Two Essays on Monetary Policy under Parameter Uncertainty

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Abstract of the Dissertation

Two Essays on Monetary Policy under Parameter Uncertainty by

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I present in this dissertation two topics of monetary policy making under model parameter uncertainty. The first topic is an assessment of the Brainard's principle in monetary policy making in a new context. Brainard (1967) proposes that policy making under model uncertainty should be cautious in the sense that the policy maker would move his instrument less aggressively than in the absence of uncertainty. I assess this proposal in Chapter 2 by considering monetary policy in a New Keynesian economy with the cost channel developed by Ravenna and Walsh (2006). Uncertainty in the model comes from a coefficient that governs the direct effect of interest rate and output gap on inflation in the Phillips curve. The loss function is endogenous to the structural parameters. My results show that the interest rate response to a shock under model uncertainty is not necessarily stronger than that in the absence of model uncertainty. These results imply that the Brainard's principle does not apply in this framework.

The second topic is on the optimal delegation of monetary policy under parameter uncertainty. In Chapter 3, I model an economy in which there is policy planner who faces uncertainty about the slope of the Phillips curve and a central banker who believes he knows the economy with certainty. The policy planner makes use of a delegation method similar to the one initiated by Woodford (1999) to induce his central banker to implement his min-max commitment policy. The policy planner then solves for the parameters of the delegated loss criterion by matching his min-max optimal policy, in term of a targeting rule, with that of the discretionary policy conducted by his central banker. Delegation under parameter uncertainty requires choosing a central banker with a preference for an output gap stabilization weight of less than one. In this delegation framework, the robust policy can save up to an additional loss equivalent to a permanent increase in inflation of 0.06 percentage point from its target, compared to the non-robust policy. In addition, the robust delegation is found to dominate standard discretionary policy.

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CHAPTER 1
INTRODUCTION AND SUMMARY

In the last three decades, the New Keynesian framework has emerged as a standard paradigm for monetary policy analysis. Among its many research areas, monetary policy making under uncertainty has been under extensive investigation. The importance of uncertainty in policy making is emphasized by the former FED Chairman, Greenspan (2003): "Uncertainty is not just an important feature of the monetary policy landscape; it is the defining characteristic of that landscape".

A main theme in this research area that has received much discussion in recent articles is the assessment of the Brainard (1967)'s principle about the cautiousness of policy making under model uncertainty. The motivation for this research program, as stated in Tetlow and von zur Muehlen (2002), is the search for a theoretical explanation of the observed lack of aggressiveness of interest-rate reaction to output and inflation as recommended by theoretical models. Research in this area has been undertaken in several directions. Some continue with the approach laid down by Brainard (1967), assuming a well-defined prior distribution of uncertainty and tackling the problem using Bayesian optimal control methods. Other research, notably Hansen and Sargent (2007), explore the robust-control approach and define the problem as finding the best solution when the worst-case scenario is assumed for each possible policy. Although being fruitful, this research area presents mixed conclusions depending on the assumptions regarding the transmission mechanisms of the models being used and the nature of model uncertainty. However, despite these mixed conclusions, the important message to be taken from this more recent literature is that the Brainard's principle does not always hold.

Another important area in the New Keynesian paradigm, for which there is some common agreement, is regarding the superiority of the commitment policy over discretionary policy and the infeasibility of a commitment policy, which is based on conditional expectation, due to its innate time-inconsistency. A development which results from these findings has been the consequent search for the implementation of a commitment policy outcome which uses discretionary policy by methods of delegation. Rather surprising, however, is that the connection between policy making under model uncertainty and delegation seems to have been almost ignored in the profession. Studies in this area are rare and there are very few papers that can be searched and cited in this regard. Some examples though include Kilponen (2003) who studies base money growth targeting under model uncertainty; it is found that delegation of the Freidman's k-percent rule for base money growth targeting can improve social welfare over inflation targeting. Gaspar and Vestin (2004) work on the inflation bias problem with an assumption of output gap uncertainty. They find that the degree of inflation conservatism of the central bank delegated with monetary policy increases with uncertainty. Somehow similar to Gaspar and Vestin (2004), Tillmann (2009a, 2009b) investigates how the degree of central bank inflation conservatism varies with model structure uncertainty. Although these studies integrate model uncertainty into the delegation problem, using different approaches ranging from comparing targeting regimes to assuming different sources of uncertainty, they all model the central bank, the agent in delegation problems, as facing uncertainty. Therefore, changing the focus of uncertainty could make a progress on this research area.

This thesis aims to contribute to these areas of policy making and delegation under model uncertainty in the following two self-contained chapters. I first revisit the issue of monetary policy decision under parameter uncertainty in Chapter 2. Here I focus on a line of research which emphasizes uncertainty about the model parameters in a New Keynesian model, as explored by Giannoni (2002, 2007), Tillman (2009) amongst others. This line of research assumes policy makers face Knightian uncertainty (Knight, 1921) about the model parameters, uncertainty over which they have no prior distribution; they therefore follow a min-max strategy to formulate their optimal policy decision. The overall findings of these studies imply that uncertainty about different parameters may lead to either supporting or rejecting Brainard's principle. In this chapter, I make use of a New Keynesian model augmented with a cost channel as developed by Ravena & Walsh (2006). In this class of models, the nominal interest rate can directly influence inflation through the Phillips curve channel. As step further from Giannoni (2002, 2007) and Tillman (2009), however, I use a derived loss function whose coefficients depend on the model's parameters including κ, the one that governs the direct effects of output gap and nominal interest rate on inflation. It is rather common in the literature to assume that the weights on inflation and output gap variation in the loss function are fixed by policy makers. However, Woodford (2003a) has developed the theoretical foundations for the integration of model structure and objectives. This integration has important implications for the study of models under parameter uncertainty. This is because it means that uncertainty about parameters implies uncertainty about the objective function.

With an assumption of uncertainty about a coefficient that affects the influence of both output gap and interest rate in the Phillips curve, I examine the applicability of Brainard's principle. I compare robust and non-robust monetary policy in terms of interest rate responses that result from the use of these two policies with respective shocks. The results show that Brainard's principle does not apply in this model.

The issue of monetary policy delegation under uncertainty is studied in Chapter 3. Since the work of Kydland and Presscott (1977) and subsequently Woodford (1999), it has been commonly accepted that discretionary monetary policy results in a suboptimal outcome compared to that produced by a full commitment policy. The supporting argument is that discretionary policy produces equilibrium where current and future actions are not optimally combined, thus not optimally manipulating private sector expectations in a way that supports current policy decisions. On the other hand, a full commitment policy results in an equilibrium that does implement optimal manipulation of expectations. However, a full commitment policy for some objective functions causes time inconsistency.

Although the superiority of a full commitment policy is obvious and desirable, in reality central banks conduct discretionary policy due to the absence of a commitment technology. This raises the question as to whether we can replicate the welfare outcome of a full commitment policy with that of a discretionary policy. In his pioneering work, Woodford

(1999, 2003) suggests assigning the central bank a distorted loss function that is different from the true loss of the society. In this way the central bank may be induced to generate an appropriate inertial equilibrium with discretionary optimization.

Technically, it is obviously feasible to replicate full commitment equilibrium with the above approach given the linear-quadratic framework used widely in the New Keynesian monetary policy literature, since the solution to linear-quadratic optimization is invariably a linear function of the state variables. If the delegation scheme can somehow be involved in the quadratic objective function or in structural constraints appropriate lagged endogenous variables, the desired equilibrium can be produced.

Given this feasibility, delegation schemes have been designed, focusing on different aspects of the central banks daily practice to formulate loss functions to be assigned. For example, Woodford (2003) looks at the inertial behavior of short-term interest rate setting and suggests including in the assigned loss function an interest rate smoothing objective. Walsh (2003) suggests an output-gap-change objective in the loss function so as to replace the usual output gap objective. Vestin (2006) suggests that price-level targeting can act as an alternative to inflation targeting. Jensen (2002) considers a situation in which the central bank is required to care about nominal income growth stability, disregarding inflation variability, but not neglecting variations in the output gap. Bilbiie (2009), following Walsh (1995)'s linear contract approach, proposes signing with the central bank a contract that specifies the central bank's reward or penalty which is contingent on the current state of the economy thus introducing lagged endogenous variables into the central bank's loss function.

As already mentioned, research on the optimal delegation of monetary policy has left unexhausted an area regarding model uncertainty; this is where Chapter 3 makes its contribution. In this chapter, I study a problem whereby a policy planner who is uncertain about his economy wants the commitment policy to be implemented. Due to the lack of a commitment policy, it is impossible for the policy planner to implement the commitment policy himself. The policy planner therefore needs to induce a central banker, who is assumed to believe that he knows the economy with certainty, to implement this policy. This delegation scheme presents the policy planner with two problems. He first needs to figure out which policy he wants implemented under model uncertainty, and then how to induce the central banker to do it given the fact that the policy planner and the central banker view the economy differently.

As in Chapter 2, I propose that the policy planner derives a min-max policy when faced with model uncertainty. On the delegation scheme, although studies in the current literature recommend delegation schemes that result in identical solutions of the policy maker and the central banker's optimization problems, they skip over the process by which the central banker delivers the commitment outcome. In fact, there is no need to care about this process when the policy planner and his central banker are both certain about their economy. Things may be different however, if one is certain and the other is uncertain about the model of economy. Technically, both actors are assumed to face different structural constraints, hence the importance of the implementation process. Here, I propose that the monetary policy conducted by the central bank - understood as the central bank's maintaining a certain relationship amongst observable variables - be identical to that for the min-max policy that the policy planner wants to implement.

CHAPTER 2
OPTIMAL MONETARY POLICY WITH PARAMETER UNCERTAINTY AND ENDOGENOUS LOSS FUNCTION IN A MODEL WITH COST CHANNEL

Abstract

Brainard (1967) proposes that policy making under model uncertainty should be cautious in the sense that the policy maker would move his instrument less aggressively than in the absence of uncertainty. I reassess this proposal by considering monetary policy in a New Keynesian economy model with a cost channel as developed by Ravenna and Walsh (2006). Uncertainty in the model comes from a coefficient that governs the direct effect of interest rate and output gap on inflation in the Phillips curve. The loss function is assumed to be endogenous to the structural parameters. Results show that the interest rate response to shock under model uncertainty is not necessarily stronger than that in the absence of model uncertainty. These results imply that Brainard's principle does not apply in this framework.

1. Introduction

In practice, monetary policy making is subject to considerable uncertainty about the true functioning of the economy. The policy maker does not know for sure how their target variables will be affected by their policy action. This reality calls for the development of methods that guide policy making under uncertainty. Brainard (1967) is considered to be one of the early important formal theoretical frameworks to consider policy making under model uncertainty. Brainard’s principle, as interpreted by Blinder (1998) under model uncertainty, is that the policy maker should be more cautious in the sense that he would move his instrument by less than what he computes about the economy without model uncertainty.

This chapter intends to investigate Brainard’s principle in a New Keynesian model using a cost channel proposed by Ravena and Walsh (2006). In this model, the interest rate enters directly into the Phillips equation and influences inflation. Uncertainty comes from the coefficients in the Phillips curve through which the interest rate and output gap exert their influence on inflation. The difference between this study and others that follow a similar approach line, for example Giannoni (2002, 2007) and Tillmann (2009), is that the setup in this framework allows the inflation stabilization weight in the loss function to be affected by the uncertain parameter. The re-assessment of the Brainard principle is thus undertaken by examining the optimal policy response to shock in the framework with and without parameter uncertainty.

For policy formulation under uncertainty, the policy maker is assumed to follow a min-max strategy, finding the best policy for the worst-case parameter when he does not have a prior on the uncertainty. To define the min-max strategy, Giannoni (2002) sees the system as a zero-sum simultaneous-move game between the policy maker and Nature. He then finds the min-max strategy. First, he solves for the max-min solution and numerically verifies that a Nash equilibrium exists at the max-min equilibrium. The structure of a zero-sum simultaneous-move game then implies that the solution to the max-min is the Nash equilibrium of the game and thus also solution to the min-max problem. Contrary to Giannoni (2002, 2007), I do not solve for the max-min equilibrium and resort to a numerical method to verify for the Nash equilibrium existence. Instead, I show analytically that there is always a Nash equilibrium for the zero-sum game between the policy maker and Nature within the setup considered; therefore the min-max solution is simply the Nash equilibrium. The worst-case value for the parameter in consideration is then shown to be always at its upper bound.

The paper's results show that, under parameter uncertainty, the policy maker will plan for weaker inflation suppression in response to a rise in output gap that is caused by a demand shock. However, actual policy action in terms of the interest rate response to a shock is inconclusive. This result denies the Brainard's principle but is not on the same grounds as Giannoni (2002) who finds that the policy response is more aggressive under uncertainty.

The rest of the paper is structured as follows. First, I review the related literature on monetary policy under uncertainty and monetary policy with a cost channel. Section 3 then introduces the New Keynesian model with a cost channel. Section 4 derives the min-max policy. Section 5 examines policy maker's actual interest rate response to shock under uncertainty Section 6 then experiments with some policy analysis issues; finally in Section 7, I conclude.

2. Literature review

The importance of the cost channel in New Keynesian models has been admitted given the increasing number of studies on issues related to it. In the following section, I review some of the most cited and relevant literature to this study.

Barth and Ramey (2001) present evidence that the cost channel may be an important part of the monetary transmission mechanism when working capital is an essential component of production and distribution; hence monetary contractions can affect output through a supply channel as well as the traditional demand-type channels. They find that following a monetary contraction, many industries exhibit periods of falling output and rising price-wage ratios, consistent with a supply shock in our model.

Ravena and Walsh (2006) formally develop a model with a cost channel and show that when the cost channel is introduced, an endogenous cost-push shock arises and produces a trade-off between stabilizing output gap and inflation. Surico (2008) studies the conditions that guarantee equilibrium determinacy in a standard sticky price model augmented with a cost channel. Llosa (2009) analyses how monetary policy may affect determinacy and expectational stability (E-stability) of the rational expectations equilibrium when the cost channel of monetary policy matters. Tillman (2008) empirically assesses the impact of the cost channel of monetary transmission on the dynamics of inflation within a New Keynesian Phillips curve framework. He shows that the cost channel significantly contributes to the explanation of inflation dynamics in forward-looking sticky-price models for the US, the UK, and the Euro area. Moreover, the cost channel can explain inflation episodes that cannot be accounted for by the standard New Keynesian model.

On optimal policy under model uncertainty, the literature can be classified into data uncertainty, structure uncertainty and parameter uncertainty. This paper restricts our discussion to parameter uncertainty and the more relevant topic of structure uncertainty. These two types of uncertainty can be group into a topic called model uncertainty.

Research on policy decision under model uncertainty has generally developed through three general approaches. The first approach is called model averaging. This approach is to find a policy that performs well across a wide range of model. Research using this approach include McCallum (1988, 1999), Taylor (1999), Levin *et al*. (1999, 2003), and Levin and Williams (2003). Brock et al (2003, 2004) formalizes the framework with the Bayesian model averaging method. The advantage of this approach is that it can study the effect of a special rule for a wide range of structurally different models. However, it can not specify an optimal rule in an uncertainty environment.

The second approach focuses on a special class of models and studies different aspects of uncertainty. This approach makes use of a Bayesian method to identify the policy that minimizes the expected loss criteria given a prior distribution of the parameters. This approach was initiated by Brainard (1967), followed by Clarida *et al*. (1999), Wieland (1998), Soderstrom (2000, 2002), and Kurozumi (2003) amongst others.

The third approach is to find a policy that minimizes the loss criteria in a worst case scenario that is drawn out of a set of possible scenarios. This approach is to solve a problem in which uncertainty cannot be characterized by any prior distribution – this is a form of uncertainty called Knightian uncertainty (Knight, 1921), and therefore the Bayesian method cannot be used. This approach was initiated by Gilboa and Schmeidler (1989) who show that if the policymaker has no priors on the set of alternative models, and his preferences satisfy uncertainty aversion in addition to the axioms of standard expected utility theory, then the policymaker's decision is to minimize his loss in the worst-case scenario. A thrust of this approach - which is advocated by Sargent (1999) and Hansen and Sargent (2006) - includes introducing an additive stochastic term into the structural equations of the economy so as to represent deviation from the true model, and applying robust control theory so as to find the robust policy. This approach therefore considers an unstructured form of uncertainty. The advantage of this approach is that it can specify the degree of uncertainty by imposing a limit on the statistical distance between the true model and the deviated model. Another direction of this approach is to consider uncertainty in more structural form such as Giannoni (2002, 2007) in which uncertainty is narrowed to parameter uncertainty. The advantage of this approach is its potential to study the effect on policy given uncertainty about the deep structural parameters of an economy.

In his standard New Keynesian model, Giannoni (2002, 2007) concludes that the robust optimal policy rule is likely to involve a stronger response of the interest rate to fluctuations in inflation and the output gap than is the case in the absence of uncertainty. Similarly, Kara (2002) finds that when doubts take the form of uncertainty about the slope of the Phillips curve, the robust policy rule prescribes a less aggressive response to deviations of inflation from the target. On the other hand, if the source of uncertainty is imperfect knowledge of the persistence of shocks, then robust monetary policy calls for a more aggressive response to inflation. Tillmann (2009) shows that Brainard's principle holds when a range of uncertain parameters is being considered.

Building on the works of Giannoni (2002) and Tillmann (2009), this paper incorporates endogeneity of the loss function to the uncertain parameter. This development is important because endogeneity may create conflicting effects on the loss value of the uncertain parameter. On one hand, the uncertain parameter leads to a perceived wider variation in output gap and inflation. On the other hand, the worst-case value of the uncertain parameter may lead policy makers to either increase or decrease the weights of the output gap or inflation in the loss function. That would attenuate the damaging effects that may result from more severe output gaps and inflation instability. The consequence may be a policy stance that either supports or goes against what Brainard proposed.

3. The model

3.1. The economy

The model I use is a standard New Keynesian model that is augmented with a cost channel which is used by Surico (2008), Llosa (2009) and Tillmann (2009). The micro-founded version of the model can be found in Ravena and Walsh (2006). Here I begin with a log-linearized system of equations characterizing the equilibrium.

$$x_t = E_t x_{t+1} - \frac{1}{\sigma}\ i_t - E_t \pi_{t+1} - r_t^n \qquad \text{The IS curve} \tag{3.1}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa(\sigma + \eta) x_t + \kappa i_t \qquad \text{The Phillips curve} \tag{3.2}$$

$$r_t^n = \rho r_{t-1}^n + \varepsilon_t \tag{3.3}$$

In this model, the IS curve is derived from log-linearizing the Euler equation of household's utility maximization problem. It relates the output gap with the expected output gap and the real interest rate differential which is defined as the difference between the real interest rate $(i_t - E_t \pi_{t+1})$ and the natural interest rate r_t^n. The Phillips curve relates current inflation to expected inflation, the output gap and the nominal interest rate. It is derived from the firm's profit maximization problem. The interest rate enters to influence price and inflation because of interest cost incurred by firms financing their labor wage with short-term borrowing and this is considered as the cost channel. Here, the uncertain parameter κ governs both the direct effect of output gap and that of nominal interest rate on inflation. In the micro-founded version of the above log-linearized model, κ is an increasing function of the fraction of firms able to adjust each period. The natural interest rate has a persistent coefficient of $|\rho| < 1$ and ε_t is i.i.d. Other parameters include the discount rate $\beta \in (0,1)$, the intertemporal elasticity of substitution $\sigma > 0$ and the inverse of labor supply elasticity $\eta > 0$.

3.2. Monetary policy

There is a policy maker (the central bank) who controls nominal interest rates. Instead of explicitly stating the interest rate rule, I follow Giannoni and Woodford (2003) and assume that the policy maker commits to a specific targeting rule which describes monetary policy as maintaining a linear relation between target variables of the following form:

$$\pi_t = \phi x_t \tag{3.4}$$

Here, it is assumed that the policy maker simply commit to a rule that has the same form as the targeting rule that implements optimal discretionary equilibrium. The rule is maintained by the policy maker when adjusting his instrument, the short-term nominal interest rate. As advocated in Kara (2002) and Svensson (2002), in terms of communication to the public, the description of monetary policy with a specific targeting rule is an advantage over the general targeting regime which involves specifying a set of targets, target variables and a loss function. The specific targeting rule is also more robust to structural change of an economy compared to Taylor rules.

In (3.4), the parameter ϕ which characterizes the rule is chosen to minimize an infinite sum of period loss:

$$L = E\sum_{t=0}^{\infty}\left[\left(\frac{\theta}{\kappa}\right)\pi_t^2 + \ \sigma + \eta \ x_t^2\right] \tag{3.5}$$

where $E(.)$ is the unconditional expectation operator. Woodford (2003a) has shown that in the micro-founded version of the model being used here, (3.5) can be derived as the second order approximation of the representative consumer's utility function which represents the true social loss function. The unconditional expectation, as explained by Woodford (1999, 2003b), makes the choice of the optimal equilibrium independent of the state of the economy at the time the commitment is made.

4. Parameter uncertainty and the robust optimal policy

When the policy maker is uncertain about κ in the Phillips curve, he is uncertain about the effects of the output gap as well as the effect of nominal interest rate on inflation. As mentioned the policy maker does not know the distribution of κ , however he knows that κ lays somewhere in an interval $\mathrm{K} \equiv [\overline{\kappa}, \underline{\kappa}]$. I assume that in this circumstance the policy maker follows a min-max strategy, finding the policy that minimizes the consequences when κ realizes its worst-case value.

This approach is similar to Giannoni (2002, 2007) who makes use of a simultaneous-move, zero-sum game between the policy maker and an imaginary Nature in which the policy maker minimizes his loss knowing that Nature is going to maximize it. However, different from Giannoni (2002, 2007) which have to resort to a numerical verification of the existence of a Nash equilibrium, I show analytically that there exists a Nash equilibrium of the zero-sum game. The Nash equilibrium is the solution to the min-max problem.

I will model the zero-sum game and solve for the Nash equilibrium in three steps as follows:

Step 1: Solve for the unique bounded equilibrium that results from implementing the targeting rule (3.4) in the economy (3.1), (3.2), (3.3). Since this step is equivalent to solving a system of expectation difference equations, the condition for the existence of a unique bounded solution must be derived. Since the system contains no other pre-determined variables except the natural interest rate r_t^n, the unique bounded solution should be linear functions of the form:

$$x_t = f_x(\phi,\kappa) r_t^n \qquad \pi_t = f_\pi(\phi,\kappa) r_t^n \qquad i_t = f_i(\phi,\kappa) r_t^n \tag{4.1}$$

in which the coefficients f_x, f_π, f_i are functions of the targeting rule parameter ϕ and κ. I denote Φ as the set of ϕ that guarantees determinacy for each $\kappa \in [\underline{\kappa}, \overline{\kappa}]$.

Step 2: Make use of x_t and π_t in (4.1), and transform the loss function (3.5) into a function of parameter ϕ and κ.

Step 3: Formulate the zero-sum simultaneous-move game and solve for the Nash equilibrium.

