The Spectrum of Superconformal Theories

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Wenbin Yan

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Wenbin Yan

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Leonardo Rastelli – Dissertation Advisor
Professor, C.N. Yang Institute for Theoretical Physics

Peter van Nieuwenhuizen – Chairperson of Defense
Professor, C.N. Yang Institute for Theoretical Physics

Robert McCarthy
Professor, Department of Physics and Astronomy

Martin Roček
Professor, C.N. Yang Institute for Theoretical Physics

Alexander Kirillov
Professor, Department of Mathematics
Stony Brook University

This dissertation is accepted by the Graduate School

Charles Taber
Interim Dean of the Graduate School

Abstract of the Dissertation

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The spectrum is one of the basic information of any quantum field theory. In general, it is difficult to obtain the full quantum spectrum of QFT. However, in the case of four dimensional superconformal theories, certain information of the quantum spectrum can be extracted exactly. In such theories one can compute exactly certain observables containing spectral information with the help of localization technique.

One such observable is the superconformal index, which is a partition function of the 4d theory on $S^3 \times S^1$, twisted by various chemical potentials. This index counts the states of the $4d$ theory belonging to short multiplets, up to equivalent relations that set to to zero all sequences of short multiplets that may in principle recombine into long ones. By construction, the index is invariant under continuous deformations of the theory. The superconformal index is studied for the class of $\mathcal{N} = 2$ $4d$ superconformal field theories introduced by Gaiotto. These theories are defined by compactifying the $(2, 0)$ 6d theory on a Riemann surface with punctures. The index of the $4d$ theory associated to an n-punctured Riemann

surface can be interpreted as the n-point correlation function of a $2d$ topological QFT living on the surface, which can also be identified as a certain deformation of two-dimensional Yang-Mills theory. With the help of different symmetric polynomials, even explicit formulae are conjectured for all A-type quivers of such class of theories, which in general do not have Lagrangian description. Besides the $\mathcal{N}=2$ theories, the superconformal index of the $\mathcal{N}=1$ $Y^{p,q}$ quiver gauge theories is also evaluated using Römeslberger's prescription. For the conifold quiver $Y^{1,0}$ the result agrees exactly at large N with a previous calculation in the dual $AdS_5 \times T^{1,1}$ supergravity.

The superconformal index of a $4d$ gauge theory is computed by a matrix integral arising from localization of the supersymmetric path integral on $S^3 \times S^1$ to the saddle point. As the radius of the circle goes to zero, it is natural to expect that the $4d$ path integral becomes the partition function of dimensionally reduced gauge theory on S^3. We show that this is indeed the case and recover the matrix integral of Kapustin, Willett and Yaakov from the matrix integral that computes the superconformal index. Remarkably, the superconformal index of the "parent" $4d$ theory can be thought of as the q-deformation of the $3d$ partition function.

To my parents and Luyi.

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Chapter 1

Introduction

The spectrum is always the fundamental information one wants to understand first when studying any quantum theory, of which the most classic example might be Bohr's computation of hydrogen levels. Nevertheless, it is always difficult to obtain the exact spectrum of quantum theories beyond certain simplest cases. For a long time, perturbation theory is the only way to tackle this kind of problems. Although bosonic symmetries can simplify the study of any dynamical system, they are still not constraining enough to prevent quantum corrections from rapidly becoming intractable with increasing loop order. However, supersymmetry is able to help keep the quantum corrections under control, thus becomes a powerful tool for extracting exact information about quantum field theories. In recent years, using methods based on localization, several exact quantities in supersymmetric gauge theories have been computed. This thesis is devoted to one of such observables - the superconformal index.

Superconformal algebras that have R-charges on the right hand side contain special BPS multiplets, which occur only at certain values of energies or conformal dimensions fixed by their charges and have few states than the generic multiplets. An infinitesimal change in the energy (or conformal dimension) of a BPS multiplet turns it into a generic multiplet with finite increase of the number of states. Thus the BPS states are guaranteed to be protected excluding the ones which may combine into a generic representation. The superconformal index [2] is constructed to receive contributions only from those protected states of a superconformal field theory (SCFT). It is a weighted sum over the states of the theory, which by construction evaluates to zero on a generic (long) multiplet. It follows that the index is invariant under exactly marginal deformations, since it is not affected by the recombinations of short multiplets into long ones (or viceversa) that may occur as parameters are varied. For SCFTs admitting a weakly-coupled limit, the index can then be evaluated exactly in free-field theory by a straightforward counting procedure, and takes the form of a matrix integral. However, the superconformal index does not limit itself into SCFTs admitting a weakly-coupled limit only. Some strongly coupled SCFTs may be related to other weakly coupled ones by dualities, thus the indices of the former can be computed by the indices of the later. With the help of suitable dualities and mathematical identities, we can even obtain the superconformal indices of theories without Lagrangian description. The results definitely improve our knowledge on the (protected) spectrum of such theories.

For example, there is a very general web of duality connections relating $\mathcal{N}=2$ $4d$ superconformal field theories, the vast majority of which do not have a weakly-coupled regime nor a conventional Lagrangian description. This fact, which may have been suspected since the early days of string dualities, has taken center stage after the more explicit construction of the $\mathcal{N}=2$ superconformal theories of "class $\mathcal{S}$" [3, 4], most of which are not Lagrangian.[1] Class $\mathcal{S}$ theories arise by compactification of the six-dimensional $(2,0)$ theory on a punctured Riemann surface $\mathcal{C}$. There is a growing dictionary relating four-dimensional quantities with quantities associated to the surface $\mathcal{C}$. A basic entry of the dictionary identifies the exactly marginal couplings of the $4d$ theory with the complex structure moduli of $\mathcal{C}$.[2] According to the celebrated AGT conjecture [9–11], the $4d$ partition functions on the Ω-background [12] and on S^4 [13] are computed by Liouville/Toda theory on $\mathcal{C}$. An analogous relation exists between the $4d$ superconformal index [2, 14] (which can also be viewed as a supersymmetric partition function on $S^3\times S^1$) and topological quantum field theory (TQFT) on $\mathcal{C}$ [15–18]. This last relation is a main topic of this thesis.

The superconformal index is a simpler observable than the S^4 partition function, and it should be a good starting point for a microscopic derivation of the $4d/2d$ dictionary from the $6d$ (2,0) theory. Being coupling-independent, the index is computed by a topological correlator on $\mathcal{C}$ [15], as opposed to a CFT correlator as in the AGT correspondence. For the subset of class $\mathcal{S}$ theories that have a Lagrangian description, it can be easily evaluated in the free-field limit, unlike the S^4 partition function, which is sensitive to non-perturbative physics and requires a sophisticated localization calculation [12, 13].

Despite these simplifying features, the index of class $\mathcal{S}$ theories is still a

[1]Though very large, class $\mathcal{S}$ does not cover the full space of $\mathcal{N}=2$ SCFTs. Counterexamples can be found *e.g.* in [3, 5]. See [5–7] for the beginning of a classification program for $\mathcal{N}=2$ $4d$ SCFTs.

[2]On the other hand, the conformal factor of the metric on $\mathcal{C}$ is irrelevant (in the RG sense) and its memory lost in the IR SCFT. See [8] for a recent holographic check of this fact.

very non-trivial observable with remarkable mathematical structure. First of all, there is no direct way to compute it for the non-Lagrangian SCFTs, which by definition are not continuously connected to free-field theories.[3] An indirect route is to use the generalized S-dualities [3, 28] that relate non-Lagrangian with Lagrangian theories. This is the strategy used in [16] to evaluate the index of the strongly-coupled SCFT with E_6 flavor symmetry [29]. In principle this procedure could be carried out recursively to find the index of all the non-Lagrangian theories, but it suffers from two drawbacks: conceptually, one would rather use the index to test dualities, than assume dualities to compute the index; and practically, this program gets quickly too complicated to be useful.

What one should aim for is a direct algorithm that applies to all class $\mathcal{S}$ theories – one would like to identify and solve the $2d$ TQFT that computes the index. The first step in this direction has been recently taken in [17]: in a limit where a single superconformal fugacity is kept (out of the original three) the $2d$ topological theory is recognized as the zero-area limit of q-deformed Yang-Mills theory. In [18] we generalize this result to a two-parameter slice (q, t) of the three-dimensional fugacity space, which reduces to the limit considered in [17] for $t = q$. We give a fully explicit prescription to compute this limit of the index for the most general[4] A-type generalized quiver of class $\mathcal{S}$. The principle that selects this particular fugacity slice is supersymmetry enhancement, which leads to simplifications. We study systematically the limits where the index receives contributions only from states annihilated by more than one supercharge. The (q, t) slice is the most general limit of this kind sensitive to the flavor fugacities associated to the punctures. We also study another interesting slice (Q, T), where the index receives contribution only from "Coulomb-branch" operators, which are flavor-neutral, so the flavor dependence is lost.

On the other hand, some of the most important examples of interacting $4d$ SCFTs do not have a (known) weakly-coupled description in any duality frame. A large class are the $\mathcal{N} = 1$ SCFTs that arise as IR fixed points of renormalization group flows, whose UV starting points are weakly-coupled theories. A prescription to evaluate the index of such SCFTs was formulated by Römelsberger [14, 19]. This prescription has so far been checked indirectly,

[3] We should mention that for $\mathcal{N} = 1$ SCFTs obtained as IR points of an RG flow, a prescription to compute the index in terms of the UV field content and the charges of the anomaly free R-symmetry was put forward by Romelsberger [14, 19] and recently revisited with more rigor in [20]. Following the seminal work of Dolan and Osborn [21] there have been many checks and implications of this conjecture, see *e.g.* [22–27].

[4] In particular in [17] certain overall normalization factors were determined only for theories with special types of punctures. Here we fill this gap and work in complete generality.

by showing in several examples that it gives the same result for different RG flows that end in the same IR fixed point (*i.e.* the UV theories are Seiberg dual). This was originally observed by Römelsberger, who performed a few perturbative checks in a chemical potential expansion [14, 19]. Invariance of the $\mathcal{N} = 1$ index under Seiberg duality was systematically demonstrated by Dolan and Osborn [21], in a remarkable paper that first applied the elliptic hypergeometric machinery to the evaluation of the superconformal index. These results were extended and generalized in [22–24, 26].

In [25] we apply Römelsberger's prescription to a class of $\mathcal{N} = 1$ SCFTs that admit AdS duals. The canonical example is the conifold theory of Klebanov and Witten [30]. There are infinitely many generalizations: the families of toric quivers $Y^{p,q}$ [1] and $L^{p,q,r}$ [31]. We focus on $Y^{p,q}$. In all these examples there is in principle an independent way to determine the index (at large N) from the dual supergravity. We explicitly show agreement between the gravity calculation of Nakayama [32] and our field theory calculation for the case of the conifold quiver $Y^{1,0}$. According to taste, this can be viewed either as a check of Römelsberger's prescription, or as yet another check of AdS/CFT. The upshot is a sharper bulk/boundary dictionary.

As the superconformal index is successfully applied to both $\mathcal{N} = \in$ and $\mathcal{N} = \infty$ theories, the partition function of supersymmetric gauge theories on S^3 has been used to check a variety of $3d$ dualities including mirror symmetry [33] and Seiberg-like dualities [34]. Remarkably, the exact partition function has also allowed for a direct field theory computation of $N^{3/2}$ degrees of freedom of ABJM theory [35, 36]. The S^3 partition function of $\mathcal{N} = 2$ theories is extremized by the exact superconformal R-symmetry [37–39] so just like the a-maximization in $4d$, the $3d$ partition function can be used to determine the exact R-charges at interacting fixed points.

The superconformal index of a $4d$ gauge theory can be computed as a path integral on $S^3 \times S^1$ with supersymmetric boundary conditions along S^1. All the modes on the S^1 contribute to this path integral. In a limit with the radius of the circle shrinking to zero the higher modes become very heavy and decouple. The index is then given by a path integral over just the constant modes on the circle. In other words, the superconformal index of the $4d$ theory reduces to a partition function of the dimensionally reduced $3d$ gauge theory on S^3. The $3d$ theory preserves all the supersymmetries of the "parent" $4d$ theory on $S^3 \times S^1$. More generally, for any d dimensional manifold M^d, one would expect the index of a supersymmetric theory on $M^d \times S^1$ to reduce to the exact partition function of dimensionally reduced theory on M^d. This idea was applied by Nekrasov to obtain the partition function of $4d$ gauge theory on Ω-deformed background as a limit of the index of a $5d$ gauge theory [12].

A crucial property of the four dimensional index that facilitates its computation is the fact that it can be computed exactly by a saddle point integral. In the limit of vanishing circle radius, this matrix integral reduces to the one that computes the partition function of $3d$ gauge theories on S^3 [33, 40]. It doesn't come as a surprise as the path integral of the $\mathcal{N} = 2$ supersymmetric gauge theory on S^3 was also shown to localize on saddle points of the action.

The rest of the thesis is organized as follows: in chapter 2 we review the definition of the superconformal index, the computation of the index in weakly coupled SCFTs and the re-expression of the index by the special function called the elliptic gamma function. Chapter 3 and 4 deal with the application of the index in $\mathcal{N} = \in$ class $\mathcal{S}$ theories and 2d topological field theory structure behind. The reduction of the 4d index into a 3d partition function will be discussed in chapter 5. Chapter 6 focuses mainly on the index of $Y^{p,q}$ quiver gauge theories and a preliminary holographic check.

Chapter 2

Review of The Superconformal Index

The superconformal index [2] encodes the information about the protected spectrum (which means its conformal dimension does not get quantum corrections) of a superconformal field theory (SCFT) that can be obtained from representation theory alone. The index of a $4d$ SCFT is defined as the Witten index of the theory in radial quantization. Let $\mathcal{Q}$ be one of the Poincaré supercharges in the superconformal algebra of the theory, and $\mathcal{Q}^\dagger = \mathcal{S}$ the conjugate conformal supercharge. Schematically, the index with respect to $\mathcal{Q}$ is defined as [2, 14, 19]

$$\mathcal{I}(\mu_i) = \mathrm{Tr}\,(-1)^F\, e^{-\beta\,\delta}\, e^{-\mu_i \mathcal{M}_i}\,, \qquad \delta = 2\left\{\mathcal{Q}, \mathcal{Q}^\dagger\right\}\,, \tag{2.1}$$

where the trace is over the Hilbert space of the theory on S^3 (or S^{d-1} in general d dimensions) in the usual radial quantization, F is the Fermion number operator, $\mathcal{M}_i$ are $\mathcal{Q}$-closed conserved charges and μ_i the associated chemical potentials. Since states with $\delta > 0$ come in boson/fermion pairs, only the $\delta = 0$ states contribute (the "harmonic representatives" of the cohomology classes of $\mathcal{Q}$), and the index is independent of β. There are infinitely many states with $\delta = 0$ – this is true even for a single short irreducible representation of the superconformal algebra, because some of the non-compact generators (some of the spacetime derivatives) have $\delta = 0$. The introduction of the chemical potentials μ_i serves both to regulate this divergence and to achieve a more refined counting. From the index one can reconstruct the spectrum of short multiplets, up to the equivalence relations that set to zero the combinations of short multiplets that may a priori recombine into long ones [2].

$\mathcal{Q}$	$SU(2)_1$	$SU(2)_2$	$SU(2)_R$	$U(1)_r$	δ	Commuting δs
$\mathcal{Q}_{1-}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\delta_{1-} = E - 2j_1 - 2R - r$	δ_{2+}, $\tilde{\delta}_{1\dot{+}}$, $\tilde{\delta}_{1\dot{-}}$
$\mathcal{Q}_{1+}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\delta_{1+} = E + 2j_1 - 2R - r$	δ_{2-}, $\tilde{\delta}_{1\dot{+}}$, $\tilde{\delta}_{1\dot{-}}$
$\mathcal{Q}_{2-}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\delta_{2-} = E - 2j_1 + 2R - r$	δ_{1+}, $\tilde{\delta}_{2\dot{+}}$, $\tilde{\delta}_{2\dot{-}}$
$\mathcal{Q}_{2+}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\delta_{2+} = E + 2j_1 + 2R - r$	δ_{1-}, $\tilde{\delta}_{2\dot{+}}$, $\tilde{\delta}_{2\dot{-}}$
$\widetilde{\mathcal{Q}}_{1\dot{-}}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\tilde{\delta}_{1\dot{-}} = E - 2j_2 - 2R + r$	$\tilde{\delta}_{2\dot{+}}$, δ_{1+}, δ_{1-}
$\widetilde{\mathcal{Q}}_{1\dot{+}}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\tilde{\delta}_{1\dot{+}} = E + 2j_2 - 2R + r$	$\tilde{\delta}_{2\dot{-}}$, δ_{1+}, δ_{1-}
$\widetilde{\mathcal{Q}}_{2\dot{-}}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\tilde{\delta}_{2\dot{-}} = E - 2j_2 + 2R + r$	$\tilde{\delta}_{1\dot{+}}$, δ_{2+}, δ_{2-}
$\widetilde{\mathcal{Q}}_{2\dot{+}}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\tilde{\delta}_{2\dot{+}} = E + 2j_2 + 2R + r$	$\tilde{\delta}_{1\dot{-}}$, δ_{2+}, δ_{2-}

Table 2.1: For each supercharge $\mathcal{Q}$, we list its quantum numbers, the associated $\delta \equiv 2\left\{\mathcal{Q}, \mathcal{Q}^\dagger\right\}$, and the other δs commuting with it. Here $I = 1, 2$ are $SU(2)_R$ indices and $\alpha = \pm$, $\dot{\alpha} = \pm$ Lorentz indices. E is the conformal dimension, (j_1, j_2) the Cartan generators of the $SU(2)_1 \otimes SU(2)_2$ isometry group, and (R, r), the Cartan generators of the $SU(2)_R \otimes U(1)_r$ R-symmetry group.

2.1 The $\mathcal{N} = 2$ Superconformal Index

For four dimensional $\mathcal{N} = 2$ SCFT with the superconformal algebra $SU(2, 2|2)$, which are non-chiral, different choices of $\mathcal{Q}$ lead to physically equivalent indices. The subalgebra of $SU(2, 2|2)$ commuting with a single supercharge is $SU(1, 1|2)$, which has rank three, so the $\mathcal{N} = 2$ index depends on three superconformal fugacities. In addition, there will be fugacities associated with the flavor symmetries. For definiteness we choose $\mathcal{Q} = \widetilde{\mathcal{Q}}_{1\dot{-}}$. See table 2.1 for a summary of our notations. There are three supercharges commuting with $\widetilde{\mathcal{Q}}_{1\dot{-}}$ and $(\widetilde{\mathcal{Q}}_{1\dot{-}})^\dagger$:

$$\mathcal{Q}_{1-}\,, \qquad \mathcal{Q}_{1+}\,, \qquad \widetilde{\mathcal{Q}}_{2\dot{+}}\,. \tag{2.2}$$

A useful choice is to take as a basis for the Cartan generators of the commutant subalgebra $SU(1, 1|2)$ the three δs of these supercharges. For each $\mathcal{Q}$ the associated δ is defined as

$$\delta \equiv 2\left\{\mathcal{Q}, \mathcal{Q}^\dagger\right\}\,, \tag{2.3}$$

and it has a non-negative real spectrum. We then write the index as

$$\mathcal{I}(\rho, \sigma, \tau) = \mathrm{Tr}(-1)^F \rho^{\frac{1}{2}\delta_{1-}} \sigma^{\frac{1}{2}\delta_{1+}} \tau^{\frac{1}{2}\tilde{\delta}_{2\dot{+}}} e^{-\beta\,\tilde{\delta}_{1\dot{-}}}\,. \tag{2.4}$$

In table 2.1 we give the expressions of the δ charges in terms of the more familiar Cartan generators (E, j_1, j_2, R, r) of $SU(2, 2|2)$. This parametrization of the fugacities makes it easy to consider special limits with enhanced supersymmetry, which is the goal of chapter 4.[1] Another very useful parametrization is in terms of fugacities (t, y, v), related to (σ, ρ, τ) as

$$t = \sigma^{\frac{1}{6}} \rho^{\frac{1}{6}} \tau^{\frac{1}{3}}\,, \quad y = \sigma^{\frac{1}{2}} \rho^{-\frac{1}{2}}\,, \quad v = \sigma^{\frac{2}{3}} \rho^{\frac{2}{3}} \tau^{-\frac{2}{3}}\,. \tag{2.5}$$

This choice, used mainly in previous papers [15, 16, 25], chapter 3 and chapter 5, corresponds to the $(p = t^3 y, q = t^3 y^{-1})$ labels of the elliptic Gamma function [41]. The index in terms of this parametrization reads

$$\mathcal{I}(t, y, v) = \mathrm{Tr}(-1)^F t^{2(E+j_2)} y^{2j_1} v^{-(r+R)} e^{-\beta' \tilde{\delta}_{1\dot{-}}}\,. \tag{2.6}$$

The third parametrization used in this thesis is in terms of fugacities (p, q, t), related to (σ, ρ, τ) as

$$p = \tau\sigma\,, \quad q = \tau\rho\,, \quad t = \tau^2\,. \tag{2.7}$$

This choice corresponds both to the (p, q) labels of the elliptic Gamma function [41], and also, as we shall see in chapter 4, to the (t, q) labels of Macdonald polynomials[2]. In terms of these fugacities, the definition of the index reads

$$\mathcal{I}(p, q, t) \;\; = \;\; \mathrm{Tr}(-1)^F \, p^{\frac{1}{2}\delta_{1+}} \, q^{\frac{1}{2}\delta_{1-}} \, t^{R+r} \, e^{-\beta'' \tilde{\delta}_{1\dot{-}}} \tag{2.8}$$

In appendix B.2 we review the shortening conditions of the $\mathcal{N} = 2$ superconformal algebra and give the expression of the index for the various short multiplets. Given the index of a SCFT, the formulae of appendix B.2 allow to determine its spectrum of short multiplets, up to the usual recombination ambiguities (spelled out in section 5.2 of [42]).

For a theory with a weakly-coupled description the index can be explicitly computed as a matrix integral,

$$\mathcal{I}(V, \rho, \sigma, \tau) = \int [dU] \, \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \, \sum_j f^{\mathcal{R}_j}(\rho^n, \sigma^n, \tau^n) \cdot \chi_{\mathcal{R}_j}(U^n, \, V^n) \right)\,, \tag{2.9}$$

[1]Although at first glance the trace formula (2.4) may seem to depend symmetrically on four equivalent δs, this is not the case. The charge $\tilde{\delta}_{1\dot{-}}$ is special: the associated supercharge $\widetilde{\mathcal{Q}}_{1\dot{-}}$ commutes with all the four δs, but the supercharges associated to the other three δs do not. This is then the index "computed with respect to $\widetilde{\mathcal{Q}}_{1\dot{-}}$", and it is independent of β, which we will usually omit.

[2]Note that while the fugacities (q, p) have exactly the same meaning in previous parametrization, the fugacity t is different from the one introduced before. We made this change of notations to make contact with the Macdonald literature, where t has a canonical definition that one wishes to respect.

Letters	E	j_1	j_2	R	r	$\mathcal{I}(\sigma,\rho,\tau)$	$\mathcal{I}(t,y,v)$	$\mathcal{I}(p,q,t)$
ϕ	1	0	0	0	-1	$\sigma\rho$	$t^2 v$	pq/t
$\lambda_{1\pm}$	$\frac{3}{2}$	$\pm\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\sigma\tau,\ -\rho\tau$	$-t^3 y,\ -t^3 y^{-1}$	$-p,\ -q$
$\bar{\lambda}_{1\dot{+}}$	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\tau^2$	$-t^4/v$	$-t$
$\bar{F}_{\dot{+}\dot{+}}$	2	0	1	0	0	$\sigma\rho\tau^2$	t^6	pq
$\partial_{-\dot{+}}\lambda_{1+} + \partial_{+\dot{+}}\lambda_{1-} = 0$	$\frac{5}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\sigma\rho\tau^2$	t^6	pq
q	1	0	0	$\frac{1}{2}$	0	τ	$t^2/\sqrt{v}$	$\sqrt{t}$
$\bar{\psi}_{\dot{+}}$	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\sigma\rho\tau$	$-t^4\sqrt{v}$	$-pq/\sqrt{t}$
$\partial_{\pm\dot{+}}$	1	$\pm\frac{1}{2}$	$\frac{1}{2}$	0	0	$\sigma\tau,\ \rho\tau$	$t^3 y,\ t^3 y^{-1}$	$p,\ q$

Table 2.2: Contributions to the index from "single letters". We denote by $(\phi, \bar{\phi}, \lambda_{I,\alpha}, \bar{\lambda}_{I\,\dot{\alpha}}, F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}})$ the components of the adjoint $\mathcal{N}=2$ vector multiplet, by $(q, \bar{q}, \psi_\alpha, \bar{\psi}_{\dot{\alpha}})$ the components of the $\mathcal{N}=1$ chiral multiplet, and by $\partial_{\alpha\dot{\alpha}}$ the spacetime derivatives.

or in the other parametrization,

$$\mathcal{I}(V,t,y,v) = \int [dU] \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_j f^{\mathcal{R}_j}(t^n, y^n, v^n) \cdot \chi_{\mathcal{R}_j}(U^n,\, V^n)\right) . \tag{2.10}$$

Here U denotes an element of the gauge group, with $[dU]$ the invariant Haar measure, and V an element of the flavor group. We will discuss briefly the derivation of this equation in section 2.5. The sum is over the different $\mathcal{N}=2$ supermultiplets appearing in the Lagrangian, with $\mathcal{R}_j$ the representation of the j-th multiplet under the flavor and gauge groups and $\chi_{\mathcal{R}_j}$ the corresponding character. The Haar measure has the following property

$$\int [dU] \prod_{j=1}^{n} \chi_{\mathcal{R}_j}(U) = \#\text{of singlets in } \mathcal{R}_1 \otimes \cdots \otimes \mathcal{R}_n \,. \tag{2.11}$$

The functions $f^{(j)}$ are the "single-letter" partition functions, $f^{(j)} = f^V$ or $f^{(j)} = f^{\frac{1}{2}H}$ according to whether the j-th multiplet is an $\mathcal{N}=2$ vector or $\mathcal{N}=2$ $\frac{1}{2}$-hypermultiplet. The "single letters" of an $\mathcal{N}=2$ gauge theory contributing to the index obey $\tilde{\delta}_{1\dot{-}} = E - 2j_2 - 2R + r = 0$ and are enumerated in table 2.2. The first block of table 2.2 shows the contributing letters from the $\mathcal{N}=2$ vector multiplet, including the equations of motion constraint. The second block shows the contributions from the half-hypermultiplet (or $\mathcal{N}=1$ chiral multiplet). The last line shows the spacetime derivatives contributing to the index. Since each field can be hit by an arbitrary number of derivatives, the derivatives give a multiplicative contribution to the single-letter partition functions of the form

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\rho\tau)^m\,(\sigma\tau)^n = \frac{1}{(1-\rho\tau)(1-\sigma\tau)}\,. \tag{2.12}$$

The single-letter partition functions of the $\mathcal{N}=2$ vector and $\mathcal{N}=1$ chiral multiplets are thus given by

$$\begin{aligned} f^V &= -\frac{\sigma\tau}{1-\sigma\tau}-\frac{\rho\tau}{1-\rho\tau}+\frac{\sigma\rho-\tau^2}{(1-\rho\tau)(1-\sigma\tau)} \qquad (2.13)\\ &= \frac{t^2v-\frac{t^4}{v}-t^3(y+y^{-1})+2t^6}{(1-t^3\,y)(1-t^3y^{-1})} \\ &= -\frac{p}{1-p}-\frac{q}{1-q}+\frac{pq/t-t}{(1-q)(1-p)}\,, \\ f^{\frac{1}{2}H} &= \frac{\tau}{(1-\rho\tau)(1-\sigma\tau)}(1-\rho\sigma) \qquad (2.14)\\ &= \frac{\frac{t^2}{\sqrt{v}}-t^4\sqrt{v}}{(1-t^3\,y)(1-t^3y^{-1})}=\frac{\sqrt{t}-pq/\sqrt{t}}{(1-q)(1-p)}\,. \end{aligned}$$

For general values of the three fugacities the explicit expression for the index of a Lagrangian theory is most elegantly expressed [21] in terms of the elliptic Gamma functions (see [41] for a nice review of these special functions). We will postpone the discussion of this topic in section 2.4 and review the index in $\mathcal{N}=1$ theories first.

2.2 The $\mathcal{N}=1$ Superconformal Index

For four dimensional $\mathcal{N}=1$ theories, the supercharges in $SU(2,2|1)$ algebra are

$$\{\mathcal{Q}_\alpha\,,\mathcal{S}^\alpha\equiv\mathcal{Q}^{\dagger\,\alpha}\,,\widetilde{\mathcal{Q}}_{\dot\alpha}\,,\widetilde{\mathcal{S}}^{\dot\alpha}\equiv\widetilde{\mathcal{Q}}^{\dagger\,\dot\alpha}\}, \tag{2.15}$$

where $\alpha=\pm$ and $\dot\alpha=\dot\pm$ are respectively $SU(2)_1$ and $SU(2)_2$ indices, with $SU(2)_1\times SU(2)_2=Spin(4)$ the isometry group of the S^3. The relevant anticommutators are

$$\begin{aligned} \{\mathcal{Q}_\alpha,\,\mathcal{Q}^{\dagger\,\beta}\} &= E+2M_\alpha^\beta+\frac{3}{2}r \qquad (2.16)\\ \{\widetilde{\mathcal{Q}}_{\dot\alpha}\,,\,\widetilde{\mathcal{Q}}^{\dagger\,\dot\beta}\} &= E+2\widetilde{M}_{\dot\alpha}^{\dot\beta}-\frac{3}{2}r\,, \qquad (2.17) \end{aligned}$$

where E is the conformal Hamiltonian, M_α^β and $\widetilde{M}_{\dot\alpha}^{\dot\beta}$ the $SU(2)_1$ and $SU(2)_2$ generators, and r the generator of the $U(1)_r$ R-symmetry. In our conventions, the $\mathcal{Q}$s have $r=-1$ and $\widetilde{Q}$s have $r=+1$, and of course the dagger operation flips the sign of r.

One can define two inequivalent indices, a "left-handed" index $\mathcal{I}^{\mathrm{L}}(t,y)$ and a "right-handed" index $\mathcal{I}^{\mathrm{R}}(t,y)$. For the left-handed index, we pick say[3] $\mathcal{Q}\equiv\mathcal{Q}_-$:

$$\begin{aligned}\mathcal{I}^{\mathrm{L}}(t,y)\equiv \mathrm{Tr}\,(-1)^F t^{2(E+j_1)}y^{2j_2} = \mathrm{Tr}\,(-1)^F t^{3(2j_1-r)}y^{2j_2}\,,\\ \delta = E-2j_1+\frac{3}{2}r\,,\end{aligned} \tag{2.18}$$

where j_1 and j_2 are the Cartan generators of $SU(2)_1$ and $SU(2)_2$. The two ways of writing the exponent of t are equivalent since they differ by a $\mathcal{Q}$-exact term. For the right-handed index, we pick say $\mathcal{Q}\equiv\widetilde{\mathcal{Q}}_{\dot -}$

$$\begin{aligned}\mathcal{I}^{\mathrm{R}}(t,y)\equiv \mathrm{Tr}\,(-1)^F t^{2(E+j_2)}y^{2j_1} = \mathrm{Tr}\,(-1)^F t^{3(2j_2+r)}y^{2j_1}\,,\\ \delta = E-2j_2-\frac{3}{2}r\,.\end{aligned} \tag{2.19}$$

One may also introduce chemical potentials for additional global symmetries of the theory.

2.2.1 Romelsberger's prescription

The expression (2.1) makes sense for a general supersymmetric QFT on $S^3\times\mathbb{R}$. In particular we can consider a theory that flows between two conformal fixed points in the UV and in the IR. At a fixed point (and only at a fixed point), the theory on $S^3\times\mathbb{R}$ is equivalent to a superconformal theory on $\mathbb{R}^4$, and $Q^\dagger$ can be interpreted as a conformal supercharge on $\mathbb{R}^4$. By the usual formal arguments, the index is invariant along the flow (it is independent of the dimensionless coupling RM, where R is the S^3 radius and M the renormalization group scale). For this procedure to make sense, clearly the Q-closed charges $\mathcal{M}_i$ must be well-defined (in particular non-anomalous) all along the RG flow. If the UV fixed point is a free theory, we can compute its index by a matrix integral that counts the gauge-invariant words with $\delta_{UV}=0$. We can then re-intepret the result as the superconformal index of the IR fixed point, which would be difficult to evaluate directly. This leads to the following prescription

[3]Picking $\mathcal{Q}\equiv\mathcal{Q}_+$ would amount to the replacement $j_1\leftrightarrow -j_1$, which is an equivalent choice because of $SU(2)_1$ symmetry. The same consideration applies to the right-handed index, which can be defined either choosing $\widetilde{\mathcal{Q}}_{\dot -}$ or $\widetilde{\mathcal{Q}}_{\dot +}$.

[14, 19]

1. Consider the UV starting point. Write down the "letters" contributing to the index of the free theory, *i.e.* the letters with $\delta_{UV} = 0$.

2. Assign to the letters the quantum numbers corresponding to the anomaly-free symmetries of the interacting theory. In the presence of $U(1)$ global symmetries, follow the usual a-maximization procedure [43] to single-out the anomaly-free R-symmetry that in the IR becomes the $U(1)_r$ of the superconformal algebra.

3. Compute the index in terms of the matrix integral which enumerates gauge-invariant words.

The considerations leading to this recipe are somewhat formal. One direction in which they could be made more precise is to discuss ultraviolet regularization and renormalization. It is not difficult to find a perturbative regulator that preserves one complex $\mathcal{Q}$, and in fact two of them, either the two left-handed charges $\mathcal{Q}_\alpha$, or the two right-handed charges $\widetilde{Q}_{\dot\alpha}$. To preserve say the left-handed charges, we can Kaluza-Klein expand the fields on the S^3, and truncate the theory by keeping all the modes whose right-handed spin $J_2 \leq J_2^{max}$. This truncation is a UV regulator since the left-handed modes will also be cut-off[4], and has the virtue of preserving the left-handed supersymmetry, since the action of $\mathcal{Q}_\alpha$ commutes with the cut-off. A similar regulator (but performed symmetrically on the left-handed and right-handed spins, which in general breaks susy) has been discussed at length in [44–47]. This style of regularization is only perturbative because it breaks the gauge symmetry, which can however be restored order by order in perturbation theory by adding counterterms [44–47]. We see no obstacle in choosing the counterterms so that they preserve one copy of the susy algebra.

We are not aware of a fully non-perturbative regulator that preserves supersymmetry on $S^3 \times \mathbb{R}$ – finding such a regulator would be very interesting in its own right. In any case ultraviolet issues are not expected to affect the play an important role for the index, much as they don't for the usual Witten index on the torus [48].

2.2.2 Computing the index

The "letters" of an $N = 1$ chiral multiplet are enumerated in table 2.3. We assume that in the IR the $U(1)_r$ charge of the lowest component of the multiplet

[4]This is clear from the structure of harmonics on S^3. Scalar harmonics have $SU(2)_1 \times SU(2)_2$ quantum numbers (J, J), spinor harmonics $(J - 1/2, J)$ and $(J, J - 1/2)$ and so on.

ϕ is some arbitrary $r_{IR} = r$ (determined in a concrete theory by anomaly cancellation and in subtle cases a-maximization). According to the prescription we have just reviewed, the index receives contributions from the letters with $\delta_{UV} = 0$, and each letter contributes as $(-1)^F t^{3(2j_1 - r_{IR})} y^{2j_2}$ to the left-handed index and as $(-1)^F t^{3(2j_2 + r_{IR})} y^{2j_1}$ to the right-handed index. To keep track of

Letters	E_{UV}	j_1	j_2	r_{UV}	r_{IR}	$\delta^{\mathtt{L}}_{UV}$	$\mathcal{I}^{\mathtt{L}}$	$\delta^{\mathtt{R}}_{UV}$	$\mathcal{I}^{\mathtt{R}}$
ϕ	1	0	0	$\frac{2}{3}$	r	2	$-$	0	t^{3r}
ψ	$\frac{3}{2}$	$\pm\frac{1}{2}$	0	$-\frac{1}{3}$	$r-1$	$0^+, 2^-$	$-t^{3(2-r)}$	2	$-$
$\partial\psi$	$\frac{5}{2}$	0	$\pm\frac{1}{2}$	$-\frac{1}{3}$	$r-1$	2	$-$	$4^+, 2^-$	$-$
$\Box\phi$	3	0	0	$\frac{2}{3}$	r	4	$-$	2	$-$
$\bar{\phi}$	1	0	0	$-\frac{2}{3}$	$-r$	0	t^{3r}	2	$-$
$\bar{\psi}$	$\frac{3}{2}$	0	$\pm\frac{1}{2}$	$\frac{1}{3}$	$-r+1$	2	$-$	$2^+, 0^-$	$-t^{3(2-r)}$
$\partial\bar{\psi}$	$\frac{5}{2}$	$\pm\frac{1}{2}$	0	$\frac{1}{3}$	$-r+1$	$2^+, 4^-$	$-$	2	$-$
$\Box\bar{\phi}$	3	0	0	$-\frac{2}{3}$	$-r$	2	$-$	4	$-$
$\partial_{\pm\pm}$	1	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	0	0	$0^{\pm+}, 2^{\pm-}$	$t^3 y^{\pm1}$	$0^{+\pm}, 2^{-\pm}$	$t^3 y^{\pm1}$

Table 2.3: The "letters" of an $\mathcal{N} = 1$ chiral multiplet and their contributions to the index. Here $\delta^{\mathtt{L}} = E - 2j_1 + \frac{3}{2} r_{UV}$ and $\delta^{\mathtt{R}}_{UV} = E - 2j_2 - \frac{3}{2} r_{UV}$. A priori we have to take into account the free equations of motion $\partial\psi = 0$ and $\Box\phi = 0$, which imply constraints on the possible words, but we see that in this case equations of motions have $\delta_{UV} \neq 0$ so they do not change the index. Finally there are two spacetime derivatives contributing to the index, and their multiple action on the fields is responsible for the denominator of the index, $1/(1 - t^3 y^{\pm1}) = \sum_{n=0}^{\infty} (t^3 y^{\pm1})^n$.

the gauge and flavor quantum numbers, we introduce characters. We assume that the chiral multiplet transforms in the representation $\mathcal{R}$ of the gauge $\times$ flavor group, and denote by $\chi_{\mathcal{R}}(U, V)$, $\chi_{\bar{\mathcal{R}}}(U, V)$ the characters of $\mathcal{R}$ and and of the conjugate representation $\bar{\mathcal{R}}$, with U and V gauge and flavor group matrices respectively. All in all, the single-letter left- and right-handed indices for a chiral multiplet are [21]

$$i^{\mathtt{L}}_{\chi(r)}(t, y, U, V) = \frac{t^{3r}\, \chi_{\bar{\mathcal{R}}}(U, V) - t^{3(2-r)}\, \chi_{\mathcal{R}}(U, V)}{(1 - t^3 y)(1 - t^3 y^{-1})} \tag{2.20}$$

$$i^{\mathtt{R}}_{\chi(r)}(t, y, U, V) = \frac{t^{3r}\, \chi_{\mathcal{R}}(U, V) - t^{3(2-r)}\, \chi_{\bar{\mathcal{R}}}(U, V)}{(1 - t^3 y)(1 - t^3 y^{-1})}\,. \tag{2.21}$$

The denominators encode the action of the two spacetime derivatives with $\delta = 0$. Note that the left-handed and right-handed indices differ by conjugation of the gauge and flavor quantum numbers. As a basic consistency check [19], consider a single free massive chiral multiplet (no gauge or flavor indices). In the UV, we neglect the mass deformation and as always $r_{UV} = \frac{2}{3}$. In the IR, the quadratic superpotential implies $r_{IR} = 1$, and one finds $i^{\mathtt{L}}_{r=1} = i^{\mathtt{R}}_{r=1} \equiv 0$. As expected, a massive superfield decouples and does not contribute to the IR index.

Finding the contribution to the index of an $\mathcal{N} = 1$ vector multiplet is even easier, since the R-charge of a vector superfield W_α is fixed at the canonical value $+1$ all along the flow. For both left- and the right-handed index, the single-letter index of a vector multiplet is [2]

$$i_V(t, y, U) = \frac{2t^6 - t^3(y + \frac{1}{y})}{(1 - t^3 y)(1 - t^3 y^{-1})}\, \chi_{adj}(U)\,. \tag{2.22}$$

Armed with the single-letter indices, the full index is obtained by enumerating all the words and then projecting onto gauge-singlets by integrating over the Haar measure of the gauge group. Schematically,

$$\mathcal{I}(t, y, V) = \int [dU] \prod_k \mathrm{PE}[i_k(t, y, U, V)]\,, \tag{2.23}$$

where k labels the different supermultiplets, and $\mathrm{PE}[i_k]$ is the plethystic exponential of the single-letter index of the k-th multiplet. The pletyhstic exponential,

$$\mathrm{PE}[i_k(t, y, U, V)] \equiv \exp\left\{\sum_{m=1}^{\infty} \frac{1}{m} i_k(t^m, y^m, V^m) \chi_{\mathcal{R}_k}(U^m, V^m)\right\}\,, \tag{2.24}$$

implements the combinatorics of symmetrization of the single letters, see *e.g.* [49–51]. As usual, one can gauge fix the integral over the gauge group and reduce it to an integral over the maximal torus, with the usual extra factor arising of van der Monde determinant.

In chapter 6 we focus on quiver gauge theories. The gauge group will be taken to be a product of $SU(N)$ factors, with the chiral matter transforming in bifundamental representations. The gauge characters factorize into products of fundamental and anti-fundamental characters of the relevant factors, $\chi_{\mathcal{R}_{a\bar{b}}}(U^m) \to \mathrm{tr}(u_a^m)\mathrm{tr}\left(u_b^{\dagger m}\right)$. For $SU(N)$ the adjoint character is $\chi_{adj}(U^m) \equiv \mathrm{tr}(u_a^m)\mathrm{tr}(u_a^{\dagger m}) - 1$.

2.3 A universal result about $\mathcal{N}=2 \to \mathcal{N}=1$ flows

Consider an $\mathcal{N}=2$ gauge theory where all the gauge couplings are exactly marginal. Upon turning on a mass term for the adjoint chiral multiplet inside the $\mathcal{N}=2$ vector multiplet, supersymmetry is broken to $\mathcal{N}=1$ and the theory flows in the IR to an $\mathcal{N}=1$ superconformal field theory with a quartic superpotential. The simplest example is the flow between the $\mathcal{N}=2$ $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ and the Klebanov-Witten theory. A large class of examples have been discussed in [52]. For this general class of flows, there is a universal linear relation between the a and c conformal anomaly coefficients of the UV and IR theories [53].

It turns out that the superconformal indices of the UV and IR theories are also related in a simple universal way, namely

$$\mathcal{I}_{IR}^{\mathcal{N}=1}(t,y) = \mathcal{I}_{UV}^{\mathcal{N}=2}(t,y,v=t)\,. \tag{2.25}$$

Choosing for definiteness the right-handed index, the definition of the $\mathcal{N}=2$ superconformal index is

$$\mathcal{I}^{\mathcal{N}=2} \equiv \mathrm{Tr}\,(-1)^F t^{2(E+j_2)} y^{2j_1} v^{-(r_{\mathcal{N}=2}+R)}\,, \tag{2.26}$$

where R and $r_{\mathcal{N}=2}$ are the quantum numbers under the $SU(2)_R \times U(1)_r$ R-symmetry.[5] The $\mathcal{N}=1$ and $\mathcal{N}=2$ R-symmetry quantum numbers are related as

$$r_{\mathcal{N}=1} = \frac{2}{3}(2R_{\mathcal{N}=2} - r_{\mathcal{N}=2})\,. \tag{2.27}$$

Our claim is easily proved by recalling the single-letter indices of the $\mathcal{N}=2$ vector multiplet (2.13) and of the chiral multiplet (half-hypermultiplet) (2.14) computed in section 2.1

$$i_V^{\mathcal{N}=2}(t,y,v) \;=\; f^V(t,y,v) = \frac{t^2 v - \frac{t^4}{v} - t^3(y+y^{-1}) + 2t^6}{(1-t^3 y)(1-t^3 y^{-1})} \tag{2.28}$$

$$i_\chi^{\mathcal{N}=2}(t,y,v) \;=\; f^{\frac{1}{2}H}(t,y,v) = \frac{\frac{t^2}{\sqrt{v}} - t^4\sqrt{v}}{(1-t^3 y)(1-t^3 y^{-1})}\,. \tag{2.29}$$

[5] In our conventions, the bottom component ϕ of the $\mathcal{N}=2$ vector multiplet has $r_{\mathcal{N}=2} = -1$ (and of course $R=0$), while the scalar doublet in the hypermultiplet has $r_{\mathcal{N}=2}=0$ and $R=\pm 1/2$.

Comparing with (2.21) and (2.22), we see that

$$i_V^{\mathcal{N}=2}(t,y,v=t) = i_V^{\mathcal{N}=1}(t,y) \tag{2.30}$$

$$i_\chi^{\mathcal{N}=2}(t,y,v=t) = i^{\mathcal{N}=1}_{\chi(r=\frac{1}{2})}(t,y)\,. \tag{2.31}$$

So setting $v=t$ has the effect of converting the $\mathcal{N}=2$ vector multiplet to $\mathcal{N}=1$ vector multiplets, and of changing the R-charge of the chiral multiplets from $r_{\mathcal{N}=1}=2/3$ to $r_{\mathcal{N}=1}=1/2$, which is the correct IR value since a quartic superpotential is generated from the decoupling of the adjoint chiral multiplets. Since both the conformal anomaly coefficients and the index undergo a universal transformation between the UV and IR of this class of RG flows, one may wonder whether there is any simple connection between the index and the anomaly coefficients.

2.4 Elliptic Hypergeometric Expressions for the Index

As was observed by Dolan and Osborn [21] the expressions for the index can be recast in an elegant way in terms of special functions. First, recall the definition of the elliptic Gamma function,

$$\Gamma(z;p,q) \equiv \prod_{j,k\geq 0} \frac{1-z^{-1}\,p^{j+1}q^{k+1}}{1-z\,p^j q^k}\,. \tag{2.32}$$

For reviews of the elliptic Gamma function and of elliptic hypergeometric mathematics the reader can consult [54–56]. Throughout this paper we will use the standard condensed notations

$$\Gamma(z_1,\ldots,z_k;p,q) \equiv \prod_{j=1}^{k}\Gamma(z_j;p,q), \quad \Gamma(z^{\pm 1};p,q) \equiv \Gamma(z;p,q)\Gamma(1/z;p,q)\,. \tag{2.33}$$

Basic identities satisfied by the elliptic Gamma function that will be of use to us are

$$\Gamma\left(pq/z;p,q\right)\Gamma\left(z;p,q\right) = 1\,, \tag{2.34}$$

$$\lim_{z\to a}\left(1-z/a\right)\Gamma(z/a;p,q) = \frac{1}{(p;p)(q;q)}\,, \tag{2.35}$$

with the bracket defined as

$$(a;\, b) \equiv \prod_{k=0}^{\infty} \left(1 - a\, b^k\right) \,. \tag{2.36}$$

From the definition (2.32), it is straightforward to show [21]

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\, \frac{t^{2n} z^n - t^{4n} z^{-n}}{(1-t^{3n}y^n)(1-t^{3n}y^{-n})}\right) = \Gamma(t^2\, z; p, q), \tag{2.37}$$

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\, \frac{2t^{6n} - t^{3n}(y^n + y^{-n})}{(1-t^{3n}y^n)(1-t^{3n}y^{-n})} (z^n + z^{-n})\right) = -\frac{z}{(1-z)^2}\, \frac{1}{\Gamma(z^{\pm 1}; p, q)},$$

where

$$p = t^3 y, \qquad q = t^3 y^{-1} \,. \tag{2.38}$$

Using the above identities the basic building blocks of the superconformal index computation can be written as follows.

