Pricing European and American Options in FronTier Framework and Other Applications

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Abstract of the Dissertation

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This thesis is concerned with the numerical solution of the American option valuation problem formulated as a free boundary/initial value model. While other studies have focused on modified pricing model (Jamshidian, 1996), formulating the problem as a non-linear model (Kholodnyi, 1997), using the front-fix method (Crank, 1984) to fix the moving boundary (Wu and Kwok, 1996) (Pantazopoulos et al., 1998), or trying to find semi-/analytical solutions to the problem (Sevcovic, 2001), we introduce and analyze a front- tracking (FT) finite difference method (FDM) based on original Black- Scholes Model (Black and Scholes, 1973) (Merton, 1973). The basis of the B-S Model, FDM, FT and options theory will be introduced. The numerical experiments performed indicate that the front tracking method considered is an efficient alternative for approximating simultaneously the option value and optimal exercise boundary functions associated with the valuation problem. We also extend the study to pricing options with stochastic volatility (Heston, 1993), as well as valuation of multi-asset options.

Dedication

To my parents, Zhang Ting-Shuang and Liu Li-Ying.

To my lovely wife, Chen Lin-Chen.

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1

Options

Options have been around for many years, but it was only on 26th April 1973 that they were first traded on an exchange. It was then that The Chicago Board Options Exchange (CBOE) first created standardized, listed options. Initially there were just calls on 16 stocks. Puts werent introduced until 1977. In the US, options are traded on CBOE, the American Stock Exchange, the Pacific Exchange and the Philadelphia Stock Exchange. Worldwide, there are over 50 exchanges on which options are traded.(Wilmott, 2006)

1.1 European Options

European Options give the holder the right to trade the underlying asset in the future at a previously agreed price. This could for example be the right to buy or sell stocks at a particular strike price. The option would of course only be exercised if it was in the owner's interest to do so.

A European option taken out at current time t gives the owner the right to do something when the option expires at time T. For example a single asset European **put** option, with strike price E and expiry time T, gives the holder the right at time T to sell a particular asset for E. If the asset is worth S_T at maturity then the value of the put option at expiry time, known as the **payoff**, is thus max $(E - S_T, 0)$. By contrast a single asset European **call** option, with strike price E and expiry time T, gives the owner the right at time T to buy an asset for E; the payoff at expiry time for a call option is max $(S_T - E, 0)$.

As an example, consider the following call option on Google stock. It gives the holder the right to buy one share of Google stock for an amount \$100 in one month's time. Today's stock price is \$90. The amount \$100 which we can pay for the stock is called the **exercise price** or **strike price**. The date on which we must exercise our option, if we decide to, is called the **expiry** or **expiration date**. The stock on which the option is based is known as the **underlying asset**.

Many European options take the form of a relatively easy definite integral from which it is possible to compute a closed form solution. The valuation of multi-asset European options, depending on a large number of underlying assets, is more complicated but can conveniently be achieved by using **Monte Carlo** simulation to compute the required multidimensional definite integral. The expected current value of a single asset European option will depend on the current asset price at time t, S, the duration of the option, T, the strike price, E, the risk-less interest rate, r, and the probability density function of the underlying asset price at expiry, $P(S_T)$.

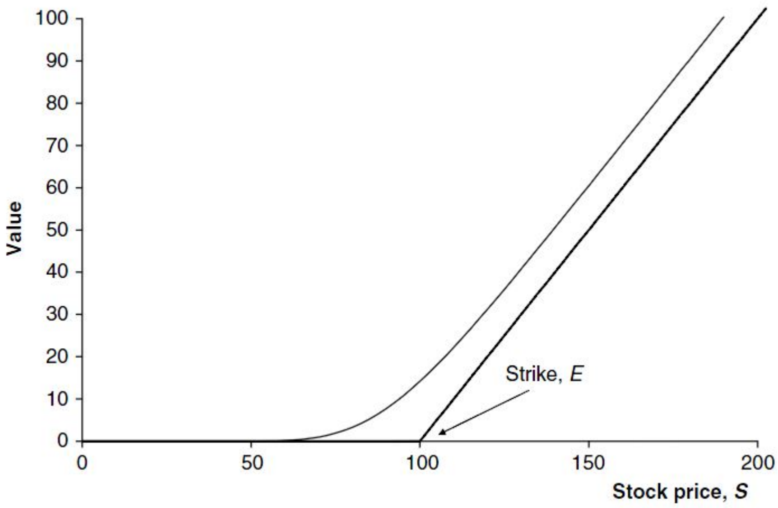

Figure 1.1: Pay-off for call option

Before we go any further into the financial (and numerical) world, here are some of the definitions that will be mentioned throughout the thesis.

- **Premium**: The amount paid for the contract initially.

- **Underlying (asset)**: The financial instrument on which the option value depends. Stocks, commodities, currencies and indices are going to be denoted by S. The option payoff is defined as some function of the underlying asset at expiry.

- **Strike (price)** or **Exercise price**: The amount for which the underlying can be bought (call) or sold (put). This will be denoted by E. This definition only really applies to the simple calls and puts. We will see more complicated contracts in later chapters and the definition of strike or exercise price will be extended.

- **Expiration (date)** or **Expiry**: Date on which the option can be exercised or date on which the option ceases to exist or give the holder any rights. This will be denoted by T.

- **Intrinsic value**: The payoff that would be received if the underlying is at its current level when the option expires.

- **In the money**: An option with positive intrinsic value. A call option when the asset price is above the strike, a put option when the asset price is below the strike.

- **Out of the money**: An option with no intrinsic value, only time value. A call option when the asset price is below the strike, a put option when the asset price is above the strike.

- **Long position**: A positive amount of a quantity, or a positive exposure to a quantity.

- **Short position**: A negative amount of a quantity, or a negative exposure to a quantity. Many assets can be sold short, with some constraints on the length of time before they must be bought back.

1.2 American Options

The options described above are examples of **European options** because exercise is only permitted at expiry. Some contracts allow the holder to exercise at any time before expiry, and these are called **American options**. American options give the holder more rights than their European equivalent and can therefore be more valuable, and they can never be less valuable. The main point of interest with American-style contracts is deciding when to exercise. The details of American options will be discussed in chapter 5 when we talk about the valuation of American options.

1.3 Put-Call Parity

Let's say that someone buys one European call option with a strike of E and an expiry of T, and that the same person writes a European put option with the same strike and expiry. Today's date is t. The payoff you receive at T for the call will look like the line in figure 1.1. The payoff for the portfolio of the two options is the sum of the individual payoffs. The payoff for this portfolio of options is

$$\max\left(S\left(T\right)-E,0\right)-\max\left(E-S\left(T\right),0\right)=S\left(T\right)-E$$

where $S\left(T\right)$ is the value of the underlying asset at time T.

To lock in a payment of E at time T involves a cash flow of $Ee^{-r(T-t)}$ at time t. The conclusion is that the portfolio of a long call and a short put gives the holder exactly the same payoff as a long asset, short cash position. The equality of these cash flows is independent of the future behavior of the stock and is model independent:

$$C - P = S - Ee^{-r(T-t)}$$

where C and P are today's values of the call and the put respectively. This relationship holds at any time up to expiry and is known as **put-call parity**.

1.4 Other Types of Options

In finance, an **exotic option** is a derivative which has features making it more complex than commonly traded products (i.e. European or American options). An exotic option could have one or more of the following features:

- The payoff at expiry depends not only on the value of the underlying asset at maturity, but also at its value at several times during the contract's life.
- It could depend on more than one asset.
- There could be call ability and put ability rights.
- It could involve foreign exchange rates in various ways.

Here are a few examples of such options.

- **Barrier option** is a type of financial option where the option to exercise depends on the underlying crossing or reaching a given barrier level.

- **Asian option** is a special type of option contract where its payoff is determined by the average underlying asset price over some pre-set period of time.

- **Compound option** is option on an option. The exercise payoff of a compound option involves the value of another option. A compound option then has two expiration dates and two strike prices.

- **Lookback option** is a type of exotic options with path dependency, among many other kind of options. The payoff depends on the optimal (maximum or minimum) underlying asset's price occurring over the life of the option. The option allows the holder to "look back" over time to determine the payoff. There exist two kinds of Lookback options: with floating strike and with fixed strike.

- **Rainbow option** is an option exposed to two or more sources of uncertainty, as opposed to a simple option that is exposed to one source of uncertainty, such as the price of underlying asset. Rainbow options are usually calls or puts on the best or worst of n underlying assets, or options which pay the best or worst of n assets.

2

Black-Scholes Model

This is an important chapter in the thesis. In it I describe and explain the basics of the Black-Scholes theory (Black and Scholes, 1973) (Merton, 1973). These basics are delta hedging and no arbitrage. They form a moderately sturdy foundation to the subject and have performed well since 1973 when the ideas became public (Wilmott, 2006).

In this chapter I will skip the basics of stochastic calculus, and begin with the stochastic differential equation model for equities and exploit the correlation between this asset and an option on this asset to make a perfectly risk-free portfolio.

The arguments are trivially modified to incorporate dividends on the underlying and also to price commodity and currency options and options on futures. Even though all of the assumptions can be shown to be wrong to a greater or lesser extent, the Black Scholes model is profoundly important both in theory and in practice.

2.1 Introduction

In Chapter 1 we discussed some of the characteristics of options and options markets. I introduced the idea of call and put options, amongst others. The value of a call option is clearly going to be a function of various parameters in the contract, such as the strike price E and the time to expiry $T - t$, where T is the date of expiry and t is the current time. The value will also depend on properties of the asset itself, such as its price, its drift and its volatility, as well as the risk-free rate of interest. We can write the option value as

$$V\left(S, t, \sigma, E, \mu, T, r\right)$$

Now we use Π to denote the value of a portfolio of one long option position and a short position in some quantity Δ of the underlying:

$$\Pi = V\left(S, t\right) - \Delta S$$

The first term on the right is the option and the second term is the short asset position. Notice the minus sign in front of the second term. The quantity Δ will for the moment be some constant quantity of our choosing. We will assume that the underlying follows a log-normal random walk (geometric Brownian motion):

$$dS = \mu S dt + \sigma S dW$$

Recall Itō's lemma, given a process $dX_t = \mu_t dt + \sigma_t dB_t$,

$$df\left(t, X_t\right) = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu_t + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x} dB_t$$

Applying Itō's lemma to $V\left(S_t, t\right)$ gives

$$dV = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right) dt + \sigma S \frac{\partial V}{\partial S} dW$$

Which is

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt$$

The change in the portfolio value is due partly to the change in the option value and partly to the change in the underlying

$$d\Pi = dV - \Delta dS$$

Notice that Δ has not changed during the time step; we have not anticipated the change in S.

$$d\Pi = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt - \Delta dS$$

Now applying no-arbitrage assumption, $d\Pi$ is risk-less. If we have a completely risk-free change $d\Pi$ in the portfolio value Π, then it must be the same as the growth we would get if we put the equivalent amount of cash in a risk-free interest-bearing account ($B_t = e^{rt} \to dB_t = rB_t dt$).

$$d\Pi = r\Pi dt$$

By delta-hedging, we choose $\Delta = \dfrac{\partial V}{\partial S}$ to eliminate randomness. Now

$$d\Pi = \left(\frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$$

Plugging in everything we have, we get

$$\left(\frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt = r\left(V - S\frac{\partial V}{\partial S}\right) dt$$

Now we have the famous Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Notice that a few assumptions have been made in this section, such as the underlying follows a log-normal random walk; interest rate and volatility are constant; delta hedging is continuous; there is no arbitrage. These assumptions are important for B-S model to be true. Some of them can be dropped (or loosened). We will discuss those in latter chapters.

2.2 The Black-Scholes PDE

Now let's consider a European option with dividend D.(Wilmott et al., 1997)

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r - D)\, S\frac{\partial V}{\partial S} + rV = 0 \tag{2.1}$$

Note τ is the time to expiry (expiry $T -$ current time t). The domain is $S \in [0, \infty]$ and $\tau \in [0, T]$. The initial conditions are

$$C\,(S, 0) = max\,(S - E, 0) \;\text{ for call options}$$

$$P\,(S, 0) = max\,(E - S, 0) \;\text{ for put options}$$

And the boundary conditions are

$$\text{for call options }\; C\,(0, \tau) = 0 \;\; C\,(S, \tau) \to S \text{ as } S \to \infty$$

$$\text{for put options } P(0,\tau) = Ee^{-r\tau} \quad P(S,\tau) \to 0 \text{ as } S \to \infty$$

Note that the analytic solution for Black-Scholes formula for European options is well known. We will not discuss it here. Please see chapter 4 for the result, and check (Wilmott et al., 1997) for the detailed derivation.

2.3 Multi-Dimension Black-Scholes Model

2.3.1 General Black-Scholes PDE Model in d-dimension

The d-asset European option can be priced using the following d-dimension Black-Scholes equation.

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i S_j} - \sum_{i=1}^{d}(r-\delta_i)\, S_i \frac{\partial V}{\partial S_i} + rV = 0 \tag{2.2}$$

Again we let $\tau = T - t$. The solution V is the option price based on the underlying assets S_i with $i = 1, \ldots, d$, σ_i is the volatility of asset i, ρ_{ij} is the correlation coefficient between the assets i and j with $\rho_{ii} = 1$, r is the risk-free interest rate, and δ_i is a continuous dividend yield. Equation 2.2 comes with a pay-off that determines the type of the option. We will assume a *put basket option* for now, whose payoff function is

$$V(T, S_1, \ldots, S_d) = \max\left[0, K - \sum_{i=1}^{d} w_i S_i\right]$$

where w_i is weight of asset i and $\sum_{i=1}^{d} w_i = 1$.

2.3.1.1 Initial Condition

Since $\tau = T - t$, the final condition (i.e. the payoff function) is our initial condition. When $\tau = 0$ ($t = T$), we have

$$V(0, S_1, \ldots, S_d) = \max\left[0, K - \sum_{i=1}^{d} w_i S_i\right] \tag{2.3}$$

2.3.1.2 Boundary Conditions

The boundary conditions require a little bit more analysis. For most financial problems, the domain is semi-infinite or infinite. However, for most of the financial models, the diffusion term is normally the dominate one. So the disturbance from the hyperbolic terms due to imperfect boundary conditions will be minimized. If we place the boundaries "far enough", the artificial boundary conditions won't affect the solution.

