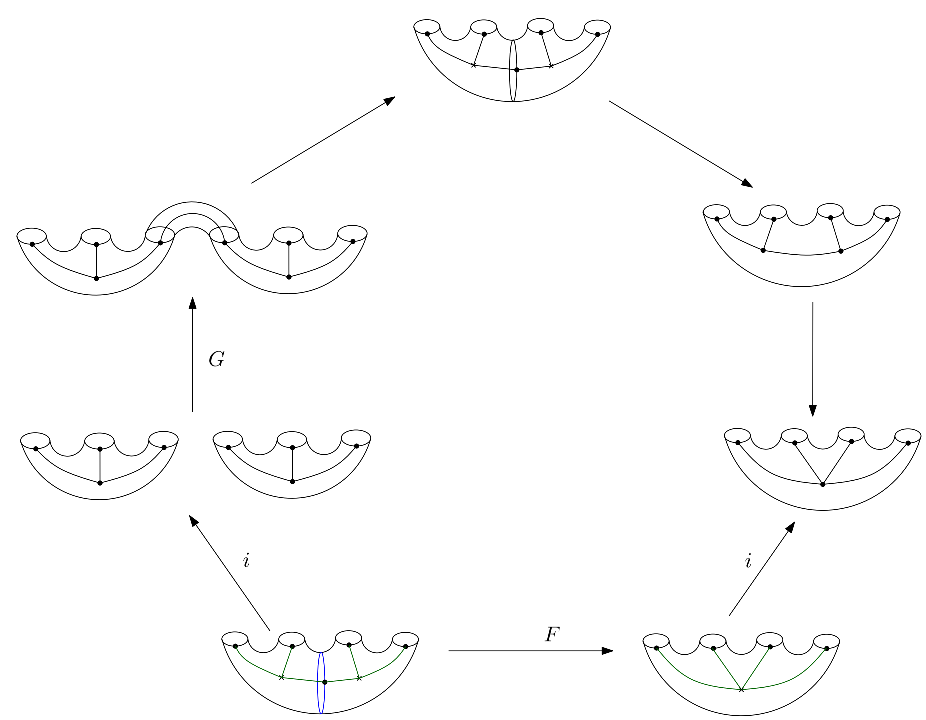


Figure 5.16: The F-move fuses together two spheres along a boundary component. In terms of parametrized surfaces this is realized by simply removing a cut separating the spheres. At the level of cell decompositions we want to identify the result with the standard sphere decomposition (Figure 5.1). We include several intermediate steps to make the computation more transparent.

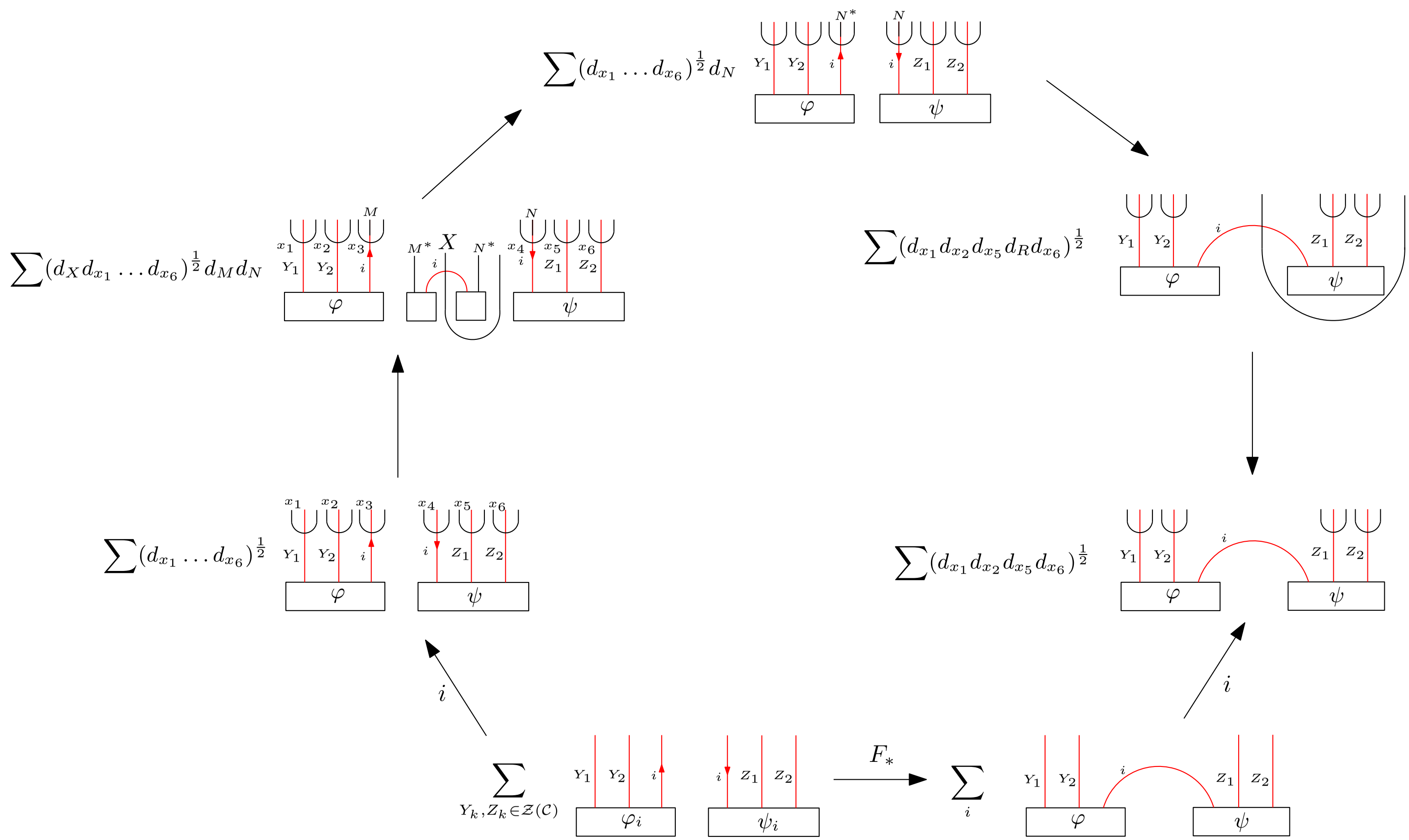


Figure 5.17: A verification that Figure 5.16 commutes. All unlabeled arrow represent cylinder maps. We start on the lower left and proceed in two ways around the diagram.

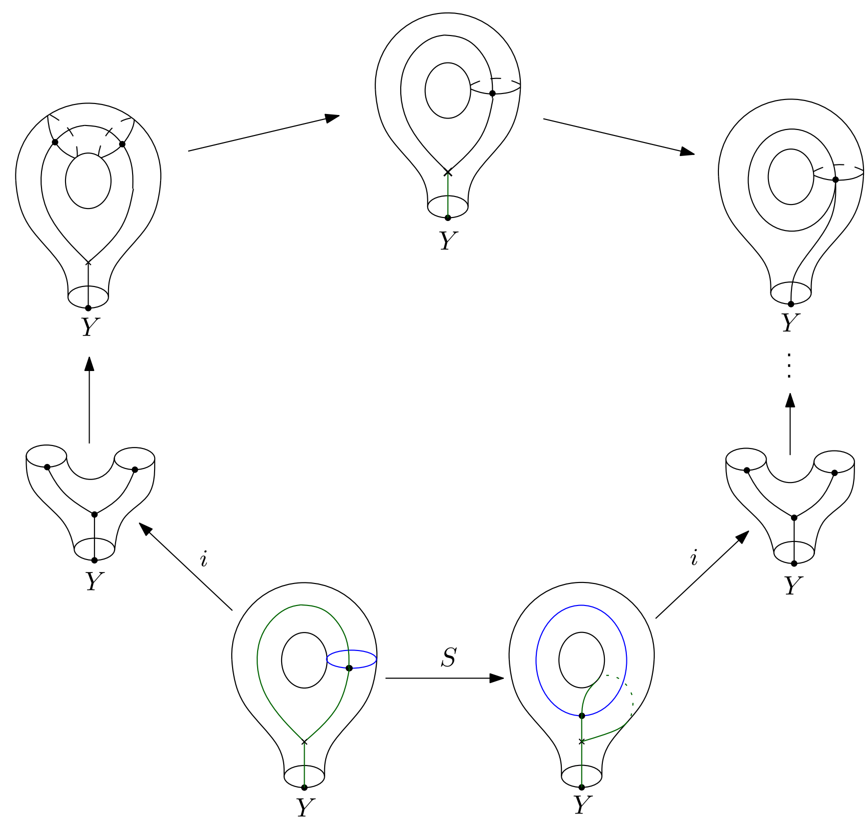


Figure 5.18: The S-move interchanges meridians and longitudes of the 1-punctured torus. Thus, the cut (blue) and the parametrizing graph (green) exchange places under the application of S.