For $\mathcal{N} = 2$ index, the contribution to the integrand of (2.10) from hypers in a fundamental representation of an $SU(n)$ gauge group is

$$\exp\left(\sum_{k=1}^{\infty} \frac{1}{k}\, f^{\frac{1}{2}H}\left(t^k,\, v^k,\, y^k\right)\, \left[\chi_f(U^k) + \chi_{\bar f}(U^k)\right]\right) = \prod_{i=1}^{n} \Gamma\left(\frac{t^2}{\sqrt{v}} a_i^{\pm 1}; p, q\right) \,. \tag{2.39}$$

The contribution to the integrand of (2.10) from the vector multiplet of $SU(n)$ is

$$\begin{aligned} &\exp\left(\sum_{k=1}^{\infty} \frac{1}{k}\, f^{V}\left(t^k,\, v^k,\, y^k\right)\, \chi_{adj}(U^k)\right) \\ &\qquad = \frac{\left[\Gamma(t^2\, v; p, q)\, (p;p)(q;q)\right]^{n-1}}{\Delta(\mathbf{a})\Delta(\mathbf{a}^{-1})} \prod_{i \neq j} \frac{\Gamma(t^2\, v\, a_i/a_j; p, q)}{\Gamma(a_i/a_j; p, q)} \,. \end{aligned} \tag{2.40}$$

We have defined the characters of the fundamental representation to be

$$\chi_f = \sum_{i=1}^{n} a_i, \qquad \chi_{\bar f} = \sum_{i=1}^{n} \frac{1}{a_i}, \qquad \prod_{i=1}^{n} a_i = 1 \,. \tag{2.41}$$

The character of the adjoint representation is

$$\chi_{adj} = \chi_f \, \chi_{\bar{f}} - 1 = \sum_{i \neq j} a_i/a_j + n - 1 \,. \tag{2.42}$$

We have also defined

$$\Delta(\mathbf{a}) = \prod_{i \neq j} (a_i - a_j) \,. \tag{2.43}$$

The Haar measure is given by

$$\oint_{SU(n)} d\mu(\mathbf{a}) f(\mathbf{a}) = \frac{1}{n!} \oint_{\mathbb{T}^{n-1}} \prod_{i=1}^{n-1} \frac{da_i}{2\pi i \, a_i} \Delta(\mathbf{a}) \Delta(\mathbf{a}^{-1}) f(\mathbf{a}) \Bigg|_{\prod_{i=1}^{n} a_i = 1} \,, \tag{2.44}$$

where $\mathbb{T}$ is the unit circle. Whenever we gauge a symmetry we have a vector multiplet associated to the integrated group and thus we will use the following notation

$$\begin{aligned} \mathcal{F}_{\mathbf{a}} \, \mathcal{G}^{\mathbf{a}} \equiv & \, \frac{\left[2 \, \Gamma(t^2 \, v; p, q) \, \kappa\right]^{n-1}}{n!} \\ & \times \oint_{\mathbb{T}_{n-1}} \prod_{i=1}^{n-1} \frac{da_i}{2\pi i \, a_i} \prod_{i \neq j} \frac{\Gamma(t^2 \, v \, a_i/a_j; p, q)}{\Gamma(a_i/a_j; p, q)} \mathcal{F}\left(\mathbf{a}\right) \, \mathcal{G}\left(\mathbf{a}^{-1}\right) \Bigg|_{\prod_{i=1}^{n} a_i = 1} \,, \end{aligned} \tag{2.45}$$

where $\kappa \equiv (p;p)(q;q)/2$. In what follows for the sake of brevity we will omit the parameters p and q from the elliptic Gamma function, i.e. $\Gamma(x)$ should always be understood as $\Gamma(x;\, p, q)$.

Similarly in $\mathcal{N} = 1$ theories, a chiral superfield in the bifundamental representation $\square\overline{\square}$ of $SU(N_1) \times SU(N_2)$, and with IR R-charge equal to r can be rewritten as

$$\mathrm{PE}[i_r(t, y, U)] \;\; \equiv \;\; \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \Gamma(t^{3r} \, z_i w_j^{-1};\, t^3 y, t^3/y), \tag{2.46}$$

Here $\{z_k\}$, $k = 1, \ldots N_1\}$, and $\{w_k\}$, $k = 1, \ldots N_2\}$, are complex numbers of unit modulus, obeying $\prod_{k=1}^{N_1} z_k = \prod_{k=1}^{N_2} w_k = 1$, which parametrize the Cartan subalgebras of $SU(N_1)$ and $SU(N_2)$. The multi-letter contribution of a vector multiplet in the adjoint of $SU(N)$ combines with the $SU(N)$ Haar measure to give the compact expression [15, 21]

$$\frac{\kappa^{N-1}}{N!}\oint_{\mathbb{T}_{n-1}}\prod_{i=1}^{N-1}\frac{dz_i}{2\pi i\, z_i}\prod_{k\neq \ell}\frac{1}{\Gamma(z_k/z_\ell;p,q)}\cdots. \tag{2.47}$$

The dots indicate that this is to be understood as a building block of the full matrix integral. This equation can also be obtained by setting $v = t$ in (2.40). The numerator factor $\prod_{i\neq j}\Gamma(t^3\, a_i/a_j;p,q)$ becomes 1 becuase of the property of elliptic Gamma function (2.35).

2.5 $4d$ Index as a path integral on $S^3\times S^1$

The superconformal index,

$$\mathcal{I}(t,y,v) = \mathrm{Tr}(-1)^F t^{2(E+j_2)} y^{2\,j_1} v^{-(r+R)}\,, \tag{2.48}$$

doesn't depend on the couplings of the theory and hence it can be calculated in the weak coupling limit. The entire contribution to the supersymmetric partition function on $S^3\times S^1$ thus comes from the saddle point approximation. One loop partition function of a $4d$ gauge theory on $S^3\times S^1$ was computed in [51] in the presence of fugacities associated with various conserved charges. To compute the superconformal index, we only allow fugacities for charges which commute with $\mathcal{Q}$; i.e. t, y and v.

For the one loop computation in $SU(N)$ gauge theory, it is convenient to use the Coulomb gauge $\partial_i A^i = 0$ where i,j,k are S^3 coordinates and ∂_i are covariant derivatives. The residual gauge freedom is fixed by imposing $\partial_0\alpha = 0$ where $\alpha = \frac{1}{V}\int_{S^3} A_0$ and V is the volume of S^3. The partition function is then written as

$$Z = \int d\alpha\Delta_2\int \mathcal{D}A\Delta_1 e^{-S(A,\alpha)}\,, \tag{2.49}$$

where Δ_1 and Δ_2 are Fadeev-Popov determinants associated with the first and second gauge fixing conditions respectively. For a charge s that commutes with $\mathcal{Q}$, we can add a supersymmetric coupling with a constant background gauge field as

$$S\to S + \int d^4x\, s^\mu\chi_\mu, \tag{2.50}$$

where s^μ is associated conserved current. χ_μ is take to be a $(\chi,0,0,0)$ and χ

is identified with the chemical potential for charge s. The chemical potential is related to the fugacity, say x, of the Hamiltonian formalism as $x = e^{-\beta\chi}$. In our case, x can be any of the t, y and v.

After performing $\int \mathcal{D}A$, one gets an $SU(N)$ unitary matrix model

$$Z = \int [dU] e^{-S_{eff}[U]}\,, \tag{2.51}$$

where $U = e^{i\beta\alpha}$ and β is the circumference of the circle, $[dU]$ is the invariant Haar measure on the group $SU(N)$. The S_{eff} is just the one appears in (2.10)

$$S^{eff}[U] = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{j} f^{\mathcal{R}_j}(t^m, y^m, v^m) \chi_{\mathcal{R}_j}(U^m, V^m)\,. \tag{2.52}$$

Here, V denotes the chemical potential that couples to the Cartan of the flavor group; $\mathcal{R}_j$ labels the representation of the fields under gauge and flavor groups and $f^{\mathcal{R}_j}$ is the single letter index of the fields in representation $\mathcal{R}_j$.

The circumference β of the circle is related to the fugacity t as $t = e^{-\beta/3}$. To produce the partition function of dimensionally reduced gauge theory on S^3 [33, 40] we also scale $v = e^{-\beta/3}$, $y = 1$, and take the limit $\beta \to 0$. In appendix C we restore the additional deformations by defining $v = e^{-\beta(1/3+u)}$ and set $y = e^{-\beta\eta}$ where u and η are chemical potentials for fugacities v and y respectively. The partition function of $3d$ gauge theories on squashed S^3 was computed in [57], the η deformation is related to the squashing parameter of S^3.

Chapter 3

S-duality and Two Dimensional Topological Field Theory

Electric-magnetic duality (S-duality) in four-dimensional gauge theory has a deep connection with two-dimensional modular invariance. The canonical example is the $SL(2,\mathbb{Z})$ symmetry of $\mathcal{N}=4$ super-Yang-Mills, which can be interpreted as the modular group of a torus. A physical picture for this correspondence is provided by the existence of the six-dimensional $(2,0)$ superconformal field theory, whose compactification on a torus of modular parameter τ yields $\mathcal{N}=4$ SYM with holomorphic coupling τ (see [58] for a recent discussion).

Gaiotto [3] has discovered a beautiful generalization of this construction. A large class of $\mathcal{N}=2$ SCFTs (class $\mathcal{S}$ SCFTs) in 4d is obtained by compactifying a twisted version of the $(2,0)$ theory on a Riemann surface Σ, of genus g and with n punctures. The complex structure moduli space $\mathcal{T}_{\mathsf{g},n}/\Gamma_{\mathsf{g},n}$ of Σ is identified with the space of exactly marginal couplings of the 4d theory. The mapping class group $\Gamma_{\mathsf{g},n}$ acts as the group of generalized S-duality transformations of the 4d theory. A striking correspondence between the Nekrasov's instanton partition function [12] of the 4d theory and Liouville field theory on Σ has been conjectured in [9] and further explored in [10, 59–69]. According to the celebrated AGT conjecture [9–11], the $4d$ partition functions on the Ω-background [12] and on S^4 [13] are computed by Liouville/Toda theory on $\mathcal{C}$. Relations to string/M theory have been discussed in [70–73]. See also [52, 74, 75].

In this chapter we apply the superconformal index to this class of 4d SCFTs. The index is invariant under continuous deformations of the theory, and is also expected to be invariant under the S-duality group $\Gamma_{\mathsf{g},n}$. *Assuming* S-duality, this implies that the index must be computed by a topological QFT living on Σ. The usual physical arguments involving the $(2,0)$ theory give a "proof" of this assertion, as follows. The path integral representation of the index (2.49) uplifts to a (suitably twisted) path integral of the $(2,0)$ theory on $S^3 \times S^1 \times \Sigma$. This path integral must be independent of the metric on Σ. In the limit of small Σ we recover the 4d definition; in the opposite limit of large Σ we expect a purely 2d description. Each puncture on Σ should be regarded as an operator insertion. By this logic, the index must be equal to the n-point correlation function of some TQFT on Σ. The question is whether one can describe this TQFT more directly, and in the process *check* the S-duality of the index.

It is likely that a "microscopic" Lagrangian formulation of the 2d TQFT may be derived from the dimensional reduction of the twisted $(2,0)$ theory that we have just described, but we will postpone the discussion in chapter 4. We will first write the concrete expression of the index for class $\mathcal{S}$ A_1 theories in this chapter, which always have a 4d Lagrangian description. We show in section 3.1 that the index does indeed take the form expected for a correlator in a 2d TQFT[1]. We then evaluate explicitly the structure constants and metric of the TQFT operator algebra, and check its associativity, which is the 2d counterpart of S-duality (section 3.1.2). The metric and structure constants have elegant expressions in terms of elliptic Gamma functions and the index in terms of elliptic Beta integrals, a set of special functions which are a new and active branch of mathematical research, see *e.g.* [54–56] and references therein. For A_1 theories associativity of the topological algebra (and thus S-duality) hinges on the invariance of a special case of the $E^{(5)}$ elliptic Beta integral under the Weyl group of F_4. A proof of this symmetry can be found on the math ArXiv [76]. In a related physical context, elliptic identities have been used in [21] (following [19]) to prove equality of the superconformal index for Seiberg-dual pairs of $\mathcal{N}=1$ gauge theories.

What distinguishes the A_1 theories from their counterparts with $A_{n\geq 2}$ is that in all duality frames they have a Lagrangian description. This makes it easy to compute their superconformal index explicitly and to identify the structure constants of the $2d$ TQFT. The situation for the generalized quiver theories with higher rank gauge groups is qualitatively different: in some duality frames the quivers contain intrinsically strongly-coupled blocks with no Lagrangian description. The prototypical example of this phenomenon was discussed by Argyres and Seiberg [28][2]: the SYM theory with $SU(3)$ gauge group and $N_f = 6$ fundamental hypermultiplets has a dual description involving the strongly-coupled SCFT with E_6 flavor symmetry [29]. In the absence

[1]We thank Abhijit Gadde, Elli Pomoni, Shlomo Razamat and Leonardo Rastelli for letting us use material from their paper [15].

[2]See also [77] for more examples.

of a Lagrangian description for the E_6 SCFT, it seems difficult to compute its superconformal index and to define the TQFT structure for generalized quivers with $SU(3)$ gauge groups.

We solve this problem in the second half of this chapter. By demanding consistency with Argyres-Seiberg duality, we are able to write down an explicit integral expression for the index of the E_6 SCFT (equation (3.41)). Technically, this is possible thanks to a remarkable inversion formula for a class of integral transforms [78]. By construction, the resulting expression for the index is guaranteed to be invariant under an $SU(6) \otimes SU(2)$ subgroup of the E_6 flavor symmetry. The index is seen *a posteriori* to be invariant under the full E_6 symmetry, providing an independent check of Argyres-Seiberg duality itself.[3] We proceed to define a TQFT structure for generalized quivers with $SU(3)$ gauge symmetries. We check associativity of the operator algebra in section 3.2.3, which is equivalent to a check of S-duality for class $\mathcal{S}$ A_2 theories. Most of our checks are performed perturbatively, to several orders in an expansion in the chemical potentials that enter the definition of the index. Conversely, S-duality implies that associativity must hold exactly, so as a by-product of our analysis we conjecture new identities between integrals of elliptic Gamma functions.

3.1 The Index in A_1 Theories

We start this section by recalling the basics of Gaiotto's analysis [3]. The main achievement of [3] is a purely four-dimensional construction of the SCFT implicitly defined by compactifying the A_{N-1} $(2,0)$ theory on Σ. In the A_1 case an explicit Lagrangian description is available, in terms of a generalized quiver with gauge group $SU(2)^{N_G}$, see figure 3.1 for examples. The internal edges of a diagram correspond to the $SU(2)$ gauge groups, the external legs to $SU(2)$ flavor groups and the the cubic vertices to chiral fields in the trifundamental representation (fundamental under each of the groups joining at the vertex). The corresponding Riemann surface is immediately pictured by thickening the lines of the graph into tubes – with the external tubes assumed to be infinitely long, so that they can be viewed as punctures. The plumbing parameters τ_i of the tubes are identified with the holomorphic gauge couplings; the degeneration limit when the surface develops a long tube corresponds to the weak coupling limit $\tau \to +i\infty$ of the corresponding gauge group (figure 3.2). The different patterns of degenerations (pair-of-pants decompositions) of a surface Σ of genus g and N_F punctures give rise to the different connected diagrams

[3] For earlier checks of Argyres-Seiberg duality see [79] and [80].

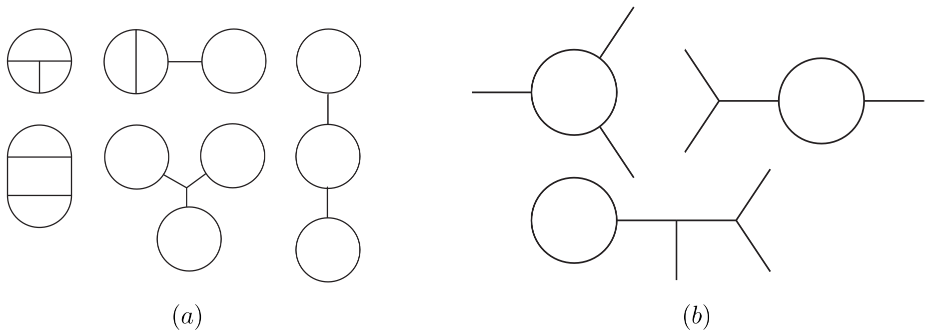


Figure 3.1: **(a)** Generalized quiver diagrams representing $\mathcal{N} = 2$ superconformal theories with gauge group $SU(2)^6$ and no flavor symmetries ($N_G = 6$, $N_F = 0$). There are five different theories of this kind. The internal lines of a diagram represent and $SU(2)$ gauge group and the trivalent vertices the trifundamental chiral matter. **(b)** Generalized quiver diagrams for $N_G = 3$, $N_F = 3$. Each external leg represents an $SU(2)$ flavor group. The upper left diagram corresponds the $\mathcal{N} = 2$ $\mathbb{Z}_3$ orbifold of $\mathcal{N} = 4$ SYM with gauge group $SU(2)$.

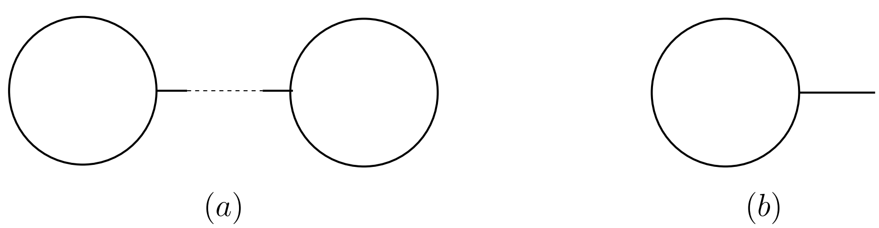


Figure 3.2: An example of a degeneration of a graph and appearance of flavour punctures. As one of the gauge coupling is taken to zero the corresponding edge becomes very long. Cutting the edge, each of the two resulting semi-infinite open legs will be associated to chiral matter in an $SU(2)$ flavor representation. In this picture setting the coupling of the middle legs in **(a)** to zero gives two copies of the theory represented in **(b)**, namely an $SU(2)$ gauge theory with a chiral field in the bifundamental representation of the gauge group and in the fundamental of a flavour $SU(2)$.

with N_F external legs ($SU(2)$ flavor groups) and $N_G = N_F + 3(\mathtt{g}-1)$ internal lines ($SU(2)$ gauge groups). Since the mapping class group permutes the diagrams, the associated field theories must be related by generalized S-duality transformations [3]. In the higher A_{N-1} cases the 4d theories are generically described by more complicated quivers that involve new exotic isolated SCFTs as elementary building blocks.

3.1.1 2d TQFT from the Superconformal Index

For the A_1 generalized quivers the index can be explicitly computed as a matrix integral (2.10),

$$\mathcal{I} = \int \prod_{\ell\in\mathcal{G}} [dU_\ell] \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \left[\sum_{i\in\mathcal{G}} f_n^V \chi_{adj}(U_i^n) + \sum_{(i,j,k)\in\mathcal{V}} f_n^{\frac{1}{2}H} \chi_{3f}(U_i^n, U_j^n, U_k^n)\right]\right) . \tag{3.1}$$

Here $f_n^V = f^V(t^n, y^n, v^n)$ and $f_n^{\frac{1}{2}H} = f^{\frac{1}{2}H}(t^n, y^n, v^n)$, with $f^V(t,y,v)$ and $f^{\frac{1}{2}H}(t,y,v)$ the "single-letter partition functions" for respectively the adjoint and trifundamental degrees of freedom, multiplying the corresponding $SU(2)$ characters. The explicit expressions for f^V and $f^{\frac{1}{2}H}$ hav already been given by (2.13) and (2.14). The $\{U_i\}$ are $SU(2)$ matrices. Their index i run over the $N_G + N_F$ edges of the diagram, both internal ("Gauge") and external ("Flavor"). The set $\mathcal{G}$ is the set of N_G internal edges while the set $\mathcal{V}$ is the set of trivalent vertices, each vertex being labelled by the triple (i,j,k) of incident edges. The integral over $\{U_\ell\,, \ell \in \mathcal{P}\}$, with $[dU]$ being the Haar measure, enforces the gauge-singlet condition. All in all, the index $\mathcal{I}$ depends on the chemical potentials t, y, v (through f^V and $f^{\frac{1}{2}H}$) and on (the eigenvalues of) the N_F unintegrated flavor matrices.

The characters depend on a single angular variable α_i for each $SU(2)$ group U_i. Writing

$$U_i = V_i^\dagger \begin{pmatrix} e^{i\alpha_i} & 0 \\ 0 & e^{-i\alpha_i} \end{pmatrix} V_i \,, \tag{3.2}$$

we have

$$\begin{aligned} \chi_{adj}(U_i) &= \mathrm{Tr}U_i\,\mathrm{Tr}U_i - 1 = e^{2i\alpha_i} + e^{-2i\alpha_i} + 1 \equiv \chi_{adj}(\alpha_i)\,, && (3.3) \\ \chi_{3f}(U_i,U_j,U_k) &= \mathrm{Tr}U_i\,\mathrm{Tr}U_j\,\mathrm{Tr}U_k && (3.4) \\ &= (e^{i\alpha_i} + e^{-i\alpha_i})(e^{i\alpha_j} + e^{-i\alpha_j})(e^{i\alpha_k} + e^{-i\alpha_k}) \\ &\equiv \chi_{3f}(\alpha_i, \alpha_j, \alpha_k)\,, \end{aligned}$$

where we have used the fact that $2 \sim \bar{2}$. Integrating over V_i, the Haar measure simplifies to

$$\int [dU_i] = \frac{1}{\pi} \int_0^{2\pi} d\alpha_i \; \sin^2 \alpha_i \equiv \int d\alpha_i \, \Delta(\alpha_i) \,. \tag{3.5}$$

We now define

$$\begin{aligned} C_{\alpha_i \alpha_j \alpha_k} &\equiv \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f_n^{\frac{1}{2}H} \cdot \; \chi_{3f}(n\alpha_i, n\alpha_j, n\alpha_k) \right) , \\ \eta^{\alpha_i \alpha_j} &\equiv \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f_n^{V} \cdot \; \chi_{adj}(n\alpha_i) \right) \hat{\delta}(\alpha_i, \alpha_j) \equiv \eta^{\alpha_i} \, \hat{\delta}(\alpha_i, \alpha_j), \end{aligned} \tag{3.6}$$

where $\hat{\delta}(\alpha, \beta) \equiv \Delta^{-1}(\alpha)\delta(\alpha - \beta)$ (with the understanding that α and β are defined modulo 2π) is the delta-function with respect to the measure (3.5). Further define the "contraction" of an upper and a lower α labels as

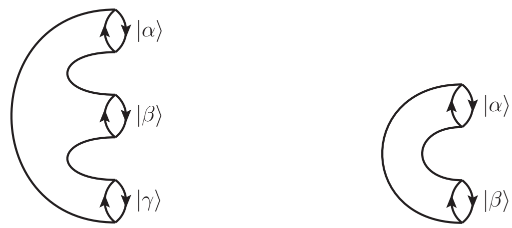
$$A^{\cdots\alpha\cdots} \, B_{\ldots\alpha\ldots} \equiv \int_0^{2\pi} d\alpha \, \Delta(\alpha) \, A^{\cdots\alpha\cdots} \, B_{\ldots\alpha\ldots} \,. \tag{3.7}$$

The superconformal index (3.1) can then be suggestively written as

$$\mathcal{I} = \prod_{\{i,j,k\} \in \mathcal{V}} C_{\alpha_i \alpha_j \alpha_k} \prod_{\{m,n\} \in \mathcal{G}} \eta^{\alpha_m \alpha_n} \,. \tag{3.8}$$

The internal labels $\{\alpha_i \, , i \in \mathcal{G}\}$ associate to the gauge groups are contracted, while the N_F external labels associated to the flavor groups are left open. The expression (3.8) is naturally interpreted as an N_F-point "correlation function" $\langle \alpha_1 \ldots \alpha_{N_F} \rangle_{\mathtt{g}}$, evaluated by regarding the generalized quiver as a "Feynman diagram". The Feynman rules assign to each trivalent vertex the cubic coupling $C_{\alpha\beta\gamma}$, and to each internal propagator the inverse metric $\eta^{\alpha\beta}$. S-duality implies that the superconformal indices calculated from two diagrams with the same (N_F, N_G) must be equal. These properties can be summarized in the statement that the superconformal index is evaluated by a 2d Topological QFT (TQFT).

At the informal level sufficient for our discussion, a 2d TQFT [81, 82] can be characterized in terms of the following data: a space of states $\mathcal{H}$; a non-degenerate, symmetric metric η: $\mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$; and a completely symmetric triple product C: $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$. The states in $\mathcal{H}$ are understood physically as wavefunctionals of field configurations on the "spatial" manifold S^1. The metric and triple product are evaluated by the path integral over field con-



(a) (b)

Figure 3.3: **(a)** Topological interpretation of the structure constants $C_{\alpha\beta\gamma} \equiv \langle C| \, |\alpha\rangle|\beta\rangle|\gamma\rangle$. The path integral over the sphere with three boundaries defines $\langle C| \in \mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{H}^*$. **(b)** Analogous interpretation of the metric $\eta_{\alpha\beta} \equiv \langle\eta||\alpha\rangle|\beta\rangle$, with $\langle\eta| \in \mathcal{H}^* \otimes \mathcal{H}^*$, in terms of the sphere with two boundaries.

figurations on the sphere with respectively two and three boundaries (figure 3.3). The 2d surfaces where the TQFT is defined are assumed to be oriented, so the S^1 boundaries inherit a canonical orientation. To a boundary of inverse orientation (with respect to the canonical one) is associated the *dual* space $\mathcal{H}^*$. Choosing a basis for $\mathcal{H}$, we can specify the metric and triple product in terms of $\eta_{\alpha\beta} \equiv \eta(|\alpha\rangle, |\beta\rangle)$ and $C_{\alpha\beta\gamma} \equiv C(|\alpha\rangle, |\beta\rangle, |\gamma\rangle)$, or

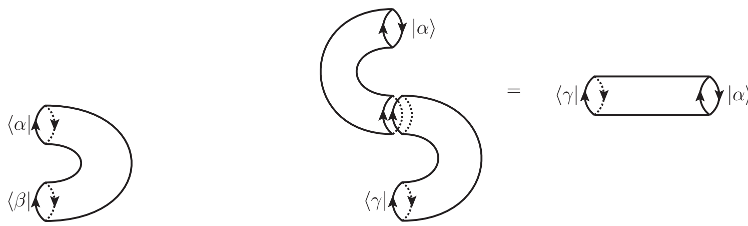
$$\eta = \sum_{\alpha,\beta} \eta_{\alpha\beta} \langle\alpha|\langle\beta| \,, \quad C = \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma} \langle\alpha|\langle\beta|\langle\gamma| \,. \tag{3.9}$$

The inverse metric $\eta^{\alpha\beta}$ is associated to the sphere with two boundaries of inverse orientation, and as its name suggests it obeys $\eta^{\alpha\beta}\eta_{\beta\gamma} = \delta^{\alpha}_{\gamma}$, see figure 3.4. Index contraction corresponds geometrically to gluing of S^1 boundary of compatible orientation.

The metric and triple product obey natural compatibility axioms which can be simply summarized by the statement that the metric and its inverse are used to lower and raise indices in the usual fashion. Finally the crucial requirement: the structure constants ${C_{\alpha\beta}}^{\gamma} \equiv C_{\alpha\beta\epsilon}\eta^{\epsilon\gamma}$ define an associative algebra

$${C_{\alpha\beta}}^{\delta} \, {C_{\delta\gamma}}^{\epsilon} = {C_{\beta\gamma}}^{\delta} \, {C_{\delta\alpha}}^{\epsilon} \,, \tag{3.10}$$

as illustrated in figure 3.5. From these data, arbitrary n-point correlators on a genus g surface can be evaluated by factorization (= pair-of-pants decomposition of the surface). The result is guaranteed to be independent of the specific decomposition.



(a) (b)

Figure 3.4: Topological interpretation of **(a)** the inverse metric $\eta^{\alpha\beta}$, **(b)** the relation $\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta_\alpha^\gamma$. By convention, we draw the boundaries associated with upper indices facing left and the boundaries associated with the lower indices facing right.

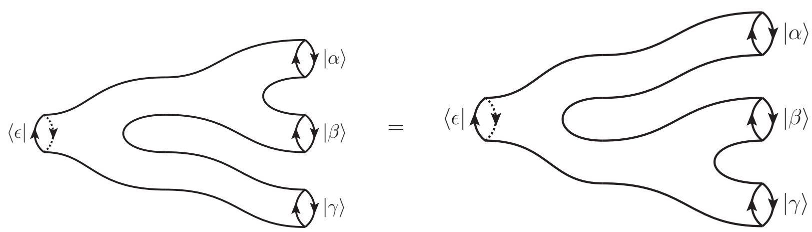


Figure 3.5: Pictorial rendering of the associativity of the algebra.

In our case the space $\mathcal{H}$ is spanned by the states $\{|\alpha\rangle, \alpha \in [0, 2\pi)\}$, where α parametrizes the $SU(2)$ eigenvalues, equ.(3.2). Alternatively we may "Fourier transform" to the basis of irreducible $SU(2)$ representations, $\{|R_K\rangle, K \in \mathbb{Z}_+\}$, see appendix A.1. We have concrete expressions (3.6, 3.7) for the cubic couplings $C_{\alpha\beta\gamma}$ and for the *inverse* metric $\eta^{\alpha\beta}$, which are manifestly symmetric under permutations of the indices. Formal inversion of (3.7) gives the metric $\eta_{\alpha\beta} \equiv (\eta^\alpha)^{-1}\hat{\delta}(\alpha, \beta)$. Finally with the help of (3.7) we can raise, lower and contract indices at will. On physical grounds we expect these formal manipulations to make sense, since the superconformal index is well-defined as a series expansion in the chemical potential t, which should have a finite radius of convergence [2]. The explicit analysis of sections 3.1.2 will confirm these expectations. We will find expressions for the index as analytic functions of the chemical potentials. Our definitions satisfy the axioms of a 2d TQFT by construction, and independently of the specific form of the functions $f^V(t, y, v)$ and $f^{\frac{1}{2}H}(t, y, v)$, except for the associativity axiom, which is completely non-trivial. Associativity of the 2d topological algebra is equivalent to 4d S-duality, and it can only hold for very special choices of field content, encoded in the single-letter partition functions f^V and $f^{\frac{1}{2}H}$.

3.1.2 Associativity of the Algebra

In this section we determine explicitly the structure constants and the metric of the TQFT and write them in terms of elliptic Beta integrals. With the help of a mathematical result [76] we prove analytically the associativity of the topological algebra.

Explicit Evaluation of the Index

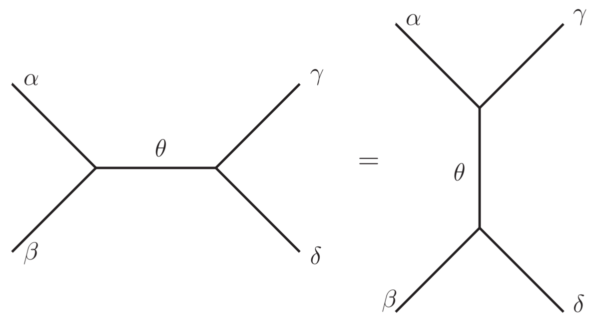


Figure 3.6: The basic S-duality channel-crossing. The two diagrams are two equivalent (S-dual) ways to represent the $\mathcal{N} = 2$ gauge theory with a single gauge group $SU(2)$ and four $SU(2)$ flavour groups, which is the basic building block of the A_1 generalized quiver theories. The indices on the edges label the eigenvalues of the corresponding $SU(2)$ groups.

As shown in (2.13) and (2.14), the single letter partition function in (t, y, v) parametrization are given by

$$\text{adjoint} \quad : \quad f^V(t,y,v) = \frac{t^2 v - \frac{t^4}{v} - t^3(y + y^{-1}) + 2t^6}{(1 - t^3\, y)(1 - t^3 y^{-1})}\,, \tag{3.11}$$

$$\text{trifundamental} \quad : \quad f^{\frac{1}{2}H}(t,y,v) = \frac{\frac{t^2}{\sqrt{v}} - t^4\sqrt{v}}{(1 - t^3\, y)(1 - t^3 y^{-1})}\,. \tag{3.12}$$

We are now ready to check explicitly the basic S-duality move – S-duality with respect to one of the $SU(2)$ gauge groups, represented graphically as channel-crossing with respect to one of the edges of the graph (figure 3.6). The full S-duality group of a graph is generated by repeated applications of the basic move to different edges. The contribution to the index from the left graph in figure 3.6 is

$$\mathcal{I} = \int d\theta\, \Delta(\theta) e^{\sum_{n=1}^{\infty} \frac{1}{n}\left[f_n^V \chi_{adj}(n\theta) + f_n^{\frac{1}{2}H}\chi_{3f}(n\alpha,n\beta,n\theta) + f_n^{\frac{1}{2}H}\chi_{3f}(n\theta,n\gamma,n\delta)\right]}\,. \tag{3.13}$$

Substituting the expressions for the characters,

$$\mathcal{I} = \frac{e^{\sum_{n=1}^{\infty}\frac{f_n^V}{n}}}{\pi}\int_0^{2\pi} d\theta \sin^2\theta\, e^{\sum_{n=1}^{\infty}\left[\frac{2f_n^V}{n}\cos 2n\theta + \frac{8f_n^{\frac{1}{2}H}}{n}(\cos n\alpha\cos n\beta + \cos n\gamma\cos n\delta)\cos n\theta\right]}\,, \tag{3.14}$$

where $f_n^V \equiv f^V(t^n, y^n, v^n)$ and $f_n^{\frac{1}{2}H} \equiv f^{\frac{1}{2}H}(t^n, y^n, v^n)$. S-duality of the index is the statement this integral is invariant under permutations of the external labels $\alpha, \beta, \gamma, \delta$. Since symmetries under $\alpha \leftrightarrow \beta$ and (independently) under $\gamma \leftrightarrow \delta$ are manifest, the non-trivial requirement is symmetry under $\beta \leftrightarrow \gamma$, which gives the index associated to the crossed graph on the right of figure 3.6.

The integrand of (3.14) is *not* invariant under $\beta \leftrightarrow \gamma$, but the integral is, as once can check order by order in a series expansion in the chemical potential t. Here is how things work to the first non-trivial order. We expand the integrand in t around $t = 0$, and set $y = v = 1$ for simplicity. The single-letter partition functions behave as

$$f^V(t, y=1, v=1) \sim t^2 - 2\,t^3\,, \qquad f^{\frac{1}{2}H}(t, y=1, v=1) \sim t^2 - t^4\,. \tag{3.15}$$

The first non-trivial check is for the coefficient of $\mathcal{I}$ of order $O(t^4)$,

$$\begin{aligned}\mathcal{I} \sim t^4 \int_0^{2\pi} d\theta \sin^2\theta \Big(& \cos 4\theta + 2\cos^2 2\theta + 4A_2\cos 2\theta \\ & + 32A_1^2\cos^2\theta - 2\cos 2\theta + 16A_1\cos\theta\cos 2\theta - 8A_1\cos\theta\Big)\,,\end{aligned} \tag{3.16}$$

where $A_n \equiv \cos n\alpha\cos n\beta + \cos n\gamma\cos n\delta$. Performing the elementary integrals,

$$\mathcal{I} \sim t^4\left[6\pi + 2\pi\ (\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \cos 2\delta + 8\cos\alpha\cos\beta\cos\gamma\cos\delta)\right]\,, \tag{3.17}$$

which is indeed symmetric under $\alpha \leftrightarrow \beta \leftrightarrow \gamma \leftrightarrow \delta$. We stress that crossing symmetry depends crucially on the specific form of the single-letter partition functions (2.13) and (2.14) and thus on the specific field content. We have performed systematic checks by calculating the series expansion to several higher orders using *Mathematica*. Fortunately it is possible to give an analytic proof of crossing symmetry of the index, as we now describe.

Elliptic Beta Integrals and S-duality

The fundamental integral (3.14) can be recast in an elegant way in terms of special functions known as elliptic Beta integrals. We start by rewriting the building blocks (3.6) for the index in the following compact form as in section 2.4

$$\begin{aligned} C_{\alpha_i\alpha_j\alpha_k} &= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f_n^{\frac{1}{2}H}\ \chi_{3f}(n\alpha_i,n\alpha_j,n\alpha_k)\right) \\ &= \Gamma(\frac{t^2}{\sqrt{v}}a_i^{\pm1}a_j^{\pm1}a_k^{\pm1};p,q), \\ \eta^{\alpha_i} &= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f_n^{V}\ \chi_{adj}(n\alpha_i)\right) \\ &= \frac{1}{\Delta(\alpha_i)}\frac{(p;p)(q;q)}{4\pi}\Gamma(t^2\,v;p,q)\,\frac{\Gamma(t^2\,v\,a_i^{\pm2};p,q)}{\Gamma(a_i^{\pm2};p,q)}\,. \end{aligned} \tag{3.18}$$

Here we have defined $a_i = \exp(i\alpha_i)$ and used

$$\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f_n\right) = (p;p)(q;q)\Gamma(t^2v;p,q), \qquad (a;b)\equiv\prod_{k=0}^{\infty}(1-ab^k) \tag{3.19}$$

Again, One should keep in mind that the *rhs* of the first line in (3.18) is a product of eight elliptic Gamma functions according to the condensed notation (2.33).

Collecting all these definitions the fundamental integral (3.14) becomes

$$\kappa\,\Gamma\left(t^2v;p,q\right)\oint\frac{dz}{z}\,\frac{\Gamma(t^2\,v\,z^{\pm2};p,q)}{\Gamma(z^{\pm2};p,q)}\,\Gamma(\frac{t^2}{\sqrt{v}}a^{\pm1}b^{\pm1}z^{\pm1};p,q)\Gamma(\frac{t^2}{\sqrt{v}}c^{\pm1}d^{\pm1}z^{\pm1};p,q),$$
$$pq=t^6\,, \tag{3.20}$$

with $\kappa\equiv(p;p)(q;q)/4\pi i$. As it turns out, this integral fits into a class of integrals which are an active subject of mathematical research, the elliptic Beta integrals

$$E^{(m)}(t_1,\dots,t_{2m+6})\sim\oint\frac{dz}{z}\frac{\Gamma(t_1z,\dots t_{2m+6}z;p,q)}{\Gamma(z^{\pm2};p,q)}\,,\qquad\qquad \prod_{k=1}^{2m+6}t_k=(pq)^{m+1}\,. \tag{3.21}$$

Our integral is a special case of $E^{(5)}$. Elliptic Beta integrals have very interesting

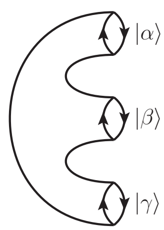
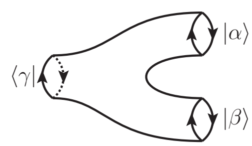
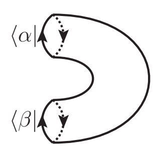
Symbol	Surface	Value
$C_{\alpha\beta\gamma}$		$\Gamma(\frac{t^2}{\sqrt{v}}a^{\pm 1}b^{\pm 1}c^{\pm 1})$
$C_{\alpha\beta}{}^{\gamma}$		$\frac{i\kappa}{\Delta(\gamma)}\,\Gamma(t^2\,v)\,\frac{\Gamma(t^2\,v\,c^{\pm 2})}{\Gamma(c^{\pm 2})}\Gamma(\frac{t^2}{\sqrt{v}}a^{\pm 1}b^{\pm 1}c^{\pm 1})$
$\eta^{\alpha\beta}$		$\frac{i\,\kappa}{\Delta(\alpha)}\,\Gamma(t^2\,v)\,\frac{\Gamma(t^2\,v\,a^{\pm 2})}{\Gamma(a^{\pm 2})}\,\hat{\delta}(\alpha,\beta)$

Table 3.1: The structure constants and the metric in terms of elliptic Gamma functions. For brevity we have left implicit the parameters of the Gamma functions, $p = t^3 y$ and $q = t^3 y^{-1}$. We have defined $a \equiv \exp(i\alpha)$, $b \equiv \exp(i\beta)$, and $c \equiv \exp(i\gamma)$. Recall also $\kappa \equiv (p;p)(q;q)/4\pi i$ and $\Delta(\alpha) \equiv (\sin^2\alpha)/\pi$.

symmetry properties. For instance the symmetry of $E^{(2)}$ is related to the Weyl group of E_7. Very recently van de Bult proved [76] that special cases of the $E^{(5)}$ integral, which are equivalent to (3.20), are invariant under the Weyl group of F_4. In particular (3.20) is invariant under $b \leftrightarrow c$. This is theorem 3.2 in [76], with the parameters $\{t_{1,2,3,4},\, b\}$ of [76] related to the parameters $\{a,\, b,\, c,\, d,\, t^2 v\}$ in our equation (3.20) by the substitution

$$t_1 \to \frac{t^2}{\sqrt{v}}\, a\, b, \;\; t_2 \to \frac{t^2}{\sqrt{v}}\, a/b, \;\; t_3 \to \frac{t^2}{\sqrt{v}}\, c\, d, \; t_4 \to \frac{t^2}{\sqrt{v}}\, c/d, \;\; b \to t^2\, v. \qquad (3.22)$$

This completes the proof of crossing symmetry of the fundamental integral (3.14).

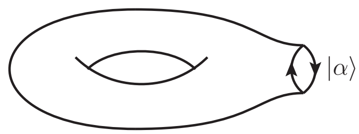


Figure 3.7: Handle-creating operator $\mathcal{J}_\alpha$

The expressions for the structure constants and metric of the topological algebra in terms of the elliptic Gamma functions are summarized in table 3.1. We will give a more general table 4.1 later. These expressions are analytic functions of their arguments, except for for the metric $\eta^{\alpha\beta}$ which contains a delta-function. One can try and use the results of the theory of elliptic Beta integrals to represent the delta-function in a more elegant way, indeed such a representation is sometimes available in terms of a contour integral [78]. However, for generic choices of the parameters, the definition of [78] involves contour integrals not around the unit circle and thus using this representation one presumably should also change the prescription (3.7) for contracting indices. In the limit $v \to t$ the relevant contours do approach the unit circle and the formalism of [78] yields elegant expressions. This limit is however slightly singular. We discuss it in appendix A.2.

As a simple illustration of the use of the expressions in table 3.1 let us compute the superconformal index of the theory associated to diagram **(b)** in figure 3.2. This is essentially the "handle-creating" vertex $\mathcal{J}_\alpha$ of the TQFT, figure 3.7. We have

$$\mathcal{J}_\alpha = C_{\alpha\beta\gamma}\, \eta^{\beta\gamma} = \kappa\, \Gamma\left(t^2 v\right)\, \Gamma\left(\frac{t^2}{\sqrt{v}} a^{\pm 1}\right)^2 \oint \frac{dz}{z} \frac{\Gamma(t^2 v\, z^{\pm 2})}{\Gamma(z^{\pm 2})} \Gamma\left(\frac{t^2}{\sqrt{v}} z^{\pm 2}\, a^{\pm 1}\right). \qquad (3.23)$$

Multivariate extensions of elliptic Beta integrals have appeared in the calculation of the superconformal index for pairs of $\mathcal{N} = 1$ theories related by Seiberg duality [21]. Unlike our $\mathcal{N} = 2$ superconformal cases, there is no continuous deformation relating two Seiberg-dual theories, and it is not a priori obvious that their indices, evaluated at the free UV fixed points, should coincide – but it turns out that they do, thanks to identities satisfied by these multivariate integrals [83]. See also [22].

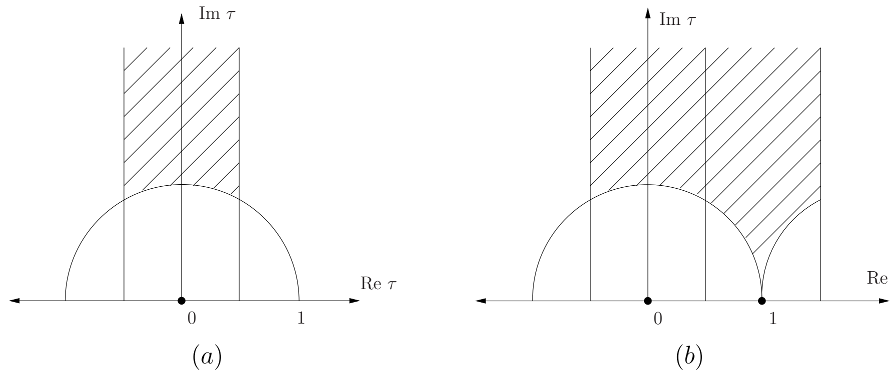


Figure 3.8: Moduli spaces for $\mathcal{N} = 2$ $SU(n)$ gauge theory with $2n$ flavors, (a) for $n = 2$ and (b) for $n = 3$ (in fact, for any $n > 2$). The shaded region in (a) is $H/SL(2,\mathbb{Z})$ while in (b) it is $H/\Gamma^0(2)$, where H is the upper half plane.

3.2 Argyres-Seiberg duality and the index of E_6 SCFT

The S-duality group of the $\mathcal{N} = 2$ $SU(2)$ gauge theory with four flavors is $SL(2,\mathbb{Z})$. The action of this group on the gauge coupling is generated by $\tau \to \tau + 1$ and $\tau \to -1/\tau$. In Gaiotto's description [3] this theory is constructed by compactification of the $6d$ $(2,0)$ theory on a sphere with four punctures of the *same* kind. Then, the S-duality group could be understood as the mapping class group of this Riemann surface. The moduli space of the gauge coupling is shown in figure 3.8 (a). We can see that a fundamental domain can be chosen such that nowhere in the moduli space does the coupling take an infinite value.

For the case of $\mathcal{N} = 2$ $SU(3)$ gauge theory with 6 flavors, however, the S-duality group is $\Gamma^0(2)$. The action of the S-duality on the complex coupling is generated by the transformations $\tau \to \tau + 2$ and $\tau \to -1/\tau$. In Gaiotto's setup this theory is obtained by compactifying the $(2,0)$ theory on the sphere with two punctures of one type and two of another. The mapping class group of such a sphere is $\Gamma^0(2)$. The fundamental domain of this group is shown in the figure 3.8 (b) and, unlike the $SU(2)$ case, this does unavoidably contain a point with infinite coupling. In [28], it was shown that this infinitely coupled cusp could be described in terms of an $SU(2)$ gauge group weakly-coupled to a single hypermultiplet and a rank 1 interacting SCFT with E_6 flavor symmetry. Figure 3.10 describes this duality pictorially. The $SU(2)$ subgroup of the flavor symmetry of the SCFT that is gauged commutes with the $SU(6)$ subgroup of E_6. This $SU(6)$ combined with $SO(2)$ flavor symmetry of the single hypermultiplet generates the full $U(6)$ flavor symmetry of the original $SU(3)$ gauge theory. In other words, the $SO(2)$ flavor symmetry of the

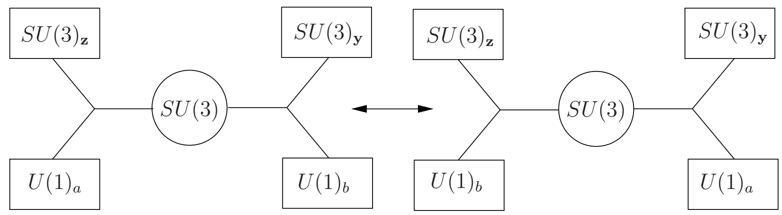


Figure 3.9: $SU(3)$ SYM with $N_f = 6$. The $U(6)$ flavor symmetry is decomposed as $SU(3)_{\mathbf{z}} \otimes U(1)_a \oplus SU(3)_{\mathbf{y}} \otimes U(1)_b$. S-duality $\tau \to -1/\tau$ interchanges the two $U(1)$ charges.

single hypermultiplet corresponds to the baryon number of the original $SU(3)$ gauge theory. The quarks of the $SU(3)$ theory are charged ± 1 under this $U(1)_B$ while the quarks of the $SU(2)$ theory are charged ± 3 under the same.

The E_6 SCFT has a Coulomb branch parametrized by the expectation value of a dimension 3 operator u which is identified with $\mathrm{Tr}\phi^3$ of the dual $SU(3)$ theory, while the $\mathrm{Tr}\phi^2$ of the $SU(3)$ theory corresponds to the Coulomb branch parameter of the $SU(2)$ gauge theory. The E_6 CFT also has a Higgs branch parametrized by the expectation value of dimension 2 operators $\mathbb{X}$, which transform in the adjoint representation of E_6 (**78**). As shown in [80] the Higgs branch operators obey a Joseph relation at quadratic order which leaves a 22 complex dimensional Higgs branch. When coupled to the $SU(2)$ gauge group, the resulting Higgs branch has complex dimension 20. The dual $SU(3)$ theory also has a Higgs branch of complex dimension 20 and its Higgs operators can be easily constructed by combination of squark fields. See appendix A.5 for more details.

The moduli space might contain also other infinitely coupled cusps which however are S-dual to the weakly-coupled cusp $\tau = i\infty$. This is the usual S-dualty mapping the $N_f = 6$ $SU(3)$ gauge theory to itself with some of the $U(1)$ flavor factors interchanged. This duality is represented in figure 3.9.

We proceed to compute the superconformal index of the $SU(3)$ theory and, by using the Argyres-Seiberg duality, of the interacting E_6 SCFT.

3.2.1 Weakly-coupled frame

We take the chiral multiplets to be in the fundamental and antifundamental of the color and flavor. $U(1)_B$ rotates them into each other. The vector multiplet is in the adjoint of the color. The $SU(3)$ characters of the relevant representations are:

$$\chi_f = z_1 + z_2 + z_3 \quad \chi_{\bar{f}} = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \quad \text{and} \quad \chi_{adj} = \chi_f \chi_{\bar{f}} - 1 \tag{3.24}$$

while writing down these characters, we have to impose $z_1 z_2 z_3 = 1$.

