Transversal String Topology & Invariants of Manifolds

A Dissertation Presented

by

Somnath Basu

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

August 2011

Stony Brook University
The Graduate School

Somnath Basu

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Dennis P. Sullivan - Advisor
Distinguished Professor, Department of Mathematics

John Morgan - Committee Chair
Director, Simons Center for Geometry & Physics

Oleg Viro
Professor, Department of Mathematics

Mahmoud Zeinalian
Associate Professor of Mathematics, Long Island University

This dissertation is accepted by the Graduate School.

Lawrence Martin
Dean of the Graduate School

Abstract of the Dissertation

Transversal String Topology & Invariants of Manifolds

by

Somnath Basu

Stony Brook University

2011

Loop spaces have played a recurring and important role in mathematics - from closed geodesics in differential geometry to its related variant of based loop space which plays a central role in homotopy theory. The subject of string topology focuses on topological aspects of the free loop space of manifolds; it's the study of the algebraic structures present therein and it originated in the seminal work of Chas and Sullivan.

From the point of view of computations, several techniques of algebraic topology may apply. We show, using rational homotopy theory and minimal models, that the Lie algebra structure on the (circle) equivariant homology of a product of odd spheres is highly non-trivial although the same structure for an odd sphere is trivial. Similar smaller (related) computational results are presented.

In the main result of this work, we define and study certain geometric loops, called transversal strings, which satisfy some specific boundary conditions. The relevant algebraic backdrop happens to be the category of bicomodules and algebra objects in this setting. Using the machinery of minimal models and homological algebra in this setting, we show that via transversal string topology it's possible to distinguish non-homeomorphic but homotopy equivalent Lens spaces.

To my family.

"I see it but I don't believe it!" - George Cantor

Contents

List of Figures

Acknowledgements

Little did I know what it will take to get through a PhD program when I joined Stony Brook. Looking back, it is my pleasure to thank everyone who made this thesis possible and facilitated my personal growth.

I would like to thank my family for their unwavering support for my academic pursuits. I would especially like to thank my parents and several of my uncles, aunts and cousins. Their constant support has often helped me sustain myself throughout. My mother and father separately deserves a very special mention - this wouldn't be possible without your belief in me! I wish to thank Hari *kaku* whose words of wisdom helped me ease through some really strenuous times.

At this point, when nostalgic associations come to the fore, I must thank the faculty at Indian Statistical Institute, Bangalore, who are responsible for my academic path that has led me here. I wish to thank Alladi Sitaram for providing excellent social support during my undergraduate years. Gadadhar Misra deserves a special mention for being supportive, caring and most importantly for sharing his philosophy of mathematics.

It has been enjoyable to be around friends at Stony Brook. In particular, I have fond memories of delectable mathematical discussions, often of no consequence, or discussions bordering on the philosophy of mathematics. I have Andrew Stimpson, Benjamin, Marcelo, Matt Young, Paul, Pedro and Vamsi to thank for that. The Mathematics Commons Room also deserves a big thanks - for being an omnipresent entity by itself and where discussions and gatherings at tea were often delightful! I also wish to thank the department secretaries, especially Lucille and Nancy, at Stony Brook and Karen at CUNY, who are excellent at taking care of graduate students.

Any one who has successfully been through a PhD program would attest to the existence of a core group of individuals who provide mutual energy and support for sustenance. This is an absolutely crucial group of friends whose importance couldn't be stressed further! I gladly thank Aprameyan, Abhijit, Fred, Luoy, Manas, Pankaj, Prasad, Rik, Ritwik, Shubhroshekhar, Soumya and Sucharit. The experiences range from culinary experiments to the Scottish pleasures of life, from just being there for each other to playing strategic board games, from playing squash to walking across Manhattan from north to south, from US open visits to those occasional lunch on campus, from discussing scientific gossips to playing Halo, from humming old, forgotten songs to discussing human psyche and possibly from Dunkin Donuts to Seawolves marketplace.

As we finally arrive at the more mathematical part of this journey, I wish to thank all the mathematicians with whom I have enjoyed discussing mathematics. Apart from providing words of encouragement and showing interest in my work, they have made me feel welcome to the mathematical community. This list includes Kevin Costello, Mahan Maharaj, John Morgan, Tony Phillips, Graeme Segal, Michael Sullivan, Oleg Viro and Mahmoud Zeinalian. Blaine Lawson and Moira Chas deserve to be thanked specifically. Interesting mathematical discussions with Moira aside, she thought I was a good enough teacher who could possibly transmit the passion of mathematics to the younger generation. This teaching experience has immensely improved my mathematical communication skills and has developed a keen interest in teaching mathematics. My experience with Blaine mostly centers around his courses and discussions following those lectures. Apart from being one of the best lecturers I have ever seen, Blaine is amazing at explaining things in such a lucid way that sometimes things look deceivingly simple! I really wish I could keep taking his courses.

It's time to finally thank my advisor, Dennis Sullivan. It's hard to describe but I'll try nonetheless. Dennis has a very special way of doing mathematics, one marked with extreme conceptual clarity, geometric intuition and beautiful imagination. He has been very patient and attentive, simultaneously providing guidance and sharing his mathematical insights very generously. My debt to him is immense and as I gradually imbibe his philosophy, I couldn't imagine someone better to learn mathematics from! Being around Dennis, due to his mathematically active nature, also has the significant benefits of interacting with other mathematicians. It's been a privilege to know Dennis and he'll continue to be an inspiration!

Chapter 1

Introduction

There is geometry in the humming of the strings,
here is music in the spacing of the spheres.

Pythagoras

There is no precise criterion for determining what should be called *String Topology*. This emerging field was kick-started by the seminal paper *String Topology* by Chas and Sullivan. A lot of mathematical research has since been done on this and the most modern techniques of algebraic topology are often used in its study. However, in its current amorphous state, this field lacks a definitive shape and direction, although there are suggestive pointers. We shall be lax and adopt the point of view that string topology is the study of algebraic topology of loop spaces. This immediately begs the following questions :

What kind of loops on what type of spaces do we aim to study?

What new information are we are hoping to glean form this theory?

Before answering these let's make a note of the origin of the name. Loops in a target space can be thought of closed strings in this background space and since we aim to study the topology of this space of strings (loops or arcs), string topology seems a very inspired name! Presumably, this is what transpired in the minds of the authors of *String Topology*.

We now answer the questions raised before. Smooth loops (or geodesics in the presence of a metric) have been studied since the days of Cayley and Maxwell. Later in the 30's Morse developed a theory for studying the topology of a manifold by analyzing differentiable functions on it. This theory came to bear his name afterwards. The existence of smooth geodesics, Bott's celebrated proof of the periodicity theorem, Smale's proof of the h-cobordism theorem all resulted from the use of this theory, embellished along the way by the work of various people. Since, our primary aim is to study the topology using algebraic methods, we focus on continuous loops. One can specify slightly more geometric variants of continuous loops (for example, allowing singularities of certain type(s)) but at the cost of being more sensitive to the relevant algebraic structure. The approach to string topology via Morse theory has been taken up by Goresky and Hingston. However, most of the research has primarily employed methods from algebraic topology. One should keep in mind that to define the structures that arise in string topology, transversality is a necessary ingredient. As for the spaces where string topology can be defined, manifolds are the first choice. More general spaces where appropriate notions of Poincaré duality holds also work.

String topology started as an attempt to understand a naturally present Lie bialgebra structure on surfaces. On the underlying vector space of free homotopy classes of non-trivial curves on the surface, there is a Lie bracket defined by Goldman and a cobracket defined by Turaev. String topology, as defined originally, is presumably a homotopy invariant and much of it has been proven to be so. However, imposing geometric conditions on the loops that we study may result in interesting invariants which distinguish non-diffeomorphic manifolds. There is evidence towards developing powerful knot invariants using ideas arising from string topology applied with a geometric sensibility.

Among the various structures that arise in string topology, the moduli space of Riemann surface plays a central role. In a sense, one can think of this moduli space governing the natural *gluing* laws that present themselves when loops interact in an ambient space. Some of these structures admit alternative algebraic descriptions via Hochschild homology, cyclic homology which tie up with various modern developments in algebraic topology, for e.g., properad and algebras over properads and categorification of topological quantum field theory. There is also an emerging body of evidence for strong connections between symplectic topology and string topology. If one were to believe this, then string topology, which is highly calculable in certain aspects, could provide answers to some of the computations arising from analytical constructions on the symplectic side, which are considerably more difficult.

1.1 The History in a Nutshell

Loop spaces have played a recurring and important role in mathematics. To a geometer, the free loop space codifies closed geodesics while the based loop space is an object of utmost importance in homotopy theory. In the able hands of Bott, it was used in conjunction with Morse theory to prove the celebrated periodicity theorem. In this work we focus on topological aspects of the free loop space of manifolds. More specifically, we show that transversal string topology, a geometric variant of string topology, distinguishes non-homeomorphic but homotopy equivalent lens spaces. As yet another application of transversal string topology, we can recover Ng's cord algebra of the knot complement which arose from the symplectic topology of the conormal bundle associated to the knot.

Almost a decade back, Chas and Sullivan [4] defined and studied algebraic operations on the free loop space LM of an oriented manifold M. Very briefly, using the fibration $\Omega M \hookrightarrow LM \to M$ one can define a *loop product* on the homology of LM using the classical intersection product on M and the Pontrjagin product on ΩM. There are other associated operations on $H_*(LM)$ which make it into a BV algebra. Some of these operations have their equivariant versions via the circle action on LM. In fact, $H_*^{S^1}(LM, M)$ is an involutive Lie bialgebra. There is a homotopy theoretic version [8] of this story too.