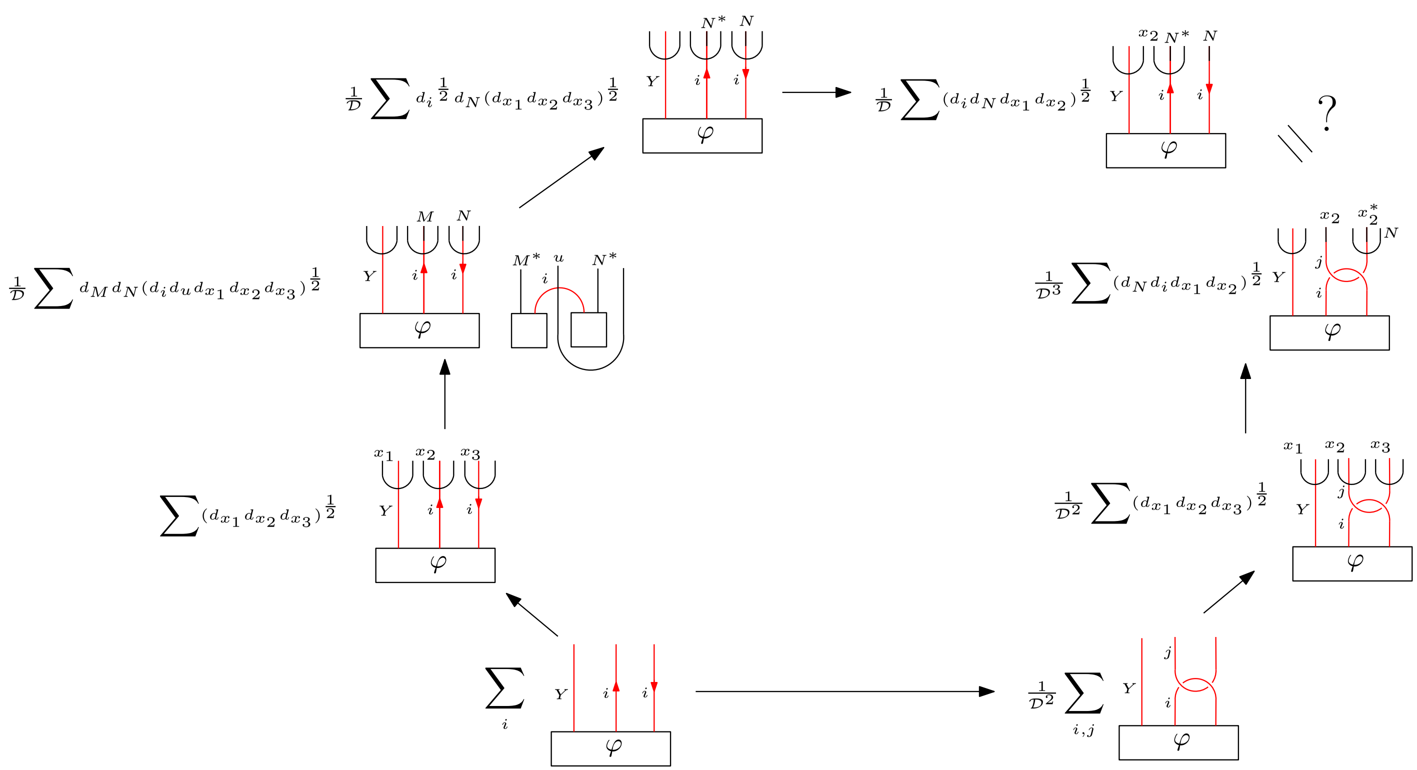


Figure 5.19: A demonstration that Figure 5.18 commutes. Notice that when we proceed around the diagram we get two different pictures (separated by " $\overset{?}{=}$ "), but can verify that they represent the same vector.

Chapter 6

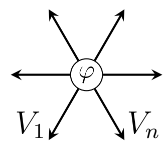
The String-net Model

The string-net model was introduced by Levin and Wen in 2004 (34]) as a model in condensed matter physics. It was independently discovered and studied by Walker (50]) in much greater generality as a framework for describing TQFTs. Given a closed surface Σ and a spherical fusion category $\mathcal{C}$, the string-net space $H^{str}(\Sigma)$ is constructed by considering all embedded graphs in Σ suitably colored by data from $\mathcal{C}$ modulo some local relations. Levin and Wen derive these local relations by requiring that any vector $\varphi \in H^{str}(\Sigma)$ be a fixed point under so-called renormalization group (RG) flow. An equivalent description can be obtained as the ground state of some Hamiltonian on a Hilbert space. In this work we will use the names Levin-Wen model and string-net model interchangeably.

It has long been proposed that the string-net model is equivalent two the two-dimensional part of Turaev-Viro theory. It has been partially proved in 25] and 32] and has been recently proved in full generality in the recent paper by Kirillov 29]. In particular, the subtleties of the model on surfaces with boundary are completely new to Kirillov's paper.

In this chapter, we present a quick overview of the string-net model as described in 29]. We mostly state the results without proof. The reader is encouraged to consult 29] for a much more detailed presentation.

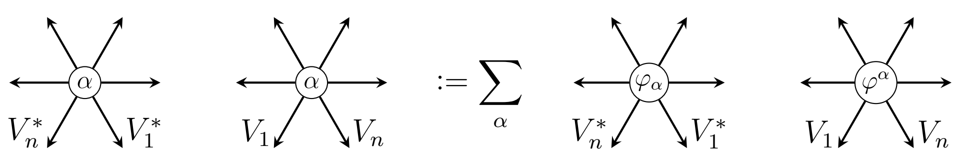
Like in Turaev-Viro theory, we begin with a spherical fusion category $\mathcal{C}$ and consider colored graphs Γ on Σ. Edges of the graph should be oriented and colored by objects of $\mathcal{C}$ (not necessarily simple); vertices are colored by morphisms $\psi_v \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_1 \otimes \cdots \otimes V_n)$, where V_i are colors of edges incident to v taken in clockwise order and with outward orientation (if some edges come with inward orientation, the corresponding V_i should be replaced by V_i^*).



$\varphi \in \mathrm{Hom}(\mathbf{1}, V_1 \otimes \cdots \otimes V_n)$

Figure 6.1: Labeling of colored graphs

We will follow the conventions of 29]; in particular, if a graph contains a pair of vertices, one with outgoing edges labeled $V_1, \ldots, V_n$ and the other with edges labeled $V_n^*, \ldots, V_1^*$, and the vertices are labeled by the same letter α (or β, or ...) it will stand for summation over the dual bases:



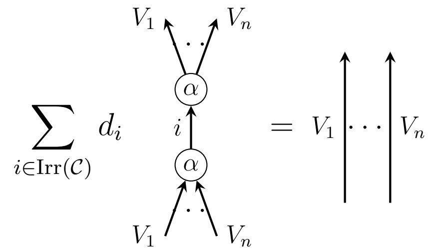
(6.0.1)

where $\varphi_\alpha \in \langle V_1, \ldots, V_n \rangle$, $\varphi^\alpha \in \langle V_n^*, \ldots, V_1^* \rangle$ are dual bases with respect to pairing (2.4.4).

We then define the string-net space

$$H^{str}(\Sigma) = \text{Formal linear combinations of colored graphs on } \Sigma/\text{Local relations}$$

Local relations come from embedded disks in Σ; the precise definition can be found in 29]. Here we only give one local relation which will be useful in the future:



(6.0.2)

The following result has been stated in a number of papers; a rigorous proof can be found in 29].

Theorem 6.0.1. *Let $\mathcal{C}$ be a spherical fusion category and Σ – a closed oriented surface. Then one has a canonical isomorphism $Z_{TV,\mathcal{C}}(\Sigma) \simeq H^{str}(\Sigma)$.*

In fact, we will need a more detailed version of the theorem above. Namely, let Δ be a cell decomposition of Σ. Let $\Sigma - \Delta^0$ be the surface with punctures obtained by removing from Σ all vertices of Δ and let $H^{str}_\Delta = H^{str}(\Sigma - \Delta^0)$ be the corresponding string-net space. Then one has the following results.

Theorem 6.0.2.

1. *The natural map $H^{str}_\Delta \to H^{str}$ induces an isomorphism*

$$H^{str} \simeq \mathrm{Im}(B^s) = \{\psi \in H^{str}_\Delta \mid B^s_p\psi = \psi \ \forall p\} \subset H^{str}_\Delta$$

where $B^s = \prod_p B^s_p$, p runs over the set of vertices of Δ and $B^s_p \colon H^{str}_\Delta \to H^{str}_\Delta$ is the operator which adds to a colored graph a small loop around puncture p as shown below. (The superscript s is introduced to avoid *confusion with the plaquette operators B_p in Kitaev's model; relation between the two operators is clarified below.)*

$$\sum_i \frac{d_i}{\mathcal{D}^2}$$

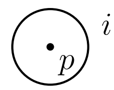


Figure 6.2: Operator B_p

2. *One has a natural isomorphism $H_{TV}(\Sigma, \Delta) \simeq H^{str}_{\Delta}$*

3. *Under the isomorphism of the previous part, the operator associated to the cylinder $Z_{TV}(\Sigma \times I)\colon H_{TV} \to H_{TV}$ is identified with the projector $B^s = \prod_p B_p \colon H^{str}_{\Delta} \to H^{str}_{\Delta}$.*

The proof of this theorem can be found in 29]; obviously, it implies Theorem 6.0.1.