Let z's stand for the eigenvalues of the flavor group and x's be the eigenvalues of the color group. The $U(1)_B$ charge is counted by the variable a. Let us write down the characters of the representation of the matter

$$\chi_{hyp} = \sum_{i=1}^{3}\sum_{j=1}^{3} a\, z_i\, x_j + \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{1}{a\, z_i\, x_j}\,. \tag{3.25}$$

Using (2.39) the index contributed by the matter can be written in a closed form as

$$C_{a,\mathbf{x},\mathbf{y}} = \prod_{i=1}^{3}\prod_{j=1}^{3} \Gamma\left(\frac{t^2}{\sqrt{v}}\,\left(a\, x_i\, y_j\right)^{\pm 1}\right)\,. \tag{3.26}$$

The index for the $SU(3)$ gauge theory with six hypermultiplets is then given by the following contour integral.

$$\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}} = C_{b,\mathbf{y},\mathbf{x}}\, C_{a,\mathbf{z}}{}^{\mathbf{x}} =$$
$$\frac{2}{3}\kappa^2\Gamma(t^2 v)^2 \oint_{\mathbb{T}^2} \prod_{i=1}^{2}\frac{dx_i}{2\pi i\, x_i} \frac{\prod_{i=1}^{3}\prod_{j=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{a z_i}{x_j}\right)^{\pm 1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\left(b\, y_i\, x_j\right)^{\pm 1}\right)\prod_{i\neq j}\Gamma\left(t^2 v\,\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}. \tag{3.27}$$

By expanding this integral in t one can show that it is symmetric under interchanging the two $U(1)$ factors (see appendix A.3),

$$a \quad \leftrightarrow \quad b\,. \tag{3.28}$$

Interchanging the two $U(1)$s is equivalent to performing a usual S-duality between a weakly-coupled and infinitely-coupled points of the moduli space and thus we expect the index to be invariant under this operation.[4]

One can analytically prove this statement in a special case. Notice that if $t = v$, the integral (3.27) is given by

$$\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}}\big|_{v=t} = I_{A_2}^{(2)}\left(1|\, t^{\frac{3}{2}}a^{-1}\mathbf{z}^{-1}, t^{\frac{3}{2}}b\mathbf{y}; t^{\frac{3}{2}}a\mathbf{z}, t^{\frac{3}{2}}b^{-1}\mathbf{y}^{-1}\right)\,, \tag{3.29}$$

where [83]

$$I_{A_n}^{(m)}(Z|t_0,\ldots,t_{n+m+1};u_0,\ldots,u_{n+m+1};\,p,q) = \tag{3.30}$$
$$\frac{2^n}{n!}\kappa^n \oint_{\mathbb{T}^{n-1}} \prod_{i=1}^{n-1}\frac{dx_i}{2\pi i\, x_i}\, \frac{\prod_{i=1}^{n}\prod_{j=0}^{m+n+1}\Gamma(t_j\, x_i,\, u_j/x_i;\, p,q)}{\prod_{i\neq j}\Gamma(x_i/x_j;\, p,q)}\Bigg|_{\prod_{i=1}^{n} x_i = Z}\,.$$

[4]The integral (3.27) is an $SU(3)$ generalization of the $SU(2)$ integral in [15] for which the analogous statement to (3.28) has an analytic proof [76]. It is easy to generalize [3.27,3.28] for $SU(n)$ theories with arbitrary n, see appendix A.6.

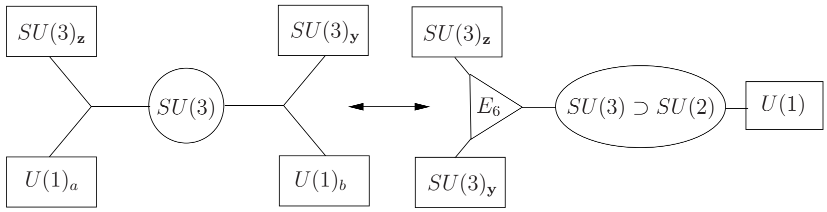


Figure 3.10: Argyres-Seiberg duality for $SU(3)$ SYM with $N_f = 6$.

If the integral $I^{(m)}_{A_n}(Z|\dots t_i \dots; \dots u_i \dots)$ satisfies the condition that $\prod_{i=1}^{m+n+2} t_i u_i = (pq)^{m+1}$ then due to [83], the following theorem holds

$$I^{(m)}_{A_n}\left(Z|\dots t_i \dots; \dots u_i \dots\right) = I^{(n)}_{A_m}\left(Z|\dots \frac{T^{\frac{1}{m+1}}}{t_i} \dots; \dots \frac{U^{\frac{1}{m+1}}}{u_i} \dots\right) \prod_{r,s=1}^{m+n+2} \Gamma\left(t_r u_s\right) , \tag{3.31}$$

where $T \equiv \prod_{r=1}^{m+n+2} t_r$ and $U \equiv \prod_{r=1}^{m+n+2} u_r$.[5] Coincidently, our integral (3.27) satisfies the above requirement and applying the theorem we can transform it into

$$I^{(2)}_{A_2}\left(1|t^{\frac{3}{2}} b\mathbf{z}, t^{\frac{3}{2}} a^{-1}\mathbf{y}^{-1}; t^{\frac{3}{2}} b^{-1}\mathbf{z}^{-1}, t^{\frac{3}{2}} a\mathbf{y}\right) = I^{(2)}_{A_2}\left(1|t^{\frac{3}{2}} b^{-1}\mathbf{z}^{-1}, t^{\frac{3}{2}} a\mathbf{y}; t^{\frac{3}{2}} b\mathbf{z}, t^{\frac{3}{2}} a^{-1}\mathbf{y}^{-1}\right) . \tag{3.32}$$

Note that the factor $\prod_{r,s=1}^{m+n+2} \Gamma(t_r u_s)$ in (3.31) reduces to 1 after pairwise cancelations using the property (2.35). What we have effectively achieved through this transformation is that we have exchanged the $U(1)$ quantum numbers of the matter charged under the $SU(3)^2$ flavor. This in particular implies that both the $SU(3)$ flavor groups are on the same footing and are not associated with separate $U(1)$'s.

3.2.2 Strongly-coupled frame and the index of E_6 SCFT

In the strongly-coupled S-duality frame, figure 3.10, we have a fundamental hypermultiplet coupled to an $SU(2)$ gauge theory. This gauge group is identified with an $SU(2)$ subgroup of the E_6 flavor symmetry of a strongly-coupled rank one SCFT. We do not know the field content of the strongly-coupled rank 1 E_6 SCFT. This implies that we can not write down the "single letter" partition function for that theory and, a-priori, can not directly compute its index. In what follows we will use the index computed in the weakly-coupled frame (3.27) and the above statements about Argyres-Seiberg duality to infer the index of the E_6 SCFT.

Let $C^{(E_6)}$ denote the index of rank 1 E_6 SCFT [29]. The maximal subgroup of E_6 is $SU(3)^3$. Two among these three $SU(3)$'s are identified with the two $SU(3)$ factors

[5] This identity was extensively used in [21] to show that certain theories related by Seiberg duality have equal superconformal indices [19]. In this context the authors of [22, 23] applied the elliptic hypergeometric techniques to a large class of Seiberg dualities.

in the flavor group of the weakly-coupled theory, see figure 3.10. Let the additional $SU(3)$ be denoted by $\mathbf{w}$. The fundamental representation of E_6 is decomposed under $SU(3)_{\mathbf{w}} \otimes SU(3)_{\mathbf{y}} \otimes SU(3)_{\mathbf{z}}$ as,

$$\mathbf{27}_{E_6} = (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) \oplus (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})\,. \tag{3.33}$$

Thus, the character of the E_6 fundamental fields is,

$$\chi_{\mathbf{27}} = \sum_{i,j=1}^{3} \left(\frac{w_i}{y_j} + \frac{z_i}{w_j} + \frac{y_i}{z_j} \right), \qquad \prod_{i=1}^{3} y_i = \prod_{i=1}^{3} z_i = \prod_{i=1}^{3} w_i = 1\,. \tag{3.34}$$

The index $C^{(E_6)}$ is thus a function of $\mathbf{w}$, $\mathbf{y}$, and $\mathbf{z}$. The S-duality picture suggests that we should decompose $SU(3)_{\mathbf{w}}$ as $SU(2)_e \otimes U(1)_r$. This amounts to the change of variables $\{w_1, w_2, w_2\} \to \{er, \frac{r}{e}, \frac{1}{r^2}\}$, for which the character of the fundamental of E_6 becomes

$$\chi_{\mathbf{27}} = (er + \frac{r}{e} + \frac{1}{r^2})(\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}) + (\frac{1}{er} + \frac{e}{r} + r^2)(z_1 + z_2 + z_3) + \sum_{i,j=1}^{3} \frac{y_i}{z_j}\,. \tag{3.35}$$

Thus, the index of the E_6 SCFT can be denoted as $C^{(E_6)}\left((e,\, r), \mathbf{y}, \mathbf{z}\right)$. In the above notations the index of the additional hypermultiplet of the theory is

$$C_{s,\,e} = \Gamma\left(\frac{t^2}{\sqrt{v}}\, e^{\pm 1}\, s^{\pm 1} \right)\,. \tag{3.36}$$

Thus, one can write the superconformal index of the theory in the strongly-coupled frame as

$$\begin{aligned} \hat{\mathcal{I}}\left(s, r; \mathbf{y}, \mathbf{z}\right) &= C_s{}^e\, C^{(E_6)}_{(e,r), \mathbf{y}, \mathbf{z}} \qquad (3.37)\\ &= \kappa\, \Gamma(t^2 v) \oint_{\mathbb{T}} \frac{de}{2\pi i\, e} \frac{\Gamma(t^2 v e^{\pm 2})}{\Gamma(e^{\pm 2})} \Gamma(\frac{t^2}{\sqrt{v}} e^{\pm 1}\, s^{\pm 1})\, C^{(E_6)}\left((e,\, r), \mathbf{y}, \mathbf{z}\right)\,. \end{aligned}$$

By Argyres-Seiberg duality we have to equate

$$\hat{\mathcal{I}}\left(s, r; \mathbf{y}, \mathbf{z}\right) = \mathcal{I}_{a, \mathbf{z}; b, \mathbf{y}}\,, \tag{3.38}$$

where $\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}}$ is given in (3.27), and we appropriately identify the $U(1)$ charges,

$$s = (a/b)^{3/2}, \qquad r = (a\, b)^{-1/2}\,. \tag{3.39}$$

It so happens that the integral of equation (3.37) has special properties which allow us to invert it (see appendix A.4 and [78] for the details). One can write the following

$$\kappa \oint_{C_w} \frac{ds}{2\pi i\, s} \frac{\Gamma(\frac{\sqrt{v}}{t^2} w^{\pm 1}\, s^{\pm 1})}{\Gamma(\frac{v}{t^4},\ s^{\pm 2})} \hat{\mathcal{I}}\,(s, r; \mathbf{y}, \mathbf{z}) \quad = \quad \Gamma(t^2 v\, w^{\pm 2})\, C^{(E_6)}\,((w,\, r), \mathbf{y}, \mathbf{z})\ , \tag{3.40}$$

where the contour C_w is a deformation of the unit circle such that it encloses $s = \frac{\sqrt{v}}{t^2}\, w^{\pm 1}$ and excludes $s = \frac{t^2}{\sqrt{v}}\, w^{\pm 1}$ (for precise definition and details see appendix A.4 and [78]). The above expression for the index $C^{(E_6)}$ does satisfy (3.37), but *a-priori* does not uniquely follow from it. However, as we will explicitly see below, (3.40) is consistent with what is expected from E_6 SCFT. We will comment on this issue in the end of this section. We can thus use the Argyres-Seiberg duality (3.38) to write a closed form expression for the E_6 index

$$\boxed{\begin{aligned} C^{(E_6)}\,((w,\, r), \mathbf{y}, \mathbf{z}) = \frac{2\,\kappa^3 \Gamma(t^2 v)^2}{3\,\Gamma(t^2 v\, w^{\pm 2})} \oint_{C_w} \frac{ds}{2\pi i\, s} \frac{\Gamma(\frac{\sqrt{v}}{t^2} w^{\pm 1}\, s^{\pm 1})}{\Gamma(\frac{v}{t^4},\ s^{\pm 2})} \times \\ \times \oint_{\mathbb{T}^2} \prod_{i=1}^{2} \frac{dx_i}{2\pi i\, x_i} \frac{\prod_{i=1}^{3}\prod_{j=1}^{3} \Gamma\left(\frac{t^2}{\sqrt{v}} \left(\frac{s^{\frac{1}{3}}\, z_i}{x_j\, r}\right)^{\pm 1}\right) \Gamma\left(\frac{t^2}{\sqrt{v}} \left(\frac{s^{-\frac{1}{3}}\, y_i\, x_j}{r}\right)^{\pm 1}\right) \prod_{i \neq j} \Gamma\left(t^2 v \frac{x_i}{x_j}\right)}{\prod_{i \neq j} \Gamma\left(\frac{x_i}{x_j}\right)}\,. \end{aligned}} \tag{3.41}$$

One can rewrite the above expression without using the special integration contour. The integration contour C_w can be split into five pieces: a contour around the unit circle $\mathbb{T}$, two contours encircling the simple poles of $\Gamma(\frac{\sqrt{v}}{t^2} w^{\pm 1}\, s^{\pm 1})$ at $s = \frac{\sqrt{v}}{t^2} w^{\pm 1}$, and two contours encircling in the opposite direction the simple poles of $\Gamma(\frac{\sqrt{v}}{t^2} w^{\pm 1}\, s^{\pm 1})$ at $\frac{t^2}{\sqrt{v}} w^{\pm 1}$. Using the fact that elliptic Gamma function satisfies (2.35) we have

$$\begin{aligned} C^{(E_6)}\,((w,\, r), \mathbf{y}, \mathbf{z}) &= \frac{\kappa}{\Gamma(t^2 v w^{\pm 2})} \oint_{\mathbb{T}} \frac{ds}{s} \frac{\Gamma(\frac{\sqrt{v}}{t^2} w^{\pm 1}\, s^{\pm 1})}{\Gamma(\frac{v}{t^4},\ s^{\pm 2})} \hat{\mathcal{I}}\,(s, r; \mathbf{y}, \mathbf{z}) \\ &+ \frac{1}{2} \frac{\Gamma(w^{-2})}{\Gamma(t^2 v w^{-2})} \left[\hat{\mathcal{I}}\left(s = \frac{\sqrt{v} w}{t^2}, r; \mathbf{y}, \mathbf{z}\right) + \hat{\mathcal{I}}\left(s = \frac{t^2}{\sqrt{v} w}, r; \mathbf{y}, \mathbf{z}\right)\right] \\ &+ \frac{1}{2} \frac{\Gamma(w^{2})}{\Gamma(t^2 v w^{2})} \left[\hat{\mathcal{I}}\left(s = \frac{\sqrt{v}}{t^2 w}, r; \mathbf{y}, \mathbf{z}\right) + \hat{\mathcal{I}}\left(s = \frac{t^2 w}{\sqrt{v}}, r; \mathbf{y}, \mathbf{z}\right)\right]\,. \end{aligned} \tag{3.42}$$

The index (3.41) encodes some information about the matter content of the E_6 theory. To extract this information it is useful to expand the index (3.41) in the chemical potentials. We define an expansion in t as

$$C^{(E_6)} \equiv \sum_{k=0}^{\infty} a_k\, t^k\,. \tag{3.43}$$

The first several orders in this expansion have the following form

$$\begin{aligned}
a_0 =& 1\\
a_1 t =& a_2 t^2 = a_3 t^3 = 0\\
a_4 t^4 =& \frac{t^4}{v}\chi^{E_6}_{\mathbf{78}}\\
a_5 t^5 =& 0\\
a_6 t^6 =& -t^6\chi^{E_6}_{\mathbf{78}} - t^6 + t^6 v^3\\
a_7 t^7 =& \frac{t^7}{v}\left(y+\frac{1}{y}\right)\chi^{E_6}_{\mathbf{78}} + \frac{t^7}{v}\left(y+\frac{1}{y}\right) - t^7 v^2\left(y+\frac{1}{y}\right)\\
a_8 t^8 =& \frac{t^8}{v^2}\left(\chi^{E_6}_{sym^2(\mathbf{78})} - \chi^{E_6}_{\mathbf{650}} - 1\right) + t^8 v + t^8 v\\
a_9 t^9 =& -t^9\left(y+\frac{1}{y}\right)\chi^{E_6}_{\mathbf{78}} - 2t^9\left(y+\frac{1}{y}\right) + t^9 v^3\left(y+\frac{1}{y}\right)\\
a_{10} t^{10} =& -\frac{t^{10}}{v}(\chi^{E_6}_{\mathbf{78}}\,\chi^{E_6}_{\mathbf{78}} - \chi^{E_6}_{\mathbf{650}} - 1) + \frac{t^{10}}{v}\left(y^2+1+\frac{1}{y^2}\right)\chi^{E_6}_{\mathbf{78}} +\\
& + \frac{t^{10}}{v}\left(y+\frac{1}{y}\right)^2 - t^{10}v^2\left(y+\frac{1}{y}\right)^2\\
a_{11} t^{11} =& \frac{t^{11}}{v^2}\left(y+\frac{1}{y}\right)(\chi^{E_6}_{\mathbf{78}}\,\chi^{E_6}_{\mathbf{78}} - \chi^{E_6}_{\mathbf{650}} - 1) + t^{11}v\left(y+\frac{1}{y}\right) + t^{11}v\left(y+\frac{1}{y}\right)\,.
\end{aligned} \tag{3.44}$$

The adjoint representation of E_6 , $\mathbf{78}$, decomposes in the following way in terms of its maximal $SU(3)^3$ subgroup

$$\mathbf{78} \;=\; (\mathbf{3},\mathbf{3},\mathbf{3}) + (\bar{\mathbf{3}},\bar{\mathbf{3}},\bar{\mathbf{3}}) + (\mathbf{8},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{8})\,, \tag{3.45}$$

and $\mathbf{650}$ of E_6 is composed as

$$\mathbf{650} = \mathbf{27}\times\overline{\mathbf{27}} - \mathbf{78} - \mathbf{1}\,. \tag{3.46}$$

The Higgs branch operators $\mathbb{X}$ of E_6 theory are in the adjoint ($\mathbf{78}$) representation of E_6 flavor algebra. The terms of the index proportional to $\chi^{E_6}_{\mathbf{78}}$ are forming the following series,

$$\left[\frac{t^4}{v} - t^6 + \frac{t^7}{v}\left(y+\frac{1}{y}\right) - t^9\left(y+\frac{1}{y}\right) + \cdots\right]\chi^{E_6}_{\mathbf{78}}\,, \tag{3.47}$$

which is the index of a multiplet with $\Delta = 2$, $j = \bar{j} = 0$ and $r = 0$ and of its derivatives (see appendix C.2 of [42]). Taken as a "letter" this multiplet has the following "single letter" partition function

$$\frac{t^4/v - t^6}{(1-t^3y)(1-t^3/y)}\,, \tag{3.48}$$

which matches the quantum numbers of the Higgs branch operators on the weakly-coupled side of the Argyres-Seiberg duality if we follow the identifications listed in [80].

The E_6 singlet part of the index contains yet another series,

$$t^6v^3 - t^7v^2\left(y+\frac{1}{y}\right) + t^8v + t^9v^3\left(y+\frac{1}{y}\right) + \cdots\,. \tag{3.49}$$

This series forms the index of a chiral multiplet with $\Delta = 3$, $j = \bar{j} = 0$ and $r = 3$ together with its derivatives (appendix C.1 of [42])

$$\frac{t^6v^3 - t^7v^2\left(y+\frac{1}{y}\right) + t^8v}{(1-t^3y)(1-t^3/y)}\,. \tag{3.50}$$

Since the Coulomb branch operator, u, of E_6 theory (which is identified as $\mathrm{Tr}\phi^3$ of the dual $SU(3)$ theory) has exactly the same quantum numbers, this multiplet is identified as the Coulomb branch operator.

The remaining singlet part of the index,

$$-t^6 + \frac{t^7}{v}\left(y+\frac{1}{y}\right) + t^8v - 2t^9\left(y+\frac{1}{y}\right) + \cdots\,, \tag{3.51}$$

is just the index of the stress tensor multiplet and its derivatives (appendix C.3 of [42])

$$\frac{-t^6 + \frac{t^7}{v}\left(y+\frac{1}{y}\right) + t^8v - t^9\left(y+\frac{1}{y}\right)}{(1-t^3y)(1-t^3/y)}\,. \tag{3.52}$$

Besides the matter content, the index also provides possible constraints among operators. For example, it was argued [80] that the Higgs branch operators of the E_6 theory should obey the Joseph relations,

$$(\mathbb{X}\otimes\mathbb{X})|_{\mathcal{I}_2} = 0\,, \tag{3.53}$$

where the representation $\mathcal{I}_2$ is defined as

$$sym^2(V(\mathbf{adj})) = V(2\mathbf{adj}) \oplus \mathcal{I}_2\,. \tag{3.54}$$

For E_6, $\mathbf{adj} = \mathbf{78}$, $2\mathbf{adj} = \mathbf{2430}$ and then $sym^2(\mathbf{78}) = \mathbf{2430}\oplus\mathbf{650}\oplus\mathbf{1}$. Thus, in our case

$$\mathcal{I}_2 = \mathbf{650} \oplus \mathbf{1}\,. \tag{3.55}$$

The Joseph relation in E_6 theory reads,

$$(\mathbb{X} \otimes \mathbb{X})|_{\mathbf{650}\oplus\mathbf{1}} = 0\,, \tag{3.56}$$

which means that these operators should not appear in the index. The index of $\mathbb{X}$ is t^4/v, then the index of $\mathbb{X}\otimes\mathbb{X}$ is t^8/v^2. (3.44) shows that our index is consistent with the Joseph relation.

Further constraints can also be derived from the higher order terms in (3.44). Let us consider the index at order t^{10}. The meaning of each term is clear. The first term corresponds to operators $\mathbb{X}\otimes(Q\mathbb{X})$ with the constraint $Q(\mathbb{X}\otimes\mathbb{X})_{\mathbf{650}+1} = 0$ which is a descendant of Joseph relation above (3.56). The last three terms are derivative descendants of $\frac{t^4}{v}\chi^{E_6}_{\mathbf{78}}$, $\frac{t^7}{v}\left(y+\frac{1}{y}\right)$ and $-t^7v^2\left(y+\frac{1}{y}\right)$ respectively. However, terms of the form

$$t^{10}v^2\chi^{E_6}_{\mathbf{78}}\,, \tag{3.57}$$

which would be corresponding to the *Higgs* $\otimes$ *Coulomb* operators are absent. This fact implies the constraint

$$\mathbb{X}\otimes u = 0\,. \tag{3.58}$$

This is consistent with the fact that the E_6 theory has rank 1. The absence of $-\frac{t^{10}}{v}\chi^{E_6}_{\mathbf{78}}$ also implies the constraint

$$\mathbb{X}\otimes T = 0\,, \tag{3.59}$$

where T is the stress tensor. The structure of the index at order t^{11} is consistent with these two constraints.

Finally, let us comment on the uniqueness of our proposal. In principle, the index (3.41) produced by the construction of this section might differ from the true index of the E_6 SCFT: $C^{(E_6)}_{true}((e,\,r),\mathbf{y},\mathbf{z}) = C^{(E_6)}((e,\,r),\mathbf{y},\mathbf{z}) + \delta C((e,\,r),\mathbf{y},\mathbf{z})$, with δC satisfying

$$\oint_{\mathbb{T}} \frac{de}{2\pi i\, e}\frac{\Gamma(\frac{t^2}{\sqrt{v}}\,e^{\pm1}s^{\pm1})\Gamma(t^2 v\, e^{\pm2})}{\Gamma(e^{\pm2})}\,\delta C((e,\,r),\mathbf{y},\mathbf{z}) = 0\,. \tag{3.60}$$

At this stage we are not able to rigorously rule out such a possibility. However, the E_6 covariance of our proposal, its consistency with physical expectations about protected operators and the further S-duality checks performed in the following section, make us confident that we have identified the correct index of the E_6 SCFT.

Note that the expression for the index (3.41) is not explicitly given in terms of E_6 characters. However, as one learns from the perturbative expansion (3.44), the characters of $SU(3)_{\mathbf{y}}\otimes SU(3)_{\mathbf{z}}\otimes SU(2)_w\otimes U(1)_r$ always combine into E_6 characters. Essentially, since the weakly-coupled frame has really $SU(6)\otimes U(1)$ flavor symmetry we can write an expression for the E_6 index which has a manifest $SU(6)\otimes SU(2)$ symmetry,[6] but not the full E_6. The fact that just by assuming Argyres-Seiberg duality we obtain an index for a theory with an E_6 flavor symmetry and with a consistent spectrum of operators is a non-trivial check of Argyres-Seiberg duality.

3.2.3 S-duality checks of the E_6 index

In the previous section we have discussed the superconformal index of the $N_f=6$ $SU(3)$ theory and of its strongly-coupled dual. One can obtain this theory by compactifying a $(2,0)$ $6d$ theory on a sphere with four punctures, two $U(1)$ punctures and two $SU(3)$ punctures. The different S-duality frames are then given by the different degeneration limits of this Riemann surface. The weakly-coupled frames are obtained by bringing together one of the $U(1)$ punctures and one of the $SU(3)$ punctures, and the strongly-coupled frame is obtained by colliding the two $SU(3)$ $(U(1))$ punctures. The coupling constant of the theory is related to the cross ratio of the four punctured sphere.

In [3] Gaiotto suggested to generalize this picture by considering general Riemann surfaces with an arbitrary numbers of punctures of different types (two types in case of the $SU(3)$ theories). The claim is that all theories with the same number and type of punctures and same topology of the Riemann surface are related by S-dualities. The immediate consequence of this claim for the superconformal index is that all such theories have to have the same index as it is independent of the values of the coupling, i.e. the moduli of the Riemann surface. This implies that the superconformal index is a topological invariant of the punctured Riemann surface. It was claimed in [15] that the superconformal index can be actually interpreted as a correlator in a two dimensional topological quantum field theory. The structure constants of this TQFT are given by the index of the three punctured sphere and the contraction of indices (i.e. metric) is gauging of the flavor symmetries. The associativity of the algebra generated by the structure constants is equivalent to the invariance of the index of four punctured spheres under pair-of-pants decomposition into two three punctured spheres. The structure constants and the metric were constructed and the associativity was explicitly verified for the $SU(2)$ case.

In this section we will make the same analysis for the $SU(3)$ case. We have two types of punctures, associated to $U(1)$ and $SU(3)$ flavor symmetries. There are thus different three point functions one can construct. The index of the theory on a sphere with three $SU(3)$ punctures, i.e. the index of the E_6 theory, is a structure constant which we will denote by $C^{(333)}_{\mathbf{x},\mathbf{y},\mathbf{z}}$ and it is just given by (3.41),

$$C^{(333)}_{\mathbf{x},\mathbf{y},\mathbf{z}}=C^{(E_6)}\left(\left(\sqrt{\frac{x_1}{x_2}},\sqrt{x_1x_2}\right),\mathbf{y},\mathbf{z}\right)\,. \tag{3.61}$$

[6] The fact that this symmetry can be manifestly seen in the expression for the index is very reminiscent of the construction of the E_6 symmetry using multi-pronged strings in [84]. It is very interesting to understand whether these facts are related.

This vertex corresponds to the E_6 theory which has rank one, and thus we will refer to it as a rank **1** vertex. We will denote by $C^{(133)}_{\mathbf{x},\mathbf{y},a}$ the index of the sphere with two $SU(3)$ punctures and one $U(1)$ puncture. This is a free theory consisting of a hypermultiplet in fundamental of two $SU(3)$ flavor groups and its value is given by (3.26),

$$C^{(133)}_{a,\mathbf{x},\mathbf{y}} = \prod_{i,j=1}^{3} \Gamma\left(\frac{t^2}{\sqrt{v}}\,(a x_i y_j)^{\pm}\right) . \tag{3.62}$$

This vertex corresponds to a free, rank **0**, theory and we will refer to it as rank zero structure constant. Later on we will define yet another three point function, formally associated to a sphere with two $U(1)$ punctures and one $SU(3)$ puncture. This vertex will have effective rank $-\mathbf{1}$. The metric of the model, $\eta^{\mathbf{x},\mathbf{y}}$, is defined as

$$\eta^{\mathbf{x},\mathbf{y}} = \frac{2}{3}\,\kappa^2\,\Gamma^2(t^2 v) \prod_{1\leqslant i<j\leqslant 3} \frac{\Gamma\left(t^2 v\left(\frac{x_i}{x_j}\right)^{\pm}\right)}{\Gamma\left(\left(\frac{x_i}{x_j}\right)^{\pm}\right)} \hat{\Delta}(\mathbf{x}^{-1},\mathbf{y})\,, \tag{3.63}$$

where $\hat{\Delta}(\mathbf{x}^{-1},\mathbf{y})$ is a δ-function kernel defined by

$$\oint_{\mathbb{T}^2} \prod_{i=1}^{2} \frac{dx_i}{2\pi i\, x_i}\,\hat{\Delta}(\mathbf{x},\mathbf{w})\,f(\mathbf{x}) = f(\mathbf{w})\,, \qquad \mathbf{w}\in\mathbb{T}^2\,. \tag{3.64}$$

The indices are contracted as follows

$$A^{\cdots\mathbf{u}\cdots}B_{\cdots\mathbf{u}\cdots} \equiv \oint_{\mathbb{T}^2} \prod_{i=1}^{2} \frac{du_i}{2\pi i u_i}\,A^{\cdots\mathbf{u}\cdots}B_{\cdots\mathbf{u}\cdots}\Bigg|_{\prod_{i=1}^{3} u_i=1} . \tag{3.65}$$

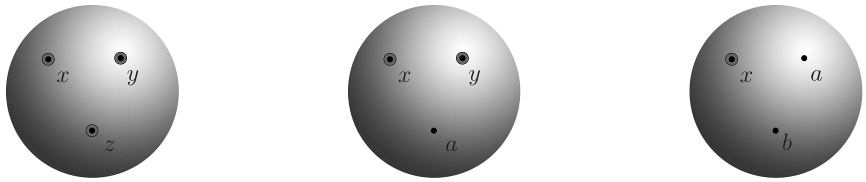
Following these definitions the superconformal indices of all the $SU(3)$ generalized quivers are obtained by contracting the structure constants in different ways.

For the S-duality to hold, and subsequently for the structure constants to have a TQFT interpretation, the algebra generated by these objects has to be associative. We proceed to verify this fact.

$(333)-(333)$ associativity

Let us consider the generalized quiver with genus zero and four $SU(3)$ punctures. The index should be invariant under the permutation of the four $SU(3)$ characters,

$$\mathcal{I}_{3333}(\mathbf{x},\mathbf{y};\mathbf{w},\mathbf{z}) \;=\; C^{(333)}_{\mathbf{x},\mathbf{y},\mathbf{u}}\eta^{\mathbf{u},\mathbf{v}}C^{(333)}_{\mathbf{v},\mathbf{z},\mathbf{w}} = C^{(333)}_{\mathbf{x},\mathbf{z},\mathbf{u}}\eta^{\mathbf{u},\mathbf{v}}C^{(333)}_{\mathbf{v},\mathbf{y},\mathbf{w}}\,. \tag{3.66}$$



$C^{(333)}_{\mathbf{x},\mathbf{y},\mathbf{z}}$ $C^{(133)}_{a,\mathbf{x},\mathbf{y}}$ $C^{(113)}_{a,b,\mathbf{x}}$

Figure 3.11: The three structure constants of the TQFT. The dots represent $U(1)$ punctures and the circled dots $SU(3)$ punctures.

At order $O(t^4)$ we find ,

$$\mathcal{I}_{3333} \sim t^4 \left[\frac{1}{v}(\chi_{\mathbf{8}}(\mathbf{x}) + \chi_{\mathbf{8}}(\mathbf{y}) + \chi_{\mathbf{8}}(\mathbf{z}) + \chi_{\mathbf{8}}(\mathbf{w})) + v^2\right] , \tag{3.67}$$

and at order $O(t^6)$,

$$\mathcal{I}_{3333} \sim t^6 \left[-(\chi_{\mathbf{8}}(\mathbf{x}) + \chi_{\mathbf{8}}(\mathbf{y}) + \chi_{\mathbf{8}}(\mathbf{z}) + \chi_{\mathbf{8}}(\mathbf{w})) + 3v^3\right] . \tag{3.68}$$

These axpressions are symmetric under the exchange $\mathbf{x} \leftrightarrow \mathbf{y} \leftrightarrow \mathbf{z} \leftrightarrow \mathbf{w}$. The associativity can be checked to hold to higher orders as well.

$(333) - (331)$ associativity

Let us consider the generalized quiver with genus zero, three $SU(3)$ punctures and one $U(1)$ puncture. The index should be invariant under permutations of the three $SU(3)$ characters

$$\mathcal{I}_{3331}(a, \mathbf{x}; \mathbf{y}, \mathbf{z}) \;=\; C^{(133)}_{a,\mathbf{x},\mathbf{u}}\eta^{\mathbf{uv}}C^{(333)}_{\mathbf{v},\mathbf{y},\mathbf{z}} = C^{(133)}_{a,\mathbf{y},\mathbf{u}}\eta^{\mathbf{uv}}C^{(333)}_{\mathbf{v},\mathbf{x},\mathbf{z}} . \tag{3.69}$$

We also expand the integrand in t around $t = 0$. The first non-trivial check is for the coefficient of $\mathcal{I}_{3331}$ at order $O(t^4)$,

$$\mathcal{I}_{3331} \sim t^4 \left[\frac{1}{v}\left(\chi_{\mathbf{8}}(\mathbf{x}) + \chi_{\mathbf{8}}(\mathbf{y}) + \chi_{\mathbf{8}}(\mathbf{z}) + 1\right) + v^2\right] , \tag{3.70}$$

which is indeed symmetric under $\mathbf{x} \leftrightarrow \mathbf{y} \leftrightarrow \mathbf{z}$. At order $O(t^6)$,

$$\begin{aligned}\mathcal{I}_{3331} \;\sim\; & \frac{t^6}{v^{3/2}}\left(a^{-3} + a^{-1}\chi_{\bar{\mathbf{3}}}(\mathbf{x})\chi_{\bar{\mathbf{3}}}(\mathbf{y})\chi_{\bar{\mathbf{3}}}(\mathbf{z}) + a\chi_{\mathbf{3}}(\mathbf{x})\chi_{\mathbf{3}}(\mathbf{y})\chi_{\mathbf{3}}(\mathbf{z}) + a^3\right) \\ & -t^6\left(\chi_{\mathbf{8}}(\mathbf{x}) + \chi_{\mathbf{8}}(\mathbf{y}) + \chi_{\mathbf{8}}(\mathbf{z}) + 1\right) + 2t^6v^3 ,\end{aligned} \tag{3.71}$$

which is also symmetric under $\mathbf{x} \leftrightarrow \mathbf{y} \leftrightarrow \mathbf{z}$. Again, we can perform systematic checks to arbitrary high order in t.

The (311) three point function and $(311)-(331)$ associativity

The index of the $N_f = 6$ $SU(3)$ theory in the strongly-coupled frame is given in terms of an integral over an $SU(2)$ character. Thus, we can not write it using the structure constants and the metric we defined in the beginning of this section. The strongly-coupled frame is obtained when two $U(1)$ punctures collide and thus in what follows we will formally define a structure constant with two $U(1)$ characters and an $SU(3)$ character such that when contracted with the E_6 structure constant using the metric above it will produce the index of the strongly-coupled frame.

Let us rewrite the index in the strongly-coupled frame,

$$\hat{\mathcal{I}}\left(s,r;\mathbf{y},\mathbf{z}\right) \;\;=\;\; \kappa\,\Gamma(t^2v)\oint_{\mathbb{T}}\frac{de}{2\pi i\,e}\frac{\Gamma(\frac{t^2}{\sqrt{v}}e^{\pm}\,s^{\pm})}{\Gamma(e^{\pm2})}\,\Gamma(t^2v\,e^{\pm2})\,C\left((e,\,r),\mathbf{y},\mathbf{z}\right)\,, \tag{3.72}$$

as rank one (E_6) (333) and rank -1 (113) vertices contracted

$$\begin{aligned}\hat{\mathcal{I}}\left(a,b;\mathbf{y},\mathbf{z}\right) =& C^{(113)}_{a,b,\mathbf{x}}\,\eta^{\mathbf{x},\mathbf{x}'}\,C^{(333)}_{\mathbf{x}',\mathbf{y},\mathbf{z}}\\ =&\frac{2}{3}\,\kappa^2\,\Gamma(t^2v)^2\,\oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i\,x_i}\prod_{i\neq j}\frac{\Gamma(t^2v\,x_i/x_j)}{\Gamma(x_i/x_j)}\,C^{(113)}\left(a,b,\mathbf{x}^{-1}\right)\,C^{(333)}\left(\mathbf{x},\mathbf{y},\mathbf{z}\right)\,.\end{aligned} \tag{3.73}$$

For this we define

$$C^{(113)}\left(a,b,\mathbf{x}^{-1}\right) =\frac{3}{2\kappa\Gamma(t^2v)}\oint_{\mathbb{T}}\frac{de}{2\pi i\,e}\frac{\Gamma(\frac{t^2}{\sqrt{v}}\,e^{\pm1}\,s^{\pm1})\,\Gamma(t^2v\,e^{\pm2})}{\Gamma(e^{\pm2})}\prod_{i\neq j}\frac{\Gamma(x_i/x_j)}{\Gamma(t^2v\,x_i/x_j)}\,\hat{\Delta}(\mathbf{x},\mathbf{w})\,. \tag{3.74}$$

Here, $\mathbf{w} = (e,\,r)$ with e an $SU(2)$ character and r a $U(1)$ character. The $U(1)$ charges are related as in (3.39), $s = (a/b)^{3/2}$ and $r = (a\,b)^{-1/2}$. $\hat{\Delta}(\mathbf{x},\mathbf{w})$ is a δ-function kernel defined in (3.64). The (113) vertex has effective rank $-\mathbf{1}$. Using the above definition the TQFT algebra is well defined with all the contractions being $SU(3)$ integrals.

The associativity of (311) vertex contracted with a (333) vertex is achieved by construction: remember that we obtained the index of E_6 SCFT by requiring this property. Let us check the associativity of (331) contracted with (113)

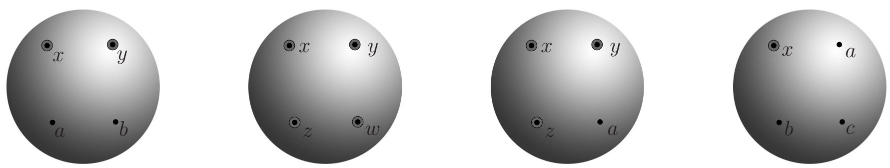
$$
\begin{aligned}
&\mathcal{I}(a,b;\,c,\mathbf{y}) = C^{(113)}_{a,b,\mathbf{x}}\,\eta^{\mathbf{x},\mathbf{x}'}\,C^{(331)}_{\mathbf{x}',\mathbf{y},c} = \\
&\quad \frac{2}{3}\,\kappa^2\,\Gamma(t^2v)^2\,\oint\prod_{i=1}^{2}\frac{dx_i}{2\pi i\,x_i}\prod_{i\neq j}\frac{\Gamma(t^2v\,x_i/x_j)}{\Gamma(x_i/x_j)}\,C^{(113)}\left(a,b,\mathbf{x}^{-1}\right)\,\prod_{i,j}\Gamma\left(\frac{t^2}{\sqrt{v}}\,(c\,x_i\,y_j)^{\pm1}\right) \\
&\qquad\qquad = \prod_{i=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{c\,y_i}{r^2}\right)^{\pm1}\right) \\
&\qquad \times\,\kappa\Gamma(t^2v)\oint\frac{de}{2\pi i\,e}\,\frac{\Gamma(t^2v\,e^{\pm2})}{\Gamma(e^{\pm2})}\Gamma\left(\frac{t^2}{\sqrt{v}}\,s^{\pm1}\,e^{\pm1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\,(c\,r\,y_i)^{\pm1}\,e^{\pm1}\right)\,.
\end{aligned}
\tag{3.75}
$$

This is exactly the index of $SU(2)$ $N_f = 4$ (the fourth line in (3.75)) with a decoupled hypermultiplet in the fundamental of an $SU(3)$ flavor (the third line in (3.75)). Remembering (3.39) and the results of [15, 76] it is easy to show that there is a permutation symmetry between the three $U(1)$ punctures a, b and c,

$$
a \quad\leftrightarrow\quad b \quad\leftrightarrow\quad c\,. \tag{3.76}
$$

Using the definition (3.74) the index of a sphere with four $U(1)$ punctures is singular. However, we do not have a physical interpretation of this surface and it does not appear in any decoupling limit of a physical theory. Thus, making sense of this surface is not essential.

We have shown that the structure constants define an associative algebra and thus define a TQFT. In particular the superconformal index of theories with equal genus and equal number/type of punctures is the same in agreement with S-duality.



(a) (b) (c) (d)

Figure 3.12: The relevant four-punctured spheres for A_2 theories. The three different degeneration limits of a four-punctured sphere correspond to different S-duality frames. For example, in (a) two of the degeneration limits (when a $U(1)$ puncture collides with an $SU(3)$ puncture) correspond to the weakly-coupled $N_f = 6$ $SU(3)$ theory, the third limit (when two like punctures collide) corresponds to the Argyres-Seiberg theory. In (d) the degeneration limits correspond to the different duality frames of $SU(2)$ SYM with $N_f = 4$ theory plus a decoupled hypermultiplet.

Chapter 4

TQFT Structure of the Index for A_n-Type Quivers

In the previous chapter we have showed that the superconformal index is a useful observable of class $\mathcal{S}$ theories. With the concrete expression of the index for class $\mathcal{S}$ A_1 theories, we demonstrate in section 3.1 that the index does indeed take the form of a correlator in a 2d TQFT and in the process check the S-duality of the index. Then by demanding consistency with Argyres-Seiberg duality, we further write down an explicit integral expression for the index of the E_6 SCFT which has no Lagrangian description. The spectral information predicted by the E_6 index coincides with other works. With the index of the E_6 SCFT as one of the basic components, we proceed to define a TQFT structure for generalized quivers with $SU(3)$ gauge symmetries and check the associativity and S-duality.

General Four-dimensional superconformal field theories of $\mathcal{S}$ [3, 4] arise from partially-twisted compactification of the six-dimensional $(2,0)$ theory on a punctured Riemann surface $\mathcal{C}$. The complex-structure moduli of $\mathcal{C}$ are identified with the exactly marginal couplings of the $4d$ SCFT, while the punctures are associated to flavor symmetries.

Any punctured surface can be obtained, usually in more than one way, by gluing three-punctured spheres (pairs of pants) with cylinders. The three-punctured spheres are then the elementary building blocks. They correspond to isolated $4d$ SCFTs with flavor symmetry $G_1 \otimes G_2 \otimes G_3$, where each factor G_I is associated to one of the three punctures.[1] The cylinders correspond to $\mathcal{N}=2$ vector multiplets, and the gluing operation amounts to gauging a common $SU(k)$ symmetry of two punctures. The gluing parameter is interpreted as the complexified gauge coupling, with zero coupling corresponding to an infinitely long cylinder – a degeneration limit of the surface. Different pairs-of-pants decompositions of the same surface

[1] In this chapter we focus on class $\mathcal{S}$ theories that descend from the $(2,0)$ theory of type A_{k-1}. Then the punctures are classified by the possible embeddings of $SU(2)$ into $SU(k)$ and $G_I \subset SU(k)$ is the commutant of the chosen embedding.

$\mathcal{C}$ correspond to different descriptions of the same SCFT, related by generalized S-dualities.

As showen in previous chapters the index is naturally viewed as a correlator in a $2d$ topological QFT living on $\mathcal{C}$. Let us review how this works. We parametrize the index of a three-punctured sphere as $\mathcal{I}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$, where $\mathbf{a}_I$ are fugacities dual to the Cartan subgroup of G_I: except in special cases these are *a priori* unknown functions. On the other hand we can easily write down the "propagator" associated to a cylinder,

$$\eta(\mathbf{a}, \mathbf{b}) = \Delta(\mathbf{a})\mathcal{I}^V(\mathbf{a})\,\delta(\mathbf{a}, \mathbf{b}^{-1})\,, \tag{4.1}$$

where $\Delta(\mathbf{a})$ is the Haar measure and $\mathcal{I}^V(\mathbf{a})$ the index of a vector multiplet, which is known explicitly. The index of a generic theory of class $\mathcal{S}$ can be written in terms of the index of these elementary constituents. As the simplest example, gluing two three-punctured spheres with one cylinder one obtains the index of a four-punctured sphere,

$$\begin{aligned} \mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) &= \oint [d\mathbf{a}] \oint [d\mathbf{b}]\, \mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a})\, \eta(\mathbf{a}, \mathbf{b})\, \mathcal{I}(\mathbf{b}, \mathbf{a}_3, \mathbf{a}_4) \\ &= \oint [d\mathbf{a}]\; \Delta(\mathbf{a})\, \mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a})\, \mathcal{I}^V(\mathbf{a})\, \mathcal{I}(\mathbf{a}^{-1}, \mathbf{a}_3, \mathbf{a}_4)\,, \end{aligned} \tag{4.2}$$

where we have introduced the notation

$$\oint [d\mathbf{a}] \equiv \oint \prod_{i=1}^{k-1} \frac{da_i}{2\pi i a_i}\,. \tag{4.3}$$

If we expand the index in a convenient basis of functions $\{f^\alpha(\mathbf{a})\}$, labeled by $SU(k)$ representations $\{\alpha\}$,[2] we can associate to each three-punctured sphere "structure constants" $C_{\alpha\beta\gamma}$ and to each propagator a metric $\eta^{\alpha\beta}$,

$$\mathcal{I}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma}\, f^\alpha(\mathbf{a})\, f^\beta(\mathbf{b})\, f^\gamma(\mathbf{c}) \tag{4.4}$$

$$\eta^{\alpha\beta} = \oint [d\mathbf{a}] \oint [d\mathbf{b}]\; \eta(\mathbf{a}, \mathbf{b})\, f^\alpha(\mathbf{a})\, f^\beta(\mathbf{b})\,. \tag{4.5}$$

Invariance of the index under the different ways to decompose the surface is tantamount of saying that $C_{\alpha\beta\gamma}$ and $\eta^{\alpha\beta}$ define a two-dimensional topological QFT.[3]

[2]For theories of type A, $\{f^\alpha(\mathbf{a})\}$ are symmetric functions of their arguments, which are fugacities dual to the Cartan generators of $SU(k)$. More generally, for theories of type D and E, $\{f^\alpha(\mathbf{a})\}$ are invariant under the appropriate Weyl group.

[3]We are using this term somewhat loosely. As axiomatized by Atiyah, a TQFT is understood to have a finite-dimensional state-space, while in our case the state-space will be infinite-dimensional. The best-understood example of a $2d$ topological theory with an infinite-dimensional state-space is the zero-area limit of $2d$ Yang-Mills theory [85, 86] (see

The crucial property is associativity,

$$C_{\alpha\beta\gamma}C^{\gamma}{}_{\delta\epsilon} = C_{\alpha\delta\gamma}C^{\gamma}{}_{\beta\epsilon}\,, \tag{4.6}$$

where indices are raised with the metric $\eta^{\alpha\beta}$ and lowered with the inverse metric $\eta_{\alpha\beta}$.

It is very natural to choose the complete set of functions $\{f^{\alpha}(\mathbf{a})\}$ to be orthonormal under the measure that appears in the propagator,

$$\oint [d\mathbf{a}]\ \Delta(\mathbf{a})\,\mathcal{I}^{V}(\mathbf{a})\,f^{\alpha}(\mathbf{a})f^{\beta}(\mathbf{a}^{-1}) = \delta^{\alpha\beta}\,. \tag{4.7}$$

Then the metric $\eta^{\alpha\beta}$ is trivial,

$$\eta^{\alpha\beta} = \delta^{\alpha\beta}\,. \tag{4.8}$$

Condition (4.7) still leaves considerable freedom, as it is obeyed by infinitely many bases of functions related by orthogonal transformations. The real simplification arises if we can find an *explicit* basis $\{f^{\alpha}(\mathbf{a})\}$, such that the structure constants are *diagonal*,

$$C_{\alpha\beta\gamma} \neq 0 \quad \rightarrow \quad \alpha = \beta = \gamma\,. \tag{4.9}$$

Associativity (4.6) is then automatic. For structure constants satisfying (4.6) one can always find a basis in which they are diagonal: we give a detailed example of such a diagonalization procedure in appendix B.1 for the simplest limit of the index. The challenge is to describe the basis in concrete form.

In general the measure appearing in the propagator is complicated and no explicit set of orthonormal functions is available. We find it very useful to consider an ansatz

$$f^{\alpha}(\mathbf{a}) = \mathcal{K}(\mathbf{a})P^{\alpha}(\mathbf{a})\,, \tag{4.10}$$

for some function $\mathcal{K}(\mathbf{a})$. Clearly, from (4.7), the functions $\{P^{\alpha}(\mathbf{a})\}$ are orthornormal under the new measure $\hat{\Delta}(\mathbf{a})$,

$$\oint [d\mathbf{a}]\ \hat{\Delta}(\mathbf{a})\,P^{\alpha}(\mathbf{a})P^{\beta}(\mathbf{a}^{-1}) = \delta^{\alpha\beta}\,, \qquad \hat{\Delta}(\mathbf{a}) \equiv \mathcal{I}^{V}(\mathbf{a})\,\mathcal{K}(\mathbf{a})^{2}\,\Delta(\mathbf{a})\,. \tag{4.11}$$

(Recall that $\Delta(\mathbf{a})$ always denotes the Haar measure). The name of the game is to find a clever choice of $\mathcal{K}(\mathbf{a})$, for which $\hat{\Delta}(\mathbf{a})$ is a simple known measure and the orthonormal basis $\{P^{\alpha}(\mathbf{a})\}$ an explicit set of functions such that (4.9) holds.

Once the diagonal basis $\{f^{\alpha}(\mathbf{a})\}$ and the structure constant $C_{\alpha\alpha\alpha}$ are known,

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e.g. [87] for a comprehensive review). Happily, the $2d$ topological theory associated to the index turns out to be closely related to $2d$ Yang-Mills.

one can easily calculate the index of the SCFT associated to the genus $\mathfrak{g}$ surface with s punctures. Such a surface can be built by gluing $2\mathfrak{g}-2+s$ three-punctured spheres, so we have[4]

$$\mathcal{I}_{\mathfrak{g},s}(\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_s)=\sum_{\alpha}(C_{\alpha\alpha\alpha})^{2\mathfrak{g}-2+s}\prod_{I=1}^{s}f^{\alpha}(\mathbf{a}_I)\,. \tag{4.12}$$

In the rest of this chapter we implement the following strategy. We start by considering the generalized $SU(2)$ quivers. Since they have a Lagrangian description, closed form expressions for the index (as matrix integrals) are readily available. We then look for a basis of functions $\{f^{\alpha}(\mathbf{a})\}$ that diagonalizes the structure constants. Fortunately, for each special limit of the index that we consider, the diagonal basis is of the form (4.10), with $\{P^{\alpha}(\mathbf{a})\}$ well-known symmetric *polynomials*: Hall-Littlewood, Schur or Macdonald polynomials. (The first two are in fact special cases of Macdonald polynomials). Since these polynomials are defined for arbitrary rank, we can extrapolate from the $SU(2)$ case and formulate compelling conjectures for the index of *all* generalized quivers of type A. (This approach readily generalizes to all ADE theories, but in this paper we focus on the A series). Finally we check our conjectures against expected symmetry enhancements and S-dualities.