A little later Cohen, Klein and Sullivan [10] showed that the loop product and the Lie algebra structure is an invariant of the homotopy type of the manifold M. Cohen and Godin [7] used homotopy theory to define k-to-l operations on $H_*(LM)$. These operations agree with those previously defined by Chas and Sullivan [5]. General string topology operations can be thought of as maps $H_*(LM)^k \to H_*(LM)^l$, encoded by a map of a Riemann surface with k input and l output punctures. It was shown by Godin [16], [15] that the homology of $\mathscr{M}$, the moduli space of Riemann surfaces, acts on the homology of the loop space LM. It has been unknown for a while whether this structure is an invariant of the homotopy type of the manifold or whether it is sensitive to some finer structure. Based on the work of Costello [11], Godin conjectured that it is a homotopy invariant. Lurie's recent work [24] also support the claim. The action of the open part can be extended to certain compactifications as worked out by Poirier [30] in her thesis. It is likely that these extended operations encode non-homotopy invariants because it deals with small loops which are geometric in nature. In fact, it has been a long standing open problem of whether any one of these invariants is a not a homotopy invariant.

We adopt the view point that closed loops in M are open strings in $M \times M$ that start and end on the diagonal. We study transversal open strings and ways to combine them. Transversality is the simplest kind of (vacuous) singularity that there is and to combine two such strings we need to make use of the normal bundle of the diagonal of M in $M \times M$. One should be able to probe how the locally-supported Thom class is used by the construction to determine whether the algebraic structure is a homotopy invariant [32]. It is known that [26] that such constructions may produce non-homotopy invariants. In fact, in our case that's exactly what we show.

1.2 Outline of Chapter 2

For the most part, the material in this chapter is essentially an overview of what is known in string topology. The ideas and results described are originally due to various authors. At times we have made an attempt to elucidate parts of the theory by presenting explicit computations or a different viewpoint. While this may draw the reader's interest and clarify their understanding, the experts can skip the details.

§2.1 : **A Primer on String Topology**

The study of string topology involves intersection of chains on loop spaces. One can consider two families of closed oriented curves in a manifold M of dimension d. At each point of intersection of a curve from one family with a curve from the other family, form a new curve by going along the first curve followed by the second. Typically, an i-dimensional family and a j-dimensional family will produce an $(i + j + 2 - d)$-dimensional family. The mathematical structures behind such interactions was described and studied under the heading of “String Topology" by M. Chas and D. Sullivan. We follow their paper [4] throughout §2.1.1 and §2.1.3 briefly reviewing the main theorems of string topology. In the intermediate §2.1.2 we discuss the coproduct structures on the free loop space, reviewing a construction outlined in [33]. We also invite the interested reader to look up [18] for a detailed but different view of the same construction.

§2.2 : **Immersed Loops and String Topology**

A simple construction built out of S. Smale's work allows us to identify the free loop space of the unit tangent bundle of M with the space of immersed loops in M. The algebraic structures present in the free loop space can then be transferred to the space of immersed loops. We briefly analyze these structures in §2.2.2 and provide a few examples in §2.2.3. During the time of writing this, a slightly general approach has been outlined in [6] which addresses the same construction.

§2.3 : **String Topology of Surfaces**

We give a flavour of the computational aspects of string topology and review the origins of string topology in §2.3. In §2.3.1 we review the Goldman bracket (also called the *string bracket*) on the torus. It's interesting to note that the string bracket on $S^1 \times S^1$ is non-trivial while the string bracket on S^1 is essentially trivial. We'll have more to say about this issue in §3.2. We also review in §2.3.2 what is known for other surfaces, i.e., with possible boundaries and having any genus. Finally, in §2.3.3 we provide a nice interpretation of the fact the free homotopy classes of curves on a surface is a Lie bialgebra. In fact, as shown in [5], this Lie bialgebra structure extends to an infinity Lie bialgebra. We restrict ourselves to the Lie bialgebra part of the structure and interpret it as a solution to $D^2 = 0$ for some operator D in the space of *multicurves*.

1.3 Outline of Chapter 3

The theory of minimal models was discovered by Sullivan as an attempt to study the rational homotopy type of spaces. It has been a very successful theory with lots of applications, [12] and [34] to name a few. These minimal models can be applied to fibrations which lead to models of the loop space fibration [34]. We review these constructions in §3.1 and provide computations for spheres and projective spaces. Armed with these calculations, we then apply this machinery to products of manifolds in §3.2 and arrive at interesting conclusions.

§3.1 : **String Topology via Minimal Models**

Rational homotopy theory, as approached by Sullivan via minimal models, is highly computable and robust. In §3.1.1 we review the relevant minimal models for the fibration $\Omega M \hookrightarrow LM \to M$. We take up the study of these models for manifolds with monogenic cohomology ring in §3.1.2 and §3.1.3. Although the algebraic structures on the loop space are non-trivial, the equivariant operations of string bracket and cobracket all turn out to be zero. It's worth mentioning (although we don't specify the details in this work) and perhaps of some interest that Massey products abound in the minimal models for the free loop space of even spheres or complex projective spaces.

§3.2 : **String Topology of Product Manifolds**

We apply the theory of minimal models developed in the previous section to a product manifold. It is known that the string bracket for the torus is the Goldman bracket which is non-trivial although the string bracket for the circle (§2.1.4), appropriately interpreted, is trivial. This phenomenon persists (as worked out in §3.2.2) for the product of two odd spheres. As remarked before, the Lie algebra structure on the equivariant loop homology of an odd sphere is trivial while it is highly non-trivial for products (refer §3.2.2 and §3.2.3). This interestingly suggests a possibility of using the string bracket to detect certain factorization of manifolds. One of our future goals is to use the Lie bialgebra structure for a product of odd spheres to make a rational association of a Lie bialgebra to a Lie group and study its properties. We end with a non-equivariant discussion of the loop homology of some of the classical Lie groups in §3.2.4.

1.4 Outline of Chapter 4

This chapter is about the main result of the thesis. It is highly likely, in view of [10] and by the work of Lurie, that the string topology operations as defined originally in [4] are homotopy invariants. We would like to change our point of view and study smooth loops motivated by geometric intuition. This is the subject of *transversal string topology* which is introduced here. The relevant algebraic setting is that of algebras and coalgebras in the category of differential graded bicomodules. We review the background and derive a bar-cobar adjunction in this new setup. Combining geometric ideas with homological algebra we can recover, starting from transversal strings, a model for the based loop space of the complement of the diagonal. However, the product on this based loop space is not the Pontrjagin product but a twisted version of it! We then use the computational power of rational homotopy theory to show, by recasting [23], that this twisted Pontrjagin product is not a homotopy invariant. This approach can be modified and applied to embedding spaces like 3-manifolds in S^5 or knots in S^3.

§4.1 : **Transversal String Topology**

We define the notion of transversal strings in §4.1.1 and lay down its basic properties. These constructions are motivated by geometric considerations aimed at probing the diagonal M inside $M \times M$. We, however, work in the general setup of Y embedded in X for the most part and study open strings in X that start and end on Y and is non-tangential to Y otherwise. The various algebraic structures that manifest itself on the chains of such strings need to be treated in the context of objects in the category of bicomodules. We define this category in §4.1.2 and observe its relevance in our case. Finally, we describe some of the algebraic structures on transversal strings in §4.1.3.

§4.2 : **Bar and Cobar Construction**

At the expense of the reader, who may be familiar with the classical bar-cobar adjunction, we briefly review the material starting with geometric realization of simplicial sets in §4.2.1. This construction generalizes to a bigger context and in particular, applies in our setting of bicomodules. We prove (rather check an existing proof in this general setup) a bar-cobar adjunction in §4.2.2. The proof has a deformation theory flavour to it. In §4.2.3 we apply this adjunction with the machinery developed in §4.1 to get a model for the based loop space of the complement of Y in X, equipped with a twsited version of the Pontrjagin product.

§4.3 : **Detecting Non-homotopy Invariants**

Configuration spaces have been classically studied in its own right and otherwise. We review what is known about configuration spaces that applies to us, viz., it was shown in [23] that the configuration space of n-points is not a homotopy invariant. We review this material in §4.3.1 with an eye on rational homotopy theory and minimal models. In §4.3.2 we use the minimal model version of this result to show that the twisted Pontrjagin product on the homology of the based loop space of the universal cover of $X \setminus Y$ is not a homotopy invariant. The proof, apart from the computational aspects, involves Minor-Moore's classic theorem on the structure on Hopf algebras. We finish off the proof of our main result in §4.3.3 by showing the simple fact that the twisted Pontrjagin product on the based loop space of $X \setminus Y$ still is a non-homotopy invariant when applied to $Y = M$ and $X = M \times M$.

Chapter 2

What is String Topology?

2.1 A Primer on String Topology

The study of string topology involves intersection of chains on loop spaces. One can consider two families of closed oriented curves in a manifold M of dimension d. At each point of intersection of a curve from one family with a curve from the other family, form a new curve by going along the first curve followed by the second. Typically, an i-dimensional family and a j-dimensional family will produce an $(i+j+2-d)$-dimensional family. The mathematical structures behind such interactions was described and studied under the heading of *String Topology* by M. Chas and D. Sullivan. We follow their paper [4] throughout §2.1.1 and §2.1.3 briefly reviewing the main theorems of string topology. In the intermediate §2.1.2 we discuss the coproduct structures on the free loop space, reviewing a construction outlined in [33]. We also invite the interested reader to look up [18] for a detailed but different view of the same construction.