In one dimension, three equations are needed in addition to the payoff function (interior function and two boundary functions). In two dimensions, eight equations are needed. In general d-dimension cases, 3^d are needed. We will discuss it later in 2-D and 3-D cases.

2.3.2 2-D Case

Let's consider a *2-asset basket put option.* Let's assume correlations and risk-free rate are constant. Let's also assume zero dividend ($\delta = 0$) and $\rho_{12} = \rho_{21} = 1$. The 2-D Black-Scholes equation is

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} - \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 S_2} - r S_1 \frac{\partial V}{\partial S_1} - r S_2 \frac{\partial V}{\partial S_2} + rV = 0$$

2.3.2.1 Initial Condition

The payoff function is (when $\tau = 0$)

$$V(0, S_1, S_2) = \max\left[0, K - (w_1 S_1 + w_2 S_2)\right]$$

2.3.2.2 Boundary Conditions

If $S_1 = S_2 = 0$, then $V(\tau, 0, 0) = 0$.

If S_1 (or S_2) $= 0$, the 2-D equation reduces to a standard 1-D Black-Scholes equation. Hence the boundary conditions are $V(\tau, 0, S_2) = P_{1D}(\tau, S_2)$ and $V(\tau, S_1, 0) = P_{1D}(\tau, S_1)$.

Now we put the artificial upper boundaries "far enough" at 50 (normally 2 or 3 times of the strike price is considered “far enough”). A commonly used Dirichlet boundary condition is to set option price to zero at far field (Duffy, 2006). So we have $V(\tau, 50, S_2) = 0$ and $V(\tau, S_1, 50) = 0$.

Now we have boundary conditions on all four boundary edges and four boundary vertices. That is $3^2 - 1 = 8$ boundary conditions.

Note that sometime a more economical solution is to apply the pricing equation itself as the B.C., but with the second derivative term set to zero (Tavella and Randall, 2000). For example, the 1-D BSEQ has this “linear” B.C.

$$\frac{\partial V}{\partial \tau} + (r - \delta)\frac{\partial V}{\partial S} - rV = 0 \tag{2.4}$$

where S is understood to be the boundary $S_{\max}$. A discretization of the equation 2.4 will give us an approximation on the boundary.

In any case, the sensitivity of the boundary conditions should be small due to the strong diffusion.

2.3.3 3-D Case

Now consider the 3-D case. Again, the black-scholes equation is

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i S_j} - \sum_{i=1}^{3} r S_i \frac{\partial V}{\partial S_i} + rV = 0$$

All coefficients are constant and dividend is zero.

2.3.3.1 Initial Condition

The payoff function is (when $\tau = 0$)

$$V(0, S_1, S_2, S_3) = \max\left[0, K - (w_1 S_1 + w_2 S_2 + w_3 S_3)\right]$$

2.3.3.2 Boundary Conditions

The boundary conditions for the 2-D case can be extended into 3-D.
If $S_i = 0$, $i = 1, 2, 3$, the 3-D equation reduces to 2-D equation. If any two of the three assets are zero, then it becomes a 1-D equation.

Now consider the value in the far field. We can set the value to zero at a "far enough" point. Or we can apply the "linear" boundary conditions, eliminate all second order derivative terms from the equation on the boundaries and discretize the drift terms using one-sided difference operators.

Again, we will have equations on all six boundary surfaces, all twelve boundary edges and eight boundary vertices. So we have a total of $3^3 - 1$ B.C. equations.

3

Finite Difference Method

3.1 Introduction

This chapter introduces the finite difference methods (FDM). The sections in this chapter focus on producing accurate and robust schemes for second-order parabolic and first-order hyperbolic partial differential equations in two independent variables, usually called x and t. The first variable x plays the role of a space coordinate and the second variable t plays the role of time. We will introduce the concept of divided differences and how use them to approximate the first- and second-order derivatives of real-valued functions of one variable. We then model the partial differential equations by approximating the derivatives using divided differences. These are defined at so-called discrete mesh points. Having set up FDM, we will then apply the resulting finite difference schemes to the single asset Black-Scholes model.

3.2 FDM Schemes for Hyperbolic Equation

There is a vast literature on first-order hyperbolic equations. Much effort has gone into devising robust approximate schemes in application areas such as gas and fluid dynamics, chemical reactor theory and wave phenomena. We consider first-order partial differential equations in two independent variables x and t. The first variable is typically space (or some other dimension) and the second variable usually represents time. The following is an initial boundary value problem (IBVP) on an interval

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad 0 < x < 1, \quad t > 0$$

$$u(x, 0) = f(x), \quad 0 < x < 1$$

$$u(0, t) = g(t), \quad t \geq 0$$

Determining boundary conditions is essential to solving these problems. We will discuss it in detail in latter chapters. Before we go into details on differencing schemes, let's define

$$\delta^+ u_m = u_{m+1} - u_m$$

$$\delta^- u_m = u_m - u_{m-1}$$

$$\delta^0 u_m = u_{m+1} - u_{m-1}$$

3.2.1 Explicit Upwind Scheme

In following scheme, we assume $a > 0$ and $R = a\frac{\Delta t}{\Delta x}$.

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0\,, \quad u_j^{n+1} = u_j^n - R\delta^- u_j^n \tag{3.1}$$

Note for an explicit scheme to be stable, we need the CFL number $|R| < 1$.

3.2.2 Central Implicit Scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} = 0\,, \quad u_j^{n+1} + \frac{1}{2}R\delta^0 u_j^{n+1} = u_j^n \tag{3.2}$$

3.2.3 Crank-Nicholson Scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{2}a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{1}{2}a\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} = 0\,, \quad u_j^{n+1} + \frac{1}{4}R\delta^0 u_j^{n+1} = u_j^n - \frac{1}{4}R\delta^0 u_j^n \tag{3.3}$$

Both of the implicit schemes are **unconditionally stable**. With exception of Crank-Nicholson being 2nd order accurate in terms of time and space, all other schemes introduced here are 1st order accurate.

3.3 FDM Schemes for Parabolic Equation

Considering the following parabolic equation

$$u_t = bu_{xx} \quad b > 0 \quad r = b\frac{\Delta t}{\Delta x^2}$$

And define

$$\delta^2 u_m = u_{m+1} - 2u_m + u_{m-1}$$

3.3.1 Central Explicit Scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = b\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}, \quad u_j^{n+1} = u_j^n + r\delta^2 u_j^n \tag{3.4}$$

This scheme is 1st order accurate in terms of time and 2nd order accurate in terms of space. CFL condition is $r \leq \frac{1}{2}$.

3.3.2 Central Implicit Scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = b\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}, \quad \left(1 - r\delta^2\right) u_j^{n+1} = u_j^n \tag{3.5}$$

This scheme is 1st order accurate in terms of time and 2nd order accurate in terms of space. And it is unconditionally stable.

3.3.3 Crank-Nicholson Scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{1}{2}b\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \frac{1}{2}b\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}, \tag{3.6}$$
$$\left(1 - \frac{1}{2}r\delta^2\right) u_j^{n+1} = \left(1 + \frac{1}{2}r\delta^2\right) u_j^n$$

This scheme is 2nd order accurate in terms of time and space. And it is unconditionally stable.

Since the B-S PDE is a mixture of hyperbolic and parabolic equations, we will put our attention on these schemes primarily.

3.4 Front-Tracking Method

Front tracking method (Crank, 1984) is a Lagrangian method for the propagation of a moving manifold. Front tracking works by moving marker particles which represent the interface. It is distinguished from the marker particle method in that the particles are located only on the interface, rather than in a volume region near the interface, and in that the particles are connected to each other, to form a triangulated mesh (3D) or piecewise linear segments (2D) of the interface. It is significantly faster than other particle methods, since fewer particles (one or two in 2D) are used per cell in front tracking than the number used in typical particle method simulations. 2D bifurcations of interface topology in front tracking are resolved accurately through detection of interface intersections. In three dimensions, we use a local Eulerian reconstruction method. This method has the robustness of the Eulerian method while it maintains the high resolution and accuracy of the Lagrangian method.

The front tracking method has showed its advantage in the computation of several important physical problems such as the study of fluid interface instabilities, providing the first or the only physically validated simulation for the solution of turbulent mixing.

Free and moving boundary value problems have their origins in the physical sciences. Problems in which the solution of differential equations must satisfy certain conditions on the boundary of a prescribed domain are called boundary value problems. In many cases the boundary of the domain is not known a priori but it must be determined as part of the problem. We partition such problems into two groups: first, the term 'free boundary problem' is used when the boundary is stationary and a steady-state solution exists (for example, the solution of an elliptic problem). We then have the class of moving boundary value problems that are associated with time-dependent problems (for example, defined by a parabolic partial differential equation). The unknown boundaries in the latter case are a function of both space and time. In all cases we must specify two conditions on the free or moving boundary. Of course, the usual boundary conditions are specified on the fixed boundary as well as some appropriate initial conditions, as already discussed. In general, we can classify free and moving boundary value problems into different categories depending on the types of problem that they model. For example, a one-phase problem is one where we model a PDE in a single domain with an unknown boundary. The solution on the other side of the unknown boundary is known. With two-phase problems we model different PDEs, that is, defined in two domains that are separated by a free or moving boundary. Most problems in financial engineering at the moment of writing are described as one-phase problems. In this case the solution is zero on one side of the moving boundary and it satisfies the BlackScholes equation on the other side of the boundary.

Moving boundary value problems are sometimes called **Stefan problems** in honor of the Austrian mathematician, **J. Stefan**, who in 1890 studied the melting of the polar ice cap.

4

European Options

As we mentioned in chapter 1, a European option taken out at current time t gives the holder the right to do something when the option expires at time T. This could for example be the right to buy or sell stocks at a particular strike price. The option would of course only be exercised if it was in the holder's interest to do so.

4.1 European Put Options

A single asset European put option, with strike price E and expiry time T,gives the owner the right at time T to sell a particular asset for E. If the asset is worth S_T at maturity then the value of the put option at maturity, known as the payoff, is thus max $(E - S_T$, 0).

4.1.1 Black-Scholes PDE

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r - D)\, S \frac{\partial V}{\partial S} + rV = 0 \tag{4.1}$$

Note τ is the time to expiry $(T - t)$. The domain is $S \in [0, \infty]$ and $\tau \in [0, T]$. The initial condition is

$$P\,(S, 0) = max\,(E - S, 0) \;\text{ for put options}$$

4.1.2 Exact Solution

The Black-Scholes formula for a European put

$$P(S,\tau) = Ee^{-r\tau}N(-d_2) - SN(-d_1)$$

where

$$d_1 = \frac{log(S/E) + (r + \sigma^2/2)}{\sigma\sqrt{\tau}} \text{ and } d_2 = d_1 - \sigma\sqrt{\tau}$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy$$

4.1.3 Crank-Nicholson Finite-Differencing Formulation

Following the finite differencing formulas 3.3 and 3.7, we can get the Crank-Nicholson finite differencing formulation for Black-Scholes PDE 4.1. After rearranging the terms, the formulation becomes

$$\begin{aligned}
& P_{i-1}^{n+1}\left[\frac{1}{4}\Delta t\left((r-D)S - \sigma^2 S^2\right)\right] \\
+ \quad & P_{i}^{n+1}\left[1 + \frac{1}{2}\Delta t\left(\sigma^2 S^2 + r\right)\right] \\
+ \quad & P_{i+1}^{n+1}\left[-\frac{1}{4}\Delta t\left((r-D)S + \sigma^2 S^2\right)\right] \\
= \quad & \\
& P_{i-1}^{n}\left[\frac{1}{4}\Delta t\left((r-D)S - \sigma^2 S^2\right)\right] \\
+ \quad & P_{i}^{n}\left[1 + \frac{1}{2}\Delta t\left(\sigma^2 S^2 + r\right)\right] \\
+ \quad & P_{i+1}^{n}\left[-\frac{1}{4}\Delta t\left((r-D)S + \sigma^2 S^2\right)\right]
\end{aligned}$$

Note that there are only three unknowns on the left side of the equation, which means the coefficient matrix of this system of linear equations is a tri-diagonal matrix. And we can solve such system easily and efficiently using Thomas algorithm (Wilmott et al., 1997).

4.2 European Call Options

A single asset European call option, with strike price E and expiry time T, gives the owner the right at time T to buy (instead of sell) a particular asset for E. If the asset is worth S_T at maturity then the value of the put option at maturity, known as the payoff, is thus max $(S_T - E,\ 0\ )$.

4.2.1 Black-Scholes PDE

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r - D)\, S \frac{\partial V}{\partial S} + rV = 0 \tag{4.2}$$

Note τ is the time to expiry $(T - t)$. The domain is $S \in [0, \infty]$ and $\tau \in [0, T]$. The initial condition is

$$C\,(S, 0) = max\,(S - E, 0) \;\text{ for call options}$$

4.2.2 Exact Solution

The Black-Scholes formula for a European call

$$C\,(S, \tau) = SN\,(d_1) - Ee^{-r\tau} N\,(d_2)$$

where

$$d_1 = \frac{log\,(S/E) + (r + \sigma^2/2)}{\sigma\sqrt{\tau}} \;\text{ and }\; d_2 = d_1 - \sigma\sqrt{\tau}$$

and

$$N\,(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy$$

4.2.3 FDM Formulation

The FDM is the same as the one for European put option, the only changes are boundary conditions and initial conditions.

4.3 Multi-Asset European Option

Let's consider a *2-asset basket put option.* Assume correlations and risk-free rate are constant. Let's also assume zero dividend ($\delta = 0$) and $\rho_{12} = \rho_{21} = 1$. The 2-D Black-Scholes equation is

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} - \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 S_2} - rS_1 \frac{\partial V}{\partial S_1} - rS_2 \frac{\partial V}{\partial S_2} + rV = 0 \quad (4.3)$$

4.3.1 Initial Condition

The payoff function is (when $\tau = 0$)

$$V(0, S_1, S_2) = \max\left[0, K - (w_1 S_1 + w_2 S_2)\right] \quad (4.4)$$

4.3.2 Boundary Conditions

If $S_1 = S_2 = 0$, then $V(\tau, 0, 0) = 0$.

If S_1 (or S_2) $= 0$, the 2-D equation reduces to a standard 1-D Black-Scholes equation. Hence the boundary conditions are $V(\tau, 0, S_2) = P_{1-D}(\tau, S_2)$ and $V(\tau, S_1, 0) = P_{1-D}(\tau, S_1)$.