4.1 Limits of the index with additional supersymmetry

We now consider several limits of the superconformal index, such that the states contributing to it are annihilated by more than one supercharge. Recall that before taking any limit the index receives contributions only from states with

$$\tilde{\delta}_{1\dot{-}}=E-2j_2-2R+r=0\,, \tag{4.13}$$

which are annihilated by $\widetilde{\mathcal{Q}}_{1\dot{-}}$. We tend to refer to the different limits of the index by the type of symmetric polynomials relevant for their evaluation. In appendix B.2 we discuss which short multiplets of the superconformal algebra are counted by the index in each of these limits.

[4]Here for simplicity we are considering the case where all external punctures are "maximal", *i.e.* they have flavor symmetry $SU(k)$. The prescription for punctures with reduced symmetry is discussed in detail in sections 4.2, 4.3 and 4.4.

Macdonald index

We first consider the limit[5]

$$\sigma \to 0\,, \qquad \rho\,,\tau \text{ fixed}\,, \tag{4.14}$$

(which is the same as $p \to 0$ with q and t fixed). The limit is well-defined since the power of σ in the trace formula (2.4) is given by $\frac{1}{2}\delta_{1+} \geq 0$. The index is given by

$$\begin{aligned} \mathcal{I}_M &= \mathrm{Tr}_M(-1)^F \rho^{\frac{1}{2}(E-2j_1-2R-r)}\, \tau^{\frac{1}{2}(E+2R+2j_2+r)} \\ &= \mathrm{Tr}_M(-1)^F q^{\frac{1}{2}(E-2j_1-2R-r)}\, t^{R+r}\,, \end{aligned} \tag{4.15}$$

where Tr_M denotes the trace restricted to states with $\delta_{1+} = E + 2j_1 - 2R - r = 0$. Such states are annihilated by $\mathcal{Q}_{1+}$. All in all $\mathcal{I}_M$ is a $\frac{1}{4}$-BPS object receiving contributions only from states annihilated by two supercharges, one chiral ($\mathcal{Q}_{1+}$) and one anti-chiral ($\widetilde{\mathcal{Q}}_{1\dot{-}}$). The single letter partition functions of the half-hypermultiplet and the vector simplify to

$$f^{\frac{1}{2}H} = \frac{\tau}{1-\rho\tau} = \frac{\sqrt{t}}{1-q}\,, \quad f^V = \frac{-\tau^2 - \rho\tau}{1-\rho\tau} = \frac{-t-q}{1-q}\,. \tag{4.16}$$

Hall-Littlewood index

We further specialize the index by sending $\rho \to 0$, so we are taking the limit

$$\sigma \to 0\,, \qquad \rho \to 0\,, \qquad \tau \text{ fixed}\,, \tag{4.17}$$

(equivalently, $q\,,p \to 0$ with t fixed), which is well-defined thanks to $\delta_{1\pm} \geq 0$. The index is given by

$$\mathcal{I}_{HL} = \mathrm{Tr}_{HL}(-1)^F \tau^{\frac{1}{2}(E+2R+2j_2+r)} = \mathrm{Tr}_{HL}(-1)^F \tau^{2(E-R)}\,, \tag{4.18}$$

where Tr_{HL} denotes the trace restricted to states with $\delta_{1\pm} = E \pm 2j_1 - 2R - r = 0$. All in all, taking (4.13) into account, the states contributing to the index obey

$$j_1 = 0\,, \qquad j_2 = r\,, \qquad E = 2R + r\,, \tag{4.19}$$

and are annihilated by three supercharges: $\mathcal{Q}_{1+}$, $\mathcal{Q}_{1-}$ and $\widetilde{\mathcal{Q}}_{1\dot{-}}$.

Let us consider the Hall-Littlewood (HL) index for a theory with a Lagrangian description. From table 2.2, we see that it gets contributions only from the scalar q of the hypermultiplet and from the fermion $\bar{\lambda}_{1\dot{+}}$ of the vector multiplet. The single

[5] An equivalent limit can be obtained by sending ρ to zero.

letter partition function of the half-hypermultiplet and the vector multiplet is then

$$f^{\frac{1}{2}H} = \tau, \quad f^V = -\tau^2 \,. \tag{4.20}$$

Remarkably, for generalized quivers with a sphere topology the computation of the HL index is equivalent to the computation of the partition function over the Higgs branch discussed in [88, 89] (the Hilbert series of the Higgs branch).[6] This can be shown as follows. To compute the partition function of [88, 89] for the Higgs branch of an $\mathcal{N}=2$ gauge theory one counts all the possible gauge invariant operators built from the scalar components of the hypermultiplets taking into account the F-term superpotential constraints. In an $\mathcal{N}=2$ gauge theory with M $SU(2)$ gauge factors the superpotential takes the form

$$W = \sum_{i=1}^{M} \sum_{\alpha \in \{i\}} Q^{(\alpha)}_{a_i a_k a_l}\, \Phi^{a_i}{}_{b_i}\, Q^{(\alpha)\, b_i a_k a_l} \,, \tag{4.21}$$

where the summation over i is over the gauged groups. The set $\{i\}$ is the set of (at most two) trifundamental hypermultiplets transforming non-trivially under gauge group i. The F-term constraints then read

$$Q^{(\alpha_1)}_{a_i a_k a_l}\, Q^{(\alpha_1) b_i a_k a_l} + Q^{(\alpha_2)}_{a_i a_m a_n}\, Q^{(\alpha_2) b_i a_m a_n} = 0 \,. \tag{4.22}$$

If the quiver diagram does not have loops, *i.e.* the corresponding Riemann surface has a topology of a sphere, this is a set of M independent constraints. It then follows that the computation of this partition function is the same as the computation of the index. Indeed, one associates a fugacity τ for each scalar component of Q. The constraint (4.22) is quadratic in Q and is in the adjoint representation of the gauge group. It is implemented by multiplying the unconstrained partition function with the following factor [88, 89],

$$\exp\left[-\sum_{n=1}^{\infty} \frac{1}{n} \tau^{2n} \left(a_i^{2n} + a_i^{-2n} + 1\right)\right] = (1-\tau^2)(1-\tau^2\, a_i^2)(1-\tau^2\, a_i^{-2}) \,. \tag{4.23}$$

This factor is the same as the index of the letter $\bar{\lambda}_{1\dot{+}}$. Thus, one can think of the letter $\bar{\lambda}_{1\dot{+}}$ in the calculation of the index as playing the same role as the superpotential constraint in the calculation of the Higgs partition function! This logic can be extended to higher-rank theories, where not all the building blocks have Lagrangian description, but the Higgs branch can still be described in terms of operators obeying certain constraints. This concludes the argument that the HL index is the same as the Higgs partition function for theories with sphere topology. Our derivation

[6] A relation of a similar limit of the $\mathcal{N}=1$ index with the counting problems discussed in [90, 91] was mentioned in [23]. We thank V. Spiridonov for bringing this reference to our attention.

also makes it clear that this correspondence fails for higher-genus theories.

In [88] non-trivial very explicit expressions for the Higgs branch partition function of the SCFTs with exceptional flavor symmetry groups [29, 92] were conjectured. We will see that they are exactly reproduced by the HL index.

Schur index

The Schur index is defined by specializing the fugacities to $\rho = \tau$ with σ arbitrary (equivalently $q = t$ with p arbitrary). It reads

$$\mathcal{I}_S = \mathrm{Tr}(-1)^F \, \sigma^{\frac{1}{2}(E+2j_1-2R-r)} \, \rho^{E-j_1+j_2} \, e^{-\beta(E-2j_2-2R+r)} \,. \tag{4.24}$$

By construction, all charges in the trace formula commute with the supercharge $\widetilde{\mathcal{Q}}_{1\dot{-}}$ "with respect to which" the index is evaluated. From table 2.1, we observe that the charges in (4.24) *also* commute with $\mathcal{Q}_{1+}$. Thus the index receives contributions from states with $\delta_{1+} = \tilde{\delta}_{1\dot{-}} = 0$ (the intersection of the cohomologies of $\mathcal{Q}_{1+}$ and of $\widetilde{\mathcal{Q}}_{1\dot{-}}$) and it is independent of *both* σ and β. We can then write

$$\mathcal{I}_S = \mathrm{Tr}(-1)^F \, \rho^{2(E-R)} = \mathrm{Tr}(-1)^F \, q^{E-R} \,. \tag{4.25}$$

The Schur index can also be obtained as a special case of the Macdonald index by setting $\rho = \tau$ (equivalently $q = t$); we have just seen that for $\rho = \tau$ the index becomes independent of σ so the limit $\sigma \to 0$ that we take to obtain the Macdonald index is immaterial.

The single letter partition functions of the half-hypermultiplets and the vector multiplet are given by

$$f^{\frac{1}{2}H} = \frac{\rho}{1-\rho^2} = \frac{\sqrt{q}}{1-q} \,, \quad f^V = \frac{-2\rho^2}{1-\rho^2} = \frac{-2q}{1-q} \,. \tag{4.26}$$

The Schur index is the same as the index studied in [17], where we referred to it as the *reduced index*.

Coulomb-branch index

Finally we consider the limit

$$\tau \to 0 \,, \qquad \rho \,, \sigma \text{ fixed} \,, \tag{4.27}$$

which is well-defined thanks to $\tilde{\delta}_{2\dot{+}} \geq 0$. The trace formula becomes

$$\mathcal{I}_C = \mathrm{Tr}_C(-1)^F \, \sigma^{\frac{1}{2}(E+2j_1-2R-r)} \, \rho^{\frac{1}{2}(E-2j_1-2R-r)} \, e^{-\beta(E-2j_2-2R+r)} \,, \tag{4.28}$$

where Tr_C denotes the trace over the states with $\tilde{\delta}_{2\dot{+}} = E + 2j_2 + 2R + r = 0$, which are annihilated by $\widetilde{\mathcal{Q}}_{2\dot{+}}$. All in all, the index gets contributions from states annihilated by two antichiral supercharges, $\widetilde{\mathcal{Q}}_{1\dot{-}}$ and $\widetilde{\mathcal{Q}}_{2\dot{+}}$.

In this limit the single-letter partition function of the half-hypermultiplet and the vector multiplet are

$$f^{\frac{1}{2}H} = 0, \qquad f^V = \sigma\rho \equiv T\,. \tag{4.29}$$

From the viewpoint of the the single-letter partition functions one can take an interesting less restrictive limit,

$$\tau\,, \sigma \to 0\,, \qquad \rho \to \infty \qquad \text{with} \quad Q \equiv \tau\rho \ \text{ and } \ T \equiv \sigma\rho \ \text{ fixed.} \tag{4.30}$$

In this limit we have

$$f^{\frac{1}{2}H} = 0, \qquad f^V = \frac{T - Q}{1 - Q}\,. \tag{4.31}$$

We recover (4.29) for $Q \to 0$. In terms of the new fugacities Q and T the index reads

$$\mathcal{I}_{CM} = \mathrm{Tr}_{CM}(-1)^F \, T^{\frac{1}{2}(E+2j_1-2R-r)} \, Q^{\frac{1}{2}(E+2j_2+2R+r)}\,, \tag{4.32}$$

where Tr_{CM} denotes the trace restricted to states satisfying $E + 2j_1 + r = 0$. This index is well-defined for Lagrangian theories and for theories related to them by dualities.

We now describe the explicit evaluation of these special limits of the index for the SCFTs of class $\mathcal{S}$.

4.2 Hall-Littlewood index

We begin with the Hall-Littlewood index,

$$\mathcal{I}_{HL}(\tau) = \mathrm{Tr}_{HL}(-1)^F \, \tau^{2E-2R}\,, \tag{4.33}$$

where Tr_{HL} denotes the trace restricted to states with $j_1 = 0$ and $E - 2R - r = 0$. This is the limit that leads to the greatest simplifications.

4.2.1 $SU(2)$ quivers

Let us start from the $SU(2)$ generalized quivers, for which the basic building blocks are known explicitly. There is only one type of non-trivial puncture, the maximal puncture with $SU(2)$ flavor symmetry. The SCFT corresponding the three-punctured sphere, denoted by T_2 in [3], is the theory of free hypermultiplets in the trifundamental representation of $SU(2)$. Its index is immediately evaluated,

$$\mathcal{I}(a,b,c) = PE\left[\tau\chi_1(a)\chi_1(b)\chi_1(c)\right]_{a,b,c,\tau} = \frac{1}{\prod_{s_a,s_b,s_c=\pm1}(1-\tau\, a^{s_a}\, b^{s_b}\, c^{s_c})}\,, \quad (4.34)$$

where the fugacities a, b, and c label the Cartans of the three $SU(2)$ flavor groups. The plethystic exponent PE is defined as

$$PE\left[f(x_i)\right]_{x_i} \equiv \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f(x_i^n)\right)\,. \quad (4.35)$$

We will often omit the subscript x_i in the expressions for $PE[\dots]$. $\chi_1(a)$ is the character of fundamental representation of $SU(2)$. More generally the $SU(2)$ Schur polynomials χ_λ are given by

$$\chi_\lambda(a) = \frac{a^{-1-\lambda}-a^{1+\lambda}}{a^{-1}-a}\,. \quad (4.36)$$

The propagator $\eta(a,b)$ is also easily evaluated:

$$\eta(a,b) = \Delta(a)\mathcal{I}^V(a)\delta(a,b^{-1})\,, \quad (4.37)$$

where $\mathcal{I}^V(a)$ is the index of the vector multiplet,

$$\mathcal{I}^V(a) = PE[-\tau^2\chi_2(a)]_{a,\tau} = (1-\tau^2)\,(1-\tau^2\,a^2)\,(1-\tau^2\,a^{-2})\,, \quad (4.38)$$

and $\Delta(a)$ the $SU(2)$ Haar measure,

$$\Delta(a) = \frac{1}{2}(1-a^2)(1-\frac{1}{a^2})\,. \quad (4.39)$$

Following the strategy outlined in the beginning of this chapter, we look for a complete set of functions $\{f^\lambda(a)\}$ orthonormal under the propagator measure such that the structure constants are diagonal,

$$\mathcal{I}(a,b,c) = \sum_{\lambda=0}^{\infty} C_{\lambda\lambda\lambda}\, f^\lambda(a)f^\lambda(b)f^\lambda(c)\,. \quad (4.40)$$

We describe this calculation in appendix B.1. We find the remarkable result

$$\begin{aligned} f^\lambda(a) \quad &= \quad \mathcal{K}(a)\; P^\lambda_{HL}(a,a^{-1}|\tau), & (4.41)\\ C_{\lambda\lambda\lambda} \quad &= \quad \frac{\sqrt{1-\tau^2}\,(1+\tau^2)}{P^\lambda_{HL}(\tau,\tau^{-1}|\tau)}\,. & (4.42)\end{aligned}$$

Here P^{λ}_{HL} are the $SU(2)$ Hall-Littlewood polynomials,

$$P^{\lambda}_{HL}(a,a^{-1}|\tau) \;=\; \chi_{\lambda}(a)-\tau^2\chi_{\lambda-2}(a) \quad \text{for } \lambda\geq 1\,, \quad P^{\lambda=0}_{HL}(a,a^{-1}|\tau)=\sqrt{1+t^2}\,, \tag{4.43}$$

which are orthonormal under the measure

$$\hat{\Delta}(a)=\Delta_{HL}(a)=\frac{1}{2}\frac{(1-a^2)(1-a^{-2})}{(1-\tau^2a^2)(1-\tau^2a^{-2})}\,. \tag{4.44}$$

The requirement that $\{f^{\lambda}(a)\}$ be orthonormal under the propagator measure $\Delta(a)\mathcal{I}^V(a)$ fixes the prefactor $\mathcal{K}(a)$,

$$\mathcal{K}(a)=\left(\frac{\Delta_{HL}(a)}{\Delta(a)\mathcal{I}^V(a)}\right)^{\frac{1}{2}}=\frac{1}{\sqrt{1-\tau^2}}\frac{1}{(1-\tau^2a^2)(1-\tau^2a^{-2})}\,. \tag{4.45}$$

We can now immediately write down an explicit formula for the index of any generalized $SU(2)$ quiver associated to a genus $\mathfrak{g}$ Riemann surface with s punctures. From (4.12),

$$\begin{aligned}\mathcal{I}_{\mathfrak{g},s}(a_1,a_2,\dots,a_s) \;=\;& \left(1-\tau^2\right)^{\mathfrak{g}-1}\left(1+\tau^2\right)^{2\mathfrak{g}-2+s}\cdot \\ & \sum_{\lambda=0}^{\infty}\frac{1}{\left[P^{\lambda}_{HL}(\tau,\tau^{-1}|\,\tau)\right]^{2\mathfrak{g}-2+s}}\prod_{I=1}^{s}\frac{P^{\lambda}_{HL}(a_I,a_I^{-1}|\,\tau)}{(1-\tau^2a_I^2)(1-\tau^2a_I^{-2})}\,.\end{aligned} \tag{4.46}$$

In particular for genus $\mathfrak{g}$ with no punctures the sum over the $SU(2)$ irreducible representations in (4.46) can be explicitly performed and one gets

$$\mathcal{I}^{(2)}_{\mathfrak{g}}=\frac{\left(1-\tau^2\right)^{\mathfrak{g}-1}\left(\tau^{2\mathfrak{g}-2}+\left(1+\tau^2\right)^{\mathfrak{g}-1}\left(1-\tau^{2\mathfrak{g}-2}\right)\right)}{1-\tau^{2\mathfrak{g}-2}}\,. \tag{4.47}$$

We observe that setting a flavor fugacity $a=\tau$ we "close" the corresponding puncture. For example we can go from the three-punctured sphere to the two-punctured sphere (=cylinder),

$$\mathcal{I}(a_1,a_2,\tau)\sim\sum_{\lambda}P^{\lambda}_{HL}(a_1,a_1^{-1}|\tau)P^{\lambda}_{HL}(a_2,a_2^{-1}|\tau)=\eta(a_1,a_2)\,. \tag{4.48}$$

(There is an overall divergent proportionality factor). This procedure of (partially) closing punctures by trading (some of) the flavor fugacities with τ plays an important role, as it will allow us to construct the index for theories with arbitrary types of punctures. For $SU(k)$ theories the punctures are classified by the different embeddings of $SU(2)$ inside $SU(k)$ [3, 4], which are conveniently labelled by auxiliary Young diagrams with k boxes. For $SU(2)$ we get only two possibilities: (i) a row with two boxes corresponding to the "maximal" puncture with $SU(2)$ flavor symmetry, (ii) a column with two boxes corresponding to the absence of a puncture. For higher-rank theories the space of possibilities will be more interesting.

4.2.2 Higher rank: preliminaries

For higher-rank quivers the situation is more complicated since the basic building blocks are given by strongly-interacting SCFTs for which direct computations are not possible. However, the expressions that we obtained for the index of the $SU(2)$ quivers can be naturally extrapolated to higher rank. The basic conjecture is that the set of functions $\{f^{\alpha}(\mathbf{a})\}$ that diagonalize the structure constants are related to Hall-Littlewood polynomials for higher-rank as well.

The Hall-Littlewood (HL) polynomials associated to $U(k)$ are a set of orthogonal polynomials labeled by Young diagrams with at most k rows, $\lambda = (\lambda_1, \dots, \lambda_k)$, $\lambda_j \geq \lambda_{j+1}$. They are given by [93]

$$P^{\lambda}_{HL}(x_1, \dots, x_k | \, \tau) = \mathcal{N}_{\lambda}(\tau) \sum_{\sigma \in S_k} x^{\lambda_1}_{\sigma(1)} \dots x^{\lambda_k}_{\sigma(k)} \prod_{i<j} \frac{x_{\sigma(i)} - \tau^2 x_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}} \,, \tag{4.49}$$

and they are orthonormal under the measure

$$\Delta_{HL} = \frac{1}{k!} \prod_{i \neq j} \frac{1 - x_i/x_j}{1 - \tau^2 x_i/x_j} \,. \tag{4.50}$$

The normalization $\mathcal{N}_{\lambda}(t)$ is given by

$$\mathcal{N}^{-2}_{\lambda_1, \dots \lambda_k}(\tau) = \prod_{i=0}^{\infty} \prod_{j=1}^{m(i)} \left(\frac{1 - \tau^{2j}}{1 - \tau^2} \right) \,, \tag{4.51}$$

where $m(i)$ is the number of rows in the Young diagram $\lambda = (\lambda_1, \dots, \lambda_k)$ of length i. For $SU(k)$ groups we take Young diagrams with $\lambda_k = 0$ and the product of x_k in (4.49) is constrained as $\prod_{i=1}^{k} x_k = 1$.

Let us also quote from the outset the expression for the $SU(k)$ propagator,

$$\eta(\mathbf{a}, \mathbf{b}^{-1}) = \Delta(\mathbf{a}) \mathcal{I}^V(\mathbf{a}) \delta(\mathbf{a}, \mathbf{b}^{-1}) \,, \tag{4.52}$$

where $\Delta(\mathbf{a})$ is the $SU(k)$ Haar measure,

$$\Delta(\mathbf{a}) = \frac{1}{k!} \prod_{i \neq j} \left(1 - \frac{a_i}{a_j} \right) \,, \qquad \prod_{i}^{k} a_i = 1 \,, \tag{4.53}$$

and $\mathcal{I}^V(\mathbf{a})$ the vector multiplet index,

$$\mathcal{I}^V = \frac{1}{1-\tau^2} \prod_{j,i=1}^{k} (1-\tau^2 a_j/a_i)\,. \tag{4.54}$$

4.2.3 $SU(3)$ quivers – the E_6 SCFT

We now focus on the $SU(3)$ theories. There are two kinds of non-trivial punctures: the maximal puncture, associated to the Young diagram $(3,0,0)$, which carries the full $SU(3)$ flavor symmetry; the puncture associated with the Young diagram $(2,1,0)$, which carries $U(1)$ flavor symmetry. The elementary building blocks are the 333 vertex and the 331 vertex, where 3 and 1 are shorthands for the $SU(3)$ and $U(1)$ punctures, respectively.

The 333 vertex corresponds to the E_6 SCFT of [29], denoted by T_3 in [3]. A maximal subgroup of the E_6 flavor symmetry is given by $SU(3)^3$ and we parametrize the Cartans of the three $SU(3)$s by $\mathbf{a}_I$. Guided by the expression of the T_2 index obtained in the previous subsection, we conjecture that the index of T_3 is given by

$$\begin{aligned}
\mathcal{I}(\mathbf{a_1},\mathbf{a_2},\mathbf{a_3}) &= \sum_{\lambda_1,\lambda_2} \frac{\mathcal{A}(\tau)}{P_{HL}^{\lambda_1,\lambda_2}(\tau^2,\tau^{-2},1|\,\tau)} \prod_{I=1}^{3} \mathcal{K}(\mathbf{a}_I)\, P_{HL}^{\lambda_1,\lambda_2}(\mathbf{a}_I|\,\tau) && (4.55)\\
\mathcal{K}(\mathbf{a}) &= \frac{1}{1-\tau^2} \prod_{i,j=1,\, i\neq j}^{3} \frac{1}{(1-\tau^2 a_i/a_j)}\,, \qquad \prod_{i=1}^{3} a_i = 1 && (4.56)\\
\mathcal{A}(\tau) &= (1-\tau^4)(1+\tau^2+\tau^4)\,. && (4.57)
\end{aligned}$$

The function $\mathcal{K}(\mathbf{a})$ is fixed as always by (4.11), with $\hat{\Delta} = \Delta_{HL}$, while the overall fugacity-independent normalization factor $\mathcal{A}(\tau)$ was fixed by comparing with the known result for this index (3.41). We expanded the above expression in power series in τ and found a perfect match with (3.41).[7] In [88] an explicit expression was conjectured for the partition function over the Higgs branch of the E_6 SCFT, which we argued in section 4.1 to be equivalent to the Hall-Littlewood index. This expression has a very simple form [88],

$$\mathcal{I}(\mathbf{z}_{E_6}) = \sum_{k=0}^{\infty} [0,k,0,0,0,0]_{\mathbf{z}}\, \tau^{2k}\,, \tag{4.58}$$

where $\mathbf{z}$ is an E_6 fugacity and $[0,k,0,0,0,0]_{\mathbf{z}}$ are the characters of the irreducible representation of E_6 with Dynkin labels $[0,k,0,0,0,0]$. This expression is manifestly

[7] All the expressions for the HL index we obtain here are geometric progressions which in principle can be explicitly summed. However, for the purposes of this paper we often found it computationally more feasible and insightful to perform perturbative checks to high order in expansion in τ.

E_6 covariant while (4.55) is not: however, order by order in the τ-expansion we find that the fugacities of $SU(3)^3$ combine to label representations of E_6 and we obtain perfect agreement. We emphasize that for this to happen the overall factors $\mathcal{K}(\mathbf{a}_i)$ are absolutely crucial – without taking them into account the flavor-symmetry enhancement to E_6 does not occur.

We can define an *unrefined* index by setting all the flavor fugacities to one. In this case the series can be easily summed up in closed form and we obtain that the unrefined index is given by

$$\mathcal{I} = \frac{1+\tau^{20}+55(\tau^2+\tau^{18})+890(\tau^4+\tau^{16})+5886(\tau^6+\tau^{14})+17929(\tau^8+\tau^{12})+26060\tau^{10}}{(1+\tau^2)^{-1}(1-\tau^2)^{22}}\,, \tag{4.59}$$

in complete agreement with [88].

The 331 vertex corresponds to the SCFT of a free hypermultiplet in the bifundamental of $SU(3)$ and charged under $U(1)$. Its index is given by

$$\begin{aligned}\mathcal{I}(\mathbf{a_1},\mathbf{a_2},c) &= PE\left[\tau\chi_1(\mathbf{a})\chi_1(\mathbf{b})c\right]_{\mathbf{a},\mathbf{b},c} PE\left[\tau\chi_1(\mathbf{a}^{-1})\chi_1(\mathbf{b}^{-1})c^{-1}\right]_{\mathbf{a},\mathbf{b},c} \\ &= \prod_{i,j=1}^{3}\frac{1}{1-\tau a_i b_j c}\,\frac{1}{1-\tau\frac{1}{a_i b_j c}}, \qquad \prod_{i=1}^{3} a_i = \prod_{i=1}^{3} b_i = 1\,.\end{aligned} \tag{4.60}$$

It can be rewritten by partially closing a puncture of the E_6 vertex (4.55), as

$$\mathcal{I}(\mathbf{a_1},\mathbf{a_2},c) = \frac{1-\tau^6}{1-\tau^2}\frac{\mathcal{K}(\mathbf{a}_1)\mathcal{K}(\mathbf{a}_2)}{(1-\tau^3c^3)(1-\tau^3c^{-3})}\sum_{\lambda_1,\lambda_2}\frac{P_{HL}^{\lambda_1,\lambda_2}(\tau c,\tau^{-1}c,c^{-2}|\,\tau)}{P_{HL}^{\lambda_1,\lambda_2}(\tau^2,\tau^{-2},1|\,\tau)}\prod_{I=1}^{2}P_{HL}^{\lambda_1,\lambda_2}(\mathbf{a}_I|\,\tau)\,. \tag{4.61}$$

The sum over representations here is a geometric progression and can be easily performed establishing the equivalence of (4.60) and (4.61) (in the process we have fixed the overall τ-dependent factor).

We can use the above expressions to write the index of any $SU(3)$ quiver. Let us give again the example of the genus $\mathfrak{g}$ theory,

$$\begin{aligned}\mathcal{I}_{\mathfrak{g}}^{(3)} &= \left(1-\tau^4\right)^{\mathfrak{g}-1}\left(1-\tau^6\right)^{\mathfrak{g}-1} \\ &\quad+\frac{\left(1+2\left(1+\tau^{-2}\right)^{\mathfrak{g}-1}\left(\tau^{2-2\mathfrak{g}}-\tau^{2\mathfrak{g}-2}\right)\right)\tau^{4(\mathfrak{g}-1)}\left(1-\tau^2\right)^{2\mathfrak{g}-2}}{\left(\tau^{2-2\mathfrak{g}}-\tau^{2\mathfrak{g}-2}\right)^2}\,.\end{aligned} \tag{4.62}$$

We can subject (4.55) and (4.61) to a further non-trivial check. The channel-crossing duality of the four-punctured sphere with two $SU(3)$ and two $U(1)$ punctures corresponds to Argyres-Seiberg duality [28]. In one channel we glue together two 331 vertices along two 3 punctures, while in the other channel the 333 vertex (index of T_3) is (formally) glued to a 311 vertex. Requiring equality of the two

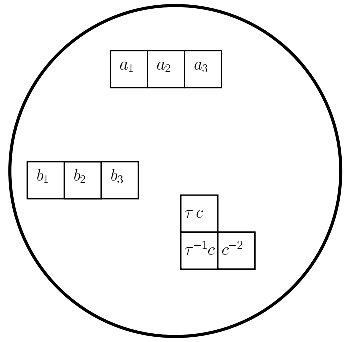


Figure 4.1: Association of flavor fugacities for the vertex corresponding to the 331 of the $SU(3)$ quivers. Here $a_1a_2a_3 = 1$ and $b_1b_2b_3 = 1$.

channels we find the index of the 311 vertex,

$$\mathcal{I}_{311}(\mathbf{a}, c, d) = \frac{1-\tau^6}{(1-\tau^2)(1-\tau^4)}\,\frac{\mathcal{K}(\mathbf{a})}{(1-\tau^3c^3)(1-\tau^3c^{-3})(1-\tau^3d^3)(1-\tau^3d^{-3})} \tag{4.63}$$
$$\sum_{\lambda_1,\lambda_2}\frac{P_{HL}^{\lambda_1,\lambda_2}(\tau c, \tau^{-1}c, c^{-2}|\,\tau)P_{HL}^{\lambda_1,\lambda_2}(\tau d, \tau^{-1}d, d^{-2}|\,\tau)P_{HL}^{\lambda_1,\lambda_2}(\mathbf{a}|\,\tau)}{P_{HL}^{\lambda_1,\lambda_2}(\tau^2, \tau^{-2}, 1|\,\tau)}\,.$$

In the expression above the sum over representations diverges. The 311 should be regarded as a formal construct that only makes sense as a part of the larger theory. It can be interpreted as implementing a δ-function constraint on the flavor indices. The non-singular way to view the gluing of 333 vertex with 311 vertex is as gauging an $SU(2)$ subgroup of E_6, as opposed to an $SU(3)$ subgroup [28]. With this interpretation of the 311 vertex, equality of the two channels amounts to

$$(1-\tau^2)\oint\frac{da}{4\pi i a}P_{HL}^{\lambda_1,\lambda_2}(ar, a^{-1}r, r^{-2}|\tau)\prod_{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5=\pm1}\frac{1}{1-\tau s^{\sigma_1}a^{\sigma_2}}\,\frac{1}{1-\tau^2r^{3\sigma_3}a^{\sigma_4}}(1-a^{2\sigma_5})$$
$$= \frac{1-\tau^6}{1-\tau^4}\,\frac{\prod_{\sigma=\pm1}P_{HL}^{\lambda_1,\lambda_2}(\tau\frac{s^{\sigma/3}}{r}, \tau^{-1}\frac{s^{\sigma/3}}{r}, \frac{s^{-2\sigma/3}}{r^{-2}}|\tau)}{P_{HL}^{\lambda_1,\lambda_2}(\tau^2, \tau^{-2}, 1|\tau)\prod_{\sigma_1,\sigma_2=\pm1}(1-\tau^3s^{\sigma_1}/r^3)(1-\tau^3s^{\sigma_2}r^3)}\,. \tag{4.64}$$

In the first line we gauge an $SU(2)$ subgroup of E_6 and couple it to a single hypermultiplet, and in the second line a 311 vertex is glued to 333 vertex by gauging an $SU(3)$ flavor group. This is a non-trivial identity involving HL polynomials which we have checked to very high order in a perturbative expansion in τ.

4.2.4 A conjecture for the structure constants with generic punctures

Extrapolating from the $SU(2)$ and $SU(3)$ cases, we are now formulate a complete conjecture for the index of all building blocks of $SU(k)$ quivers. The building blocks are classified by a triple of Young diagrams $(\Lambda_1, \Lambda_2, \Lambda_3)$. We conjecture

$$\mathcal{I}_{\Lambda_1,\Lambda_2,\Lambda_3} = \frac{\prod_{j=2}^{k}(1-\tau^{2j})}{(1-\tau^2)^{-k-2}} \prod_{I=1}^{3} \hat{\mathcal{K}}_{\Lambda_I}(\mathbf{a}_I) \sum_{\lambda} \frac{\prod_{I=1}^{3} P_{HL}^{\lambda}(\mathbf{a_I}(\Lambda_I)|\tau)}{P_{HL}^{\lambda}(\tau^{k-1},\tau^{k-3},\ldots,\tau^{1-k}|\tau)}\,. \quad (4.65)$$

Here the assignment of fugacities according to the Young diagram labelling the type of the puncture, $\mathbf{a}(\Lambda)$, is as illustrated in figure 4.2. The summation over λ is over the Young diagrams with $k-1$ rows, $\lambda = (\lambda_1, \ldots, \lambda_{k-1})$, $\lambda_j \geq \lambda_{j+1}$. The factors

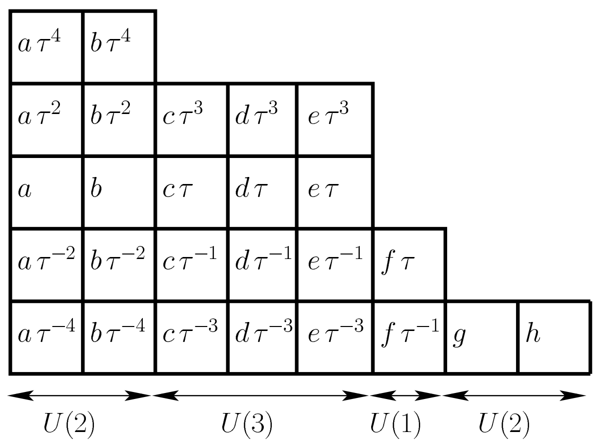


Figure 4.2: Association of the flavor fugacities for a generic puncture. Punctures are classified by embeddings of $SU(2)$ in $SU(k)$, so they are specified by the decomposition of the fundamental representation of $SU(k)$ into irreps of $SU(2)$, that is, by a partition of k. Graphically we represent the partition by an auxiliary Young diagram Λ with k boxes, read from left to right. In the figure we have the fundamental of $SU(26)$ decomposed as $\mathbf{5}+\mathbf{5}+\mathbf{4}+\mathbf{4}+\mathbf{4}+\mathbf{2}+\mathbf{1}+\mathbf{1}$. The commutant of the embedding gives the residual flavor symmetry, in this case $S(U(3)\times U(2)\times U(2)\times U(1))$, where the $S(\ldots)$ constraint amounts to removing the overall $U(1)$. The τ variable is viewed here as an $SU(2)$ fugacity, while the Latin variables are fugacities of the residual flavor symmetry. The $S(\ldots)$ constraint implies that the flavor fugacities satisfy $(ab)^5(cde)^4f^2gh = 1$.

$\hat{\mathcal{K}}_{\Lambda}(\mathbf{a})$ are defined as

$$\hat{\mathcal{K}}_{\Lambda}(\mathbf{a}) = \prod_{i=1}^{row(\Lambda)} \prod_{j,k=1}^{l_i} \frac{1}{1-\mathfrak{a}_j^i \bar{\mathfrak{a}}_k^i}\,. \quad (4.66)$$

Here $row(\Lambda)$ is the number of rows in Λ and l_i is the length of ith row. The coefficients $\mathfrak{a}_k^i$ are associated to the Young diagram as illustrated in figure 4.3. Our conjecture is consistent with the $SU(2)$ and $SU(3)$ cases seen previously as well as with all other examples discussed below.

For three maximal punctures (the T_k theory), (4.65) becomes

$$\mathcal{I}_{T_k}(\mathbf{a_1},\mathbf{a_2},\mathbf{a_3}) = \sum_{\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_{k-1}} \frac{\mathcal{A}(\tau)}{P_{HL}^{\lambda_1,..,\lambda_{k-1}}(\tau^{k-1},..,\tau^{1-k}|\,\tau)} \prod_{I=1}^{3} \mathcal{K}(\mathbf{a}_I)\, P_{HL}^{\lambda_1,..\lambda_{k-1}}(\mathbf{a}_I|\,\tau)\,,$$

$$\mathcal{K}(\mathbf{a}) = \frac{1}{(1-\tau^2)^{\frac{k-1}{2}}} \prod_{i,j=1,\, i\neq j}^{k} \frac{1}{(1-\tau^2 a_i/a_j)}\,, \qquad \prod_{i=1}^{k} a_i = 1\,,$$

$$\mathcal{A}(\tau) = \frac{\prod_{j=2}^{k}(1-\tau^{2j})}{(1-\tau^2)^{\frac{k-1}{2}}}\,.$$

Let us illustrate the power of these TQFT expressions by computing the index of the genus $\mathfrak{g}$ $SU(k)$ theory. It is given by

$$\mathcal{I}_{\mathfrak{g}}^{(k)} = \frac{\left(\prod_{j=2}^{k}(1-\tau^{2j})\right)^{2\mathfrak{g}-2}}{(1-\tau^2)^{(k-1)(\mathfrak{g}-1)}} \sum_{\lambda} \frac{1}{P_{HL}^{\lambda}(\tau^{k-1},\tau^{k-3},\ldots,\tau^{1-k}|\tau)^{2\mathfrak{g}-2}}\,, \tag{4.67}$$

where the summation is over all Young diagrams with $k-1$ rows, *i.e.* over the finite irreducible representations of $SU(k)$.

The sum over representations in (4.65) does not converge for arbitrary choices of the three Young diagrams Λ_I. We have already encountered an example in the last subsection: the 311 vertex of $SU(3)$ theories has a divergent expression. There is no actual SCFT corresponding to the 311 vertex, but one can glue this vertex to a larger quiver and obtain meaningful results. There are cases however where the divergent vertex cannot appear as a piece of a larger quiver and thus the expression (4.65) for its index does not have a clear physical interpretation. An example of such a vertex is the index of an $SU(6)$ theory with three $SU(3)$ punctures. We have checked in several cases that a divergence in (4.65) correlates with the fact that the graded rank of the Coulomb branch (as defined in [94]) of the putative SCFT has negative components. This is an indication that associating field theories to such punctured surfaces may be delicate. Punctured surfaces of this type were recently considered in [95] and subtleties associated with them addressed in [96].

4.2.5 $SU(4)$ quivers – T_4 and the E_7 SCFT

Let us use the general expressions of the previous section to discuss some of the features of $SU(4)$ quivers. First, from (4.67) we can compute the unrefined index

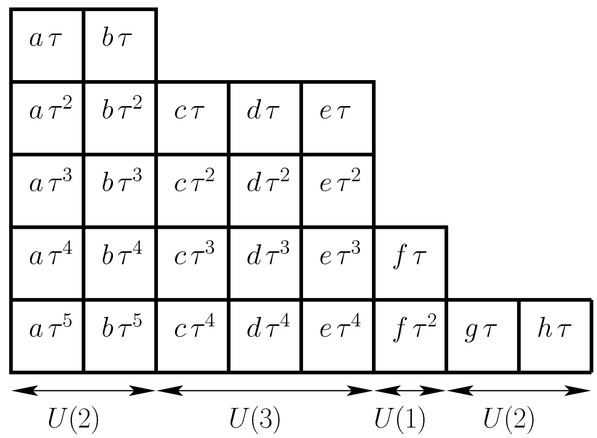


Figure 4.3: The factors $\mathfrak{a}^i_k$ associated to a generic Young diagram. The upper index is the row index and the lower is the column index. In $\bar{\mathfrak{a}}^i_k$ one takes the inverse of flavor fugacities while τ is treated as real number. As before, the flavor fugacities in this example satisfy $(ab)^5(cde)^4f^2gh = 1$.

of T_4,

$$\begin{aligned}\mathcal{I}_{T_4} &= 1 + 45\tau^2 + 128\tau^3 + 1249\tau^4 + 5504\tau^5 + 30786\tau^6 \\ &\quad +136832\tau^7 + 623991\tau^8 + \dots .\end{aligned} \tag{4.68}$$

We present a closed form expression for it in appendix B.4. Refining with the flavor fugacities one gets

$$\begin{aligned}\mathcal{I}_{T_4} =&1 + [(\mathbf{15},1,1) + (1,\mathbf{15},1) + (1,1,\mathbf{15})]\,\tau^2 + [(\mathbf{4},\mathbf{4},\mathbf{4}) + (\bar{\mathbf{4}},\bar{\mathbf{4}},\bar{\mathbf{4}})]\,\tau^3 + \\ &[1 + (\mathbf{15},1,1) + (1,\mathbf{15},1) + (1,1,\mathbf{15}) + (\mathbf{20},1,1) + (1,\mathbf{20},1) + (1,1,\mathbf{20}) + \\ &+(\mathbf{15},\mathbf{15},1) + (1,\mathbf{15},\mathbf{15}) + (\mathbf{15},1,\mathbf{15}) + (\mathbf{84},1,1) + (1,\mathbf{84},1) + (1,1,\mathbf{84}) + \\ &+(\mathbf{6},\mathbf{6},\mathbf{6})]\,\tau^4 + \dots .\end{aligned} \tag{4.69}$$

In terms of Young diagrams $\mathbf{84} = (4,2,2)$, $\mathbf{6} = (1,1,0)$, $\mathbf{20} = (2,2,0)$. The symmetric product of the τ^2 term reproduces all the terms at the τ^4 order except for the $(\mathbf{6},\mathbf{6},\mathbf{6})$ term, and for the fact that two singlets are missing (the symmetric product contains three singlets while only one is present at order τ^4). We deduce that the $(\mathbf{6},\mathbf{6},\mathbf{6})$ state is an additional generator of the Higgs branch, and that there is a constraint allowing only for one singlet in the symmetric product of the τ^2 states to appear at τ^4 order. Unlike the situation for the E_6 SCFT where the Higgs branch is generated by a single scalar transforming as $\mathbf{78}$ of E_6 [80] here one has new generators appearing at higher orders in the τ expansion and thus having different $E - R$ quantum numbers.

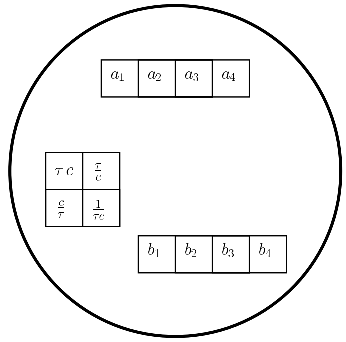


Figure 4.4: Association of the flavor fugacities for the E_7 vertex. Here $\prod_{i=1}^4 b_i = \prod_{i=1}^4 a_i = 1$.

Next, we can partially close a puncture to obtain the index of the 441 vertex. On one hand, the 441 vertex correspond to the free hypermultiplet SCFT in the bifundamental of two $SU(4)$s and charged under the $U(1)$, so its index can be evaluated by direct counting,

$$\mathcal{I}(\mathbf{a_1}, \mathbf{a_2}, c) = \prod_{i,j}^{4} \frac{1}{1-\tau a_i b_j c} \frac{1}{1-\tau \frac{1}{a_i b_j c}}, \qquad \prod_{i=1}^{4} a_i = \prod_{i=1}^{4} b_i = 1\,. \tag{4.70}$$

On the other hand, from (4.65),

$$\begin{aligned}\mathcal{I}(\mathbf{a_1}, \mathbf{a_2}, c) = \frac{1-\tau^8}{(1-\tau^2)} \frac{\mathcal{K}(\mathbf{a}_1)\mathcal{K}(\mathbf{a}_2)}{(1-\tau^4 c^4)(1-\tau^4 c^{-4})} \\ \sum_{\lambda_1,\lambda_2,\lambda_3} \frac{P_{HL}^{\lambda_1,\lambda_2,\lambda_3}(\tau^2 c, c, \tau^{-2} c, c^{-3}|\, \tau)}{P_{HL}^{\lambda_1,\lambda_2,\lambda_3}(\tau^3, \tau, \tau^{-1}, \tau^{-3}|\, \tau)} \prod_{i=1}^{2} P_{HL}^{\lambda_1,\lambda_2,\lambda_3}(\mathbf{a_i}|\, \tau)\,.\end{aligned} \tag{4.71}$$

We have checked the equivalence of these two expressions perturbatively to very high order in τ. Finally, let us look at the vertex with two maximal punctures and one puncture corresponding to a square Young diagram, which carries an $SU(2)$ flavor symmetry, see figure 4.4. The flavor symmetry of this theory is known to enhance to E_7 [28]. From (4.65), the Hall-Littlewood index of this SCFT is given by

$$\begin{aligned}\mathcal{I}_{E_7}(\mathbf{a_1}, \mathbf{a_2}, c) =& \frac{(1+\tau^2+\tau^4)(1+\tau^4)}{(1-\tau^2)} \frac{\mathcal{K}(\mathbf{a}_1)\mathcal{K}(\mathbf{a}_2)}{(1-\tau^2 c^{\pm 2})(1-\tau^4 c^{\pm 2})} \\ &\times \sum_{\lambda_1,\lambda_2,\lambda_3} \frac{P_{HL}^{\lambda_1,\lambda_2,\lambda_3}(\tau c, \frac{c}{\tau}, \frac{\tau}{c}, \frac{1}{\tau c}|\, \tau)}{P_{HL}^{\lambda_1,\lambda_2,\lambda_3}(\tau^3, \tau, \tau^{-1}, \tau^{-3}|\, \tau)} \prod_{i=1}^{2} P_{HL}^{\lambda_1,\lambda_2,\lambda_3}(\mathbf{a_i}|\, \tau)\,.\end{aligned} \tag{4.72}$$

In [88] an explicit expression for the Higgs partition function was conjectured,

$$\mathcal{I}(\mathbf{z}_{E_7}) = \sum_{k=0}^{\infty} [k,0,0,0,0,0,0]_{\mathbf{z}}\, \tau^{2k}\,, \tag{4.73}$$

where $\mathbf{z}$ is an E_7 fugacity and $[k,0,0,0,0,0,0]_{\mathbf{z}}$ are the characters of the irreducible representation of E_7 with Dynkin labels $[k,0,0,0,0,0,0]$. We have checked also here (4.72) is in complete agreement with [88], and thus in particular is secretly E_7 covariant: the check can be done analytically for the unrefined index and perturbatively in τ to high order for the refined one.