We can now describe the Levin-Wen model as an extended theory, in which we allow surfaces with boundary. We give an overview of the theory, referring the reader to 29] for a detailed description.

Recall that given a spherical category $\mathcal{C}$, we defined the notion of a colored graph Γ on an oriented surface Σ_0. For a surface with boundary, we consider colored graphs which may terminate on the boundary, and the legs terminating on the boundary should be colored by objects of $\mathcal{C}$. Thus, every colored graph Γ defines a collection of points $B = \{b_1, \ldots, b_n\} \subset \partial\Sigma_0$ (the endpoints of the legs of Γ) and a collection of objects $V_b \in \mathrm{Obj}\ \mathcal{C}$ for every $b \in B$: the colors of the legs of Γ taken with outgoing orientation. We will denote the pair $(B, \{V_b\})$ by $\mathbf{V} = \Gamma \cap \partial\Sigma$ and call it *boundary value*. Similar to the closed case, we can define, for a fixed boundary value $\mathbf{V}$, the string-net space

$$H^{str}(\Sigma_0,\mathbf{V}) = \begin{pmatrix}\text{formal combinations of colored}\\ \text{graphs with boundary value } \mathbf{V}\end{pmatrix} /\text{local relations}$$

It was shown in 29] that boundary conditions actually form a category $\widehat{\mathcal{A}}(\partial\Sigma_0)$ so that $H^{str}(\Sigma_0,\mathbf{V})$ is functorial in $\mathbf{V}$. Moreover, if we denote by $\mathcal{A}(\partial\Sigma_0)$ the pseudo-abelian completion of this category, then one has an equivalence

$$\begin{aligned} J\colon \mathcal{A}(S^1) &\simeq \mathcal{A}\\ \{V_1,\dots,V_n\} &\mapsto I(V_1\otimes\dots\otimes V_n)\end{aligned}$$

where $\mathcal{A} = Z(\mathcal{C})$ is the Drinfeld center of $\mathcal{C}$ and $I\colon \mathcal{C}\to\mathcal{A}$ is the adjoint of the forgetful functor $Z(\mathcal{C})\to\mathcal{C}$. Thus, if $\partial\Sigma_0$ is a union of n circles, then a choice of parametrization $\psi\colon \partial\Sigma_0 \simeq S^1\sqcup\dots\sqcup S^1$ gives rise to an equivalence of categories $\mathcal{A}(\partial\Sigma_0)\simeq\mathcal{A}^{\boxtimes n}$.

Since any functor $\widehat{\mathcal{A}}\to\mathcal{V}ec$ naturally extends to a functor of the pseudoabelian completion $\mathcal{A}\to\mathcal{V}ec$, we can define the string-net space $H^{str}(\Sigma_0,\mathbf{Y})$ for any $\mathbf{Y}\in\mathcal{A}(\partial\Sigma_0)$. Equivalently, given a surface Σ_0 together with a parametrization ψ of the boundary components, we can define the vector space $H^{str}(\Sigma_0,\psi,\mathbf{Y})$, where $\mathbf{Y}=\{Y_1,\dots,Y_n\})$, $Y_a\in Z(\mathcal{C})$.

The space $H^{str}(\Sigma_0,\psi,\mathbf{Y})$ admits an alternative definition. Namely, let Σ be the closed surface obtained by gluing to Σ_0 a copy of the standard 2-disk D along each boundary circle $(\partial\Sigma_0)_a$ of Σ_0, using parametrization ψ_a. So defined, the surface comes with a collection of marked points $p_a=\psi_a^{-1}(p)$, where $p = (1, 0)$ is the marked point on S^1. Moreover, for every point p_a we also have a distinguished "tangent direction" v_a at p_a (in PL setting, we understand it as a germ of an arc staring at p_a), namely the direction of the radius connecting p with the center of the disk D. We will refer to the collection $(\Sigma, \{p_a\}, \{v_a\})$ as an *extended surface*. It is easy to see that given $(\Sigma, \{p_a\}, \{v_a\})$, the original surface Σ_0 and parametrizations ψ_a are defined uniquely up to a contractible set of choices.

For such an extended surface and a choice of collection of objects $\mathbf{Y} = \{Y_1, \dots, Y_n\})$, $Y_a \in Z(\mathcal{C})$, define

$$\hat{H}^{str}(\Sigma, \mathbf{Y}) = \mathrm{VGraph}'(\Sigma, \mathbf{Y})/(\text{Local relations}) \tag{6.0.3}$$

where $\mathrm{VGraph}'(\Sigma, \mathbf{Y})$ is the vector space of formal linear combinations of colored graphs on Σ such that each colored graph has an uncolored one-valent vertex at each point p_a, with the corresponding edge coming from direction v_a (i.e., in some neighborhood of p_a, the edge coincides with the corresponding arc) and colored by the object $F(Y_a)$ as shown in Figure 6.3, and local relations are defined in the same way as before: each embedded disk $D \subset \Sigma$ **not containing the special points** p_a gives rise to local relations.

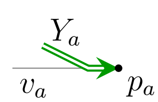


Figure 6.3: Colored graphs in a neighborhood of marked point

The following lemma is a reformulation of results of 29].

Lemma 6.0.3. *Let Σ_0 be a compact surface with n boundary components,*

$\psi\colon \partial\Sigma_0 \simeq S^1 \sqcup \cdots \sqcup S^1$ *— a parametrization of the boundary, and* $\mathbf{Y} = \{Y_1, \ldots, Y_n\}$*),* $Y_a \in Z(\mathcal{C})$ *— a choice of boundary conditions. Then one has a canonical isomorphism*

$$H^{str}(\Sigma_0, \psi, \mathbf{Y}) = \{x \in \hat{H}^{str}(\Sigma, \mathbf{Y}) \mid B_{p_a} x = x \ \forall a\}$$

where for each marked point p_a*, the operator* B_{p_a} *is defined by*

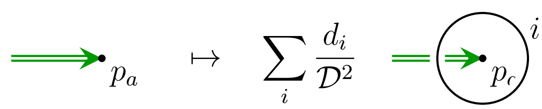


Figure 6.4: Operator B_p for a marked point

The following is the main result of 29].

Theorem 6.0.4. *Let* Σ_0 *be a compact surface with* n *boundary components,* $\psi\colon \partial\Sigma_0 \simeq S^1 \sqcup \cdots \sqcup S^1$ *— a parametrization of the boundary. Then for any* $\mathbf{Y} = \{Y_1, \ldots, Y_n\} \in Z(\mathcal{C})^{\boxtimes n}$*, one has a canonical functorial isomorphism*

$$Z_{TV}(\Sigma_0, \psi, \mathbf{Y}) \cong H^{str}(\Sigma, \mathbf{Y}).$$

where, as before, Σ *is obtained from* Σ_0 *by gluing disks along the boundary.*

As before, we will need a more detailed construction of the isomorphism of this theorem, parallel to the description for closed surfaces given in Theorem 6.0.2. Namely, let Δ_0 be a cell decomposition of Σ_0 such that for every boundary component $(\partial\Sigma_0)_a$, the corresponding marked point $p_a = \psi^{-1}(0, 1)$ is a vertex of Δ_0. By adding to Δ_0 a disk D_a for each boundary component, we get a cell decomposition Δ of closed surface Σ.

Let $\Sigma - \Delta^0$ be the surface with punctures obtained by removing from Σ all vertices of Δ (this includes the marked points p_a). Let $\hat{H}^{str}(\Sigma - \Delta^0, \mathbf{Y})$ be the string-net space defined by boundary condition Y_a near puncture p_a (and trivial boundary conidition near all other punctures). Then one has the following results.

Theorem 6.0.5.

1. *One has an isomorphism*

$$H^{str}(\Sigma, \mathbf{Y}) \simeq \{x \in \hat{H}^{str}(\Sigma - \Delta^0, \mathbf{Y}) \mid B^s_p x = x \; \forall p \in \Delta^0\} \subset \hat{H}^{str}(\Sigma - \Delta^0, \mathbf{Y})$$

   where $B^s_p \colon H^{str}_\Delta \to H^{str}_\Delta$ is the operator which adds to a colored graph a small loop around puncture p as shown in Figure 6.2, Figure 6.4

2. *One has a natural isomorphism $H_{TV}(\Sigma_0, \Delta_0, \mathbf{Y}^*) \simeq \hat{H}^{str}(\Sigma - \Delta^0_0, \mathbf{Y})$*

3. *Under the isomorphism of the previous part, the operator associated to the cylinder $Z_{TV}(\Sigma \times I) \colon H_{TV} \to H_{TV}$ is identified with the projector $B^s = \prod_p B_p \colon \hat{H}^{str}(\Sigma - \Delta^0, \mathbf{Y}) \to \hat{H}^{str}(\Sigma - \Delta^0, \mathbf{Y})$.*

The proof of this theorem can be found in 29]; obviously, it implies Theorem 6.0.4.