2.1.1 Products in loop homology

The free loop space LM associated to a space M is the continuous mapping space $\mathrm{Map}(S^1, M^d)$. By the *loop homology* we mean the ordinary homology of LM and by the *string homology* we mean the equivariant homology of LM with the circle symmetry of rotating the domain. Recall that an i-chain in LM is a linear combination of oriented i-dimensional simplices of loops in M. Such a chain naturally gives rise to a chain in M by the image of $1 \in S^1$. The *loop product* $\bullet$ is transversely defined at the chain level as follows : given x, an i-chain of loops in M, and y, a j-chain of loops in M, first intersect the i-chain (of M) of marked points of x with the j-chain (of M) of marked points of y, to obtain an $(i+j-d)$-chain c (of M) along which the marked points of x coincide with that of y. Now define $x \bullet y$ by putting at each point of c the composed loop that goes around the loop of x and then around the loop of y.

Remark 2.1. *To make this definition precise, we need to work over transversal pairs of chains and also have to adopt a consistent orientation convention. This is taken up in* [4] *following their Remark* 2.1*. This necessitates M to be a manifold or more generally a Poincaré duality space.*

The loop product passes to loop homology, following Lemma 2.3 in [4], and defines a product

$$\mathbb{H}_i \otimes \mathbb{H}_j \xrightarrow{\bullet} \mathbb{H}_{i+j},$$

where $\mathbb{H}_* = H_{*+d}(LM)$ is the homology of LM with degrees shifted down by d. As is known, for e.g., S^2, the loop product may be non-zero way above the dimension of the manifold.

The loop product is associative since classical based loop composition is associative up to homotopy. The commutativity follows from homotopy commutativity at the chain level via a homotopy defined as $*$ in [4]. It was proved (Theorem 3.3 in [4]) that

Theorem 2.2. (Chas-Sullivan)
$(\mathbb{H}_*, \bullet)$ *is an associative graded commutative algebra.*

There is a way to compare the loop product with two other canonical products - cup product and Pontrjagin product. There are natural maps

$$H_{*+d}(M) \xrightarrow{\iota} \mathbb{H}_* \xrightarrow{\cap} H_*(\Omega M),$$

where ι is the natural inclusion of constant loops into all loops and $\cap$ is the transversal intersection with one fibre for the fibration

$$\Omega M \longrightarrow LM \xrightarrow{\mathrm{ev}} M.$$

The homology of the based loop space has the based loop product (also called the *Pontrjagin product*) while the homology of M has the classical intersection product. If we use the usual grading on the homology of ΩM and the shifted grading on the homology of LM and M then the two maps above preserve products, i.e.,

$$(H_{*+d}(M), \wedge) \xrightarrow{\iota} (\mathbb{H}_*, \bullet) \xrightarrow{\cap} (H_*(\Omega M), \times) \tag{2.1.1}$$

is a map of rings.

We note that $\mathrm{ev} \circ \iota : M \to M$ is just identity, whence ι is an injection onto a direct summand. Further, if M is a Lie group then LM is homeomorphic to $\Omega M \times M$, whence by the Künneth formula $H_*(LM)$ is the product of $H_*(M)$ and $H_*(\Omega M)$. Then the map $\cap$ is just the projection onto the second factor. Therefore, $\cap$ is a surjection for Lie groups.

Remark 2.3. *The 7-sphere S^7 is an example of a space that is not a Lie group but the map $\cap$ is a surjection with integer coefficients. This follows from the observation that S^7, thought of as unit octonions, is an H-space and admits inverses, whence left (or right) translations are homeomorphisms. Consequently, there is a homeomorphism $LS^7 \cong S^7 \times \Omega S^7$ which implies the surjectivity of $\cap$.*

Let $i : \Omega M \to LM$ be the inclusion map and $\cap$ denote the intersection with the fibre over $m_0 \in M$. Then $[m_0]$, the class of the constant loop at m_0, can be thought of as an element of $\mathbb{H}_{-d}(LM)$. It then follows that $i_* \circ \cap$ is just the map sending $x \in \mathbb{H}_*(LM)$ to $x \bullet [m_0] \in \mathbb{H}_{*-d}(LM)$.

The *loop bracket* is defined transversely on $C_*(LM)$ by anti-symmetrizing $*$, i.e.,

$$\{x, y\} := x * y - (-1)^{(|x|+1)(|y|+1)} y * x.$$

This defines a Lie bracket and this can be proved by showing that $*$ defines a pre-Lie algebra, as in Lemma 4.2 in [4]. Since $\{\ ,\ \}$ is a chain map, this Lie algebra structure passes to loop homology. The compatibility of the loop product and the loop bracket is valid because $*$ is a left derivation of $\bullet$ exactly and is a right derivation of $\bullet$ up to chain homotopy. Lemma 4.6 of [4] is precisely this. This leads us to one of the main results (Thereom 4.7 in [4]) of string topology :

Theorem 2.4. (Chas-Sullivan)
The loop product with the loop bracket turns the loop homology into a Gerstanhaber algebra.

There is an operator Δ of degree 1 on the chains which passes to loop homology. It is defined by the rotation action of the circle on any i-chain, i.e., it is geometrically erasing the marked points and then putting markings everywhere.

It satisfies $\Delta^2 = 0$ and it was shown that (Corallary 5.3 in [4]) the loop bracket $\{\ ,\ \}$ is the deviation of Δ from being a derivation of the loop product $\bullet$. This in turn implies (Theorem 5.4 in [4]) :

Theorem 2.5. (Chas-Sullivan)
The loop homology with the loop product and the operator Δ forms a Batalin-Vilkovisky algebra.

2.1.2 Coproducts in loop homology

We'll outline constructions for two different coproducts on the loop homology. On the level of chains of the loop space, under certain transversality conditions, there is a coproduct $\tau_t, t \in [0,1]$ which commutes with the boundary. This is a coproduct of degree $-d$, where d is the dimension of the manifold. There is also a coproduct τ of degree $1-d$, which doesn't commute with the boundary. We shall be interchangeably using the equivalent descriptions of LM as the space of loops in M or as the space of *open strings* in $M \times M$ which start and end on the diagonal. This point of view will be crucial for what we do in Chapter 4.

Remark 2.6. *In contrast to the traditional use of the word open when we say an open string we mean a closed interval while a closed string is a circle. This is a terminology frequently used in theoretical physics.*

We shall review these coproduct structures that were originally defined in [33] and connect it to other relevant works.

Definition 2.7. (Coproducts of degree $-d$) Let $x : \Delta \times [0,1] \to M \times M$ be an i-chain which satisfies $x(\cdot, 0), x(\cdot, 1) \in M$. Assuming that $x(\cdot, t) \pitchfork M$ we get an $(i-d)$-chain in $\Delta \times [0,1]$ defined by $K := x^{-1}(x(\cdot, t) \pitchfork M)$. Fix an orientation on M, which induces one on $M \times M$ and we can assign each point $p \in K$ with a sign $\varepsilon(p) = \pm 1$ depending on whether the oriented intersection of x and M concur with that of $M \times M$ or not. This sign function is a constant on each simplex of K.

Write $K = \sum_{i=1}^{n} \Delta_j$ and let ε_j denote the common sign value on Δ_j. We then define

$$x_j^t : \Delta_j \times [0,1] \longrightarrow (M \times M) \times (M \times M)$$

$$x_j^t(\cdot, s) = (x(\cdot, (1-s)t), x(\cdot, t + (1-t)s)). \tag{2.1.2}$$

is an $(i-d)$-chain in $LM \times LM$.

Let $C_*^{\pitchfork}(LM)$ denote the space of all chains in LM that are transversal to M. We then have a map

$$\tau_t : C_*^{\pitchfork}(LM) \longrightarrow C_*^{\pitchfork}(LM) \otimes C_*^{\pitchfork}(LM), \;\; \tau_t(x) = \sum_{j=1}^{n} \varepsilon_j x_j^t. \tag{2.1.3}$$

Tha map τ_t is called the *coproduct* at time t.

It is clear from the definition that τ_t commutes with the boundary map and there is a well defined induced map on homology. Since the homology of transversal chains is the same as the homology of all chains, we get a map

$$\tau_t : H_*(LM) \longrightarrow H_*(LM) \otimes H_*(LM).$$

Notice that $\tau_t([M]) = \chi(M)(\mathrm{p} \otimes \mathrm{p})$, where p denotes the constant loop at $p \in M$. Moreover, for $0 \leq t_1 \leq t_2 \leq 1$ the map given by

$$h : C_*^{\pitchfork}(LM) \times [t_1, t_2] \longrightarrow C_*^{\pitchfork}(LM) \otimes C_*^{\pitchfork}(LM), \;\; h(x,s) = \tau_s(x)$$

is a chain homotopy between τ_{t_1} and τ_{t_2}. Therefore, the induced coproduct map τ_t of degree $-d$ on loop homology are all the same for any $t \in [0,1]$.

For any closed manifold of zero Euler characteristic (which is tantamount to having a nonwhere zero vector field) the coproduct τ_t is trivial. As sketched in [33], put a Riemannian metric on M and choose a unit vector field V. This induces a map $\widetilde{V} : LM \to L(T_1M)$. Compounded with S. Smale's work [31] on immersed loops, which says that the space $\mathrm{Imm}(S^1, M)$, consisting of smooth immersed loops, is homotopy equivalent to $L(T_1M)$, we conclude that an i-chain in LM can actually be replaced by an i-chain of immersed loops. Any immersed loop in M or alternatively, any open string γ in $(M \times M, M)$ transversal to the diagonal has a smallest $t_\gamma \in (0,1]$ such that $\gamma(t_\gamma) \in M$. Since any chain

$$x : \Delta \times [0,1] \to M \times M$$

is compact, if $x \pitchfork M$ then

$$T = \inf\{t_{x(\cdot)} \,|\, \cdot \in \Delta\}$$

is a positive number for otherwise there will be a limiting open string which will not be immersed. Therefore, for any $t < T$ we conclude that $\tau_t(x) = x_t = 0$.