The expression (4.72) can be also checked by the Argyres-Seiberg duality between $USp(4)$ theory coupled to six fundamental hypermultiplets and E_7 theory with an $SU(2)$ subgroup gauged [28]. The former has a weakly-coupled description and its index can be computed directly,

$$\mathcal{I}_{USp(4)} = 1 + \chi^{\mathbf{66}}_{SO(12)}(u,v,w,x,y,z)\tau^2 + \cdots\,. \tag{4.74}$$

Since there are six fundamental hypermultiplets the flavor group is $SO(12)$. On the other hand, gauging an $SU(2)$ inside one $SU(4)$ subgroup of the E_7 index (4.72) gives

$$\mathcal{I} = \oint \frac{de}{4\pi i e}\,(1-e^2)\,(1-e^{-2})\,PE\left[-\tau^2\chi_2(e)\right]_{t,e}\mathcal{I}_{E_7}(\mathbf{a},\{es, s/e, b/s, 1/bs\}, c)\,. \tag{4.75}$$

We have checked perturbatively in τ that (4.74) and (4.75) coincide under the following identification of the fugacities:

$$u \to \frac{a_1}{s},\quad v \to \frac{a_2}{s},\quad w \to \frac{a_3}{s},\quad x \to \frac{1}{a_1 a_2 a_3 s},\quad y \to bc,\quad z \to \frac{b}{c}\,. \tag{4.76}$$

4.2.6 $SU(6)$ quivers – the E_8 SCFT

As our last example, we consider the index of the E_8 SCFT [92]. This theory corresponds to a sphere with a maximal $SU(6)$ puncture and two non-maximal punctures with $SU(3)$ and $SU(2)$ flavor symmetries, see figure 4.5. The group $SU(6) \times SU(3) \times SU(2)$ is a maximal subgroup of E_8. Following the general pre-

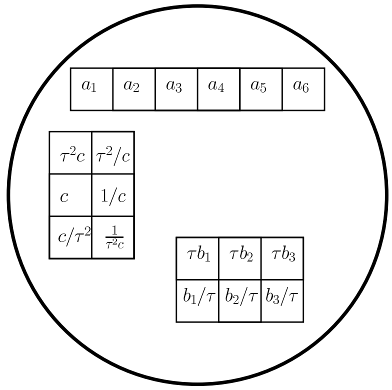


Figure 4.5: Association of the flavor fugacities for the E_8 vertex. Here $\prod_{i=1}^{3} b_i = \prod_{i=1}^{6} a_i = 1$.

scription (4.65) the index of E_8 SCFT is given by

$$\mathcal{I}_{E_8}(\mathbf{a}, b_1, b_2, c) = \frac{(1-\tau^8)(1-\tau^{10})(1-\tau^{12})}{(1-\tau^2)^{1/2}(1-\tau^4)^4(1-\tau^6)} \times \tag{4.77}$$
$$\frac{\mathcal{K}(\mathbf{a})}{(1-\tau^2 c^{\pm 2})(1-\tau^4 c^{\pm 2})(1-\tau^6 c^{\pm 2})\prod_{i\neq j}(1-\tau^2 b_i/b_j)(1-\tau^4 b_i/b_j)} \times$$
$$\sum_{\lambda_1,\dots,\lambda_5\equiv\lambda} \frac{P_{HL}^{\lambda}(\tau b_1, \tau b_2, \tau b_3, \frac{b_1}{t}, \frac{b_2}{\tau}, \frac{b_3}{\tau}|\,\tau) P_{HL}^{\lambda}(\tau^2 c, c, \frac{c}{\tau^2}, \frac{\tau^2}{c}, \frac{1}{c}, \frac{1}{\tau^2 c}, |\,\tau) P_{HL}^{\lambda}(\mathbf{a_i}|\,\tau)}{P_{HL}^{\lambda}(\tau^5, \tau^3, \tau, \tau^{-1}, \tau^{-3}, \tau^{-5}|\,\tau)}\,.$$

In [88] it was conjectured that the Higgs partition function has the following E_8 covariant expansion,

$$\mathcal{I}(\mathbf{z}_{E_8}) = \sum_{k=0}^{\infty} [k,0,0,0,0,0,0,0]_{\mathbf{z}}\, \tau^{2k}\,, \tag{4.78}$$

where $\mathbf{z}$ is an E_8 fugacity and $[k,0,0,0,0,0,0,0]_{\mathbf{z}}$ are the characters of the irreducible representation of E_8 with Dynkin labels $[k,0,0,0,0,0,0,0]$. We have again checked equivalence of (4.77) and (4.78) in the τ-expansion, though in this case due to computational complexity we could perform the expansion only up to order τ^8. The size of representations of E_8 contributing to the index grows very fast with the order of τ, *e.g.* the unrefined index is given by

$$\mathcal{I}_{E_8} = 1 + 245\tau^2 + 26255\tau^4 + 1681887\tau^6 + 73829103\tau^8 + \dots\,. \tag{4.79}$$

4.2.7 Large k limit

It is not difficult to evaluate the large k limit of the HL index of $SU(k)$ generalized quivers.[8] For instance, for the index of the theory corresponding to a genus $\mathfrak{g}$ surface without punctures (4.67),

$$\mathcal{I}_{\mathfrak{g}}^{(k\to\infty)} = \prod_{j=2}^{\infty}(1-\tau^{2j})^{\mathfrak{g}-1} = PE\left[-(\mathfrak{g}-1)\frac{\tau^4}{1-\tau^2}\right] . \tag{4.80}$$

In appendix B.3 we give a short derivation of this expression. In the large k limit only the singlet in the sum over the representations of (4.67) contributes. Since (4.80) is of order one for large k it is expected to be matched by counting the appropriate supergravity modes in the dual AdS background [70]. We can also compute the index of the T_k theories in the large k limit,

$$\begin{aligned}\mathcal{I}_{T_{k\to\infty}}(\mathbf{a_1},\mathbf{a_2},\mathbf{a_3}) &= \prod_{j=2}^{\infty}\frac{1}{1-\tau^{2j}}\prod_{I=1}^{3}\prod_{j\neq i}^{\infty}\frac{1}{1-\tau^2 a_i^I/a_j^I} \\ &= PE\left[\frac{\tau^4}{1-\tau^2}\right]\prod_{I=1}^{3}PE\left[\tau^2\sum_{i\neq j}a_i^I/a_j^I\right] .\end{aligned} \tag{4.81}$$

From here the large k index of any generalized quiver is trivial to compute; in particular (4.80) can be obtained by gluing together the index of (4.81). Unrefining the index of T_k by setting all the flavor fugacities $a_j^I = 1$ we see that it has a non-trivial k dependence for large k limit. Taking the plethystic log of (4.81) (that is, considering the index of single-particle states) we find

$$\mathcal{I}_{T_{k\to\infty}}^{s.p.} = 3\,\tau^2\,(k^2-k) + \frac{\tau^4}{1-\tau^2} + O\left(\frac{1}{k}\right) . \tag{4.82}$$

The term of order k^2 on the right-hand-side comes from states in the adjoint representation of the flavor group, while the term of order k comes from neutral states. At least some of $O(k^2)$ states in the adjoint representation must correspond to modes of the AdS gauge fields that couple to the flavor currents of the boundary theory. It would be interesting to check whether all the $O(k^2)$ and $O(k)$ states can be accounted for by supergravity states. If not, the extra states could arise as non-perturbative states in the bulk geometry (*e.g.* wrapped branes or black holes). In all cases studied so far the index is of order one in the large k limit and thus cannot capture the non-perturbative states of the bulk theory [2, 25, 32, 97]. This is not a contradiction, since the index only counts protected states with signs. The index vanishes on combinations of short multiplets that can in principle recombine

[8] We thank Davide Gaiotto and Juan Maldacena for discussions on issues related to this section.

into long ones, even when such kinematically-allowed recombination do not actually happen [2]. However, for linear quivers (in particular for the T_k theories) the HL index has the meaning of a Hilbert series over the Higgs branch, so it is expected to capture *all* the relevant $\frac{3}{8}$-BPS states of the dual theory. We leave the very interesting comparison with the bulk theory for future research.

4.3 Schur index

We turn to the Schur index,

$$\mathcal{I}_S = \mathrm{Tr}(-1)^F\, q^{E-R}\,, \tag{4.83}$$

which is the same as the *reduced index* considered in [17]. Let us first recall the expression for the $SU(k)$ propagator. It is of the usual form

$$\eta(\mathbf{a}, \mathbf{b}^{-1}) = \Delta(\mathbf{a})\mathcal{I}^V(\mathbf{a})\delta(\mathbf{a}, \mathbf{b}^{-1})\,, \tag{4.84}$$

where $\Delta(\mathbf{a})$ is the Haar measure (4.53), and $\mathcal{I}^V(\mathbf{a})$ the index of the vector multiplet, given by

$$\mathcal{I}^V_q(\mathbf{a}) = PE\left[\frac{-2q}{1-q}\chi_{adj}(\mathbf{a})\right]_{q,\mathbf{a}}\,. \tag{4.85}$$

The set of functions $\{f^\lambda_q(\mathbf{a})\}$ that diagonalize the structure constants are proportional to the Schur polynomials [17],

$$f^\lambda_q(\mathbf{a}) = \mathcal{K}_q(\mathbf{a})\;\chi^\lambda(\mathbf{a})\,. \tag{4.86}$$

The Schur polynomials are orthonormal under the Haar measure, so in this case $\hat{\Delta}(\mathbf{a}) = \Delta(\mathbf{a})$ (recall (4.10)) and the factor $\mathcal{K}_q(\mathbf{a})$ is given by

$$\mathcal{K}_q(\mathbf{a}) = \frac{1}{[\mathcal{I}^V_q(\mathbf{a})]^{\frac{1}{2}}}\,. \tag{4.87}$$

Generalizing our results in [17], we conjecture the following expression for the Schur index of a three-punctured sphere with generic punctures,

$$\mathcal{I}_{\Lambda_1,\Lambda_2,\Lambda_3} = \frac{(q;q)^{k+2}}{\prod_{j=1}^{k-1}(1-q^j)^{k-j}}\prod_{I=1}^{3}\hat{\mathcal{K}}_{\Lambda_I}(\mathbf{a}_I)\sum_\lambda\frac{\prod_{I=1}^3\chi^\lambda(\mathbf{a_I}(\Lambda_I))}{\chi^\lambda(q^{\frac{k-1}{2}},q^{\frac{k-3}{2}},\ldots,q^{\frac{1-k}{2}})}\,. \tag{4.88}$$

Here the sum is over the finite-dimensional irreducible representations of $SU(k)$. The assignment of fugacities according to the Young diagram, $\mathbf{a}(\Lambda)$, is again as in figure 4.2, with $\tau \to q^{1/2}$. The Pochhammer symbol $(a;\ b)$ is defined by

$$(a;\ b) = \prod_{i=0}^{\infty} (1 - a\, b^i)\,. \tag{4.89}$$

The character of the representation corresponding to Young diagram $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, 0)$ is given by a Schur polynomial,

$$\chi^\lambda(\mathbf{a}) = \frac{\det(a_i^{\lambda_j + k - j})}{\det(a_i^{k-j})}\,. \tag{4.90}$$

The $\hat{\mathcal{K}}_\Lambda$ prefactors are given by

$$\hat{\mathcal{K}}_\Lambda(\mathbf{a}) = \prod_{i=1}^{row(\Lambda)} \prod_{j,k=1}^{l_i} PE\left[\frac{\mathfrak{a}_j^i \bar{\mathfrak{a}}_k^i}{1-q}\right]_{\mathfrak{a}_i, q}\,, \tag{4.91}$$

where $row(\Lambda)$ is the number of rows in Λ and l_i is the length of ith row. The coefficients $\mathfrak{a}_k^i$ are associated to the Young diagram again as in figure 4.3, with $\tau \to q^{1/2}$. Note that the quantity appearing in the denominator of (4.88) is the quantum dimension of the representation λ of $SU(k)$,

$$\dim_q \lambda = \chi^\lambda(q^{\frac{k-1}{2}}, q^{\frac{k-3}{2}}, \ldots, q^{\frac{1-k}{2}})\,. \tag{4.92}$$

For $SU(2)$ the quantum dimension is also known as the q-number $[\lambda]_q$.

We have subjected (4.88) to similar checks as the one described for the Hall-Littlewood index, finding complete agreement with expectations; a few such checks were reported in [17]. Let us only mention here the basic identity following from compatibility of (4.88) with the index of the $SU(2)$ trifundamental hypermultiplet,

$$\begin{aligned} &PE\left[\frac{q^{1/2}}{1-q}\left(a_1 + \frac{1}{a_1}\right)\left(a_2 + \frac{1}{a_2}\right)\left(a_3 + \frac{1}{a_3}\right)\right]_{a_i,\tau} = \\ &\quad (q;q)^3 (q^2;q) \prod_{i=1}^{3} PE\left[\frac{q}{1-q}\left(a_i^2 + a_i^{-2} + 2\right)\right]_{a_i,\tau} \sum_{\lambda=0}^{\infty} \frac{\prod_{i=1}^{3} \chi^\lambda(a_i, a_i^{-1})}{\chi^\lambda(q^{\frac{1}{2}}, q^{-\frac{1}{2}})}\,. \end{aligned} \tag{4.93}$$

A proof of this identity is outlined in appendix B.5.

4.4 Macdonald index

We are now ready to combine and generalize the results of the two previous sections. The Hall-Littlewood and Schur polynomials are special cases of a two-parameter family of polynomials discovered by Macdonald [93]. One naturally expects Macdonald polynomials to be relevant for the calculation of the index in a two-dimensional slice of the full three-dimensional fugacity space. The precise confirmation of this idea is our main result. Identifying the correct slice is by no means obvious, but at this point it will come as no great surprise that it is given by the limit that we have called the Macdonald index in section 4.1,

$$\mathcal{I}_M = \mathrm{Tr}_M(-1)^F \, q^{E-2R-r} \, t^{R+r} = \mathrm{Tr}_M(-1)^F \, q^{-2j_1} \, t^{R+r} \,, \tag{4.94}$$

where Tr_M denotes the trace restricted to states with $\delta_{1+} = E + 2j_1 - 2R - r = 0$. For $q = t$ Macdonald polynomials reduce to Schur polynomials, while for $q = 0$ they reduce to Hall-Littlewood polynomials. By design, the Macdonald trace formula (4.94) reproduces respectively the Schur and Hall-Littlewood trace formulae in the same limits.

Our basic ansatz is that the complete set of functions $\{f^{\lambda}_{q,t}(\mathbf{a})\}$ that diagonalize the structure constants are proportional to Macdonald polynomials with parameters q and t,

$$f^{\lambda}_{q,t}(\mathbf{a}) = \mathcal{K}_{q,t}(\mathbf{a}) \, P^{\lambda}(\mathbf{a}|q,t) \,. \tag{4.95}$$

The Macdonald polynomials [93][9] $\{P^{\lambda}(\mathbf{a})\}$ are defined as the set of polynomials labeled by Young diagrams λ, orthonormal under the measure

$$\Delta_{q,t}(\mathbf{a}) = \frac{1}{k!} \, PE\left[-\frac{1-t}{1-q} \, (\chi_{adj}(\mathbf{a}) - k + 1)\right]_{q,t,\mathbf{a}} = \frac{1}{k!} \prod_{n=0}^{\infty} \prod_{i \neq j} \frac{1 - q^n a_i/a_j}{1 - t\, q^n a_i/a_j} \,, \tag{4.96}$$

and having the expansion

$$P^{\lambda} = \mathcal{N}_{\lambda}(q,t) \left\{ m_{\lambda} + \sum_{\mu < \lambda} h_{\lambda\mu}(q,t) \, m_{\mu} \right\} \,. \tag{4.97}$$

Here we define

$$m_{\lambda=(\lambda_1,..,\lambda_k)}(\mathbf{a}) = \sum_{\sigma \in S'_k} \prod_{i=1}^{k} a_i^{\sigma(\lambda_i)} \,, \tag{4.98}$$

[9] Macdonald polynomials appear in physics in many different contexts. Some recent papers on subjects related to $\mathcal{N} = 2$ gauge theories that discuss Macdonald polynomials are [98–100].

where S'_k denotes the set of distinct permutations of $(\lambda_1, ..., \lambda_k)$.

The factor $\mathcal{K}_{q,t}(\mathbf{a})$ is again fixed by requiring orthonormality of $\{f^{\lambda}_{q,t}(\mathbf{a})\}$ under the propagator measure. The propagator takes the standard form

$$\eta(\mathbf{a}, \mathbf{b}^{-1}) = \Delta(\mathbf{a})\mathcal{I}^V(\mathbf{a})\delta(\mathbf{a}, \mathbf{b}^{-1}) \,, \tag{4.99}$$

where as always $\Delta(\mathbf{a})$ is the Haar measure (4.53), while the index of the vector multiplet is in this case given by

$$\mathcal{I}^V_{q,t}(\mathbf{a}) = PE\left[\frac{-q-t}{1-q}\chi_{adj}(\mathbf{a})\right]_{q,\mathbf{a}} \,. \tag{4.100}$$

We then have

$$\mathcal{K}_{q,t}(\mathbf{a}) \quad = \quad \left(\frac{\Delta_{q,t}(\mathbf{a})}{\Delta(\mathbf{a})\,\mathcal{I}^V_{q,t}(\mathbf{a})}\right)^{\frac{1}{2}} \,. \tag{4.101}$$

We can finally state our main conjecture. The Macdonald index of the $SU(k)$ quiver theory associated to a sphere with three punctures of generic type is

$$\boxed{\mathcal{I}_{\Lambda_1,\Lambda_2,\Lambda_3} = (t;q)^{k+2}\prod_{j=2}^{k}\frac{(t^j;q)}{(q;q)}\prod_{I=1}^{3}\hat{\mathcal{K}}_{\Lambda_I}(\mathbf{a}_I)\sum_{\lambda}\frac{\prod_{I=1}^{3}P^{\lambda}(\mathbf{a}_I(\Lambda_I)|q,t)}{P^{\lambda}(t^{\frac{k-1}{2}},t^{\frac{k-3}{2}},\ldots,t^{\frac{1-k}{2}}|q,t)}} \,. \tag{4.102}$$

The assignment of fugacities according to the Young diagram $\mathbf{a_i}(\Lambda_i)$ is again as in figure 4.2, with $\tau \to t^{1/2}$. The $\hat{\mathcal{K}}$ prefactors are

$$\hat{\mathcal{K}}_{\Lambda}(\mathbf{a}) = \prod_{i=1}^{row(\Lambda)}\prod_{j,k=1}^{l_i} PE\left[\frac{\mathfrak{a}^i_j\bar{\mathfrak{a}}^i_k}{1-q}\right]_{\mathfrak{a}_i,q} \,, \tag{4.103}$$

with the coefficients $\mathfrak{a}^i_k$ associated to the Young diagram again as in figure 4.3, with $\tau \to t^{1/2}$. It is immediate to check that (4.102) reduces to the HL and Schur expressions in the respective limits. For three maximal punctures (4.102) becomes,

$$\begin{aligned}\mathcal{I}_{T_k}(\mathbf{a_1},\mathbf{a_2},\mathbf{a_3}) = &\sum_{\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_{k-1}}\frac{\mathcal{A}(q,t)}{P^{\lambda_1,..,\lambda_{k-1}}(t^{\frac{k-1}{2}},..,t^{\frac{1-k}{2}}\,|\;q,\;t)}\prod_{I=1}^{3}\mathcal{K}_{q,t}(\mathbf{a}_I)\,P^{\lambda_1,..\lambda_{k-1}}(\mathbf{a}_I|\;q,\;t)\,,\\ \mathcal{A}(q,t) = &PE\left[\frac{1}{2}(k-1)\frac{t-q}{1-q}\right]\prod_{j=2}^{k}(t^j;q)\,.\end{aligned} \tag{4.104}$$

For $k = 2$, this expression must agree with the index of the hypermultiplet in the trifundamental representation of $SU(2)$,

$$PE\left[\frac{t^{1/2}}{1-q}\left(a_1+\frac{1}{a_1}\right)\left(a_2+\frac{1}{a_2}\right)\left(a_3+\frac{1}{a_3}\right)\right]_{a_i,q,t} \tag{4.105}$$
$$=\frac{(t;q)^4(t^2;q)}{(q;q)}\prod_{i=1}^{3}PE\left[\frac{t}{1-q}\left(a_i^2+a_i^{-2}+2\right)\right]_{a_i,q,t}\sum_{\lambda=0}^{\infty}\frac{\prod_{i=1}^{3}P^{\lambda}(a_i,a_i^{-1}|q,t)}{P^{\lambda}(t^{\frac{1}{2}},t^{-\frac{1}{2}}|q,t)}\,.$$

We have verified this identity in the t and q expansions. It helps that for $SU(2)$ one can write an explicit form for the Macdonald polynomials,

$$P^{\lambda}(a,a^{-1}|q,t)=\mathcal{N}_{\lambda}(q,t)\sum_{i=0}^{\lambda}\prod_{j=0}^{i-1}\frac{1-t\,q^j}{1-q^{j+1}}\prod_{j=0}^{\lambda-i-1}\frac{1-t\,q^j}{1-q^{j+1}}\,a^{2i-\lambda}\,, \tag{4.106}$$

where $\mathcal{N}_{\lambda}(q,t)$ is a normalization constant rendering the Macdonald polynomials orthonormal under the measure (4.96). More generally, equating the index for the $(nn1)$ vertex from (4.102) with the index of a hypermultiplet in the bifundamental representation of $SU(k)$ and charged under $U(1)$, we obtain the identity

$$\begin{aligned}&PE\left[\frac{t^{1/2}}{1-q}\left(c\sum_{i,j=1}^{k}a_ib_j+\frac{1}{c}\sum_{i,j=1}^{k}a_i^{-1}b_j^{-1}\right)\right]_{a,b,c,q,t}=\frac{(t;q)^k}{(q;q)^{k-1}}(t^k;q)\\&\quad\times PE\left[\frac{t}{1-q}\sum_{i,j=1}^{k}a_i\,a_j^{-1}\right]_{a,q,t}PE\left[\frac{t}{1-q}\sum_{i,j=1}^{k}b_i\,b_j^{-1}\right]_{b,q,t}PE\left[\frac{t^{\frac{k}{2}}}{1-q}(c^k+c^{-k})\right]_{c,q,t}\\&\quad\times\sum_{\lambda}\frac{P^{\lambda}(c\,t^{\frac{k-2}{2}},c\,t^{\frac{k-4}{2}},\ldots,c\,t^{\frac{2-k}{2}},c^{1-k}|q,t)P^{\lambda}(a_i|q,t)P^{\lambda}(b_i|q,t)}{P^{\lambda}(t^{\frac{k-1}{2}},t^{\frac{k-3}{2}},\ldots,t^{\frac{1-k}{2}}|q,t)}\,.\end{aligned} \tag{4.107}$$

It would be interesting to have an analytic proof of these identities.

From (4.12) we can readily calculate the index of the genus $\mathfrak{g}$ theory with s punctures,

$$\mathcal{I}_{\mathfrak{g},s}(\mathbf{a}_I;q,t)=\prod_{j=2}^{k}(t^j;q)^{2\mathfrak{g}-2+s}\frac{(t;q)^{(k-1)(1-\mathfrak{g})+s}}{(q;q)^{(k-1)(1-\mathfrak{g})}}\sum_{\lambda}\frac{\prod_{i=1}^{s}\hat{\mathcal{K}}_{\Lambda_i}(\mathbf{a}_i)\;P^{\lambda}(\mathbf{a}_i(\Lambda_i)|q,t)}{\left[P^{\lambda}(t^{\frac{k-1}{2}},t^{\frac{k-3}{2}},\ldots,t^{\frac{1-k}{2}}|q,t)\right]^{2\mathfrak{g}-2+s}}\,. \tag{4.108}$$

Let us dwell upon this result. Let us first consider the genus $\mathfrak{g}$ partition function (no punctures) in the Schur limit, $q=t$. We can write it as

$$\mathcal{I}_{\mathfrak{g}}(q)=\left[(q;q)^{2\mathfrak{g}-2}\right]^{k-1}\;S_{00}(q)^{2-2\mathfrak{g}}\sum_{\lambda}\frac{1}{[dim_q(\lambda)]^{2\mathfrak{g}-2}}\,. \tag{4.109}$$

Here S_{00} is the partition function of $SU(k)$ level ℓ Chern-Simons theory on S^3 if we

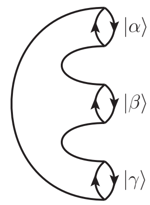
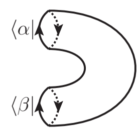
Symbol	Surface	Value
$C_{\alpha\beta\gamma}$		$\frac{\mathcal{A}(q,t)}{dim_{q,t}(\alpha)}\ \delta_{\alpha\beta}\ \delta_{\alpha\gamma}$
V^{α}		$\frac{dim_{q,t}(\alpha)}{\mathcal{A}(q,t)}$
$\eta^{\alpha\beta}$		$\delta^{\alpha\beta}$

Table 4.1: The structure constants, the cap, and the metric for the TQFT of the Macdonald index.

formally identify $q = e^{\frac{2\pi i}{\ell+k}}$,

$$S_{00}(q) = \prod_{j=2}^{k} \frac{(q;q)}{(q^j;q)}\,. \tag{4.110}$$

The expression (4.109), up to the simple factor $\left[(q;q)^{2\mathfrak{g}-2}\right]^{k-1}$, is the genus $\mathfrak{g}$ partition function of q-deformed $2d$ Yang-Mills theory in the zero area limit [101], which is in fact the same as the partition function of $SU(k)$ level ℓ Chern-Simons theory on $\mathcal{C}_{\mathfrak{g}} \times S^1$ with $q = e^{\frac{2\pi i}{\ell+k}}$ [101].[10] If we reintroduce punctures, the index is related to a correlator the q-deformed $2d$ Yang-Mills theory; the relation involving both a flavor independent factor and flavor-dependent factors $\hat{\mathcal{K}}_{\Lambda}$ associated to the punctures. We have recovered in more generality the relation found in [17] between the Schur index and $2d$ q-deformed Yang-Mills theory.[11]

In the more general case of $q \neq t$ the genus $\mathfrak{g}$ partition function can be written

[10] More precisely, q-deformed Yang-Mills theory in the zero area limit can be viewed as an analytical continuation of Chern-Simons theory, or equivalently of the G/G WZW model (see [102] for a review of the latter), to non-integer rank ℓ.

[11] Ordinary $2d$ Yang-Mills theory [85, 86] is obtained by sending $q \to 1$. From the index perspective, because of the additional overall factors, this is a singular limit. However, with proper regularization this limit can be understood as reducing the $4d$ index to a $3d$ partition function [103–105]. See also [106] for yet another 3d/4d connection.

as

$$\mathcal{I}_{\mathfrak{g}}(q,t) = \left[(t;q)^{\mathfrak{g}-1}\,(q;q)^{\mathfrak{g}-1}\right]^{k-1}\ \hat{S}_{00}(q,t)^{2-2\mathfrak{g}}\ \sum_{\lambda}\frac{1}{\left[dim_{q,t}(\lambda)\right]^{2\mathfrak{g}-2}}\,, \tag{4.111}$$

where the generalized quantum dimension is given by

$$dim_{q,t}(\lambda) = P^{\lambda}(t^{\frac{k-1}{2}}, t^{\frac{k-3}{2}}, \ldots, t^{\frac{1-k}{2}}|q,t) \tag{4.112}$$

and we have defined

$$\hat{S}_{00}(q,t) = \prod_{j=2}^{k}\frac{(t;q)}{(t^j;q)}\,. \tag{4.113}$$

This result appears to be closely related to the *refinement* of Chern-Simons theory recently discussed by Aganagic and Shakirov [107]. Up to the overall factor $\left[(t;q)^{\mathfrak{g}-1}\,(q;q)^{\mathfrak{g}-1}\right]^{k-1}$, $\mathcal{I}_{\mathfrak{g}}(q,t)$ is equal to the partition function of refined Chern-Simons on $\mathcal{C}_{\mathfrak{g}}\times S^1$, and $\hat{S}_{00}(q,t)$ to the partition function on S^3. In terms of the Chern-Simons matrix model the refinement of [107] amounts to changing the matrix integral measure from Haar to Macdonald. We can thus identify the $2d$ theory whose correlators give the Macdonald index as the theory obtained from q-Yang-Mills theory by deforming in the same way the path integral measure. It would be interesting to find a more conventional Lagrangian description of this $2d$ theory, for example the deformed measure could arise by integrating out some matter fields. It would also be desirable to have a better understanding of the flavor-independent factors needed to relate $2d$ Yang-Mills (q-deformed or (q,t)-deformed) to the index. They can be formally associated to a decoupled TQFT with a single operator (the identity). Perhaps this decoupled TQFT plays a similar role as the decoupled $U(1)$ factor in the AGT correspondence [9].

4.5 Coulomb-branch index

Finally we consider the index

$$\mathcal{I}_{CM}(T,Q) = \mathrm{Tr}_{CM}(-1)^F\,T^{\frac{1}{2}(E+2j_1-2R-r)}\,Q^{\frac{1}{2}(E+2j_2+2R+r)}\,, \tag{4.114}$$

where Tr_{CM} stands for the trace over states with $E+2j_1+r=0$. This limit of the full index makes sense for theories with a Lagrangian description, since the single-letter partition functions have well-defined expressions,

$$f^{\frac{1}{2}H} = 0, \qquad f^V = \frac{T-Q}{1-Q}\,. \tag{4.115}$$

Theories connected to Lagrangian theories by dualities also have a well-defined $\mathcal{I}_{CM}(T,Q)$. As discussed in section 4.1, the further limit $Q \to 0$ leads to the $\mathcal{I}_C(T)$ index, which is guaranteed to be well-defined for any $\mathcal{N}=2$ SCFT.

We refer to (4.114) as the "Coulomb-branch" index, or Coulomb index for short, because in a Lagrangian theory it receives contributions only from the $\bar{\mathcal{E}}$-type short-multiplets (see appendix B.2), whose bottom components are the gauge-invariant operators that parametrize the Coulomb branch, for example

$$\mathrm{Tr}\,\phi^2,\ \mathrm{Tr}\,\phi^3, \ldots,\ \mathrm{Tr}\,\phi^k \tag{4.116}$$

for a theory with $SU(k)$ gauge group. Since the hypermultiplets do not contribute, the Coulomb index is independent of the flavor fugacities and the TQFT structure is very simple. The structure constants associated to a three-punctured sphere depend only on T and Q, and so does the propagator, since the gauge-group matrix integral can be carried out independently of what the propagator connects to. The index of a quiver is then just a product over the indices of its constituents (propagators and vertices).

The index of a vector multiplet in the adjoint representation of a simple gauge group $\mathcal{G}$ is

$$\mathcal{I}^V_{(\mathcal{G})}(Q,T) = \oint_{\mathbb{T}^{r_\mathcal{G}}} \prod_{i=1}^{r_\mathcal{G}} \frac{da_i}{2\pi i} \Delta_\mathcal{G}(\mathbf{a})\, \exp\left[-\sum_{n=1}^{\infty} \frac{1}{n} \frac{Q^n - T^n}{1-Q^n} \chi^{(\mathcal{G})}_{adj}(\mathbf{a}^n) \right], \tag{4.117}$$

where $r_\mathcal{G}$ is the rank of $\mathcal{G}$ and $\Delta_\mathcal{G}(\mathbf{a})$ the Haar measure,

$$\Delta_\mathcal{G}(\mathbf{a}) = \frac{1}{|W_\mathcal{G}|} \exp\left[-\sum_{n=1}^{\infty} \frac{1}{n} (\chi_{adj}(\mathbf{a}^n) - r_\mathcal{G}) \right], \tag{4.118}$$

with $|W_\mathcal{G}|$ the order of the Weyl group. We recognize the integrand in (4.117) as the Macdonald measure (4.96) with parameters Q and T. The integral can be evaluated explicitly thanks to Macdonald's celebrated constant-term identities (see *e.g.* [93, 108] for pedagogic expositions and [109] for a brief review),

$$\mathcal{I}^V_\mathcal{G} = PE\left[\mathrm{rank}(\mathcal{G}) \tilde{\mathcal{I}}_1 + \sum_{\alpha \in R^+} \tilde{\mathcal{I}}_{1+\mathcal{C}_\alpha} - \tilde{\mathcal{I}}_{\mathcal{C}_\alpha} \right], \qquad \mathcal{C}_\alpha \equiv \sum_{\beta \in R^+} \frac{(\alpha,\beta)}{(\beta,\beta)}, \tag{4.119}$$

where R^+ is the collection of positive roots of $\mathcal{G}$ and

$$\tilde{\mathcal{I}}_\ell = T^{\ell-1} \frac{T-Q}{1-Q}. \tag{4.120}$$

We recognize $\tilde{\mathcal{I}}_\ell$ as the index of the $\bar{\mathcal{E}}_{-\ell(0,0)}$ superconformal multiplet, which satisfies the shortening condition $E = \ell$ (see appendix B.2). By a Lie-algebraic identity, ([4.119](#)) can be rewritten more succinctly as [109]

$$\mathcal{I}^V_{\mathcal{G}} = PE\left[\sum_{j\in\exp(\mathcal{G})} \tilde{\mathcal{I}}_{j+1}\right] , \tag{4.121}$$

where $\exp(\mathcal{G})$ stands for the set of exponents of the Lie group $\mathcal{G}$. This result has an immediate physical interpretation. The Coulomb index is saturated by the $\bar{\mathcal{E}}$-multiplets, whose bottom components are the gauge-invariant operators made of ϕs. The single-particle index (the argument of the plethystic exponential in (4.121)) then counts the independent gauge-invariant operators made of ϕs, which are in 1-1 correspondence with the Casimirs of the group, that is with $\exp(\mathcal{G})$. For example, for $\mathcal{G} = SU(k)$, $\exp(\mathcal{G}) = \{1, 2, \ldots, k-1\}$, and we see that the Coulomb index counts the independent single-trace operators (4.116) that parametrize the Coulomb branch. Turning the logic around, we can view this as a "physical" (or perhaps, combinatorial) proof of Macdonald's constant term identities. The integral over the Macdonald measure (4.117) counts gauge-invariant words built from certain letters of the vector multiplet; from superconformal representation theory we can identify which short multiplets are relevant for this counting problem, and deduce (4.121).

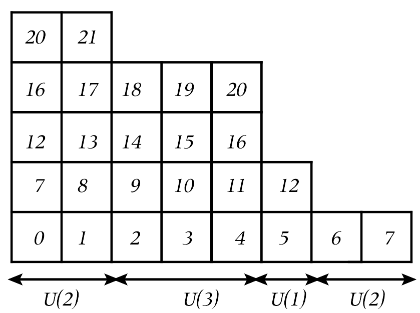


Figure 4.6: The bottom left box is assigned 0. The assigned integer increases from left to right. As we move up, the first box of each row is assigned the same number as the last box in the row below.

Though the TQFT structure for the Coulomb index is very simple, it is not entirely trivial. We can deduce the Coulomb index of strongly-coupled theories by using dualities, and check that different routes to obtain the index give the same result. For example, using Argyres-Seiberg duality [28]

$$\mathcal{I}_{E_6} = \frac{\mathcal{I}^V_{SU(3)}}{\mathcal{I}^V_{SU(2)}} = PE[\tilde{\mathcal{I}}_3] , \tag{4.122}$$

which is the expected result since the Coulomb branch of the E_6 SCFT is generated by an operator with $E = |r| = 3$. Strongly coupled SCFTs are sometimes obtained using S-dualities in more than one way [77] but all the dualities yield the same index, for example

$$\begin{aligned} \mathcal{I}_{E_6} &= \frac{\mathcal{I}^V_{SU(3)}}{\mathcal{I}^V_{SU(2)}} = \frac{\mathcal{I}^V_{SU(4)}}{\mathcal{I}^V_{USp(4)}} = PE[\tilde{\mathcal{I}}_3]\,, \\ \mathcal{I}_{E_7} &= \frac{\mathcal{I}^V_{SU(4)}}{\mathcal{I}^V_{SU(3)}} = \frac{\mathcal{I}^V_{USp(4)}}{\mathcal{I}^V_{SU(2)}} = \frac{\mathcal{I}^V_{SO(7)}}{\mathcal{I}^V_{G_2}} = \frac{\mathcal{I}^V_{SO(8)}}{\mathcal{I}^V_{SO(7)}} = PE[\tilde{\mathcal{I}}_4]\,, \\ \mathcal{I}_{E_8} &= \frac{\mathcal{I}^V_{SU(6)}}{\mathcal{I}^V_{SU(5)}} = \frac{\mathcal{I}^V_{USp(6)}}{\mathcal{I}^V_{SO(5)}} = PE[\tilde{\mathcal{I}}_6]\,. \end{aligned} \tag{4.123}$$

The index of the T_k theory is also obtained easily from the generalized Argyres-Seiberg duality,

$$\mathcal{I}_{T_k} = \frac{(\mathcal{I}^V_{SU(k)})^{k-2}}{\prod_{j=2}^{k-1} \mathcal{I}^V_{SU(j)}} = PE\left[\sum_{j=3}^{k} (j-2)\tilde{\mathcal{I}}_j\right]\,. \tag{4.124}$$

This is again as expected, since the Coulomb branch of the T_k theory is spanned by $(j-2)$ operators with $E = |r| = j$, for $j = 3, \ldots, k$ (see *e.g.* [73]).

Extrapolating from these examples let us conjecture the Coulomb index of the theory corresponding to a sphere with three generic punctures. For a general puncture I in the A_{k-1} theory, we associate the set of k numbers $\{p_j^{(I)} : j = 1, \ldots k\}$ from the corresponding auxiliary Young diagram. The assignment is illustrated in figure 4.6. The Coulomb branch index of the theory corresponding to a sphere with three punctures $p^{(1)}, p^{(2)}, p^{(3)}$ is then

$$\mathcal{I}_{p^{(1)},p^{(2)},p^{(3)}} = PE\left[d_j\ \tilde{\mathcal{I}}_j\right]\,, \qquad d_j \equiv \sum_{j=2}^{k} (1 - 2j + p_j^{(1)} + p_j^{(2)} + p_j^{(3)})\,. \tag{4.125}$$

The dimension d_j of the Coulomb branch spanned by operators with $E = |r| = j$ agrees with the dimension of the space of meromorphic j-differentials on the Riemann surface having poles of order at most $p_j^{(I)}$ at puncture I [3, 94].

Let us finally observe that the Coulomb index (4.117) discussed in this section can also be interpreted as the index of $\mathcal{N} = 4$ SYM in a certain limit of the $\mathcal{N} = 4$ superconformal fugacities, such that the index of the $\mathcal{N} = 4$ vector multiplet reduces to the index of $\mathcal{N} = 2$ vector multiplet. The authors of [110] noticed the appearance of the Macdonald measure in this context.

Chapter 5

Reducing the 4d Index to the S^3 Partition Function

String/M theory has led to a rich web of non-perturbative dualities between supersymmetric field theories. Checking/exploiting/extending these dualities requires exact computations in field theories. In recent years, using methods based on localization, several exact quantities in supersymmetric gauge theories have been computed. The connection between two of such quantities, the superconformal index of $4d$ gauge theories [2, 14] and the partition function of supersymmetric gauge theories on S^3 [33, 40], is the main focus of this chapter.

The superconformal index of $\mathcal{N}=1$ IR fixed points was first computed in [21–23], there it served as a check of Seiberg duality. The indices of $\mathcal{N}=4$ SYM and type IIB supergravity in AdS_5 were computed and matched in [2]. The superconformal index of $\mathcal{N}=2$ supersymmetric gauge theories was used to check $\mathcal{N}=2$ S-dualities conjectured by Gaiotto and to define a $2d$ topological field theory in the process as discussed in chapter 3 and 4. Recently the partition function of supersymmetric gauge theories on S^3 has been used to check a variety of $3d$ dualities including mirror symmetry [33] and Seiberg-like dualities [34]. Remarkably, the exact partition function has also allowed for a direct field theory computation of $N^{3/2}$ degrees of freedom of ABJM theory [35, 36]. The S^3 partition function of $\mathcal{N}=2$ theories is extremized by the exact superconformal R-symmetry [37–39] so just like the a-maximization in $4d$, the $3d$ partition function can be used to determine the exact R-charges at interacting fixed points. The purpose of this chapter is to relate these two interesting and useful exactly calculable quantities in 3 and 4 dimensions.

As mentioned in section 2.5 the superconformal index of a $4d$ gauge theory can be computed as a path integral on $S^3 \times S^1$ with supersymmetric boundary conditions along S^1 (2.49). All the modes on the S^1 contribute to this path integral. In a limit with the radius of the circle shrinking to zero the higher modes become very heavy and decouple. The index is then given by a path integral over just the constant modes on the circle. In other words, the superconformal index of the $4d$ theory reduces to a partition function of the dimensionally reduced $3d$ gauge theory on S^3. The $3d$ theory preserves all the supersymmetries of the "parent" $4d$ theory on $S^3 \times S^1$.

More generally, for any d dimensional manifold M^d, one would expect the index of a supersymmetric theory on $M^d \times S^1$ to reduce to the exact partition function of dimensionally reduced theory on M^d. This idea was applied by Nekrasov to obtain the partition function of $4d$ gauge theory on Ω-deformed background as a limit of the index of a $5d$ gauge theory [12].

A crucial property of the four dimensional index that facilitates its computation is the fact that it can be computed exactly by a saddle point integral. We show that in the limit of vanishing circle radius, this matrix integral reduces to the one that computes the partition function of $3d$ gauge theories on S^3 [33, 40]. It doesn't come as a surprise as the path integral of the $\mathcal{N} = 2$ supersymmetric gauge theory on S^3 was also shown to localize on saddle points of the action.

The chapter is organized a follows. We have already written the superconformal index of $4d$ theory as a saddle point integral and describe the limit in which this integral reduces to the S^3 partition function in section 2.5. The limit is performed in section 5.1. In particular, we show that the building blocks of the matrix model that computes the superconformal index in $4d$ map separately to the building blocks of the $3d$ partition function matrix model. In section 5.2, we comment on the connections between $4d$ and $3d$ dualities. We conclude with an appendix that generalizes the Kapustin et. al. matrix model for $\mathcal{N} = 4$ gauge theories with two supersymmetric deformations. One such deformation involving squashed S^3 was studied in [57]. Discussion of the similar material can also be found in [103, 105].

5.1 $4d$ Index to $3d$ Partition function on S^3

A matrix model for computing the partition function of $3d$ gauge theories on S^3 (S^3 matrix model) was obtained in [33, 40]. In this section, we will derive this matrix model as a $\beta \to 0$ limit of the matrix model that computes the superconformal index (2.52) (index matrix model) of the $4d$ gauge theories. Both matrix models involve integrals over gauge group parameters and their integrand contains one-loop contributions from vector- and hyper-multiplets. We will show that the gauge group integral together with the contribution from the vector multiplet map nicely from the index model to the S^3 model. The contributions of the hypermultiplets match up separately. We also show that the superconformal index is the q-deformation of the S^3 partition function of the daughter theory.

5.1.1 Building blocks of the matrix models

For concreteness, let us consider $4d$ $\mathcal{N} = 2$ $SU(N)$ gauge theory. It is constructed using two basic building blocks: hyper-multiplets and vector multiplets.

Hyper-multiplet

As was first observed in [21], the index of the hypermultiplet can be written elegantly in terms of a special function [15]

$$\mathcal{I}^{hyp} = \prod_i \Gamma\left(\frac{t^2}{\sqrt{v}} a_i; t^3 y, t^3 y^{-1}\right), \tag{5.1}$$

where Γ is the elliptic gamma function [111] defined to be

$$\Gamma(z; r, s) = \prod_{j,k \geq 0} \frac{1 - z^{-1} r^{j+1} s^{k+1}}{1 - z r^j s^k}, \tag{5.2}$$

and a_i are eigenvalues of the maximal torus of the gauge/flavor group satisfying $\prod_{i=1}^{N} a_i = 1$. In this section, for the sake of simplicity, we set $v = t$ and $y = 1$ and will discuss the general assignment of chemical potentials in appendix C. We choose a convenient variable $q \equiv e^{-\beta}$ to parametrize the chemical potentials of the Cartan of the flavor group as $a_i = q^{-i\alpha_i}$, and the chemical potential t as $t = q^{\frac{1}{3}}$. The index of the hyper-multiplet then becomes

$$\mathcal{I}^{hyp} = \prod_i \prod_{j,k \geq 0} \frac{1 - q^{-\frac{1}{2} + i\alpha_i} q^{j+1} q^{k+1}}{1 - q^{\frac{1}{2} - i\alpha_i} q^j q^k} = \prod_i \prod_{n \geq 1} \left(\frac{[n + \frac{1}{2} + i\alpha_i]_q}{[n - \frac{1}{2} - i\alpha_i]_q}\right)^n, \tag{5.3}$$

where $[n]_q \equiv \frac{1-q^n}{1-q}$ is the *q-number*. It has the property $[n]_q \xrightarrow{q \to 1} n$. So far we have fixed the chemical potentials v and y that couple to $-(R + r)$ and j_1 respectively. To recover $3d$ partition function on S^3 we should take the radius of S^1 to be very small, which corresponds to the limit $q \to 1$.

$$\mathcal{I}^{hyp} = \prod_i \prod_{n \geq 1} \left(\frac{n + \frac{1}{2} + i\alpha_i}{n - \frac{1}{2} - i\alpha_i}\right)^n = \prod_i (\cosh \pi\alpha_i)^{-\frac{1}{2}}. \tag{5.4}$$

One can find a proof of the second equality in [40]. From the limiting procedure, it is clear that the superconformal index of the hypermultiplet is the q-deformation of the $3d$ hypermultiplet partition function.

Vector multiplet

The index of an $\mathcal{N} = 2$ vector multiplet is given by

$$\mathcal{I}^{vector} = \prod_{i<j} \frac{1}{(1 - a_i/a_j)(1 - a_j/a_i)} \frac{\Gamma(t^2 v (a_i/a_j)^{\pm}; t^3 y, t^3 y^{-1})}{\Gamma((a_i/a_j)^{\pm}; t^3 y, t^3 y^{-1})}, \tag{5.5}$$

Here we have dropped an overall a_i-independent factor. We use the condensed notation, $\Gamma(z^{\pm 1}; r, s) = \Gamma(z^{-1}; r, s)\Gamma(z; r, s)$. With the same variable change as above we get

$$
\begin{aligned}
\mathcal{I}^{vector} &= \prod_{i<j} \frac{1}{1-q^{i(\alpha_i-\alpha_j)}} \frac{1}{1-q^{-i(\alpha_i-\alpha_j)}} \frac{1}{\Gamma(q^{\pm i(\alpha_i-\alpha_j)};q,q)} \\
&= \prod_{i<j} \frac{1}{1-q^{i(\alpha_i-\alpha_j)}} \frac{1}{1-q^{-i(\alpha_i-\alpha_j)}} \prod_{n\geq 1} \left(\frac{1-q^{n+i(\alpha_i-\alpha_j)+1}}{1-q^{n-i(\alpha_i-\alpha_j)-1}} \frac{1-q^{n-i(\alpha_i-\alpha_j)+1}}{1-q^{n+i(\alpha_i-\alpha_j)-1}} \right)^{-n} \\
&\overset{reg}{=} \prod_{i<j} \prod_{n\geq 1} \left(\frac{[n-i(\alpha_i-\alpha_j)]_q}{[n]_q} \frac{[n+i(\alpha_i-\alpha_j)]_q}{[n]_q} \right)^2 .
\end{aligned}
\tag{5.6}
$$

The last line involves regulating the infinite product in a way that doesn't depend on α. In the limit $q \to 1$, i.e. the radius of the circle goes to zero, we get

$$
\mathcal{I}^{vector} = \prod_{i<j} \prod_{n\geq 1} \left(1 + \frac{(\alpha_i-\alpha_j)^2}{n^2} \right)^2 = \prod_{i<j} \left(\frac{\sinh \pi(\alpha_i-\alpha_j)}{\pi(\alpha_i-\alpha_j)} \right)^2 . \tag{5.7}
$$

The last equality again is explained in [40]. Again, the we see that the index of the vector multiplet is the q-deformation of the $3d$ vector partition function. Most general expression for the one-loop contribution of the vector multiplet with u and η turned on is obtained in appendix C.

Gauge group integral

The gauge group integral in the $4d$ index matrix model is done with the invariant Haar measure

$$
[dU] = \prod_i d\alpha_i \prod_{i<j} \sin^2 \left(\frac{\beta(\alpha_i-\alpha_j)}{2} \right) \overset{\beta\to 0}{\longrightarrow} \prod_i d\alpha_i \prod_{i<j} \left(\frac{\beta(\alpha_i-\alpha_j)}{2} \right)^2 . \tag{5.8}
$$

After appropriate regularization, the measure factor precisely cancels the weight factor in the denominator of the vector multiplet one-loop determinant. The unitary gauge group integral in the index matrix model can be done as a contour integral over a variables parametrizing the Cartan sub-group, i.e. $a \in \mathbb{T}$. After the change variables $a = q^{-i\alpha}$ the contour integral becomes a line integral as follows. We write $a = q^{-i\alpha} = e^{i\beta\alpha}$. The contour integral around the unit circle is then

$$
\oint_{\mathbb{T}} \frac{da}{a} \cdots = \int_{-\pi/\beta}^{\pi/\beta} d\alpha \ldots \qquad : \qquad \beta \to 0, \qquad \oint_{\mathbb{T}} \frac{da}{a} \cdots = \int_{-\infty}^{\infty} d\alpha \ldots . \tag{5.9}
$$

5.2 $4d \leftrightarrow 3d$ dualities

S duality

Let us illustrate the reduction of a four dimensional index to three dimensional partition function with a simple example. Consider $\mathcal{N}=2$ $SU(2)$ gauge theory with four hypermultiplets in four dimensions. The index of this theory is given by the following expression (up to overall normalization constants)

$$\oint \frac{dz}{z}\,\frac{\Gamma(t^{3/2}a^{\pm1}b^{\pm1}z^{\pm1};t^3,t^3)\,\Gamma(t^{3/2}c^{\pm1}d^{\pm1}z^{\pm1};t^3,t^3)}{\Gamma(z^{\pm2};t^3,t^3)}\,. \tag{5.10}$$

Here, a, b, c and d label the Cartans of $SU(2)^4 \subset SO(8)$ flavor group. The Gamma functions in the numerator come from the four hyper-multiplets; the Gamma functions in the denominator come from the $\mathcal{N}=2$ vector multiplet.

From the results of the previous section this expression for the index gives rise to the partition function of $\mathcal{N}=2$ $SU(2)$ gauge theory in three dimensions. We scale $t\to1$ and rewrite this as

$$\mathcal{Z}(\alpha,\beta,\gamma,\delta)=\int d\sigma\frac{\sinh^2 2\pi\sigma}{\cosh\pi(\sigma\pm\alpha\pm\beta)\cosh\pi(\sigma\pm\gamma\pm\delta)}\,, \tag{5.11}$$

where, each cosh is product of four factors with all sign combinations. The flavor (now mass) parameters α, β, γ and δ are related to the flavor parameters in $4d$ as before.

The superconformal index of the $\mathcal{N}=2$ $SU(2)$ gauge theory with four hypermultiplets in four dimensions is expected to be invariant under the action of an S-duality group which permutes the four hypermultiplets. The expression above can be explicitly shown to exhibit this property [15]. The four dimensional S-duality implies that the three dimensional partition function is invariant under permuting α, β, γ, and δ. One can show (e.g. numerically or order by order expansion in α) that this is indeed true. Note that this implies a new kind of Seiberg-like duality in three dimensions. This computation can be generalized to any of the theories recently discussed by Gaiotto [3] in four dimensions. In particular the index of these theories was claimed to posses a TQFT structure [15]; and this structure is inherited by the three dimensional partition functions after doing the dimensional reduction. The reasoning in four dimensions and three dimensions is however different. In four dimensions one can associate a punctured Riemann surface to each of the superconfomal theories with the modular parameters of the surface related to the gauge coupling constants. The index does not depend on the coupling constants and thus is independent of the moduli giving rise to a topological quantity associated to the Riemann surface. After dimensionally reducing to three dimensions the theories cease to be conformal invariant and flow to a fixed point in the IR. The statement is then that at the IR fixed point the information about the original coupling constant is "washed away" and theories originally associated to punctured Riemann surfaces of the same topology flow to an equivalent fixed point in the IR.