Chapter 7

The Kitaev Model

In this chapter, we discuss Kitaev's toric code and its generalizations. The toric code was introduced by Kitaev in 31]. One starts with a finite group G and a closed surface with cell-decomposition. Using these data, one can construct a Hilbert space and Hamiltionian whose ground states are topological invariants of the surface. In this chapter, we show that the ground state, (which is a vector subspace of the beginning Hilbert space) is naturally isomorphic to the Turaev-Viro space associated to that surface with spherical fusion category $\mathcal{C} = Rep(G)$. Much more interesting is the collection of excited states (higher eigenstates of the Hamiltonian). These correspond to a generalization of Kitaev's theory corresonding to surfaces with boundary. As we'll see in this chapter, the excitations in these theories are local— they reside at vertex-plaquette combinations of the cell decomposition called *sites*. Such excitations have characteristic similar to particles, and are called *anyons* in physics literature. The term anyons was chosen to signify that the phase shift when two identical particles are switched can be an arbitrary unitary transormation (on the other hand, fermions and bosons just inherit phase changes of -1 and +1 respectively). Since unitary transformations form the basis for quantum computation, there is hope that the toric code is a suitable model for a quantum computer. And since the ground state is a topological invariant of the surface, the state of such a quantum computer would be *fault-tolerant*; small amounts of noise would not affect the underlying state of the system. The creation of a fault-tolerant quantum computer was the main motivation of Kitaev's work.

The excitations in the Kitaev model based on a group G correspond to irreducible representations of $D(G)$, the Drinfeld Double of G. It is known that there is a braided equivalence $Rep(D(G)) \cong Z(Rep(G))$: the category of representations of the double is braided equivalent to the Drinfeld Center of G (see 27],8.5). Using this theorem, we can say that excitations in Kitaev's model are given by objects of $Z(\mathcal{C})$, where $\mathcal{C} = Rep(G)$. This should look very similar to the boundary conditions in the Turaev-Viro and string-net models! In this chapter, we provide a detailed proof that these theories are, in fact, equivalent.

There have been a lot of papers giving mathematical treatments and generalizations of Kitaev's model. Of particular note is 11], where the authors carefully describe the model generalized to a semisimple Hopf algebra R and, in particular, prove that the excitations correspond to irreducible representations of the double of R. Another comprehensive paper is 8] which discusses the model for a finite group and includes an analysis of ribbon operators, condensation and confinement.

It is important to note that the string-net and Kitaev models assign vector spaces to oriented surfaces, but do not *a priori* give invariants of 3-manifolds. They are examples of 2-dimensional modular functors, whereas Turaev-Viro is a 3d TQFT. It is well-known (see 5]) that these two notions are equivalent. In particular, the 2-dimensional part of Turaev-Viro TQFT is the same as Kitaev and Levin-Wen, so they define the same TQFT. In general, there is no obvious way to construct a TQFT from its underlying modular functor (see 5]). Abusing terminology slightly, we will often refer to the Kitaev and string-net model as TQFTs.

In this section, we look at Kitaev's lattice model. This model is a generalization of the well-known toric code; we get a theory for any finite-dimensional semisimple Hopf algebra R over $\mathbb{C}$. We begin with a compact, oriented surface Σ with a fixed cell decomposition Δ. [1]. We will assign to (Σ, Δ) a finite-dimensional Hilbert space $\mathcal{H}_K(\Sigma)$ and introduce a Hamiltonian consisting of local operators. The ground state of this Hamiltonian is useful for quantum computation. We will later show see that this ground state can be identified with the Turaev-Viro vector space $Z_{TV}(\Sigma)$. This obviously implies that the ground state is a topological invariant of Σ: in particular, it does not depend on the cell decomposition Δ.

From now on, we fix a choice of finite-dimensional, semisimple Hopf algebra R.

7.1 Description of the model

Crude Hilbert space

Given a compact oriented surface Σ with a cell decomposition Δ, we denote by E the set of (unoriented) edges of Δ. Then for any choice $\mathbf{o}$ of orientation of each edge of Σ, we define the space

$$\mathcal{H}_K(\Sigma, \Delta, \mathbf{o}) = \bigotimes_E R \tag{7.1.1}$$

[1]Some papers begin instead with a surface Σ with an embedded graph Γ. This is clearly equivalent data; the graph Γ corresponds to the 1-skeleton of the cell decomposition.

We will graphically represent a vector $\bigotimes x_e \in \mathcal{H}_K$ by writing the vector x_e next to each edge e.

As defined, the above vector space appears to depend on the choice of edge orientation. However, in fact, vector spaces coming from different orientations are canonically isomorphic. Namely, if $\mathbf{o}$ and $\mathbf{o}'$ are two orientations that differ by reversing orientation of a single edge e, we identify

$$\begin{aligned} \mathcal{H}_K(\mathbf{o}) &\to \mathcal{H}_K(\mathbf{o}') \\ x_e &\mapsto S(x_e) \end{aligned} \tag{7.1.2}$$

(see Figure 7.1). Note that since $S^2 = \mathrm{id}$, this isomorphism is well defined. This shows that all spaces $\mathcal{H}_K(\mathbf{o})$ for different choices of orientation are canonically isomorphic to each other; thus, we will drop the choice of orientation from our formulas writing just $\mathcal{H}_K(\Sigma, \Delta)$.

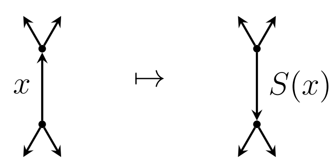


Figure 7.1: The antipode allows us to identify the Hilbert spaces obtained via any two choices of edge orientation.

The Hilbert space $\mathcal{H}_K$ is clearly not a topological invariant; in particular, its dimension depends on number of edges in Δ.

Vertex and plaquette operators

We now define a collection of operators on $\mathcal{H}_K$; in the next section, we will use them to construct the Hamiltonian on $\mathcal{H}_K$. As before, we fix a closed oriented surface Σ and a choice of cell decomposition Δ.

Definition 7.1.1. Let (Σ, Δ) be a surface with a cell decomposition. A *site* s is a pair (v, p) where v is a vertex of Δ and p is an adjacent plaquette (face).

A typical site is shown in Figure 7.2. We will depict a site as a green line connecting a vertex to the center of an adjacent plaquette. Equivalently, if we superimpose the dual lattice, a site connects a vertex in the lattice to an adjacent vertex in the dual lattice.

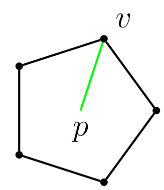
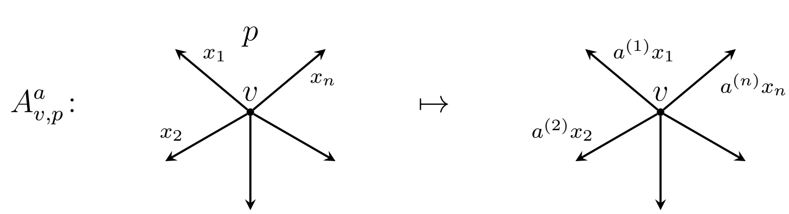


Figure 7.2: The site $s = (v, p)$ is drawn as a green line connecting v and the center of p

At each vertex v, we have a natural counterclockwise cyclic ordering of edges incident to v. Similarly, given a plaquette (2-cell) p, we have the **clockwise** cyclic ordering of edges on ∂p.

Definition 7.1.2. Given a site $s = (v, p)$ of the cell decomposition Δ and an element $a \in R$, the vertex operator $A^a_{v,p} \colon \mathcal{H}_K(\Sigma, \Delta) \to \mathcal{H}_K(\Sigma, \Delta)$ is defined by

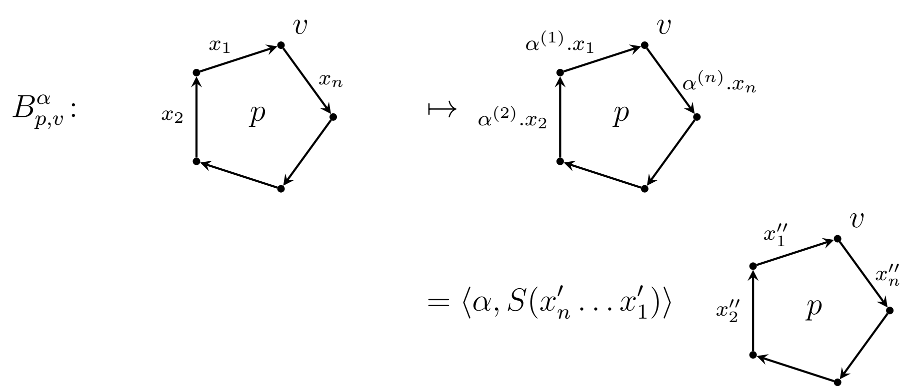


where the edges incident to v are indexed **counterclockwise** starting from p.

In the definition above, the edges incident to v are all pointing away from the vertex. It is easy to see, using (7.1.2), that if any edge is oriented towards the vertex, then left action would be replaced by the right action: instead of $a^{(i)}x_i$, we would have $x_i S(a^{(i)})$.

In a similar way, one defines the plaquette operators.

Definition 7.1.3. Given a site $s = (v, p)$ of the cell decomposition Δ and an element $\alpha \in \overline{R}$, the plaquette operator $B^{\alpha}_{p,v} \colon \mathcal{H}_K(\Sigma, \Delta) \to \mathcal{H}_K(\Sigma, \Delta)$ is defined by



where $\alpha.x$ stands for left action of $\overline{R}$ on R as defined in Section 2.4.