Now we put the artificial upper boundaries "far enough" at 50 (normally 2 or 3 times of the strike price is considered "far enough"). A commonly used Dirichlet boundary condition is to set option price to zero at far field (Duffy, 2006). So we have $V(\tau, 50, S_2) = 0$ and $V(\tau, S_1, 50) = 0$.

Now we have boundary conditions on all four boundary edges and four boundary vertices. That is $3^2 - 1 = 8$ boundary conditions.

Note that sometime a more economical solution is to apply the pricing equation itself as the B.C., but with the second derivative term set to zero (Tavella and Randall, 2000). For example, the single asset Black-Scholes PDE 4.1 has this "linear" B.C.

$$\frac{\partial V}{\partial \tau} + (r - \delta)\frac{\partial V}{\partial S} - rV = 0$$

where S is understood to be the boundary $S_{\max}$. A discretization of the above equation will give us an approximation on the boundary.

In any case, the sensitivity of the boundary conditions should be small due to the strong diffusion.

4.3.3 A 2-D Crank-Nicholson Finite-Differencing Example

Here is an example of two dimensional Crank-Nicholson finite-differencing method for 2-asset Black-Scholes PDE. We will introduce the details in Chapter 6.

$$
\begin{aligned}
& V_{i-1,j-1}^{n+1}\left[\frac{1}{8}\sigma_1\sigma_2\left(\Delta S_1\cdot i\right)\left(\Delta S_2\cdot j\right)\frac{\Delta\tau}{\Delta S_1\Delta S_2}\right] \\
+\quad & V_{i,j-1}^{n+1}\left[\frac{1}{4}\sigma_2^2\left(\Delta S_2\cdot j\right)^2\frac{\Delta\tau}{\Delta S_2^2}-\frac{1}{4}r\left(\Delta S_2\cdot j\right)\frac{\Delta\tau}{\Delta S_2}\right] \\
+\quad & V_{i+1,j-1}^{n+1}\left[-\frac{1}{8}\sigma_1\sigma_2\left(\Delta S_1\cdot i\right)\left(\Delta S_2\cdot j\right)\frac{\Delta\tau}{\Delta S_1\Delta S_2}\right] \\
+\quad & V_{i-1,j}^{n+1}\left[\frac{1}{4}\sigma_1^2\left(\Delta S_1\cdot i\right)^2\frac{\Delta\tau}{\Delta S_1^2}-\frac{1}{4}r\left(\Delta S_1\cdot i\right)\frac{\Delta\tau}{\Delta S_1}\right] \\
+\quad & V_{i,j}^{n+1}\left[1-\frac{1}{2}\sigma_1^2\left(\Delta S_1\cdot i\right)^2\frac{\Delta\tau}{\Delta S_1^2}-\frac{1}{2}\sigma_2^2\left(\Delta S_2\cdot j\right)^2\frac{\Delta\tau}{\Delta S_2^2}\right] \\
+\quad & V_{i+1,j}^{n+1}\left[\frac{1}{4}\sigma_1^2\left(\Delta S_1\cdot i\right)^2\frac{\Delta\tau}{\Delta S_1^2}+\frac{1}{4}r\left(\Delta S_1\cdot i\right)\frac{\Delta\tau}{\Delta S_1}\right] \\
+\quad & V_{i-1,j+1}^{n+1}\left[-\frac{1}{8}\sigma_1\sigma_2\left(\Delta S_1\cdot i\right)\left(\Delta S_2\cdot j\right)\frac{\Delta\tau}{\Delta S_1\Delta S_2}\right] \\
+\quad & V_{i,j+1}^{n+1}\left[\frac{1}{4}\sigma_2^2\left(\Delta S_2\cdot j\right)^2\frac{\Delta\tau}{\Delta S_2^2}+\frac{1}{4}r\left(\Delta S_2\cdot j\right)\frac{\Delta\tau}{\Delta S_2}\right] \\
+\quad & V_{i+1,j+1}^{n+1}\left[\frac{1}{8}\sigma_1\sigma_2\left(\Delta S_1\cdot i\right)\left(\Delta S_2\cdot j\right)\frac{\Delta\tau}{\Delta S_1\Delta S_2}\right] \\
=\quad & \\
& V_{i-1,j-1}^{n}\left[-\frac{1}{8}\sigma_1\sigma_2\left(\Delta S_1\cdot i\right)\left(\Delta S_2\cdot j\right)\frac{\Delta\tau}{\Delta S_1\Delta S_2}\right] \\
+\quad & V_{i,j-1}^{n}\left[-\frac{1}{4}\sigma_2^2\left(\Delta S_2\cdot j\right)^2\frac{\Delta\tau}{\Delta S_2^2}+\frac{1}{4}r\left(\Delta S_2\cdot j\right)\frac{\Delta\tau}{\Delta S_2}\right] \\
+\quad & V_{i+1,j-1}^{n}\left[\frac{1}{8}\sigma_1\sigma_2\left(\Delta S_1\cdot i\right)\left(\Delta S_2\cdot j\right)\frac{\Delta\tau}{\Delta S_1\Delta S_2}\right] \\
+\quad & V_{i-1,j}^{n}\left[-\frac{1}{4}\sigma_1^2\left(\Delta S_1\cdot i\right)^2\frac{\Delta\tau}{\Delta S_1^2}+\frac{1}{4}r\left(\Delta S_1\cdot i\right)\frac{\Delta\tau}{\Delta S_1}\right] \\
+\quad & V_{i,j}^{n}\left[1+r+\frac{1}{2}\sigma_1^2\left(\Delta S_1\cdot i\right)^2\frac{\Delta\tau}{\Delta S_1^2}+\frac{1}{2}\sigma_2^2\left(\Delta S_2\cdot j\right)^2\frac{\Delta\tau}{\Delta S_2^2}\right] \\
+\quad & V_{i+1,j}^{n}\left[-\frac{1}{4}\sigma_1^2\left(\Delta S_1\cdot i\right)^2\frac{\Delta\tau}{\Delta S_1^2}-\frac{1}{4}r\left(\Delta S_1\cdot i\right)\frac{\Delta\tau}{\Delta S_1}\right] \\
+\quad & V_{i-1,j+1}^{n}\left[\frac{1}{8}\sigma_1\sigma_2\left(\Delta S_1\cdot i\right)\left(\Delta S_2\cdot j\right)\frac{\Delta\tau}{\Delta S_1\Delta S_2}\right] \\
+\quad & V_{i,j+1}^{n}\left[-\frac{1}{4}\sigma_2^2\left(\Delta S_2\cdot j\right)^2\frac{\Delta\tau}{\Delta S_2^2}-\frac{1}{4}r\left(\Delta S_2\cdot j\right)\frac{\Delta\tau}{\Delta S_2}\right] \\
+\quad & V_{i+1,j+1}^{n}\left[-\frac{1}{8}\sigma_1\sigma_2\left(\Delta S_1\cdot i\right)\left(\Delta S_2\cdot j\right)\frac{\Delta\tau}{\Delta S_1\Delta S_2}\right]
\end{aligned}
$$

There should be 9 non-zero entries in each row of the coefficient matrix.

5

American Options

5.1 American Put Options

5.1.1 Introduction

The majority of traded options are of American type. However their valuation, even in the standard case of a log-normal process for the underlying asset, remains a topic of active research. This situation stems from the nature of the solution which requires the determination of the optimal exercise strategy as well as the value of the option. In contrast to the European option, which can only be exercised at its expiration date, has been valued by the celebrated BlackScholes formula (Black and Scholes, 1973) (Merton, 1973) for the standard financial model.

Due to a lack of closed-form solutions to American option valuation problems, a vast array of approximation schemes has been advanced. While other studies have focused on modified pricing model (Jamshidian, 1996), formulating the problem as a non-linear model (Kholodnyi, 1997), using front-fix methods (Crank, 1984) to fix the moving boundary (Wu and Kwok, 1996) (Pantazopoulos et al., 1998), or trying to find semi-/analytical solutions to the problem (Sevcovic, 2001), we introduce and analyze a front-tracking (FT) finite difference method (FDM) based on original Black-Scholes Model with free moving boundary.

5.1.2 Black-Scholes PDE

Let $P = P(S, \tau)$ be the put option price. Then put satisfies the PDE:

$$\frac{\partial P}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - (r - D) S \frac{\partial P}{\partial S} + rP = 0$$

Note τ is the time to expiry ($\tau = T - t$). The domain is $S \in (S_f(\tau), S_{max}]$ and $\tau \in [0, T]$. The initial condition is

$$P(S, 0) = max(E - S, 0), \quad S > 0 \quad S_f(0) = E$$

And the boundary conditions are

$$P(0, \tau) = E$$

$$P(S_f(\tau), \tau) = E - S_f(\tau), \quad \frac{\partial P(S_f(\tau), \tau)}{\partial S} = -1 \tag{5.1}$$

For $S \in [0, S_f(\tau))$, the value of the put is equal to the payoff function $\max(E - S, 0)$.

5.1.3 Tracking the Front Point

In order to approximation the moving boundary, we assume that the stock price S lies on a quadratic curve near the moving front S_f. As shown in figure 5.1, we will use three points $S_i, \quad i = 1,\ 2,\ 3$ and the corresponding P_i to compute the coefficients a, b and c.

$$aS_i^2 + bS_i + c = P_i \quad i = 1,\ 2,\ 3$$

Note that P_2 and P_3 can be obtained from the current time step. In order to get a better estimation for the next S_f, we will try to approximate P_1 differently from other P_i. First let's consider the Black-Scholes equation

$$\frac{\partial P}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - (r - D) S \frac{\partial P}{\partial S} + rP = 0$$

S_1 is in the exercise region, which means $P_1 = E - S_1$. We know the 2nd order term $\frac{\partial^2 P}{\partial S^2} = 0$, and $\frac{\partial P}{\partial S} = -1$. Furthermore, we let $\frac{\partial P}{\partial \tau} = \frac{P_1^{n+1} - P_1^n}{\Delta \tau}$. So the Black-Scholes equation becomes

$$P_1^{n+1} = -\Delta\tau (r - D) S + (1 - r\Delta\tau) P_1^n$$

After obtaining P_1, the linear system can be easily solved since we have 3 unknowns and 3 equations. Once we have a, b and c, and we know $P\left(S_f^{n+1}\right) = E - S_f^{n+1}$, we can plug

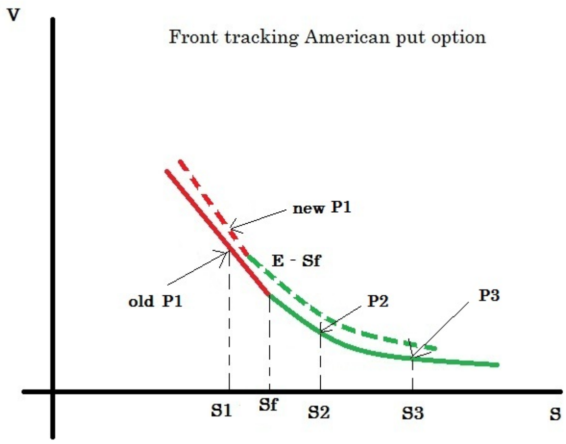

Figure 5.1: Tracking American Put Option Front

all these into $aS_f^2 + bS_f + c = E - S_f$ and solve for S_f at the next time step.

After getting S_f at the next time step, we can set up the boundary condition and solve the Black-Scholes PDE equation.

5.1.4 Crank-Nicholson Finite-Differencing Formulation

Similar to the Crank-Nicholson finite-differencing formulation for European options, within the domain $S \in (S_f(\tau), S_{max}]$ we will have

$$\begin{aligned} & P_{i-1}^{n+1}\left[\frac{1}{4}\Delta t\left((r-D)S-\sigma^2S^2\right)\right] \\ + \quad & P_{i}^{n+1}\left[1+\frac{1}{2}\Delta t\left(\sigma^2S^2+r\right)\right] \\ + \quad & P_{i+1}^{n+1}\left[-\frac{1}{4}\Delta t\left((r-D)S+\sigma^2S^2\right)\right] \\ = \quad & \\ & P_{i-1}^{n}\left[\frac{1}{4}\Delta t\left((r-D)S-\sigma^2S^2\right)\right] \\ + \quad & P_{i}^{n}\left[1+\frac{1}{2}\Delta t\left(\sigma^2S^2+r\right)\right] \\ + \quad & P_{i+1}^{n}\left[-\frac{1}{4}\Delta t\left((r-D)S+\sigma^2S^2\right)\right] \end{aligned}$$

Given the initial conditions and boundary conditions, after obtaining the moving front at each time step we will be able to solve this system of linear equations. Outside of the domain $S \in (S_f(\tau), S_{max}]$ the value of the put option is just the pay-off function $P(S,0) = max(E-S,0)$. Hence the option prices over the entire domain $S \in (0, S_{max}]$ are solved.

5.2 American Call Options

5.2.1 Black-Scholes PDE

We already know that a European option can be exercised only at the expiry date. American options, on the other hand, can be exercised at any time before, or up to, the expiry date. In this section we concentrate on a call option with an early exercise feature with dividend D. Let $C = C(S,\tau)$ be the call option price. Then C satisfies the PDE:

$$\frac{\partial C}{\partial \tau} - \frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2} - (r-D)S\frac{\partial C}{\partial S} + rC = 0$$

Note τ is the time to expiry ($\tau = T-t$). The domain is $S \in [0, S_f(\tau)]$ and $\tau \in [0,T]$. The initial condition is

$$C(S,0) = max(S-E,0), \quad S>0 \quad S_f(0) = \frac{rE}{D}$$

And the boundary conditions are

$$C(0,\tau) = 0$$

$$C(S_f(\tau),\tau) = S_f(\tau) - E, \quad \frac{\partial C(S_f(\tau),\tau)}{\partial S} = 1 \tag{5.2}$$

For $S \in [S_f(\tau), \infty)$, the value of the call is equal to the payoff function $\max(S-E,0)$.