4.1. Solving for the unique bounded equilibrium.

The system (3.1) - (3.4) can be reduced to a single difference equation in inflation (4.2) by using the IS equation and the targeting rule to substitute out the nominal interest rate and output gap in the Phillips curve:

$$\pi_t = \left[\frac{(\beta+\kappa)\phi+\kappa\sigma}{\phi-\kappa\eta}\right]E_t\pi_{t+1} + \left(\frac{\kappa\phi}{\phi-\kappa\eta}\right)r_t^n \tag{4.2}$$

The condition for equation (4.2) to have a unique bounded solution is:

$$-1 < \frac{(\beta+\kappa)\phi+\kappa\sigma}{\phi-\kappa\eta} < 1 \tag{4.3}$$

Here, the inequalities (4.3) define Φ, the set of ϕ that guarantees the determinacy for each value of $\kappa \in [\underline{\kappa}, \overline{\kappa}]$ as mentioned in Step 1 above. When κ is known with certainty, the policy maker is required to restrict his choice of the policy rule in Φ for the loss function defined in (3.5) to be well-defined.

Construct the determinacy set Φ:

In constructing Φ, I maintain a general assumption that $\kappa > 1-\beta$ as is the approach adopted in most of the literature regarding parameterization.[1] There are two cases for consideration, $\phi > \kappa\eta$ and $\phi < \kappa\eta$.

Case 1: If $\phi > \kappa\eta$, then $\frac{(\beta+\kappa)\phi+\kappa\sigma}{\phi-\kappa\eta} > 1$ since $(\beta+\kappa) > 1$, so that (4.3) is not satisfied.

Case 2: If $\phi < \kappa\eta$, then (4.3) is manipulated to:

$$(1-\beta-\kappa)\phi < \kappa(\eta+\sigma) \tag{4.4}$$

and

$$(1+\beta+\kappa)\phi < \kappa(\eta-\sigma) \tag{4.5}$$

[1] For example, see Woodford (1999, 2003a,b), Tillmann (2009a), see also Llosa (2009), Table 2 for a brief survey of the literature.

with the assumption that $\kappa > 1-\beta$, the set Φ is defined as:

$$\Phi \equiv \left\{ \phi \in R : \frac{\kappa(\eta+\sigma)}{(1-\beta-\kappa)} < \phi < \frac{\kappa(\eta-\sigma)}{(1+\beta+\kappa)} \right\} \text{ for } \kappa > 1-\beta \tag{4.6}$$

The determinacy set Φ for a given κ is illustrated by the yellow segments in Figure 2.1 for two situations when $\eta < \sigma$ and when $\eta > \sigma$.

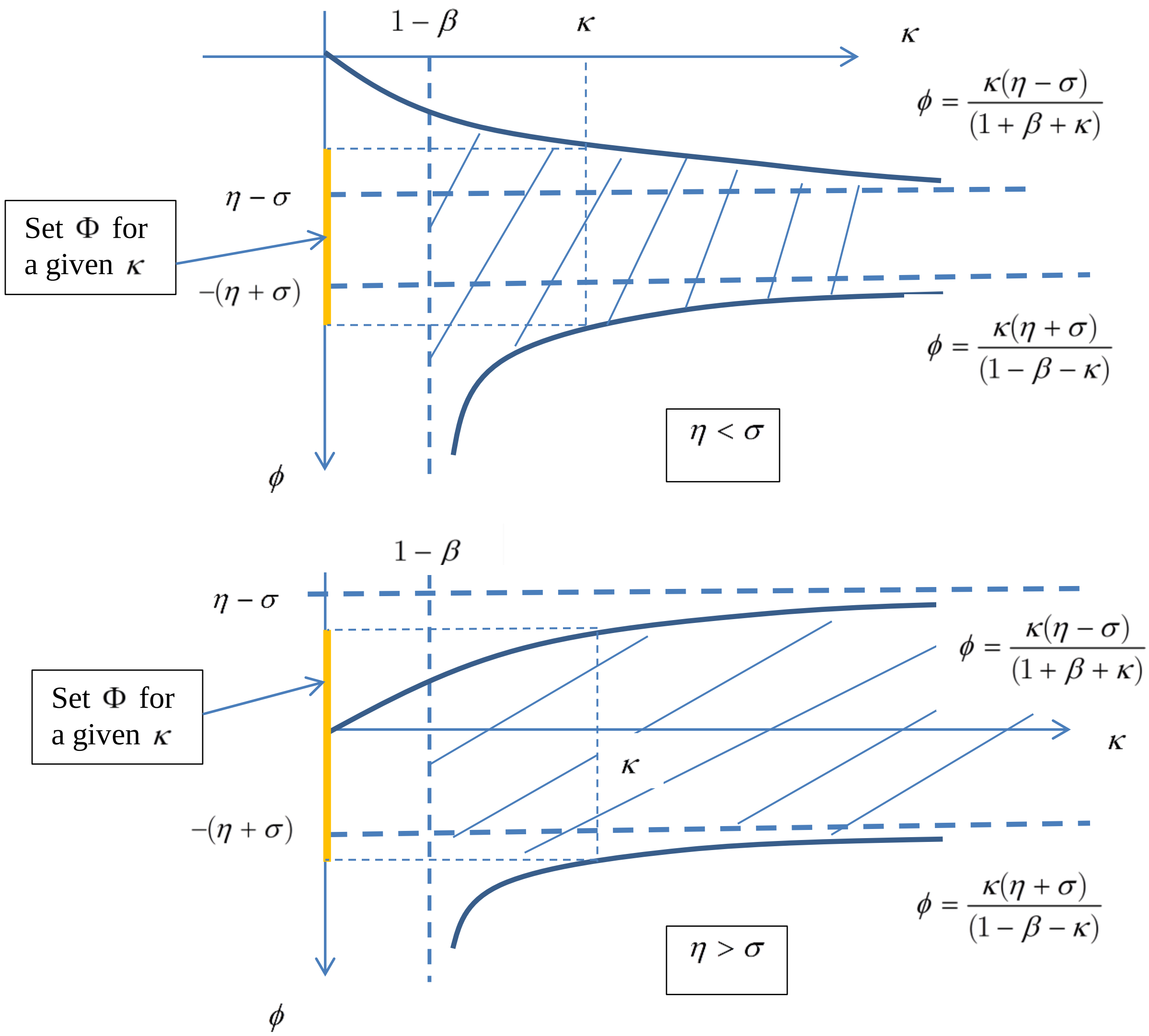

Figure 2.1 The determinacy set Φ for a given κ .

The unique bounded solution:

Solving (4.2) forward for π_t:

$$\pi_t = \left[\frac{(\beta+\kappa)\phi + \kappa\sigma}{\phi - \kappa\eta}\right] E_t \pi_{t+1} + \left(\frac{\kappa\phi}{\phi - \kappa\eta}\right) r_t^n$$

$$\Leftrightarrow \pi_t = \left(\frac{(\beta+\kappa)\phi + \kappa\sigma}{\phi - \kappa\eta}\right)^n E_t \pi_{t+n} + \left(1 + \rho\left(\frac{(\beta+\kappa)\phi + \kappa\sigma}{\phi - \kappa\eta}\right) + \ldots + \right)\left(\frac{\kappa\phi}{\phi - \kappa\eta}\right) r_t^n$$

$$\Leftrightarrow \pi_t = \left(\frac{\kappa\phi}{\phi[1-\rho(\kappa+\beta)] - \kappa(\sigma\rho+\eta)}\right) r_t^n \tag{4.7}$$

From the targeting rule (3.4):

$$x_t = \frac{\pi_t}{\phi} = \left(\frac{\kappa}{\phi[1-\rho(\kappa+\beta)] - \kappa(\sigma\rho+\eta)}\right) r_t^n \tag{4.8}$$

From the Phillips curve:

$$i_t = \frac{f_\pi(1-\beta\rho) + \kappa(\sigma+\eta)f_x}{\kappa} = \frac{(1-\rho\beta)\phi - (\sigma+\eta)\kappa}{\phi\ \ 1-\rho(\beta+\kappa)\ \ - \kappa(\rho\sigma+\eta)} r_t^n \tag{4.9}$$

Thus the unique bounded solution defined in (4.1) is characterized by:

$$f_\pi = \frac{\kappa\phi}{\phi[1-\rho(\kappa+\beta)] - \kappa(\sigma\rho+\eta)} \tag{4.10}$$

$$f_x = \frac{\kappa}{\phi[1-\rho(\kappa+\beta)] - \kappa(\sigma\rho+\eta)} \tag{4.11}$$

$$f_i = \frac{(1-\rho\beta)\phi - (\sigma+\eta)\kappa}{\phi\big(1-\rho(\beta+\kappa)\big) - \kappa(\rho\sigma+\eta)} \tag{4.12}$$

4.2. Transforming the loss function

Under the unconditional expectation, loss (3.5) can be reduced to:

$$L = E\sum_{t=0}^{\infty}\beta^t\left[\left(\frac{\theta}{\kappa}\right)\pi_t^2 + \ \sigma+\eta \ x_t^2\right] = \frac{\beta}{1-\beta}E\left[\left(\frac{\theta}{\kappa}\right)\pi_t^2 + \ \sigma+\eta \ x_t^2\right] \tag{4.13}$$

$$= \frac{\beta}{1-\beta}E\left[\left(\frac{\theta}{\kappa}\right)\pi_t^2 + \ \sigma+\eta \ x_t^2\right] = \frac{\beta}{1-\beta}\left[\left(\frac{\theta}{\kappa}\right)\mathrm{var}(\pi_t) + \ \sigma+\eta \ \mathrm{var}(x_t)\right] \tag{4.14}$$

The loss (4.14) can be transformed into a function of ϕ and κ by substituting (4.10) and (4.11) into (4.14):

$$\begin{aligned} L &= \frac{\beta}{1-\beta}\left[\left(\frac{\theta}{\kappa}\right)f_\pi^2 + \ \sigma+\eta \ f_x^2\right]\mathrm{var}(r_t^n) \\ &= \frac{\beta}{1-\beta}\left[\frac{\theta\kappa\phi^2 + \ \sigma+\eta \ \kappa^2}{\phi[1-\rho(\kappa+\beta)]-\kappa(\sigma\rho+\eta) \ ^2}\right]\mathrm{var}(r_t^n) \end{aligned} \tag{4.15}$$

Since $\mathrm{var}(r_t^n)$ are exogenous constants, minimizing (4.14) is equivalent to minimizing:

$$\tilde{L}(\phi,\kappa) = \left[\frac{\theta\kappa\phi^2 + \ \sigma+\eta \ \kappa^2}{\phi[1-\rho(\kappa+\beta)]-\kappa(\sigma\rho+\eta) \ ^2}\right] \tag{4.16}$$

4.3.Finding the Nash equilibrium.

<u>The zero-sum simultaneous-move game</u>

Given the transformed loss function (4.16), the policy maker's situation can now be modeled as a game between the policy maker and imaginary Nature in which the policy maker chooses a policy to minimize his loss while Nature chooses a value $\kappa \in [\underline{\kappa},\overline{\kappa}]$ to maximize his loss.

Here, the set of the policy maker's strategies needs some discussion. Since the set Φ specifies the allowable ϕ that results in determinacy for a given κ but the policy maker does not know κ, he needs to choose his policy from a set $\bar{\Phi}$ that guarantees him determinacy for all $\kappa \in [\underline{\kappa},\overline{\kappa}]$ values that may be realized.

The zero-sum game can be defined as $\Gamma = \langle (P, N), (\bar{\Phi}, \mathrm{K}), -\tilde{L}(\phi,\kappa), \tilde{L}(\phi,\kappa) \rangle$ in which *P* stands for the policy maker, *N* stands for Nature, $\bar{\Phi}$ and $K \equiv [\underline{\kappa}, \bar{\kappa}]$ are the separate strategy sets of the policy maker and Nature. As shown in Giannoni (2002, Proposition 1), if the game Γ has a Nash equilibrium, the profile $\langle \phi^*, \kappa^* \rangle$ is a Nash equilibrium of Γ, if and only if the choice of each player is a max-minimizer:

$$\phi^* = \arg\max_{\phi\in\bar{\Phi}} \min_{\kappa\in\mathrm{K}} [-\tilde{L}(\phi,\kappa)] = \arg\min_{\phi\in\bar{\Phi}} \max_{\kappa\in\mathrm{K}} [\tilde{L}(\phi,\kappa)]$$

$$\kappa^* = \arg\max_{\kappa\in\mathrm{K}} \min_{\phi\in\bar{\Phi}} [\tilde{L}(\phi,\kappa)]$$

The above proposition guarantees that the solution to the min-max problem is the Nash equilibrium, if there is one that exists.

<u>Nash equilibrium</u>

The policy maker's problem: The policy maker chooses a policy rule ϕ from a set $\bar{\Phi} \equiv \left\{\phi \in R : \phi_{\min} \le \phi \le \phi_{\max}\right\}$ to minimize the loss defined in (4.16) for a given value $\kappa \in [\underline{\kappa}, \bar{\kappa}]$:

$$\min_{\phi\in\bar{\Phi}} \left[\frac{\theta\kappa\phi^2 + \left(\sigma+\eta\right)\kappa^2}{\left(\phi[1-\rho(\kappa+\beta)] - \kappa(\sigma\rho+\eta)\right)^2} \right] \tag{4.17}$$

Here, some discussion on the construction of the set $\bar{\Phi}$ is needed before any possible solution is to be found, since it is not always possible for the policy maker to find the set $\bar{\Phi}$ if the interval of κ that he should consider from empirical estimates has too high an upper bound. To show this situation, suppose for the moment that the policy maker was solving an unconstrained problem.

The first order derivative is as follows:

$$\frac{d\tilde{L}}{d\phi} = \frac{2\kappa^2\left(-\theta\phi(\sigma\rho+\eta) - (\sigma+\eta)[1-\rho(\kappa+\beta)]\right)}{\left[\phi\left(1-\rho(\beta+\kappa)\right) - \kappa(\rho\sigma+\eta)\right]^3} \tag{4.18}$$

and the interior solution: $$\phi^*(\kappa) = \frac{-(\sigma+\eta)\left(1-\rho(\beta+\kappa)\right)}{\theta(\rho\sigma+\eta)} \tag{4.19}$$

The interior solution line is illustrated in Figure 2.2 for the case $\eta < \sigma$.

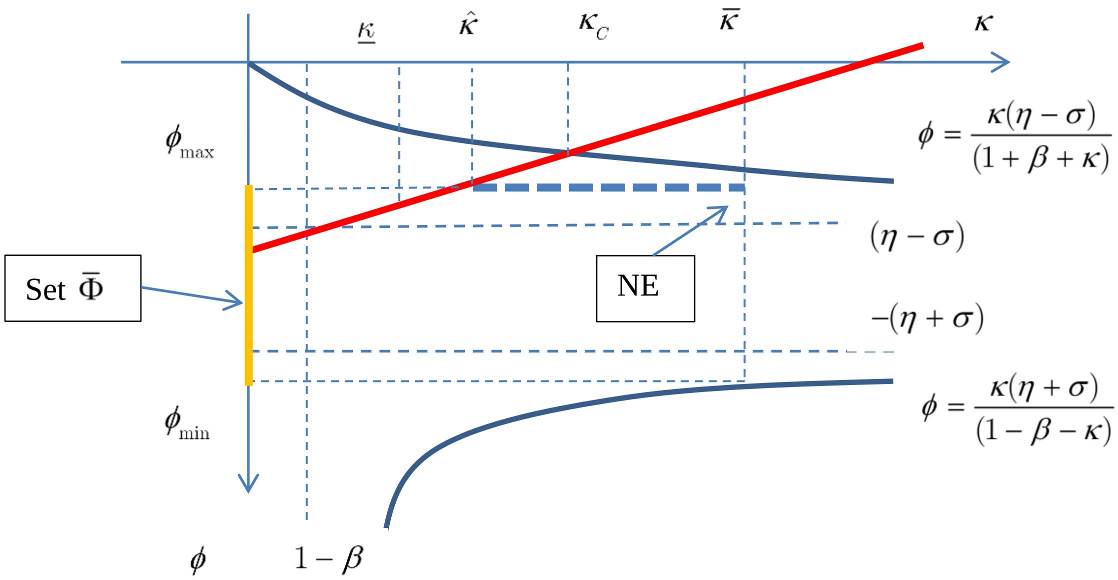

Figure 2.2 Spurious solution

Suppose now for a given interval $[\underline{\kappa}, \bar{\kappa}]$, the policy maker defines the set $\bar{\Phi}$ by choosing a $\phi_{\max}$ and a $\phi_{\min}$ that are distant from the upper and lower determinacy bounds, defined by (4.6), at $\kappa = \bar{\kappa}$ by a sufficiently small ε. The set $\bar{\Phi}$ is illustrated by the yellow segment in Figure 2.2.

$$\bar{\Phi} \equiv \left\{\phi \in R : \frac{\bar{\kappa}(\eta+\sigma)}{(1-\beta-\bar{\kappa})} + \varepsilon \leq \phi \leq \frac{\bar{\kappa}(\eta-\sigma)}{(1+\beta+\bar{\kappa})} - \varepsilon\right\} \tag{4.20}$$

Denote κ_C and $\hat{\kappa}$ the values of κ at which the interior solution line intersects with the upper determinacy bound and with the $\phi_{\max}$ line.

Consider the first order derivative (4.18). It can be seen that in the region containing combinations of (ϕ, κ) in which $\kappa > 1-\beta$ and $\phi \in \Phi$ for a given κ (the shaded area in

Figure 2.1), the denominator of (4.18) is negative. Indeed, the determinacy condition (4.3) implies that in this region:

$$\rho\left(\frac{(\beta+\kappa)\phi+\kappa\sigma}{\phi-\kappa\eta}\right)<1$$
$$\Rightarrow \phi[1-\rho(\kappa+\beta)]-\kappa(\sigma\rho+\eta)<0 \tag{4.21}$$

Also, in this region, the area below the interior solution line (4.19) is associated with combinations (ϕ,κ) that make the nominator of (4.18) positive.

The above analysis implies that if $\bar{\kappa}>\kappa_C$ and with the set $\bar{\Phi}$ so defined, the policy maker's solution is always:

$$\phi^*(\kappa) \text{ for } \underline{\kappa}\le\kappa\le\hat{\kappa} \text{ and } \phi_{\max} \text{ for } \hat{\kappa}<\kappa\le\bar{\kappa} \tag{4.22}$$

The best response (4.22) may lead the policy maker to commit to a spurious solution in the sense that the solution he chooses may turn out to result in infinite loss. Indeed, suppose there exists a Nash equilibrium where Nature chooses $\bar{\kappa}$ as her best response. The Nash equilibrium is denoted NE in Figure 4.2, a point very close to the upper determinacy bound. Although other model parameters (the $\beta,\sigma,\eta,\theta,\rho$) are assumed to be known, the policy maker never knows their true values. There might be a possibility that these parameters realize values that are different from those values the policy maker uses to compute his upper determinacy bound in a way in which the true bound becomes low enough to send the Nash equilibrium point into the true indeterminacy region. In this situation, there exist an infinite number of equilibriums, some of which have a very large variance. What this means is that the policy maker is in fact facing an infinite loss.

If $\bar{\kappa}$ is sufficiently lower than κ_C, then the policy maker can avoid the above situation as he will be guaranteed with an interior solution for all values of $\kappa\in[\underline{\kappa},\bar{\kappa}]$. The set $\bar{\Phi}$ is also easily defined as:

Case $\eta<\sigma$: $$\bar{\Phi}\equiv\left\{\phi\in R:\frac{\bar{\kappa}(\eta+\sigma)}{(1-\beta-\bar{\kappa})}\le\phi\le\frac{\bar{\kappa}(\eta-\sigma)}{(1+\beta+\bar{\kappa})}\right\}$$

Case $\eta > \sigma$:
$$\overline{\Phi} \equiv \left\{ \phi \in R : \frac{\overline{\kappa}(\eta+\sigma)}{(1-\beta-\overline{\kappa})} \le \phi \le \frac{\underline{\kappa}(\eta-\sigma)}{(1+\beta+\underline{\kappa})} \right\}$$

The set $\overline{\Phi}$ for the case $\overline{\kappa} < \kappa_C$ is illustrated in Figure 2.3 and Figure 2.4.

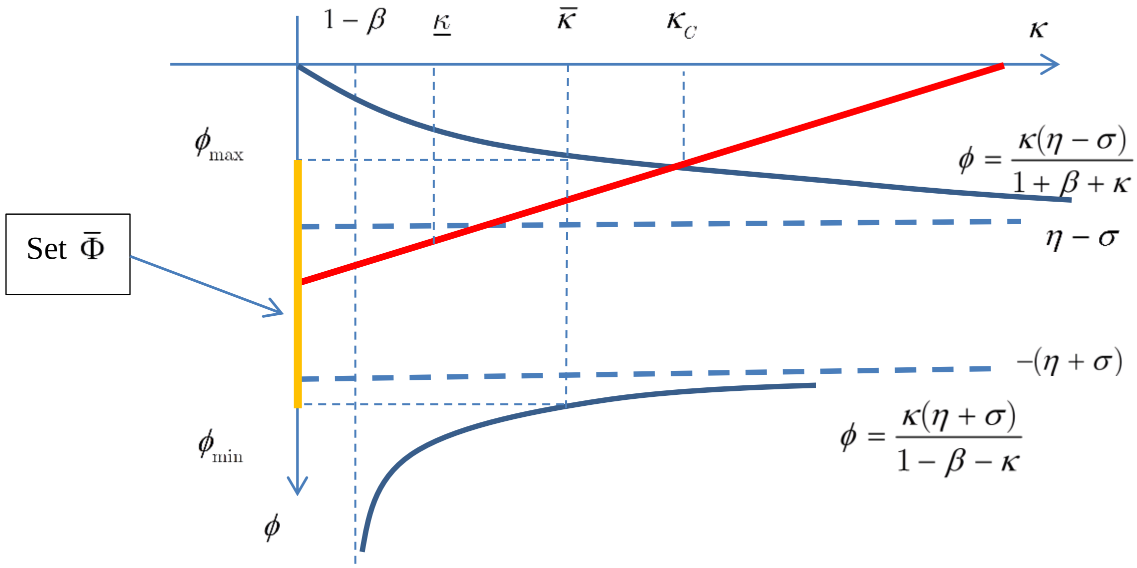

Figure 2.3 The set $\overline{\Phi}$, the case $\eta < \sigma$

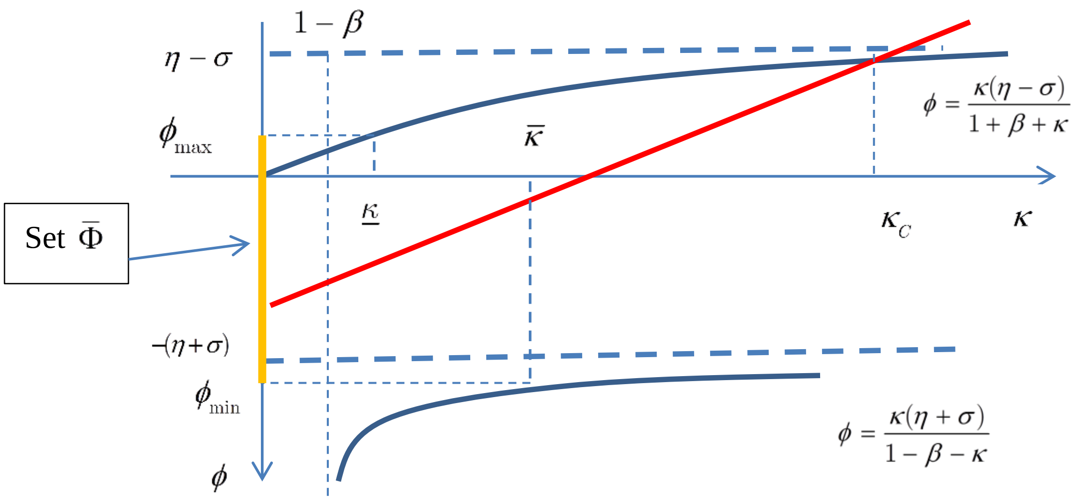

Figure 2.4 The set $\overline{\Phi}$, the case $\eta > \sigma$

The above analysis presents the policy maker with another difficulty which includes in practice whether or not the empirical estimates of κ support the case $\bar{\kappa} < \kappa_C$. To examine all of the relative positions of $\bar{\kappa}$ and κ_C, I consider a wide range of parameterization for those parameters that determine κ_C and then compare the minimum value of κ_C with the maximum value of κ which prevails empirically.