In the definition above, the edges surrounding p are all given a **clockwise** orientation (even though the indices go counterclockwise). It is easy to see, using (7.1.2), that if any edge is oriented counterclockwise, then left action would be replaced by the right action: instead of $\alpha^{(i)}.x_i$, we would have $x_i.S(\alpha^{(i)})$.

Theorem 7.1.4.

1. *If v, w are distinct vertices, then the operators A^a_v, A^b_w commute for any $a, b \in R$.*

Similarly, if p, q are distinct plaquettes, then the operators B_p^α, B_q^β commute for any $\alpha, \beta \in \overline{R}$.

2. *If v, p are not incident to one another, then operators A_v^a, B_p^α commute.*

3. *For a given site $s = (v, p)$, the operators $A_{v,p}^a$, $B_{p,v}^\alpha$ satisfy the commutation relations of Drinfeld double of R: the map*

$$\rho_s \colon D(R) \to \mathrm{End}(\mathcal{H}_K(\Sigma, \Delta)) \tag{7.1.3}$$

$$a \otimes \alpha \mapsto A_v^a B_p^\alpha \tag{7.1.4}$$

is an algebra morphism.

Proof. 1. The operators A_v, A_w obviously commute if the edges incident to v and those incident to w are disjoint. We therefore assume that v and w are adjacent, i.e. at least one edge connects them. Clearly, we need only to check that the actions of A_v, A_w commute on their common support. Suppose such an edge e is oriented so that it points from v to w. Then A_v acts on the corresponding copy of R via the left regular representation, and A_w acts on e via the right regular representation. These are obviously commuting actions. The proof for plaquette operators is similar.

2. Obvious.

3. Follows from the following generalization of Lemma 2.5.3, proof of which we leave to the reader.

Lemma 7.1.5. *Let X be a representation of R, and Y – a representation of $\overline{R}$. For $a \in R$, $\alpha \in \overline{R}$, define the operators $p_a, q_\alpha \in \mathrm{End}(R \otimes X \otimes Y \otimes R)$ by*

$$p_a(u \otimes x \otimes y \otimes v) = a'u \otimes a''x \otimes y \otimes vS(a''')$$
$$q_\alpha(u \otimes x \otimes y \otimes v) = \alpha'''.u \otimes x \otimes \alpha''.y \otimes \alpha'.v$$

Then these operators satisfy the commutation relations of $D(R)$: the map

$$D(R) \to \mathrm{End}(R \otimes X \otimes Y \otimes R)$$
$$a \otimes \alpha \mapsto p_a q_\alpha$$

is a morphism of algebras.

□

Duality

The A and B projectors are dual to one another in the following sense. Consider a *dual* theory, in which we begin with the dual cell decomposition Δ^* with edge orientation inherited from Δ as shown in Figure 7.3 and the dual Hopf algebra $\overline{R}$.

We get the Hilbert space $\overline{\mathcal{H}_K}$ which may be identified with $\mathcal{H}_K^*$ using the evaluation pairing $\mathrm{ev}\colon \overline{R} \otimes R \to \mathbb{C}$. Note that the vertices in Δ^* correspond to plaquettes in Δ and vice versa. The following lemma shows that the vertex operators from one theory correspond naturally to the plaquette operators from the other. .

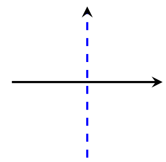


Figure 7.3: The convention for orienting edges of the dual graph is shown above. Here the solid black edge is from the original cell decomposition Δ and the dashed blue arrow belongs to the dual one Δ^*.

Lemma 7.1.6. *Under the natural pairing $\langle , \rangle$ of $\overline{\mathcal{H}_K}$ and $\mathcal{H}_K$, we have*

1. $\langle A_s^\alpha \psi, \varphi \rangle = \langle \psi, B_{s'}^\alpha \varphi \rangle$

2. $\langle B_s^a \psi, \varphi \rangle = \langle \psi, A_{s'}^a \varphi \rangle$

where $\psi \in \mathcal{H}^*, \varphi \in \mathcal{H}$, $a \in R, \alpha \in R^*$ and s and s' denote dual sites in the decomposition Γ and its dual.

Proof. $\langle A_s^\alpha (\beta_1 \otimes \cdots \otimes \beta_n), a_1 \otimes \cdots \otimes a_n \rangle$

$$= \langle \alpha_{(1)} \beta_1, a_1 \rangle \dots \langle \alpha_{(n)} \beta_n, a_n \rangle = \langle \beta_1, \alpha_1 . a_1 \rangle \dots \langle \beta_n, \alpha_n . a_n \rangle$$

$$= \langle \beta_1, \alpha_1 (S(a_{1_{(1)}})) a_{1_{(2)}} \rangle \dots \langle \beta_n, \alpha_n (S(a_{n_{(1)}}) a_{n_{(2)}} \rangle$$

$$= \langle \beta_1 \otimes \cdots \otimes \beta_n, \alpha (S(a_{1_{(1)}}) S(a_{2_{(1)}}) \dots S(a_{n_{(1)}})) \rangle$$

$$= \langle \beta_1 \otimes \cdots \otimes \beta_n, B_s^\alpha (a_1 \otimes \cdots \otimes a_n) \rangle$$

□

The groundspace

Let $\mathcal{H}_K(\Sigma, \Delta)$ be as in Section 7.1. Consider the following special case of the vertex and plaquette operators:

$$A_v = A_v^h$$
$$B_p = B_p^{\bar{h}} \tag{7.1.5}$$

where $h \in R$, $\bar{h} \in \overline{R}$ are the Haar integrals of $R, \overline{R}$.

Note that since $\Delta^n(h)$ is cyclically invariant (see Theorem 2.4.2), the operator A_v only depends on the vertex v and not on the choice of the adjacent plaquette p (which was used before to construct the linear ordering of the edges adjacent to v); similarly, B_p only depends on the choice of p.

Using these operators, we define the Hamiltonian $H\colon \mathcal{H}_K \to \mathcal{H}_K$ by

$$H = \sum_v (1 - A_v) + \sum_p (1 - B_p) \tag{7.1.6}$$

The most important property of this Hamiltonian is that it consists of *commuting* operators.

Theorem 7.1.7.

1. *All operators A_v, B_p commute with each other.*

2. *Each of these operators is idempotent: $A_v^2 = A_v$, $B_p^2 = B_p$.*

Proof. Immediately follows from Theorem 7.1.4 and $h^2 = h$, h is central (Theorem 2.4.2). □

The Hamiltonitan (7.1.6) is a sum of these local projectors and since they all commute, H is diagonalizable.

Definition 7.1.8. The *ground state* $K_R(\Sigma, \Delta)$ of Kitaev's model is the zero eigenspace of H:

$$K_R(\Sigma, \Delta) = \{\psi \in \mathcal{H}_K(\Sigma, \Delta) | H\psi = 0\}.$$

It is easy to see that $\psi \in K_R(\Sigma)$ iff $A_v\psi = B_p\psi = \psi$ for every vertex v and plaquette p.

We will show that up to a canonical isomorphism, the groundspace does not depend on the choice of the cell decomposition.

7.2 The main theorem: closed surface

In this section, we prove the first main result in the paper, identifying the ground space $K_R(\Sigma, \Delta)$ of Kitaev model with the vector space $Z^{\mathcal{A}}_{TV}$ of the Turaev–Viro TQFT with the category $\mathcal{A} = \mathrm{Rep}(R)$.

Theorem 7.2.1. *Let R a finite-dimensional semisimple Hopf algebra. Then for any closed, oriented surface Σ with a cell decomposition Δ one has a canonical isomorphism $K_R(\Sigma, \Delta) \cong Z^{\mathcal{A}}_{TV}(\Sigma)$, where $Z^{\mathcal{A}}_{TV}$ is the Turaev-Viro TQFT based on the category $\mathcal{A} = \mathrm{Rep}(R)$.*

For example, on the sphere S^2, the ground state is one-dimensional, or *non-degenerate* in physics terminology.

The proof of this theorem occupies the rest of this section. For brevity, we will denote the Hilbert space of Kitaev model just by $\mathcal{H}_K$, dropping Σ, Δ from the notation.

Recall that $K_R \subset \mathcal{H}_K$ was defined as the ground space of the Hamiltonian. We begin by introducing an intermediate vector space $\mathcal{H}_A$ such that $K_R \subset \mathcal{H}_A \subset \mathcal{H}_K$. Namely, we let

$$\mathcal{H}_A = \ker(\sum(1 - A_v)) = \{\psi \in \mathcal{H}_K \mid A_v\psi = \psi \ \forall \ v\} \subset \mathcal{H}_K$$

where A_v are vertex operators (7.1.5) and the sum is over all vertices v of Δ.

Since A_v, B_p commute, the B_p operators preserve subspace $\mathcal{H}_A \subset \mathcal{H}_K$. The following equality is obvious from the definitions:

$$K_R = \{\psi \in \mathcal{H}_A \mid B_p\psi = \psi \ \forall \ p\} \tag{7.2.1}$$

where p ranges over all 2-cells of Δ.

We can now formulate the first lemma relating Kitaev's model with the Turaev–Viro TQFT.