5.2.2 Tracking the Front Point

Similar to American put option, we will use three points near the moving boundary S_f at time n to approximate S_f at time $n+1$. The procedure is merely identical to what we have done for put option, with the exception of $C\left(S_f^{n+1}\right) = S_f^{n+1} - E$. Consequently

$$C_1^{n+1} = +\Delta\tau(r-D)S + (1-r\Delta\tau)C_1^n$$

5.2.3 Crank-Nicholson Finite-Differencing Formulation

Again the same finite-differencing formulation holds. So along with the proper initial and boundary conditions we can solve the system.

$$\begin{aligned}
& C_{i-1}^{n+1}\left[\frac{1}{4}\Delta t\left((r-D)S - \sigma^2 S^2\right)\right] \\
+ \quad & C_i^{n+1}\left[1 + \frac{1}{2}\Delta t\left(\sigma^2 S^2 + r\right)\right] \\
+ \quad & C_{i+1}^{n+1}\left[-\frac{1}{4}\Delta t\left((r-D)S + \sigma^2 S^2\right)\right] \\
= \quad & \\
& C_{i-1}^{n}\left[\frac{1}{4}\Delta t\left((r-D)S - \sigma^2 S^2\right)\right] \\
+ \quad & C_i^{n}\left[1 + \frac{1}{2}\Delta t\left(\sigma^2 S^2 + r\right)\right] \\
+ \quad & C_{i+1}^{n}\left[-\frac{1}{4}\Delta t\left((r-D)S + \sigma^2 S^2\right)\right]
\end{aligned}$$

5.3 Numerical Experiments

5.3.1 Numerical Results Comparison

In this section we present a series of numerical results of the front-tracking method. Another front-tracking method (EFT/IFT) was discussed in (Pantazopoulos et al., 1998). The method-of-lines with invariant imbedding (MLII) was discussed in (Goldenberg and Schmidt, 1994), and the linear complementarity method (LC) was presented extensively in (Wilmott et al., 1993). Since the American pricing problem has no analytic solution, for comparison purposes we use the binomial method which is known to converge to the true value (Amin and Khanna, 1994) for a large number of time steps. We refer to the results given by the binomial method as benchmark (BENCH) (see Table 5.1).

Stock Price	EFT	IFT	MLII	LC	BENCH	FT
7.00000	0.003939	0.004025	0.003977	0.004057	0.003988	0.003995
7.54545	0.016429	0.016526	0.016450	0.016577	0.016450	0.016459
8.09091	0.051271	0.051412	0.051360	0.051353	0.051323	0.051258
8.63636	0.127828	0.127973	0.128069	0.127895	0.127893	0.127901
9.18182	0.266722	0.266247	0.266645	0.266366	0.266390	0.266331
9.72727	0.481699	0.481072	0.481230	0.480727	0.481106	0.481026
10.27273	0.776188	0.775478	0.775818	0.775283	0.775587	0.775938
10.81818	1.143544	1.142981	1.143191	1.142845	1.142953	1.142812
11.36364	1.569059	1.569793	1.569901	1.569956	1.569856	1.569819
11.90909	2.040716	2.040556	2.040546	2.040665	2.040508	2.040312
13.00000	3.059954	3.060005	3.059906	3.060106	3.059931	3.059884

Table 5.1: American Call Results Comparison

5.3.2 Computation Consistency and Efficiency

In this section we will exam the consistency and efficiency of our method. If we consider Binomial Method with very large time steps as benchmarks (i.e. “exact” solution), we demonstrate that as mesh size increases, the difference between our results and Binomial method decreases (as shown in figure 5.2). However, In figure 5.6 we can see in Binomial method time step doubles from 5000 to 10000, the running time goes from 45 seconds to 185 seconds. On the other hand, in FT method in order to produce similar numerical results (within 10^{-3}) we only need mesh size of 100. Moreover when mesh size doubles from 100 to 200, the running time only goes from 3 to 6 seconds. In figures 5.3 and 5.4, we show the option price differences (from one mesh size to a bigger mesh size) as mesh size increases (and cell size ΔS decreases). In figure 5.5, we can see as mesh size increases the running time increases substantially. However, even with smaller mesh size (100 or 200) we can obtain relatively accurate results.

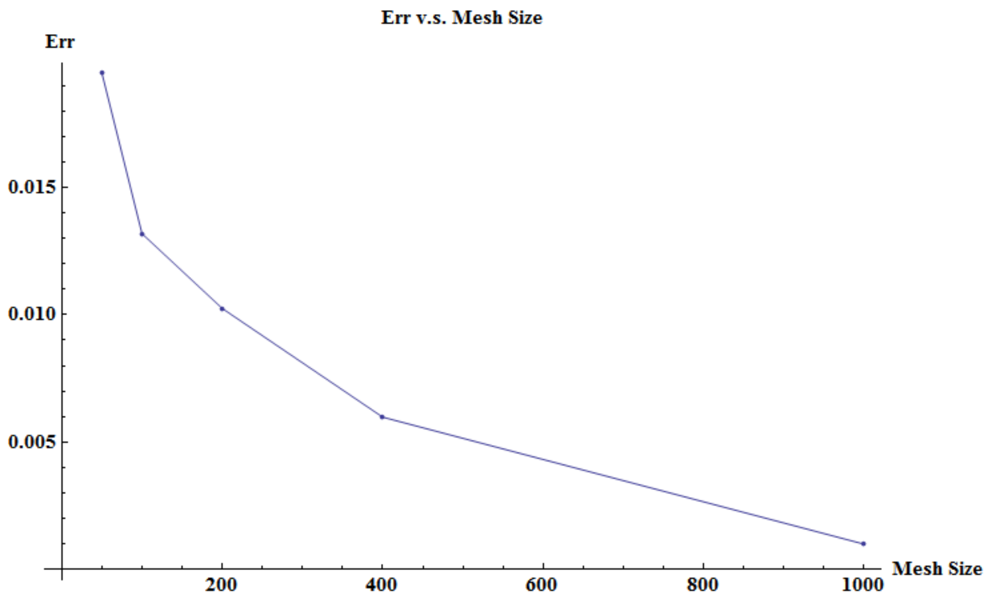

Figure 5.2: Error vs Mesh Size

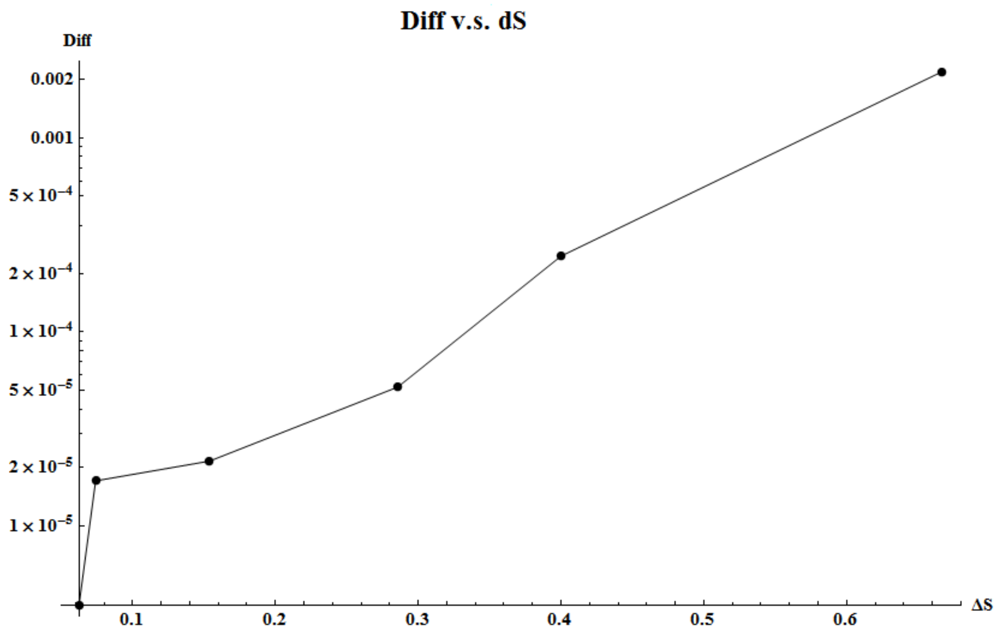

Figure 5.3: Difference vs dS

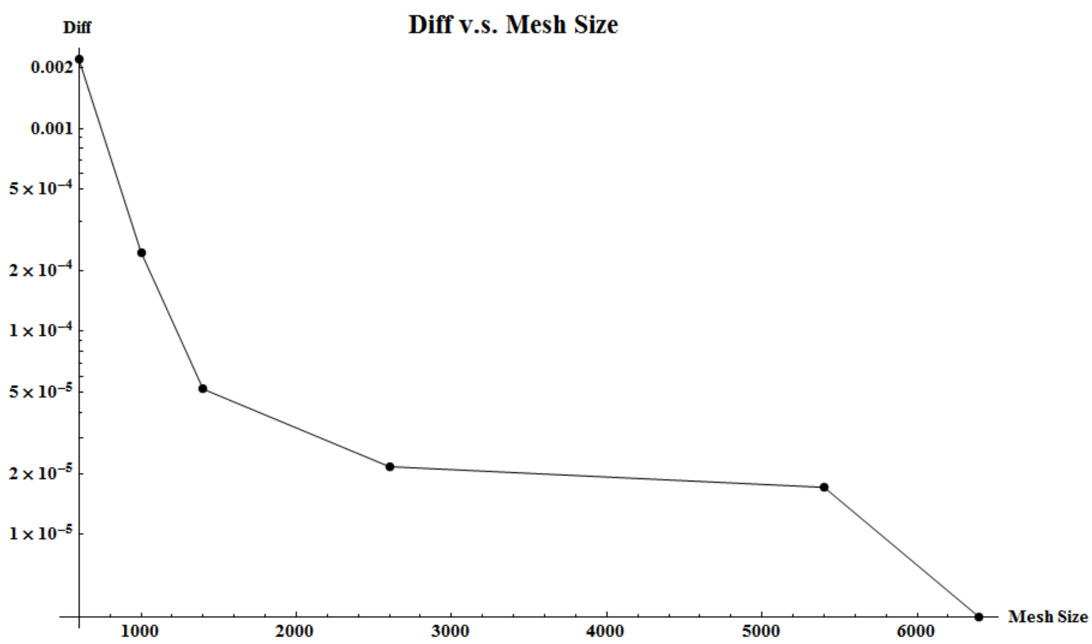

Figure 5.4: Difference vs Mesh size

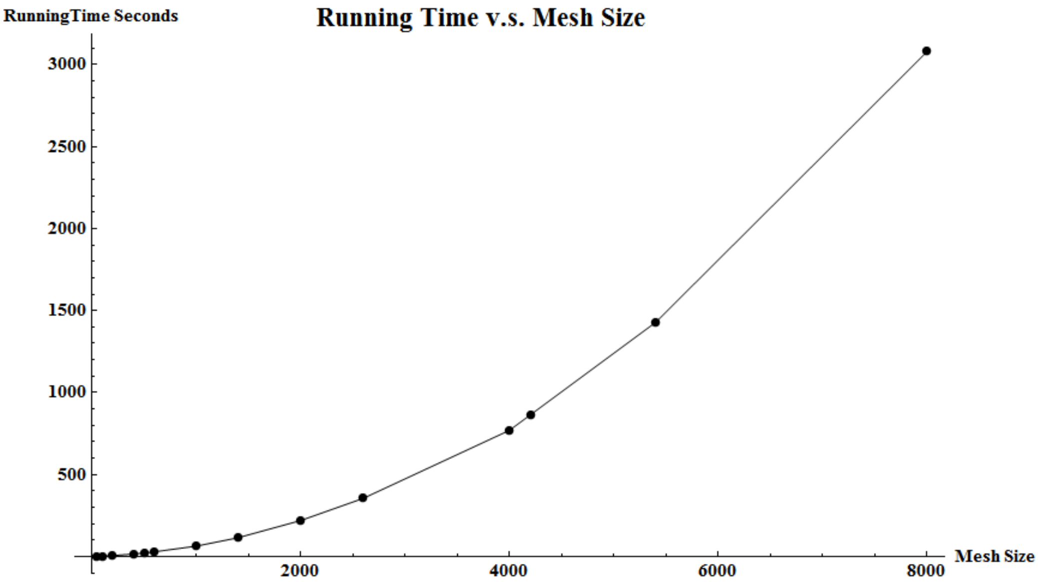

Figure 5.5: Running time vs Mesh size

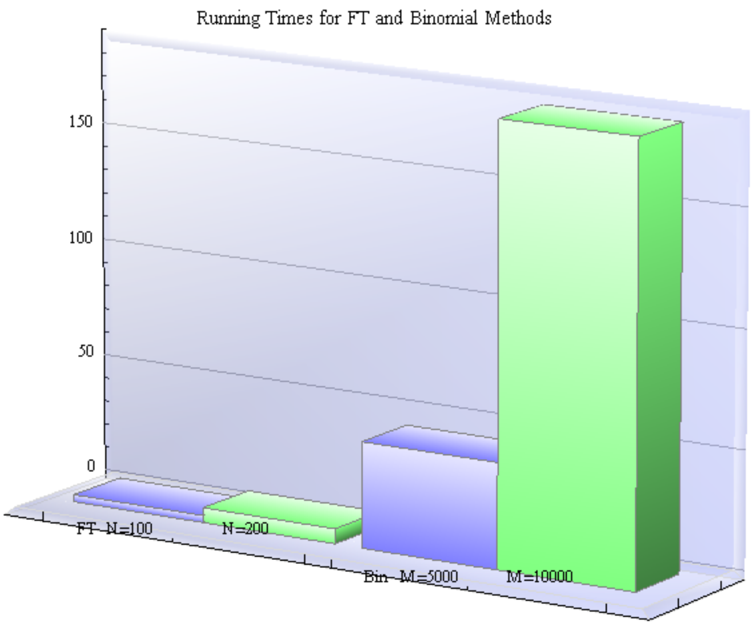

Figure 5.6: Running time for FT and Binomial Methods

6

Multi-Asset Options

6.1 Introduction

In this chapter we give an introduction to option problems with two or more correlated underlying assets. We will focus on deriving finite difference methods (FDM) for these problems, not using ADI or splitting. Because of the size of linear system for 2-D or 3-D models, we will utilize the PDE solver in **FronTier** *PETSC*. One of the goals of this chapter is to provide a setting so that financial models for correlation options can be posed and then mapped to a PDE formulation. We then approximate the corresponding initial boundary value problem using finite differences. Finally, we solve the discrete sets of equations using *PETSC*.

We will provide a detailed derivation for Crank-Nicholson and Implicit method, although Crank-Nicholson method doesn't not always produce accurate results (see (Duffy, 2004b) for details).