Mirror symmetry

In principle one can try to use relations special to field theories in three dimensions to gain information about the four dimensional theories. Let us comment how this can come about. In three dimensions certain classes of theories are related by mirror symmetry. for example, in [112] it is claimed that mirror duals of T_N [3] theories have Lagrangian description and are certain star shaped quiver gauge theories. Let us see if the partition function of T_2 (free hyper-multiplet in trifundamental of $SU(2)^3$) matches with the partition function of its mirror dual:

$$\begin{aligned} \mathcal{Z}_{T_2} &= \frac{1}{\cosh\pi(\alpha\pm\beta\pm\gamma)}, \\ \mathcal{Z}_{\tilde{T}_2} &= \int d\sigma d\mu d\nu d\rho \frac{\sinh^2 2\pi\sigma\, e^{2\pi i(\mu\alpha+\nu\beta+\rho\gamma)}}{\cosh\pi(\sigma\pm\mu)\cosh\pi(\sigma\pm\nu)\cosh\pi(\sigma\pm\rho)}. \end{aligned} \tag{5.12}$$

In $\mathcal{Z}_T$, the parameters α, β, γ appear as masses while in $\mathcal{Z}_{\tilde{T}}$ they appear as FI terms. Let us compute $\mathcal{Z}_{\tilde{T}_2}$. One can perform the $\mathcal{Z}_{\tilde{T}_2}$ integrations. First we work out

$$\int d\mu \frac{e^{2\pi i\alpha\mu}}{\cosh\pi(\mu\pm\sigma)} = \frac{2\sin 2\pi\alpha\sigma}{\sinh\pi\alpha\sinh 2\pi\sigma}\,.$$

Then we find that

$$\mathcal{Z}_{\tilde{T}_2} = \int d\sigma \frac{8\sin 2\pi\alpha\sigma\sin 2\pi\beta\sigma\sin 2\pi\gamma\sigma}{\sinh\pi\alpha\sinh\pi\beta\sinh\pi\gamma\sinh 2\pi\sigma} = \frac{1}{\cosh\pi(\alpha/2\pm\beta/2\pm\gamma/2)}\,. \tag{5.13}$$

$\mathcal{Z}_{\tilde{T}_2}$ is actually $\mathcal{Z}_{T_2}$ if we rescale α, β and γ in $\mathcal{Z}_{\tilde{T}_2}$ by a factor of 2. This fact can be in principle use to investigate the index of the strongly coupled SCFTs in four dimensions which do not have Lagrangian description. One can dimensionally reduce these theories to three dimensions, consider their mirror dual and compute its $3d$ partition function; finally, one can try to uplift this result to $4d$ and obtain thus the superconformal index of the original four dimensional theory. The further analytical proof can be found in [113, 114]

Chapter 6

The Superconformal Index of $\mathcal{N} = 1$ IR Fixed Points

In section 2.2 we review the Römelsberger prescription [14, 19] to evaluate the index of such SCFTs the $\mathcal{N} = 1$ SCFTs that arise as IR fixed points of renormalization group flows, whose UV starting points are weakly-coupled theories. This prescription has so far been checked indirectly, by showing in several examples that it gives the same result for different RG flows that end in the same IR fixed point (*i.e.* the UV theories are Seiberg dual). This was originally observed by Römelsberger, who performed a few perturbative checks in a chemical potential expansion [14, 19]. Invariance of the $\mathcal{N} = 1$ index under Seiberg duality was systematically demonstrated by Dolan and Osborn [21], in a remarkable paper that first applied the elliptic hypergeometric machinery to the evaluation of the superconformal index. These results were extended and generalized in [22–24, 26].

In this chapter we apply Römelsberger's prescription to a class of $\mathcal{N} = 1$ SCFTs that admit AdS duals. The canonical example is the conifold theory of Klebanov and Witten [30]. There are infinitely many generalizations: the families of toric quivers $Y^{p,q}$ [1] and $L^{p,q,r}$ [31]. We focus on $Y^{p,q}$. In all these examples there is in principle an independent way to determine the index (at large N) from the dual supergravity. We will explicitly show agreement between the gravity calculation of Nakayama [32] and our field theory calculation for the case of the conifold quiver $Y^{1,0}$. According to taste, this can be viewed either as a check of Römelsberger's prescription, or as yet another check of AdS/CFT. The upshot is a sharper bulk/boundary dictionary.

In section 6.1 we review basic facts about the $Y^{p,q}$ family of toric quivers (the conifold being a special case $Y^{1,0}$). From the quiver diagrams, it is immediate to write integral expressions for the superconformal index, at finite N. We show that the indices of toric-dual theories are equal, as expected. In section 6.2 we consider the large N limit. We conjecture a simple closed form expression for the large N index of the $Y^{p,q}$ quivers. In section 6.3 we review the gravity computation of the index for the conifold [32] and find exact agreement with the large N limit of our field theory result. appendix D collects useful material about $\mathcal{N} = 1$ superconformal representation theory and the index of the different short and semishort supermultiplets.

6.1 The $Y^{p,q}$ quiver gauge theories

Let us begin by recalling the basic facts about the $Y^{p,q}$ quiver gauge theories [1]. The fields are of four types: $U_{\alpha=1,2}$, $V_{\alpha=1,2}$, Y and Z. There are $2p$ gauge groups, and $4p+2q$ bifundamental fields: p fields of type U, q fields of type V, $p-q$ fields of type Z, and $p+q$ fields of type Y. The $Y^{p,q}$ quiver diagram is obtained by a recursive procedure starting with $Y^{p,p}$, which is a familiar $\mathbb{Z}_{2p}$ orbifold of $\mathcal{N} = 4$ SYM. The superpotential takes the form

$$\mathbf{W} = \sum \epsilon^{\alpha\beta}\mathrm{Tr}\left(U_\alpha^k\, V_\beta^k\, Y^{2k+2} + V_\alpha^k\, U_\beta^{k+1}\, Y^{2k+3}\right) + \epsilon_{\alpha\beta}\sum \mathrm{Tr}\left(Z^k\, U_\alpha^{k+1}\, Y^{2k+3}\, U_\beta^k\right)\,,$$

where the cubic and quartic gauge-invariant terms are read off from the quiver diagram. There are $2q$ terms in the first sum and $p-q$ terms in the second sum. For the Klebanov-Witten theory, $T^{1,1} = Y^{1,0}$ has only quartic terms.

The R-charges are determined as follows [1, 115]. Requiring the vanishing of the NSVZ beta functions and that each term of the superpotential has R-charge 2, the R-charges of all the fields are fixed in terms of two independent parameters x and y,

$$r_{Z^k} = x, \qquad r_{Y^k} = y, \qquad r_{U_\alpha^k} = 1 - \frac{1}{2}(x+y), \qquad r_{V_\alpha^k} = 1 + \frac{1}{2}(x-y)\,. \tag{6.1}$$

This twofold ambiguity is related to the existence of two $U(1)$ global symmetries, and is resolved by a-maximization. One finds [1]

$$\begin{aligned} y_{p,q} &= \frac{1}{3q^2}\left\{-4p^2 + 2pq + 3q^2 + (2p-q)\sqrt{4p^2-3q^2}\right\}\,, \\ x_{p,q} &= \frac{1}{3q^2}\left\{-4p^2 - 2pq + 3q^2 + (2p+q)\sqrt{4p^2-3q^2}\right\}\,. \end{aligned} \tag{6.2}$$

For any p, there are simple special cases. The $Y^{p,p}$ quiver corresponds to the $\mathbb{Z}_{2p}$ orbifold of $\mathbb{C}^3$. In this case all the superpotential terms are cubic, the theory is exactly conformal and all R-charges are equal to $\frac{2}{3}$. This theory has $\mathcal{N} = 1$ supersymmetry for general p while for $p = 1$ the supersymmetry is enhanced to $\mathcal{N} = 2$. At the other extreme, the $Y^{p,0}$ quiver corresponds to a $\mathbb{Z}_p$ orbifold of the conifold. All the R-charges are equal to $\frac{1}{2}$ and the superpotential is quartic. The associated quiver diagrams for $p = 4$ are shown in figure 6.1.

The charges of the fields under the global symmetries $U(1)_B$, $U(1)_s$ and $SU(2)_l$

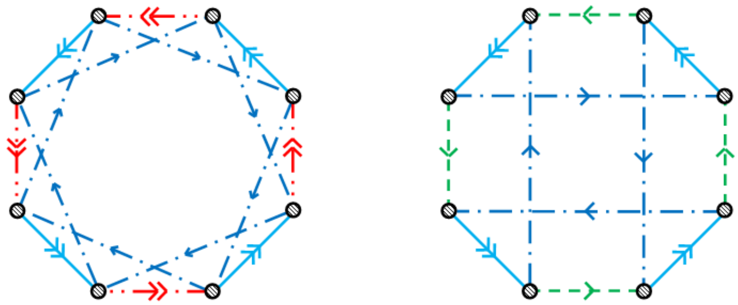


Figure 6.1: Left: quiver diagram for $Y^{4,4}$. Right: quiver diagram for $Y^{4,0}$.

and the color-coding of the arrows are indicated below.

$$\begin{array}{|c|c|c|c|c|}
\hline
 & U(1)_B & U(1)_s & SU(2)_l & \text{Arrows} \\
\hline
U & -p & 0 & \pm\frac{1}{2} & \\
V & q & \frac{1}{2} & \pm\frac{1}{2} & \\
Z & p+q & \frac{1}{2} & 0 & \\
Y & p-q & -\frac{1}{2} & 0 & \\
\hline
\end{array} \tag{6.3}$$

We can refine the index by chemical potentials for the global symmetries,

$$\mathcal{I}^{\mathrm{L}}(t,y,a,b,h) = \mathrm{Tr}(-1)^F\, t^{3(2j_1-r)}\, y^{2j_2}\, a^{2s}\, b^{2l}\, h^{Q_B} \tag{6.4}$$

$$\mathcal{I}^{\mathrm{R}}(t,y,a,b,h) = \mathrm{Tr}(-1)^F\, t^{3(2j_2+r)}\, y^{2j_1}\, a^{2s}\, b^{2l}\, h^{Q_B}\,. \tag{6.5}$$

In practice we can focus on say the left-handed index. The right-handed index of a given theory is obtained from the left-handed index of the same theory by conjugation of the flavor quantum numbers, $a \to 1/a$, $h \to 1/h$.

Given a $Y^{p,q}$ quiver diagram, it is immediate to combine the chiral and vector building blocks (2.46), (2.47) and construct the matrix integral that calculates the corresponding index. We illustrate the procedure in the two simplest examples.

• $Y^{1,0}$ ($T^{1,1}$)

The quiver of $T^{1,1}$ is shown in figure 6.2. The index can be simply read from the quiver diagram,

$$\begin{aligned}
\mathcal{I}_{1,0} = \prod_{k=1}^{2}\left[\frac{\kappa^{N-1}}{N!}\oint_{\mathbb{T}}\prod_{i=1}^{N-1}\frac{dz_i^{(k)}}{2\pi i z_i^{(k)}}\frac{1}{\prod_{i\neq j}\Gamma(z_i^{(k)}/z_j^{(k)})}\right] \\
\times \prod_{i,j=1}^{N}\Gamma(t^{3r_U}b^{\pm}z_i^{(2)}/z_j^{(1)})\prod_{i,j=1}^{N}\Gamma(t^{3r_Y}a^{-1}z_i^{(1)}/z_j^{(2)})\prod_{i,j=1}^{N}\Gamma(t^{3r_Z}a\,z_i^{(1)}/z_j^{(2)})
\end{aligned} \tag{6.6}$$

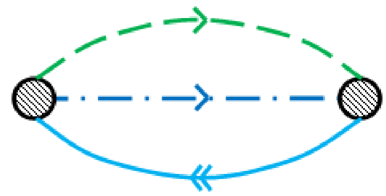


Figure 6.2: Quiver diagram for $Y^{1,0}$ (the conifold theory $T^{1,1}$). The solid (cyan) arrow represents the U field, the dash-dot (blue) arrow represents the Y field and the dashed (green) arrow represents the Z field.

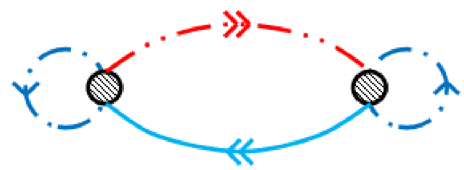


Figure 6.3: Quiver of $Y^{1,1}$ theory. Solid (cyan) arrow represents U field, dash-dot-dot arrow (red) represents V field and dash-dot arrow (blue) represents Y field.

where the R-charges are

$$r_U = r_Y = r_Z = \frac{1}{2}. \tag{6.7}$$

The fact that Y and Z share the same R-charge leads to the symmetry enhancement $U(1)_s \to SU(2)_s$.

- $Y^{1,1}$ $(\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C})$

The quiver corresponding to $Y^{1,1}$ is shown in figure 6.3. This theory is the familiar $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM which in fact preserves $\mathcal{N}=2$ supersymmetry, but we write its $\mathcal{N}=1$ index for a uniform analysis,[1]

$$\begin{aligned}\mathcal{I}_{1,1} = \prod_{k=1}^{2}&\left[\frac{\kappa^{N-1}}{N!}\oint_{\mathbb{T}}\prod_{i=1}^{N-1}\frac{dz_i^{(k)}}{2\pi i z_i^{(k)}}\frac{1}{\prod_{i\neq j}\Gamma(z_i^{(k)}/z_j^{(k)})}\right]\\ &\times\left[\prod_{i,j=1}^{N}\Gamma(t^{3r_U}b^{\pm}z_i^{(2)}/z_j^{(1)})\prod_{i,j=1}^{N}\Gamma(t^{3r_V}b^{\pm}a\,z_i^{(1)}/z_j^{(2)})\right]\\ &\times\left[\prod_{i\neq j}\Gamma(t^{3r_Y}a^{-1}z_i^{(1)}/z_j^{(1)})\Gamma(t^{3r_Y})^{N-1}\prod_{i\neq j}\Gamma(t^{3r_Y}a^{-1}z_i^{(2)}/z_j^{(2)})\Gamma(t^{3r_Y})^{N-1}\right].\end{aligned} \tag{6.8}$$

[1]The index for this theory has been already calculated at large N [42, 97].

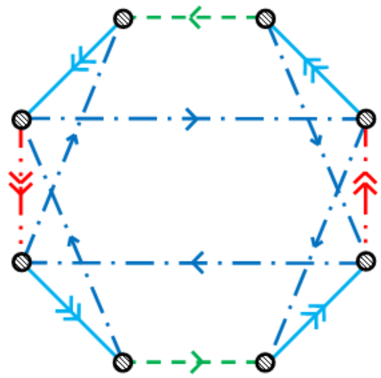


Figure 6.4: Quiver diagram for $Y^{4,2}$, obtained from $Y^{4,4}$ by using the procedure in [1].

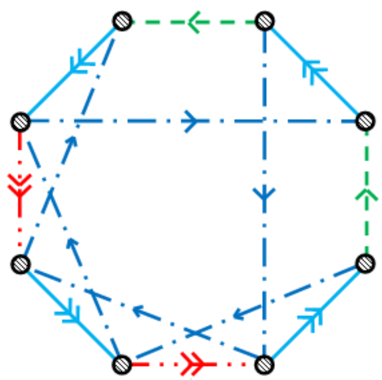


Figure 6.5: A different quiver diagram for $Y^{4,2}$, related to the diagram above by toric duality.

6.1.1 Toric Duality

A toric Calabi-Yau singularity may have several equivalent quiver representations, related by what has been called "toric duality" [116]. In terms of the gauge theories on D3 branes probing the singularity, two toric-dual quiver diagrams define two UV theories that flow to the same IR superconformal fixed point. Toric duality can in fact be understood in terms of the usual Seiberg duality of super QCD [117–123]. In particular, the prescription [1] for finding the quiver theory associated $Y^{p,q}$ does not lead to unique answer, rather to a family of quivers related by toric duality. The simplest example occurs for $Y^{4,2}$: the pair of toric-dual quivers associated to $Y^{4,2}$ is shown in figures 6.4 and 6.5.

We are now going to check the equality of the indices of two dual theories using an identity between elliptic hypergeometric integrals.

Consider the k-th node of a $Y^{p,q}$ quiver with one incoming Y, one incoming Z and an outgoing U doublet (see the first diagram in figure 6.6). Its contribution to the index (suppressing global symmetry charges) is

$$\begin{aligned}\mathcal{I}^k_{p,q} =&\frac{\kappa^{N-1}}{N!}\oint_{\mathbb{T}}\prod_{i=1}^{N-1}\frac{dz_i^{(k)}}{2\pi i z_i^{(k)}}\frac{1}{\prod_{i\neq j}\Gamma(z_i^{(k)}/z_j^{(k)})}\\ &\times\prod_{i,j}\Gamma(t^{3r_Z}\frac{z_i^k}{z_j^Z})\prod_{i,j}\Gamma(t^{3r_Y}\frac{z_i^k}{z_j^Y})\prod_{i,j}\Gamma(t^{3r_U}\frac{z_j^U}{z_i^k})^2\prod_{i,j}\Gamma(t^{3r_V}\frac{z_i^Z}{z_j^U})^2.\end{aligned} \tag{6.9}$$

where z^U, z^Y and z^Z represents the "flavor" group of U, Y and Z. This is precisely the A_n-type integral defined in [83],

$$\mathcal{I}^k_{p,q} = I^{(N-1)}_{A_{N-1}}(Z|t^{3r_Z}/z_j^Z, t^{3r_Y}/z_j^Y; t^{3r_U}z_j^U, t^{3r_U}z_j^U; p,q)\prod_{i,j}\Gamma(t^{3r_V}\frac{z_i^Z}{z_j^U})^2\,. \tag{6.10}$$

This integral obeys the balancing condition

$$\prod_{j=1}^{N}\frac{t^{3r_Z}}{z_j^Z}\frac{t^{3r_Y}}{z_j^Y}t^{3r_U}z_j^U t^{3r_U}z_j^U = (pq)^N\,, \tag{6.11}$$

thanks to the relation

$$r_Y + r_Z + 2r_U = y_{p,q} + x_{p,q} + 2[1-\frac{1}{2}(x_{p,q}+y_{p,q})] = 2\,. \tag{6.12}$$

Then the following identity holds [83]:

$$I^{(m)}_{A_n}(Z|t_i\ldots,u_i\ldots) = \prod_{r,s=1}^{m+n+2}\Gamma(t_r u_s)I^{(n)}_{A_m}(Z|\frac{T^{\frac{1}{m+1}}}{t_i}\ldots,\frac{U^{\frac{1}{m+1}}}{u_i}\ldots). \tag{6.13}$$

So we have

$$\begin{aligned}\mathcal{I}^k_{p,q} =&I^{(N-1)}_{A_{N-1}}(Z|t^{3r_Z}/z_j^Z, t^{3r_Y}/z_j^Y; t^{3r_U}z_j^U, t^{3r_U}z_j^U; p,q)\prod_{i,j}\Gamma(t^{3r_V}\frac{z_i^Z}{z_j^U})^2\\ =&\prod_{i,j=1}^{N}\Gamma(t^{3(r_Z+r_U)}\frac{z_i^U}{z_j^Z})^2\prod_{i,j=1}^{N}\Gamma(t^{3(r_Y+r_U)}\frac{z_i^U}{z_j^Y})^2\\ &\times I^{(N-1)}_{A_{N-1}}(Z|t^{3r_Y}z_j^Z, t^{3r_Z}z_j^Y; t^{3r_U}/z_j^U, t^{3r_U}/z_j^U; p,q)\prod_{i,j=1}^{N}\Gamma(t^{3r_V}\frac{z_j^Z}{z_i^U})^2\\ =&\prod_{i,j=1}^{N}\Gamma(t^{3r_V}\frac{z_i^U}{z_j^Y})^2 I^{(N-1)}_{A_{N-1}}(Z|t^{3r_Y}z_j^Z, t^{3r_Z}z_j^Y; t^{3r_U}/z_j^U, t^{3r_U}/z_j^U; p,q),\end{aligned} \tag{6.14}$$

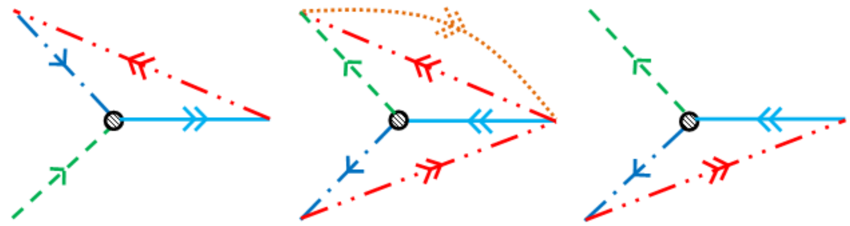


Figure 6.6: Left: initial quiver. The node represents a $SU(N)$ gauge group. The effective number of flavors is $N_f = 2N$. Middle: quiver after Seiberg duality. The node represents the Seiberg dual gauge group $SU(2N-N) = SU(N)$. All arrows are reversed and mesons (with appropriate R-charges) are added. Right: the dash-dot-dot (red) and dot (orange) mesons cancel each other out by equ.(4.15). This can be understood physically in terms of integrating out massive degrees of freedom [31].

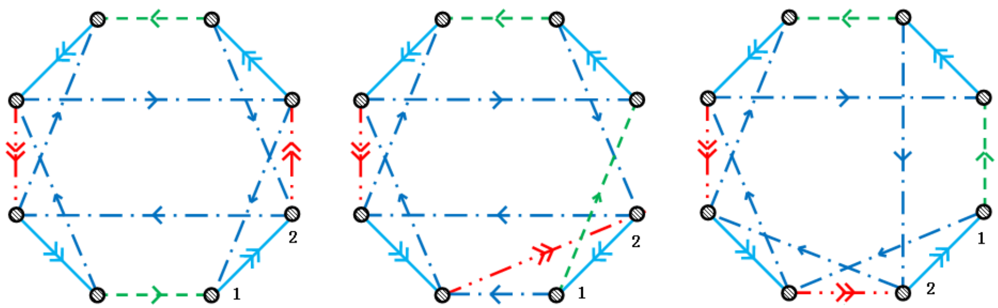


Figure 6.7: Left: $Y^{4,2}$ quiver in figure 6.4. Middle: Seiberg dual on node 1. Right: the quiver in figure 6.5 is obtained by swap node 1 and 2 in the middle figure.

where we have used $r_V = r_Z + r_U$ and

$$\Gamma(t^{3r_V}\frac{z_i^Z}{z_j^U})\Gamma(t^{3(r_Y+r_U)}\frac{z_j^U}{z_i^Z}) = 1\,. \tag{6.15}$$

For example, one can perform this duality on one of the $YZ\bar{U}$ nodes of the $Y^{4,2}$ quiver in figure 6.4 and obtain the quiver in figure 6.5. The procedure is illustrated in figure 6.7.

This transformation can be represented on a quiver as a local graph transformation of figure 6.6. It has the interpretation of Seiberg duality on the node. (In fact the same elliptic hypergeometric identity was used in [21] to demonstrate the equality of the index under Seiberg duality.) Iterating this step, we can reach all the toric phases of any $Y^{p,q}$ gauge theory.

6.2 Large N evaluation of the index

In the large N limit the leading contribution to the index is evaluated using matrix models techniques (see e.g. [2, 51]). Let $\{e^{\alpha_{ai}}\}_{i=1}^{N_a}$ denote the N_a eigenvalues of u_a. Then the matrix model integral (2.23) is,

$$\mathcal{I}(x) = \int \prod_{a,i} [d\alpha_{ai}] \exp\left\{ - \sum_{ai\neq bj} V^a_b(\alpha_{ai} - \alpha_{bj}) \right\} . \tag{6.16}$$

Here, the potential V is the following function

$$V^a_b(\theta) = \delta^a_b(\ln 2) + \sum_{n=1}^{\infty} \frac{1}{n}[\delta^a_b - i^a_b(x^n)] \cos n\theta \,, \tag{6.17}$$

where, $i^a_b(x)$ is the total single letter index in the representation $r^a \otimes r_b$. Writing the density of the eigenvalues $\{e^{\alpha_{ai}}\}$ at the point θ on the circle as $\rho_a(\theta)$, we reduce it to the functional integral problem,

$$\mathcal{I}(x) = \int \prod_a [d\rho_a] \exp\{- \int d\theta_1 d\theta_2 \sum_{a,b} n_a n_b \rho_a(\theta_1) V^a_b(\theta_1 - \theta_2) \rho^\dagger_b(\theta_2)\} . \tag{6.18}$$

For large N, we can evaluate this expression with the saddle point approximation,

$$\mathcal{I}(x) \;\; = \;\; \prod_k \frac{1}{\det(1 - i(x^k))} .$$

For $SU(N)$ gauge groups instead of $U(N)$, the result is modified as follows,

$$\mathcal{I}(x) = \prod_k \frac{e^{-\frac{1}{k}\mathrm{tr}\ i(x^k)}}{\det(1 - i(x^k))} . \tag{6.19}$$

Here $i(x)$ is the matrix with entries $i^a_b(x)$. We will see examples of such matrices below.

The single-trace partition function can be obtained from the full partition function,

$$\begin{aligned} \mathcal{I}_{s.t.} &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \mathcal{I}(x^n) \\ &= -\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log[\det(1 - i(x^k))] - \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{k=1}^{\infty} \frac{\mathrm{tr}\ i(x^{nk})}{k} \\ &= -\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log[\det(1 - i(x^k))] - \mathrm{tr}\ i(x) . \end{aligned} \tag{6.20}$$

The second term in the summation would be absent for the $U(N)$ gauge theories. Here $\mu(n)$ is the Möbius function ($\mu(1) \equiv 1$, $\mu(n) \equiv 0$ if n has repeated prime factors and $\mu(n) \equiv (-1)^k$ if n is the product of k distinct primes) and $\varphi(n)$ is the Euler Phi function, defined as the number of positive integers less than n that are coprime to n. We have used the properties

$$\sum_{d|n} d\mu(\frac{n}{d}) = \varphi(n), \qquad \sum_{d|n} \mu(d) = \delta_{n,1}. \tag{6.21}$$

After deriving the general expression for the superconformal index of a quiver gauge theory let us study some concrete examples. Recall the single-letter indices

$$\begin{aligned} i_V(t,y) =& \frac{2t^6 - t^3(y+\frac{1}{y})}{(1-t^3y)(1-t^3y^{-1})}\,, \\ i_{\bar{\chi}(r)}(t,y) =& \frac{t^{3r}}{(1-t^3y)(1-t^3y^{-1})} \quad i_{\chi(r)}(t,y) = -\frac{t^{3(2-r)}}{(1-t^3y)(1-t^3y^{-1})}\,, \end{aligned} \tag{6.22}$$

where for future convenience we have split the index of the matter multiplet into a chiral and an antichiral contribution. Let us write down explicit expressions for the index in some examples

- $Y^{1,0}\,(T^{1,1})$

For the conifold gauge theory, $U(1)_s$ is enhanced to $SU(2)_s$ so the global symmetry is $SU(2)_s \times SU(2)_l$. Assigning the chemical potentials a and b, for the two $SU(2)$s, the single letter index matrix $i_{1,0}(t,y)$ is

$$i_{1,0} = \begin{pmatrix} i_V & (a+\frac{1}{a})(i_{\chi(\frac{1}{2})} + i_{\bar{\chi}(\frac{1}{2})}) \\ (b+\frac{1}{b})(i_{\chi(\frac{1}{2})} + i_{\bar{\chi}(\frac{1}{2})}) & i_V \end{pmatrix}, \tag{6.23}$$

and the single-trace index evaluates to

$$\begin{aligned} \mathcal{I}_{s.t.} &= -\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log[(1-i_V(x^k))^2 - (a+\frac{1}{a})(b+\frac{1}{b})(i_{\chi(\frac{1}{2})}(x^k) + i_{\bar{\chi}(\frac{1}{2})}(x^k))^2] - 2i_V(x) \\ &= \frac{t^3 ab}{1-t^3ab} + \frac{t^3\frac{a}{b}}{1-t^3\frac{a}{b}} + \frac{t^3\frac{b}{a}}{1-t^3\frac{b}{a}} + \frac{t^3\frac{1}{ab}}{1-t^3\frac{1}{ab}}\,. \end{aligned} \tag{6.24}$$

This is the index for the theory where *both* the overall and the relative $U(1)$ degrees of freedom have been removed. The overall $U(1)$ is completely decoupled, while the relative $U(1)$ has positive beta function and decouples in the IR. The removal of the relative $U(1)$ introduces certain double-trace terms in the superpotential which are important to achieve exact conformality [124]. We have used the following property of Euler Phi function.

$$\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log(1-x^k) = \frac{-x}{1-x}. \tag{6.25}$$

We will match the expression (6.24) to the gravity computation.

• $Y^{1,1}$ ($\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$)

The index for this theory was already obtained in [42, 97]. The single letter index matrix $i_{1,1}(t,y)$ is given by

$$i_{1,1} = \begin{pmatrix} i_V + a^{-1} i_{\chi(\frac{2}{3})} + a i_{\bar{\chi}(\frac{2}{3})} & (b+\frac{1}{b})(a i_{\chi(\frac{2}{3})} + i_{\bar{\chi}(\frac{2}{3})}) \\ (b+\frac{1}{b})(i_{\chi(\frac{2}{3})} + a^{-1} i_{\bar{\chi}(\frac{2}{3})}) & i_V + a^{-1} i_{\chi(\frac{2}{3})} + a\, i_{\bar{\chi}(\frac{2}{3})} \end{pmatrix}, \tag{6.26}$$

and the index evaluates to

$$\begin{aligned} \mathcal{I}_{s.t.} =& -\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log[(1 - i_V(x^k) - a^{-1} i_{\chi(\frac{2}{3})}(x^k) - a\, i_{\bar{\chi}(\frac{2}{3})}(x^k))^2 - (b+\frac{1}{b})^2 \frac{1}{a} (a\, i_{\chi(\frac{2}{3})}(x^k) + i_{\bar{\chi}(\frac{2}{3})}(x^k))^2] \\ & - 2(i_V(x) + a^{-1}\, i_{\chi(\frac{2}{3})}(x) + a\, i_{\bar{\chi}(\frac{2}{3})}(x)) \\ =& 2\frac{t^2 a}{1-t^2 a} + \frac{t^4 b^2 a^{-1}}{1-t^4 b^2 a^{-1}} + \frac{t^4 b^{-2} a^{-1}}{1-t^4 b^{-2} a^{-1}} - 2\frac{a\, t^2 - a^{-1}\, t^4}{(1-t^3 y)(1-t^3 y^{-1})}. \end{aligned} \tag{6.27}$$

Again, we have subtracted both the overall and relative $U(1)$ degrees of freedom (in this case it is appropriate to subtract $\mathcal{N}=2$ vector multiplets).

• General $Y^{p,q}$

A simple generalization gives the index for $Y^{p,0}$ ($T^{1,1}/\mathbb{Z}_p$) and for $Y^{p,p}$ ($\mathbb{C}^3/\mathbb{Z}_{2p}$),

$$\begin{aligned} Y^{p,p} \quad &: \qquad \det(1 - i(t)) = \frac{(1-t^{4p})^2(1-t^{2p})^2}{(1-t^3 y)^{2p}(1-t^3 y^{-1})^{2p}}, \\ Y^{p,0} \quad &: \qquad \det(1 - i(t)) = \frac{(1-t^{3p})^4}{(1-t^3 y)^{2p}(1-t^3 y^{-1})^{2p}}. \end{aligned} \tag{6.28}$$

In fact the determinant of the adjacency matrix appears to factorize for general $Y^{p,q}$, to give[2]

$$\det(1 - i(t)) = \frac{\left[1 - t^{3p\left(1+\frac{1}{2}(x_{p,q}-y_{p,q})\right)}\right]^2 \left[1 - t^{3p+\frac{3q}{2}\left(1-\frac{1}{2}(x_{p,q}+y_{p,q})\right)}\right]^2}{(1-t^3 y)^{2p}(1-t^3 y^{-1})^{2p}}. \tag{6.29}$$

[2] We have checked this result in several cases but have not attempted an analytic proof.

Thus the single-trace partition function is[3]

$$\mathcal{I}^{s.p.}_{p,q} = 2\left[\frac{t^{\frac{p(3q+2p-\sqrt{4p^2-3q^2})}{q}}}{1-t^{\frac{p(2p+3q-\sqrt{4p^2-3q^2})}{q}}} + \frac{t^{\frac{p(3q-2p+\sqrt{4p^2-3q^2})}{q}}}{1-t^{\frac{p(2p+3q-\sqrt{4p^2-3q^2})}{q}}}\right]$$

Again, this is the result with all $U(1)$ factors subtracted. If one introduces a chemical potential b^{2l} for the global $SU(2)_l$ and a chemical potential a^{2s} for the global $U(1)_s$ of table 6.3 the index becomes

$$\begin{aligned}\mathcal{I}^{s.p.}_{p,q} =& \frac{a^{-p}b^{p+q}t^{\frac{p(3q+2p-\sqrt{4p^2-3q^2})}{q}}}{1-a^{-p}b^{p+q}t^{\frac{p(2p+3q-\sqrt{4p^2-3q^2})}{q}}} + \frac{a^{-p}b^{-p-q}t^{\frac{p(3q+2p-\sqrt{4p^2-3q^2})}{q}}}{1-a^{-p}b^{-p-q}t^{\frac{p(2p+3q-\sqrt{4p^2-3q^2})}{q}}} \\ &+ \frac{a^{p}b^{p-q}t^{\frac{p(3q-2p+\sqrt{4p^2-3q^2})}{q}}}{1-a^{p}b^{p-q}t^{\frac{p(3q-2p+\sqrt{4p^2-3q^2})}{q}}} + \frac{a^{p}b^{q-p}t^{\frac{p(3q-2p+\sqrt{4p^2-3q^2})}{q}}}{1-a^{p}b^{q-p}t^{\frac{p(3q-2p+\sqrt{4p^2-3q^2})}{q}}}.\end{aligned} \tag{6.30}$$

This is the left-handed index. The right-handed index is obtained by letting $a \to 1/a$.

6.3 $T^{1,1}$ Index from Supergravity

On the dual supergravity side, the index of the conifold theory was computed by Nakayama [32], using the results of [128–130] for the KK reduction of IIB supergravity on $AdS_5 \times T^{1,1}$.

Let us briefly review the structure of the calculation. For a general $AdS_5 \times Y^{p,q}$ background, the KK spectrum organizes itself in three types of multiplets [129, 130]: graviton $((\frac{1}{2},\frac{1}{2}))$, LH-gravitino $((\frac{1}{2},0))$, RH-gravitino $((0,\frac{1}{2}))$, and vector $((0,0))$. The details of the specific background manifest themselves in the possible spectrum of the R-charges and their multiplicities. This information can be obtained by solving the spectrum of relevant differential operators, *e.g.* scalar Laplacian and Dirac operators. For the $Y^{p,q}$ geometries the scalar Laplacian is given by Heun's differential equation spectrum of which is hard to obtain in closed form, see *e.g.* [126]. For the $T^{1,1}$ background these data were carefully computed in [128–130]. A generic multiplet of the KK spectrum does not obey shortening conditions and thus does not contribute to the index. Table 6.1 summarizes the multiplets which do contribute of the index. The eigenvalue of the scalar laplacian is denoted by $H_0(s,l,r)$,

$$H_0(s,l,r) = 6(s(s+1) + l(l+1) - \frac{r^2}{8}). \tag{6.31}$$

[3] Curiously, this is exactly twice the index of the chiral mesons denoted $\mathcal{L}_+$ (first term) and $\mathcal{L}_-$ (second term) in [125]. We don't have a deeper understanding of this observation. On the gravity sides, the chiral mesons of $\mathcal{L}_{+/-}$ were identified in [126] (see also [127] with the zero modes of the scalar Laplacian on the $Y^{p,q}$ manifold.).

Fields	Shortening Cond.	s	l	Mult.	$\mathcal{I}^{\mathtt{L}}(t,y)$	$\sum_{\tilde r}(\mathcal{I}^{\mathtt{L}}_{[\tilde r, j_2]^{\mathtt{L}}_+}\times\dots)$
Graviton	$E=1+\sqrt{H_0+4}$	$\frac{r}{2}$	$\frac{r}{2}$	$\mathcal{C}_{r(\frac12,\frac12)}$	$\mathcal{I}^{\mathtt{L}}_{[r+1,\frac12]^{\mathtt{L}}_-}$	$-\chi_{\frac{\tilde r-1}{2}}(a)\chi_{\frac{\tilde r-1}{2}}(b)$
Gravitino$_{\mathrm{I}}$	$E=-\frac12+\sqrt{H_0^-+4}$	$\frac{r-1}{2}$	$\frac{r-1}{2}$	$\mathcal{B}_{r(0,\frac12)}$	$\mathcal{I}^{\mathtt{L}}_{[r-2,\frac12]^{\mathtt{L}}_-}$	$-\chi_{\frac{\tilde r+1}{2}}(a)\chi_{\frac{\tilde r+1}{2}}(b)$
		$\frac{r-1}{2}$	$\frac{r+1}{2}$	$\mathcal{C}_{r(0,\frac12)}$	$\mathcal{I}^{\mathtt{L}}_{[r,\frac12]^{\mathtt{L}}_+}$	$+\chi_{\frac{\tilde r-1}{2}}(a)\chi_{\frac{\tilde r+1}{2}}(b)$
		$\frac{r+1}{2}$	$\frac{r-1}{2}$	$\mathcal{C}_{r(0,\frac12)}$	$\mathcal{I}^{\mathtt{L}}_{[r,\frac12]^{\mathtt{L}}_+}$	$+\chi_{\frac{\tilde r+1}{2}}(a)\chi_{\frac{\tilde r-1}{2}}(b)$
Gravitino$_{\mathrm{III}}$	$E=-\frac12+\sqrt{H_0^++4}$	$\frac{r+1}{2}$	$\frac{r+1}{2}$	$\mathcal{C}_{r(\frac12,0)}$	$\mathcal{I}^{\mathtt{L}}_{[r+1,0]^{\mathtt{L}}_-}$	$-\chi_{\frac{\tilde r}{2}}(a)\chi_{\frac{\tilde r}{2}}(b)$
Gravitino$_{\mathrm{IV}}$	$E=\frac52+\sqrt{H_0^-+4}$	$\frac{r-1}{2}$	$\frac{r-1}{2}$	$\mathcal{C}_{r(\frac12,0)}$	$\mathcal{I}^{\mathtt{L}}_{[r+1,0]^{\mathtt{L}}_-}$	$-\chi_{\frac{\tilde r}{2}-1}(a)\chi_{\frac{\tilde r}{2}-1}(b)$
Vector$_{\mathrm{I}}$	$E=-2+\sqrt{H_0+4}$	$\frac{r}{2}$	$\frac{r}{2}$	$\mathcal{B}_{r(0,0)}$	$\mathcal{I}^{\mathtt{L}}_{[r-2,0]^{\mathtt{L}}_-}$	$-\chi_{\frac{\tilde r}{2}+1}(a)\chi_{\frac{\tilde r}{2}+1}(b)$
		$\frac{r}{2}$	$\frac{r+2}{2}$	$\mathcal{C}_{r(0,0)}$	$\mathcal{I}^{\mathtt{L}}_{[r,0]^{\mathtt{L}}_+}$	$+\chi_{\frac{\tilde r}{2}}(a)\chi_{\frac{\tilde r}{2}+1}(b)$
		$\frac{r+2}{2}$	$\frac{r}{2}$	$\mathcal{C}_{r(0,0)}$	$\mathcal{I}^{\mathtt{L}}_{[r,0]^{\mathtt{L}}_+}$	$+\chi_{\frac{\tilde r}{2}+1}(a)\chi_{\frac{\tilde r}{2}}(b)$
Vector$_{\mathrm{IV}}$	$E=1+\sqrt{H_0^{--}+4}$	$\frac{r-2}{2}$	$\frac{r-2}{2}$	$\mathcal{B}_{r(0,0)}$	$\mathcal{I}^{\mathtt{L}}_{[r-2,0]^{\mathtt{L}}_-}$	$-\chi_{\frac{\tilde r}{2}}(a)\chi_{\frac{\tilde r}{2}}(b)$
		$\frac{r-2}{2}$	$\frac{r}{2}$	$\mathcal{C}_{r(0,0)}$	$\mathcal{I}^{\mathtt{L}}_{[r,0]^{\mathtt{L}}_+}$	$+\chi_{\frac{\tilde r}{2}-1}(a)\chi_{\frac{\tilde r}{2}}(b)$
		$\frac{r}{2}$	$\frac{r-2}{2}$	$\mathcal{C}_{r(0,0)}$	$\mathcal{I}^{\mathtt{L}}_{[r,0]^{\mathtt{L}}_+}$	$+\chi_{\frac{\tilde r}{2}}(a)\chi_{\frac{\tilde r}{2}-1}(b)$

Table 6.1: Short multiplets appearing in the KK reduction of Type IIB supergravity on $AdS_5\times T^{1,1}$. In the last column, we summarize the full index contributions of multiplets by listing the $SU(2)_s\times SU(2)_l$ characters multiplying $\mathcal{I}^{\mathtt{L}}_{[\tilde r,\frac12]^{\mathtt{L}}_+}$ for first four rows and $\mathcal{I}^{\mathtt{L}}_{[\tilde r,0]^{\mathtt{L}}_+}$ for remaining rows. The range of $\tilde r$ is specified by the two conditions that $\tilde r\geq -1$ *and* that the $SU(2)_s\times SU(2)_l$ representation makes sense. The chemical potentials a and b couple to $SU(2)_s\times SU(2)_l$ flavor charges respectively. Exception: The first row of Gravitino$_{\mathrm{I}}$ starts from $\tilde r=0$. The $\tilde r=-1$ state of Gravitino$_{\mathrm{I}}$ gives rise to the Dirac multiplet $\mathcal{D}_{(0,\frac12)}$ due to additional shortening. It corresponds in the dual field theory to a decoupled $U(1)$ vector multiplet.

$H_0^{\pm}$ and $H_0^{\pm\pm}$ are shorthands for $H_0(s,l,r\pm1)$ and $H_0(s,l,r\pm2)$ respectively. Besides the KK modes of table 6.1, there are additional Betti multiplets, arising from the non-trivial homology of $T^{1,1}$. Their contribution to the index is found to vanish [32].

The $T^{1,1}$ manifold has $SU(2)_s\times SU(2)_l$ isometry. We refine the index by adding chemical potentials a and b that couple respectively to $SU(2)_s$ and $SU(2)_l$. Simply reading off the R-charges and the multiplicities of the different modes, we can write down the index as [32][4]

$$\begin{aligned}\mathcal{I}^{\mathtt{L}} = &-\sum_{\tilde r\geq 0}\mathcal{I}^{\mathtt{L}}_{[\tilde r,\frac{1}{2}]^{\mathtt{L}}_{+}}[(ab)^{\tilde r+1}+(\frac{a}{b})^{\tilde r+1}+(\frac{b}{a})^{\tilde r+1}+(\frac{1}{ab})^{\tilde r+1}]\\ &-\sum_{\tilde r\geq -1}\mathcal{I}^{\mathtt{L}}_{[\tilde r,0]^{\mathtt{L}}_{+}}[(ab)^{\tilde r}+(\frac{a}{b})^{\tilde r}+(\frac{b}{a})^{\tilde r}+(\frac{1}{ab})^{\tilde r}+(ab)^{\tilde r+2}+(\frac{a}{b})^{\tilde r+2}+(\frac{b}{a})^{\tilde r+2}+(\frac{1}{ab})^{\tilde r+2}]\\ &+\mathcal{I}^{\mathtt{L}}_{[-1,0]^{\mathtt{L}}_{+}}\chi_{-\frac{3}{2}}(a)\chi_{-\frac{3}{2}}(b)-\mathcal{I}^{\mathtt{L}}_{[0,0]^{\mathtt{L}}_{+}}[-\chi_{-1}(a)\chi_{-1}(b)+\chi_{-1}(a)\chi_{0}(b)+\chi_{0}(a)\chi_{-1}(b)]\end{aligned}\tag{6.32}$$

The definition of the index building blocks $\mathcal{I}^{\mathtt{L}}_{[\tilde r,j_2]^{\mathtt{L}}_{\pm}}$ is given in appendix D, while the symbol $\chi_j(x)$ stands for the standard character of the spin-j representation of $SU(2)$,

$$\chi_j(x)\equiv\frac{x^{2j+1}-x^{-(2j+1)}}{x-x^{-1}}\,.\tag{6.33}$$

After simplification,

$$\mathcal{I}^{\mathtt{L}}=\frac{t^3ab}{1-t^3ab}+\frac{t^3\frac{a}{b}}{1-t^3\frac{a}{b}}+\frac{t^3\frac{b}{a}}{1-t^3\frac{b}{a}}+\frac{t^3\frac{1}{ab}}{1-t^3\frac{1}{ab}}\,,\tag{6.34}$$

which precisely agrees with the large N index (6.24) computed from gauge theory using Römelsberger's prescription.

[4]On the field theory side, we subtracted both $U(1)$ factors. Correspondingly, on the gravity side we should subtract all singleton degrees of freedom, and thus omit the $\tilde r=-1$ mode of the Gravitino$_{\mathrm{I}}$ tower, which corresponds to a $\mathcal{D}_{(0,1/2)}$ multiplet.

Chapter 7

Discussion

In this thesis we have discussed an important observable related to the spectrum of superconformal field theories - the superconformal index. Computed exactly by localization technique, the superconformal index captures of the information about the protected spectrum of the theory.

We have defined and studied several limits of the $\mathcal{N} = 2$ superconformal index. They are characterized by enhanced supersymmetry and depend at most on two superconformal fugacities, out of the possible three. We have given a prescription to calculate these limits for all A-series superconformal quivers of class $\mathcal{S}$, even when they lack a Lagrangian description. Thanks to the topological QFT structure of the index, it suffices to find a formula for the elementary three-valent building blocks. For the $SU(2)$ quivers, which do have a Lagrangian description, the building blocks can be written in terms of algebraic objects that admit a natural extrapolation to higher rank, leading to a compelling general conjecture that passes many tests. These objects are the Macdonald polynomials, tailor-made for our purposes as they depend on two fugacities, and for which a beautiful general theory is already available. We expect the generalization of our results to the D-series quivers of class $\mathcal{S}$ (and possibly to the E-series as well) to be straightforward.

The TQFT that calculates the index of the A_{k-1} quivers is closely related to two-dimensional Yang-Mills theory with gauge group $SU(k)$. An immediate qualitative hint, of course, is that the state-space of the index TQFT is the space of irreducible $SU(k)$ representations. As first discussed in [17], and confirmed here in more generality, there is in fact a precise quantitative correspondence between the limit of the index that we have dubbed the "Schur index", which depends on a single fugacity q, and correlators of q-deformed $2d$ Yang-Mills theory [101] in the zero-area limit. In turn, the zero-area limit of q-deformed $2d$ Yang-Mills on the Riemann surface $\mathcal{C}$ can be viewed as an analytic continuation of Chern-Simons theory on $\mathcal{C} \times S^1$ [101].

Recently, a "refinement" of Chern-Simons theory on three-manifolds admitting a circle action was defined in [107], via the relation with topological string theory and its embedding into M-theory. Taking the three-manifold to be of the form $\mathcal{C} \times S^1$, and reducing on the S^1, one obtains an indirect definition of "refined q-deformed Yang-Mills theory" on $\mathcal{C}$, which depends on two parameters q and t. (The definition is indirect because unlike the purely q-deformed case no Lagrangian description is available for the refined theory.) The refinement essentially amounts to trading Schur polynomials with Macdonald polynomials, and we have found a precise relation between our (q,t) "Macdonald index" and correlators of this (q,t)-Yang-Mills theory. It is natural to ask whether this is pointing to a direct connection between topological string theory and the superconformal index. At first sight the geometries involved appear to be quite different, since to obtain the superconformal index we must consider the $(2,0)$ theory on $S^3 \times S^1 \times \mathcal{C}$, with appropriate twists induced by the fugacities, while in the setup of [6, 107] the relevant geometry is $(\mathbb{C} \times S^1 \times M_3)_{q,t}$, where one may take $M_3 = S^1 \times \mathcal{C}$ (we refer to the cited papers for a proper explanation). Moreover while the index admits a further refinement for a total of three fugacities, it seems difficult to introduce a third parameter in the framework of [6, 107] while preserving supersymmetry. Nevertheless, at least for the special case of the Macdonald index, there should be a deeper way to understand the striking similarity of the two results.