For the interval of κ, I consider two estimates by Woodford (1999) for a model without a cost channel and that of Tillman (2008) for a model with a cost channel. The two estimates are presented in Table 2.1. The interval of κ can be chosen with a lower bound that is two-sigma below Woodford's mean estimate and a maximum value two-sigma above Tillman's mean estimate of κ. The κ interval is then computed as [0.017, 0.037] .

	Mean	Standard deviation
Woodford (1999)	0.024	0.0035
Tillman (2008)	0.035	0.001

Table 2-1. Estimates of κ

σ	θ	ρ	β	η
[0.157,2]	[1.01,7.61]	[0.35,0.8]	0.99	1

Table 2-2.Parameterization of $\beta, \eta, \sigma, \theta, \rho$

Since κ_C is determined at the intersection of the upper determinacy bound and the interior solution line by the equation:

$$\frac{-(\sigma+\eta)\ 1-\rho(\beta+\kappa)}{\theta(\rho\sigma+\eta)} = \frac{\kappa(\eta-\sigma)}{(1+\beta+\kappa)}$$

Denote: $\kappa_C = f(\beta, \eta, \sigma, \theta, \rho)$.

The parameterization to compute the minimum value of κ_C is presented in Table 2.2. In this parameterization, I choose $\beta = 0.99$, $\eta = 1$ as is standard practice in the literature. The above ranges are supported by those parameterizations presented in Rabanal (2006), Woodford (1999), Giannoni (2002, 2007), Lubik (2004), Surico (2010). With this parameterization, the minimum value of κ_C is found at $\min(\kappa_C) = 0.0516$. It can be seen that the value of $\bar{\kappa}$ that is supported by empirical evidence is significantly less than the computed κ_C. The policy maker is guaranteed with interior solutions and his best response is defined by (4.19).

I consider how sensitive this conclusion is to some changes in the parameters β and η by doing the following experiments. I will first consider lower values for β while keep other parameters unchanged and then repeat the same experiment with η, however considering values of η that are both lower and higher than unity. The results in Table 2-3 and 2-4 show that $\min(\kappa_C)$ tends to increase with lower β and higher η. In particular, when η is smaller than 0.5 then $\min(\kappa_C)$ goes below $\bar{\kappa}$. If we maintain the assumption that $\eta > 0.5$, the parameterization in Table 2-2 can guarantee the above relative positions of $\bar{\kappa}$ and $\min(\kappa_C)$.

β	0.9	0.95	0.99
$\min(\kappa_C)$	0.0674	0.0588	0.0516

Table 2-3. Sensitivity of $\min(\kappa_C)$ to β

η	0.5	1	1.5
$\min(\kappa_C)$	0.0376	0.0516	0.0858

Table 2-4. Sensitivity of $\min(\kappa_C)$ to η

Nature's best response:

On the side of Nature, it is not necessary to find her best response to the whole set $\bar{\Phi}$ but only to those $\phi's$ which are the policy maker's best response for some κ. The reason is that there is no justification for the policy maker to use any strategy which is not a best response to any $\kappa \in [\underline{\kappa}, \bar{\kappa}]$. In other words, Nature only cares about those policy rules that might be in a Nash equilibrium. Denote such a set of ϕ the $\hat{\Phi} \equiv \left\{ \begin{array}{l} \phi \in R : \phi = \dfrac{-(\sigma+\eta)\left(1-\rho(\beta+\kappa)\right)}{\theta(\rho\sigma+\eta)} \\ \kappa \in [\underline{\kappa}, \bar{\kappa}] \end{array} \right\}$

Nature's problem can then be stated as follows:

$$\max_{\kappa \in [\underline{\kappa}, \bar{\kappa}]} \left[\frac{\theta\kappa\phi^2 + \left(\sigma+\eta\right)\kappa^2}{\left(\phi[1-\rho(\kappa+\beta)] - \kappa(\sigma\rho+\eta)\right)^2} \right] \text{ for a given } \phi \in \hat{\Phi}.$$

The first order derivative:

$$\frac{\partial \tilde{L}(\phi,\kappa)}{\partial \kappa} = \frac{\phi \;\; \theta\phi^2 \;\; 1-\beta\rho+\rho\kappa \;\; + \kappa\theta\phi(\sigma\rho+\eta) + 2\kappa(\sigma+\eta) \;\; 1-\rho\beta}{\phi[1-\rho(\kappa+\beta)] - \kappa(\sigma\rho+\eta) \;\;^3} \tag{4.23}$$

As shown in (4.21) for $\phi \in \Phi$ and $\kappa > 1-\beta$, the denominator of (4.23):

$$\phi[1-\rho(\kappa+\beta)] - \kappa(\sigma\rho+\eta) \;\;^3 < 0 \tag{4.24}$$

The sign of the first order derivative (4.23) is determined by the sign of its nominator. Since the set $\hat{\Phi}$ contains only those choices defined by the policy maker as his best response $\phi^*(\kappa)$ as represented by (4.19), the sign of the nominator can be found by substituting $\phi^*(\kappa)$ into the nominator of (4.23):

The nominator of (4.23) =

$$\phi^*(\kappa)\left(\theta \;\; \phi^*(\kappa) \;\;^2 \;\; 1-\beta\rho+\rho\kappa \;\; + \kappa\theta(\sigma\rho+\eta)\left(\frac{-(\sigma+\eta) \;\; 1-\rho(\beta+\kappa)}{\theta(\rho\sigma+\eta)}\right) + 2\kappa(\sigma+\eta) \;\; 1-\rho\beta \;\;\right)$$

$$= \underbrace{\phi^*(\kappa)}_{<0}\left(\underbrace{\theta\ \phi^*(\kappa)^{\ 2}\ 1-\beta\rho+\rho\kappa\ +(\sigma+\eta)\kappa+\kappa(\sigma+\eta)\ 1-\rho\beta}_{>0}\right)<0 \tag{4.25}$$

From (3.24) and (3.25), the first order derivative $\dfrac{\partial\tilde{L}(\phi,\kappa)}{\partial\kappa}>0$. The best response of Nature to the policy maker's best response is always $\bar{\kappa}$. The Nash equilibrium $\left(\dfrac{-(\sigma+\eta)\ 1-\rho(\beta+\bar{\kappa})}{\theta(\rho\sigma+\eta)},\bar{\kappa}\right)$ is illustrated in Figure 2.5.

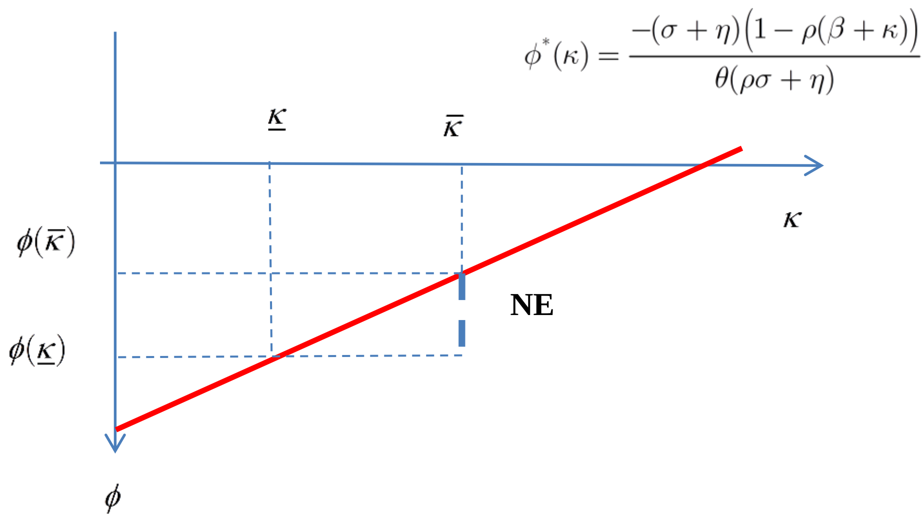

Figure 2.5 Nash Equilibrium

5. Model Implications regarding Brainard's principle

Under parameter uncertainty with $\bar{\kappa}$ being considered the worst-case value for κ, the robust targeting rule:

$$\pi_t = \phi^*(\bar{\kappa})x_t \tag{5.1}$$

with $$\phi^*(\bar{\kappa}) = \frac{-(\sigma+\eta)\left(1-\rho(\beta+\bar{\kappa})\right)}{\theta(\rho\sigma+\eta)}$$

It can be seen that $\phi^*(\bar{\kappa}) \geq \phi^*(\kappa)$ for all values of $\kappa \in [\underline{\kappa}, \bar{\kappa}]$. Under uncertainty about κ, the policy maker is willing to take less variation in inflation relative to variation in output compared to the "certainty" case. The intuition for this is that when the economy is hit by a positive demand shock, the policy maker needs to raise nominal interest rates to suppress output. Since the interest rate exists in the Phillips curve, this creates two conflicting effects on inflation. On one hand, reducing the output gap helps to decrease inflation. On the other hand, raising interest rate increases inflation. When $\bar{\kappa}$ is considered the worst-case value, the net effect on inflation is amplified. To be safe, therefore the policy maker would engineer a smaller change in inflation in response to a given change in output gap.

However, this may not be understood as an attenuated monetary policy. In the sense of the Brainard's principle, when faced with model uncertainty, a cautious policy maker would move his interest rate by less than what he computes if he knew the economy with certainty. To reassess the Brainard's principle, I compare the interest rate response to a shock that results from the implementation of the robust targeting rule and the non-robust targeting rule.

To be more precise, consider the economy for a given true value for κ. Equation (4.9) gives the interest rate response to r_t^n given κ and ϕ:

$$i_t = f_i(\phi, \kappa) r_t^n = \frac{(1-\rho\beta)\phi - (\sigma+\eta)\kappa}{\phi \;\; 1-\rho(\beta+\kappa) \;\; -\kappa(\rho\sigma+\eta)} r_t^n$$

If the policy maker is uncertain he uses a value of $\phi = \phi^*(\bar{\kappa})$ and the interest rate response is given by:

$$i_t^R = f_i^R r_t^n = \frac{(1-\rho\beta)\phi^*(\bar{\kappa}) - (\sigma+\eta)\kappa}{\phi^*(\bar{\kappa})\big(1-\rho(\beta+\kappa)\big) - \kappa(\rho\sigma+\eta)} r_t^n \tag{5.2}$$

If the policy maker knows the true value for certain, he uses a value of $\phi = \phi^*(\kappa)$ and the interest rate response is given by:

$$i_t^{NR} = f_i^{NR} r_t^n = \frac{(1-\rho\beta)\phi^*(\kappa) - (\sigma+\eta)\kappa}{\phi^*(\kappa)\big(1-\rho(\beta+\kappa)\big) - \kappa(\rho\sigma+\eta)} r_t^n \tag{5.3}$$

where f_i^R and f_i^{NR} are the coefficients in the case of robust and non-robust rules respectively. Consider the difference $f_i^R - f_i^{NR}$:

$$f_i^R - f_i^{NR} = \frac{\kappa\left(\phi^*(\bar{\kappa}) - \phi^*(\kappa)\right)\left((\sigma+\eta)(1-\rho\beta-\rho\kappa) - (1-\rho\beta)(\sigma\rho+\eta)\right)}{\left(\phi^*(\bar{\kappa})(1-\rho\beta-\rho\kappa) - \kappa(\sigma\rho+\eta)\right)\left(\phi^*(\kappa)(1-\rho\beta-\rho\kappa) - \kappa(\sigma\rho+\eta)\right)} \tag{5.4}$$

(5.4) shows that f_i^R can be lower than or greater than f_i^{NR}.

Indeed, the denominator is positive because $\left(\phi^*(\kappa)(1-\rho\beta-\rho\kappa) - \kappa(\sigma\rho+\eta)\right) < 0$ for all values of $\kappa \in [\underline{\kappa}, \bar{\kappa}]$ as shown in (4.21). Given that $\phi^*(\bar{\kappa}) - \phi^*(\kappa) > 0$, there is a threshold value of κ to determine the sign of $f_i^R - f_i^{NR}$ that can be computed as follows:

$$\tilde{\kappa} = \frac{\sigma(1-\rho)(1-\rho\beta)}{\rho(\sigma+\eta)} \tag{5.5}$$

With the first order derivatives of $\tilde{\kappa}$ with respect to σ and ρ, it can be verified that $\tilde{\kappa}$ is increasing in σ and decreasing in ρ. For σ being constant, when ρ tends to zero, $\tilde{\kappa}$ approaches infinity while $\tilde{\kappa}$ equals zero when ρ equals to one. This means, for a given σ, if the cost shock becomes very persistent, f_i^R can be greater than f_i^{NR}. Similarly, f_i^R can be smaller than f_i^{NR} if the cost shock is very transitory. The same analysis is applied to the influence of σ and it is found that if σ keeps increasing or decreasing, there will be a possibilities for which $f_i^R > f_i^{NR}$ or $f_i^R < f_i^{NR}$.

When σ and ρ vary in the ranges chosen in Table 4.1, the minimum value of $\tilde{\kappa}$ is reached at $\min(\tilde{\kappa}) = 0.0179$, smaller than the $\bar{\kappa}$ considered by the policy maker. This implies that there are possibilities that $\tilde{\kappa}$ is less than $\bar{\kappa}$ for some parameterizations of σ and ρ. In this situation, $f_i^R > f_i^{NR}$ for $\underline{\kappa} \le \kappa < \tilde{\kappa}$ and $f_i^R < f_i^{NR}$ for $\tilde{\kappa} \le \kappa < \bar{\kappa}$. This means the Brainard's principle does not apply. Figure 2.6 illustrates a situation in which $\sigma = 1$ and $\rho = 0.8$. The threshold value in this case is $\tilde{\kappa} = 0.026$.

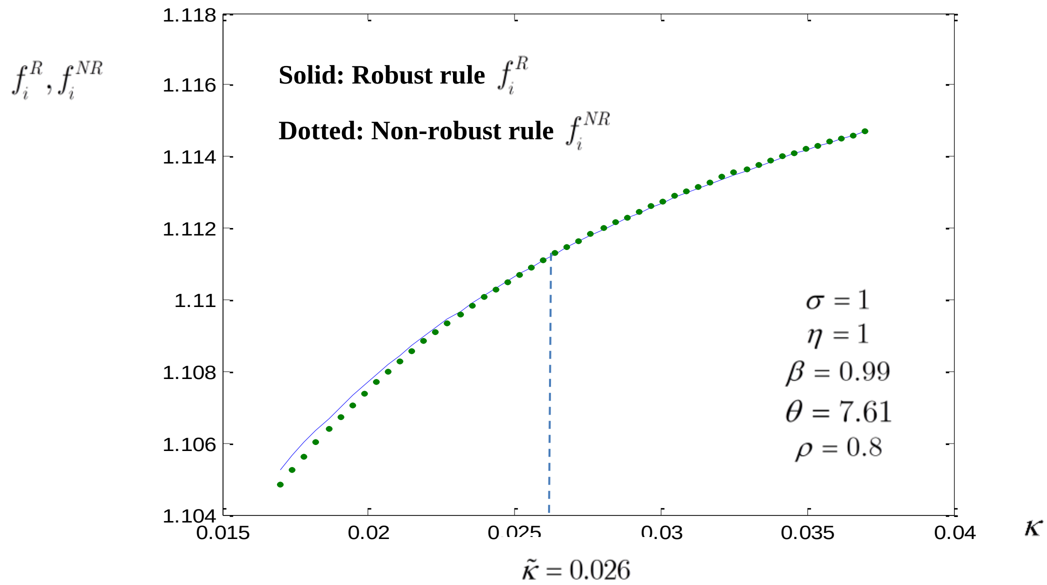

Figure 2.6. Interest rate response: Robust rule vs. Non-robust rule

6. Some welfare analysis.

6.1. How large should the κ interval be?

It is obvious that in the min-max equilibrium, the policy maker always considers $\bar{\kappa}$ to be the worst-case value so long as $\bar{\kappa} < \kappa_C$. This gives rise to the question as to how high the value of $\bar{\kappa}$ should be since it would be natural for policy maker to want to guard against larger uncertainty. This presents the policy maker with a choice of decisions and a cost/benefit analysis.

Suppose the policy maker wants to guard against an initial worst-case value of κ,denoted the $\bar{\kappa}$. However, it is considered that the true value of κ lies somewhere at the further right of this current $\bar{\kappa}$. Therefore he should consider a larger κ interval with higher upper bound, the $\bar{\kappa} + \delta\kappa$, where $\delta\kappa$ is the increment in the upper bound. By following the min-max strategy, he should move to protect himself against the new worst-case κ, which is $\bar{\kappa} + \delta\kappa$. Doing so, he may reduce the loss if the true κ value turns out to be the new worst-case of $\bar{\kappa} + \delta\kappa$;this is the benefit of considering the higher upper bound value for κ. However, this also entails the risk that the true κ value is not at that new upper bound. As a result, by using this new policy rule, his actual loss might increase - this is the cost of considering a higher upper bound. Since the policy maker does not know the true value of κ, he needs to benchmark the value of κ for the estimation of this possible increase in loss. Formally, the analysis can be undertaken as follows.

Denote:

- $\tilde{L}\left(\phi^*(\bar{\kappa}+\delta\kappa), \bar{\kappa}+\delta\kappa\right)$: The loss that results from the new robust policy if the true value of κ turns out to be at the new upper bound $\bar{\kappa}+\delta\kappa$.
- $\tilde{L}\left(\phi^*(\bar{\kappa}), \bar{\kappa}+\delta\kappa\right)$: The loss that results from the initial robust policy when the true value of κ turns out to be at the new upper bound $\bar{\kappa}+\delta\kappa$.
- The loss saved by using the new rule $\phi^*(\bar{\kappa}+\delta\kappa)$ instead of the initial rule $\phi^*(\bar{\kappa})$ when the true κ value turns out to be at the new upper bound $\bar{\kappa}+\delta\kappa$:

$$B = \tilde{L}\left(\phi^*(\bar{\kappa}), \bar{\kappa}+\delta\kappa\right) - \tilde{L}\left(\phi^*(\bar{\kappa}+\delta\kappa), \bar{\kappa}+\delta\kappa\right)$$

- $\tilde{L}\left(\phi^*(\bar{\kappa}+\delta\kappa), \kappa_E\right)$ and $\tilde{L}\left(\phi^*(\bar{\kappa}), \kappa_E\right)$: The estimated losses that result from the new robust and the initial robust rule, and are based on the benchmark κ_E.
- The loss that is increased by using the new rule $\phi^*(\bar{\kappa}+\delta\kappa)$ instead of the initial rule $\phi^*(\bar{\kappa})$ when the loss estimation is based on the benchmark κ_E is as follows:

$$C = \tilde{L}\left(\phi^*(\bar{\kappa}+\delta\kappa), \kappa_E\right) - \tilde{L}\left(\phi^*(\bar{\kappa}), \kappa_E\right)$$

Figure 2.7 shows the cost/benefit ratio C/B when the policy maker considers the value of $\kappa_E = 0.024$, the mean value of the estimates undertaken by Woodford (1999). The initial maximum value is $\bar{\kappa}_{initial} = 0.037$, which is the two-sigma upper bound of Tillman's (2008) estimate. The policy maker then expands the maximum value to $\bar{\kappa}_{final} = 0.037$ which is the two-sigma upper bound of the estimate undertaken by Tillman (2008). In the graph, values for $(\bar{\kappa}+\delta\kappa)$ run from $\bar{\kappa}_{initial}$ to $\bar{\kappa}_{final}$ when $\delta\kappa$ assumes an increasing value from zero to $(\bar{\kappa}_{final} - \bar{\kappa}_{initial})$. The cost/benefit analysis shows that his consideration of a higher upper bound value for is not supported up to some κ value near 0.05 when the cost/benefit ratio goes beyond one.

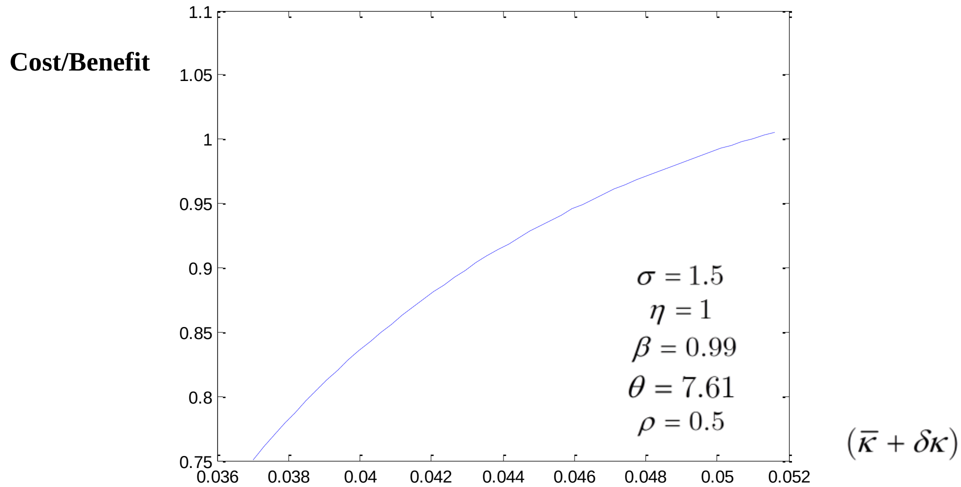

Figure 2.7. Protecting against larger uncertainty: A cost/benefit analysis

6.2. Robust rule vs. non-robust rule: A welfare comparison.

It is obvious that the robust rule results in the lowest loss value if the worst case scenario value of κ is realized. However, the robust rule may perform poorly for the rest of the κ values. In this section, I compare the welfare outcomes of the robust rule used by a policy maker who cares about the uncertainty about κ and the welfare outcome of the non-robust rule used by a policy maker who is not concerned with parameter uncertainty. His policy is based on an assumed value of κ_b, which is assumed to be at the middle of the κ interval.