6.2 Multi-Dimension Black-Scholes Model

The d-asset European option can be priced using the following d-dimension Black-Scholes equation.

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i S_j} - \sum_{i=1}^{d}\left(r-\delta_i\right) S_i \frac{\partial V}{\partial S_i} + rV = 0 \tag{6.1}$$

We let $\tau = T - t$. The solution V is the option price based on the underlying assets S_i

with $i = 1, \ldots, d$, σ_i is the volatility of asset i, ρ_{ij} is the correlation coefficient between the assets i and j, and with $\rho_{ii} = 1$, r is the risk-free interest rate, and δ_i is a continuous dividend yield. Equation (1) comes with a pay-off that determines the type of the option. We will assume a *put basket option* for now, whose payoff function is

$$V(T, S_1, \ldots, S_d) = \max\left[0, K - \sum_{i=1}^{d} w_i S_i\right] \tag{6.2}$$

where w_i is weight of asset i and $\sum_{i=1}^{d} w_i = 1$.

6.2.1 Initial Condition

Since $\tau = T - t$, the final condition (i.e. the payoff function) is our initial condition. When $\tau = 0$ $(t = T)$, we have

$$V(0, S_1, \ldots, S_d) = \max\left[0, K - \sum_{i=1}^{d} w_i S_i\right] \tag{6.3}$$

6.2.2 Boundary Conditions

The boundary conditions require a little bit more. For most financial problems, the domain is semi-infinite or infinite. However, for most of the financial models, the diffusion term is normally the dominate one. So the disturbance from the hyperbolic terms due to imperfect boundary conditions will be minimized. If we place the boundaries "far enough", the artificial boundary conditions won't affect the solution.

In one dimension, three equations are needed in addition to the payoff function (interior function and two boundary functions). In two dimensions, eight equations are needed. In general, 3^n are needed. We will discuss it later in 2-D and 3-D cases.

6.3 2-D Case

Let's consider a *2-asset basket put option.* Assume correlations and risk-free rate are constant. Let's also assume zero dividend ($\delta = 0$) and $\rho_{12} = \rho_{21} = 1$. The 2-D Black-

Scholes equation is

$$\frac{\partial P}{\partial \tau}-\frac{1}{2}\sigma_1^2 S_1^2\frac{\partial^2 P}{\partial S_1^2}-\frac{1}{2}\sigma_2^2 S_2^2\frac{\partial^2 P}{\partial S_2^2}-\sigma_1\sigma_2 S_1 S_2\frac{\partial^2 P}{\partial S_1 S_2}-(r-D_1)\,S_1\frac{\partial P}{\partial S_1}-(r-D_2)\,S_2\frac{\partial P}{\partial S_2}+rP=0 \tag{6.4}$$

6.3.1 Initial Condition

The payoff function is (when $\tau = 0$)

$$V\left(0, S_1, S_2\right) = \max\left[0, K - \left(w_1 S_1 + w_2 S_2\right)\right] \tag{6.5}$$

6.3.2 Boundary Conditions

If $S_1 = S_2 = 0$, then $V(\tau, 0, 0) = 0$.

If S_1 (or S_2) $= 0$, the 2-D equation reduces to a standard 1-D Black-Scholes equation. Hence the boundary conditions are $V(\tau, 0, S_2) = P_{1-D}(\tau, S_2)$ and $V(\tau, S_1, 0) = P_{1-D}(\tau, S_1)$.

Now we put the artificial upper boundaries "far enough" at 50 (normally 2 or 3 times of the strike price is considered "far enough"). A commonly used Dirichlet boundary condition is to set option price to zero at far field (Duffy, 2006). So we have $V(\tau, 50, S_2) = 0$ and $V(\tau, S_1, 50) = 0$.

Now we have boundary conditions on all four boundary edges and four boundary vertices. That is $3^2 - 1 = 8$ boundary conditions.

Note that sometime a more economical solution is to apply the pricing equation itself as the B.C., but with the second derivative term set to zero (Tavella and Randall, 2000). For example, the single-asset Black-Scholes PDE **??** has this "linear" B.C.

$$\frac{\partial V}{\partial \tau} + (r - \delta)\frac{\partial V}{\partial S} - rV = 0 \tag{6.6}$$

where S is understood to be the boundary $S_{\max}$. A discretization of the equation 6.6 will give us an approximation on the boundary.

In any case, the sensitivity of the boundary conditions should be small due to the strong diffusion.

6.3.3 Finite-Differencing Examples

6.3.3.1 2-D Crank-Nicholson Method

Let n, i and j be indices for τ, S_1 and S_2 respectively.

$$\frac{\partial P}{\partial \tau} = \frac{P_{i,j}^{n+1} - P_{i,j}^{n}}{\Delta \tau}$$

$$\frac{\partial^2 P}{\partial S_1^2} = \frac{1}{2}\frac{1}{\Delta S_1^2}\left[\left(P_{i+1,j}^{n+1} - 2P_{i,j}^{n+1} + P_{i-1,j}^{n+1}\right) + \left(P_{i+1,j}^{n} - 2P_{i,j}^{n} + P_{i-1,j}^{n}\right)\right]$$

$$\frac{\partial^2 P}{\partial S_2^2} = \frac{1}{2}\frac{1}{\Delta S_2^2}\left[\left(P_{i,j+1}^{n+1} - 2P_{i,j}^{n+1} + P_{i,j-1}^{n+1}\right) + \left(P_{i,j+1}^{n} - 2P_{i,j}^{n} + P_{i,j-1}^{n}\right)\right]$$

$$\frac{\partial P}{\partial S_1} = \frac{1}{4\Delta S_1}\left[\left(P_{i+1,j}^{n+1} - P_{i-1,j}^{n+1}\right) + \left(P_{i+1,j}^{n} - P_{i-1,j}^{n}\right)\right]$$

$$\frac{\partial P}{\partial S_2} = \frac{1}{4\Delta S_2}\left[\left(P_{i,j+1}^{n+1} - P_{i,j-1}^{n+1}\right) + \left(P_{i,j+1}^{n} - P_{i,j-1}^{n}\right)\right]$$

$$rP = \frac{r}{2}\left(P_{i,j}^{n+1} + P_{i,j}^{n}\right)$$

$$\begin{aligned}\frac{\partial^2 P}{\partial S_1 \partial S_2} = {} & \frac{1}{8\Delta S_1 \Delta S_2} \\ & [(P_{i+1,j+1}^{n+1} - P_{i+1,j-1}^{n+1}) + (P_{i-1,j+1}^{n+1} - P_{i-1,j-1}^{n+1}) \\ & +(P_{i+1,j+1}^{n} - P_{i+1,j-1}^{n}) + (P_{i-1,j+1}^{n} - P_{i-1,j-1}^{n})]\end{aligned}$$

Now (6.4) becomes

$$
\begin{aligned}
& \frac{P_{i,j}^{n+1} - P_{i,j}^{n}}{\Delta\tau} \\
- \quad & \frac{1}{2}\sigma_1^2 S_1^2 \frac{1}{2}\frac{1}{\Delta S_1^2}\left[\left(P_{i+1,j}^{n+1} - 2P_{i,j}^{n+1} + P_{i-1,j}^{n+1}\right) + \left(P_{i+1,j}^{n} - 2P_{i,j}^{n} + P_{i-1,j}^{n}\right)\right] \\
- \quad & \frac{1}{2}\sigma_2^2 S_2^2 \frac{1}{2}\frac{1}{\Delta S_2^2}\left[\left(P_{i,j+1}^{n+1} - 2P_{i,j}^{n+1} + P_{i,j-1}^{n+1}\right) + \left(P_{i,j+1}^{n} - 2P_{i,j}^{n} + P_{i,j-1}^{n}\right)\right] \\
- \quad & \sigma_1\sigma_2 S_1 S_2 \frac{1}{8\Delta S_1 \Delta S_2}\left[\left(P_{i+1,j+1}^{n+1} - P_{i+1,j-1}^{n+1}\right) + \left(P_{i-1,j+1}^{n+1} - P_{i-1,j-1}^{n+1}\right)\right] \\
- \quad & \sigma_1\sigma_2 S_1 S_2 \frac{1}{8\Delta S_1 \Delta S_2}\left[\left(P_{i+1,j+1}^{n} - P_{i+1,j-1}^{n}\right) + \left(P_{i-1,j+1}^{n} - P_{i-1,j-1}^{n}\right)\right] \\
- \quad & \left(r - D_1\right) S_1 \frac{1}{4\Delta S_1}\left[\left(P_{i+1,j}^{n+1} - P_{i-1,j}^{n+1}\right) + \left(P_{i+1,j}^{n} - P_{i-1,j}^{n}\right)\right] \\
- \quad & \left(r - D_2\right) S_2 \frac{1}{4\Delta S_2}\left[\left(P_{i,j+1}^{n+1} - P_{i,j-1}^{n+1}\right) + \left(P_{i,j+1}^{n} - P_{i,j-1}^{n}\right)\right] \\
+ \quad & r P_{i,j}^{n+1} \\
= \quad & 0
\end{aligned}
$$

Let

$$
\begin{aligned}
\mathrm{coeff}\,[0] = \quad & \frac{1}{8}\sigma_1\sigma_2 i j \Delta\tau \\
\mathrm{coeff}\,[1] = \quad & \left(-\frac{1}{4}\sigma_2^2 j^2 + \frac{1}{4}\left(r - D_2\right) j\right)\Delta\tau \\
\mathrm{coeff}\,[2] = \quad & \mathrm{coeff}\,[0] \\
\mathrm{coeff}\,[3] = \quad & \left(-\frac{1}{4}\sigma_1^2 i^2 + \frac{1}{4}\left(r - D_1\right) i\right)\Delta\tau \\
\mathrm{coeff}\,[4] = \quad & 1 + \left(0.5r + \sigma_1^2 i^2 + \sigma_2^2 j^2\right)\Delta\tau \\
\mathrm{coeff}\,[5] = \quad & \left(-\frac{1}{4}\sigma_1^2 i^2 - \frac{1}{4}\left(r - D_1\right) i\right)\Delta\tau \\
\mathrm{coeff}\,[6] = \quad & -\mathrm{coeff}\,[0] \\
\mathrm{coeff}\,[7] = \quad & \left(-\frac{1}{4}\sigma_2^2 j^2 - \frac{1}{4}\left(r - D_2\right) j\right)\Delta\tau \\
\mathrm{coeff}\,[8] = \quad & -\mathrm{coeff}\,[0]
\end{aligned}
\tag{6.7}
$$

Then we have

$$
\begin{array}{cc}
P^{n+1}_{i-1,j-1} & \cdot \ \mathrm{coeff}\,[0] \\
P^{n+1}_{i,j-1} & \cdot \ \mathrm{coeff}\,[1] \\
P^{n+1}_{i+1,j-1} & \cdot \ \mathrm{coeff}\,[2] \\
P^{n+1}_{i-1,j} & \cdot \ \mathrm{coeff}\,[3] \\
P^{n+1}_{i,j} & \cdot \ \mathrm{coeff}\,[4] \\
P^{n+1}_{i+1,j} & \cdot \ \mathrm{coeff}\,[5] \\
P^{n+1}_{i-1,j+1} & \cdot \ \mathrm{coeff}\,[6] \\
P^{n+1}_{i,j+1} & \cdot \ \mathrm{coeff}\,[7] \\
P^{n+1}_{i+1,j+1} & \cdot \ \mathrm{coeff}\,[8] \\
 & \\
= & \\
 & \\
P^{n}_{i-1,j-1} & \cdot \ (-\mathrm{coeff}\,[0]) \\
P^{n}_{i,j-1} & \cdot \ (-\mathrm{coeff}\,[1]) \\
P^{n}_{i+1,j-1} & \cdot \ (-\mathrm{coeff}\,[2]) \\
P^{n}_{i-1,j} & \cdot \ (-\mathrm{coeff}\,[3]) \\
P^{n}_{i,j} & \cdot \ (-\mathrm{coeff}\,[4]+2) \\
P^{n}_{i+1,j} & \cdot \ (-\mathrm{coeff}\,[5]) \\
P^{n}_{i-1,j+1} & \cdot \ (-\mathrm{coeff}\,[6]) \\
P^{n}_{i,j+1} & \cdot \ (-\mathrm{coeff}\,[7]) \\
P^{n}_{i+1,j+1} & \cdot \ (-\mathrm{coeff}\,[8])
\end{array}
$$

For convection-dominated problems some papers suggest that they have difficulty with CrankNicholson (time-averaging), in particular they experience spurious oscillations and spikes in the solution and the Greeks as well as near barriers Tavella and Randall (2000). A remedy is to use the exponentially fitted schemes in each underlying direction (see

(Duffy, 2004b) for details).