An obvious direction for future work is the generalization of our results to the full three-parameter index. The Haar measure together with the index of the $\mathcal{N} = 2$ vector multiplet combine to [16, 21]

$$\frac{1}{k!} \prod_{i,j=1, i\neq j}^{k} \frac{1}{\Gamma\left(x_i/x_j;\ q,p\right) \Gamma\left(t\, x_i/x_j;\ q,p\right)}\,, \tag{7.1}$$

where $\Gamma(z;p,q)$ is the elliptic Gamma function

$$\Gamma(z;p,q) = \prod_{i,j=0}^{\infty} \frac{1-p^{i+1}q^{j+1}/z}{1-p^i q^j z}\,. \tag{7.2}$$

A natural speculation is that the functions $f^{\lambda}_{p,q,t}(\mathbf{a})$ that diagonalize the structure constants of the full index should be proportional to elliptic extensions of the Macdonald polynomials, to which they should reduce in the limit $p \to 0$ (or $q \to 0$). Various proposals for elliptic Macdonald functions have appeared in the mathematical literature, see *e.g.* [83, 131, 132]. We can in fact formulate a more precise conjecture, motivated by the relation between two-dimensional gauge theories and integrable quantum mechanical models of Calogero-Moser (CM) type, see *e.g.* [133–138]. The reduction of ordinary $2d$ Yang-Mills theory to one dimension yields the rational (non-relativistic) CM model [133]. One can consider the trigonometric and elliptic generalizations of the non-relativistic model, as well as their relativistic cousins (the relativistic versions are also known as Ruijsenaars-Schneider (RS) models). The relativistic trigonometric model (trigonometric RS) depends on two parameters (q,t), has Macdonald polynomials as its eigenfunctions, and is closely related to the two-dimensional G/G WZW model[1] or equivalently to Chern-Simons theory on $\mathcal{C} \times S^1$. At the summit of this hierarchy is the elliptic relativistic model (elliptic RS), which depends on three parameters, analogous to (p, q, t) of the full index. Our conjecture is then that the symmetric functions relevant for the computation of the full index are the eigenfunctions of the elliptic RS model. Not too much is known about them, see [140] for a review.[2]

Perhaps the most interesting open problem is to give a "microscopic" derivation of the two-dimensional TQFT of the index from the six-dimensional $(2, 0)$ theory. A promising shortcut, which exploits the mentioned connection between $2d$ gauge theories and $1d$ Calogero-Moser models, is along the following lines. Consider the $(2, 0)$ theory on $S^3 \times S^1_{(1)} \times \mathcal{C}_{\mathfrak{g},s}$. The Riemann surface $\mathcal{C}_{\mathfrak{g},s}$ can be viewed as a circle, $S^1_{(2)}$, times a graph $I_{\mathfrak{g},s}$ By first reducing the $(2, 0)$ theory on $S^1_{(2)}$ (note that there is no twist around this circle) one obtains $5d$ super Yang-Mills on $S^3 \times S^1_{(1)} \times I_{\mathfrak{g},s}$. We propose that the further reduction of $5d$ SYM on $S^1_{(1)} \times S^3$ (with the fugacity twists) yields the elliptic RS model on the graph $I_{\mathfrak{g},s}$, with appropriate boundary conditions at the s external punctures and at the internal junctures. In a suitable limit, which corresponds to taking $S^1_{(1)}$ to be small, the $4d$ index becomes the $3d$ partition function [103–105], and our proposal reduces to the one of [114] (see also [113]). These authors show how to interpret such $3d$ partition functions as overlaps of quantum mechanical wave functions. We are suggesting that a similar idea may apply to the $4d$ index, and that the relevant quantum mechanical model is the elliptic RS model. More work is going to be done along these lines.

[1] See [137] for a review and [139] for recent relevant work.

[2] Quantum mechanical integrable models have been recently related to the problem of counting vacua of $\mathcal{N} = 2$ supersymmetric theories in the Ω-background [141, 142]. See also [143, 144] for connections of elliptic Gamma functions to integrable systems.

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Appendix A

Duality

A.1 The Representation Basis

The labels of the topological algebra as we have defined in (3.6) are (compact) continuous parameters $\alpha_i \in [0, 2\pi)$. We can "Fourier" transform to the discrete basis of irreducible $SU(2)$ representations. We denote by R_K the irreducible representation of $SU(2)$ of dimension $K + 1$. The integrals over characters translate into sums over representations. The structure constants in the discrete basis are given by

$$\begin{aligned} C_{\alpha\beta\gamma} &= \sum_{K,L,M=0}^{\infty} \frac{\sin(K+1)\alpha}{\sin\alpha}\,\frac{\sin(L+1)\beta}{\sin\beta}\,\frac{\sin(M+1)\gamma}{\sin\gamma}\,\hat{C}_{KLM} \qquad \text{(A.1)} \\ &= \sum_{K,L,M=0}^{\infty} \chi_K(\alpha)\chi_L(\beta)\chi_M(\gamma)\,\hat{C}_{KLM}, \end{aligned}$$

where $\chi_K(\alpha)$ is the character of R_K,

$$\chi_K(\alpha) = \frac{\sin(K+1)\,\alpha}{\sin\alpha}\,. \qquad \text{(A.2)}$$

Similarly the metric in the discrete basis is given by

$$\eta^{\alpha\beta} = \sum_{K,\,L\,=\,0}^{\infty} \chi_K(\alpha)\chi_L(\beta)\,\hat{\eta}^{KL}\,. \qquad \text{(A.3)}$$

Further, we define the scalar product of characters[1]

$$
\begin{aligned}
\langle \chi_K \, \chi_M \rangle &= \frac{1}{2\pi i} \oint \frac{dz}{z} \, (1 - z^2) \, \chi_K(z) \, \chi_M(z) \\
&= -\frac{1}{4\pi i} \oint \frac{dz}{z} \, (z - \frac{1}{z})^2 \, \chi_K(z) \, \chi_M(z) = \int_0^{2\pi} d\theta \Delta(\theta) \chi_K(\theta) \chi_M(\theta) = \delta_{K,M}
\end{aligned} \tag{A.4}
$$

In the second equality we have introduced the measure (3.5) and used the fact that $\chi(z) = \chi(z^{-1})$. Thus we have

$$
\sum_{K=0}^{\infty} \chi_K(\alpha) \, \chi_K(\beta) = \hat{\delta}(\alpha, \beta), \qquad \int_0^{2\pi} d\theta \Delta(\theta) \, \hat{\delta}(\theta, \alpha) \, f(\theta) = f(\alpha), \tag{A.5}
$$

for any f obeying $f(\theta) = f(-\theta)$. Using (3.6) we can write

$$
\hat{\eta}^{KL} = \eta^I \langle \chi^I \chi^K \chi^L \rangle, \qquad \eta^I = \int d\alpha \, \Delta(\alpha) \, \eta^\alpha \, \chi_I(\alpha). \tag{A.6}
$$

Finally with the help of these definitions, we can rewrite (3.8) as

$$
\mathcal{I} = \prod_{\{i,j,k\} \in \mathcal{V}} \hat{C}_{L_i L_j L_k} \prod_{\{m,n\} \in \mathcal{G}} \hat{\eta}^{L_m L_n} \,, \tag{A.7}
$$

where index contractions now indicate sums over the non-negative integers.

A.2 TQFT Algebra for $v = t$

For $v = t$ we can rewrite the algebra of the topological quantum field theory (3.6) in a more elegant way, removing the delta-functions by making use of identities obeyed by elliptic Beta integrals. This does not appear to be a preferred limit physically, except for the fact that the contribution to the index of the chiral superfield in the $\mathcal{N} = 2$ vector multiplet vanishes, see (2.13). Our manipulations will be slightly formal since the limit $v = t$ of the formulae we will use is somewhat singular. We start by quoting the important identity

$$
E^{(m=0)}(t_1, \ldots, t_6) = \kappa \oint \frac{dz}{z} \frac{\prod_{k=1}^{6} \Gamma\left(t_k \, z^{\pm 1}; p, q\right)}{\Gamma\left(z^{\pm 2}; p, q\right)} = \prod_{1 \le j < k \le 6} \Gamma\left(t_j \, t_k; p, q\right), \qquad \prod_{k=1}^{6} t_k = pq \tag{A.8}
$$

[1] We have a slightly different convention for the characters and thus the expression of the scalar product differs from the one in [145].

This is a vast generalization to elliptic Gamma functions of that seminal object in string theory, the classic Beta integral of Euler,

$$B(\alpha,\beta)=\int_0^1 dt\, t^{\alpha-1}(1-t)^{\beta-1}=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,, \tag{A.9}$$

which is recovered as a special limit, see *e.g.* [54]. Applying (A.8) we have

$$\begin{aligned}\kappa\oint\frac{dz}{z}\frac{\Gamma\left(\tau\sqrt{\nu}\,a^{\pm1}b^{\pm1}z^{\pm1}\right)\,\Gamma\left(\frac{\tau}{\nu}\,z^{\pm1}y^{\pm1}\right)}{\Gamma\left(z^{\pm2}\right)}&= \\ \Gamma\left(\frac{\tau^2}{\sqrt{\nu}}\,a^{\pm1}b^{\pm1}y^{\pm1}\right)\,\Gamma\left(\tau^2\nu\,a^{\pm2}\right)\Gamma\left(\tau^2\nu\,b^{\pm2}\right)\,&\Gamma\left(\frac{\tau^2}{\nu^2}\right)\,\Gamma\left(\tau^2\,\nu\right)^2\,.\end{aligned} \tag{A.10}$$

For brevity we have omitted the p and q parameters in the Gamma functions. We assume $pq=\tau^6$. For these values of p and q, $\Gamma(\tau^3z^{\pm1})=1$. Now if we take $\nu=\tau$,

$$\kappa\oint\frac{dz}{z}\frac{\Gamma\left(\tau^{3/2}\,a^{\pm1}b^{\pm1}z^{\pm1}\right)\,\Gamma\left(z^{\pm1}y^{\pm1}\right)}{\Gamma\left(z^{\pm2}\right)}=\Gamma\left(\tau^{3/2}\,a^{\pm1}b^{\pm1}y^{\pm1}\right)\,\Gamma\left(1\right)\,. \tag{A.11}$$

Strictly speaking the elliptic Beta integral formula (A.8) holds when $|t_k|<1$ for all $k=1\ldots6$. For $\nu=\tau$ some of the t_ks in (A.10) saturate this bound. The elliptic Beta integral (A.10) is proportional to $\Gamma(\frac{\tau^2}{\nu^2};p,q)\to\Gamma(1;p,q)$. Since the elliptic Gamma function has a simple pole when its argument approaches $z=1$ (see (2.32)), (A.10) diverges in the limit. We will proceed by keeping formal factors of $\Gamma(1)$ in all the expressions. Thanks to (A.11), the expression

$$\frac{\Gamma(z^{\pm1}y^{\pm1})}{\Gamma(z^{\pm2})\Gamma(1)}\equiv\delta_y^z \tag{A.12}$$

acts as a formal identity operator. All factors of $\Gamma(1)$ will cancel in the final expression for the index.

For $t=v$ we can write the building blocks of the topological algebra in the form summarized in table A.1. Contraction of indices is defined as

$$A^{..a..}\,B_{..a..}\to\kappa\oint\frac{d\,a}{a}\,A^{..a..}\,B_{..a..}. \tag{A.13}$$

We now proceed to perform a few sample calculations and consistency checks.

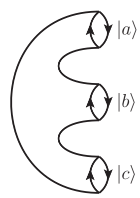
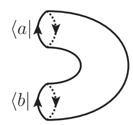
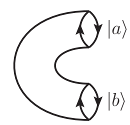
Symbol	Surface	Value	Symbol	Surface	Value
C_{abc}		$\Gamma(t^{\frac{3}{2}}a^{\pm1}b^{\pm1}c^{\pm1})$	V^a		$\frac{1}{\Gamma(1)^2}\,\frac{\Gamma(t^{\pm\frac{3}{2}}a^{\pm1})}{\Gamma(a^{\pm2})}$
η^{ab}		$\frac{1}{\Gamma(1)}\,\frac{\Gamma(a^{\pm1}b^{\pm1})}{\Gamma(a^{\pm2},b^{\pm2})}$	η_{ab}		$\frac{1}{\Gamma(1)}\Gamma\left(a^{\pm1}b^{\pm1}\right)$

Table A.1: The basic building blocks of the topological algebra in the $v = t$ case.

We can raise an index of the structure constants to obtain

$$C_{abe}\,\eta^{ec} = \frac{\kappa}{\Gamma(1)}\oint\frac{d\,e}{e}\Gamma(t^{\frac{3}{2}}a^{\pm1}b^{\pm1}e^{\pm1})\frac{\Gamma(e^{\pm1}c^{\pm1})}{\Gamma(e^{\pm2},c^{\pm2})} = \frac{\Gamma(t^{\frac{3}{2}}a^{\pm1}b^{\pm1}c^{\pm1})}{\Gamma(c^{\pm2})} = C_{ab}{}^{c}\,. \tag{A.14}$$

In particular we see that the index (3.20) is finite and is simply given by $C_{ab}{}^{c}\,C_{cde}$. The "vacuum state" $|V\rangle \equiv V^a|a\rangle$ satisfies by definition (see *e.g.* [82]) $C_{abc}\,V^c = \eta_{ab}$, as illustrated in figure A.1. This determines V^a to be the expression in table A.1,

$$C_{abc}\,V^c = \frac{\kappa}{\Gamma(1)^2}\oint\frac{dz}{z}\Gamma(t^{\frac{3}{2}}a^{\pm1}b^{\pm1}z^{\pm1})\,\frac{\Gamma(t^{\pm\frac{3}{2}}z^{\pm1})}{\Gamma(z^{\pm2})} = \frac{1}{\Gamma(1)}\Gamma(a^{\pm1}b^{\pm1}) = \eta_{ab}\,. \tag{A.15}$$

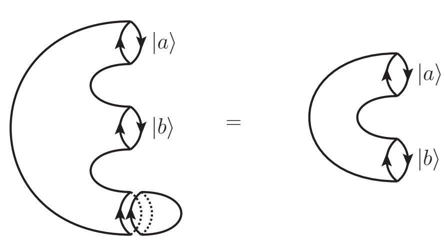


Figure A.1: Constructing the metric by capping off the trivalent vertex.

Further, we can check that η_{ab} and η^{ab} in table A.1 are one the inverse of the other,

$$\eta^{ae}\,\eta_{ec} = \frac{\kappa}{\Gamma(1)^2}\oint \frac{d\,e}{e}\,\frac{\Gamma(a^{\pm1}e^{\pm1})}{\Gamma(a^{\pm2},e^{\pm2})}\,\Gamma\left(e^{\pm1}c^{\pm1}\right) = \frac{1}{\Gamma(1)}\,\frac{\Gamma(a^{\pm1}c^{\pm1})}{\Gamma(a^{\pm2})} = \delta^a_c\,. \quad \text{(A.16)}$$

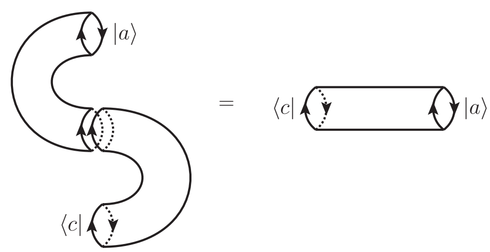


Figure A.2: Topological interpretation of the property $\eta^{ce}\,\eta_{ea} = \delta^c_a$.

As a consistency check one can verify in examples that δ^a_b is indeed an identity. For instance

$$\delta^z_a\,C_{zbc} = \frac{\kappa}{\Gamma(1)}\oint \frac{dz}{z}\,\frac{\Gamma(a^{\pm1}z^{\pm1})}{\Gamma(z^{\pm2})}\,\Gamma(t^{\frac{3}{2}}z^{\pm1}b^{\pm1}c^{\pm1}) = \Gamma(t^{\frac{3}{2}}a^{\pm1}b^{\pm1}c^{\pm1}) = C_{abc}\,, \quad \text{(A.17)}$$

as illustrated in figure A.3. For completeness we can also compute the sphere

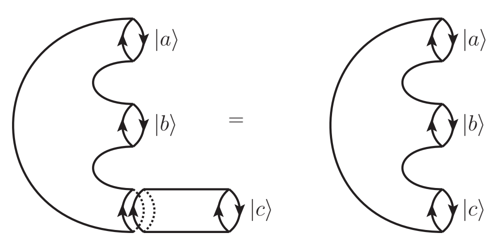


Figure A.3: The consistency requirement $\delta^z_c\,C_{abz} = C_{abc}$.

and the torus partition functions. (These partition functions do not appear in any index computation of a 4d superconformal theory so their physical interpretation is unclear.)

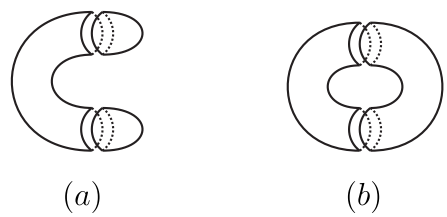


Figure A.4: The sphere **(a)** and the torus **(b)** partition functions.

The sphere partition function is given by

$$V^c\, V^e\, \eta_{ce} \;=\; \frac{\kappa^2}{\Gamma(1)^5}\oint \frac{de}{e}\oint \frac{dc}{c}\frac{\Gamma\left(c^{\pm1}e^{\pm1}\right)\Gamma\left(t^{\pm3/2}\,c^{\pm1}\right)\,\Gamma\left(t^{\pm3/2}\,e^{\pm1}\right)}{\Gamma\left(c^{\pm2}\right)\Gamma\left(e^{\pm2}\right)}$$
$$=\; \frac{\kappa}{\Gamma(1)^4}\oint \frac{de}{e}\frac{\Gamma\left(t^{\pm3/2}\,e^{\pm1}\right)^2}{\Gamma\left(e^{\pm2}\right)} = \Gamma(t^{-3})\,\frac{1}{\Gamma(1)}. \qquad \text{(A.18)}$$

The torus partition function is given by

$$\eta_{ab}\eta^{ab} = \frac{\kappa}{\Gamma(1)}\oint \frac{d\,a}{a}\frac{\Gamma(a^{\pm1}a^{\pm1})}{\Gamma(a^{\pm2})} = \kappa\,\Gamma(1)\oint \frac{d\,a}{a} = 2\,\pi\,i\,\kappa\,\Gamma(1). \qquad \text{(A.19)}$$

Since $\Gamma(1) = \infty$ the sphere partition function vanishes and the torus partition function diverges.

A.3 t expansion in the weakly-coupled frame

We expand the index (3.27) in t as

$$\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}} = \sum_{k=0}^{\infty} b_k\, t^k. \qquad \text{(A.20)}$$

The first few orders are

$$
\begin{aligned}
b_0 &= 1, \\
b_1 &= b_2 = b_3 = 0, \\
b_4 &= \frac{1}{v}\chi^{SU(6)}_{\mathbf{35},adj} + \frac{1}{v} + v^2, \\
b_5 &= -v\left(y + \frac{1}{y}\right), \\
b_6 &= \frac{1}{v^{3/2}}\chi^{SU(6)}_{\mathbf{20}}\left(\left(\frac{a}{b}\right)^{3/2} + \left(\frac{b}{a}\right)^{3/2}\right) - \chi^{SU(6)}_{\mathbf{35},adj} + v^3 - 1, \\
b_7 &= \frac{1}{v}\left(y + \frac{1}{y}\right)\chi^{SU(6)}_{\mathbf{35},adj} + \frac{2}{v}\left(y + \frac{1}{y}\right), \\
b_8 &= \frac{1}{v^2}\chi^{SU(6)}_{sym^2\mathbf{35}} + v\chi^{SU(6)}_{\mathbf{35},adj} - \frac{1}{\sqrt{v}}\chi^{SU(6)}_{\mathbf{20}}\left(\left(\frac{a}{b}\right)^{3/2} + \left(\frac{b}{a}\right)^{3/2}\right) + v^4 - v\left(y + \frac{1}{y}\right)^2 + 2v, \\
b_9 &= -2\left(y + \frac{1}{y}\right)\chi^{SU(6)}_{\mathbf{35},adj} + \frac{1}{v^{3/2}}\left(y + \frac{1}{y}\right)\chi^{SU(6)}_{\mathbf{20}}\left(\left(\frac{a}{b}\right)^{3/2} + \left(\frac{b}{a}\right)^{3/2}\right) - 2\left(y + \frac{1}{y}\right).
\end{aligned}
\tag{A.21}
$$

In the above equation we decomposed $SU(6) \supset SU(3)_z \otimes SU(3)_{y^{-1}} \otimes U(1)$. The branching of **35** and **20** of $SU(6)$ is given by (see [146]),

$$
\begin{aligned}
\mathbf{35} &= (\mathbf{1},\mathbf{1})_0 + (\mathbf{8},\mathbf{1})_0 + (\mathbf{1},\mathbf{8})_0 + (\bar{\mathbf{3}},\mathbf{3})_2 + (\mathbf{3},\bar{\mathbf{3}})_{-2}\,, \\
\mathbf{20} &= (\mathbf{1},\mathbf{1})_3 + (\mathbf{1},\mathbf{1})_{-3} + (\bar{\mathbf{3}},\mathbf{3})_{-1} + (\mathbf{3},\bar{\mathbf{3}})_1\,.
\end{aligned}
\tag{A.22}
$$

For example, the character of the adjoint is

$$
\begin{aligned}
\chi^{SU(6)}_{\mathbf{35},adj} &= \left[(a\,b)^{1/2}\,(z_1+z_2+z_3) + (a\,b)^{-1/2}\,(\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3})\right] \\
&\quad \times\left[(a\,b)^{-1/2}\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right) + (a\,b)^{1/2}\,(y_1+y_2+y_3)\right] - 1\,.
\end{aligned}
\tag{A.23}
$$

We conclude that the $U(1)$ charge in $SU(6)$ can be identified as $(a\,b)^{-1/2}$.

A.4 Inversion theorem

In this appendix we quote the inversion theorem [78], which we use in section 3.2.2 to obtain the index of the E_6 theory. Define

$$
\delta(z,\,w;T) \equiv \frac{\Gamma(T\,z^{\pm1}\,w^{\pm1};\,p,q)}{\Gamma(T^2,\,z^{\pm2};\,p,q)}. \tag{A.24}
$$

If T, p and q are such that

$$|\max(p,\, q)| < |T| < 1\,, \tag{A.25}$$

then the following theorem holds true. For fixed w on the unit circle we define a contour C_w (see figure A.5) in the annulus $\mathbb{A} = \{|T| - \epsilon < |z| < |T|^{-1} + \epsilon\}$ with small but finite $\epsilon \in \mathbb{R}^+$, such that the points $T^{-1} w^{\pm 1}$ are in its interior and $C_w = C_w^{-1}$ (i.e. an inverse of the point in the interior of C_w is in the exterior of C_w). Let $f(z) = f(z^{-1})$ be a holomorphic function in $\mathbb{A}$. Then for $|T| < |x| < |T|^{-1}$,

$$\hat{f}(w) = \kappa \oint_{C_w} \frac{dz}{2\pi i\, z}\, \delta(z, w; , T^{-1})\, f(z) \quad \longrightarrow \quad f(x) = \kappa \oint_{\mathbb{T}} \frac{dw}{2\pi i\, w}\, \delta(w, x; , T)\, \hat{f}(w)\,. \tag{A.26}$$

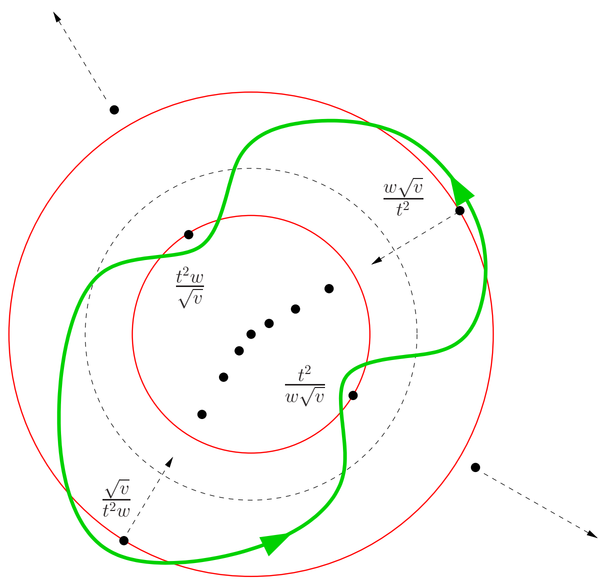


Figure A.5: The integration contour C_w (green). The dashed (black) circle is the unit circle $\mathbb{T}$. Black dots are poles of $\Gamma\left(\frac{\sqrt{v}}{t^2}\, w^{\pm 1}\, z^{\pm 1}\right)$. There are four sequences of poles: two sequences starting at $\frac{\sqrt{v}}{t^2} w^{\pm 1}$ and converging to $z = 0$, and two sequences starting at $\frac{t^2}{\sqrt{v}} w^{\pm 1}$ and converging to $z = \infty$. The contour encloses the two former sequences.

Our expression for the index in the strongly-coupled frame (3.37) is of the form of the right hand side of (A.26). Thus, to use the inversion theorem to obtain the index of E_6 theory we *assume* that this index can be written as

$$\Gamma(t^2 v\, w^{\pm 2})\, C^{(E_6)}\left((w,\, r), \mathbf{y}, \mathbf{z}\right) = \kappa \oint_{C_w} \frac{ds}{2\pi i\, s} \frac{\Gamma(\frac{\sqrt{v}}{t^2} w^{\pm 1}\, s^{\pm 1})}{\Gamma(\frac{v}{t^4},\; s^{\pm 2})}\, F\left(s, r; \mathbf{y}, \mathbf{z}\right)\,, \tag{A.27}$$

for some function F. The theorem (A.26) then implies that $F\left(s, r; \mathbf{y}, \mathbf{z}\right) = \hat{\mathcal{I}}\left(s, r; \mathbf{y}, \mathbf{z}\right)$ with $\mathcal{I}\left(s, r; \mathbf{y}, \mathbf{z}\right)$ given in (3.37).

A.5 The Coulomb and Higgs branch operators of E_6 SCFT

We collect here a few facts about the Coulomb and the Higgs branches of E_6 SCFT, following the analysis of [80]. Argyres-Seiberg duality can be used to determine the quantum numbers of protected operators of E_6 theory if their dual operators in the dual $SU(3)$ theory are known. The Coulomb branch operator u of the E_6 theory (the operator whose vev parametrized the Coulomb branch) is identified as $\mathrm{Tr}\,\phi^3$ in the $SU(3)$ theory. Since ϕ has quantum numbers $(E, j_1, j_2, R, r) = (1, 0, 0, 0, -1)$, u should have quantum numbers $(3, 0, 0, 0, -3)$ and contribute to the superconformal index as $t^6 v^3$.

The operator $\mathbb{X}$ whose vev parametrized the Higgs branch transforms in the adjoint representation of E_6. Under the $SU(2) \otimes SU(6)$ subgroup of E_6 it decomposes as

$$X^i_j, \quad Y^{[ijk]}_\alpha, \quad Z_{\alpha\beta}\,, \tag{A.28}$$

where $i, j, k = 1, \ldots, 6$ are the $SU(6)$ indices, and $\alpha, \beta = 1, 2$ are the $SU(2)$ indices. At the same time, the $SU(2)$ gauge theory provides the quarks q_α, $\tilde{q}_\alpha$ and the F-term constraint

$$Z_{\alpha\beta} + q_{(\alpha}\tilde{q}_{\beta)} = 0\,. \tag{A.29}$$

Thus the gauge-invariant operators are

$$(q\tilde{q}), \quad X^i_j, \quad (Y^{ijk} q), \quad (Y_{ijk}\tilde{q})\,. \tag{A.30}$$

On the $SU(3)$ side, the Higgs branch is parameterized by gauge invariant operators

$$M^i_j = Q^i_a \tilde{Q}^a_j, \quad B^{ijk} = \epsilon^{abc} Q^i_a Q^j_b Q^k_c, \quad \tilde{B}_{ijk} = \epsilon_{abc} \tilde{Q}^a_i \tilde{Q}^b_j \tilde{Q}^c_k\,, \tag{A.31}$$

where Q^i_a and $\tilde{Q}^a_i$ are the squark fields, $i = 1, \ldots, 6$ are flavor indices, and

$a = 1, 2, 3$ the color indices.

The duality of the two sides suggests the following identification

$$\mathrm{Tr} M \leftrightarrow (q\tilde{q}), \qquad \hat{M}^i_j \leftrightarrow X^i_j, \tag{A.32}$$

$$B^{ijk} \leftrightarrow (Y^{ijk} q), \qquad \tilde{B}_{ijk} \leftrightarrow (Y_{ijk}\tilde{q}) \tag{A.33}$$

where $\hat{M}^i_j$ is the traceless part of M^i_j. Since the quantum numbers of Q are $(1, 0, 0, 1/2, 0)$, the quantum numbers of $\mathbb{X}$ should be $(2, 0, 0, 1, 0)$, and contribute to the index as t^4/v.

A.6 Identities from S-duality

In this appendix we summarize identities of integrals of elliptic Gamma functions implied by S-duality of the $SU(3)$ quiver theories.

Generalization of [76]

We define

$$\mathcal{I}^{(n)}\left(a\,, \mathbf{z}_{SU(n)}; b\,, \mathbf{y}_{SU(n)}\right) \equiv \frac{2^{n-1}}{n!}\kappa^{n-1}\Gamma(t^2 v)^{n-1}\times$$
$$\oint_{\mathbb{T}^{n-1}} \prod_{i=1}^{n-1} \frac{dx_i}{2\pi i\, x_i} \frac{\prod_{i=1}^n \prod_{j=1}^n \Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{a z_i}{x_j}\right)^{\pm 1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\left(b\, y_i\, x_j\right)^{\pm 1}\right)\prod_{i\neq j}\Gamma\left(t^2 v\,\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}\Bigg|_{\prod_{j=1}^n x_j = 1}. \tag{A.34}$$

The claim is that

$$\mathcal{I}^{(n)}\left(a\,, \mathbf{z}_{SU(n)}; b\,, \mathbf{y}_{SU(n)}\right) = \mathcal{I}^{(n)}\left(b\,, \mathbf{z}_{SU(n)}; a\,, \mathbf{y}_{SU(n)}\right). \tag{A.35}$$

For $SU(2)$ this identity was proven in [76], and for $SU(3)$ we have performed perturbative checks. The usual S-duality of $N_f = 2n$ $SU(n)$ theories implies that this identity should be true for any n. Note that for $t = v$ this is a special case of identities discussed in [83].

E_6 Integral

We define

$$C^{(E_6)}\left((w,\,r),\mathbf{y},\mathbf{z}\right)\equiv\frac{2\,\kappa^3\Gamma(t^2v)^2}{3\,\Gamma(t^2v\,w^{\pm2})}\oint_{C_w}\frac{ds}{2\pi i\,s}\,\frac{\Gamma(\frac{\sqrt{v}}{t^2}w^{\pm1}\,s^{\pm1})}{\Gamma(\frac{v}{t^4},\;s^{\pm2})}\times$$
$$\times\oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i\,x_i}\frac{\prod_{i=1}^{3}\prod_{j=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{s^{\frac13}\,z_i}{x_j\,r}\right)^{\pm1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{s^{-\frac13}\,y_i\,x_j}{r}\right)^{\pm1}\right)\prod_{i\neq j}\Gamma\left(t^2v\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}\,. \tag{A.36}$$

This integral has manifest symmetry under $SU(2)_w\otimes SU(6)$, where the $SU(6)$ has been decomposed as $SU(3)_{\mathbf{z}}\otimes SU(3)_{\mathbf{y}^{-1}}\otimes U(1)_r$. The identification with the index of the E_6 SCFT implies that there must be a symmetry enhancement $SU(2)_w\otimes SU(6)\to E_6$. Two properties that are sufficient to guarantee E_6 covariance are: first,

$$C^{(E_6)}\left((w,\,r),\mathbf{y},\mathbf{z}\right)=C^{(E_6)}\left(\left(\frac{w^{1/2}}{r^{3/2}},\,\frac{1}{w^{1/2}\,r^{1/2}}\right),\mathbf{y},\mathbf{z}\right)\,, \tag{A.37}$$

which is the statement that (w,r) combine into a character of $SU(3)$ (which we shall denote by $\mathbf{w}$); second,

$$C^{(E_6)}(\mathbf{w},\mathbf{y},\mathbf{z})=C^{(E_6)}(\mathbf{y},\mathbf{w},\mathbf{z})\,. \tag{A.38}$$

We presented perturbative evidence for the full E_6 symmetry in the text.

S-dualities of $SU(3)$ quivers

Define

$$\mathcal{I}_{3333}\left(\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{s}\right)\equiv\oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i x_i}\prod_{i\neq j}\frac{\Gamma\left(t^2vx_i/x_j\right)}{\Gamma\left(x_i/x_j\right)}C^{(E_6)}\left(\mathbf{y},\mathbf{z},\mathbf{x}\right)C^{(E_6)}\left(\mathbf{u},\mathbf{s},\mathbf{x}^{-1}\right)\,,$$
$$\mathcal{I}_{3331}\left(\mathbf{y},\mathbf{z},\mathbf{u},a\right)\equiv\oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i x_i}\prod_{i\neq j}\frac{\Gamma\left(t^2vx_i/x_j\right)}{\Gamma\left(x_i/x_j\right)}C^{(E_6)}\left(\mathbf{y},\mathbf{z},\mathbf{x}\right)\prod_{i,j=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(a\,x_i^{-1}\,u_j\right)^{\pm}\right)\,. \tag{A.39}$$

The S-dualities of the $SU(3)$ quivers imply

$$\mathcal{I}_{3333}\left(\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{s}\right)=\mathcal{I}_{3333}\left(\mathbf{y},\mathbf{u},\mathbf{z},\mathbf{s}\right)\,, \tag{A.40}$$
$$\mathcal{I}_{3331}\left(\mathbf{y},\mathbf{z},\mathbf{u},a\right)=\mathcal{I}_{3331}\left(\mathbf{y},\mathbf{u},\mathbf{z},a\right)\,.$$

Appendix B

MacDonald

B.1 Construction of the diagonal expression for the $SU(2)$ HL index

In this appendix we diagonalize the structure constants of the $SU(2)$ quivers in the $\rho \to 0$, $\sigma \to 0$ limit. With hindsight, we have dubbed this limit the Hall-Littlewood (HL) index, since the diagonal functions turn out to be closely related to the Hall-Littlewood polynomials. This is precisely what we show in this appendix.

For $SU(2)$, the SCFT associated to three-punctured sphere is the the free hypermultiplet in the trifundamental representation. In the limit of interest, its index reads

$$\mathcal{I}(a,b,c) = PE\left[\tau\chi_1(a)\chi_1(b)\chi_1(c)\right]_{a,b,c,\tau} = \frac{1}{\prod_{s_a,s_b,s_c=\pm 1}(1-\tau\, a^{s_a}\, b^{s_b}\, c^{s_c})}\,, \tag{B.1}$$

where the fugacities a, b, and c label the Cartans of the three $SU(2)$ flavor groups. The index of the vector multiplet and the $SU(2)$ Haar measure combine to

$$\Delta(a)\mathcal{I}^V(a,\tau) = (1-\tau^2)\Delta_{\tau^2,\tau^4}(a)\,, \tag{B.2}$$

where $\Delta_{\tau^2,\tau^4}(a)$ is the Macdonald measure (4.96) with $q=\tau^2$ and $t=\tau^4$,

$$\Delta_{\tau^2,\tau^4}(a) = \frac{1}{2}(1-a^2)(1-\frac{1}{a^2})(1-\tau^2 a^2)(1-\frac{\tau^2}{a^2})\,. \tag{B.3}$$

The corresponding Macdonald polynomials $P^\lambda(a,a^{-1};q,t)$, normalized to be orthonormal under (B.2), are[1]

$$P^\lambda(a;\tau^2,\tau^4) \quad = \quad \frac{\tau}{\sqrt{1-\tau^2}\left(1-\frac{1}{a^2}\tau^2\right)(1-a^2\tau^2)}\sqrt{\chi_\lambda(\tau)\chi_{\lambda+2}(\tau)}\left\{\frac{\chi_\lambda(a)}{\chi_\lambda(\tau)}-\frac{\chi_{\lambda+2}(a)}{\chi_{\lambda+2}(\tau)}\right\}. \tag{B.4}$$

By choosing $\{P^\lambda(a,a^{-1};q,t)\}$ as a basis, the metric of the TQFT is then trivial, $\eta^{\lambda\mu}=\delta^{\lambda\mu}$. On the other hand, the projection of $\mathcal{I}(a,b,c)$ into the basis functions gives the structure constants $C_{\mu\nu\lambda}$,

$$\mathcal{I}(a,b,c)=\sum_{\mu,\nu,\lambda=0}^{\infty}C_{\mu\nu\lambda}P^\mu(a;\tau^2,\tau^4)P^\nu(b;\tau^2,\tau^4)P^\lambda(c;\tau^2,\tau^4)\,. \tag{B.5}$$

We find that while the structure constants are not diagonal, they take a relatively simple "upper triangular" form. The only non-vanishing coefficients are

$$C_{\lambda\lambda\lambda}\equiv\Psi_\lambda\,,\qquad C_{\lambda\mu\mu}=C_{\mu\lambda\mu}=C_{\mu\mu\lambda}\equiv\Omega_\lambda\quad\text{for }\mu<\lambda\,,\;(-1)^{\lambda+\mu}=1\,, \tag{B.6}$$

where

$$\begin{aligned}\Psi_\lambda(\tau)&=\frac{\sqrt{1-\tau^2}}{\sqrt{\chi_{\lambda+2}(\tau)}}\left(\frac{\tau^{-1}+\tau}{\sqrt{\chi_\lambda(\tau)}}-\tau^{\lambda+3}\sqrt{\chi_\lambda(\tau)}\right)\,,\\ \Omega_\lambda(\tau)&=\sqrt{1-\tau^2}(\tau^{-1}+\tau)\frac{1}{\sqrt{\chi_\lambda(\tau)\chi_{\lambda+2}(\tau)}}\,.\end{aligned} \tag{B.7}$$

Associativity is easy to check. It is trivial for most choices of external states, the one interesting case being the four-point function $\mu\mu\nu\nu$ with $\mu<\nu$ and having the same parity (both even or both odd). Equality of the two channels reads

$$\sum_{\lambda\geq\nu,(-1)^{\lambda+\mu}=1}C_{\mu\mu\lambda}C_{\lambda\nu\nu}=\left[C_{\mu\mu\nu}\right]^2\,, \tag{B.8}$$

which amounts to (no sum on ν)

$$\sum_{\lambda>\nu,(-1)^{\lambda+\mu}=1}\Omega_\lambda(\tau)^2+\Omega_\nu(\tau)\Psi_\nu(\tau)=\Omega_\nu(\tau)^2\,. \tag{B.9}$$

One can verify that this property is satisfied for the particular values of the

[1] This normalization is only used in this appendix. In the rest of the paper Macdonald polynomials are taken to have unit norm with respect to the Macdonald measure.

coefficients given in (B.7).[2]

Let us now perform an orthogonal transformation that diagonalizes the structure constants. From (B.6) we see even and odd Macdonald polynomials do not mix with each other and thus can carry our the diagonalization separately for each parity; the discussion below is restricted to the even parity case for definiteness. The Latin letter indices below, $j, \ldots$, run over the integers and correspond to half the value of the Greek indices used above.

We define real symmetric matrices N_i as[3]

$$(N_i)_{jk} \equiv C_{ijk}\,. \tag{B.10}$$

Associativity implies that they commute, $[N_i, N_j] = 0$, so they can be simultaneously diagonalized. Recall that the structure of each matrix N_j is

$$(N_j)_{ik} = \begin{cases} i<j,\ i=k & \Omega_j \\ i=k=j & \Psi_j \\ i>j,\ k=j & \Omega_i \\ k>j,\ i=j & \Omega_k \\ \text{other} & 0 \end{cases} \tag{B.11}$$

The non-zero eigenvalues of this matrix are Ω_j with multiplicity j, and $\Psi_j - \Omega_j$ with multiplicity one. The unique eigenvector with eigenvalue $\Psi_j - \Omega_j$ is

$$\mathbf{e}_{j+1} = (0,\,\ldots,\,0,\,\Psi_j - \Omega_j,\,\Omega_{j+1},\,\Omega_{j+2},\,\ldots)\,, \tag{B.12}$$

where there are j zeros in the beginning of the vector. Note that the $\mathbf{e}_j$s are orthogonal to each other,

$$\mathbf{e}_{j+1}\cdot\mathbf{e}_{k+1} = (\Psi_j - \Omega_j)\Omega_j + \sum_{i>j}[\Omega_i]^2 = 0\,, \tag{B.13}$$

where we took $j > k$ without loss of generality and used the associativity constraint (B.9). Moreover, the vectors $\mathbf{e}_i$ turn out to be eigenvectors of *all*

[2]One needs the identity $\sum_{k=0}^{\infty}\frac{1}{\sinh\alpha(2k+3)\sinh\alpha(2k+1)} = \frac{e^{-\alpha}}{2\sinh^2\alpha\cosh\alpha}$ and induction on ν.

[3]Note that since the metric is trivial, $\eta^{ij} = \delta^{ij}$, the upper or lower position of the indices is immaterial.

the matrices N_j,

$$\begin{array}{lcl} i < j & : & N_j \cdot \mathbf{e}_{i+1} = \Omega_j \, \mathbf{e}_{i+1} \,, \\ i = j & : & N_j \cdot \mathbf{e}_{i+1} = (\Psi_j - \Omega_j) \, \mathbf{e}_{i+1} \,, \\ i > j & : & N_j \cdot \mathbf{e}_{i+1} = 0 \,. \end{array} \tag{B.14}$$

This can be shown from the definitions with the help of the associativity constraint (B.9). To complete this set of vectors to a basis we have to add one more vector, orthogonal to all $\mathbf{e}_j$,

$$\mathbf{e}_0 = (\Omega_1, \, \Omega_2, \, \ldots \,) \,. \tag{B.15}$$

This is an eigenvector of all the matrices N_j with eigenvalue Ω_j. We have thus managed to diagonalize the matrices N_i. In the diagonal basis $\{\mathbf{e}_j\}$ the matrices are given by (we use hatted indices to represent components in the new basis)

$$(N_j)_{\hat{i}\hat{k}} = \begin{cases} j > \hat{i} \,, & \Omega_j \, \delta_{\hat{i}\hat{j}} \\ \hat{i} = j \,, & (\Psi_j - \Omega_j) \, \delta_{\hat{i}\hat{j}} \\ j < \hat{i}, & 0 \end{cases} \tag{B.16}$$

Finally we perform the orthogonal transformation to the new basis also for the matrix label j of N_j, and find constants in the new basis read

$$C_{\hat{j}\hat{i}\hat{k}} = \frac{1}{n_{\hat{j}}} \sum_l (\mathbf{e}_{\hat{j}})_l \cdot (N_l)_{\hat{i}\hat{k}} \,, \tag{B.17}$$

where $n_{\hat{j}}$ is the normalization of $\mathbf{e}_{\hat{j}}$,

$$\begin{array}{lcll} n_{\hat{j}} & = & \sqrt{\mathbf{e}_{\hat{j}} \cdot \mathbf{e}_{\hat{j}}} = \tau^{2\hat{j}} \sqrt{1 - \tau^2} & \text{for } \hat{j} > 0 \,, \\ n_{\hat{0}} & = & \sqrt{\mathbf{e}_{\hat{0}} \cdot \mathbf{e}_{\hat{0}}} = \sqrt{(1 - \tau^2)(1 + \tau^2)} \,. & \end{array} \tag{B.18}$$

A little calculation gives

$$C_{\hat{i}\hat{i}\hat{i}} = n_{\hat{i}} \,, \tag{B.19}$$

and zero for the other choices of the indices. So far we have restricted attention to even parity (in terms of the original Greek labels). The case of odd parity works along completely parallel lines.

We can now explicitly compute the functions that diagonalize the structure constants, by contacting the normalized vectors $\mathbf{e}_\mu / n_\mu$ with the Macdonald polynomials (B.4). A useful identity is ($\lambda > 0$)

$$\sum_{\mu=\lambda,\,(-1)^{\lambda+\mu}=1}^{\infty} P^{\mu}\,\Omega_{\mu} = \frac{1+\tau^2}{(1-\tau^2 a^2)\,(1-\tau^2/a^2)}\frac{\chi_\lambda(a)}{\chi_\lambda(\tau)}\,. \tag{B.20}$$

One finds that the diagonal basis is given by

$$\begin{aligned} f^{\lambda}(a,\tau) =& \frac{1}{\sqrt{1-\tau^2}}\frac{1}{(1-\tau^2 a^2)\,(1-\tau^2/a^2)}\left\{\chi_\lambda(a)-\tau^2\chi_{\lambda-2}(a)\right\} \quad \text{for } \lambda>0\,, \\ f^{0}(a,\tau) =& \frac{1}{\sqrt{1-\tau^2}}\frac{1}{(1-\tau^2 a^2)\,(1-\tau^2/a^2)}\sqrt{1+\tau^2}\,. \end{aligned} \tag{B.21}$$

It is straightforward to verify that this basis is orthonormal under the measure (B.2). Remarkably, the functions $f^{\lambda}(a,\tau)$ are proportional to the $SU(2)$ Hall-Littlewood polynomials $P^{\lambda}_{HL}(a,a^{-1}|\tau)$, see (4.43), with a λ-independent proportionality factor $\mathcal{K}(a,\tau)$.

Finally we can write the diagonalized form for the index,

$$\begin{aligned} &\mathcal{I}(a_1,a_2,a_3) = \\ &\quad \frac{1}{1-\tau^2}\prod_{i=1}^{3}\frac{1}{(1-\tau^2 a_i^2)\,(1-\tau^2/a_i^2)}\left\{(1+\tau^2)^2+\sum_{\lambda=1}^{\infty}\tau^{\lambda}\prod_{i=1}^{3}\left(\chi_\lambda(a_i)-\tau^2\chi_{\lambda-2}(a_i)\right)\right\}. \end{aligned} \tag{B.22}$$

The equality of this expression with (B.1) can be proven directly by elementary means since the sum above is a geometric sum. By noting that

$$\chi_\lambda(\tau)-\tau^2\chi_{\lambda-2}(\tau) = \tau^{-\lambda}(1+\tau^2)\,, \tag{B.23}$$

and recalling the definition (4.43) of the HL polynomials we can also write

$$\mathcal{I}(a_1,a_2,a_3) = \frac{1+\tau^2}{1-\tau^2}\prod_{i=1}^{3}\frac{1}{(1-\tau^2 a_i^2)\,(1-\tau^2/a_i^2)}\sum_{\lambda=0}^{\infty}\frac{1}{P^{HL}_{\lambda}(\tau,\tau^{-1}|\,\tau)}\prod_{i=1}^{3}P^{HL}_{\lambda}(a_i,a_i^{-1}|\,\tau) \tag{B.24}$$

B.2 Index of short multiplets of $\mathcal{N}=2$ superconformal algebra

A generic long multiplet $\mathcal{A}^{E}_{R,r(j_1,j_2)}$ of the $\mathcal{N}=2$ superconformal algebra is generated by the action of the eight Poincaré supercharges $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ on a superconformal primary, which by definition is annihilated by all conformal

Shortening Conditions				Multiplet
$\mathcal{B}_1$	$\mathcal{Q}_{1\alpha}\lvert R,r\rangle^{h.w.}=0$	$j_1=0$	$E=2R+r$	$\mathcal{B}_{R,r(0,j_2)}$
$\bar{\mathcal{B}}_2$	$\tilde{\mathcal{Q}}_{2\dot\alpha}\lvert R,r\rangle^{h.w.}=0$	$j_2=0$	$E=2R-r$	$\bar{\mathcal{B}}_{R,r(j_1,0)}$
$\mathcal{E}$	$\mathcal{B}_1\cap\mathcal{B}_2$	$R=0$	$E=r$	$\mathcal{E}_{r(0,j_2)}$
$\bar{\mathcal{E}}$	$\bar{\mathcal{B}}_1\cap\bar{\mathcal{B}}_2$	$R=0$	$E=-r$	$\bar{\mathcal{E}}_{r(j_1,0)}$
$\hat{\mathcal{B}}$	$\mathcal{B}_1\cap\bar{\mathcal{B}}_2$	$r=0,\ j_1,j_2=0$	$E=2R$	$\hat{\mathcal{B}}_R$
$\mathcal{C}_1$	$\epsilon^{\alpha\beta}\mathcal{Q}_{1\beta}\lvert R,r\rangle^{h.w.}_{\alpha}=0$		$E=2+2j_1+2R+r$	$\mathcal{C}_{R,r(j_1,j_2)}$
	$(\mathcal{Q}_1)^2\lvert R,r\rangle^{h.w.}=0$ for $j_1=0$		$E=2+2R+r$	$\mathcal{C}_{R,r(0,j_2)}$
$\bar{\mathcal{C}}_2$	$\epsilon^{\dot\alpha\dot\beta}\tilde{\mathcal{Q}}_{2\dot\beta}\lvert R,r\rangle^{h.w.}_{\dot\alpha}=0$		$E=2+2j_2+2R-r$	$\bar{\mathcal{C}}_{R,r(j_1,j_2)}$
	$(\tilde{\mathcal{Q}}_2)^2\lvert R,r\rangle^{h.w.}=0$ for $j_2=0$		$E=2+2R-r$	$\bar{\mathcal{C}}_{R,r(j_1,0)}$
	$\mathcal{C}_1\cap\mathcal{C}_2$	$R=0$	$E=2+2j_1+r$	$\mathcal{C}_{0,r(j_1,j_2)}$
	$\bar{\mathcal{C}}_1\cap\bar{\mathcal{C}}_2$	$R=0$	$E=2+2j_2-r$	$\bar{\mathcal{C}}_{0,r(j_1,j_2)}$
$\hat{\mathcal{C}}$	$\mathcal{C}_1\cap\bar{\mathcal{C}}_2$	$r=j_2-j_1$	$E=2+2R+j_1+j_2$	$\hat{\mathcal{C}}_{R(j_1,j_2)}$
	$\mathcal{C}_1\cap\mathcal{C}_2\cap\bar{\mathcal{C}}_1\cap\bar{\mathcal{C}}_2$	$R=0,r=j_2-j_1$	$E=2+j_1+j_2$	$\hat{\mathcal{C}}_{0(j_1,j_2)}$
$\mathcal{D}$	$\mathcal{B}_1\cap\bar{\mathcal{C}}_2$	$r=j_2+1$	$E=1+2R+j_2$	$\mathcal{D}_{R(0,j_2)}$
$\bar{\mathcal{D}}$	$\bar{\mathcal{B}}_2\cap\mathcal{C}_1$	$-r=j_1+1$	$E=1+2R+j_1$	$\bar{\mathcal{D}}_{R(j_1,0)}$
	$\mathcal{E}\cap\bar{\mathcal{C}}_2$	$r=j_2+1,R=0$	$E=r=1+j_2$	$\mathcal{D}_{0(0,j_2)}$
	$\bar{\mathcal{E}}\cap\mathcal{C}_1$	$-r=j_1+1,R=0$	$E=-r=1+j_1$	$\bar{\mathcal{D}}_{0(j_1,0)}$

Table B.1: Shortening conditions and short multiplets for the $\mathcal{N}=2$ superconformal algebra.

supercharges $\mathcal{S}$. If some combination of the $\mathcal{Q}$s also annihilates the primary, the corresponding multiplet is shorter and the conformal dimensions of all its members are protected against quantum corrections. The shortening conditions for the $\mathcal{N}=2$ superconformal algebra were studied in [147–149]. We follow the nomenclature of [149], whose classification scheme is summarized in table B.1. Let us take a moment to explain the notation. The state $\lvert R,r\rangle^{h.w.}_{(j_1,j_2)}$ is the highest weight state with $SU(2)_R$ spin $R>0$, $U(1)_r$ charge r, which can have either sign, and Lorentz quantum numbers (j_1,j_2). The multiplet built on this state is denoted as $\mathcal{X}_{R,r(j_1,j_2)}$, where the letter $\mathcal{X}$ characterizes the shortening condition. The left column of table B.1 labels the condition. A superscript on the label corresponds to the index $\mathcal{I}=1,2$ of the supercharge that kills the primary: for example $\mathcal{B}_1$ refers to $\mathcal{Q}_{1\alpha}$. Similarly a "bar" on the label refers to the conjugate condition: for example $\bar{\mathcal{B}}_2$ corresponds to $\tilde{Q}_{2\,\dot\alpha}$ annihilating the state; this would result in the short anti-chiral multiplet $\bar{\mathcal{B}}_{R,r(j_1,0)}$, obeying $E=2R-r$. Note that conjugation reverses the signs of r, j_1 and j_2 in the expression of the conformal dimension.