Since τ_t's induce the same operation on $H_*(LM)$ for any $t \in [0,1]$, we conclude that $\tau_t \equiv 0$ on loop homology. However, it is important to note that this depends on the choice of the unit vector field. We'll have more to say on this later on when we discuss string topology of odd spheres.

Definition 2.8. (Coproduct of degree $1-d$) Define the coproduct

$$\tau : C_*^{\pitchfork}(LM) \longrightarrow C_*(LM) \otimes C_*(LM) \tag{2.1.4}$$

as the one parameter family of operations τ_t, where $t \in [0,1]$.

Since $[\partial, \tau_t] = 0$ we have

$$[\partial, \tau] = \tau_1 - \tau_0.$$

Notice that the image of τ_0 lies in $C_*(M) \otimes C_*(LM)$ while the image of τ_1 lies in $C_*(LM) \otimes C_*(M)$. The map τ has the property that it maps $C_*(M) \subset C_*(LM)$ to $C_*(M) \otimes C_*(M)$. Therefore, there is an induced map (also denoted by τ) of degree $1-d$

$$\tau : H_*(LM, M) \longrightarrow H_*(LM, M) \otimes H_*(LM, M). \tag{2.1.5}$$

As a special case, for closed manifolds admitting a nowhere zero vector field V, we actually have

$$\tau_V : H_*(LM) \longrightarrow H_*(LM) \otimes H_*(LM). \tag{2.1.6}$$

This follows from the remark above that τ_0 is homologous to τ_t for any t and τ_t is zero on homology for sufficiently small t. Also note that τ_V depends on the choice of the nowhere zero vector field V. The coproduct τ enjoys the usual properties of cocommutativity and coassociativity.

Remark 2.9. *If we dualize τ we get an associative algebra structure on $H^*(LM, M)$. In* [18] *M. Goresky and N. Hingston use Morse theory to study the energy functional on the free loop space and get algebraic structures on the reduced cohomology of the free loop space. In particular, they define a cohomology product*

$$\circledast : H^i(LM, M) \otimes H^j(LM, M) \longrightarrow H^{i+j+d-1}(LM, M).$$

They also show that this is just the dual of the coproduct τ defined originally (as in the preceeding paragraphs) by D. Sullivan in [33]*. As expected, the product $\circledast$ is associative and commutative.*

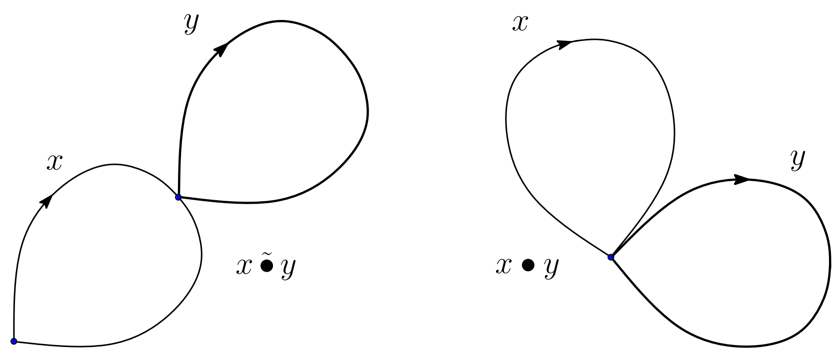

Figure 2.1: The loop coproducts

There is an important relationship between τ and $\bullet$. At the chain level, we may define $\tilde{\bullet}$ exactly as $\bullet$ with a slight change. For i and j-chains of loops x and y we intersect $x(\frac{1}{2})$ transversally with $y(0)$ and then compose the loops. This defines $x \mathbin{\tilde{\bullet}} y$. By construction, $\bullet$ and $\tilde{\bullet}$ are chain homotopic. In fact, only half of the chain homotopy $*$, which was used to show commutativity of $\bullet$, does the trick. If we follow through the definitions we see that

$$\tau(x \mathbin{\tilde{\bullet}} y) = \tau(x) \mathbin{\tilde{\bullet}} y + x \mathbin{\tilde{\bullet}} \tau(y).$$

Observe that $x \mathbin{\tilde{\bullet}} y$ consists of half a loop from x traversed twice as fast followed by a loop from y and then followed by the remaining loop of x. We see that $\tau_t(x \mathbin{\tilde{\bullet}} y) = \tau(x) \mathbin{\tilde{\bullet}} y$ as t runs through $[0, 1/2]$ while for $t \in [1/2, 1]$ we get $\tau(x \mathbin{\tilde{\bullet}} y) = x \mathbin{\tilde{\bullet}} \tau(y)$. Two of the boundary terms cancel giving the equality above. At the level of homology (either with $H_*(LM, M)$ if M doesn't admit a nowhere vanishing vector field or with $H_*(LM)$ if $\chi(M) = 0$) we have

$$\tau(x \bullet y) = \tau(x) \bullet y + x \bullet \tau(y). \tag{2.1.7}$$

In other words, τ is a derivation of the loop product.

2.1.3 String homology

There is a natural action of the unit circle on the loop space LM. However, this action is not free as constant loops are fixed points. The S^1-equivariant homology of LM can be defined by the *Borel construction*. Let ES^1 be the total space of the universal S^1-bundle and consider the fibration

$$S^1 \longrightarrow LM \times ES^1 \longrightarrow LM \times_{S^1} ES^1.$$

Here the base space is the quotient of the diagonal action of S^1 on $LM \times ES^1$. The homology of the base space is defined to be S^1-equivariant homology of LM. It is called the *string homology* of M and is denoted by $\mathcal{H}_*$. There is a long exact sequence associated with this circle bundle (called the *Gysin sequence*)

$$\ldots \longrightarrow \mathbb{H}_{i-d} \xrightarrow{\mathcal{E}} \mathcal{H}_i \xrightarrow{c} \mathcal{H}_{i-2} \xrightarrow{\mathcal{M}} \mathbb{H}_{i-d-1} \longrightarrow \ldots, \tag{2.1.8}$$

where d is the dimension of M. The map $\mathcal{E}$ forgets the marked points of each member in a family of loops, the map $\mathcal{M}$ puts markings on each circle in a family in all possible points while c is the cap product with the characteristic class of the bundle. The Batalin-Vilkovisky operator Δ can be shown to be $\mathcal{M} \circ \mathcal{E}$ while $\mathcal{E} \circ \mathcal{M} = 0$ by exactness.

Remark 2.10. *We denote by $\mathcal{S}$ the space of closed strings, which is the quotient of the space of continuous injective maps Inj$(S^1, \mathbb{R}^\infty)$ by the natural circle action. Since Inj$(S^1, \mathbb{R}^\infty)$ is contractible, we get*

$$ES^1 \times_{S^1} LM \simeq \mathit{Inj}(S^1, \mathbb{R}^\infty) \times_{S^1} LM,$$

where an element can be thought of as one in $\mathcal{S}_M :=$ Map$(\mathcal{S}, M)$. We shall call this space the string space of M and call its homology the string homology, which is the equivariant homology of LM.

Using the loop product and the maps $\mathcal{E}, \mathcal{M}$ it is possible to define n-ary string operations on string homology. For $k \geq 2$ define the operation

$$\overline{m}_k : \mathcal{H}^{\otimes k} \xrightarrow{\mathcal{M}^{\otimes k}} \mathbb{H}^{\otimes k} \xrightarrow{\bullet^{\otimes(k-1)}} \mathbb{H} \xrightarrow{\mathcal{E}} \mathcal{H}. \tag{2.1.9}$$

The operator $\overline{m}_k$ has degree $-k(d-1)+d$, where d is the dimension of M. In particular, the operation $\overline{m}_2 : \mathcal{H}_* \otimes \mathcal{H}_* \to \mathcal{H}_*$, with an added sign, is called the *string bracket*. More precisely,

$$[a, b] := (-1)^{|a|-d} \mathcal{E}(\mathcal{M}(a) \bullet \mathcal{M}(b)) \tag{2.1.10}$$

defines a Lie bracket of degree $2-d$ on $\mathcal{H}_*$.

One may make a similar construction on the equivariant homology starting with the coproduct τ. Since the space of constant loops is S^1-equivariant there is a Gysin sequence for the fibration

$$S^1 \hookrightarrow (LM, M) \times_{S^1} ES^1 \longrightarrow (LM, M).$$

Let us denote the relative string homology $\mathcal{H}_*(LM, M)$ by $\widetilde{\mathcal{H}}_*$. Now we define the operations $c_k, k \geq 2$ as

$$c_k : \widetilde{\mathcal{H}} \xrightarrow{\mathcal{M}} H(LM, M) \xrightarrow{\tau \otimes \mathrm{id}^{\otimes (k-2)} \circ \cdots \circ (\tau \otimes \mathrm{id}) \circ \tau} H(LM, M)^{\otimes k} \xrightarrow{\mathcal{E}^{\otimes k}} \widetilde{\mathcal{H}}^{\otimes k}.$$

Note that c_k had degree $k + d - dk$, the same as that of $\overline{m}_k$. We shall see later that for surfaces c_2 is the famous Turaev cobracket. Moreover, the operations $\overline{m}_2, c_2$ turn $\mathcal{H}_*(LM, M)$ into a Lie bialgebra.