6.3.3.2 2-D Implicit Method

Let n, i and j be indices for τ, S_1 and S_2 respectively.

$$\frac{\partial P}{\partial \tau} = \frac{P_{i,j}^{n+1} - P_{i,j}^{n}}{\Delta \tau}$$
$$\frac{\partial^2 P}{\partial S_1^2} = \frac{1}{\Delta S_1^2}\left(P_{i+1,j}^{n+1} - 2P_{i,j}^{n+1} + P_{i-1,j}^{n+1}\right)$$
$$\frac{\partial^2 P}{\partial S_2^2} = \frac{1}{\Delta S_2^2}\left(P_{i,j+1}^{n+1} - 2P_{i,j}^{n+1} + P_{i,j-1}^{n+1}\right)$$
$$\frac{\partial P}{\partial S_1} = \frac{1}{2\Delta S_1}\left(P_{i+1,j}^{n+1} - P_{i-1,j}^{n+1}\right)$$
$$\frac{\partial P}{\partial S_2} = \frac{1}{2\Delta S_2}\left(P_{i,j+1}^{n+1} - P_{i,j-1}^{n+1}\right)$$
$$rP = rP_{i,j}^{n+1}$$
$$\frac{\partial^2 P}{\partial S_1 \partial S_2} = \frac{1}{4\Delta S_1 \Delta S_2}\left[\left(P_{i+1,j+1}^{n+1} - P_{i+1,j-1}^{n+1}\right) + \left(P_{i-1,j+1}^{n+1} - P_{i-1,j-1}^{n+1}\right)\right]$$

Now (6.4) becomes

$$\begin{aligned}
& \frac{P_{i,j}^{n+1} - P_{i,j}^{n}}{\Delta \tau} \\
- \quad & \frac{1}{2}\sigma_1^2 S_1^2 \frac{P_{i+1,j}^{n+1} - 2P_{i,j}^{n+1} + P_{i-1,j}^{n+1}}{\Delta S_1^2} \\
- \quad & \frac{1}{2}\sigma_2^2 S_2^2 \frac{P_{i,j+1}^{n+1} - 2P_{i,j}^{n+1} + P_{i,j-1}^{n+1}}{\Delta S_2^2} \\
- \quad & \sigma_1 \sigma_2 S_1 S_2 \frac{1}{4\Delta S_1 \Delta S_2}\left[\left(P_{i+1,j+1}^{n+1} - P_{i+1,j-1}^{n+1}\right) + \left(P_{i-1,j+1}^{n+1} - P_{i-1,j-1}^{n+1}\right)\right] \\
- \quad & \left(r - D_1\right) S_1 \frac{1}{2\Delta S_1}\left(P_{i+1,j}^{n+1} - P_{i-1,j}^{n+1}\right) \\
- \quad & \left(r - D_2\right) S_2 \frac{1}{2\Delta S_2}\left(P_{i,j+1}^{n+1} - P_{i,j-1}^{n+1}\right) \\
+ \quad & rP_{i,j}^{n+1} \\
= \quad & 0
\end{aligned}$$

Let

$$
\begin{aligned}
\text{coeff}\,[0] = &\quad \frac{1}{4}\sigma_1\sigma_2 ij\Delta\tau \\
\text{coeff}\,[1] = &\quad \left(-\frac{1}{2}\sigma_2^2 j^2 + \frac{1}{2}\left(r - D_2\right) j\right)\Delta\tau \\
\text{coeff}\,[2] = &\quad \text{coeff}\,[0] \\
\text{coeff}\,[3] = &\quad \left(-\frac{1}{2}\sigma_1^2 i^2 + \frac{1}{2}\left(r - D_1\right) i\right)\Delta\tau \\
\text{coeff}\,[4] = &\quad 1 + \left(r + \sigma_1^2 i^2 + \sigma_2^2 j^2\right)\Delta\tau \\
\text{coeff}\,[5] = &\quad \left(-\frac{1}{2}\sigma_1^2 i^2 - \frac{1}{2}\left(r - D_1\right) i\right)\Delta\tau \\
\text{coeff}\,[6] = &\quad -\text{coeff}\,[0] \\
\text{coeff}\,[7] = &\quad \left(-\frac{1}{2}\sigma_2^2 j^2 - \frac{1}{2}\left(r - D_2\right) j\right)\Delta\tau \\
\text{coeff}\,[8] = &\quad -\text{coeff}\,[0]
\end{aligned}
\tag{6.8}
$$

Then we have

$$
\begin{array}{cc}
P_{i-1,j-1}^{n+1} & \cdot\ \text{coeff}\,[0] \\
P_{i,j-1}^{n+1} & \cdot\ \text{coeff}\,[1] \\
P_{i+1,j-1}^{n+1} & \cdot\ \text{coeff}\,[2] \\
P_{i-1,j}^{n+1} & \cdot\ \text{coeff}\,[3] \\
P_{i,j}^{n+1} & \cdot\ \text{coeff}\,[4] \\
P_{i+1,j}^{n+1} & \cdot\ \text{coeff}\,[5] \\
P_{i-1,j+1}^{n+1} & \cdot\ \text{coeff}\,[6] \\
P_{i,j+1}^{n+1} & \cdot\ \text{coeff}\,[7] \\
P_{i+1,j+1}^{n+1} & \cdot\ \text{coeff}\,[8] \\
= & P_{i,j}^{n}
\end{array}
$$

6.4 3-D Case

Now consider the 3-D case. Again, the Black-Scholes equation is

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i S_j} - \sum_{i=1}^{3} r S_i \frac{\partial V}{\partial S_i} + rV = 0 \tag{6.9}$$

And again all coefficients are constant and dividend is zero.

6.4.1 Initial Condition

The payoff function is (when $\tau = 0$)

$$V\left(0, S_1, S_2, S_3\right) = \max\left[0, K - \left(w_1 S_1 + w_2 S_2 + w_3 S_3\right)\right] \tag{6.10}$$

6.4.2 Boundary Conditions

The boundary conditions for the 2-D case can be extended into 3-D.
If $S_i = 0$, $i = 1, 2, 3$, the 3-D equation reduces to 2-D equation. If any two of the three assets are zero, then it becomes a 1-D equation.
Now consider the value in the far field. We can set the value to zero at a far enough" point. Or we can apply the "linear" boundary conditions, eliminate all second order derivative terms from the equation on the boundaries and discretize the drift terms using one-sided difference operators.

Again, we will have equations on all six boundary surfaces, all twelve boundary edges and eight boundary vertices. So we have a total of $3^3 - 1$ B.C. equations. Detailed study on 3-D case will be a big part of future research. Given *FronTier* package's built-in parallel computing capability, 3-D case is very much within our reach.

6.5 Numerical Experiments

In this section we will show some numerical results of 2-asset European put basket option. We set parameters as $S_{1,2} = [0,\ 400]$, mesh size $N = 200$ for both S_1 and S_2, $t = [0,\ 4]$, strike price $K = 150$, volatility $\sigma_1 = \sigma_2 = 0.4$, interest rate $r = 0.1$ (dividend $D = 0$), and correlation $\rho = 0.5$. In figure 6.1 we show the initial state of our model at $t = 0$. Figure 6.2 is the option price at expiry ($t = 4$)

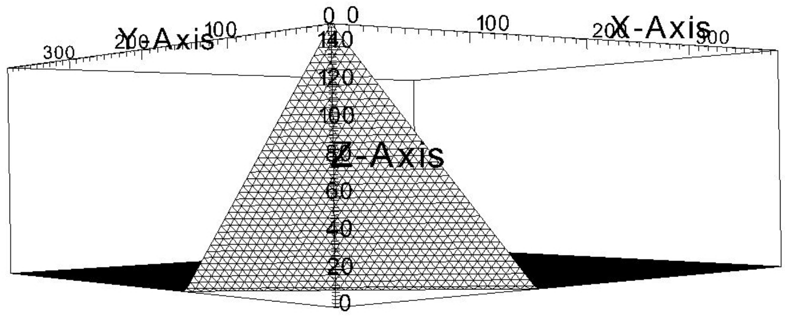

Figure 6.1: 2-asset European Put basket option t=0

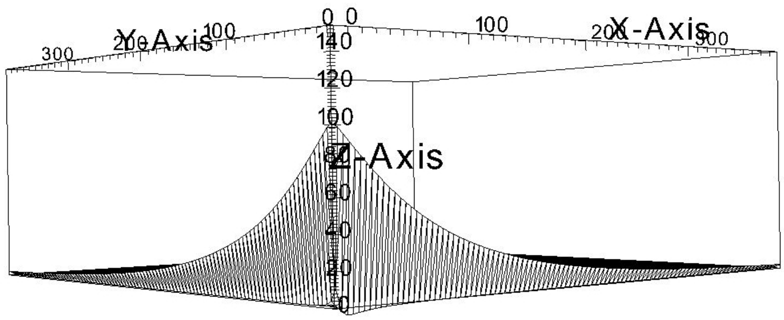

Figure 6.2: 2-asset European Put basket option t=4

7

Stochastic Volatility

7.1 Introduction

Some of the important assumptions of the Black-Scholes model (see Section 2.1) are that the underlying asset's price process is continuous and that the volatility is constant. The second assumption leads to the conclusion that if we plot volatility against the strike price we would obtain a straight line, parallel to the horizontal axis. Setting up the Black-Scholes model with the market observed option price and solving for volatility gives us the implied volatility. However, when plotting implied volatility using real market data one typically obtains a convex curve, known as the "smile curve" or the "volatility smile", with minimum price "at the money" i.e. where the strike price is equal to the underlying spot.

In order to have a more realistic approach to the problem of option pricing, jump models and stochastic volatility models have been introduced. Jump models deal with the assumption of continuity by allowing the spot asset's process to jump. When studying stochastic volatility models the volatility is described by a stochastic process. These models are used in order to price options where volatility varies over time. If we denote the underlying stock price by S, and W_1 and W_2 two wiener processes with correlation ρ, then S satisfies the stochastic differential equation (Hull and White, 1987)

$$dS = rSdt + \sigma\left(t\right)dW_1$$

where the volatility process $\sigma(t)$ satisfies some stochastic differential equation of the form

$$d(\sigma) = b(\sigma)\,dt + a(\sigma)\,dW_1$$

7.2 Heston Model

7.2.1 Heston Model

Let's assume that asset S follows

$$dS = \mu S dt + \sqrt{\upsilon(t)} S_t dW_1$$

where W_1 is a Wiener process. The volatility $\upsilon(t)$ follows an Ornstein-Unlenbeck process

$$d\sqrt{\upsilon(t)} = -\beta\sqrt{\upsilon(t)}dt + \delta dW_2$$

where W_2 is another Wiener process. And W_1 and W_2 are correlated with correlation ρ. Let $x = \sqrt{\upsilon(t)}$ and apply Ito's formula in $f(x) = x^2$. The result is

$$d\upsilon(t) = [\delta^2 - 2\beta\upsilon(t)]dt + 2\delta\sqrt{\upsilon(t)}dW_2$$

If we let $k = 2\beta$, $\theta = \dfrac{\sigma^2}{2\beta}$, and $\sigma = 2\delta$ we will have the Heston model (Heston, 1993).

$$dS = \mu S dt + \sqrt{\upsilon(t)} S dW_1 \tag{7.1}$$

$$d\upsilon(t) = k[\theta - \upsilon(t)]\,dt - \sigma\sqrt{\upsilon(t)}dW_2 \tag{7.2}$$

$$dW_1 dW_2 = \rho dt \tag{7.3}$$

The relationship between the parameter θ and the volatility $\upsilon(t)$ determines the instantaneous drift of ν.The parameter θ is the long-term variance. The parameter k shows how fast the process reverts to θ. A high k implies higher rate of reversion and vice versa. The parameter σ in 7.2 is the volatility of the volatility. Finally, the correlation between the two Wiener processes is denoted by ρ, where $\rho \in [-1, 1]$.

7.2.2 Derivation of PDE Model

Looking at the model and comparing this model with the vanilla Black-Scholes model, we will see the number of the random sources (two Wiener processes) with the number of the risky traded assets (only the underlying stock since volatility is not traded). We can easily see that the Heston model is an incomplete model. Therefore, it is not possible to obtain a unique price using only the underlying asset and a bank account, which is normally the case for complete models such as the Black-Scholes model. In order for the portfolio to be hedged we need to have equal number of random sources with risky traded assets. Now considering the portfolio as P and the relative weights of the bank account, the stock and the benchmark derivative as x, y, and z, respectively, we get

$$P = xB + yS + zC \tag{7.4}$$

where $C = C(S, v, t)$ is the derivative (for example a European Put), and B is a risk-free asset (for example government bond) which satisfying

$$dB = rBdt \tag{7.5}$$

We also assume that P is self-financing, which means

$$dP = xdB + ydS + zdC \tag{7.6}$$

Now we try to hedge a contingent claim (denoted by $U = U(S, v, t)$) using P.

$$P = U \text{ and } dP = dU$$

and applying Ito's formula, we get

$$\begin{aligned} dU = \quad & (U_t + \mu S U_s + k(\theta - v(t))U_v + \frac{1}{2}S^2 v(t) U_{ss} + \frac{1}{2}\sigma^2 v(t) U_{vv} + \sigma v(t) S \rho U_{sv})dt \\ & + \sqrt{v(t)} S U_s dW_1 + \sqrt{v(t)} \sigma U_v dW_2 \end{aligned}$$

where

$$U_t = \frac{\partial U}{\partial t}, U_s = \frac{\partial U}{\partial s}, U_v = \frac{\partial U}{\partial v}$$

$$U_s s = \frac{\partial^2 U}{\partial S^2}, U_{vv} = \frac{\partial^2 U}{\partial v^2(t)}, U_{sv} = \frac{\partial^2 U}{\partial S \partial v(t)}$$

Assuming C is at least twice differentiable, we will get

$$\begin{aligned} dC = \quad & (C_t + \mu S C_s + k(\theta - \upsilon(t))C_v + \frac{1}{2}S^2\upsilon(t)C_{ss} + \frac{1}{2}\sigma^2\upsilon(t)C_{vv} + \sigma\upsilon(t)S\rho C_{sv})dt \\ & + \sqrt{\upsilon(t)}SC_s dW_1 + \sqrt{\upsilon(t)}\sigma C_v dW_2 \end{aligned} \tag{7.7}$$

Substituting 7.1, 7.5 and 7.7 in 7.6 we will have

$$\begin{aligned} dP = \quad & z(C_t + \mu S C_s + k(\theta - \upsilon(t))C_v + \frac{1}{2}S^2\upsilon(t)C_{ss} + \frac{1}{2}\sigma^2\upsilon(t)C_{vv} + \sigma\upsilon(t)S\rho C_{sv})dt \\ & + (xrB + y\mu S)dt + (z\sqrt{\upsilon(t)}SC_s + y\sqrt{\upsilon(t)}S)dW_1 + z\sqrt{\upsilon(t)}\sigma C_v dW_2 \end{aligned} \tag{7.8}$$

Now comparing dW_1 and dW_2 terms, we will find that

$$z = \frac{U_v}{C_v} \text{ and } y = U_s - zC_s$$

In order to compare the drift terms we combine 7.4 and $P = U$

$$xB = U - yS - zC$$

Thus we have

$$\begin{aligned} & z(C_t + \mu S C_s + k(\theta - \upsilon(t))C_v + \frac{1}{2}S^2\upsilon(t)C_{ss} \\ & + \frac{1}{2}\sigma^2\upsilon(t)C_{vv} + \sigma\upsilon(t)S\rho C_{sv}) + y\mu S + r(U - yS - zC) \\ & = U_t + \mu S U_s + k(\theta - \upsilon(t))U_v + \frac{1}{2}S^2\upsilon(t)U_{ss} \\ & + \frac{1}{2}\sigma^2\upsilon(t)U_{vv} + \sigma\upsilon(t)S\rho U_{sv} \end{aligned}$$

After performing some manipulations and substitutions, we get

$$\begin{aligned} & \frac{1}{U_v}(U_t + \mu S U_s + k(\theta - \upsilon(t))U_v + \frac{1}{2}S^2\upsilon(t)U_{ss} \\ & + \frac{1}{2}\sigma^2\upsilon(t)U_{vv} + \sigma\upsilon(t)S\rho U_{sv} - (\mu - r)SU_s - rU) \\ = \quad & \frac{1}{C_v}(C_t + \mu S C_s + k(\theta - \upsilon(t))C_v + \frac{1}{2}S^2\upsilon(t)C_{ss} \\ & + \frac{1}{2}\sigma^2\upsilon(t)C_{vv} + \sigma\upsilon(t)S\rho C_{sv} - (\mu - r)SC_s - rC) \\ = \quad & \lambda(S, \upsilon, t) \end{aligned}$$

where $\lambda(S, \upsilon, t)$ is the market price of volatility. After some more manipulations, we obtain the Heston partial differential equation

$$\begin{aligned}
&\frac{\partial U}{\partial t} + rS\frac{\partial U}{\partial S} + \left(k\left(\theta - \upsilon\left(t\right)\right) - \lambda\left(S, \upsilon, t\right)\right)\frac{\partial U}{\partial \upsilon} \\
&+\frac{1}{2}S^2\nu\left(t\right)\frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma^2\upsilon\left(t\right)\frac{\partial^2 U}{\partial \upsilon^2} + \sigma\upsilon\left(t\right)S\rho\frac{\partial^2 U}{\partial S\partial \upsilon} - rU = 0 \qquad (7.9)
\end{aligned}$$

7.2.3 Boundary Conditions

We now discuss the boundary conditions for 7.9. Similarly to Black-Scholes model, we will set an artificial upper boundary for stock price S and volatility υ far enough so that they won't affect our numerical results too much. So we will have four boundary conditions to form our domain.