Lemma 7.2.2. *One has a natural isomorphism*

$$\mathcal{H}_A(\Sigma, \Delta) \simeq H_{TV}(\Sigma, \Delta^*)$$

where Δ^* is the dual cell decomposition.

Proof. Recall that we have an isomorphism $R \simeq \bigoplus V_i \otimes V_i^*$ (see (4.4.1)). Using this isomorphism, we can give an equivalent description of the vector space $\mathcal{H}_K$. Namely, let us denote by E^{or} the set of oriented edges of Δ, i.e. pairs $\mathbf{e} = (e, \text{orientation of } e)$; for such an oriented edge $\mathbf{e}$, we denote by $\bar{\mathbf{e}}$ the edge with opposite orientation.

Then we can rewrite the definition of $\mathcal{H}_K$ as follows:

$$\mathcal{H}_K = \bigoplus_l \bigotimes_{\mathbf{e}} l(\mathbf{e}) \tag{7.2.2}$$

where the sum is over all ways of colorings of oriented edges $\mathbf{e}$ of Δ by simple objects $l(\mathbf{e})$ of $\mathcal{A}$ so that $l(\bar{\mathbf{e}}) = l(\mathbf{e})^*$ and tensor product is over all *oriented* edges $\mathbf{e}$ of Δ; thus, every unoriented edge e appears in this tensor product twice, with opposite orientations. We will illustrate a vector $v = \bigotimes v_{\mathbf{e}}$ by drawing two oriented half-edges in place of every (unoriented) edge e and writing the corresponding vector $v_{\mathbf{e}}$ next to each half-edge, as shown in Figure 7.4.

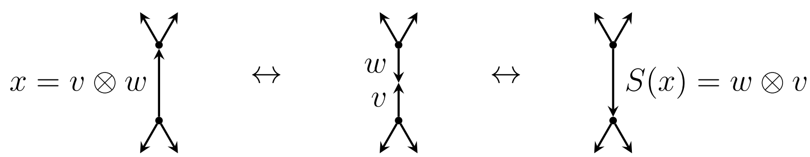


Figure 7.4: $\mathcal{H}_K = \bigoplus_l \bigotimes_{\mathbf{e}} V_{l(\mathbf{e})}$

Re-arranging the factors of (7.2.2), we can write

$$\mathcal{H}_K = \bigoplus_l \bigotimes_v \mathcal{H}_v$$

where the product is over all vertices v of the cell decomposition Δ and

$$\mathcal{H}_v = l(e_1) \otimes \cdots \otimes l(e_n)$$

where the $l(e_1), \ldots, l(e_n)$ are the colors of edges incident to v taken in counterclockwise order with outgoing orientation.

In this language, the vertex operator A_v acts on $\mathcal{H}_v$ by

$$A_v(a_1 \otimes \cdots \otimes a_n) = h^{(1)}a_1 \otimes \cdots \otimes h^{(n)}a_n$$

By Corollary 2.4.4, we see that therefore the image of $\prod A_v$ is the space

$$\mathcal{H}_A = \bigoplus_l \bigotimes_v \langle l(e_1), \dots, l(e_n) \rangle$$

where, as before, $l(e_1), \dots, l(e_n)$ are the colors of edges incident to v taken in counterclockwise order with outgoing orientation.

Since vertices of Δ correspond to 2-cells of Δ^*, this gives an isomorphism

$$\theta \colon \mathcal{H}_A \simeq \bigoplus_l \bigotimes_v H_{TV}(C_v, l) = H_{TV}(\Sigma, \Delta^*)$$

where C_v is the 2-cell of Δ^* corresponding to vertex v of Δ.

However, for reasons that will become clear in the future, we will rescale this isomorphism and define

$$\tilde{\theta} = \sqrt{\mathbf{d}_l}\theta$$

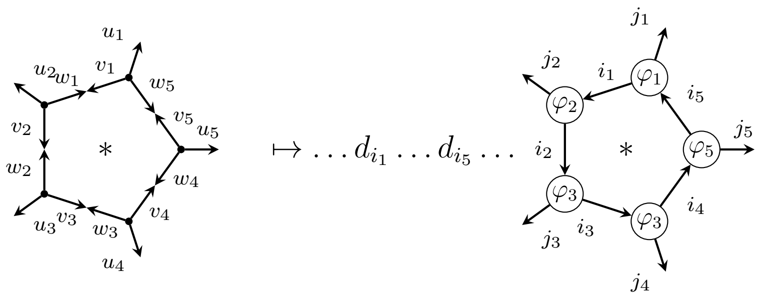
where, for any choice of simple coloring l of edges, $\sqrt{\mathbf{d}_l}$ acts on $\bigotimes_{C_v} H(C_v, l)$ by multiplication by the factor

$$\prod_e \sqrt{d_{l(e)}}$$

where the product is over all unoriented edges e of Δ^*. □

Figure 7.5 shows the composition map $\mathcal{H}_A \xrightarrow{\tilde{\theta}} H_{TV}(\Sigma, \Delta^*) \simeq H^{str}_{\Delta^*}$ (cf. Theorem 6.0.2).

Lemma 7.2.3. *Under the isomorphism of Lemma 7.2.2, the operator* $B = \prod_p B_p \colon \mathcal{H}_A \to \mathcal{H}_A$ *is identified with the operator* $Z_{TV}(\Sigma \times I) \colon H_{TV} \to H_{TV}$.



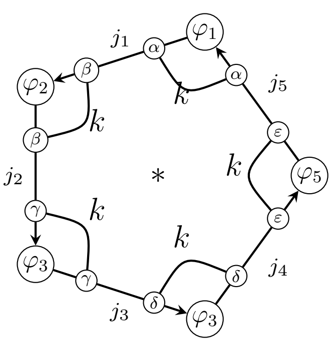
$$v_1 \in V_{i_1}, w_1 \in V_{i_1}^*, \dots$$
$$\varphi_1 = u_1 \otimes v_1 \otimes w_5, \dots$$

Figure 7.5: Isomorphism $\mathcal{H}_A \simeq H^{str}_{\Delta^*}$. Asterisk $*$ shows the puncture obtained by removing a vertex of Δ^*.

Proof. We will prove it in the language of stringnets: combining isomorphism $\mathcal{H}_A \simeq H_{TV}$ (see Lemma 7.2.2) and $H_{TV} \simeq H^{str}_{\Delta^*}$ (see Theorem 6.0.2), we get an isomorphism $\mathcal{H}_A \simeq H^{str}_{\Delta^*}$, and it suffices to prove that under this isomorphism, the plaquette projector B_p of Kitaev model is identified with the projector B^s_p of Levin-Wen model. To avoid complicated notation, we write explicitly the proof in the case shown in Figure 7.5.

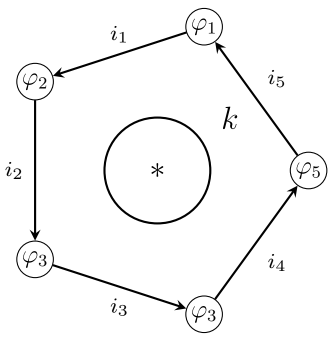
Using Lemma 2.4.8, we see that the projector B_p of Kitaev model can be described as follows: if $\psi \in \mathcal{H}_A$ is as shown in Figure 7.5, then

$$\tilde{\theta}(B_p\psi) = \sum_{k, j_1, \dots, j_5} \frac{d_k}{\dim R}\, d_{j_1} \dots d_{j_5}$$



Now we can use local relations in stringnet space to transform it as follows:

$$\tilde{\theta}(B_p\psi) = \sum_k \frac{d_k}{\dim R}$$



$$= B_p^s(\tilde{\theta}(\psi))$$

(because for $\mathcal{A} = \operatorname{Rep} R$, $\mathcal{D}^2 = \dim R$). □

Combining Lemma 7.2.2, Lemma 7.2.3, we get the statement of the theorem.

Corollary 7.2.4. *The space $K_R(\Sigma, \Delta)$ is independent of the choice of cell decomposition Δ.*

7.3 Excited states and Turaev–Viro theory with boundary

In the previous section, we constructed a Hamiltonian on the Hilbert space $\mathcal{H}_K(\Sigma, \Delta)$. The Hamiltonian had a special form; it was expressed as a sum of local commuting projectors. We saw that the ground state was naturally isomorphic to that in Turaev-Viro theory. In this section we study higher eigenstates of the Hamiltonian, which are typically called excited states. Physically, excited states are interpreted as "quasiparticles" (anyons) of various types sitting on the surface Σ. Excited states can also be described in Turaev-Viro theory, viewed as an extended 3-2-1 TQFT; a particle in this language corresponds to a puncture in the surface with certain boundary conditions.

Excited states in Kitaev model

As before, let Σ be a closed surface with a cell decomposition Δ.

Recall (see Definition 7.1.1) that a site of Δ is a pair $s = (v, p)$ of a vertex and incident edge.

Definition 7.3.1. Two sites $s = (v, p)$ and $s' = (v', p')$ are said to be *disjoint* if v is not incident to p' and v' is not incident to p (which in particular implies that $v \neq v'$ and $p \neq p'$). More generally, we call a collection of n sites *disjoint* if any two among them are disjoint.

The following result immediately follows from Theorem 7.1.4.