Remark 2.11. *We have seen that some operators from loop homology carry over to string homology via the use of the transfer maps $\mathcal{E}$ and $\mathcal{M}$. For example, $\bullet$ gives rise to the string bracket $[\ ,\ ]$ and τ gives rise to c_2 on the reduced string homology. Similar construction with Δ and $\{\ ,\ \}$ yield the trivial operation.*

It is natural to inquire if these structures on the loop spaces detect any smooth or topological structures of the manifold. Fix closed oriented manifolds M and N of the same dimension and consider $f : M \to N$ such that

$$(Lf)_* : (\mathbb{H}_*(LM; \mathbb{Z}), \bullet) \longrightarrow (\mathbb{H}_*(LN; \mathbb{Z}), \bullet)$$

is an isomorphism of algebras. Then $(Lf)_*$ restricts to an isomorphism between $(H_*(M; \mathbb{Z}), \wedge)$ and $(H_*(N; \mathbb{Z}), \wedge)$. If M and N are both simply connected then M and N are forced to be homotopy equivalent via f. Conversely, let $f : M \to N$ be an orientation preserving homotopy equivalence between closed oriented manifolds. Then there is a natural map $\Omega f : \Omega M \to \Omega N$ and it induces an isomorphism of rings (with the Pontrjagin product)

$$(\Omega f)_* : H_*(\Omega M; \mathbb{Z}) \xrightarrow{\cong} H_*(\Omega N; \mathbb{Z}).$$

Similarly, the map $Lf : LM \to LN$ induces an isomorphism in loop homology as graded abelian groups. Since the loop product is constructed using the Pontrjagin product and the intersection product, one can ask if Lf induces a ring isomorphism. As was shown (Theorem 1 in [10]), the loop product and the string bracket are homotopy invariants. We state the result for singular homology.

Theorem 2.12. (Cohen-Klein-Sullivan)
Let $f : M \to N$ be an orientation preserving homotopy equivalence between closed oriented manifolds. The induced map $Lf : LM \to LN$ induces a ring isomorphism of loop homology algebras

$$(Lf)_* : H_*(LM; \mathbb{Z}) \xrightarrow{\cong} H_*(LN; \mathbb{Z})$$

and a Lie algebra isomorphism of graded Lie algebras

$$(Lf)_* : \mathcal{H}_*(LM;\mathbb{Z}) \xrightarrow{\cong} \mathcal{H}_*(LN;\mathbb{Z}).$$

However, this leaves open the following questions :

Question 2.13. (1) Does the loop coproduct τ commute with $(Lf, f)_*$?
(2) Is the the induced map

$$(Lf, f)_* : \mathcal{H}_*(LM, M;\mathbb{Z}) \longrightarrow \mathcal{H}_*(LN, N;\mathbb{Z})$$

an isomorphism of Lie bialgebras?

Although at present we don't have an answer to this question (which is largely believed to be true) we will take up a geometric variant of the same question in Chapter 4. The answer in that case will turn out to be negative!

2.1.4 An illustrative example

The constructions reviewed in the previous sections are rich and give rise to beautiful structures on the homology of the loop space of manifolds. As a basic example, we characterize the algebraic structure of the loop homology and the string homology of S^1. We fix the counterclockwise orientation on the circle.

The fibration $\Omega S^1 \to LS^1 \to S^1$ is trivial and ΩS^1 has the homotopy type of a discrete space. In fact $\Omega S^1 \simeq \mathbb{Z}$ and $LS^1 \simeq S^1 \times \mathbb{Z}$. Let L_n denote the component of LS^1 which contains the loop $\alpha_n : z \mapsto z^n$. Each L_n deformation retracts to the circle $S^1_n \subset L_n$ given by

$$S^1_n(w) = w\alpha_n.$$

This is precisely the circle traced out by α_n as we rotate it by the circle symmetry. The loop homology (for $i = 0, 1$)

$$H_i(LS^1) = \oplus_{n\in\mathbb{Z}} H_i(S^1_n) = \oplus_{n\in\mathbb{Z}}\mathbb{Z}$$

can now be described more explicitly. The generators of $H_0(LS^1)$ are just the elements α_n of $\pi_1(S^1)$. The generators of $H_1(LS^1)$ is the collection of single generators of $H^1(S^1_n)$ for $n \in \mathbb{Z}$. A typical element of $H_1(S^1_n)$ can be thought of as a map of $S^1 \times S^1$ into the circle where $\{1\} \times S^1$ maps to L_n. Then a generator of $H_1(L_n)$ is given by the map rotating α_n once, i.e.,

$$\theta_n : S^1 \times S^1 \longrightarrow S^1,\ (w, z) \mapsto w\alpha_n(z).$$

Then $m\theta_n$ denotes the map which rotates α_n m times. It is given by the map

$$m\theta_n : S^1 \times S^1 \longrightarrow S^1,\ (w, z) \mapsto w^m\alpha_n(z).$$

Let us calculate the loop product of $m\theta_n, k\theta_l \in \mathbb{H}_0(LS^1)$. The marked points of these two 1-chains on LS^1, parametrized by w_1 and w_2 respectively, give rise to intersection points $c = w_1^m = w_2^k$. Each c has mk pre-images and at c we associate the loop $\alpha_n \cdot \alpha_l = \alpha_{n+l}$. Therefore, the loop product satisfies :

$$(m\theta_n) \bullet (k\theta_l) = mk\theta_{n+l}. \tag{2.1.11}$$

Let α_k be the loop representing a generator of $\mathbb{H}_{-1}(L_n)$. By definition the image of 1 under $m\theta_n$ intersects $\alpha_k(1)$ m times. Then the loop product associates at each point the loop $\alpha_k \cdot \alpha_n = \alpha_{n+k}$. Hence

$$\alpha_k \bullet (m\theta_n) = m\alpha_{n+k}. \tag{2.1.12}$$

We can treat θ_1 as a formal variable and θ_{-1} its inverse due to (2.1.11).

Since S^1 has a nowhere vanishing vector field, the Lie cobracket is

$$\nu : \mathcal{H}_{*-1}(LS^1) \longrightarrow \mathcal{H}_*(LS^1)^{\otimes 2}.$$

To understand this, we need to calculate the loop coproduct (of degree -1)

$$\tau : \mathbb{H}_{*+1}(LS^1) \longrightarrow \mathbb{H}_*(LS^1) \otimes \mathbb{H}_*(LS^1).$$

Recall that the orientation of S^1 determines a vector field on it and using it we perturb the diagonal in $S^1 \times S^1$. Following through the definition of τ we see that θ_1, pictured as an open string, intersects the perturbed diagonal as pictured. Therefore,

$$\tau(\theta_1) = \theta_1 \otimes \alpha_0. \tag{2.1.13}$$

Similarly, one can show that

$$\tau(\theta_{-1}) = -\alpha_0 \otimes \theta_{-1}. \tag{2.1.14}$$

It is also clear that $\tau(\theta_0) = 0 = \tau(\alpha_0)$. Using (2.1.7) we can explicitly write down τ. In conclusion, we have the following :

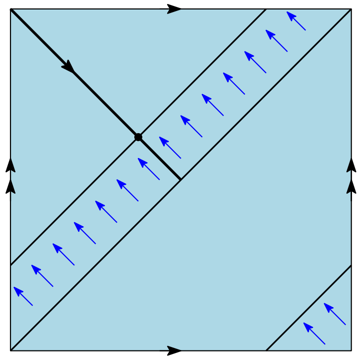

Figure 2.2: The loop coproduct via the counterclockwise rotation

Proposition 2.14. *The loop homology $\mathbb{H}_*(LS^1)$ is graded isomorphic to $\Lambda_{\mathbb{Z}}(\alpha_0) \otimes \mathbb{Z}[\theta_{\pm 1}]$, where α_0 generates $\mathbb{H}_{-1}(L^0S^1)$ and $\theta_{\pm 1} \in \mathbb{H}_0(LS^1)$. The loop bracket $\{,\}$ is non-zero and $\Delta : \mathbb{H}_{-1}(LS^1) \to \mathbb{H}_0(LS^1)$ maps $\alpha_n = \alpha_0\theta_n$ to $n\theta_n$. The coalgebra structure on $\mathbb{H}_*(LS^1)$ is generated by* (2.1.13),(2.1.14).

Proof The algebra and coalgebra structures are clear. For the other operations, observe that the operator Δ is as claimed and that

$$-\Delta(\alpha_m \bullet \theta_n) + \Delta(\alpha_m) \bullet \theta_n - \alpha_m \bullet \Delta(\theta_n) = -\Delta(\alpha_{m+n}) + m\theta_m \bullet \theta_n = -n\theta_{m+n}.$$

We also have

$$-\Delta(\alpha_m \bullet \alpha_n) + \Delta(\alpha_m) \bullet \alpha_n - \alpha_m \bullet \Delta(\alpha_n) = m\theta_m \bullet \alpha_n - n\alpha_m \bullet \theta_n = (m-n)\alpha_{m+n}.$$

Therefore, the loop bracket on $\mathbb{H}_*(LS^1)$ given by

$$\{\alpha_m, \theta_n\} = -n\theta_{m+n} = -\{\theta_n, \alpha_m\} \tag{2.1.15}$$

$$\{\alpha_m, \alpha_n\} = (m-n)\alpha_{m+n} \tag{2.1.16}$$

is clearly non-zero. $\square$

Notice that τ would change if we had chosen the clockwise orientation for the circle; we would've $\tau(\theta_1) = \alpha_0 \otimes \theta_1$.