Denote:

- $\tilde{L}\left(\phi^*(\bar{\kappa}),\kappa\right) \equiv L^R$: Loss resulted from using the robust rule
- $\tilde{L}\left(\phi^*(\kappa_b),\kappa\right) \equiv L^b$: Loss resulted from using the non-robust rule

Figure 2.8 shows the welfare comparison of L^R and L^b . As argued by Tillmann (2009b,c) and Jensen (2002), comparing the loss values directly may not have any economic meaning. The extra losses that result from the worse policy, for example $\left(L^b - L^R\right)$, should be interpreted as being equivalent to a loss that results from a permanent increase of inflation from its target due to the use of the worse policy. Jensen (2002) terms this measure “inflation equivalent” and suggests a procedure for its computation. The application of this measure is performed in Tillman (2009b,c). The formulation of this measure is as follows:

The loss with a permanent increase in inflation of π^{eqv} above its target can be computed as:

$$
\begin{aligned}
L^{eqv} &= E\left(\sum_{t=0}^{\infty}\beta^t\left(\lambda x_t^2+(\pi_t+\pi^{eqv})^2\right)\right)\\
&= \frac{1}{1-\beta}\left(E(\lambda x_t^2+\pi_t^2)+2\operatorname{cov}(\pi_t\,\pi^{eqv})+(\pi^{eqv})^2\right)\\
&= \frac{1}{1-\beta}E(\lambda x_t^2+\pi_t^2)+\frac{1}{1-\beta}(\pi^{eqv})^2
\end{aligned}
$$

With the original loss function defined in (4.16), the inflation equivalent is computed as:

$$
\begin{aligned}
&\frac{1}{1-\beta}\left(\pi^{eqv}\right)^2=\frac{\sigma_r^2}{1-\beta}\left|\tilde{L}\left(\phi^*(\kappa^b),\kappa\right)-\tilde{L}\left(\phi^*(\kappa^*),\kappa\right)\right|=\frac{\sigma_u^2}{1-\beta}\left|L^b-L^R\right|\\
&\Leftrightarrow \pi^{eqv}=\sigma_r\sqrt{\left|L^b-L^R\right|}
\end{aligned}
$$

where σ_r^2 is the demand shock variance.

It can be confirmed from the graph that except for the worst case scenario, the robust rule performs worse than the non-robust rule for a large range of κ. However, in the worst case, in comparison to the robust policy, the non-robust policy results in a loss equivalent to that resulted from a permanent increase in inflation of 0.013 percentage point.

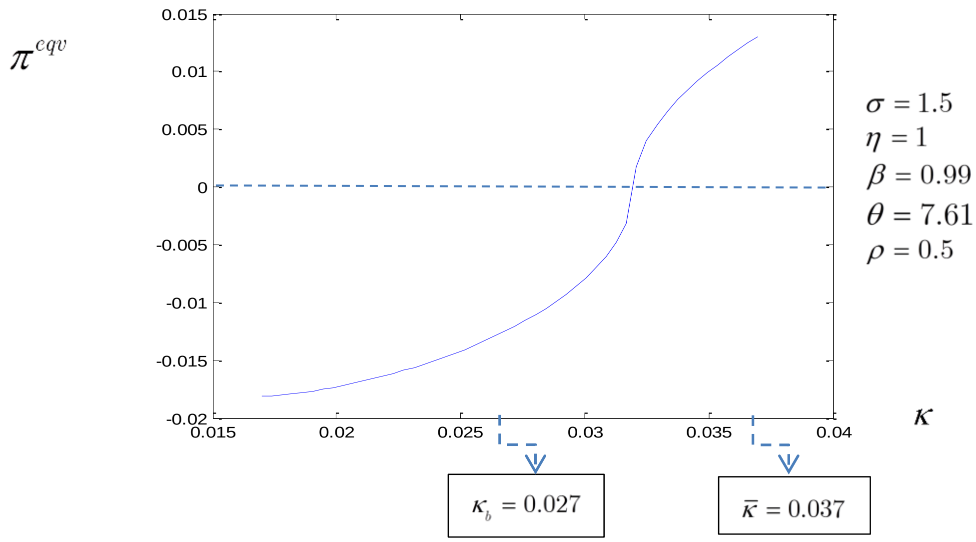

Figure 2.8. Robust vs. Non-robust rule: Welfare comparison

The loss difference in term of inflation equivalent measure is not significant in this model with its parameterization. This is due to the fact that the κ interval in consideration is not very large to allow for a significant distance between the worst-case $\bar{\kappa}$ and the benchmark κ^b which is assumed to be at the middle of the κ interval. In this case, the values of $\phi^*(\kappa^b)$ and $\phi^*(\bar{\kappa})$ are -0.0923 and -0.0913 respectively. This analysis points out an important implication that parameter uncertainty and the need of a robust policy is not very quantitatively important given this parameterization.

7. Conclusion.

Since the seminal paper by Brainard (1967) which proposes that policy response is cautious in an uncertain economic environment, a vast research program has been conducted to re-investigate the proposal using different approaches and settings. The results from this analysis are inconclusive but the prevailing evidence seems to support Brainard's principle. This chapter contributes to the literature by re-assessing the principle in a New Keynesian framework with a cost channel.

I find that the Brainard's principle does not apply in a model where the κ coefficient in the Phillips curve is assumed to be uncertain. The interest rate response that results from a concern about uncertainty and the use of robust policy is therefore not necessary stronger than that which results from using a non-robust policy which disregards uncertainty. In addition, robust policy underperforms non-robust policy for the majority of κ values in the interval considered. It is also found that, the policy maker has a tendency to consider a higher upper bound for his κ interval because the estimated cost of moving to guard him against larger uncertainty is less than the loss that might be saved if the true κ is actually in the interval with higher upper bound.

CHAPTER 3
OPTIMAL MONETARY POLICY DELEGATION UNDER PARAMETER UNCERTAINTY

Abstract

I model an economy in which there is policy planner who faces uncertainty about the slope of the Phillips curve and a central bank who knows the economy with certainty. The policy planner makes use of a delegation method similar to the one initiated by Woodford (1999) to induce his central banker to implement his min-max commitment policy. The policy planner solves for parameters of the delegated loss criterion by matching his min-max optimal policy with the discretionary policy conducted by his central banker. Delegation under parameter uncertainty specifies choosing a central bank with an output gap stabilization weight of less than one. In comparison to delegation without parameter uncertainty, this robust delegation can save a loss equivalent to that which results from a permanent increase in inflation of 0.06 percentage point from target. In addition, the robust delegation is found to dominate standard discretionary policy.

1. Introduction.

Woodford (1999) starts a vast research program on using delegation to correct for the problem of stabilization bias that results from lack of inertia in discretionary policy. However, the current literature leaves unexhausted an area of delegation under model uncertainty. This chapter therefore contributes to the literature in this regard.

I assume in this chapter a New Keynesian economy with three agents: a policy planner, a central banker and a private sector. The policy maker is assumed to know the structure of his economy but is ambiguous about its parameters. He has a set of econometric estimates of the economy's parameters. In contrast, the central banker believes there is no uncertainty and has her own beliefs about the parameters. The policy planner would like to control the economy using the best policy, which is invariably a commitment one. However, he has to deal with two problems. First, facing model uncertainty, he would need to figure out what is the best commitment policy he wants under model uncertainty. Second, how should he deliver his commitment policy outcome given the fact that his central banker always conducts discretionary policy?

To solve the first problem, I follow a similar approach to Giannoni (2002, 2007) and assume that when faced with parameter uncertainty, the policy planner would like to derive a min-max policy that results in the lowest social loss value when the worst-case parameters are realized. With regards to the second problem, the standard approach for the policy planner is to design a delegation scheme which induces the central banker to implement the commitment equilibrium. The way to solve for the delegation function parameters, for instance in Bilbiie (1999), might be to require that the first order conditions of the central's discretionary optimization be the same as the first order condition of the policy planner's commitment optimization. Things become more complicated with parameter uncertainty. This is because requiring that first order conditions be identical might be not sufficient for the commitment equilibrium to be replicated given the fact that the policy planner and his central banker are now being subject to different structural constraints.

I propose in this paper that to replicate the commitment equilibrium under uncertainty, the policy rule that the policy planner would like to conduct in his own economy is required to be the same as the policy rule that the central banker uses to implement her optimal discretionary equilibrium. By this requirement, when the central banker conducts her policy, she indeed implements the commitment equilibrium.

There are assumptions that need to be made with the above approach. First, because the discretionary equilibrium can be implemented using various policy rules, which means an additional requirement for the policy planner's delegation scheme must be that the central banker chooses only one of them and excludes all other alternatives. Although such a requirement can be satisfied with more complexity of the issues to be considered, my approach to solve for the problem is to narrow the set of policy rules that the central banker can use to a single one by making assumptions on the central bank's operation. As I make it clear in the later sections of this chapter, by assuming that the central banker stick to a policy rule that requires the least operational information, a single policy rule can be identified. Second, this delegation scheme however needs a justification for how it can be accepted by the central bank. For this reason, I assume that the policy planner may get his central banker to use the delegated loss function by signing an appropriate contract with the central banker in the sense of Walsh (1995).

The finding in this chapter is that, with a standard parameterization, delegation under parameter uncertainty requires choosing a central bank with output gap stabilization weight of less than one in order for the central bank to come up with a single policy rule. In the worst-case scenario, compared to delegation without parameter uncertainty, this robust delegation can save a loss equivalent to that which results from a permanent increase in inflation of 0.06 percent from the target of zero. Although, robust delegation is found to be worse than non-robust delegation for almost all of the uncertain parameter values, it is found to dominate the standard discretionary policy.

This chapter is organized as follows. Section 2 briefly surveys the existing literature and highlights the main differences with this study. In Section 3 I introduce the concept of commitment policies which will be applied in this study. In Section 4 I introduce delegation when there is no model uncertainty. This is compared to Section 5 when I introduce a model with uncertainty. Here I discuss in detail the min-max policy and the delegation scheme under parameter uncertainty. Finally, in Section 6 I conclude the analysis and summarize the main results.

2. Literature review

This study is related to a large literature on optimal delegation, which includes Woodford (1999, 2003), Jensen (2002), Walsh (2003), Vestin (2006), and Bilbiie (2009); robust optimal policy by Giannoni (2002), optimal delegation under model uncertainty by Tillmann (2009b, 2009c); and linear contract by Walsh (1995). It is helpful to split the aforementioned delegation literature into two main topics, namely delegation with and without uncertainty. For example, Woodford (1999, 2003), Jensen (2002), Walsh (2003), Vestin (2006), Bilbiie (2009) analyze delegation problems in economies without parameter uncertainty. In fact, these papers are variant mutations of Woodford's (1999) idea about the superiority of inertial commitment policy such as the timeless perspective policy. These papers make use of delegation to introduce lagged endogenous variables into the loss function to obtain inertial equilibriums, which are as close as possible in term of social welfare, to that produced by the timeless perspective policy. Bilbiie (2009) is the first attempt to replicate exactly the timeless perspective equilibrium with delegation.

The study in this chapter is different from the above cited literature in the following ways. In relation to the first topic of delegation without uncertainty, I do the same job of introducing inertia into discretionary policy by replicating an optimal commitment policy. However, the commitment policy to be replicated is the one that minimizes the unconditional loss, and is termed by Damjanovic (2008) the unconditionally optimal policy. In the above-mentioned papers, inertial policies are all in forms of the well-known "timeless perspective policy" initiated by Woodford (1999). With respect to the second topic of delegation with uncertainty, studies are rare but include Kilponen (2003), Dennis (2007) and more recently two studies undertaken in the New Keynesian framework by Tillmann (2009a, 2009b).

Tillmann (2009a) integrates uncertainty into the classical problem of inflation bias in discretionary monetary policy. Essentially, he investigates the degree of a central bank's inflation conservativeness when facing parameter uncertainty. Tillmann (2009b) studies a similar problem in a setup with additive model uncertainty and follows a robust-control method as used in Hansen and Sargent (2003). In both papers, the central banker is the one concerned with model uncertainty; the best the policy planner can do is to find an appropriate banker that can produce an equilibrium that is as close as possible to the commitment one. Technically, given a parameter describing inflation conservatism, the central banker finds an optimal policy, which is robust against the worst-case scenario. The worst-case equilibrium is then solved as a function of the inflation conservatism parameter. This equilibrium is then put back into a true social loss function and the inflation conservatism parameter is then varied to find the one associated with the lowest true social loss value. The delegation scheme conducted this way is named "robust delegation" since it is robust to the parameter worst-case scenario. In the sense of Rogoff (1985), these studies are limited to choosing a central banker who has an appropriate level of inflation conservatism to produce the best robust policy against model uncertainty.

This paper is different from Tillmann (2009b, 2009c) in several aspects. I assume that the policy planner faces model uncertainty and corrects for the problem of inertia in the central banker's discretionary policy by designing a delegation scheme to replicate exactly the full commitment equilibrium. Technically, to solve for the delegation parameters, I follow a process that is actually reverse to that undertaken by Tillman (2009b, 200c). First, the policy planer finds a robust commitment policy and then designs a delegation scheme for his central bank in a way that induces the central bank to implement the discretionary equilibrium with this robust commitment policy. Given the worst-case parameters identified when the policy planner solves for the robust commitment policy, the parameters of the delegation loss function can be solved for. This approach therefore makes it possible to find the delegation parameters analytically, hence it is an advancement compared to Tillmann (2009b, 2009c).

3. The model and optimal commitment equilibriums

3.1. The model

In this section, I make use of a simple New Keynesian model which consists of an IS equation and a Phillips equation as follows:

$$x_t = E_t x_{t+1} - \frac{1}{\sigma}[(i_t - E_t \pi_{t+1}) - r_t^n)] \qquad \text{the IS curve} \qquad (3.1)$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t \qquad \text{the Phillip curve} \qquad (3.2)$$

$$u_t = \rho u_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim iid(0, \sigma_\varepsilon^2) \qquad 0 < \rho < 1 \qquad (3.3)$$

$$r_t^n = \omega r_{t-1}^n + \upsilon_t \qquad \upsilon_t \sim iid(0, \sigma_\upsilon^2) \qquad 0 < \omega < 1 \qquad (3.4)$$

In the system above, the IS equation (3.1) represents a demand side. The equation is derived from log-linearizing the Euler equation of household's utility maximization problem. It relates the output gap with the expected output gap and the real interest rate differential which is defined as the difference between the real interest rate $(i_t - E_t \pi_{t+1})$ and the natural interest rate r_t^n. The Phillips curve, representing a supply side, relates current inflation to expected inflation and the output gap. It is derived from the firm's profit maximization problem.

In (3.1) and (3.2), x_t is the gap between the log-deviation of actual output around its steady state and the log-deviation of efficient output around its steady state. Similarly, π_t and i_t are log-deviations of inflation and short-term interest rate around their steady states. In the micro-founded model of the above system, κ is positive and is an increasing function of the fraction of firms that can change their prices in every period. In the IS curve, σ is the intertemporal substitution elasticity in the consumer's utility function. Finally, β is the discount rate which is positive and less than one.

There are two kinds of shock to the economy, the demand shock r_t^n and supply shock u_t which are assumed to be AR(1) with persistence coefficients of ρ and ω respectively, ε_t and υ_t are iid processes with zero mean and variance $\sigma_\varepsilon, \sigma_\upsilon$. In this model, the demand shock can be fully offset by interest rate adjustment that matches one-to-one to the shock; however, it is the supply shock that brings about fluctuation in output gap and inflation. Here, the demand shock is only relevant to setting nominal interest rate when the central bank is assumed to use an instrument rule specifying interest rate response that is consistent with the IS equation.

To maximize societal utility the policy planner stabilizes output around its efficient level (which in this simplified model is the flexible price level) and around zero inflation. The period loss function is defined as:

$$L_t = \pi_t^2 + \lambda x_t^2 \tag{3.5}$$

in which the output gap weight λ is positive and assumed to be chosen by the policy planner.

In relation to model uncertainty, I assume that κ is the only parameter that is uncertain. The policy planner has no prior knowledge of the distribution of κ, however, he has an interval of κ that is derived from empirical evidence. Although the main focus of the paper is on delegation with κ being uncertain, for the ease of illustration, I first use the case of certainty in Section 4 and then proceed to introduce uncertainty about κ in Section 5.

3.2. Commitment policies.

This paper's objective is to analyze how a policy planner can design a delegation scheme that will induce the central bank to follow the 'best' policy. In this section, I discuss three different concepts of 'best' policy: the full commitment optimal policy; the timeless perspective policy; and the unconditionally optimal policy. All of these concepts assume the availability of a commitment technology, which in general implies superior outcomes to discretionary policy. I argue that for the purposes of this paper, the last concept is the most natural. Therefore I focus on this concept for the remainder of the paper.

<u>The full commitment optimal policy:</u>

The policy planner can commit to a policy at initial time t to stabilize the economy. He can choose the entire future evolution of the economy once and for all at time t. Formally, he commits to an optimal policy to minimize a present discounted loss:

$$W_t = E_t \sum_{j=0}^{\infty} \beta^{t+j} L_{t+j}$$

subjected to structural constraints (3.1) - (3.4).

$E_t(.)$ is an expectation operator conditional on information at time t.

Since the interest rate is not an argument in the loss function, there is no cost involved in varying the nominal interest rate. In other words, the IS relation imposes no real constraint on the policy maker and therefore, the multiplier attached to the IS equation must be zero. Also, supply shock is exogenous and not a choice variable. The Phillips curve is the only relevant constraints.

The Lagrangian:

$$E_t \sum_{j=0}^{\infty} \beta^j \left[\frac{1}{2}(\pi_{t+j}^2 + \lambda x_{t+j}^2) + \Phi_{t+j}[\pi_{t+j} - \beta\pi_{t+j+1} - \kappa x_{t+j} - u_{t+j}] \right]$$

where Φ_{t+j} is the Lagrangian multiplier. The first order conditions are:

At time t:

$$\pi_t + \Phi_t = 0 \tag{3.6}$$

$$\lambda x_t - \kappa\Phi_t = 0 \tag{3.7}$$

At time $t+j$ for $j \geq 1$:

$$\pi_{t+j} + \Phi_{t+j} - \Phi_{t+j-1} = 0 \tag{3.8}$$

$$\lambda x_{t+j} - \kappa\Phi_{t+j} = 0 \tag{3.9}$$

The full commitment optimal policy is time-inconsistent since it implements (3.6) - (3.7) at time t and promises to implement (3.8) - (3.9) at time (t + 1) onward. But when (t + 1) comes, if the policy maker re-optimizes, he has an absolute motivation to renege on his words and implement (3.6) - (3.7) instead. Given this inconsistency, the full optimal policy is not a desirable equilibrium concept.

The timeless perspective policy:

Woodford (1999) recommends a policy, which is optimal from the timeless perspective, as a remedy for the problem of time-inconsistency innate in the full optimal commitment policy. Its working mechanism is to ignore the first-order conditions in the starting period and to therefore act as though the full optimal policy plan has been initiated and activated infinitely far in the past. In that way, the timeless optimal policy is formed by first eliminating the Lagrangian multipliers in (3.8)- (3.9). The resulting policy:

$$\pi_{t+j} = \frac{\lambda}{\kappa}\left(x_{t+j-1} - x_{t+j}\right) \quad \text{for } j \geq 1 \tag{3.10}$$

is then applied in every period including period t and can be re-written as:

$$\pi_t = \frac{\lambda}{\kappa}\left(x_{t-1} - x_t\right) \tag{3.11}$$

By implementing the timeless perspective policy, the policy maker can avoid the problem of time-inconsistency imposed by the difference in first order conditions of the starting period and in periods after that. However, by working in this way the timeless perspective policy fails to implement the full set of first order conditions making it a sub-optimal policy. The sub-optimality of the timeless perspective policy is evidenced, for example, in Blake (2001), Jensen and McCallum (2002) and Dennis (2010).

<u>The unconditionally optimal policy:</u>

Recently, in the literature of optimal monetary policy, there has been call for a policy that is globally optimal and timeless in the sense that it guarantees that the full set of first order conditions is implemented and that the policy is optimal "on average", unconditional on the time when the policy is evaluated. This type of policy is first mentioned in Taylor (1979), Whiteman (1986), and recently by in Blake (2001), McCallum and Nelson (2001), Jensen and McCallum (2002). Damjanovic (2008) term this type of policy as unconditionally optimal policy and propose that the policy is derived by minimizing the unconditionally expected loss criterion.

Following Damjanovic (2008), I reformulate the Lagrangian as:

$$L = E\left(E_t \sum_{j=0}^{\infty} \beta^j \left[\frac{1}{2}(\pi_{t+j}^2 + \lambda x_{t+j}^2) + \Phi_{1,t+j}[\pi_{t+j} - \beta E_t \pi_{t+j+1} - \kappa x_{t+j} - u_{t+j}] \right] \right) \tag{3.12}$$

where $E(.)$ denotes unconditional expectation operator.

Here, the loss is evaluated with asymptotic properties of the equilibrium. This requires the considering of stationary processes x_t and π_t.

Since unconditional expectation is applied, we have:

$$E\left(\Phi_{t+j} E_t \pi_{t+j+1}\right) = E\left(\Phi_{t-1} \pi_t\right) \tag{3.13}$$

The Lagrangian is reduced to:

$$\left(\frac{1}{1-\beta}\right)E\left[\frac{1}{2}(\pi_t^2+\lambda x_t^2)+\Phi_t(\pi_t-\kappa x_t-u_t)-\beta\Phi_{t-1}\pi_t\right] \tag{3.14}$$

The first order condition is therefore:

$$\frac{\partial L}{\partial \pi_t}=\pi_t+\Phi_{2,t}-\beta\Phi_{2,t-1}=0 \tag{3.15}$$

$$\frac{\partial L}{\partial x_t}=\lambda x_t-\kappa\Phi_{2,t}=0 \tag{3.16}$$

After eliminating the Lagrangian multiplier I derive a relation among endogenous variables which is termed by Giannoni and Woodford (2003) the optimal targeting rule:

$$\pi_t=\frac{\lambda\beta}{\kappa}x_{t-1}-\frac{\lambda}{\kappa}x_t \tag{3.17}$$

The resulting commitment equilibrium can be solved as a solution of a system that consists of the rule (3.17), the Phillips curve (3.2) and the shock process (3.3). It can be shown that this system results in a unique bounded solution. Indeed, this system can be reduced to:

$$-\frac{\lambda\beta}{\kappa}E_t x_{t+1}+\left(\frac{\lambda\beta^2}{\kappa}+\frac{\lambda}{\kappa}+\kappa\right)x_t-\frac{\beta\lambda}{\kappa}x_{t-1}+u_t=0$$

of which the characteristic equation is:

$$g(\mu)=-\frac{\beta\lambda}{\kappa}\mu^2+\left(\frac{\lambda\beta^2}{\kappa}+\frac{\lambda}{\kappa}+\kappa\right)\mu+\frac{\beta\lambda}{\kappa}$$

Since $g(1)>0$ and $g(-1)<0$, $g(\mu)$ has one root inside and one root outside the unit circle.