As first mentioned in (Heston, 1993) For European call options when $S = 0$ we consider the call to be worthless; when S becomes very large we use a Neumann boundary condition which more or less is the same as a linearity boundary condition. When the volatility is 0 we assume that the PDE 7.9 is satisfied on the line $\upsilon = 0$; in this case some of the terms in 7.9 fall away. Finally, when υ is very large we assume that the option behaves as a standard European option.

$$U\left(S, \upsilon, T\right) = \max\left(0, S - K\right) \text{ at t=T}$$

$$U\left(0, \upsilon, t\right) = 0 \text{ at S=0}$$

Neumann boundary conditions give the value of the derivative of the function at the boundaries. Such conditions are used at the points where the stock and the volatility take their largest values.

For large values of S the option price grows linearly. The boundary condition we use at the point S $= S_{max}$ is

$$\frac{\partial U\left(S_{max}, \upsilon, t\right)}{\partial S} = 1$$

The option price is typically increasing with volatility. It is however bounded by the stock price. When the volatility obtains its highest value the option price tends to become constant. The boundary condition we use at this point is

$$\frac{\partial U\left(S,\upsilon_{max},t\right)}{\partial \upsilon}=0$$

In our computation, we will use an alternative condition $U(S,\, \upsilon_{max},\, t) = S)$.

And at $\upsilon = 0$, we have

$$\frac{\partial U}{\partial t}+rS\frac{\partial U}{\partial S}-rU+K\theta\frac{\partial U}{\partial \upsilon}=0$$

All these conditions can be approximated numerically, hence they will be used in our model.

Now for European put, the conditions are (Ikonen and Toivanen, 2004) :

$$U(0,\, \upsilon,\, t) = K$$

$$\frac{\partial U\left(S_{max},\upsilon,t\right)}{\partial S}=0$$

$$U(S,\, 0,\, t) = \max\left(K-S,\, 0\right)$$

$$\frac{\partial U\left(S,\upsilon_{max},t\right)}{\partial \upsilon}=0$$

Again, we need to set far-field conditions to make one-sided domain into a two-sided domain. For more alternative boundary conditions, please see (Duffy, 2004a) for details.

7.2.4 A brief Explicit Finite-Differencing Example of Heston PDE Model

The details of finite difference approximation of derivatives are covered in previous chapters. Here we will give the result directly. For example, we use Central difference to approximate the mixed term $\dfrac{\partial U^2}{\partial \nu \partial S}$.

$$
\begin{aligned}
&U_{i,j}^{n-1} = \Delta t[(\frac{1}{\Delta t} - \frac{\nu S^2}{\Delta S^2} - \frac{\nu \sigma^2}{\Delta \nu^2} - r)U_{i,j}^n + (\frac{\nu S^2}{2\Delta S^2} - \frac{rS}{2\Delta S})U_{i-1,j}^n \\
&+(\frac{\nu S^2}{2\Delta S^2} + \frac{rS}{2\Delta S})U_{i+1,j}^n] \\
&+\Delta t[(\frac{\nu \sigma^2}{2\Delta \nu^2} - \frac{k(\theta - \nu) - \lambda}{2\Delta \nu})U_{i,j-1}^n \\
&+(\frac{\nu \sigma^2}{2\Delta \nu^2} + \frac{k(\theta - \nu) - \lambda}{2\Delta \nu})U_{i,j+1}^n] \\
&+\Delta t[\frac{\rho \sigma \nu S}{4\Delta S \Delta \nu}(U_{i+1,j+1}^n - U_{i-1,j+1}^n - U_{i+1,j-1}^n + U_{i-1,j-1}^n)]
\end{aligned}
$$

Where n is index for time t, i for stock S and j for volatility ν. (Note we start at $n = T$ and going backwards in time.)

7.2.5 Detailed Implicit Finite-Differencing of Heston PDE Model

Recall the Heston PDE 7.9,

$$
\begin{aligned}
&\frac{\partial U}{\partial \tau} - rS\frac{\partial U}{\partial S} - \left(k\left(\theta - \upsilon\left(\tau\right)\right) - \lambda\left(S, \upsilon, \tau\right)\right)\frac{\partial U}{\partial \upsilon} \\
&-\frac{1}{2}S^2\nu\left(\tau\right)\frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma^2\upsilon\left(\tau\right)\frac{\partial^2 U}{\partial \upsilon^2} - \sigma\upsilon\left(\tau\right)S\rho\frac{\partial^2 U}{\partial S \partial \upsilon} + rU = 0
\end{aligned}
$$

Let n, i and j be indices for τ, S and υ respectively. By standard implicit finite differencing method, we have

$$
\begin{aligned}
\frac{\partial U}{\partial \tau} &= \frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta \tau} \\
\frac{\partial^2 U}{\partial S^2} &= \frac{1}{\Delta S^2}\left(U_{i+1,j}^{n+1} - 2U_{i,j}^{n+1} + U_{i-1,j}^{n+1}\right) \\
\frac{\partial^2 U}{\partial \upsilon^2} &= \frac{1}{\Delta \upsilon^2}\left(U_{i,j+1}^{n+1} - 2U_{i,j}^{n+1} + U_{i,j-1}^{n+1}\right) \\
\frac{\partial U}{\partial S} &= \frac{1}{2\Delta S}\left(U_{i+1,j}^{n+1} - U_{i-1,j}^{n+1}\right) \\
\frac{\partial U}{\partial \upsilon} &= \frac{1}{2\Delta \upsilon}\left(U_{i,j+1}^{n+1} - U_{i,j-1}^{n+1}\right)
\end{aligned}
$$

$$rU = rU_{i,j}^{n+1}$$

$$\frac{\partial^2 U}{\partial S \partial \upsilon} = \frac{1}{4\Delta S \Delta \upsilon}\left[\left(U_{i+1,j+1}^{n+1} - U_{i+1,j-1}^{n+1}\right) + \left(U_{i-1,j+1}^{n+1} - U_{i-1,j-1}^{n+1}\right)\right]$$

Now (7.9) becomes

$$\begin{aligned}
& \frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta \tau} \\
- \quad & \frac{1}{2}\upsilon S^2 \frac{U_{i+1,j}^{n+1} - 2U_{i,j}^{n+1} + U_{i-1,j}^{n+1}}{\Delta S^2} \\
- \quad & \frac{1}{2}\sigma^2 \upsilon \frac{U_{i,j+1}^{n+1} - 2U_{i,j}^{n+1} + U_{i,j-1}^{n+1}}{\Delta \upsilon^2} \\
- \quad & \sigma \upsilon S \rho \frac{1}{4\Delta S \Delta \upsilon}\left[\left(U_{i+1,j+1}^{n+1} - U_{i+1,j-1}^{n+1}\right) + \left(U_{i-1,j+1}^{n+1} - U_{i-1,j-1}^{n+1}\right)\right] \\
- \quad & (r - D)\, S \frac{1}{2\Delta S}\left(U_{i+1,j}^{n+1} - U_{i-1,j}^{n+1}\right) \\
- \quad & (k - (\theta - \upsilon) - \lambda) \frac{1}{2\Delta \upsilon}\left(U_{i,j+1}^{n+1} - U_{i,j-1}^{n+1}\right) \\
+ \quad & rU_{i,j}^{n+1} \\
= \quad & 0
\end{aligned} \tag{7.10}$$

Define coefficients as,

$$\begin{aligned}
\text{coeff}\,[0] = \quad & \frac{1}{4}\sigma \rho i j \Delta \tau \\
\text{coeff}\,[1] = \quad & \left(-\frac{1}{2\Delta \upsilon}\sigma^2 j + \frac{1}{2\Delta \upsilon}\left(k(\theta - \upsilon) - \lambda\right)\right)\Delta \tau \\
\text{coeff}\,[2] = \quad & \text{coeff}\,[0] \\
\text{coeff}\,[3] = \quad & \left(-\frac{1}{2}\upsilon i^2 + \frac{1}{2}\left(r - D\right) i\right)\Delta \tau \\
\text{coeff}\,[4] = \quad & 1 + \left(r + \upsilon i^2 + \frac{\sigma^2 j}{\Delta \upsilon}\right)\Delta \tau \\
\text{coeff}\,[5] = \quad & \left(-\frac{1}{2}\upsilon i^2 - \frac{1}{2}\left(r - D\right) i\right)\Delta \tau \\
\text{coeff}\,[6] = \quad & -\text{coeff}\,[0] \\
\text{coeff}\,[7] = \quad & \left(-\frac{1}{2\Delta \upsilon}\sigma^2 j - \frac{1}{2\Delta \upsilon}\left(k(\theta - \upsilon) - \lambda\right)\right)\Delta \tau \\
\text{coeff}\,[8] = \quad & -\text{coeff}\,[0]
\end{aligned} \tag{7.11}$$

Then we have

$$
\begin{array}{cl}
U_{i-1,j-1}^{n+1} & \cdot\ \mathrm{coeff}\,[0] \\
U_{i,j-1}^{n+1} & \cdot\ \mathrm{coeff}\,[1] \\
U_{i+1,j-1}^{n+1} & \cdot\ \mathrm{coeff}\,[2] \\
U_{i-1,j}^{n+1} & \cdot\ \mathrm{coeff}\,[3] \\
U_{i,j}^{n+1} & \cdot\ \mathrm{coeff}\,[4] \\
U_{i+1,j}^{n+1} & \cdot\ \mathrm{coeff}\,[5] \\
U_{i-1,j+1}^{n+1} & \cdot\ \mathrm{coeff}\,[6] \\
U_{i,j+1}^{n+1} & \cdot\ \mathrm{coeff}\,[7] \\
U_{i+1,j+1}^{n+1} & \cdot\ \mathrm{coeff}\,[8] \\
= & U_{i,j}^{n}
\end{array}
$$

In order to solve Heston PDE 7.9, we need to solve a linear system with nine unknowns. With corresponding boundary conditions, we can do this efficiently. Again, this is very similar to our approach to solve 2-asset Black-Scholes PDE 6.4.

7.3 Numerical Examples

7.3.1 European Call Option with Stochastic Volatility

Let's consider a simple example with initial volatility $\upsilon = 0.4$, $K = 100$, $\sigma = 0.25$, $\rho = 0$, $\theta = 0.4$, $\lambda = 0$, $S_{max} = 400$, $\nu_{max} = 1$, $T = 0.5$, $\Delta t = 0.001$, $k = 2$, and $r = 0$. And we discretize 30 steps in S direction and 80 steps in volatility direction. The results are shown in figures 7.1, 7.2 and 7.3.

7.3.2 European Put Option with Stochastic Volatility

Here is a more detailed example on European put option. We set parameters $\upsilon = [0, 3]$, $S = [0, 400]$, $K = 100$, $\sigma = 0.25$, $\rho = -0.1$, $\theta = 0.04$, $\lambda = 0.1$, $T = 4$, $k = 2$, and $r = 0.1$. And we discretize 100 steps in both S direction and volatility υ direction. In figure 7.4 we can see the initial condition for Heston Model. At expiry $T = 4$ (as shown in figure 7.5)

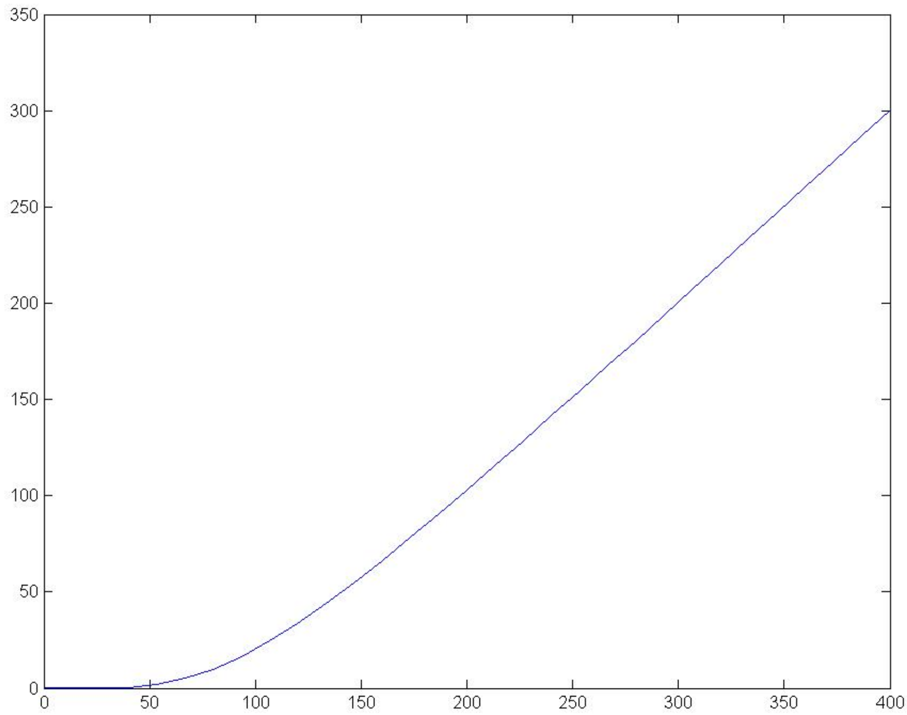

Figure 7.1: Stock S vs Option Price U at t=0

the difference in terms of option prices is obvious given different volatility values. And in figure 7.6 we can see a snapshot of option prices for a given volatility value $\upsilon = 1.5$.