To calculate the equivariant homology we need determine $ES^1 \times_{S^1} L_n, n \in \mathbb{Z}$. It is clear that the circle action is trivial on L_0. For $n \neq 0$ the subgroup $\mathbb{Z}_n \subset S^1$, consisting of nth roots of unity, fixes L_n. This gives rise to $ES^1 \times_{S^1} L_n = L(n; \ldots)$ which is just a $K(\mathbb{Z}_n, 1)$. The equivariant loop homology of S^1 can now be written as

$$\mathcal{H}_*(LS^1) = \left(H_*(S^1) \otimes H_*(\mathbb{CP}^\infty)\right) \oplus \left(\oplus_{n\neq 0} H_*(L(n; \ldots))\right).$$

Recall that $H_*(L(n;\ldots))$ is a $\mathbb{Z}_n$ in every odd dimension and zero otherwise, $H_*(\mathbb{CP}^\infty) = \Gamma_\mathbb{Z}[u], |u| = 2$ and $H_*(S^1) = \Lambda(x), |x| = 1$. Therefore, we have :

$$\begin{aligned}\mathcal{H}_{2i}(LS^1) &= \mathbb{Z}(1 \otimes u^{[i]}) \oplus \delta_{i,0}(\oplus_{n\neq 0}\mathbb{Z}) \\ \mathcal{H}_{2i+1}(LS^1) &= \mathbb{Z}(\mathbb{1} \otimes u^{[i]}) \oplus (\oplus_{n\neq 0}\mathbb{Z}_n),\end{aligned}$$

where $u^{[i]} := u^i/i!$. The element $\mathbb{1} : S^1 \to L^0S^1 \times_{S^1} ES^1$ maps z to $[\gamma_z, e]$, where e is a fixed element of ES^1 and γ_z is the constant loop at z. Although the equivariant homology is non-zero in every possible dimension, the string bracket is severely restricted since the loop homology exists only in dimensions -1 and 0. This means that the only possible non-zero brackets are between the α_i's.

We write the rational string bracket, of degree 1 as

$$[\,,\,] : \mathcal{H}_0^\mathbb{Q}(LS^1) \otimes \mathcal{H}_0^\mathbb{Q}(LS^1) \longrightarrow \mathcal{H}_1^\mathbb{Q}(LS^1).$$

If $m + n \neq 0$ then

$$-\mathcal{E}\left(\mathcal{M}\left(\alpha_m\right) \bullet \mathcal{M}\left(\alpha_n\right)\right) = -\mathcal{E}(m\theta_m \bullet n\theta_n) = -mn/(m+n)\mathcal{E} \circ \mathcal{M}(\alpha_{m+n}) = 0.$$

Therefore, the string bracket is given by

$$[\alpha_m, \alpha_n] = -\delta_{m+n,0}mn\mathbb{1}. \tag{2.1.17}$$

Exactly the same argument helps us conclude that the k-ary operations $\overline{m}_k, k \geq 2$ are

$$\overline{m}_k(\alpha_{i_1}, \ldots, \alpha_{i_k}) = (-1)^{k-1}\delta_{i_1+\ldots+i_k,0}(i_1 \ldots i_k)\mathbb{1}. \tag{2.1.18}$$

The Lie cobracket $\nu = \mathcal{E}^{\otimes 2} \circ \tau \circ \mathcal{M}$ on the reduced (rational) string homology is zero. For example, if $n > 0$ then

$$\frac{\nu(\alpha_n)}{n} = \mathcal{E}^{\otimes 2} \circ \tau(\theta_n) = \sum_{i=1}^{n} \mathcal{E}(\theta_i) \otimes \mathcal{E}(\alpha_{n-i}) = \sum_{i=1}^{n} \frac{1}{i}\left(\mathcal{E} \circ \mathcal{M}(\alpha_i)\right) \otimes \alpha_{n-i} = 0.$$

Similar considerations hold for $\alpha_n, n < 0$. Moreover, for the same reason, i.e., $\mathcal{E}(\theta_n) = 0, n \neq 0$, similar computations show that the k-ary operations $c_k, k \geq 2$ are also zero.

2.2 Immersed Loops and String Topology

We change our focus from the free loop space to a subspace which is inherently more geometric. We have a Riemannian manifold (M, g) of dimension at least two. Consider $\mathrm{Imm}(S^1, M)$, the space of all immersed loops in M. Let $T_\uparrow M := TM \setminus M$ denote the *space of directions* in M. There is a natural map

$$\Phi : \mathrm{Imm}(S^1, M) \longrightarrow L(T_\uparrow M) \tag{2.2.1}$$

defined by mapping an immersion $f : S^1 \to M$ to

$$df : TS^1 = S^1 \times \mathbb{R} \longrightarrow T_\uparrow M = T_1 M \times (0, \infty),$$

which is injective on each fibre, to its restriction on $S^1 \times \{1\}$. Notice that $T_\uparrow M \simeq T_1 M$. Our aim will be to define some structures on the free loop space of either $T_1 M$ or $T_\uparrow M$ and compare it to analogous structures on the space of immersed loops.

2.2.1 Various avatars of immersed loops

This section is expository in nature with definitions and elementary observations taking up most of the space. We start with the PhD thesis of S. Smale [31], where he proved the following result.

Theorem 2.15. (Smale)
Let M be a connected C^3-Riemannian manifold of dimension at least 2. For $(p, v) \in T_1 M$ let $\mathrm{Imm}_{(p,v)}(S^1, M)$ denote the space of immersed loops such that $\gamma'(0)$ is in the direction determined by (p, v). Then there is a homotopy equivalence

$$\Phi : \mathrm{Imm}_{(p,v)}(S^1, M) \xrightarrow{\cong} \Omega_{(p,v)}(T_1 M).$$

If we throw in all the unit directions in M then Smale tells us that

$$\Phi : \mathrm{Imm}(S^1, M) \longrightarrow L(T_\uparrow M) \tag{2.2.1}$$

is a homotopy equivalence since $L(T_\uparrow M)$ and $L(T_1 M)$ are. This implies that $\mathrm{Imm}(S^1, M)$ is not homotopy equivalent to LM. For example, if $M = \mathbb{R}^n$ then LM is contractible while $\mathrm{Imm}(S^1, M) \simeq L(S^{n-1})$ is not. More generally, whenever M is parallelizable, i.e., the tangent bundle of M is trivial, we see that $\mathrm{Imm}(S^1, M) \simeq LM \times LS^{d-1}$, where d is the dimension of M. This includes all orientable 3-manifolds and Lie groups.

We begin with a few definitions and figure out the correct spaces to work with.

Definition 2.16. For any manifold N, the *Moore free loop space* $\mathcal{L}N$ is the space

$$\mathcal{L}N := \{(f,t) \mid f : [0,t] \xrightarrow{\text{cts}} N,\ f(0) = f(t)\}. \tag{2.2.2}$$

The space $\mathcal{I}(S^1, N)$ is the space of immersed loops in N parametrized by closed intervals $[0,t]$ as t varies.

For $k > 0$ fixed, the space $\mathcal{I}_k(S^1, N)$ is the subspace of $\mathcal{I}(S^1, N)$ only containing loops of constant speed k.

The Moore loop space $\mathcal{L}N$ includes LN as subspace to which it deformation retracts. This construction is tantamount to replacing the usual fibre ΩN in the fibration $\Omega N \hookrightarrow LN \to N$ by Moore's version of the based loop space and hence the name. By rescaling it can be seen that the space $\mathcal{I}(S^1, N)$ deformation retracts to the subspace $\mathrm{Imm}(S^1, N)$. Similarly, we have a version of the well known arc-length parametrization.

Lemma 2.17. *The space $\mathcal{I}(S^1, N)$ deformation retracts to the subspace $\mathcal{I}_1(S^1, N)$.*

Proof Given an immersed loop $\gamma : [0, t_0] \to N$ we define a path

$$H : [0,1] \longrightarrow \mathcal{I}(S^1, N)$$

that starts at γ and ends in $\mathcal{I}_1(S^1, N)$. By construction, H defines a deformation retraction as needed. Let $\ell(\gamma)$ denote the length of the loop. There is the *arc-length homotopy*

$$\ell_\lambda : [0, t_0] \longrightarrow [0, (1-\lambda)t_0 + \lambda\ell(\gamma)]$$

defined by

$$\ell_\lambda(t) := \int_0^t (\lambda\|\gamma'(s)\| + 1 - \lambda)\, ds.$$

This is a diffeomorphism and we define the homotopy H by setting

$$H(\lambda) := \gamma \circ (\ell_\lambda)^{-1}.$$

It follows that H reparametrizes immersed curves and fixes $\mathcal{I}_1(S^1, N)$ pointwise. $\square$

The common intersection

$$\mathrm{Imm}_1(S^1, N) := \mathcal{I}_1(S^1, N) \cap \mathrm{Imm}(S^1, N)$$

is the space of *unit speed immersed loops parametrized by the unit interval*. With that said, observe that there is a commutative diagram extending Φ.

$$\begin{tikzcd}
& \mathrm{Imm}_1(S^1, M) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathcal{I}_1(S^1, M) \arrow[dl] \arrow[dd] \\
\mathrm{Imm}(S^1, M) \arrow[rr] \arrow[dd, "\Phi"'] & & \mathcal{I}(S^1, M) \arrow[dd] & \\
& L(T_1 M) \arrow[rr] \arrow[dl] & & \mathcal{L}(T_1 M) \arrow[dl] \\
L(T_\uparrow M) \arrow[rr] & & \mathcal{L}(T_\uparrow M) &
\end{tikzcd}$$

All the vertical arrows, essentially given by Φ, are homotopy equivalences except for $\mathrm{Imm}_1(S^1, M) \to L(T_1 M)$. The horizontal maps are natural inclusion maps. Moreover, all the four spaces in the bottom horizontal plane are homotopy equivalent. And barring $\mathrm{Imm}_1(S^1, M)$ all the other three spaces in the top horizontal plane are also homotopy equivalent. We've already noticed that $\mathrm{Imm}_1(S^1, M)$ is not closed under loop composition as rescaling the domain of a composition of two loops to $[0, 1]$ doubles it speed. Therefore, we'll be working with $\mathcal{I}_1(S^1, M)$ which is closed under loop composition.