The equilibrium has the following functional form which specifies x_t and π_t as linear functions of lagged output gap and the cost shock.

$$\pi_t=\Pi_x x_{t-1}+\Pi_u u_t \tag{3.18}$$

$$x_t=X_x x_{t-1}+X_u u_t \tag{3.19}$$

From now on, I consider the unconditionally optimal policy as the desired policy of the planner. That is, the policy planner will design a delegated loss function to induce the central bank to follow the unconditionally optimal policy.

4. Delegation without parameter uncertainty

In this section, I assume that both the policy planner and the central banker are certain about their economy. However, they each have own belief of κ which may or may not be identical. To keep this section consistent with the next, I denote the central banker and the policy planner's beliefs respectively as κ^{CB} and κ.

I now turn to the problem of how to implement the equilibrium (3.18)-(3.19) with discretionary policy. The approach here is to have the policy planner delegate to his central banker a distorted loss function designed in a way that induces the central banker to replicate the commitment policy. This delegation approach is initiated in Woodford (1999, 2003) where he shows that by appointing a central banker that cares about the variability of short term interest rate and includes in his loss criteria an "interest rate smoothing" objective, the policy planner can restore his commitment equilibrium if the weights on the stabilization objectives of the delegated loss function are appropriately chosen. When the policy planner and the central bank are both certain about their economy, these weights can be solved for by requiring that the commitment equilibrium and the central bank's discretionary equilibrium be the same without any concern about by which means these equilibriums are brought about. It is understood in this approach that their identical equilibriums must be brought about by identical policies. When the policy planner and the central bank have different, potentially incorrect, beliefs about κ, it is easier to simply require directly that their policies are identical. This requires that the policy planner and the central bank conduct the same policy and that the central bank maintains a certain relation among endogenous variables, and possibly exogenous variables, as its policy rule to induce its own discretionary equilibrium. If such a mechanism for the central bank is in place, to replicate the commitment equilibrium, the policy planner just needs to match this discretionary relation with the optimal policy rule that he uses to implement his commitment equilibrium. This approach, however, requires the policy planner to establish the relation and level the central bank is inclined to maintain since the discretionary equilibrium can be possibly delivered by various policy relations. I will argue later that if some concern about the transparency and accountability of the central bank policy is required, then it is reasonable for the central bank to decide on only one relation among the possible ones.

I now begin with a discussion on some important aspects of the delegation approach. Technically, given a linear-quadratic framework, this approach simply requires that the same set of state variables in the commitment equilibrium be included in the delegated loss function. I propose that the policy planner delegate the following loss function:

$$L_t^B = \frac{1}{2}(\lambda_x x_t^2 + \pi_t^2) + \lambda_{xx} x_{t-1} x_t + \lambda_{x\pi} x_{t-1} \pi_t \tag{4.1}$$

In (4.1), λ_x is generally different from λ in the true loss function.

One way to think about this delegated loss function is as if the planner offered a contract to the banker that specified payment according to (4.1). The contract design requires the policy planner to identify the delegation parameters $[\lambda_x, \lambda_{xx}, \lambda_{x\pi}]$ as functions of the structural parameters.

Given the fact that the central bank minimizes the loss value, the lambda coefficients need to be chosen so as to satisfy the second order condition for a minimum. In addition, the lambda coefficients needs to assure the central banker's unique bounded discretionary equilibrium.

Finally, although a robust commitment policy is derived from optimization on an unconditional loss criterion, the delegation plan does not require the central bank to undertake discretionary optimization from the unconditional perspective. The central bank is instead hired to replicate the policy planner's optimal policy rule.

In this section, I describe the problem of a central bank that has no commitment technology. I show that the discretionary (no commitment) equilibrium can be induced if the central bank follows a policy rule that maintains a given relationship of some endogenous variables. I organize the discussion into four main parts. The first part describes the central banker's problem and her policy action. This can be considered as describing a planning process when the central banker plans on what should be the optimal equilibrium given her delegated loss criterion and any constraint that might be imposed by the structure of the economy. In the second part, I discuss an implementation process when the central banker conducts a policy rule to realize what he has optimally planned. For ease of illustration, this section can be thought of as describing an alternative economy where the central bank's monetary policy is to maintain a policy rule, a given linear relationship of some selected endogenous variables. I then argue for some desired aspects and a particular form of the rule that should be maintained. This boils down to a parameterization of a linear relation with two coefficients. The equilibrium in this economy is derived given any pair of these coefficients. In the third part, I address the question of how to implement the optimal discretionary equilibrium. Here, I define what is meant to be implementable; then I show that the optimal discretionary equilibrium can be implemented through the appropriate choice of policy rule coefficients. In the last part, I show how to solve for the lambda coefficients though the simple requirement that the policy rule maintained by the central banker be equal to the planner's optimal targeting rule. I also construct a region for the lambda coefficients that guarantees the discretionary optimum be a minimum and unique.

4.1.Discretionary optimal equilibrium.

The central bank's optimization problem:

As already discussed at the beginning of this section, the central bank is delegated with a period loss function of the form (4.1). Here the discretionary optimization is fitted in a standard dynamic programming problem. The central bank's value function needs to satisfy the following Bellman equation:

$$V(x_{t-1},u_t)=\min_{\{x_t,\pi_t\}}\left\{\frac{1}{2}(\lambda_x x_t^2+\pi_t^2)+\lambda_{xx}x_{t-1}x_t+\lambda_{x\pi}x_{t-1}\pi_t+\beta E_t V(x_t,u_{t+1})\right\} \tag{4.2}$$

subject to the Phillips curve and the supply shock process:

$$\pi_t=\beta E_t\tilde{\pi}(x_t,u_{t+1})+\kappa^{CB}x_t+u_t \tag{4.3}$$
$$u_t=\rho u_{t-1}+\varepsilon_t$$

In this setup, to find the optimal decision today, the central banker needs to surmise the possible optimal policy decision function that might result from her successor's optimization and make optimal decisions consistent with her guess. Then the discretionary equilibrium requires that each central banker arrives at the same policy function as the successor one.

Since the state variables in the optimization problem include the lagged output gap and cost shock, one could assume that the future optimal inflation and output gap are some functions of these variables. In addition, since the optimization problem is linear-quadratic, one could further assume that these functions be linear. Working within these assumptions, it is reasonable for the central bank to guess that its successor will follow policy functions of the following form:

$$\tilde{x}(x_t, u_{t+1}) = \tilde{X}_x x_t + \tilde{X}_u u_{t+1} \tag{4.4}$$

$$\tilde{\pi}(x_t, u_{t+1}) = \tilde{\Pi}_x x_t + \tilde{\Pi}_u u_{t+1} \tag{4.5}$$

Given (4.4)-(4.5) the central bank restricts its optimal decision that results from current optimization to a policy which explicitly responds linearly to x_{t-1} and u_t.

Using the guess in (4.5), the central bank can transform (4.2) into a one-variable optimization problem by first substituting (4.5) into (4.3), and then making π_t a function of x_t and u_t:

$$\pi_t = (\beta\tilde{\Pi}_x + \kappa^{CB})x_t + (\beta\rho\tilde{\Pi}_u + 1)u_t \tag{4.6}$$

and then substituting (4.6) into the bracketed expression in (4.2).

Taking the first order derivative with respect to x_t, the central bankers can derive the F.O.C:

$$\begin{aligned} &\lambda_x x_t + (\beta\tilde{\Pi}_x + \kappa^{CB})\left[(\beta\tilde{\Pi}_x + \kappa^{CB})x_t + (\beta\rho\tilde{\Pi}_u + 1)u_t\right] + \lambda_{xx}x_{t-1} \\ &+ \lambda_{x\pi}(\beta\tilde{\Pi}_x + \kappa^{CB})x_{t-1} + \beta E_t \frac{\partial V(x_t, u_{t+1})}{\partial x_t} = 0 \end{aligned} \tag{4.7}$$

To deal with the expectation derivative, the Envelop theorem can be applied as follows:

$$\frac{\partial V(x_{t-1}, u_t)}{\partial x_{t-1}} = \lambda_{xx}x_t + \lambda_{x\pi}\pi_t \tag{4.8}$$

Then moved (4.8) up one period as follows:

$$E_t \frac{\partial V(x_t, u_{t+1})}{\partial x_t} = \lambda_{xx} E_t x_{t+1} + \lambda_{x\pi} E_t \pi_{t+1} \tag{4.9}$$

Again, making use of the guess (4.4) and (4.5):

$$E_t \frac{\partial V(x_t, u_{t+1})}{\partial x_t} = \lambda_{xx}\left(\tilde{X}_x x_t + \rho \tilde{X}_u u_t\right) + \lambda_{x\pi}\left(\tilde{\Pi}_x x_t + \rho \tilde{\Pi}_u u_t\right) \tag{4.10}$$

Substituting (4.10) into (4.7) and rearranging, the F.O.C at equilibrium becomes:

$$\begin{aligned}
&\left[\lambda_x + (\beta\tilde{\Pi}_x + \kappa^{CB})^2 + \beta\lambda_{x\pi}\tilde{\Pi}_x + \beta\lambda_{xx}\tilde{X}_x\right] x_t \\
&+ \left[\lambda_{x\pi}(\beta\tilde{\Pi}_x + \kappa^{CB}) + \lambda_{xx}\right] x_{t-1} \\
&+ \left[(\beta\tilde{\Pi}_x + \kappa^{CB})(\beta\rho\tilde{\Pi}_u + 1) + \beta\rho\tilde{\Pi}_u\lambda_{x\pi} + \beta\rho\tilde{X}_u\lambda_{xx}\right] u_t = 0
\end{aligned} \tag{4.11}$$

The central bank policy function for x_t can then be solved from (4.11) as follows:

$$x_t = X_x^{CB} x_{t-1} + X_u^{CB} u_t \tag{4.12}$$

In which:

$$X_x^{CB} = -\left(\frac{\lambda_{xx} + \lambda_{x\pi}(\beta\tilde{\Pi}_x + \kappa^{CB})}{\lambda_x + (\beta\tilde{\Pi}_x + \kappa^{CB})^2 + \beta\lambda_{xx}\tilde{X}_x + \beta\lambda_{x\pi}\tilde{\Pi}_x}\right) \tag{4.12.1}$$

$$X_u^{CB} = -\left(\frac{(\beta\tilde{\Pi}_x + \kappa^{CB})(\beta\rho\tilde{\Pi}_u + 1) + \beta\rho\lambda_{xx}\tilde{X}_u + \beta\rho\lambda_{x\pi}\tilde{\Pi}_u}{\lambda_x + (\beta\tilde{\Pi}_x + \kappa^{CB})^2 + \beta\lambda_{xx}\tilde{X}_x + \beta\lambda_{x\pi}\tilde{\Pi}_x}\right) \tag{4.12.2}$$

Substituting (4.12) into (4.6), the policy function for π_t can be solved as:

$$\pi_t = \Pi_x^{CB} x_{t-1} + \Pi_u^{CB} u_t \tag{4.13}$$

In which:

$$\Pi_x^{CB} = (\beta\tilde{\Pi}_x + \kappa^{CB}) X_x^{CB} \tag{4.13.1}$$

$$\Pi_u^{CB} = (\beta\tilde{\Pi}_x + \kappa^{CB}) X_u^{CB} + (\beta\rho\tilde{\Pi}_u + 1) \tag{4.13.2}$$

For the central bank's policy functions to be consistent with those that are conjectured to be derived by the successor, X_x^{CB} , X_u^{CB}, Π_x^{CB}, Π_u^{CB} in (4.12) and (4.13) are required to be the same as $\tilde{X}_x$, $\tilde{X}_u$, $\tilde{\Pi}_x$, $\tilde{\Pi}_u$. This results in a system of four equations to solve for $\tilde{X}_x$, $\tilde{X}_u$, $\tilde{\Pi}_x$, $\tilde{\Pi}_u$:

$$\tilde{X}_x = -\left(\frac{\lambda_{xx} + \lambda_{x\pi}(\beta\tilde{\Pi}_x + \kappa^{CB})}{\lambda_x + (\beta\tilde{\Pi}_x + \kappa^{CB})^2 + \beta\lambda_{xx}\tilde{X}_x + \beta\lambda_{x\pi}\tilde{\Pi}_x} \right) \tag{4.14}$$

$$\tilde{X}_u = -\left(\frac{(\beta\tilde{\Pi}_x + \kappa^{CB})(\beta\rho\tilde{\Pi}_u + 1) + \beta\rho\lambda_{xx}\tilde{X}_u + \beta\rho\lambda_{x\pi}\tilde{\Pi}_u}{\lambda_x + (\beta\tilde{\Pi}_x + \kappa^{CB})^2 + \beta\lambda_{xx}\tilde{X}_x + \beta\lambda_{x\pi}\tilde{\Pi}_x} \right) \tag{4.15}$$

$$\tilde{\Pi}_x = (\beta\tilde{\Pi}_x + \kappa^{CB})\tilde{X}_x \tag{4.16}$$

$$\tilde{\Pi}_u = (\beta\tilde{\Pi}_x + \kappa^{CB})\tilde{X}_u + (\beta\rho\tilde{\Pi}_u + 1) \tag{4.17}$$

We first solve for $\tilde{X}_x$ by deriving $\tilde{\Pi}_x$ as a function of $\tilde{X}_x$:

$$\tilde{\Pi}_x = \frac{\kappa^{CB}\tilde{X}_x}{1 - \beta\tilde{X}_x} \tag{4.18}$$

Substituting (4.18) back into (4.14):

$$\tilde{X}_x = -\left(\frac{\lambda_{xx} + \lambda_{x\pi}\left(\beta\dfrac{\kappa^{CB}\tilde{X}_x}{1-\beta\tilde{X}_x} + \kappa^{CB} \right)}{\lambda_x + \left(\beta\dfrac{\kappa^{CB}\tilde{X}_x}{1-\beta\tilde{X}_x} + \kappa^{CB} \right)^2 + \beta\lambda_{xx}\tilde{X}_x + \beta\lambda_{x\pi}\left(\dfrac{\kappa^{CB}\tilde{X}_x}{1-\beta\tilde{X}_x} \right)} \right) \tag{4.19}$$

Rearranging (4.19), a quartic equation in $\tilde{X}_x$ is derived:

$$A\tilde{X}_x^4 + B\tilde{X}_x^3 + C\tilde{X}_x^2 + D\tilde{X}_x + E = 0$$

in which:

$$A = \beta^3\lambda_{xx}$$

$$B = (\beta^2\lambda_x - 2\beta^2\lambda_{xx} - 2\beta^2\kappa^{CB}\lambda_{x\pi})$$

$$C = (\beta\lambda_{xx} + \beta\kappa^{CB}\lambda_{x\pi} - \beta^2\lambda_{xx} - 2\beta\lambda_x)$$

$$D = \beta(\lambda_{xx} + \kappa^{CB}\lambda_{x\pi}) + \beta\lambda_{xx} + \lambda_x + \left(\kappa^{CB}\right)^2$$

$$E = -(\lambda_{xx} + \kappa^{CB}\lambda_{x\pi})$$

Since $\tilde{X}_x = X_x^{CB}$, the above equation can be rewritten as:

$$A\left(X_x^{CB}\right)^4 + B\left(X_x^{CB}\right)^3 + C\left(X_x^{CB}\right)^2 + D\left(X_x^{CB}\right) + E = 0 \tag{4.20}$$

Uniqueness of the discretionary equilibrium:

The determinacy condition requires that the lambda coefficients need to be chosen so that (4.20) has only one root in (-1,1). I will tackle this problem in two steps. First, I will show that through the use of X_x^{CB} , X_u^{CB}, Π_x^{CB}, Π_u^{CB}, one can always construct a set of the lambda coefficients so that (4.20) has a root between (-1, 1). In the second step, I search (numerically) within that set for those lambda coefficients that make the root between (-1, 1) unique.

Indeed, suppose for the moment that a set of X_x^{CB} , X_u^{CB}, Π_x^{CB}, Π_u^{CB} in which $\left|X_x^{CB}\right|<1$ is given, one can plug them back into (4.14) and (4.15) to solve for the appropriate lambda coefficients. Since (4.14)-(4.15) is linear in the lambda coefficients, there is always a system of two equations with three unknown coefficients, which generically results in a solution for the lambda coefficients.

Now we come to the problem of whether the set of lambda coefficients just solved for guarantees that the set of X_x^{CB} , X_u^{CB}, Π_x^{CB}, Π_u^{CB} are unique. This boils down to the requirement that the equation (4.20) has only one root in (-1, 1). This can possibly be undertaken given the fact that we are solving for three lambda coefficients using two equations. We can try to use the free coefficient in order for this requirement to be met. Because the set of lambda coefficients also needs to satisfy the second condition for a minimum, we then need to choose the free lambda coefficient so that it satisfies both of these restrictions. I will postpone the detailed derivation of the range for the free lambda and thus the set of the lambda coefficients until I finish with showing what is the desired set of X_x^{CB} , X_u^{CB}, Π_x^{CB}, Π_u^{CB} in the following sections.

Now suppose that the lambda coefficients were appropriately chosen so that (4.20) always has only one root in (-1, 1). This guarantees a unique equilibrium for the discretionary optimization. Substituting X_x^{CB} into (4.18), we can solve for Π_x^{CB}. Substituting the solved X_x^{CB} and Π_x^{CB} into (4.15) and (4.17) we can solve for X_u^{CB} and Π_u^{CB}.

At this point, we have solved for the central bank policy functions as follows:

$$x_t = X_x^{CB} x_{t-1} + X_u^{CB} u_t \tag{4.21}$$

$$\pi_t = \Pi_x^{CB} x_{t-1} + \Pi_u^{CB} u_t \tag{4.22}$$

4.2. Implementing the discretionary equilibrium through a policy rule.

Now I turn to the implementation of the discretionary equilibrium. Consider an economy, characterized by the structural Phillips curve. To stabilize the economy, the central banker maintains a given linear relationship among endogenous variables. Here, I restrict the policy rule that the central bank aims to maintain based on some desired features. To support the transparency and accountability of the central bank to the public, the first restriction is for the policy to require the least information for its implementation. Second, the policy rule, when being announced, must uniquely determine the optimal discretionary policy function as the equilibrium outcome.

The first restriction rules out those rules that involve expectations of endogenous variables as well as exogenous shocks that are practically unobservable. Since the discretionary policy functions involve only the lagged output gap and cost shock, it is reasonable for the central bank to maintain a rule that specifies a relationship of endogenous variables in the following form:

$$\pi_t = \varphi_1 x_{t-1} + \varphi_0 x_t \tag{4.23}$$

in which $\left(\varphi_0, \varphi_1\right) \in \Phi^{\mathrm{CB}}$, the set that guarantees a unique bounded equilibrium. x_t, π_t are determined by solving the system:

$$\pi_t = \varphi_1 x_{t-1} + \varphi_0 x_t \tag{4.24}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa^{CB} x_t + u_t \tag{4.25}$$

with the shock u_t defined by (3.3). Since the predetermined variables are now x_{t-1} and u_t, we are looking for a solution in which x_t and π_t are linear functions of x_{t-1} and u_t. Substituting (4.24) into (4.25), and rearranging, this results in an expectation first order difference equation for x_t as follows:

$$\beta\varphi_0 E_t x_{t+1} + \left(\beta\varphi_1 - \varphi_0 + \kappa^{CB}\right) x_t - \varphi_1 x_{t-1} + u_t = 0 \tag{4.26}$$

of which the characteristic equation is:

$$f(\mu) = \beta\varphi_0\mu^2 + \left(\beta\varphi_1 - \varphi_0 + \kappa^{CB}\right)\mu - \varphi_1 \tag{4.27}$$

Since the relation (4.24) is required to result in a determinate solution, the equation (4.27) needs to have one real root inside the unit circle and the other one outside the unit circle given the fact that (4.26) has only one predetermined variable. Since $\left(\varphi_0, \varphi_1\right) \in \Phi^{CB}$, (4.27) has two real roots:

$$|\mu_1| < 1 \text{ and } |\mu_2| > 1. \tag{4.28}$$

To solve (4.26), I decompose (4.27) as follows:

$$\begin{aligned} \beta\varphi_0\mu^2 + \left(\beta\varphi_1 - \varphi_0 + \kappa^{CB}\right)\mu - \varphi_1 &= \beta\varphi_0(\mu - \mu_1)(\mu - \mu_2) \\ &= \beta\varphi_0\mu^2 - \beta\varphi_0(\mu_1 + \mu_2)\mu + \beta\varphi_0\mu_1\mu_2 \end{aligned} \tag{4.29}$$

With (4.29), equation (4.26) can be rewritten as:

$$\left(\beta\varphi_0 E_t x_{t+1} - \beta\varphi_0\mu_1 x_t\right) - \mu_2\left(\beta\varphi_0 x_t - \beta\varphi_0\mu_1 x_{t-1}\right) = -u_t$$

Or

$$E_t\left(\beta\varphi_0 x_{t+1} - \beta\varphi_0\mu_1 x_t\right) - \mu_2\left(\beta\varphi_0 x_t - \beta\varphi_0\mu_1 x_{t-1}\right) = -u_t$$

Rearranged:

$$\left(\beta\varphi_0 x_t - \beta\varphi_0\mu_1 x_{t-1}\right) = \frac{1}{\mu_2} E_t\left(\beta\varphi_0 x_{t+1} - \beta\varphi_0\mu_1 x_t\right) + \frac{1}{\mu_2} u_t \tag{4.30}$$

Denote $z_t = \left(\beta\varphi_0 x_t - \beta\varphi_0\mu_1 x_{t-1}\right)$ and solve (4.30) forward with u_t being an AR(1) process:

$$z_t = \left(\frac{1}{\mu_2}\right)^j E_t z_{t+j} + \left[\sum_{k=1}^{j}\left(\frac{1}{\mu_2}\right)^k \rho^{k-1}\right] u_t = \left(\frac{1}{\mu_2 - \rho}\right) u_t$$

or

$$\left(\beta\varphi_0 x_t - \beta\varphi_0\mu_1 x_{t-1}\right) = \frac{1}{\left(\mu_2 - \rho\right)} u_t$$

The stationary solution for x_t is derived as:

$$x_t = \mu_1 x_{t-1} + \frac{1}{\beta\varphi_0\left(\mu_2 - \rho\right)} u_t \tag{4.31}$$

The solution for π_t is derived by substituting x_t into (4.24):

$$\pi_t = \varphi_1 x_{t-1} + \varphi_0 \left(\mu_1 x_{t-1} + \frac{1}{\beta\varphi_0(\mu_2 - \rho)} u_t \right)$$
$$= (\varphi_1 + \varphi_0 \mu_1) x_{t-1} + \frac{1}{\beta(\mu_2 - \rho)} u_t \tag{4.32}$$

Note that the equilibrium is stationary since $|\mu_1| < 1$.