Lemma 7.3.2. *Each site* $\boldsymbol{s}$ *defines an action* ρ_s *of* $D(R)$ *on* $\mathcal{H}_K$*; if* s, s' *are disjoint sites, then these actions commute.*

From this perspective, the ground state $K_R(\Sigma)$ has the trivial representation of $D(R)$ attached to *every* site.

In physics language, a representation V of $D(R)$ at a site s models a particle of type V at s, with the trivial representation corresponding to the absence of a particle. Thus, the ground state has no particles at all at any site; it is called the vacuum state.

Now suppose we fix a collection of n disjoint sites $S = \{s_1, \dots, s_n\}$. Define the operator $H_S\colon \mathcal{H}_K \to \mathcal{H}_K$ by

$$H_S = \sum_{v\notin S}(1 - A_v) + \sum_{p\notin S}(1 - B_p) \tag{7.3.1}$$

Let

$$\mathcal{L}(\Sigma, \Delta, S) = \ker(H_S) = \{\psi \in \mathcal{H}_K \mid A_v\psi = \psi\ \forall v \notin S, \quad B_p\psi = \psi\ \forall p \notin S\} \tag{7.3.2}$$

We think of $\mathcal{L}(\Sigma, \Delta, S)$ as the space of n particles fixed at sites $s_1, \dots, s_n$ on the surface; for brevity, we will frequently drop Σ and Δ from the notation, writing just $\mathcal{L}(S)$. Our next goal is to describe this space.

By Lemma 7.3.2, we have an action of the algebra $D(R)^{\otimes S}$ on $\mathcal{L}(S)$. Since the algebra $D(R)^{\otimes S}$ is semisimple, we can write

$$\mathcal{L}(s_1, \dots, s_n) = \bigoplus_{Y_1,\dots,Y_n} (Y_1^* \boxtimes Y_2^* \boxtimes \cdots \boxtimes Y_n^*) \otimes \mathcal{M}(\Sigma, Y_1, \dots, Y_n) \tag{7.3.3}$$

where $Y_1, \dots, Y_n \in \operatorname{Irr}(D(R))$ are irreducible representations of $D(R)$ and $\mathcal{M}(\Sigma, Y_1, \dots, Y_n)$ is some vector space. (Note that $\mathcal{M}$ also depends on the cell decomposition Δ and the set of sites S;; we will usually suppress it in the notation.) The algebra $D(R)^{\otimes S}$ acts in an obvious way on the tensor product $Y_1^* \boxtimes \cdots \boxtimes Y_n^*$ and acts trivially on the space $\mathcal{M}$.

The space $\mathcal{M}(\Sigma, Y_1, \dots, Y_n)$ is called the *protected subspace* in the language of 31]. It is unaffected by local operators, as suggested above, but we can act on it (in a suitable sense), by *nonlocal* operators, such as creating, interchanging or annihilating particles. For example, there is a natural action of the braid group on $\mathcal{M}$ which, with suitable starting data, is capable of performing universal quantum computation.

Our next goal will be relating the protected space $\mathcal{M}(\Sigma, Y_1, \dots, Y_n)$ with the Turaev–Viro and string-net model for surfaces with boundary.

Rewriting the protected space

Throughout this section, $\Sigma, S, \mathbf{Y}$ are as in the previous section.

First, recall that the space $\mathcal{L}(s_1, \dots, s_n)$ has a natural structure of a $D(R)^{\otimes n}$ module, where the i-th copy of $D(R)$ acts on site s_i. Since for any collection $Y_i \in \mathrm{Irr}(D(R)$, the vector space $Y_1 \boxtimes \dots \boxtimes Y_n$ also has a natural structure of $D(R)^{\otimes n}$-module, we can define the action of a $D(R)^{\otimes n}$ on the space

$$(Y_1 \boxtimes \dots \boxtimes Y_n) \otimes \mathcal{L}(s_1, \dots, s_n)$$

using Hopf algebra structure of $D(R)^{\otimes n}$.

Using the decomposition of $\mathcal{L}(s_1, \dots, s_n)$ from (7.3.3), we can extract the protected space $\mathcal{M}$:

$$\mathcal{M}(\Sigma, \mathbf{Y}) \cong [(Y_1 \boxtimes \dots \boxtimes Y_n) \otimes \mathcal{L}(s_1, \dots, s_n)]^{D(R)^{\otimes n}} \tag{7.3.4}$$

Equivalently, consider the vector space

$$\mathcal{H}_K(\Sigma, \Delta, \mathbf{Y}) = (Y_1 \boxtimes \dots \boxtimes Y_n) \otimes \mathcal{H}_K(\Sigma, \Delta)$$

where $\mathcal{H}_K(\Sigma, \Delta)$ is the crude Hilbert space defined in Section 7.1. We will graphically represent vectors in this space by writing a vector $x_e \in R$ next to each oriented edge e, and also drawing, for every site s_i, a green segment connecting v and center of the plaquette p (as in Figure 7.2) labelled by y_i, as shown in Figure 7.6.

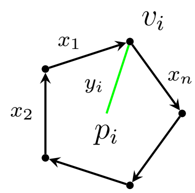
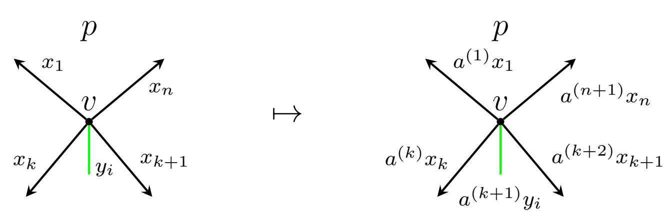


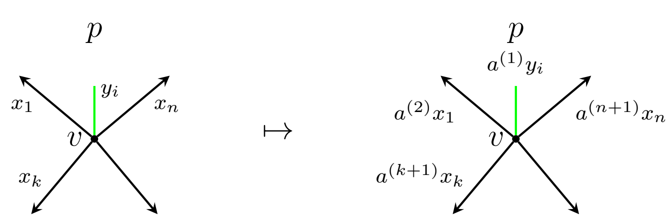
Figure 7.6: Graphical presentation of a vector in $\mathcal{H}_K(\Sigma, \Delta, \mathbf{Y})$

For every vertex v and $a \in R$, define the operator $\tilde{A}_v^a \colon \mathcal{H}_K(\Sigma, \Delta, \mathbf{Y}) \to \mathcal{H}_K(\Sigma, \Delta, \mathbf{Y})$ by $\tilde{A}_v^a = \mathrm{id}_{\mathbf{Y}} \otimes A_v^a$ if $v \notin \{s_1, \dots, s_n\}$ and by the figure below if $v = v_i \in S$:

$A^a_{v,p}$:

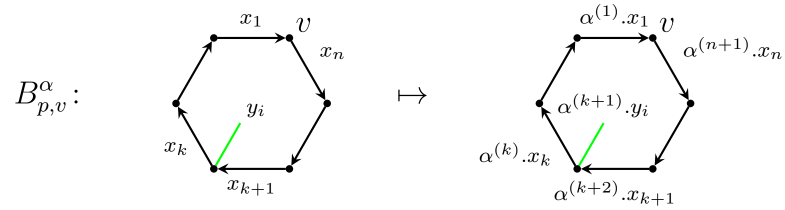


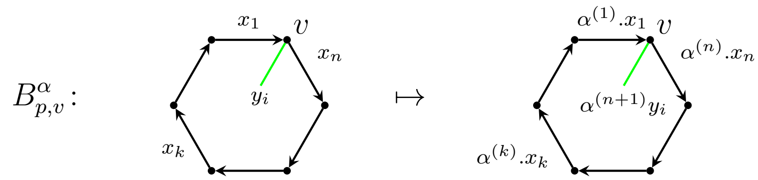
$A^a_{v,p}$:



Similarly, for any plaquette p and $\alpha \in \overline{R}$, define the operator $\tilde{B}_p^\alpha \colon \mathcal{H}_K(\Sigma, \Delta, \mathbf{Y}) \to \mathcal{H}_K(\Sigma, \Delta, \mathbf{Y})$ by $\tilde{B}_p^\alpha = \mathrm{id}_{\mathbf{Y}} \otimes B_p^\alpha$ if $p \notin \{s_1, \dots, s_n\}$ and by the figure below if

$p = p_i \in S$ (recall that comultiplication in $\overline{R}$ is given by $\Delta(\alpha) = \alpha'' \otimes \alpha'$):





It is easy to see that then the operators $\tilde{A}_v$, $\tilde{B}_p$ satisfy the relations of Theorem 7.1.4; in particular, for any site $s = (v, p)$ (including the sites $s_1, \dots, s_n$), the operators $\tilde{A}_v, \tilde{B}_p$ satisfy the relations of Drinfeld double. It follows from the definition of $\mathcal{L}$ and (7.3.4) that

$$\mathcal{M}(\Sigma, \Delta, \mathbf{Y}) = \{x \in \mathcal{H}_K(\Sigma, \Delta, \mathbf{Y}) \mid \tilde{A}^h_v x = B^{\bar{h}}_p x = x \qquad \forall v, p\} \tag{7.3.5}$$

7.4 Comparison of models for extended surfaces

In this section, we establish the relation between the protected space for Kitaev's model and the Turaev–Viro (and thus the Levin-Wen) space for surfaces with boundary, extending Theorem 7.2.1 to surfaces with boundary.