The superconformal index counts with signs the protected states of the theory, up to equivalence relations that set to zero all sequences of short multiplets that may in principle recombine into long multiplets. The recombination rules for $\mathcal{N}=2$ superconformal algebra are [149]

$$\mathcal{A}^{2R+r+2j_1+2}_{R,r(j_1,j_2)} \simeq \mathcal{C}_{R,r(j_1,j_2)} \oplus \mathcal{C}_{R+\frac{1}{2},r+\frac{1}{2}(j_1-\frac{1}{2},j_2)}\,, \tag{B.25}$$

$$\mathcal{A}^{2R-r+2j_2+2}_{R,r(j_1,j_2)} \simeq \bar{\mathcal{C}}_{R,r(j_1,j_2)} \oplus \bar{\mathcal{C}}_{R+\frac{1}{2},r-\frac{1}{2}(j_1,j_2-\frac{1}{2})}\,, \tag{B.26}$$

$$\mathcal{A}^{2R+j_1+j_2+2}_{R,j_1-j_2(j_1,j_2)} \simeq \hat{\mathcal{C}}_{R(j_1,j_2)} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(j_1-\frac{1}{2},j_2)} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(j_1,j_2-\frac{1}{2})} \oplus \hat{\mathcal{C}}_{R+1(j_1-\frac{1}{2},j_2-\frac{1}{2})}\,. \tag{B.27}$$

The $\mathcal{C}$, $\bar{\mathcal{C}}$ and $\hat{\mathcal{C}}$ multiplets obey certain "semi-shortening" conditions, while $\mathcal{A}$ multiplets are generic long multiplets. A long multiplet whose conformal dimension is exactly at the unitarity threshold can be decomposed into shorter multiplets according to (B.25, B.26, B.27). We can formally regard any multiplet obeying some shortening condition (with the exception of the $\mathcal{E}$ ($\bar{\mathcal{E}}$) types, and $\bar{\mathcal{D}}_{0(j_1,0)}$ ($\mathcal{D}_{0(0,j_2)}$) types) as a multiplet of type $\mathcal{C}$, $\bar{\mathcal{C}}$ or $\hat{\mathcal{C}}$ by allowing the spins j_1 and j_2, whose natural range is over the non-negative half-integers, to take the value $-1/2$ as well. The translation is as follows:

$$\mathcal{C}_{R,r(-\frac{1}{2},j_2)} \simeq \mathcal{B}_{R+\frac{1}{2},r+\frac{1}{2}(0,j_2)}, \quad \bar{\mathcal{C}}_{R,r(j_1,-\frac{1}{2})} \simeq \bar{\mathcal{B}}_{R+\frac{1}{2},r-\frac{1}{2}(j_1,0)}\,, \tag{B.28}$$

$$\hat{\mathcal{C}}_{R(-\frac{1}{2},j_2)} \simeq \mathcal{D}_{R+\frac{1}{2}(0,j_2)}, \qquad \hat{\mathcal{C}}_{R(j_1,-\frac{1}{2})} \simeq \bar{\mathcal{D}}_{R+\frac{1}{2}(j_1,0)}\,, \tag{B.29}$$

$$\hat{\mathcal{C}}_{R(-\frac{1}{2},-\frac{1}{2})} \simeq \mathcal{D}_{R+\frac{1}{2}(0,-\frac{1}{2})} \simeq \bar{\mathcal{D}}_{R+\frac{1}{2}(-\frac{1}{2},0)} \simeq \hat{\mathcal{B}}_{R+1}\,. \tag{B.30}$$

Note how these rules flip statistics: a multiplet with bosonic primary (j_1+j_2 integer) is turned into a multiplet with fermionic primary (j_1+j_2 half-odd), and vice versa. With these conventions, the rules (B.25, B.26, B.27) are the most general recombination rules. The $\mathcal{E}$ and $\bar{\mathcal{E}}$ multiplets never recombine.

The index of the $\mathcal{C}$ and $\mathcal{E}$ type multiplets vanishes identically (the choice of supercharge with respect to which the index is computed, $\mathcal{Q}=\widetilde{\mathcal{Q}}_{1\dot{-}}$, breaks the symmetry between $\mathcal{C}$ ($\mathcal{E}$) and $\bar{\mathcal{C}}$ ($\bar{\mathcal{E}}$) multiplets). The index of all remaining short multiplets can be specified by listing the index of $\bar{\mathcal{C}}$,$\hat{\mathcal{C}}$, $\bar{\mathcal{E}}$, $\mathcal{D}_{0(0,j_2)}$, and $\bar{\mathcal{D}}_{0(j_1,0)}$ multiplets,

$$
\begin{aligned}
\mathcal{I}_{\bar{\mathcal{C}}_{R,r(j_1,j_2)}} &= -(-1)^{2(j_1+j_2)}\tau^{2+2R+2j_2}\sigma^{j_2-r}\rho^{j_2-r} \\
&\quad \times \frac{(1-\sigma\rho)(\tau-\sigma)(\tau-\rho)}{(1-\sigma\tau)(1-\rho\tau)}\chi_{2j_1}\left(\sqrt{\frac{\sigma}{\rho}}\right), \\
\mathcal{I}_{\hat{\mathcal{C}}_{R(j_1,j_2)}} &= (-1)^{2(j_1+j_2)}\frac{\tau^{3+2R+2j_2}\sigma^{j_1+\frac{1}{2}}\rho^{j_1+\frac{1}{2}}(1-\sigma\rho)}{(1-\sigma\tau)(1-\rho\tau)} \\
&\quad \times \left(\chi_{2j_1+1}\left(\sqrt{\frac{\sigma}{\rho}}\right) - \frac{\sqrt{\sigma\rho}}{\tau}\chi_{2j_1}\left(\sqrt{\frac{\sigma}{\rho}}\right)\right), \\
\mathcal{I}_{\bar{\mathcal{E}}_{r(j_1,0)}} &= (-1)^{2j_1}\sigma^{-r-1}\rho^{-r-1}\frac{(\tau-\sigma)(\tau-\rho)}{(1-\sigma\tau)(1-\rho\tau)}\chi_{2j_1}\left(\sqrt{\frac{\sigma}{\rho}}\right), \\
\mathcal{I}_{\bar{\mathcal{D}}_{0(j_1,0)}} &= \frac{(-1)^{2j_1}(\sigma\rho)^{j_1+1}}{(1-\sigma\tau)(1-\rho\tau)} \\
&\quad \times \left((1+\tau^2)\chi_{2j_1}\left(\sqrt{\frac{\sigma}{\rho}}\right) - \frac{\tau}{\sqrt{\sigma\rho}}\chi_{2j_1+1}\left(\sqrt{\frac{\sigma}{\rho}}\right) - \tau\sqrt{\sigma\rho}\chi_{2j_1-1}\left(\sqrt{\frac{\sigma}{\rho}}\right)\right), \\
\mathcal{I}_{\mathcal{D}_{0(0,j_2)}} &= \frac{(-1)^{2j_2+1}\tau^{2j_2+2}}{(1-\sigma\tau)(1-\rho\tau)}(1-\sigma\rho).
\end{aligned}
\tag{B.31}
$$

where the Schur polynomial $\chi_{2j}\left(\sqrt{\frac{\sigma}{\rho}}\right)$ gives the character of the spin j representation of $SU(2)$.

Let us evaluate the interesting limits of the index studied in this paper on individual multiplets.

Macdonald index

This index is obtained from the general index in the limit $\sigma \to 0$. The index of the short multiplets in this limit is given by

$$
\begin{aligned}
\mathcal{I}_{\bar{\mathcal{C}}_{R,r(j_1,j_2)}} &= 0, \\
\mathcal{I}_{\hat{\mathcal{C}}_{R(j_1,j_2)}} &= (-1)^{2(j_1+j_2)}\frac{\tau^{3+2R+2j_2}\rho^{2j_1+1}}{(1-\rho\tau)}, \\
\mathcal{I}_{\bar{\mathcal{E}}_{r(j_1,0)}} &= 0, \\
\mathcal{I}_{\bar{\mathcal{D}}_{0(j_1,0)}} &= (-1)^{2j_1+1}\frac{\tau\rho^{2j_1+1}}{(1-\rho\tau)}, \qquad \mathcal{I}_{\mathcal{D}_{0(0,j_2)}} = (-1)^{2j_2+1}\frac{\tau^{2j_2+2}}{(1-\rho\tau)}.
\end{aligned}
\tag{B.32}
$$

While taking the limit of the $\bar{\mathcal{C}}$ and $\bar{\mathcal{E}}$ multiplet index we have used $j_2 - j_1 > r$ and $-r > j_1 + 1$ respectively. The first inequality follows from the bound $\delta_{1-} \geq 0$ along with $\tilde{\delta}_{1\dot{-}} = 0$ and the second one can be obtained by evaluating $\delta_{1-} \geq 0$ on the first descendant of the primary of the $\bar{\mathcal{E}}$ multiplet.

Hall-Littlewood index

This index is obtained from the Macdonald index by further taking the limit $\rho \to 0$. The index of the short multiplets is

$$\begin{aligned}
\mathcal{I}_{\bar{\mathcal{C}}_{R,r(j_1,j_2)}} &= 0\,, \\
\mathcal{I}_{\hat{\mathcal{C}}_{R(j_1,j_2)}} &= -(-1)^{2j_2}\tau^{3+2R+2j_2}\delta_{j_1,-\frac{1}{2}}\,, \\
\mathcal{I}_{\bar{\mathcal{E}}_{r(j_1,0)}} &= 0\,, \\
\mathcal{I}_{\bar{\mathcal{D}}_{0(j_1,0)}} &= 0\,, \qquad \mathcal{I}_{\mathcal{D}_{0(0,j_2)}} = (-1)^{2j_2+1}\tau^{2j_2+2}\,.
\end{aligned} \tag{B.33}$$

Schur Index

We take the limit $\tau \to \rho$. In this limit, the index becomes independent of σ and the short multiplets give

$$\begin{aligned}
\mathcal{I}_{\bar{\mathcal{C}}_{R,r(j_1,j_2)}} &= 0\,, \\
\mathcal{I}_{\hat{\mathcal{C}}_{R(j_1,j_2)}} &= (-1)^{2(j_1+j_2)}\frac{\tau^{4+2(R+j_1+j_2)}}{(1-\tau^2)}\,, \\
\mathcal{I}_{\bar{\mathcal{E}}_{r(j_1,0)}} &= 0\,, \\
\mathcal{I}_{\bar{\mathcal{D}}_{0(j_1,0)}} &= (-1)^{2j_1+1}\frac{\tau^{2j_1+2}}{(1-\tau^2)}\,, \qquad \mathcal{I}_{\mathcal{D}_{0(0,j_2)}} = (-1)^{2j_2+1}\frac{\tau^{2j_2+2}}{(1-\tau^2)}\,.
\end{aligned} \tag{B.34}$$

Coulomb Index

Finally we take $\tau \to 0$. In this limit only the $\bar{\mathcal{E}}$ multiplet have a non-vanishing index

$$\begin{aligned}
\mathcal{I}_{\bar{\mathcal{C}}_{R,r(j_1,j_2)}} &= 0\,, \\
\mathcal{I}_{\hat{\mathcal{C}}_{R(j_1,j_2)}} &= 0\,, \\
\mathcal{I}_{\bar{\mathcal{E}}_{r(j_1,0)}} &= (-1)^{2j_1}(\sigma\,\rho)^{-r}\chi_{2j_1}\left(\sqrt{\frac{\sigma}{\rho}}\right)\,, \\
\mathcal{I}_{\bar{\mathcal{D}}_{0(j_1,0)}} &= (-1)^{2j_1}(\sigma\rho)^{j_1+1}\chi_{2j_1}\left(\sqrt{\frac{\sigma}{\rho}}\right)\,, \qquad \mathcal{I}_{\mathcal{D}_{0(0,j_2)}} = 0\,.
\end{aligned} \tag{B.35}$$

The $\mathcal{N}=2$ vector multiplet is the direct sum of $\mathcal{D}_{0(0,0)}$ and $\bar{\mathcal{D}}_{0(0,0)}$, indeed (4.29) is simply $\bar{\mathcal{D}}_{0(0,0)}$.[4] In a Lagrangian theory, the only possible $\bar{\mathcal{D}}$ multiplets have $j_1 = 0$, and are obtained from the $\bar{\mathcal{D}}_{0(0,0)}$ half of the $\mathcal{N}=2$ vector

[4]Note that in this limit $\mathcal{I}_{\bar{\mathcal{E}}_{-1(0,0)}} = \mathcal{I}_{\bar{\mathcal{D}}_{0(0,0)}}$. This is also true in the less restrictive Coulomb limit (4.30).

multiplet. In the less restrictive limit of $\sigma, \tau \to 0$ and $\rho \to \infty$ the index of some of the short multiplets could potentially diverge. However, for Lagrangian theories the only contributing multiplets are $\bar{\mathcal{E}}_{r(0,0)}$ multiplets arising from tensor products of the $\bar{\mathcal{D}}_{0(0,0)}$ from the vector multiplet, whose index is finite.

B.3 Large k limit of the genus $\mathfrak{g}$ HL index

In this appendix we give some details about the large k limit of the HL index for $SU(k)$ quivers corresponding to genus $\mathfrak{g}$ surface with no punctures. For finite k, the index is given by (4.67),

$$\mathcal{I}_{\mathfrak{g}}^{(k)} = \frac{\left(\prod_{j=2}^{k}(1-\tau^{2j})\right)^{2\mathfrak{g}-2}}{(1-\tau^2)^{(k-1)(\mathfrak{g}-1)}} \sum_{\lambda} \frac{1}{P_{\lambda}^{HL}(\tau^{k-1}, \tau^{k-3}, \dots, \tau^{1-k}|\tau)^{2\mathfrak{g}-2}} \,, \tag{B.36}$$

The denominator in the sum above is explicitly given by [93],

$$P_{\lambda}^{HL}(\tau^{k-1}, \tau^{k-3}, \dots, \tau^{1-k}|\tau) = \mathcal{N}_{\lambda}(\tau)\, \tau^{\sum_{i=1}^{k-1}(2i-k-1)\lambda_i} \prod_{i=1}^{k} \frac{1-\tau^{2i}}{1-\tau^2} \,, \tag{B.37}$$

where $\mathcal{N}_{\lambda}(\tau)$ is given in (4.51),

$$\mathcal{N}_{\lambda_1,\dots\lambda_k}^{-2}(\tau) = \prod_{i=0}^{\infty} \prod_{j=1}^{m(i)} \left(\frac{1-\tau^{2j}}{1-\tau^2}\right) \,. \tag{B.38}$$

Here $m(i)$ is the number of rows in the Young diagram $\lambda = (\lambda_1, \dots, \lambda_k)$ of length i. We need to evaluate

$$\begin{aligned} \mathcal{I}_{\mathfrak{g}}^{(k)} &= (1-\tau^2)^{(k-1)(\mathfrak{g}-1)} \sum_{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{k-1}} \mathcal{N}_{\lambda_1,\dots,\lambda_{k-1},0}^{2-2\mathfrak{g}}\, \tau^{-(2\mathfrak{g}-2)\sum_{i=1}^{k-1}(2i-k-1)\lambda_i} \\ &= (1-\tau^2)^{(k-1)(\mathfrak{g}-1)} \sum_{\eta_1,\eta_2,\dots,\eta_{k-1}=0}^{\infty} \mathcal{N}_{\eta_1,\dots,\eta_{k-1}}^{2-2\mathfrak{g}}\, \tau^{(2\mathfrak{g}-2)\sum_{i=1}^{k-1}(k-i)\, i\, \eta_i} \,, \end{aligned} \tag{B.39}$$

where $\lambda_i = \sum_{j=1}^{k-i} \eta_{k-j}$. In the large k limit terms with non-zero η_i vanish since we always assume $|\tau| \ll 1$. Thus, the only contribution to the sum at leading order for large k is from the term with *all* $\eta_i = 0$,

$$\begin{aligned} \mathcal{I}_{\mathfrak{g}}^{(k\to\infty)} &= \lim_{k\to\infty}(1-\tau^2)^{(k-1)(\mathfrak{g}-1)}\mathcal{N}^{2-2\mathfrak{g}}_{\lambda_1=0,\ldots,\lambda_{k-1}=0,0} = \\ &= \prod_{j=2}^{\infty}(1-\tau^{2j})^{\mathfrak{g}-1} = PE\left[-(\mathfrak{g}-1)\frac{\tau^4}{1-\tau^2}\right]. \end{aligned} \tag{B.40}$$

The same logic applies also to the large k limit of the T_k theories: the singlet is the only term contributing to the index at leading order.

B.4 The unrefined HL index of T_4

Using the conjecture of section 4.2.4 we can write an explicit expression for the unrefined index of the T_4 theory. We find

$$\mathcal{I}_{T_4} = \frac{(1-\tau^4)(1-\tau^6)(1-\tau^8)}{(1-\tau^2)^{42}}\sum_{\lambda_1\geq\lambda_2\geq\lambda_3\geq0}\frac{\left(P^{HL}_{\lambda_1,\lambda_2,\lambda_3}(1,1,1,1|\beta)\right)^3}{P^{HL}_{\lambda_1,\lambda_2,\lambda_3}(\tau^3,\tau,\tau^{-1},\tau^{-3}|\beta)}\,. \tag{B.41}$$

The sum over the representation can be explicitly evaluated to give

$$\mathcal{I}_{T_4} = \frac{1-\tau}{(1-\tau^2)^{13}(1-\tau^3)^{17}(1-\tau^4)^{13}}\,\mathcal{P}_{86}(\tau)\,, \tag{B.42}$$

where $\mathcal{P}_{86}(\tau)$ is a polyndromic polynomial of degree 86 in τ with coefficients given in table B.2. The degree of the singularity when $\tau\to1$ has a physical meaning: since the Hall-Littlewood index computes the Hilbert series of the Higgs branch this is the complex dimension of the Higgs branch. For T_4 the HL index predicts the dimension to be 42, in agreement with [70, 73].

B.5 Proof of the $SU(2)$ Schur index identity

In this appendix we prove the basic $SU(2)$ Schur index identity (4.93),

$$\frac{PE\left[\frac{q^{1/2}}{1-q}(a_1+\frac{1}{a_1})(a_2+\frac{1}{a_2})(a_3+\frac{1}{a_3})\right]_{a_i,q}}{(q;q)^3(q^2;q)\prod_{i=1}^3 PE\left[\frac{q}{1-q}(a_i^2+a_i^{-2}+2)\right]_{a_i,q}} = \sum_{\lambda=0}^{\infty}\frac{\prod_{i=1}^3\chi_\lambda(a_i,a_i^{-1})}{\chi_\lambda(q^{\frac12},q^{-\frac12})}\,. \tag{B.43}$$

The strategy is to study the analytic properties of this expression and show that the left- and right-handed sides have the same poles and residues. Let us

1	τ	$33\tau^{2}$	$144\tau^{3}$
$873\tau^{4}$	$4169\tau^{5}$	$19486\tau^{6}$	$80693\tau^{7}$
$319237\tau^{8}$	$1165632\tau^{9}$	$4024927\tau^{10}$	$13054735\tau^{11}$
$40137244\tau^{12}$	$116876141\tau^{13}$	$323853313\tau^{14}$	$854555364\tau^{15}$
$2153519932\tau^{16}$	$5188980328\tau^{17}$	$11978372385\tau^{18}$	$26521974729\tau^{19}$
$56409853881\tau^{20}$	$115373040784\tau^{21}$	$227178289971\tau^{22}$	$431064583235\tau^{23}$
$788945072797\tau^{24}$	$1393870863434\tau^{25}$	$2379094134408\tau^{26}$	$3925581861006\tau^{27}$
$6265884973841\tau^{28}$	$9680331918067\tau^{29}$	$14483072164070\tau^{30}$	$20994033528147\tau^{31}$
$29497595795349\tau^{32}$	$40188148151858\tau^{33}$	$53110900086737\tau^{34}$	$68104402838959\tau^{35}$
$84760383950971\tau^{36}$	$102408879854636\tau^{37}$	$120143187852325\tau^{38}$	$136883008184825\tau^{39}$
$151478220483799\tau^{40}$	$162834262989902\tau^{41}$	$170047651342244\tau^{42}$	$172521386089030\tau^{43}$

Table B.2: The coefficients of $\mathcal{P}_{86}(\tau)$. The coefficient of τ^{86-k} is equal to the coefficient of τ^k.

first define

$$x=\frac{a_1}{a_2a_3},\quad y=\frac{a_2}{a_1a_3},\quad z=\frac{a_3}{a_2a_1},\quad u=a_1a_2a_3,\quad xyzu=1\,, \tag{B.44}$$

where a_i are $SU(2)$ fugacities. We also define

$$(a)\equiv(a;\,q)_\infty\equiv\prod_{i=0}^{\infty}(1-aq^i)\,. \tag{B.45}$$

We will use square brackets [ ] to denote ordinary brackets (that delimit expressions). Then, using

$$\sum_{n=1}^{\infty}\frac{x^n}{n}=-\log(1-x)\,, \tag{B.46}$$

the LHS of (B.43) is given by

$$LHS=\frac{[1-q](q)^2(qxy)(qxz)(qxu)(qyz)(qyu)(qzu)}{(q^{1/2}x)(q^{1/2}/x)(q^{1/2}y)(q^{1/2}/y)(q^{1/2}z)(q^{1/2}/z)(q^{1/2}u)(q^{1/2}/u)}\,. \tag{B.47}$$

Let us study the analytic properties of this expression as a function of x (the expression is symmetric in x,y,z,u). We have poles whenever $x=q^{1/2-l}$ with integer l (positive, zero or negative). At $x\to 0,\,\infty$ we have accumulation of poles. Let us for concreteness compute the residue with positive l

$$Res_{LHS} = \frac{[1-q](q)^2(q^{3/2-l}y)(q^{3/2-l}z)(q/(yz))(qyz)(q^{1/2+l}/z)(q^{1/2+l}/y)}{(q^{1-l})'(q^l)(q^{1/2}y)(q^{1/2}/y)(q^{1/2}z)(q^{1/2}/z)(q^l/(yz))(q^{1-l}yz)} \tag{B.48}$$

Here $(q^{1-l})'$ is $(q^{1/2}x)$ evaluated at $x = q^{1/2-l}$ with the vanishing factor removed. Now we have

$$\frac{(q^{1/2-l+1}y)(q^{1/2+l}/y)}{(q^{1/2}y)(q^{1/2}/y)} = \frac{[-y]^l}{q^{l^2/2}}\frac{1}{1-q^{1/2-l}y}\,. \tag{B.49}$$

From here we get

$$Res_{LHS} \quad = \quad \frac{[1-q](q)^2[yz]^l\prod_{i=0}^{l-2}(1-q^{1+i}/(yz))}{(q^{1-l})'(q^l)q^{l^2}[1-q^{1/2-l}y][1-q^{1/2-l}z]\prod_{i=0}^{l-1}(1-q^{-i}yz)} = \frac{q^{-1/2}-q^{1/2}}{A}\,, \tag{B.50}$$

where

$$A = x - \frac{1}{x} + y - \frac{1}{y} + z - \frac{1}{z} + u - \frac{1}{u}\,. \tag{B.51}$$

Let us now look on the RHS of (B.43), which can be written as

$$RHS = \frac{q^{-1/2}-q^{1/2}}{A}\sum_{n=1}^{\infty}\frac{q^{n/2}}{1-q^n}(x^n - \frac{1}{x^n} + y^n - \frac{1}{y^n} + z^n - \frac{1}{z^n} + u^n - \frac{1}{u^n})\,. \tag{B.52}$$

We again want to compute residues in x. To see the poles we write

$$\sum_{i=1}^{\infty}\frac{q^{n/2}}{1-q^n}x^n = \sum_{i=0}^{\infty}\sum_{n=1}^{\infty}q^{n(1/2+i)}\,x^n = \sum_{i=0}^{\infty}\frac{q^{1/2+i}\,x}{1-q^{1/2+i}\,x}\,. \tag{B.53}$$

Thus again the poles are at $x = q^{1/2-l}$ for any integer l (we have also same expression as (B.53) with $x \to 1/x$). The residue here is easily computed to give

$$Res_{RHS} = \frac{q^{-1/2}-q^{1/2}}{A}\,. \tag{B.54}$$

All in all, the LHS and RHS have the same poles and residues.

Appendix C

Refinement of $3d$ partition function

The superconformal index defined in section 2.1 is a function of fugacities t, y and v. In order to recover the matrix model of Kapustin et al. [33, 40] in section 5.1 we simply fixed the $v \to t$ and $y \to 1$. In this appendix we refine the $3d$ partition function by keeping track of all the fugacities in the index. It is convenient to define the chemical potentials

$$v = e^{-\beta(1/3+u)}, \qquad y = e^{-\beta\eta}. \tag{C.1}$$

The index, in terms of β, u and η becomes

$$\mathcal{I} = \mathrm{Tr}(-1)^F e^{-\beta[\frac{2}{3}(E+j_2)-\frac{1}{3}(r+R)-(r+R)u+2j_1\eta]}. \tag{C.2}$$

Let us compute the partition function of the hypermultiplet after turning on only u.

$$\begin{aligned}
\mathcal{I}^{hyp} &= \prod_i \Gamma\left(\frac{t^2}{\sqrt{v}}a_i; t^3 y, t^3 y^{-1}\right) = \prod_i \prod_{n\geqslant 1}\left(\frac{[n+\frac{1}{2}+\frac{u}{2}+i\alpha_i]_q}{[n-\frac{1}{2}-\frac{u}{2}-i\alpha_i]_q}\right)^n \\
&\xrightarrow{q\to 1} \prod_i \left[\cosh\pi\left(\alpha_i - i\frac{u}{2}\right)\right]^{-\frac{1}{2}} \\
\mathcal{I}^{vector} &= \prod_{i<j} \frac{1}{1-q^{-i(\alpha_i-\alpha_j)}}\frac{1}{1-q^{i(\alpha_i-\alpha_j)}}\frac{\Gamma(q^{1+u\pm i(\alpha_i-\alpha_j)};q,q)}{\Gamma(q^{\pm i(\alpha_i-\alpha_j)};q,q)} \\
&\xrightarrow{q\to 1} \prod_{i<j}\left(\frac{\sinh\pi(\alpha_i-\alpha_j)}{\pi(\alpha_i-\alpha_j)}\right)^2\left(\frac{\cosh\pi(\mp(\alpha_i-\alpha_j)+i/2)}{\cosh\pi(\mp(\alpha_i-\alpha_j)+i(u+1/2))}\right)^{1/2}.
\end{aligned} \tag{C.3}$$

Both partition functions reduce to the ones in section 5.1 as we set u to zero.

Now we restore $y = q^{-\beta\eta}$ to produce the more refined $3d$ partition function. The chemical potential η has a nice physical interpretation as the $U(1) \times U(1)$ isometry preserving squashing deformation of the S^3. The partition function of $3d$ gauge theories on this squashed background was computed in [57].

The contribution due to the hypermultiplet with η deformation turned on is

$$\begin{aligned}\mathcal{I}^{hyp} &= \prod_i \Gamma(\frac{t^2}{\sqrt{v}} a_i; t^3 y, t^3/y) \\ &\xrightarrow{y \to q^{-\beta\eta}} \prod_i \Gamma(q^{1/2-u/2-i\alpha_i}; q^{1+\eta}, q^{1-\eta}) \\ &= \prod_i \prod_{j,k\geqslant 0} \frac{1 - q^{3/2+u/2+i\alpha_i} q^{(1+\eta)j} q^{(1-\eta)k}}{1 - q^{1/2-u/2-i\alpha_i} q^{(1+\eta)j} q^{(1-\eta)k}}.\end{aligned} \tag{C.4}$$

Using the regularized infinite product representation of Barnes' double-Gamma function

$$\Gamma_2(x|\epsilon_1, \epsilon_2) \propto \prod_{m,n\geqslant 0} (x + m\epsilon_1 + n\epsilon_2)^{-1}, \tag{C.5}$$

the partition function of hyper-multiplet can be written in a compact way

$$\begin{aligned}\mathcal{I}^{hyper} &\to \prod_i \frac{\Gamma_2(1/2 - u/2 - i\alpha_i|1+\eta, 1-\eta)}{\Gamma_2(3/2 + u/2 + i\alpha_i|1+\eta, 1-\eta)} \\ &= \prod_i \frac{\Gamma_2(\frac{Q}{2}(1/2 - u/2) - i\hat{\alpha}_i|b, b^{-1})}{\Gamma_2(\frac{Q}{2}(3/2 + u/2) + i\hat{\alpha}_i|b, b^{-1})},\end{aligned} \tag{C.6}$$

where we have defined[1]

$$\hat{\alpha}_i = \frac{\alpha_i}{\sqrt{1-\eta^2}}, \qquad b = \sqrt{\frac{1-\eta}{1+\eta}}, \qquad Q = b + b^{-1}. \tag{C.7}$$

With this change of variables it is easy to see that for $u = 0$, our result is in agreement with [57]. The partition function of the vector multiplet:

$$\mathcal{I}^{vector} \to \prod_{i<j} \frac{1}{1 - q^{-i(\alpha_i - \alpha_j)}} \frac{1}{1 - q^{i(\alpha_i - \alpha_j)}} \frac{\Gamma(q^{1+u\pm i(\alpha_i-\alpha_j)}; q^{1+\eta}, q^{1-\eta})}{\Gamma(q^{\pm i(\alpha_i-\alpha_j)}; q^{1+\eta}, q^{1-\eta})}. \tag{C.8}$$

[1]We thank Davide Gaiotto for pointing out this change of variables.

reduces to

$$
\begin{aligned}
\mathcal{I}^{vector} &= \prod_{i<j} \frac{(1-\eta^2)\sinh\frac{\pi(\alpha_i-\alpha_j)}{1+\eta}\sinh\frac{\pi(\alpha_i-\alpha_j)}{1-\eta}}{\pi^2(\alpha_i-\alpha_j)^2} \frac{\Gamma_2(1+u\pm i(\alpha_i-\alpha_j)|1+\eta,1-\eta)}{\Gamma_2(1-u\pm i(\alpha_i-\alpha_j)|1+\eta,1-\eta)} \\
&= \prod_{i<j} \frac{\sinh \pi b(\hat{\alpha}_i-\hat{\alpha}_j)\sinh \pi b^{-1}(\hat{\alpha}_i-\hat{\alpha}_j)}{\pi^2(\hat{\alpha}_i-\hat{\alpha}_j)^2} \frac{\Gamma_2(\frac{Q}{2}(1+u)\pm i(\hat{\alpha}_i-\hat{\alpha}_j)|b,b^{-1})}{\Gamma_2(\frac{Q}{2}(1-u)\pm i(\hat{\alpha}_i-\hat{\alpha}_j)|b,b^{-1})}.
\end{aligned}
\tag{C.9}
$$

Again, we find a precise agreement with the partition function of the vector multiplet on squashed S^3.

Appendix D

$\mathcal{N} = 1$ superconformal shortening conditions and the index

In this appendix we summarize some basic facts about $\mathcal{N} = 1$ superconformal representation theory. A generic long multiplet $\rho A^{\Delta}_{r(j_1,j_2)}$ is generated by the action of 4 Poincaré supercharges $\mathcal{Q}_\alpha$ and $\widetilde{\mathcal{Q}}_{\dot\alpha}$ on superconformal primary which is by definition is annihilated by all conformal supercharges $\mathcal{S}$. In table D.1 we have summarized possible shortening and semishortening conditions.

Shortening Conditions				Multiplet
$\mathcal{B}$	$\mathcal{Q}_\alpha \lvert r\rangle^{h.w.} = 0$	$j_1 = 0$	$\Delta = -\frac{3}{2}r$	$\mathcal{B}_{r(0,j_2)}$
$\bar{\mathcal{B}}$	$\bar{\mathcal{Q}}_{\dot\alpha} \lvert r\rangle^{h.w.} = 0$	$j_2 = 0$	$\Delta = \frac{3}{2}r$	$\bar{\mathcal{B}}_{r(j_1,0)}$
$\hat{\mathcal{B}}$	$\mathcal{B} \cap \bar{\mathcal{B}}$	$j_1, j_2, r = 0$	$\Delta = 0$	$\hat{\mathcal{B}}$
$\mathcal{C}$	$\epsilon^{\alpha\beta}\mathcal{Q}_\beta \lvert r\rangle^{h.w.}_\alpha = 0$		$\Delta = 2 + 2j_1 - \frac{3}{2}r$	$\mathcal{C}_{r(j_1,j_2)}$
	$(\mathcal{Q})^2 \lvert r\rangle^{h.w.} = 0$ for $j_1 = 0$		$\Delta = 2 - \frac{3}{2}r$	$\mathcal{C}_{r(0,j_2)}$
$\bar{\mathcal{C}}$	$\epsilon^{\dot\alpha\dot\beta}\bar{\mathcal{Q}}_{\dot\beta} \lvert r\rangle^{h.w.}_{\dot\alpha} = 0$		$\Delta = 2 + 2j_2 + \frac{3}{2}r$	$\bar{\mathcal{C}}_{r(j_1,j_2)}$
	$(\bar{\mathcal{Q}})^2 \lvert r\rangle^{h.w.} = 0$ for $j_2 = 0$		$\Delta = 2 + \frac{3}{2}r$	$\bar{\mathcal{C}}_{r(j_1,0)}$
$\hat{\mathcal{C}}$	$\mathcal{C} \cap \bar{\mathcal{C}}$	$\frac{3}{2}r = (j_1 - j_2)$	$\Delta = 2 + j_1 + j_2$	$\hat{\mathcal{C}}_{(j_1,j_2)}$
$\mathcal{D}$	$\mathcal{B} \cap \bar{\mathcal{C}}$	$j_1 = 0, -\frac{3}{2}r = j_2 + 1$	$\Delta = -\frac{3}{2}r = 1 + j_2$	$\mathcal{D}_{(0,j_2)}$
$\bar{\mathcal{D}}$	$\bar{\mathcal{B}} \cap \mathcal{C}$	$j_2 = 0, \frac{3}{2}r = j_1 + 1$	$\Delta = \frac{3}{2}r = 1 + j_1$	$\bar{\mathcal{D}}_{(j_1,0)}$

Table D.1: Possible shortening conditions for the $\mathcal{N} = 1$ superconformal algebra.

A generic long multiplet of the $\mathcal{N}=1$ superconformal algebra $SU(2,2|1)$ is $16(2j_1+1, 2j_2+1)$ dimensional. Tables D.2, D.3, D.4 and D.5 illustrate how the $\mathcal{B}$, $\mathcal{C}$, $\hat{\mathcal{C}}$ and $\mathcal{D}$-type multiplets fit within a generic long multiplet.

	$r-2$	$r-1$	r	$r+1$	$r+2$
Δ			$\boxed{(j_1, j_2)}$		
$\Delta + \frac{1}{2}$		$(j_1 + \frac{1}{2}, j_2)$		$\boxed{(j_1, j_2 + \frac{1}{2})}$	
		$(j_1 - \frac{1}{2}, j_2)$		$\boxed{(j_1, j_2 - \frac{1}{2})}$	
$\Delta + 1$	(j_1, j_2)		$(j_1 + \frac{1}{2}, j_2 + \frac{1}{2})$ $(j_1 - \frac{1}{2}, j_2 + \frac{1}{2}), (j_1 + \frac{1}{2}, j_2 - \frac{1}{2})$ $(j_1 - \frac{1}{2}, j_2 - \frac{1}{2})$		$\boxed{(j_1, j_2)}$
$\Delta + \frac{3}{2}$		$(j_1, j_2 + \frac{1}{2})$ $(j_1, j_2 - \frac{1}{2})$		$(j_1 + \frac{1}{2}, j_2)$ $(j_1 - \frac{1}{2}, j_2)$	
$\Delta + 2$			(j_1, j_2)		

Table D.2: A long multiplet of $\mathcal{N} = 1$ superconformal algebra. The $SU(2,2)$ multiplets that are boxed form a short $\mathcal{B}_{r(0,j_2)}$ multiplet for $j_1 = 0, \Delta = -\frac{3}{2}r$. The left-handed $\bar{\mathcal{B}}$ can be obtained by reflecting the table (that is, sending $r \to -r$ and $j_1 \leftrightarrow j_2$). In general, when $j_1(j_2) = 0$, the $SU(2,2)$ multiplets $(j_1 - \frac{1}{2}, any)((any, j_2 - \frac{1}{2}))$ are set to zero, resulting in further shortening.

	$r-2$	$r-1$	r	$r+1$	$r+2$
Δ			$\boxed{(j_1, j_2)}$		
$\Delta + \frac{1}{2}$		$\boxed{(j_1 + \frac{1}{2}, j_2)}$		$\boxed{(j_1, j_2 + \frac{1}{2})}$	
		$(j_1 - \frac{1}{2}, j_2)$		$\boxed{(j_1, j_2 - \frac{1}{2})}$	
$\Delta + 1$	(j_1, j_2)		$\boxed{(j_1 + \frac{1}{2}, j_2 + \frac{1}{2})}$ $(j_1 - \frac{1}{2}, j_2 + \frac{1}{2})$ $\boxed{(j_1 + \frac{1}{2}, j_2 - \frac{1}{2})}$ $(j_1 - \frac{1}{2}, j_2 - \frac{1}{2})$		$\boxed{(j_1, j_2)}$
$\Delta + \frac{3}{2}$		$(j_1, j_2 + \frac{1}{2})$ $(j_1, j_2 - \frac{1}{2})$		$\boxed{(j_1 + \frac{1}{2}, j_2)}$ $(j_1 - \frac{1}{2}, j_2)$	
$\Delta + 2$			(j_1, j_2)		

Table D.3: A long multiplet of $\mathcal{N} = 1$ superconformal algebra. The $SU(2,2)$ multiplets that are boxed form a semi-short $\mathcal{C}_{r(j_1,j_2)}$ multiplet for $\Delta = 2 + 2j_1 - \frac{3}{2}r$. The left-handed $\bar{\mathcal{C}}$ can be obtained by reflecting the table (that is, sending $r \to -r$ and $j_1 \leftrightarrow j_2$). In general, when $j_1(j_2) = 0$, the $SU(2,2)$ multiplets $(j_1 - \frac{1}{2}, any)((any, j_2 - \frac{1}{2}))$ are set to zero, resulting in further shortening.

	$r-2$	$r-1$	r	$r+1$	$r+2$
Δ			$\boxed{(j_1,j_2)}$		
$\Delta+\frac{1}{2}$		$\boxed{(j_1+\frac{1}{2},j_2)}$ $(j_1-\frac{1}{2},j_2)$		$\boxed{(j_1,j_2+\frac{1}{2})}$ $(j_1,j_2-\frac{1}{2})$	
$\Delta+1$	(j_1,j_2)		$\boxed{(j_1+\frac{1}{2},j_2+\frac{1}{2})}$ $(j_1-\frac{1}{2},j_2+\frac{1}{2})$ $(j_1+\frac{1}{2},j_2-\frac{1}{2})$ $\boxed{-(j_1-\frac{1}{2},j_2-\frac{1}{2})}$		(j_1,j_2)
$\Delta+\frac{3}{2}$		$(j_1,j_2+\frac{1}{2})$ $\boxed{-(j_1,j_2-\frac{1}{2})}$		$(j_1+\frac{1}{2},j_2)$ $\boxed{-(j_1-\frac{1}{2},j_2)}$	
$\Delta+2$			$\boxed{-(j_1,j_2)}$		

Table D.4: Multiplet structure of $\hat{\mathcal{C}}_{(j_1,j_2)}$. The shortening conditions are $\Delta = 2 + j_1 + j_2$ and $\frac{3}{2}r = (j_1 - j_2)$.

	$r-2$	$r-1$	r	$r+1$	$r+2$
Δ			$\boxed{(j_1,j_2)}$		
$\Delta+\frac{1}{2}$		$(j_1+\frac{1}{2},j_2)$ $(j_1-\frac{1}{2},j_2)$		$\boxed{(j_1,j_2+\frac{1}{2})}$ $(j_1,j_2-\frac{1}{2})$	
$\Delta+1$	(j_1,j_2)		$(j_1+\frac{1}{2},j_2+\frac{1}{2})$ $(j_1-\frac{1}{2},j_2+\frac{1}{2}),\boxed{-(j_1+\frac{1}{2},j_2-\frac{1}{2})}$ $(j_1-\frac{1}{2},j_2-\frac{1}{2})$		(j_1,j_2)
$\Delta+\frac{3}{2}$		$(j_1,j_2+\frac{1}{2})$ $(j_1,j_2-\frac{1}{2})$		$\boxed{-(j_1+\frac{1}{2},j_2)}$ $(j_1-\frac{1}{2},j_2)$	
$\Delta+2$			$(j_1,j_2),\boxed{+(j_1,j_2-1)}$		
$\Delta+\frac{5}{2}$				$\boxed{+(j_1,j_2-\frac{1}{2})}$	

Table D.5: Multiplet structure of $\mathcal{D}_{(0,j_2)}$. The shortening conditions are $\Delta = 1+j_2 = -\frac{3}{2}r$ and $j_1 = 0$. The multiplet $\bar{\mathcal{D}}_{(j_1,0)}$ could be obtained by $j_1 \leftrightarrow j_2, r \leftrightarrow -r$ or by simply reflecting the table. The shortening conditions in that case are $\Delta = 1 + j_1 = \frac{3}{2}r$ and $j_2 = 0$.

At the unitarity threshold, a long multiplet can decompose into (semi)short multiplets. The splitting rules are:

$$
\begin{aligned}
\rho A^{2+2j_1-\frac{3}{2}r}_{r(j_1,j_2)} &\simeq \mathcal{C}_{r(j_1,j_2)} \oplus \mathcal{C}_{r-1(j_1-\frac{1}{2},j_2)} \\
\rho A^{2+2j_2+\frac{3}{2}r}_{r(j_1,j_2)} &\simeq \bar{\mathcal{C}}_{r(j_1,j_2)} \oplus \bar{\mathcal{C}}_{r+1(j_1,j_2-\frac{1}{2})} \\
\rho A^{2+j_1+j_2}_{\frac{2}{3}(j_1-j_2)(j_1,j_2)} &\simeq \hat{\mathcal{C}}_{(j_1,j_2)} \oplus \mathcal{C}_{\frac{2}{3}(j_1-j_2)-1,(j_1-\frac{1}{2},j_2)} \oplus \bar{\mathcal{C}}_{\frac{2}{3}(j_1-j_2)+1,(j_1,j_2-\frac{1}{2})}
\end{aligned}
$$

We are using a notation where the $\mathcal{B}$ and $\bar{\mathcal{B}}$ type multiplets are formally identified with special cases of $\mathcal{C}$ and $\bar{\mathcal{C}}$ multiplets, as follows

$$\mathcal{C}_{r(-\frac{1}{2},j_2)} \simeq \mathcal{B}_{r-1(0,j_2)} \qquad \bar{\mathcal{C}}_{r(j_1,-\frac{1}{2})} \simeq \bar{\mathcal{B}}_{r+1(j_1,0)}\,. \tag{D.1}$$

We define the $\mathtt{Left}$ ($\mathtt{Right}$) equivalence class of the multiplet $\mathcal{C}_{r(j_1,j_2)}(\bar{\mathcal{C}}_{r(j_1,j_2)})$ as the class of multiplets with the same $\mathtt{Left}$ ($\mathtt{Right}$) index. From the splitting rules, we see that the classes can be labeled as $[-r+2j_1,j_2]^{\mathtt{L}}_{(-)^{2j_1}}$ ($[r+2j_2,j_1]^{\mathtt{R}}_{(-)^{2j_2}}$). Moreover, $\mathcal{I}^{\mathtt{L}}_{[-r+2j_1,j_2]^{\mathtt{L}}_{-}} = -\mathcal{I}^{\mathtt{L}}_{[-r+2j_1,j_2]^{\mathtt{L}}_{+}}$ and $\mathcal{I}^{\mathtt{R}}_{[r+2j_2,j_1]^{\mathtt{R}}_{-}} = -\mathcal{I}^{\mathtt{R}}_{[r+2j_2,j_1]^{\mathtt{R}}_{+}}$. The expressions for the indices of the equivalent classes are

$$\begin{aligned}
\mathcal{I}^{\mathtt{L}}_{[\tilde{r},j_2]^{\mathtt{L}}_{\pm}} &= \pm(-)^{2j_2+1}\frac{t^{3(\tilde{r}+2)}\chi_{j_2}(y)}{(1-t^3y)(1-t^3y^{-1})}\\
\mathcal{I}^{\mathtt{R}}_{[\bar{\tilde{r}},j_1]^{\mathtt{R}}_{\pm}} &= \pm(-)^{2j_1+1}\frac{t^{3(\bar{\tilde{r}}+2)}\chi_{j_1}(y)}{(1-t^3y)(1-t^3y^{-1})}\\
\mathcal{I}^{R}[\tilde{r},j_2]^{\mathtt{L}}_{\pm} &= 0\\
\mathcal{I}^{\mathtt{L}}[\bar{\tilde{r}},j_1]^{\mathtt{R}}_{\pm} &= 0\,.
\end{aligned}$$

The situation is slightly more involved for the $\hat{\mathcal{C}}$ and $\mathcal{D}$ type multiplets. Unlike the $\mathcal{B},\mathcal{C}$ type multiplets, they contribute both to $\mathcal{I}^{\mathtt{L}}$ as well as $\mathcal{I}^{\mathtt{R}}$. The indices [21] for the different types of multiplets are collected in table D.6.

Multiplet	$\mathcal{I}^{\mathtt{L}}$	$\mathcal{I}^{\mathtt{R}}$
$\rho A^{\Delta}_{r(j_1,j_2)}$	0	0
$\mathcal{C}_{r(j_1,j_2)}$	$\mathcal{I}^{\mathtt{L}}_{[-r+2j_1,j_2]^{\mathtt{L}}_{(-)^{2j_1}}}$	0
$\bar{\mathcal{C}}_{r(j_1,j_2)}$	0	$\mathcal{I}^{\mathtt{R}}_{[r+2j_2,j_1]^{\mathtt{R}}_{(-)^{2j_2}}}$
$\hat{\mathcal{C}}_{(j_1,j_2)}$	$\mathcal{I}^{\mathtt{L}}_{[\frac{2}{3}j_2+\frac{4}{3}j_1,j_2]^{\mathtt{L}}_{(-)^{2j_1}}}$	$\mathcal{I}^{\mathtt{R}}_{[\frac{2}{3}j_1+\frac{4}{3}j_2,j_1]^{\mathtt{R}}_{(-)^{2j_2}}}$
$\mathcal{D}_{(0,j_2)}$	$\mathcal{I}^{\mathtt{L}}_{[\frac{2}{3}j_2-\frac{4}{3},j_2]^{\mathtt{L}}_{-}} + \mathcal{I}^{\mathtt{L}}_{[\frac{2}{3}j_2-\frac{1}{3},j_2-\frac{1}{2}]^{\mathtt{L}}_{-}}$	$\mathcal{I}^{\mathtt{R}}_{[\frac{4}{3}j_2-\frac{2}{3},0]^{\mathtt{R}}_{+}}$
$\bar{\mathcal{D}}_{(j_1,0)}$	$\mathcal{I}^{\mathtt{L}}_{[\frac{4}{3}j_1-\frac{2}{3},0]^{\mathtt{L}}_{+}}$	$\mathcal{I}^{\mathtt{R}}_{[\frac{2}{3}j_1-\frac{4}{3},j_1]^{\mathtt{R}}_{-}} + \mathcal{I}^{\mathtt{R}}_{[\frac{2}{3}j_1-\frac{1}{3},j_1-\frac{1}{2}]^{\mathtt{R}}_{-}}$

Table D.6: Indices $\mathcal{I}^{\mathtt{L}}$ and $\mathcal{I}^{\mathtt{R}}$ of the various short and semi-short multiplets.