Denote:

$$X_x^D = \mu_1; \qquad X_u^D = \frac{1}{\beta\varphi_0(\mu_2 - \rho)}$$
$$\Pi_x^D = \varphi_1 + \varphi_0 \mu_1; \qquad \Pi_u^D = \frac{1}{\beta(\mu_2 - \rho)}$$

The solution for system (4.24)-(4.25) can be rewritten as:

$$x_t = X_x^D x_{t-1} + X_u^D u_t \tag{4.33}$$

$$\pi_t = \Pi_x^D x_{t-1} + \Pi_u^D u_t \tag{4.34}$$

I consider the discretionary equilibrium to be implementable with the central bank maintaining the policy rule (4.24) if the unique equilibrium stochastic processes (4.33)-(4.34) that are consistent with (4.24)-(4.25) are the same as the stochastic processes that might be generated by policy functions (4.21)-(4.22). Since (4.33) and (4.34) have the same functional form as (4.21)-(4.22), if the coefficients in (4.21)-(4.22) are the same as those in (4.33)-(4.34), the stochastic processes of x_t and π_t that can be generated from both systems are the same if both systems start with the same initial value x_t and are hit by the same sequence of stochastic shock u_t. We therefore consider the discretionary equilibrium implementable (4.24) if:

$$\begin{bmatrix} X_x^D & X_u^D \\ \Pi_x^D & \Pi_u^D \end{bmatrix} = \begin{bmatrix} X_x^{CB} & X_u^{CB} \\ \Pi_x^{CB} & \Pi_u^{CB} \end{bmatrix} \tag{4.35}$$

I now propose that if the relation (4.24) is derived following a procedure of eliminating u_t in (4.21)-(4.22) then (4.35) must hold. Indeed, by eliminating u_t from (4.21)-(4.22), the relation (4.24) is determined in a particular form as:

$$\pi_t = \underbrace{\left(\Pi_x^{CB} - \frac{\Pi_u^{CB}}{X_u^{CB}} X_x^{CB} \right)}_{\varphi_1^{CB}} x_{t-1} + \underbrace{\left(\frac{\Pi_u^{CB}}{X_u^{CB}} \right)}_{\varphi_0^{CB}} x_t \tag{4.36}$$

If the policy relation (4.36) is maintained, x_t and π_t are determined by solving the system (4.36) and (4.25).

Suppose that the lambda coefficients are chosen so that the policy function coefficients results in $\left(\varphi_0^{CB}, \varphi_1^{CB}\right)$ that belongs to the set Φ^{CB}. It is obvious that the policy functions (4.21), (4.22) satisfy (4.36) by definition. In addition, the policy functions (4.21), (4.22) have to satisfy the central banker's Phillips curve. This implies that the policy functions are a solution of the system consisting of the Phillips curve (4.25) and the policy relation (4.36). This is also the unique solution since $\left(\varphi_0^{CB}, \varphi_1^{CB}\right) \in \Phi^{\mathrm{CB}}$ the set that guarantees a unique bounded solution.

4.3. Solving for the lambdas

In Section 3, we showed that the optimal targeting rule that the planner wants to use is given by (3.17). In Section 4.2, we showed that the central bank operating under discretion and with the delegated loss function (4.1) will choose the targeting rule (4.36). This targeting rule has coefficients which depend on the weights in the delegated loss function. In this section, we analyze how the planner could choose these weights so as to make (4.36) the same as (3.17). I follow three steps to solve for the weights. First, I solve for the discretionary equilibrium coefficients X_x^{CB}, $\Pi_x^{CB}, X_u^{CB}, \Pi_u^{CB}$ by requiring the rule (4.36) be the same as the optimal targeting rule (3.17) if the commitment equilibrium is to be replicated by the central bank; I then solve for the coefficients using systems (4.36) ,(4.25). In solving this system, I first show that if the central bank policy rule is chosen in this way, then $\left(\varphi_0^{CB}, \varphi_1^{CB}\right) \in \Phi^{\mathrm{CB}}$. The solved coefficients X_x^{CB}, $\Pi_x^{CB}, X_u^{CB}, \Pi_u^{CB}$ can then be substituted back into equations (4.14)-(4.15) so as to derive a linear system of two equations with three unknowns, the lambda coefficients. To solve for the system, one lambda coefficient needs to be exogenously imposed. I choose λ_x as the lambda to use. In the last step, I find a range of λ_x so that the second order condition for a minimum is satisfied. I then search within that range for those values of λ_x that guarantee a unique root of (4.20) in (-1, 1).

Step 1: The rule (4.36) is the same as the optimal targeting rule (3.17) if their coefficients are identical:

$$\left(\varphi_1^{CB}, \varphi_0^{CB}\right) = \left(\frac{\beta\lambda}{\kappa}, \frac{-\lambda}{\kappa}\right) \tag{4.37}$$

For the solution of system (4.36),(4.25) to be determinate, $\left(\frac{\beta\lambda}{\kappa}, \frac{-\lambda}{\kappa}\right)$ must be in the set Φ^{CB}. Technically, the characteristic equation (4.27) with φ_1, φ_0 being replaced by $\left(\frac{\beta\lambda}{\kappa}, \frac{-\lambda}{\kappa}\right)$ should have one root inside and one root outside the unit circle. Replacing (φ_1, φ_0) with $\left(\frac{\beta\lambda}{\kappa}, \frac{-\lambda}{\kappa}\right)$ in (4.27) we obtain:

$$f(\mu) = -\frac{\beta\lambda}{\kappa}\mu^2 + \left(\frac{\lambda\beta^2}{\kappa} + \frac{\lambda}{\kappa} + \kappa^{CB}\right)\mu + \frac{\beta\lambda}{\kappa}$$

It can be seen that $f(1) > 0$ and $f(-1) < 0$ so that $f(\mu)$ always has one root inside and one root outside the unit circle. This means that $\left(\frac{\beta\lambda}{\kappa}, \frac{-\lambda}{\kappa}\right)$ is always in Φ^{CB}.

Step 2: Now I suppose that λ_x is chosen exogenously. As a result, the two equations (4.14)-(4.15) become a linear system with two unknowns, the λ_{xx} and $\lambda_{x\pi}$. Coefficients of the system are ($\lambda_x; X_x^{CB}, \Pi_x^{CB}, X_u^{CB}, \Pi_u^{CB}$). I solve for these two unknowns as linear functions of ($\lambda_x; X_x^{CB}, \Pi_x^{CB}, X_u^{CB}, \Pi_u^{CB}$). Denoting these two functions:

$$\lambda_{xx} = \lambda_{xx}\left(\lambda_x; X_x^{CB}, X_u^{CB}, \Pi_x^{CB}, \Pi_u^{CB}\right) \tag{4.38}$$

$$\lambda_{x\pi} = \lambda_{x\pi}\left(\lambda_x; X_x^{CB}, X_u^{CB}, \Pi_x^{CB}, \Pi_u^{CB}\right) \tag{4.39}$$

Step 3: I now going to find the range of λ_x that satisfies the second order condition for minimization. Taking the derivative of the F.O.C (4.11) with respect to x_t , and replacing $\tilde{X}_x$, $\tilde{\Pi}_x$, $\tilde{X}_u$, $\tilde{\Pi}_u$ in the expression with X_x^{CB}, Π_x^{CB}, X_u^{CB} , Π_u^{CB}, we have:

$$\left[\lambda_x + (\beta\Pi_x^{CB} + \kappa)^2 + \beta\lambda_{x\pi}\Pi_x^{CB} + \beta\lambda_{xx}X_x^{CB}\right] > 0 \tag{4.40}$$

Since λ_{xx} and $\lambda_{x\pi}$ are linear in λ_x, it is easy to solve (4.40) for an inequality that specifies a range for λ_x. Compactly denoting the range as:

$$\lambda_x > R(\lambda_x; \beta, \kappa, X_x^{CB}, X_u^{CB}, \Pi_x^{CB}, \Pi_u^{CB}) \tag{4.41}$$

where $R(.)$ is a function of $(\lambda_x; \beta, \kappa, X_x^{CB}, X_u^{CB}, \Pi_x^{CB}, \Pi_u^{CB})$.

The system (4.41) and (4.38)-(4.39) defines the set of the lambda coefficients that guarantee the discretionary optimum is the minimum. I now look in this set for those values of the lambda coefficients that guarantee (4.20) is the unique root in (-1, 1). Again, using (4.38)-(4.39) to substitute out λ_{xx} and $\lambda_{x\pi}$ in the coefficients A, B, C, D, E of (4.20), the function in (4.20) can be rewritten as:

$$F(X_x^{CB}) = a(X_x^{CB})^4 + b(X_x^{CB})^3 + c(X_x^{CB})^2 + d(X_x^{CB}) + e \tag{4.55}$$

in which ***a, b, c, d, e*** are functions of λ_x only.

At this point, I can numerically vary λ_x in the range defined by (4.41) to see if we can find those values of λ_x for which $F(X_x^{CB})$ has only one root in (-1, 1). For an illustration of the computed threshold of λ_x that is determined by (4.54), I choose the following parameterization with $\beta = 0.99$ which is standard in the literature: $\rho = 0.35$ closed to the baseline value in Tillmann (2009a), and $\kappa = \kappa^{CB} = 0.075$ which is consistent with the κ value in Robert (1995) [2] . The threshold is calculated to be equal -0.2721.

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[2] In most of parameterizations in the literature, authors report κ values larger than the estimate in Woodford (1999) that I use in Chapter 1. For example, $\kappa = 0.05$ in Walsh (2003), $\kappa = 0.1$ in Jensen (2002), $\kappa = 0.2$ in

I show in Figure 3.1 and Figure 3.2 the graphs of $F(.)$ with different values of λ_x that have been used by Roberts (1995), Jensen (2002), Walsh (2003), McCallum and Nelson (2004), among others. These values range from the use of a central banker who places very little weight on stabilizing output to one for whom output stabilization is the main priority. All of the values of λ_x are positive and thus the second condition is satisfied. The resulting values of λ_{xx} and $\lambda_{x\pi}$ are presented in Table 3.1. In Figure 3.1, the equation $F(X^{CB}) = 0$ has only one root in (-1,1) when $\lambda_x = 0.01$ and $\lambda_x = 0.25$. Figure 3.2 shows that $F(X^{CB}) = 0$ has more than one root in (-1,1) when $\lambda_x = 1$ and $\lambda_x = 10$.

λ_x	0.01	0.25	1	10
λ_{xx}	0.5622	0.4062	-0.0815	-5.9333
$\lambda_{x\pi}$	-1.3750	-1.3018	-1.0732	1.6702

Table 3-1. Calculated values for λ_{xx} and $\lambda_{x\pi}$

Christiano, Eichenbaum & Evans (2005), $\kappa = 0.15$ in Surico (2008). The value provided in Robert (1995), $\kappa = 0.075$, is somewhere in the middle of the reported range.

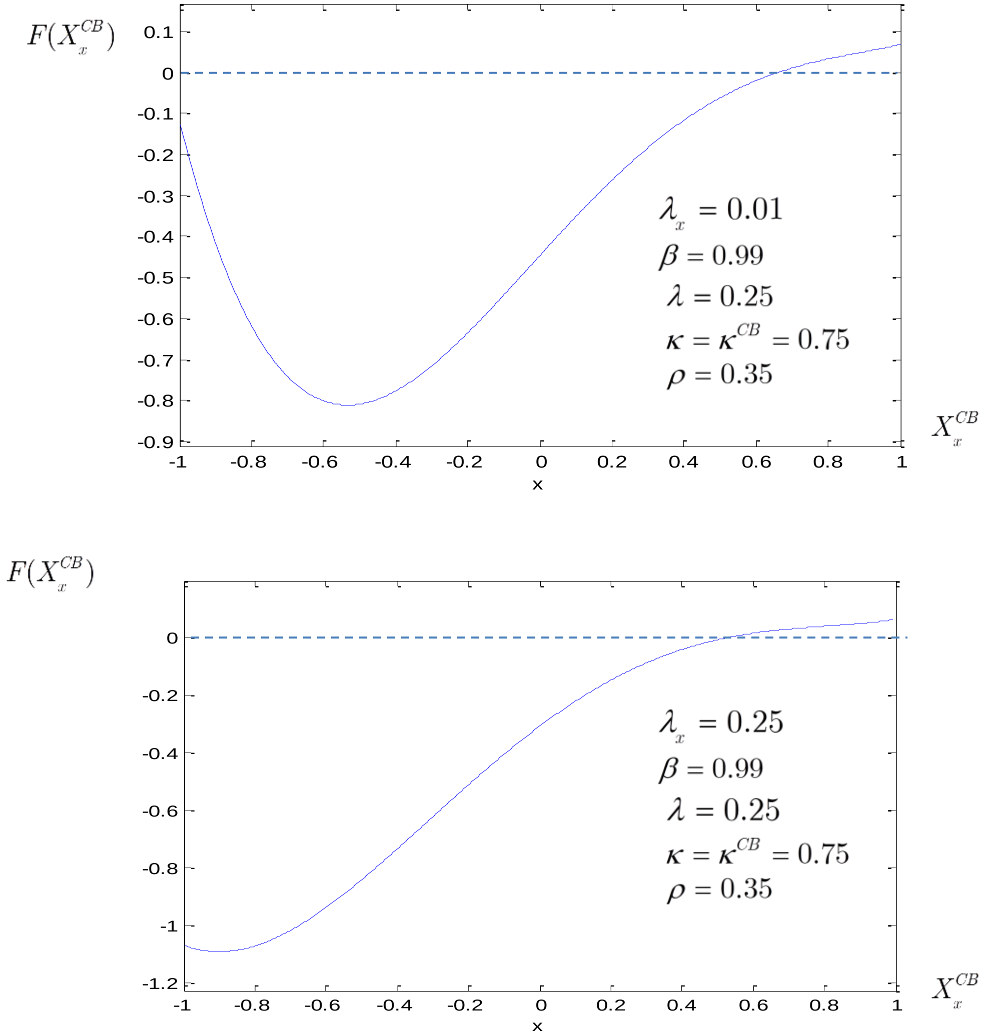


Figure 3.1. $F(X_x^{CB}) = 0$ has only one root in (-1,1).

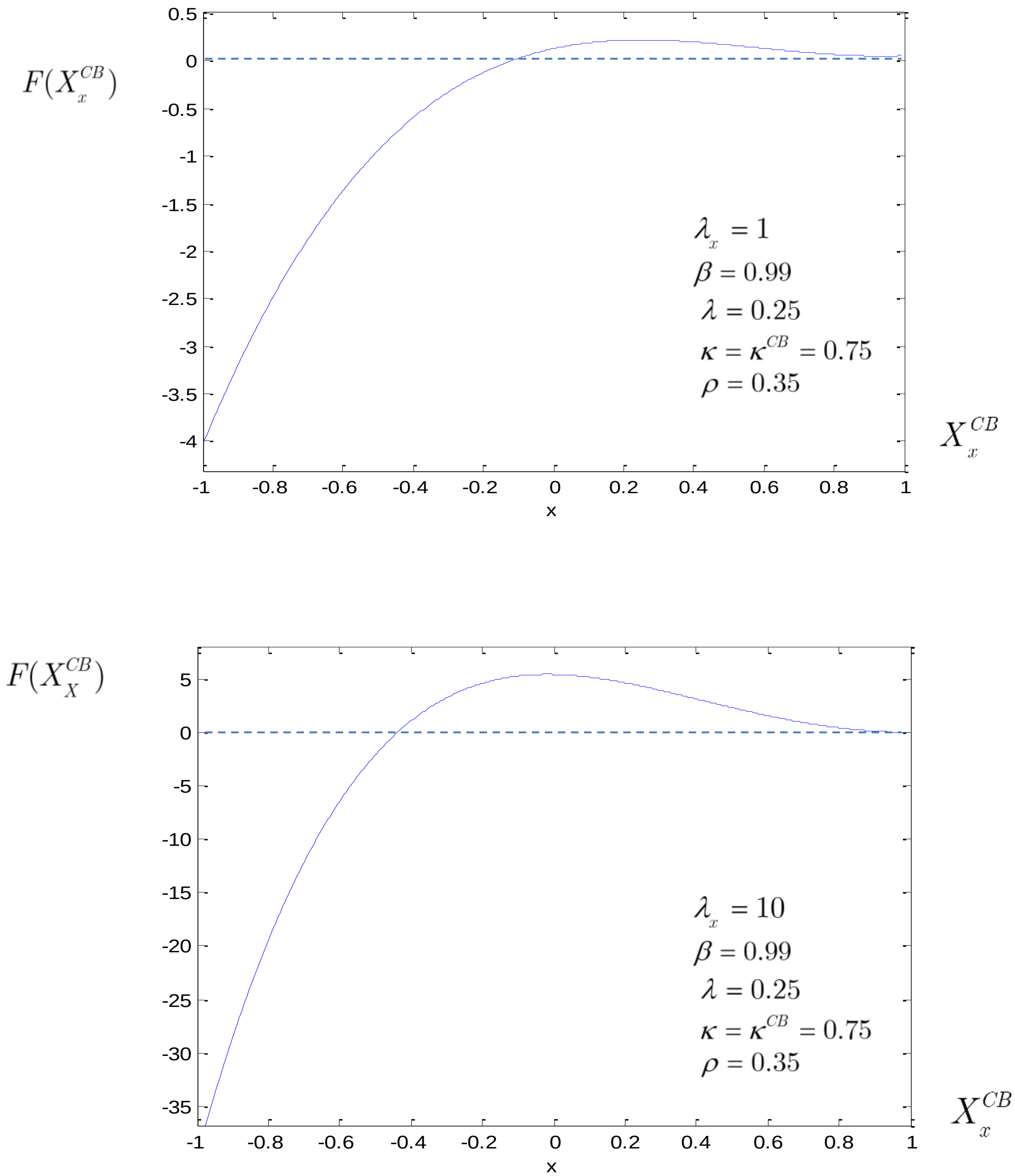


Figure 3.2. $F(X_x^{CB}) = 0$ has more than one root in (-1,1).

5. Delegation under parameter uncertainty

In this Section, I assume that the policy planner knows the structure of his economy but is ambiguous about one parameter of the economy, the κ of the Phillips curve. He does not even know the distribution of this parameter, he just has an estimated interval $[\underline{\kappa}, \overline{\kappa}]$. However, the central banker still believes in κ^{CB}.

I first discuss what should be the policy action of the policy planner under parameter uncertainty. The discussion entails a new definition of the policy planner's problem when parameter uncertainty exists; the resulting definition of an optimal policy however, turns out to have the same functional form as the optimal targeting rule under no parameter uncertainty. I then derive the optimal policy explicitly following a method proposed by Giannoni (2002). In the rest of the section, I show that delegation under parameter uncertainty is no different than delegation without uncertainty except that the optimal targeting rule is replaced by its robust version.

5.1. The policy planner's problem under uncertainty.

When κ can assume any value in $[\underline{\kappa}, \overline{\kappa}]$, the policy planner no longer knows for sure his policy transmission mechanism, the Phillips curve. He will not know which x_t and π_t might be realized. As a result, the planner's problem can no longer be defined as a minimization problem in which the planner is able to choose x_t and π_t directly. Another way to view it is that it is now impossible for him to specify a domain of stochastic processes x_t and π_t over which he optimizes, simply because there is no such x_t and π_t processes that are able to be consistent with more than one Phillips curve. In such a circumstance, what the planner can do at most is to choose, in some best way, a policy rule that produces x_t and π_t. If such a policy is in place, the policy planner's problem can be re-defined by shifting the domain of his minimization from a space of stochastic process to a space of the parameters that characterize the policy and by re-defining the objective function accordingly.

The question now is what should be the functional form of the policy for which the planner would like to derive a best version. As argued in the previous section, in order to induce the central bank to replicate the commitment equilibrium, the planner must match his policy rule with the policy relation maintained by the central bank. Since the central bank is not faced with parameter uncertainty, he must always maintain the same policy relation as specified by (4.23). Consequently, under parameter uncertainty, the functional form of the policy that the planner is seeking should be the same as (4.23) which is the same functional form as the optimal targeting rule. With things so set up, the whole idea now is that if the optimal targeting rule replicates the unconditionally optimal plan then the desired optimal equilibrium under parameter uncertainty is the one brought about by conducting a best policy rule for some criterion, which has the same functional form as the optimal targeting rule. If things can be conceptually thought of in this way then the planner's problem under parameter uncertainty can simply be defined as finding such a best targeting policy rule. His optimal plan is then defined as the outcome of conducting this policy rule.

Now I turn to specify the criterion for the best policy under uncertainty. Following Giannoni (2002) which is based on the result of Giboa and Smeidler (1989), I assume that when the policy planner has no prior on uncertainty, he chooses a min-max strategy, finding a robust policy that attains the lowest loss value if κ realizes its worst value.

Given the above argument, in what follows I construct the planner's problem. First, I restrict the planner's policy to a class of policy rules that have the same functional form as the optimal targeting rule when parameter uncertainty does not exist. In this step, the domain of choice over which the policy planner optimizes is the set of those parameters that characterize the policy. With the policy rule is in place, I re-define the objective function. I do this through transforming it from a function of the stochastic process x_t and π_t into a function of the policy rule parameters and κ. In the last step, I define what should be the robust targeting rule and the resulting equilibrium under uncertainty.

The proposed targeting policy rule now is generalized into the following form:

$$\pi_t = \phi_1 x_{t-1} + \phi_0 x_t \tag{5.1}$$

in which $(\phi_0, \phi_1) \in \Phi^{\mathrm{P}}$ a set of policy parameters that guarantees the planner a unique bounded equilibrium for all $\kappa \in [\underline{\kappa}, \overline{\kappa}]$.