Statement of the main theorem

As before, we fix a semisimple Hopf algebra R over $\mathbb{C}$ and denote $\mathcal{A} = \operatorname{Rep} R$. As was mentioned before, in this case we also have a canonical equivalence of categories $\operatorname{Rep}(D(R)) \cong Z(\mathcal{A})$.

Throughout the section, we fix a choice of a compact oriented surface Σ (without boundary) and a cell decomposition Δ of Σ. We also fix a finite collection of disjoint sites $S = \{s_1, \dots, s_n\}$ and a finite collection $\mathbf{Y} = \{Y_1, \dots, Y_n\}$ of irreducible representations of $D(R)$.

We denote by Δ^* the dual cell decomposition of Σ. Then each site $s_i = (v_i, p_i)$ defines a cell D_i (containing v_i) of Δ^* and a marked point p_i on the boundary of D_i.

Denote by Σ_0 the surface with boundary and marked points, obtained by removing from Σ the interiors of $D_1, \dots, D_n$. Clearly, in this situation Σ can be obtained from Σ_0 by gluing the disks $D_1, \dots, D_n$.

Theorem 7.4.1. *Let Σ, Σ_0, $\mathbf{Y}$ be as above. Then one has a canonical functorial isomorphism*

$$\mathcal{M}(\Sigma, \Delta, \mathbf{Y}) \cong Z_{TV}(\Sigma_0, \mathbf{Y}^*) \cong H^{str}(\Sigma_0, \{p_i\}, \mathbf{Y}),$$

where $\mathcal{M}(\Sigma, \Delta, \mathbf{Y})$ is the protected space defined by (7.3.3)*.*

The following example is instructive.

Example 7.4.2. Let Σ be the sphere with n sites labeled by $Y_1, \dots, Y_n$. Then $Z_{TV}(\Sigma_0, Y_1, \dots, Y_n) \cong \operatorname{Hom}_{D(R)}(\mathbf{1}, Y_1 \otimes \cdots \otimes Y_n)$ 2]. It follows that for $n = 1$, $\mathcal{M}(Y)$ is one-dimensional if Y is trivial one-dimensional representation of

$D(R)$, and $\mathcal{M}(Y) = 0$ if Y is non-trivial irreducible representation of $D(R)$; thus, $\mathcal{L}(s_1) = \mathcal{M}(s_1, \mathbf{1})$ is one dimensional, i.e. there are no single particle excitations on the sphere. For $n = 2$, $Z_{TV}(\Sigma_0, Y, Z) = \mathrm{Hom}_{D(R)}(\mathbf{1}, Y \otimes Z) = 0$, unless $Z \cong Y^*$. It follows that two-particle excitations on the sphere consist of a particle of type Y at one site and a particle of type Y^* at another site.

The proof of the theorem occupies the rest of this section. We begin with some preliminary results.

Lemma on Haar integral

We will need the following technical lemma.

Lemma 7.4.3. *Let Y be a representation of $D(R)$, and let $\bar{h} \in \overline{R}$ be the Haar integral of $\overline{R}$.*

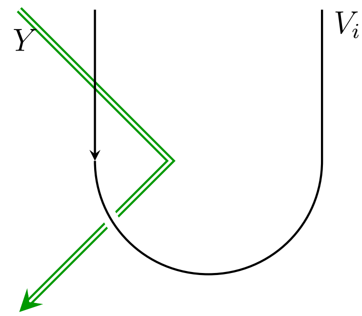
Consider the map

$$\begin{aligned} Y \otimes R &\to Y \\ y \otimes r &\mapsto \bar{h}''.y\langle \bar{h}', r\rangle \end{aligned}$$

where $\lambda.y$ stands for the action of $\overline{R}$ on Y.

Then under the isomorphism $R \simeq \bigoplus V_i \otimes V_i^*$, this map is identified with *the map pictured below*

$$\sum_i \frac{d_i}{\dim R} \tag{7.4.1}$$



where the upper crossing is just the permutation of factors (note that it is not a morphism of modules).

Proof. By definition of the R-matrix (2.5.2), the map (7.4.1) is given by

$$y \otimes v \otimes f \mapsto \frac{d_i}{\mathcal{D}^2} \sum_\alpha x^\alpha y \otimes x_\alpha v \otimes f \mapsto \frac{d_i}{\mathcal{D}^2} \sum_\alpha x^\alpha y \otimes \langle x_\alpha v, f \rangle$$
$$= \sum_\alpha x^\alpha y \langle \bar{h}, x_\alpha v \otimes f \rangle$$

where $v \in V_i$, $f \in V_i^*$, and x^α, x_α are dual bases in $\overline{R}, R$.

Since for any $\lambda \in \overline{R}$, we have $\langle \lambda, xr \rangle = \langle \lambda', x \rangle \langle \lambda'', r \rangle$, this can be rewritten as

$$\sum_\alpha x^\alpha y \langle \bar{h}', x_\alpha \rangle \langle \bar{h}'', v \otimes f \rangle = \bar{h}' y \langle \bar{h}'', v \otimes f \rangle$$

Since $\Delta(\bar{h})$ is symmetric ($\bar{h}' \otimes \bar{h}'' = \bar{h}'' \otimes \bar{h}'$), we get the statement of the lemma. □

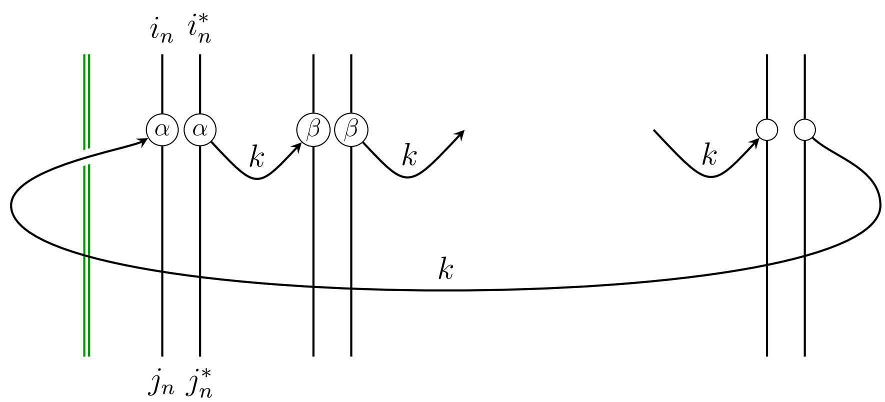
Combining this with the formula for multiplication and comultiplication in R under the isomorphims $R \simeq \bigoplus V_i \otimes V_i^*$, we get the following corollary, generalizing Lemma 2.4.8.

Corollary 7.4.4. *Consider the map*

$$\begin{aligned} Y \otimes R^{\otimes n} &\to Y \otimes R^{\otimes n} \\ y \otimes x_n \otimes \cdots \otimes x_1 &\mapsto \bar{h}^{(n+1)} y \otimes \bar{h}^{(n)}.x_n \otimes \cdots \otimes \bar{h}^{(1)}.x_1 \\ &= \bar{h}'' y \otimes \langle \bar{h}', S(x'_n \dots x'_1) \rangle x''_n \otimes \cdots \otimes x''_1 \end{aligned}$$

Then under the isomorphism $R \simeq \bigoplus V_i \otimes V_i^*$, this map is identified with the map pictured below

$$\sum_{i_1,\dots,i_n,j_1,\dots,j_n,k} \frac{d_{i_1} \dots d_{i_n} d_k}{\dim R} \sum_{\alpha,\beta,\dots}$$



7.5 Proof of the main theorem

We can now complete the proof of Theorem 7.4.1, by combining results of the two previous subsections.

By (7.3.5), the space $\mathcal{M}$ can be obtained from the space $\mathcal{H}_K(\Sigma, \Delta, \mathbf{Y})$ by applying projectors A_v, B_p. Let us consider the intermediate space obtained by A_v projectors only:

$$\mathcal{H}_A(\Sigma, \Delta, \mathbf{Y}) = \{x \in \mathcal{H}_K(\Sigma, \Delta, \mathbf{Y}) \mid \tilde{A}_v^h x = x \qquad \forall v\}$$

Lemma 7.5.1. *We have an isomorphism*

$$\mathcal{H}_A(\Sigma, \Delta, \mathbf{Y}) \cong H_{TV}(\Sigma, \Delta^*, \mathbf{Y}^*) \cong \hat{H}^{str}(\Sigma - \Delta^{*0}, \mathbf{Y})$$

where the space on the right is the string net space on Σ, with the centers of plaquettes removed, and boundary condition Y_i at site s_i (cf. (6.0.3)*).*

Proof. The proof repeats with necessary changes the proof of Lemma 7.2.2. □

Lemma 7.5.2. *Under the isomorphism of the previous lemma, the operators B_p of Kitaev's model (for all p, including $p \in S$) are identified with the operators B_p^s of stringnet model.*

Proof. For $p \notin S$, the proof is the same as in Lemma 7.2.3. For $p \in S$, it follows from Corollary 7.4.4. □

Taken together, thse two lemmas immediately imply Theorem 7.4.1.

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