2.2.2 A product on immersed loops

We would like to define a product and a coproduct on $\mathcal{I}_1(S^1, M)$ for a fixed Riemannian manifold M. Observe that an immersed smooth loop of unit speed is a curve $\gamma : [0, 2t] \to M$ such that $|\gamma'| \equiv 1$ and $\gamma'(0) = \gamma'(2t)$. We write γ as $\gamma_{\mathrm{os}} : [0, t] \to M \times M$, defined as

$$\gamma_{\mathrm{os}}(s) = (\gamma(s), \gamma(\pi(2t - s))), \quad s \in [0, t].$$

It can be checked that

$$\gamma'_{\mathrm{os}}(0) = (\gamma'(0), -\gamma'(0)), \quad \gamma'_{\mathrm{os}}(t) = (\gamma'(t), -\gamma'(t)),$$

whence both vectors above are orthogonal to $TM \subset T(M \times M)$. Conversely, given any such open string γ_{os} which is orthogonal to M at its endpoints we see that this corresponds to an immersed curve γ. For any such open string γ_{os} if we further assume that $\gamma_{\text{os}} \pitchfork M$ and it intersects perpendicularly then we call it an *ortho-immersed string*. These open strings form a closed subspace, denoted by $\mathcal{I}^{\perp}(S^1, M)$.

Definition 2.18. (Product on unit speed immersed loops) Let x, y be any two chains (of dimensions i and j respectively) in $\mathcal{I}_1(S^1, M)$. We may think of x and y as open strings in $M \times M$. Then $x(\cdot, t.)$ and $y(\cdot, 0)$ are i and j-chains respectively in $M \subset M \times M$, which are assumed to be intersecting transversely in an $(i+j-d)$-chain in M. We further reduce this to an $(i+j-2d+1)$-chain K by requiring that $p \in K$ if and only if $x'(p, t_p) = y'(p, 0)$. We compose the two open strings x and y, parametrized by K, to get another open string in $\mathcal{I}_1(S^1, M)$. This product will be denoted by $\succ(x, y)$.

At the level of transversal pairs of chains we get a map

$$\succ : C_*(\mathcal{I}_1(S^1, M)) \otimes C_*(\mathcal{I}_1(S^1, M)) \longrightarrow C_*(\mathcal{I}_1(S^1, M))$$

of degree $1 - 2d$. It is natural to ask how the induced map on homology compares to that of the loop product, also of degree $1 - 2d$,

$$\bullet : H_*(\mathcal{L}(T_1 M)) \otimes H_*(\mathcal{L}(T_1 M)) \longrightarrow H_*(\mathcal{L}(T_1 M)). \tag{2.2.3}$$

It easily follows from the definition of $\succ$, Φ and the loop product that the isomorphism

$$\Phi_* : H_*(\mathcal{I}_1(S^1, M)) \longrightarrow H_*(\mathcal{L}(T_1 M))$$

is compatible with $\succ$ and $\bullet$, whence the commutativity and associativity of $\succ$. In fact, any direct proof of this fact would essentially involve mimicking the proof for the loop product. That's exactly what we do here.

Definition 2.19. Given a chain $x : \Delta \to \mathcal{I}_1(S^1, M)$ for each $p \in \Delta$ one can associate the length of the loop at p, i.e., the domain of the definition of $x(p)$. We denote it by $2t_p$. Let

$$\widetilde{\Delta} := \{(p, t) \,|\, p \in \Delta, t \in [0, t_p]\} \subset \Delta \times [0, \infty),$$

homeomorphic to $\Delta \times [0, 1]$, denote the domain of definition of the *open string version* of x. This can be rewritten as

$$\tilde{x} : \widetilde{\Delta} \to M \times M, \;\; \tilde{x}(p, t) := (x(p, t), x(p, 2t_p - t)).$$

More generally, we denote the open string version of x *rotated by angle* s (for $s \in [0,1]$) by

$$\tilde{x}_s : \widetilde{\Delta} \to M \times M,\ \ \tilde{x}_s(p,t) := (x(p, st_p + t), x(p, (2+s)t_p - t).$$

We shall denote the open string this collection of open strings by $\{\tilde{x}_s\}$.

Proposition 2.20. *$(H_*(\mathcal{I}_1(S^1, M)), \succ)$ is a commutative and associative algebra.*

Proof Let us start with two suitably transversal (in a sense that will become clear soon) chains x and y in $\mathcal{I}_1(S^1, M)$. Let

$$\tilde{x} \times \{\tilde{y}_s\} : \widetilde{\Delta} \times \big([0,1] \times \widetilde{\Delta}'\big) \longrightarrow T_1(\Delta_M) \times T_1(\Delta_M)$$

be the map defined by

$$\tilde{x} \times \{\tilde{y}_s\}(p, \cdot, s, q, \cdot) = (\tilde{x}(p, t_p), \tilde{x}'(p, t_p), \tilde{y}_s(q, 0), \tilde{y}_s'(q, 0)).$$

Let $\widetilde{\Delta} * \widetilde{\Delta}'$ denote the transversal pre-image of the diagonal subbundle $T_1(\Delta_{M\times M})$. Then define the map

$$\tilde{x} * \tilde{y} : \widetilde{\Delta} * \widetilde{\Delta}' \longrightarrow M \times M$$

$$(\tilde{x} * \tilde{y})(p,s,q)(t) := \begin{cases} \tilde{x}(p,t) & 0 \le t \le t_p \\ \tilde{y}_s(q, t - t_p) & t_p \le t \le t_p + t_q. \end{cases}$$

It can then be shown that

$$\partial(\tilde{x} * \tilde{y}) = \partial(\tilde{x}) * \tilde{y} + (-1)^{|x|+1} \tilde{x} * \partial \tilde{y} + (-1)^{|x|} \left(\succ(\tilde{x}, \tilde{y}) - (-1)^{|x||y|} (\succ(\tilde{y}, \tilde{x}) \right).$$

This implies commutativity at the level of homology. Since intersection of chains and concatenation of loops are both strictly associative, the associativity follows. □

In analogy with usual string topology, one can introduce the BV operator on $\mathcal{I}_1(S^1, M)$ arising via the circle action.

Definition 2.21. For any $s \in [0,1]$ and $\gamma \in \mathcal{I}_1(S^1, M)$ we define the *rotation by* s of γ by

$$\gamma_s : [0, t_0] \to M,\ \ \gamma_s(t) = \gamma(st_0 + t).$$

This defines a map

(2.2.4) $$\circlearrowleft : S^1 \times \mathcal{I}_1(S^1, M) \longrightarrow \mathcal{I}_1(S^1, M),\ \ \Delta(s, \gamma) = \gamma_s.$$

It is clear from the definition that the algebra isomorphism Φ_* is compatible with the BV operators.

Proposition 2.22. *For any Riemannian closed oriented manifold* (M, g) *of dimension* $d \geq 2$*, the map* $\iota^{-1} \circ \Phi : \mathcal{I}_1(S^1, M) \to L(T_1 M)$ *induces an isomorphism of BV algebras*

$$\iota_*^{-1} \circ \Phi_* : \big(H_*(\mathcal{I}_1(S^1, M)), \succ, \oslash\big) \xrightarrow{\cong} (H_*(L(T_1 M)), \bullet, \Delta)\,.$$

In particular, for any parallelizable manifold M*, this isomorphism is of the form*

$$\iota_*^{-1} \circ \Phi_* : \big(H_*(\mathcal{I}_1(S^1, M)), \succ, \oslash\big) \xrightarrow{\cong} \big(H_*(LM) \otimes H_*(LS^{d-1}), \bullet \otimes \bullet, \Delta \otimes \Delta\big)\,.$$

One can also define a similar product, again denoted by $\succ$, on $\mathrm{Imm}(S^1, M)$. We need the space of *figure eight* immersions.

Definition 2.23. We define $\mathrm{Imm}(8, M)$ to be the space of immersions of *figure eight* into M, where *figure eight* stands for a wedge of two smooth circles of unit radius in $\mathbb{R}^2$ touching each tangentially at exactly one point.

We have a fibre bundle

$$\psi : \mathrm{Imm}(S^1, M) \times \mathrm{Imm}(S^1, M) \longrightarrow T_\uparrow M \times T_\uparrow M$$

defined by

$$\psi(\gamma_1, \gamma_2) := \big(\gamma_1^{'}(1/2), \gamma_2^{'}(0)\big).$$

The diagonal $T_\uparrow M$ is a submanifold of $T_\uparrow M \times T_\uparrow M$ of codimension $2d$. The pullback bundle over $T_\uparrow M$ is exactly $\mathrm{Imm}(8, M)$, which is actually an embedded submanifold of $\mathrm{Imm}(S^1, M) \times \mathrm{Imm}(S^1, M)$ of codimension $2d$. Let N denote the normal bundle of this submanifold. The natural inclusion map

$$(\mathrm{Imm}(S^1, M) \times \mathrm{Imm}(S^1, M)) \setminus N \hookrightarrow \mathrm{Imm}(S^1, M) \times \mathrm{Imm}(S^1, M)$$

induces a map

$$H_*(\mathrm{Imm}(S^1, M) \times \mathrm{Imm}(S^1, M)) \xrightarrow{\pi} H_*(T(N)),$$

where $T(N)$ is the Thom complex of N. There is also the Thom isomorphism

$$H_*(T(N)) \xrightarrow{\cap u} H_{*-2d}(\mathrm{Imm}(8, M)),$$

where u is the Thom class. There is also the natural loop composition map $\mathrm{Imm}(8, M) \to \mathrm{Imm}(S^1, M)$ which induces

$$\gamma_* : H_*(\mathrm{Imm}(8, M)) \longrightarrow H_*(\mathrm{Imm}(S^1, M)).$$

Definition 2.24. (Product on immersed loops parametrized by $[0,1]$) The product on $\mathrm{Imm}(S^1, M)$, of degree $-2d$,

$$\circleddash : H_*(\mathrm{Imm}(S^1, M)) \otimes H_*(\mathrm{Imm}(S^1, M)) \longrightarrow H_*(\mathrm{Imm}(S^1, M)) \tag{2.2.5}$$

is defined by composition $\gamma_* \circ (\cap u) \circ \pi$.