Consider the system consisting of the rule (5.1) and the Phillips curve (5.2):

$$\pi_t = \phi_0 x_t + \phi_1 x_{t-1} \tag{5.1}$$
$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t \tag{5.2}$$

Substituting (5.1) into (5.2) and rearranging:

$$\beta\phi_0 E_t x_{t+1} + \left(\beta\phi_1 - \phi_0 + \kappa\right)x_t - \phi_1 x_{t-1} + u_t = 0 \tag{5.3}$$

(5.3) has its characteristic equation as:

$$f(\theta) = \beta\phi_0\theta^2 + \left(\beta\phi_1 - \phi_0 + \kappa\right)\theta - \phi_1 \tag{5.4}$$

Since (5.3) has one predetermined endogenous variable, it has a determinate solution only when (5.4) has one root inside and one root outside the unit circle for all $[\underline{\kappa}, \overline{\kappa}]$. Denoting $|\theta_1| < 1$ and $|\theta_2| > 1$, then by the same process that (4.31) and (4.32) are solved for in previous section, the unique bounded solution of (5.3) is solved for as follows:

$$x_t = \theta_1 x_{t-1} + \frac{1}{\beta\varphi_0\left(\theta_2 - \rho\right)}u_t \tag{5.5}$$

$$\pi_t = \left(\varphi_1 + \varphi_0\theta_1\right)x_{t-1} + \frac{1}{\varphi_0\left(\theta_2 - \rho\right)}u_t \tag{5.6}$$

I now transform the loss function into a function of the policy parameter (ϕ_0, ϕ_1) and κ. Recall that the policy loss function has the form:

$$L = E\left(E_t\sum_{j=0}^{\infty}\beta^j(\pi_{t+j}^2 + \lambda x_{t+j}^2)\right) = \frac{1}{1-\beta}\left(E[\pi_t^2] + \lambda E[x_t^2]\right) \tag{5.7}$$

Since θ_1 and θ_2 are function of $\phi \equiv (\phi_0, \phi_1)$ and κ, the unconditional variances of x_t and π_t are derived as functions of ϕ and κ.[3]

Denote:

$$E[x_t^2] = \mathrm{X}(\phi, \kappa)\sigma_u^2 \tag{5.8}$$

$$E[\pi_t^2] = \Pi(\phi, \kappa)\sigma_u^2 \tag{5.9}$$

[3] See Appendix for the detailed derivation of (5.10)

σ_u^2 is the unconditional variance of the shock u_t

The loss function can be rewritten as:

$$L = \frac{1}{1-\beta}\left(\Pi(\phi,\kappa) + \lambda \mathrm{X}(\phi,\kappa)\right)\sigma_u^2 \equiv \frac{\sigma_u^2}{1-\beta}\tilde{L}(\phi,\kappa) \tag{5.10}$$

Minimizing (5.7) is now equivalent to minimizing $\tilde{L}(\phi,\kappa)$. With (5.10), the policy planner's problem can be re-defined using the following definitions:

(i) The robust optimal targeting rule is characterized by a vector $\phi \in \Phi^{\mathrm{P}}$ that solves:

$$\min_{\phi\in\Phi^{\mathrm{P}}} \max_{\kappa\in[\underline{\kappa},\bar{\kappa}]} \left[\tilde{L}\left(\phi,\kappa\right)\right] \tag{5.11}$$

(ii) The optimal equilibrium under parameter uncertainty is the unique bounded solution of the system:

$$\pi_t = \phi_0^*(\kappa^*)x_t + \phi_1^*(\kappa^*)x_{t-1} \tag{5.12}$$
$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t \tag{5.13}$$

in which $\left(\phi_0^*(\kappa^*), \phi_1^*(\kappa^*)\right)$ solves the policy planner's problem in (5.11)

5.2. Optimal policy under parameter uncertainty

The policy planner now faces a problem of how to choose $(\phi_0,\phi_1) \in \Phi^{\mathrm{P}}$ to minimize (5.11). We may solve the min-max problem (5.11) directly analytically or we can numerically search over Φ and K for an optimal pair of $\left(\phi^*,\kappa^*\right)$. In both of these approaches, complication may arise given the complexity of the loss function $\tilde{L}\left(\phi,\kappa\right)$. There is an easier approach however, in which we see the policy planner's problem as a two-player zero-sum game between the policy planner and Nature. In this case the policy planner chooses a policy to minimize his social loss while Nature chooses the parameter κ to maximize his loss. If a Nash equilibrium exists, then the max-min equilibrium and min-max equilibrium are identical. We can solve indirectly for the min-max equilibrium by first solving for the max-min equilibrium and then verifying that at the max-min solution, a Nash equilibrium exists. This idea can be structured using the following steps.

Step 1: Solve the policy maker constrained minimization problem to find a rule $\phi^*(\kappa)$ that minimizes the loss (5.10) for a given parameter $\kappa \in [\underline{\kappa}, \overline{\kappa}]$. We then have a policy planner's best response function $\phi^*(\kappa)$.

Step 2: Nature's best response is found by discretizing the $[\underline{\kappa}, \overline{\kappa}]$ interval and finding the result for κ^* that maximizes $L\left(\phi(\kappa), \kappa\right)$. If the optimal κ is κ^*, the policy maker's best response is fixed at $\phi^*\left(\kappa^*\right)$. I now have a candidate Nash equilibrium $\left[\phi^*(\kappa^*), \kappa^*\right]$.

Step 3: Check that the candidate NE is actually a NE equilibrium by verifying that there is no $\kappa^+ \in [\underline{\kappa}, \overline{\kappa}]$ such that:

$$L\left(\phi^*(\kappa^*), \kappa^*\right) < L\left(\phi^*(\kappa^*), \kappa^+\right) \tag{5.14}$$

Application in this model:

Step 1: In this step the policy planner's best response function is shown to be exactly the same as the optimal targeting rule. For the ease of illustration, I consider separately two problems. In the first problem the policy planner chooses stationary processes $\left\{x_{t+j}\right\}$ and $\left\{\pi_{t+j}\right\}$ to minimize the unconditional loss (5.7) subject to a Phillips curve that is characterized by any $\kappa \in [\underline{\kappa}, \overline{\kappa}]$. In the second problem, the policy planner conducts a policy rule of the form (5.1), and finds the optimal coefficients ϕ that minimize the loss (5.10). I then show that the two problems are equivalent in the sense that the optimal equilibrium derived in the first problem is the same as that of the second problem. Consequently, the policy rule that solves the second problem must be the same as the optimal targeting rule.

Original problem 1: The policy planner solves his original problem:

$$\min_{\left\{x_{t+j}, \pi_{t+j}\right\}} E \sum_{j=0}^{\infty} \beta^j \frac{1}{2}(\pi_{t+j}^2 + \lambda x_{t+j}^2)$$
$$\text{s.t: } \pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t$$

As shown in Section 3, the above problem results in the optimal targeting rule (4.18) which I rewrite as:

$$\pi_t = \frac{\lambda\beta}{\kappa} x_{t-1} - \frac{\lambda}{\kappa} x_t$$

The equilibrium is the unique bounded stationary solution of the system (3.1) and (3.17):

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t$$
$$\pi_t = \frac{\lambda\beta}{\kappa} x_{t-1} - \frac{\lambda}{\kappa} x_t$$

which has the form of (3.18)-(3.19):

$$\pi_t = \Pi_x x_{t-1} + \Pi_u u_t$$
$$x_t = X_x x_{t-1} + X_u u_t$$

Alternative problem 2:

An alternative approach for the policy planner is to find an optimal policy rule of the form (5.1) to minimize the loss (5.11):

$$\min_{\phi \in \Phi} \tilde{L}(\phi, \kappa)$$

in which Φ is the set of ϕ that guarantees determinacy for a given κ.

Suppose the rule is characterized by $(\phi_0^*(\kappa), \phi_1^*(\kappa))$. The equilibrium is solved from the system that consists of the optimal rule and the Phillips curve:

$$\pi_t = \phi_0^*(\kappa) x_t + \phi_1^*(\kappa) x_{t-1} \tag{5.15}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t \tag{5.16}$$

and has the form:

$$\pi_t = \Pi_x^R x_{t-1} + \Pi_u^R u_t \tag{5.17}$$
$$x_t = X_x^R x_{t-1} + X_u^R u_t \tag{5.18}$$

Problem 1 and 2 are considered equivalent if :

$$\begin{bmatrix} \Pi_x & \Pi_u \\ X_x & X_u \end{bmatrix} = \begin{bmatrix} \Pi_x^R & \Pi_u^R \\ X_x^R & X_u^R \end{bmatrix} \tag{5.19}$$

It can be shown that (5.19) always holds. As demonstrated in the end of section 3, the system [(3.2) - (3.17)] has unique bounded stationary solutions of the form (3.18)-(3.19). This means that $\left(\frac{\lambda\beta}{\kappa}, \frac{-\lambda}{\kappa}\right) \in \Phi$. Since (3.18)-(3.19) is the unique solution to (3.2)-(3.17), (5.17)-(5.18) is the unique solution to (5.15)-(5.16), and each targeting rule of the given functional form is uniquely determined by its system solution, then (5.19) holds only if $\left(\frac{\lambda\beta}{\kappa}, \frac{-\lambda}{\kappa}\right) = \left(\phi_0^*(\kappa), \phi_1^*(\kappa)\right)$.

Step 2: Find Nature's best response. In this step, Nature's best response can be found numerically by discretizing the $[\underline{\kappa}, \overline{\kappa}]$ interval and finding the value for κ^* that maximizes $L\left(\phi^*(\kappa), \kappa\right)$. The numerical exercise, as presented in Fig 3.3, indicates $\kappa^* = \underline{\kappa}$.

Step 3: Verify the Nash equilibrium. If Nature's best response is fixed at $\kappa^* = \underline{\kappa}$, then the policy maker's best response is fixed at $\phi^*(\underline{\kappa})$. We now have a candidate Nash equilibrium $\left[\phi^*(\underline{\kappa}), \underline{\kappa}\right]$. We then need to verify that this candidate is actually a Nash equilibrium by verifying that $\underline{\kappa}$ is actually Nature's best response. Figure 3.4 graphically represents (5.14) and confirms that a Nash equilibrium exists at $\left[\phi^*(\underline{\kappa}), \underline{\kappa}\right]$.

The above steps derive the robust targeting rule as represented by:

$$\pi_t = \frac{\lambda\beta}{\underline{\kappa}} x_{t-1} - \frac{\lambda}{\underline{\kappa}} x_t \tag{5.20}$$

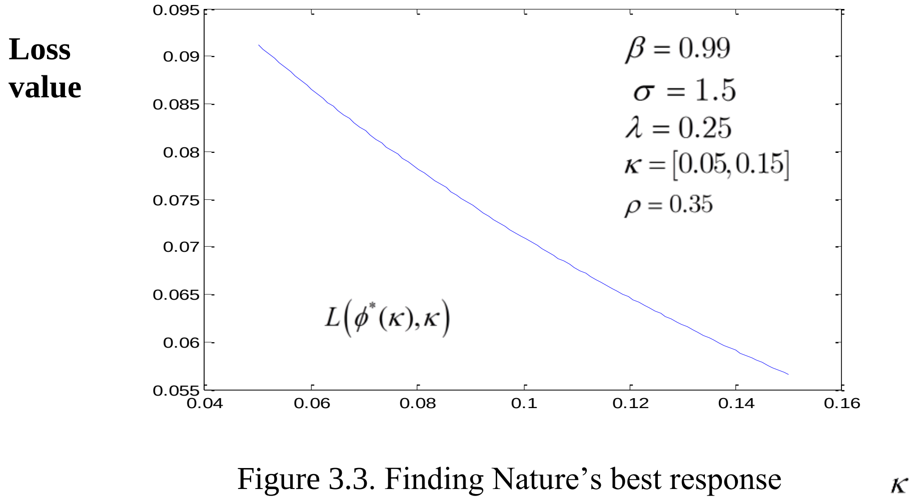


Figure 3.3. Finding Nature's best response

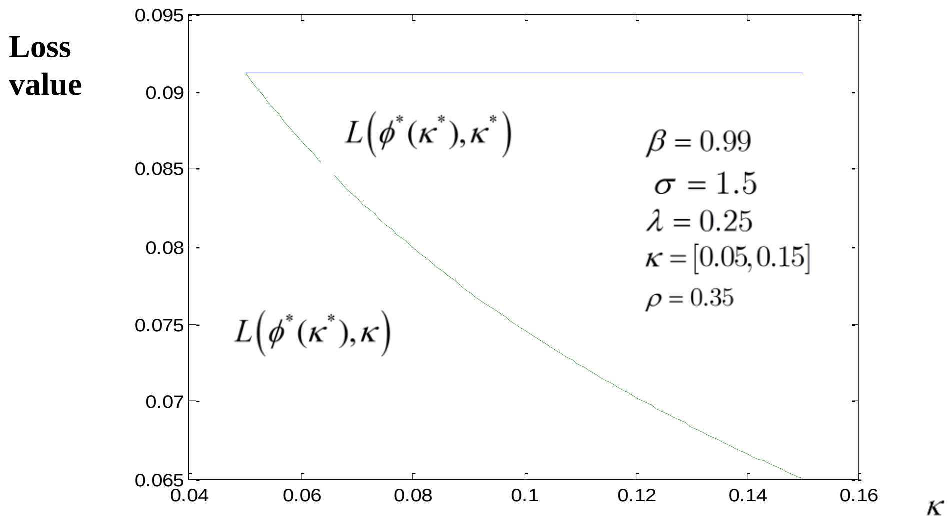


Figure 3.4. Nash equilibrium verification

It can be noted that the worst-case value for κ here is at the lower bound of the κ interval, which is opposite to what we derive in Chapter 2. A possible explanation is that, given the output gap being his policy instrument in the model, the policy maker will see his policy become more ineffective if κ become smaller because now a given reduction in inflation requires the policy maker to sacrifice more output. In Chapter 2, nominal interest rate is the policy maker's instrument. It is optimal for nominal interest rate to response more than one-to-one to demand shock, output gap is contracted to become negative. In the Phillips curve, since the positive effect of nominal interest rate dominates the negative effect of output gap, higher values for κ imply more variation in inflation and maximum κ is considered the worst-case value.

5.3. Robust policy vs. non-robust policy

This section is concerned with the question of how good the robust policy is, in comparison with the non-robust policy. We consider here the first situation whereby the policy planner takes his central banker's belief about the value of κ as the true value. In other words, he is ignorant of the fact that the true κ value can be anywhere in the κ interval. In such a situation, there is no need for a robust policy. I denote the policy rule in this situation as the benchmark one ϕ^b with a corresponding loss function L^b . I then compare the loss value when the policy planner sticks to his robust policy against the case when he just uses the benchmark policy rule. The comparison exercise is presented in Fig. 3.5 whereby the ratio of robust loss to benchmark loss is denoted by $\left(L^* / L^b \right)$. As shown in the figure, if the worst-case occurs, the loss value brought about by the robust policy is just 95 percent of the loss that results from the non-robust policy. Similar to what is done for Chapter 2, I show in the lower panel of Fig 3.5 the inflation equivalent measure. With reference to its formulation already presented in Chapter 2, the inflation equivalence is computed as:

$$\pi^{eqv} = \sigma_u \sqrt{\left| L^b - L^* \right|}$$

where σ_u^2 is the supply shock variance.

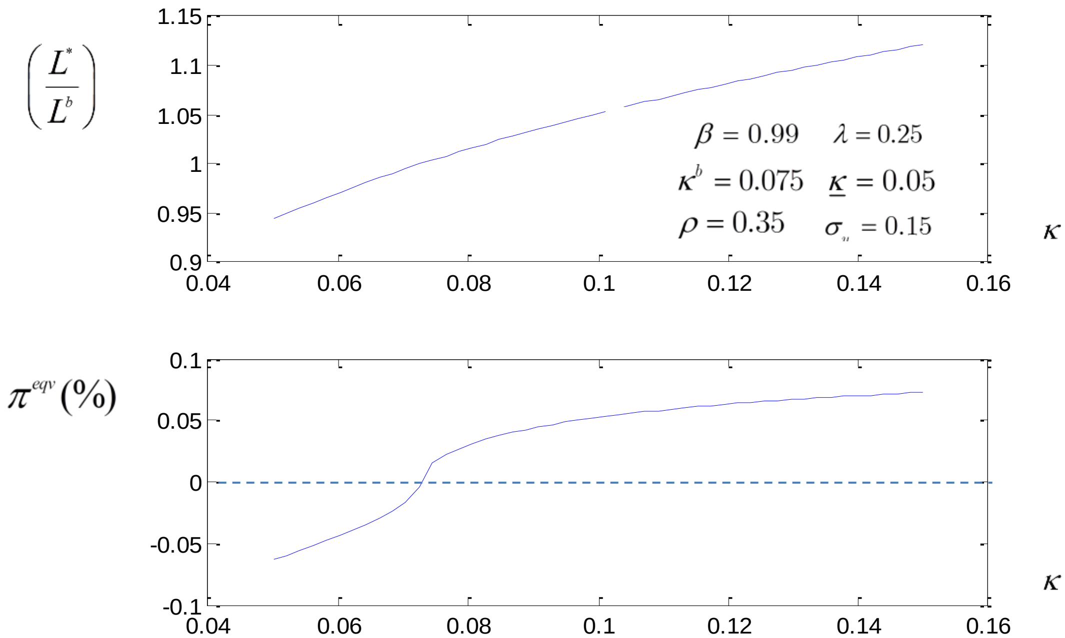


Figure 3.5. Commitment robust policy vs. commitment non-robust policy

It can be seen that in the worst-case situation, if the non-robust policy is used instead of the robust one, the extra loss $\left(L^b - L^*\right)$ is equivalent to a loss that results from a permanent increase in inflation of 0.06 percentage point. However, for the majority of κ values from minimum to maximum κ, the robust policy fails to dominate the benchmark one.

There may arise the question of whether or not the robust commitment policy is better than the standard discretionary policy when the central banker minimizes the true social loss function with her own belief about the κ value. It is obvious that in the situation of no uncertainty, the standard non-inertial discretionary policy is always dominated by inertial commitment policy. However, this might no longer hold true when the commitment policy is based on a belief about a worst-case κ value, which is not necessarily the true κ value.

Figure 3.6. shows on the upper panel the ratio $\left(L^d / L^*\right)$, the robust loss when the robust commitment policy is implemented compared to the discretionary loss when the standard discretionary policy is implemented with a benchmark κ^{CB} belief. The inflation equivalent

measure is shown on the lower panel. This exercise shows how the robust commitment policy always dominates non-robust non-inertial discretionary policy making. In the worst-case scenario, the non-robust discretionary policy results in an extra loss equivalent to that caused by a permanent increase of inflation by around 0.135 percentage point.

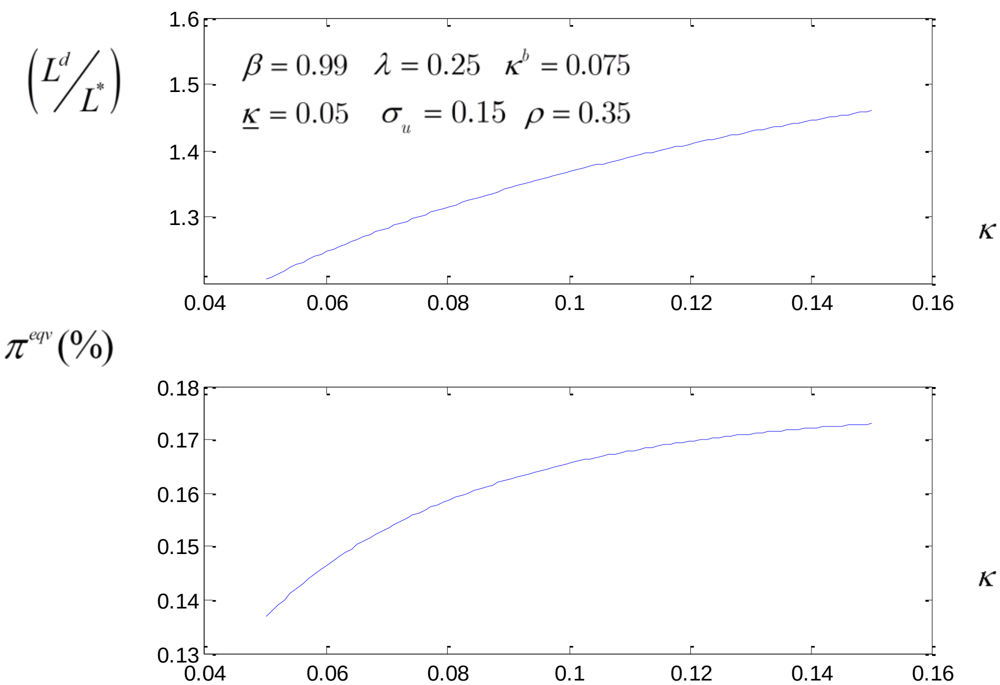


Figure 3.6. Commitment robust policy vs. discretionary non-robust policy

5.4. Delegation under parameter uncertainty

To solve for the lambda coefficients, I follow the same process that I perform in the case of delegation with no parameter uncertainty. I first match the robust targeting rule (5.20) with the policy relation (4.23) that is maintained by the central bank.

$$\left(\phi_1^*, \phi_0^*\right) = \left(\frac{\beta\lambda}{\underline{\kappa}}, \frac{-\lambda}{\underline{\kappa}}\right) \tag{5.21}$$

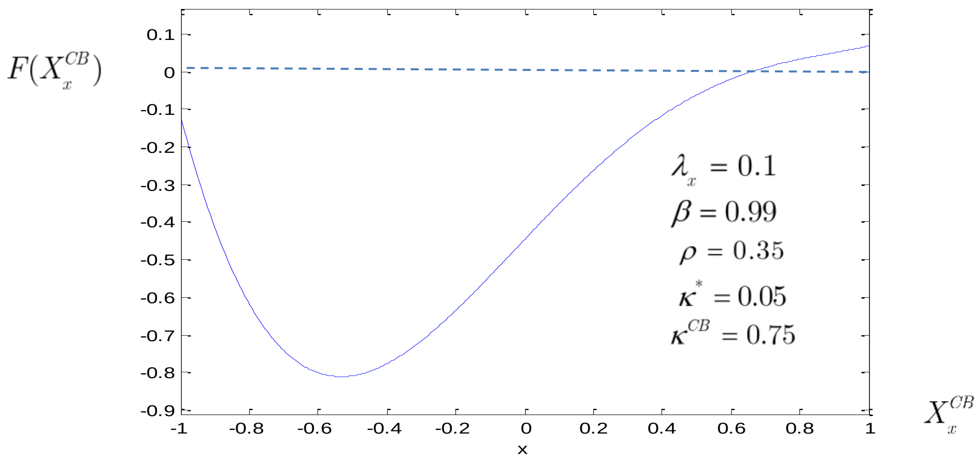
The central bank's policy functions coefficients are then derived from a system of equations that has the same form as the system (4.24)-(4.25). I then verify that $\left(\frac{\beta\lambda}{\underline{\kappa}}, \frac{-\lambda}{\underline{\kappa}}\right)$ belongs to Φ^{CB}, the set that guarantees the central bank a unique bounded equilibrium when the central bank Phillips curve is characterized by its own belief of κ. However, as long as $\left(\frac{\beta\lambda}{\underline{\kappa}}, \frac{-\lambda}{\underline{\kappa}}\right) \in \Phi^{\mathrm{P}}$ then $\left(\frac{\beta\lambda}{\underline{\kappa}}, \frac{-\lambda}{\underline{\kappa}}\right) \in \Phi^{CB}$ since Φ^{P} is the set that guarantees unique bounded equilibrium for any $\kappa \in [\underline{\kappa}, \bar{\kappa}]$.

I choose $[\underline{\kappa}, \bar{\kappa}] = [0.05, 0.15]$ from a survey of parameterizations in Walsh (2003) with $\kappa = 0.05$, Jensen (2002) with $\kappa = 0.07$, Robert (1995) and Clarida et al (2000) with $\kappa = 0.075$, Surico (2008), with $\kappa = 0.15$.[4] In the Appendix, I verify that $\left(\frac{\beta\lambda}{\underline{\kappa}}, \frac{-\lambda}{\underline{\kappa}}\right) \in \Phi^{\mathrm{P}}$. The policy functions' coefficients are then used to solve for the lambda coefficients from systems (4.14)-(4.15).

λ_x	0.01	0.25	1	10
λ_{xx}	0.5129	0.3701	-0.0761	-5.4304
$\lambda_{x\pi}$	-1.3260	-1.2780	-1.1279	0.6729

Table 3-2. Computed values for λ_{xx} and $\lambda_{x\pi}$

[4] As mentioned, in the literature, there is a wide range of the calibrated κ. These values are usually higher than the estimate in Woodford (1999) that I use in Chapter 2.