7.3.3 Heston Model versus Black-Scholes Model

Now let's take a look at option prices determined by Black-Scholes model (constant volatility) and Heston's Model (stochastic volatility). We set parameters $S = [0, 400]$, $K = 100$, $T = 4$, $r = 0.1$ and mesh size$N = 100$ for both models. Additionally we set $\upsilon = [0, 3]$, $\sigma = 0.25$, $\rho = -0.1$, $\theta = 0.04$, $\lambda = 0.1$, $k = 2$ for Heston Model and constant volatility 0.4 for Black-Scholes Model. At expiry $T = 4$, we plot option price v.s. stock price from both Models (for Heston model, we take $\upsilon = 0.4$ at $T = 4$). As we expected, since the volatility value is generally larger in Heston Model than it is in Black-Scholes Model, the result (see figure 7.7) shows that put option price is smaller in Heston Model.

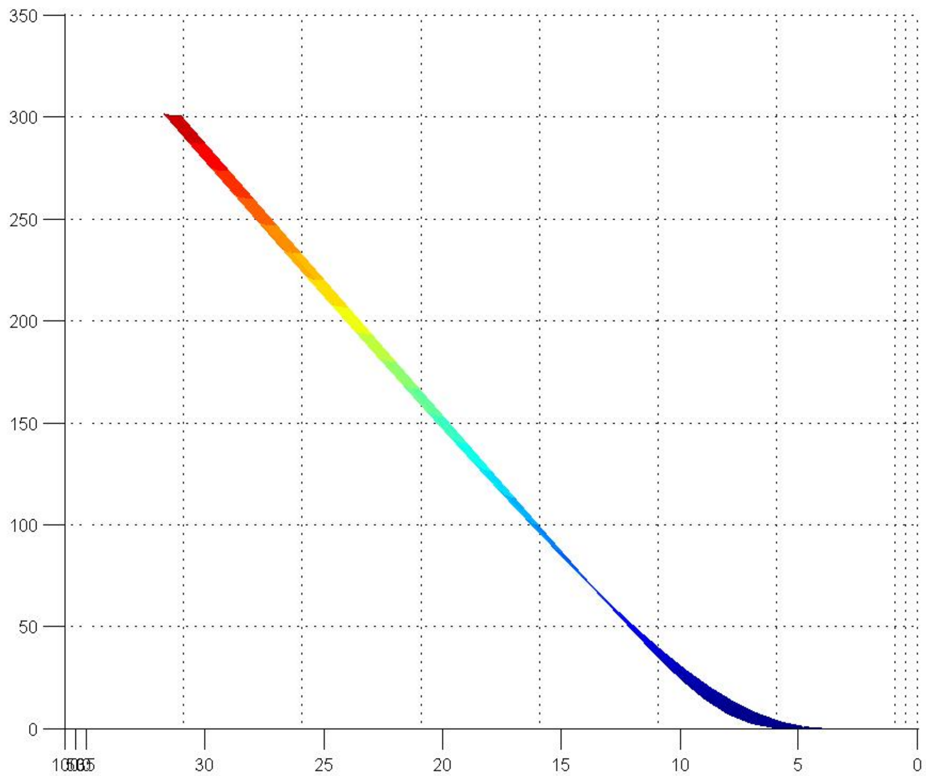

Figure 7.2: 3D Side View of Eu Call with Stoch Vol

7.4 Summary

The papers of Black-Scholes in 1973 (Black and Scholes, 1973) and (Merton, 1973) were (and still are) enormous contribution to the development of financial markets. However, the assumption of constant volatility has been a drawback to accurately model the market. Studying stochastic volatility models can be very useful since they capture the situation where volatility varies over time. In this chapter the Heston model, one of the most popular stochastic volatility models, was discussed. After solving the Heston PDE 7.9 using the Finite Difference Method 7.10 7.11 7.12, we obtained the arbitrage free price of

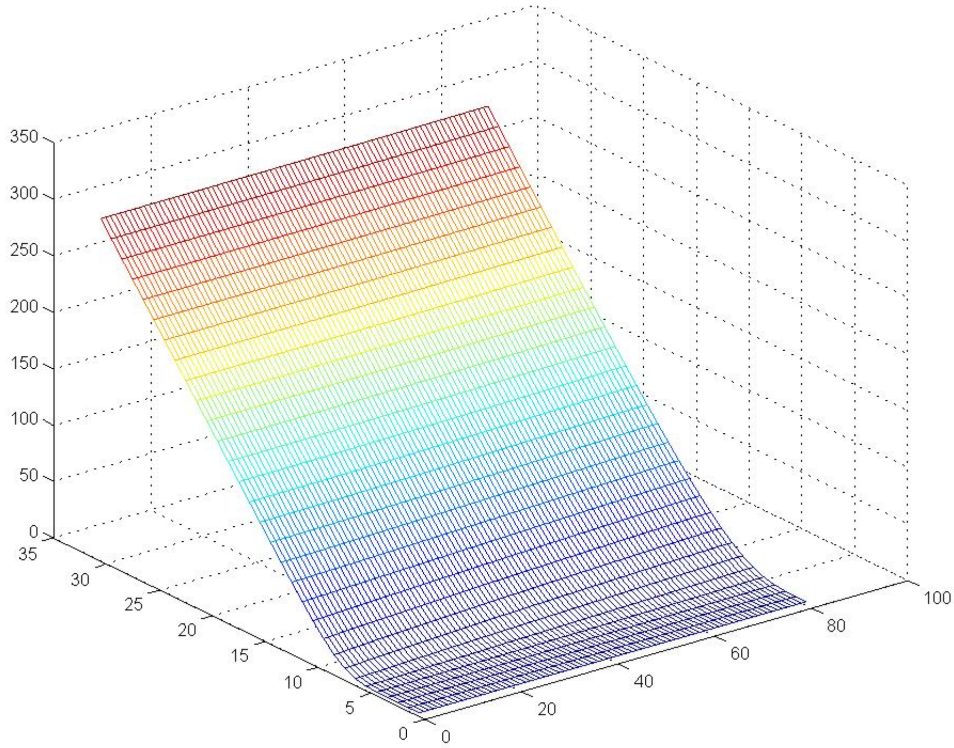

Figure 7.3: 3D plot of Eu Call with Stoch Vol

a European option at any time step.

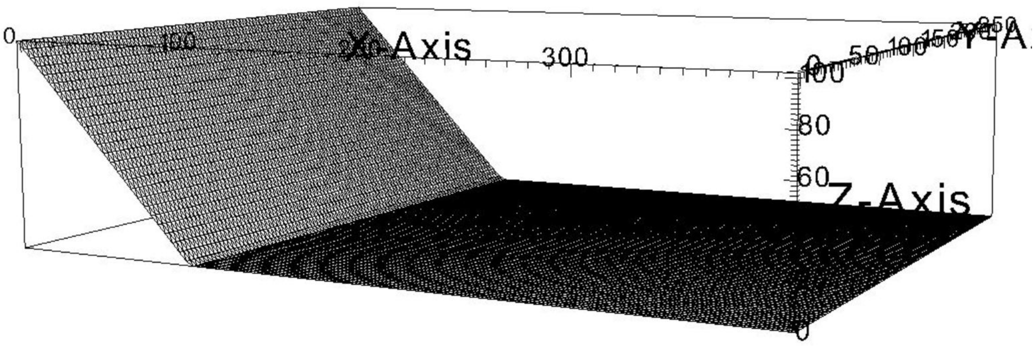

Figure 7.4: 3D View of Heston Model for European Put at time 0

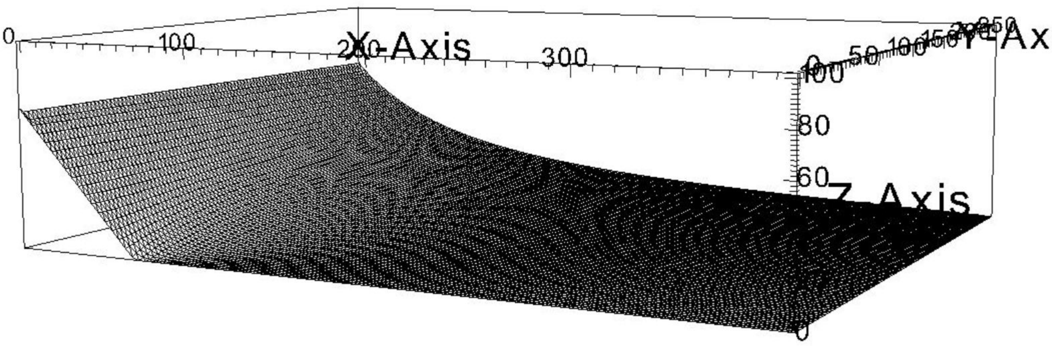

Figure 7.5: 3D View of Heston Model for European Put at expiry T=4

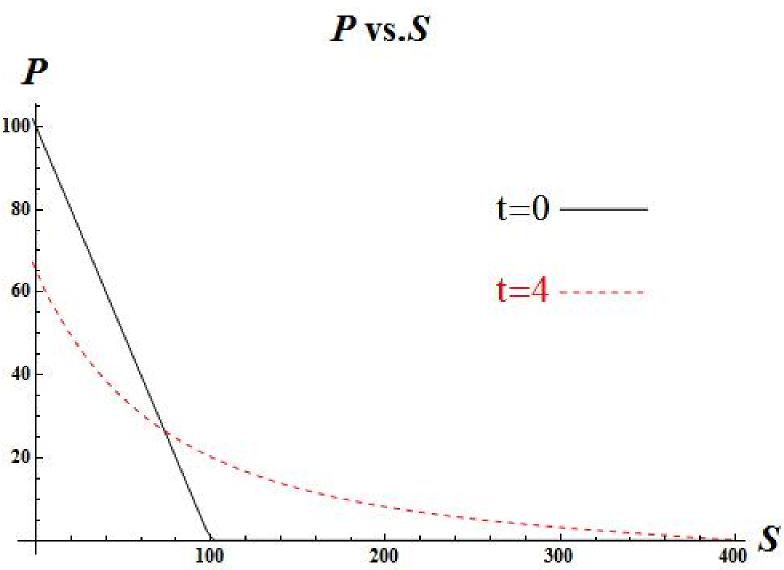

Figure 7.6: 2D Plot of Heston Model for European Put at t=0 and 4

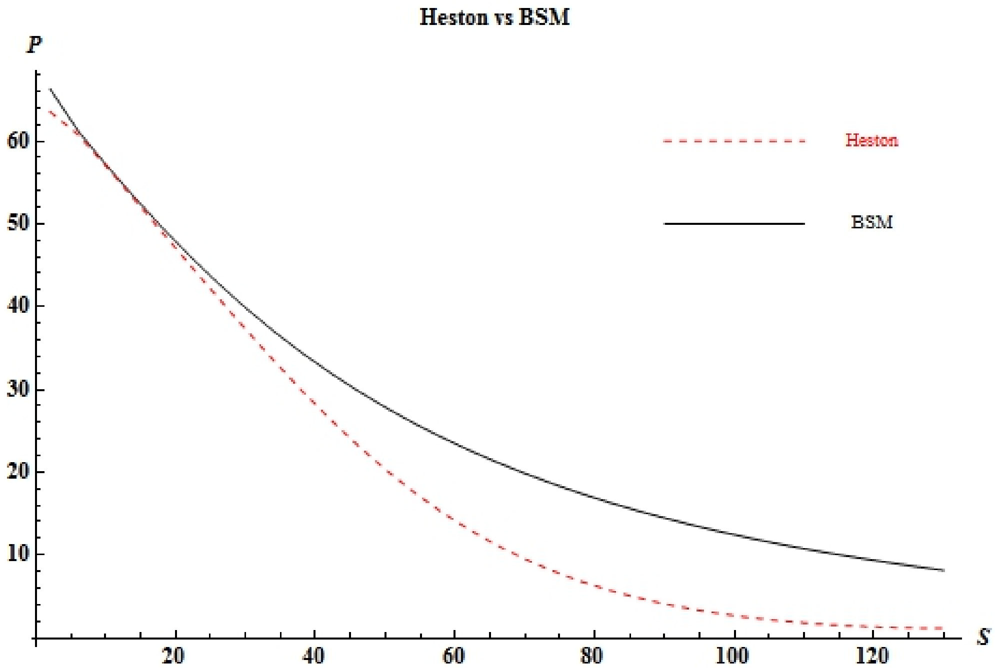

Figure 7.7: Heston Model vs Black-Scholes Model

8

Other Applications And Future Research

We shall extend this study into other areas. An immediate extension of our front-tracking method is pricing American options in Heston model. Since we only use "smooth pasting" property ($\frac{\partial U}{\partial S_f} = 1$ or -1) when deriving our front-tracking method, we can apply the same method to Heston model. At any time t and given any volatility $v(t)$, the same "smooth pasting" property holds. So at each time step t_n, we track the front (which is a line instead of a point) for all $v(t_n)$. Then after updating the front and applying proper boundary conditions, we can solve Heston PDE 7.9 for the next time step t_{n+1}.

Another application for our front-tracking method is to solve Lévy model. The same "smooth pasting" property is still true. Similarly to Black-Scholes-Merton model, we can track the front at each time step when we solve the partial integro-differential equation (PIDE). Along with proper boundary and initial conditions, such system can be solved numerical using *FronTier* package.

Lastly when we extend *FronTier* into 3-D cases, the build-in parallel computing capability in *FronTier* will be essential. Because of the advanced structure of *FronTier*, such extension can be done fairly easily. Possible application in real financial market for the *FronTier* package will also be explored.

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