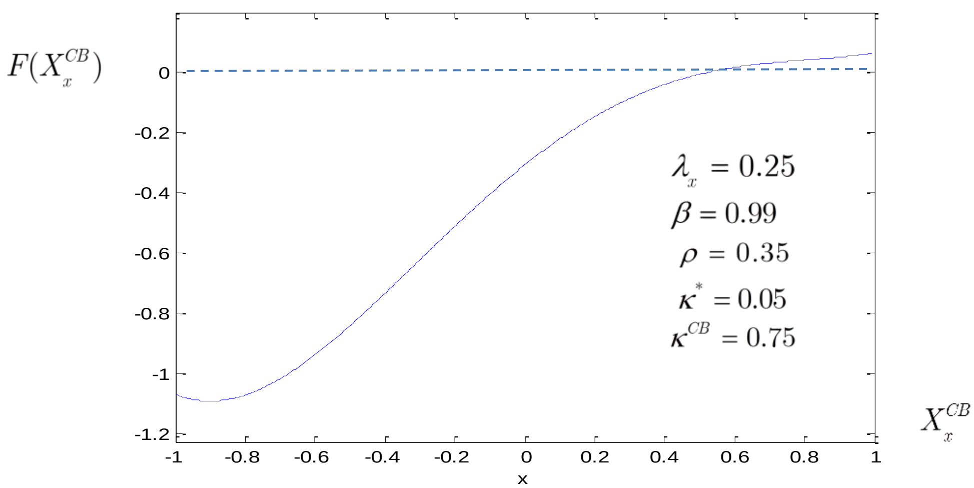
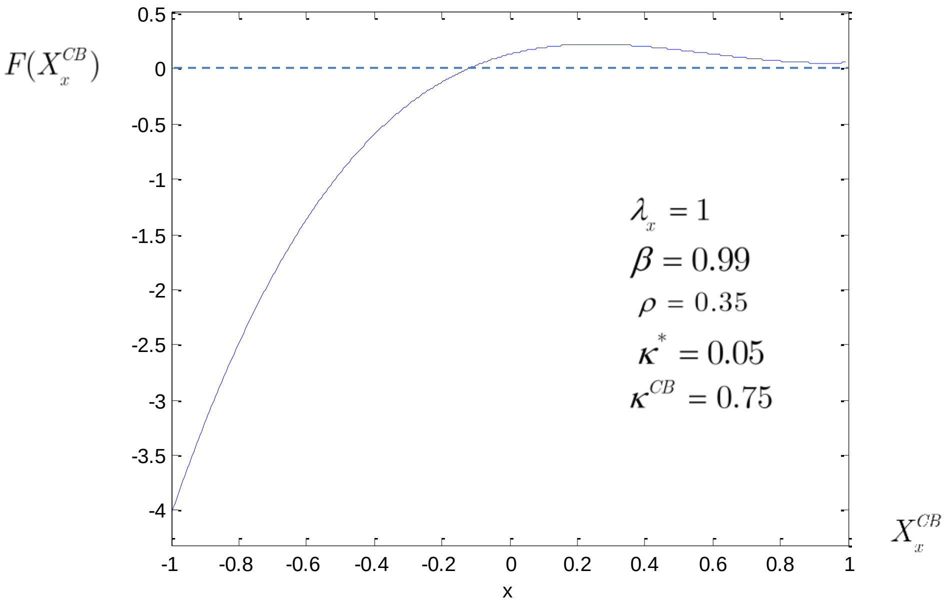


Figure 3.7. $F(X_x^{CB}) = 0$ has only one root in (-1,1).



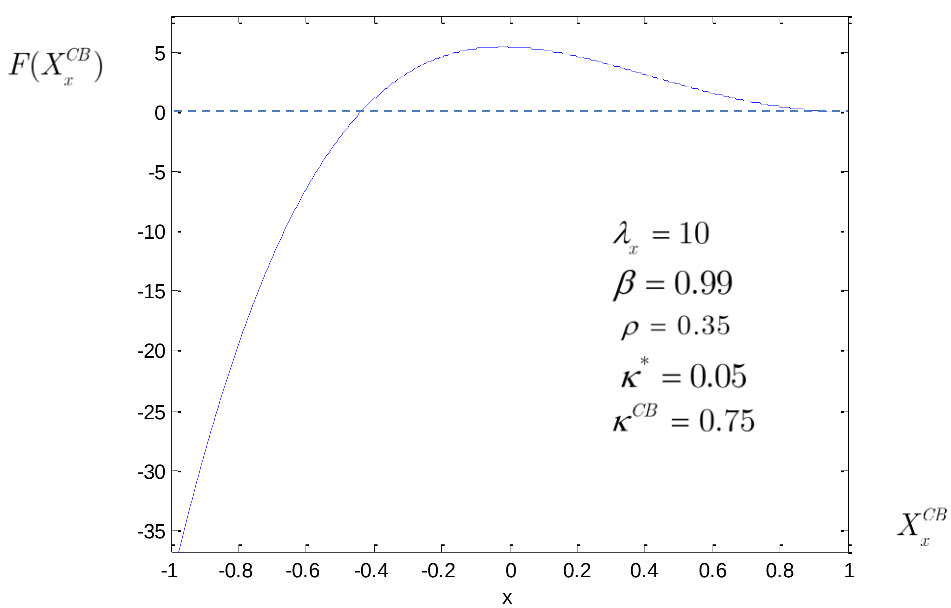


Figure 3.8. $F(X_x^{CB}) = 0$ has more than one root in (-1,1).

I choose the same values for λ_x as in case of no parameter uncertainty and keep the same parameterization with $\beta = 0.99$, $\rho = 0.35$ except that the central bank now has its own κ value which I assume takes the middle value of the κ interval, $\kappa^{CB} = 0.075$. The worst-case kappa is now at its minimum value $\underline{\kappa} = 0.05$. The threshold value of λ_x which satisfies the second order condition for a minimum is now calculated as -0.2407. Table 3.2 shows the computed results for λ_{xx} and $\lambda_{x\pi}$ respectively.

Figure 3.7 and Figure 3.8 presents a graphical verification of the optimal discretionary equilibrium uniqueness. It can be seen that for those values of λ_x that are greater than one, this does not necessarily yield a unique discretionary equilibrium.

6. Conclusions

In this paper, I study a problem of optimal monetary policy delegation given parameter uncertainty. I analyze a scenario whereby a policy planner would like to implement the commitment monetary policy equilibrium when he is uncertain about the parameters of his economy. The policy planner faces two problems in this set up. First, he needs to define a commitment policy that can be judged "optimal" under the assumption of parameter uncertainty. Second, without a commitment technology, the planner needs to devise a delegation scheme to induce his central banker to implement the commitment equilibrium with discretionary monetary policy. The delegation scheme here is considered as choosing, among central banker candidates, the one with a particular preference for output stabilization and then signing a contract specifying the banker's reward or penalty with respect to the contingent state of the economy. The delegation function parameters then characterize the type of the central banker chosen and the contract terms.

I deal with the first problem by using a method whereby the policy planner is assumed to follow a min-max strategy and has no prior on the uncertainty he faces. The "optimal" commitment policy in this case is then defined as the one that attains the lowest loss value when the parameter of interest assumes its worst-case value. Although, this min-max policy protects the policy planner in the worst case situation, it is shown to be dominated by a policy derived by assuming no parameter uncertainty for a majority of the parameter values considered in the paper.

The second problem is dealt with by using a method that requires the policy planner to match his min-max commitment policy with the discretionary policy conducted by his central banker. The delegation function parameters are then chosen so as to satisfy this requirement. Within standard parameterization, the results of this paper show that among central bankers with an output stabilization preference as currently assumed in the literature, the policy planner should choose a central banker that has an output stabilization preference of less than one.

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Appendix

A. Determinacy regions Φ^{P}.

The commitment equilibrium is the solution to a system consisting of the Phillips curve and the policy rule:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t \tag{A.1}$$
$$\pi_t = \phi_0 x_t + \phi_1 x_{t-1} \tag{A.2}$$

Since the policy planner is uncertain about κ, he would like to construct a set Φ^{P} so that (A.2) would result in a determinate equilibrium for all $\kappa \in \mathrm{K}$.

Substitute (A.2) into (A.1) :

$$\phi_0 x_t + \phi_1 x_{t-1} = \beta\left(\phi_0 E_t x_{t+1} + \phi_1 x_t\right) + \kappa x_t + u_t$$

Rearrange we have an expectation first order difference equation in x_t :

$$\beta\phi_0 E_t x_{t+1} + \left(\beta\phi_1 - \phi_0 + \kappa\right) x_t - \phi_1 x_{t-1} + u_t = 0 \tag{A.3}$$

(A.4) has its characteristic equation as:

$$f(\theta) = \beta\phi_0\theta^2 + \left(\beta\phi_1 - \phi_0 + \kappa\right)\theta - \phi_1 \tag{A.4}$$

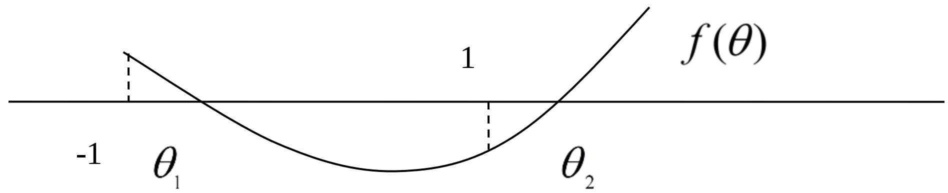
Since (A.3) has one predetermined endogenous variable, it has determinate solution only when (A.4) has one root inside and one root outside the unit circle for all $\kappa \in \mathrm{K}$.

If $|\theta_1| < 1$ and $|\theta_2| > 1$ then the policy planner has the following situations to consider:

(a) $-1 < \theta_1 < 1 < \theta_2$ and $\phi_0 > 0$
(b) $\theta_2 < -1 < \theta_1 < 1$ and $\phi_0 > 0$
(c) $-1 < \theta_1 < 1 < \theta_2$ and $\phi_0 < 0$
(d) $\theta_2 < -1 < \theta_1 < 1$ and $\phi_0 < 0$

The policy planner considers each situation above with $\forall \kappa \in \mathrm{K}$.

(a) $-1 < \theta_1 < 1 < \theta_2$ and $\phi_0 > 0$ $\forall \kappa \in \mathrm{K}$

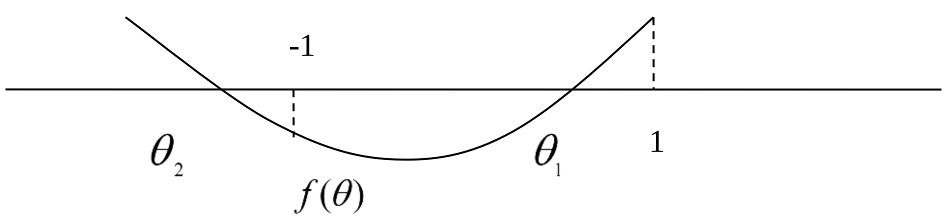


It is sufficient to show that (a) occur when:

$$\begin{Bmatrix} f(-1)>0 \\ f(1)<0 \\ \phi_0>0 \end{Bmatrix} \ \forall\kappa\in\mathrm{K} \ \Leftrightarrow \ \begin{Bmatrix} \beta\phi_0-\left(\beta\phi_1-\phi_0+\kappa\right)-\phi_1>0 \\ \beta\phi_0+\left(\beta\phi_1-\phi_0+\kappa\right)-\phi_1<0 \\ \phi_0>0 \end{Bmatrix} \forall\kappa\in\mathrm{K}$$

$$\Leftrightarrow \begin{Bmatrix} \phi_0>\phi_1+\dfrac{\kappa}{1+\beta} \\ \phi_0>-\phi_1+\dfrac{\kappa}{1-\beta} \\ \phi_0>0 \end{Bmatrix} \forall\kappa\in\mathrm{K} \quad \Leftrightarrow \quad \begin{Bmatrix} \phi_0>\phi_1+\dfrac{\kappa_{\max}}{1+\beta} \\ \phi_0>-\phi_1+\dfrac{\kappa_{\max}}{1-\beta} \\ \phi_0>0 \end{Bmatrix} \tag{A.5}$$

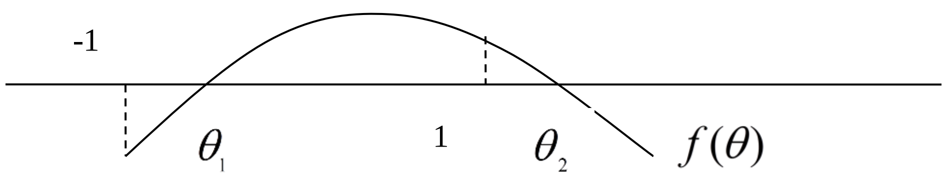
(b) $\theta_2<-1<\theta_1<1$ and $\phi_0>0 \qquad \forall\kappa\in\mathrm{K}$



$$\begin{Bmatrix} f(-1)<0 \\ f(1)>0 \\ \phi_0>0 \end{Bmatrix} \ \forall\kappa\in\mathrm{K} \ \Leftrightarrow \ \begin{Bmatrix} \beta\phi_0-\left(\beta\phi_1-\phi_0+\kappa\right)-\phi_1<0 \\ \beta\phi_0+\left(\beta\phi_1-\phi_0+\kappa\right)-\phi_1>0 \\ \phi_0>0 \end{Bmatrix} \forall\kappa\in\mathrm{K}$$

$$\Leftrightarrow \begin{Bmatrix} \phi_0 < \phi_1 + \dfrac{\kappa}{1+\beta} \\ \phi_0 < -\phi_1 + \dfrac{\kappa}{1-\beta} \\ \phi_0 > 0 \end{Bmatrix} \forall \kappa \in \mathrm{K} \quad \Leftrightarrow \quad \begin{Bmatrix} \phi_0 < \phi_1 + \dfrac{\kappa_{\min}}{1+\beta} \\ \phi_0 < -\phi_1 + \dfrac{\kappa_{\min}}{1-\beta} \\ \phi_0 > 0 \end{Bmatrix} \tag{A.6}$$

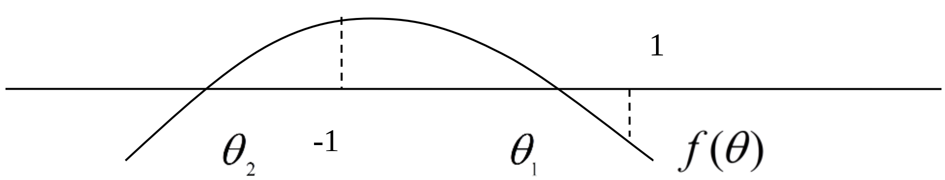
(c) $-1 < \theta_1 < 1 < \theta_2$ and $\phi_0 < 0$



$$\begin{Bmatrix} f(-1) < 0 \\ f(1) > 0 \\ \phi_0 < 0 \end{Bmatrix} \forall \kappa \in \mathrm{K} \quad \Leftrightarrow \quad \begin{Bmatrix} \beta\phi_0 - \left(\beta\phi_1 - \phi_0 + \kappa\right) - \phi_1 < 0 \\ \beta\phi_0 + \left(\beta\phi_1 - \phi_0 + \kappa\right) - \phi_1 > 0 \\ \phi_0 > 0 \end{Bmatrix} \forall \kappa \in \mathrm{K}$$

$$\Leftrightarrow \begin{Bmatrix} \phi_0 < \phi_1 + \dfrac{\kappa}{1+\beta} \\ \phi_0 < -\phi_1 + \dfrac{\kappa}{1-\beta} \\ \phi_0 > 0 \end{Bmatrix} \forall \kappa \in \mathrm{K} \quad \Leftrightarrow \quad \begin{Bmatrix} \phi_0 < \phi_1 + \dfrac{\kappa_{\min}}{1+\beta} \\ \phi_0 < -\phi_1 + \dfrac{\kappa_{\min}}{1-\beta} \\ \phi_0 > 0 \end{Bmatrix} \tag{A.7}$$

(d) $\theta_2 < -1 < \theta_1 < 1$ and $\phi_0 < 0$



$$\begin{Bmatrix} f(-1) > 0 \\ f(1) < 0 \\ \phi_0 < 0 \end{Bmatrix} \ \forall \kappa \in \mathrm{K} \quad \Leftrightarrow \quad \begin{Bmatrix} \beta\phi_0 - \left(\beta\phi_1 - \phi_0 + \kappa\right) - \phi_1 > 0 \\ \beta\phi_0 + \left(\beta\phi_1 - \phi_0 + \kappa\right) - \phi_1 < 0 \\ \phi_0 < 0 \end{Bmatrix} \forall \kappa \in \mathrm{K}$$

$$\Leftrightarrow \begin{Bmatrix} \phi_0 > \phi_1 + \dfrac{\kappa}{1+\beta} \\ \phi_0 > -\phi_1 + \dfrac{\kappa}{1-\beta} \\ \phi_0 < 0 \end{Bmatrix} \forall \kappa \in \mathrm{K} \quad \Leftrightarrow \quad \begin{Bmatrix} \phi_0 > \phi_1 + \dfrac{\kappa_{\max}}{1+\beta} \\ \phi_0 > -\phi_1 + \dfrac{\kappa_{\max}}{1-\beta} \\ \phi_0 < 0 \end{Bmatrix} \tag{A.8}$$

Since (A.8) defines a null set, Φ^{P} is constructed by three regions specified by (A.5), (A.6), (A.7). Figure 1 below represents the set that graphically consists of regions (a), (b), (c).

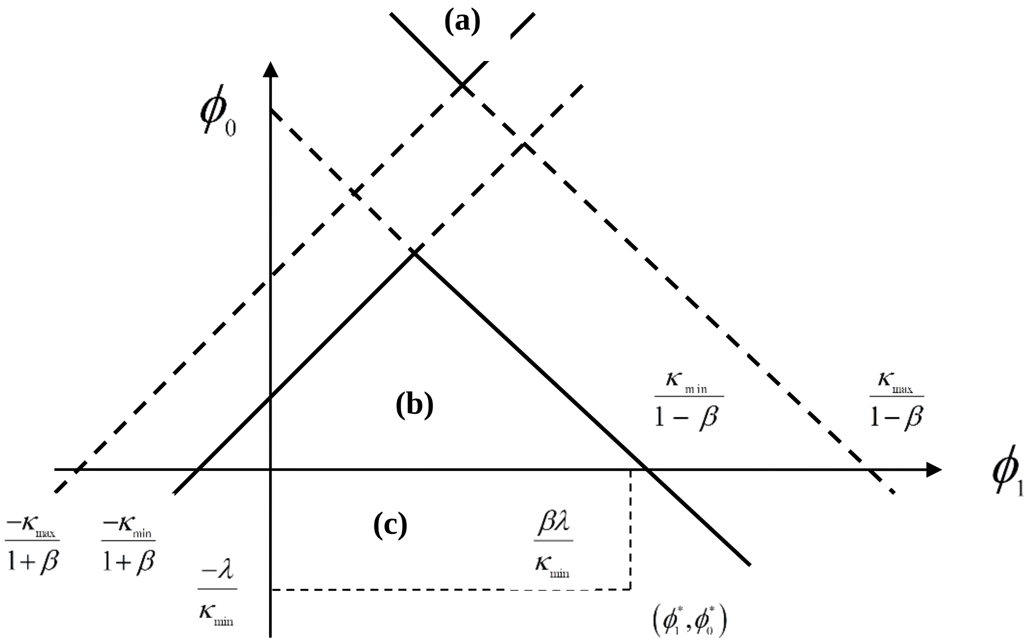


Figure 2. Determinacy region Φ^{P}

If kappa interval is chosen to be $\mathrm{K} \equiv \left[0.05, 0.15\right]$, other parameters $\lambda = 0.25$ and $\beta = 0.99$, it can be verified in Fig 1 that $\left(\dfrac{\lambda\beta}{\kappa_{\min}}, \dfrac{-\lambda}{\kappa_{\min}}\right) \in \Phi^{\mathrm{P}}$.

B. Constructing the transformed loss function

$$E\left(L_t\right) = E(\pi_t)^2 + \lambda E(x_t)^2 \tag{B.1}$$

Compute $E(x_t)^2$ and $E(\pi_t)^2$:

$$E(x_t)^2 = X_x^2 E(x_t)^2 + X_u^2 E(u_t)^2 + 2X_x X_u E(x_t,u_t) \tag{B.2}$$

$$E(x_t,u_t) = E\left[\left(X_x x_{t-1} + X_u u_t\right)\left(\rho u_{t-1} + \varepsilon_t\right)\right]$$

$$= E\left[X_x x_{t-1}\left(\rho u_{t-1} + \varepsilon_t\right) + X_u\left(\rho u_{t-2} + \varepsilon_{t-1}\right)\left(\rho u_{t-1} + \varepsilon_t\right)\right]$$

$$= X_x \rho E\left(\underbrace{x_{t-1}u_{t-1}}_{E\left(x_{t-1}u_{t-1}\right)}\right) + X_u \rho^2 E\left(\underbrace{u_{t-1}u_{t-2}}_{\rho\sigma_u^2}\right) + X_u \rho\ E\left(\underbrace{u_{t-1}\varepsilon_{t-1}}_{\sigma_\varepsilon^2}\right) \tag{B.3}$$

Since $\sigma_u^2 = \left(\dfrac{1}{1-\rho^2}\right)\sigma_\varepsilon^2$

(B.3) implies: $$E(x_t,u_t) = \left(\frac{\rho X_u}{1-\rho X_x}\right)\sigma_u^2 \tag{B.4}$$

Putting (B.4) into (B.2):

$$E(x_t)^2 = X_x^2 E(x_t)^2 + X_u^2\sigma_u^2 + 2X_x X_u\left(\frac{\rho X_u}{1-\rho X_x}\right)\sigma_u^2$$

$$E(x_t)^2 = \left(X_u^2 + \frac{2\rho X_x X_u^2}{1-\rho X_x}\right)\left(\frac{1}{1-X_x^2}\right)\sigma_u^2 = \frac{X_u^2\left(1+\rho X_x\right)}{\left(1-\rho X_x\right)\left(1-X_x^2\right)}\sigma_u^2 \tag{B.5}$$

$$E(\pi_t)^2 = \Pi_x^2 E\left(x_{t-1}^2\right) + \Pi_u^2 E\left(u_t^2\right) + 2\Pi_x\Pi_u E\left(x_{t-1}u_t\right)$$

$$= \left[\frac{\Pi_x^2 X_u^2\left(1+\rho X_x\right)}{\left(1-\rho X_x\right)\left(1-X_x^2\right)} + \Pi_u^2 + \frac{2\rho\Pi_x\Pi_u X_u}{1-\rho X_x}\right]\sigma_u^2 \tag{B.6}$$

$$L = \left[\lambda\frac{X_u^2\left(1+\rho X_x\right)}{\left(1-\rho X_x\right)\left(1-X_x^2\right)} + \frac{\Pi_x^2 X_u^2\left(1+\rho X_x\right)}{\left(1-\rho X_x\right)\left(1-X_x^2\right)} + \Pi_u^2 + \frac{2\rho\Pi_x\Pi_u X_u}{1-\rho X_x}\right]\sigma_u^2$$

$$\equiv \tilde{L}\left(\phi,\kappa\right)$$