It's rather odd to have two algebra structures on a space of degrees varying by one. In our case, $H_*(\mathcal{I}(S^1, M))$ admits two such products, one each from pulling back the products on $\mathrm{Imm}(S^1, M)$ and $\mathcal{I}_1(S^1, M)$ respectively. As it turns out, the first product $\circleddash$ is zero. Since

$$\Phi : \mathrm{Imm}(S^1, M) \longrightarrow L(T_\uparrow M)$$

induces an isomorphism which respects products, the claim follows once we show that the loop product on $L(T_\uparrow M)$ is zero on homology. Let x_1, x_2 be two chains in $L(T_\uparrow M)$ given by

$$x_j : \Delta^{i_j} \times S^1 \to T_1 M \times (0, \infty),\ j = 1, 2.$$

Since the domain is compact, by continuity, there exists $0 < t_1 < t_2$ such that the image of x_j is contained in $T_1 M \times (t_1, t_2)$. Then one may homotope x_2, via translation by t_2 in the $(0, \infty)$-direction, to get a new chain $\tilde{x}_2$ which has no intersection with x_1. In general, given any two chains x and y, one can homotope y such that the two chains don't intersect. This implies that

$$\bullet : H_*(L(T_\uparrow M)) \otimes H_*(L(T_\uparrow M)) \longrightarrow H_*(L(T_\uparrow M))$$

is the zero map. Therefore, $\circleddash$ is also trivial.

2.2.3 A few examples

We shall mainly deal with oriented surfaces and manifolds with monogenic cohomology ring. In general for any oriented closed Riemannian manifold M of dimension d we have the associated sphere bundle

$$S^{d-1} \hookrightarrow T_1 M \longrightarrow M.$$

The primary obstruction for finding a section is given by $\chi(M)[M] \in H^d(M; \mathbb{Z})$, where $[M]$ denotes the normalized volume form on M. The Gysin sequence (actually the degeneration of the Serre spectral sequence) implies

$$H_i(T_1 M; \mathbb{Z}) = \begin{cases} H_i(M; \mathbb{Z}), & \text{if } i < d-1 \\ H_{i-d+1}(M; \mathbb{Z}), & \text{if } i \geq d+1 \end{cases} \tag{2.2.6}$$

If M is simply connected or more generally, if $H_1(M;\mathbb{Z}) = 0$ then there are two cases :
(1) If $\chi = 0$ then there are isomorphisms

$$H_d(T_1M;\mathbb{Z}) \cong H_d(M;\mathbb{Z}), \ \ H_{d-1}(T_1M;\mathbb{Z}) \cong H_0(M;\mathbb{Z}).$$

(2) If $\chi \neq 0$ then there are isomorphisms

$$H_d(T_1M;\mathbb{Z}) = 0, \ \ H_{d-1}(T_1M;\mathbb{Z}) = \mathbb{Z}_\chi.$$

We're now ready to discuss some examples.

Example 2.25. (Odd spheres)

Let $d = 2k+1 \geq 3$ be an odd integer and we'll consider S^d as a Riemannian manifold. It follows from the previous discussion that $H_i(T_1S^d;\mathbb{Z})$ has no torsion and is a $\mathbb{Z}$ exactly when $i = 0, d-1, d, 2d-1$. When $d = 3$ it's clear that $T_1S^3 = S^2 \times S^3$. For $d > 3$ it follows from the Gysin sequence that T_1S^d has the same integral cohomology ring as $S^d \times S^{d-1}$. It also follows from the long exact sequence of homotopy groups that the first non-trivial homotopy group of T_1S^d is π_{d-1} and

$$\pi_{d-1}(T_1S^d) = \mathbb{Z}, \ \pi_d(T_1S^d) = \mathbb{Z} \oplus \mathbb{Z}_2.$$

Using the free generators α, β of π_{d-1}, π_d respectively we can construct a map

$$\alpha \vee \beta : S^{d-1} \vee S^d \to T_1S^d$$

which induces an isomorphism on π_i and H_i for all $i \leq d$. Replacing $\alpha \vee \beta$ by its mapping cylinder, we may assume that $\alpha \vee \beta$ is an inclusion, whence

$$H_{2d-1}(T_1S^d, S^{d-1} \vee S^d;\mathbb{Z}) \cong H_{2d-1}(T_1S^d) \cong \mathbb{Z}.$$

This means, in particular, that $\Omega(T_1S^d)$ and $L(T_1S^d)$ both behave like $\Omega S^d \times \Omega S^{d-1}$ and $LS^d \times LS^{d-1}$ respectively. By the very definition of the loop product, defined using the Pontrjagin product on ΩX and the intersection product on X, we conclude that

$$\text{(2.2.7)} \qquad \left(\mathbb{H}_*(L(T_1S^d);\mathbb{Z}), \bullet\right) \cong \left(\mathbb{H}_*(LS^d;\mathbb{Z}) \otimes \mathbb{H}_*(LS^{d-1};\mathbb{Z}), \bullet \otimes \bullet\right).$$

It's good to keep in mind that this isomorphism is abstract and except for S^3 and S^7 is not induced from a map at the level of spaces.

Example 2.26. (Even spheres)

Let $d = 2k$ be an even integer and we'll consider the Riemannian sphere S^d. It follows from the discussion at the outset that $H_i(T_1S^d;\mathbb{Z})$ only has a $\mathbb{Z}_2$ when $i = d-1$ and is a $\mathbb{Z}$ exactly when $i = 0, 2d-1$. In fact, if we write out the cohomology Serre spectral sequence for the fibration $S^{d-1} \hookrightarrow T_1S^d \to S^d$ then we see that T_1S^d has the same rational cohomology ring as S^{2d-1}. This means, in particular, that $\Omega(T_1S^d)$ and $L(T_1S^d)$ both behave rationally like ΩS^{2d-1} and LS^{2d-1} respectively. Therefore, we conclude that

$$\left(\mathbb{H}_*(L(T_1S^d);\mathbb{Q}),\bullet\right) \cong \left(\mathbb{H}_*(LS^{2d-1};\mathbb{Q}),\bullet\right). \tag{2.2.8}$$

Remark 2.27. *The isomorphisms* (2.2.7), (2.2.8) *are with rational coefficients. D. Chataur and J.-F. Le Borgne* [6] *show that these isomorphisms can be deduced from their version with $\mathbb{Z}$-coefficients. They apply a spectral sequence approach combined with the version of the loop product defined in* [18].

Based on the previous examples, it is tempting to conjecture that there is a isomorphism of algebras

$$(\mathbb{H}_*(L(T_1\mathbb{CP}^n);\mathbb{Q}),\bullet) \cong \left(\mathbb{H}_*(L\mathbb{CP}^{n-1};\mathbb{Q}) \otimes \mathbb{H}_*(LS^{2n+1};\mathbb{Q}),\bullet\otimes\bullet\right).$$

This seems highly likely to be true. The homology of the unit tangent bundle $T_1\mathbb{CP}^n$ is the same (mod $(n+1)$-torsion) as $\mathbb{CP}^{n-1}\times S^{2n+1}$. However, the author doesn't know of a proof as of this writing.

Example 2.28. (Surfaces)

Let Σ be a closed, oriented Riemann surface of genus $g > 1$. We shall consider $\mathrm{Imm}(S^1,\Sigma) \cong L(T_1\Sigma)$. Notice that $T_1\Sigma$ is a principal S^1-bundle over Σ and hence has the fundamental group as its only non-trivial homotopy group. The following is well known :

Lemma 2.29. *The fundamental group of E, the total space of a principal S^1-bundle over Σ, is a central extension of $\pi_1(\Sigma)$ by $\mathbb{Z}$ and every such central extension arises as the fundamental group of a circle bundle over Σ.*

Proof There are a lot of ways of seeing this. For example, considering E as a Seifert manifold, one can use the presentation given in [21]. If we fix an orientation of Σ, i.e., decide on a generator of $H^2(\Sigma;\mathbb{Z})$, then we have

$$\pi_1(E) \cong \{a_1,\ldots,a_g,b_1,\ldots,b_g,h \,|\, [a_i,h] = 1 = [b_i,h], [a_1,b_1]\cdots[a_g,b_g] = h^n\}$$

where n is the degree of the bundle. A more direct approach is to consider

$$\pi_1(\Sigma) \cong \{a_1, \ldots, a_g, b_1, \ldots, b_g \,|\, [a_1, b_1] \cdots [a_g, b_g] = 1\}$$

where $[a_i, b_i]$ is the commutator. Using homotopy lifting and local triviality, lift the generators a_i, b_i's to generators $\tilde{a}_i, \tilde{b}_i$'s in $\pi_1(E)$. One can notice that the fibre over the base point generates $\mathbb{Z} = \langle h \rangle$. It can be checked that

$$\tilde{a}_i h \tilde{a}_i^{-1} = h, \;\; \tilde{b}_i h \tilde{b}_i^{-1} = h$$

essentially because the an oriented circle over a circle (a_i or b_i) is trivial. Using the homotopy lifting property again, we can find a lift of the homotopy between $[a_1, b_1] \cdots [a_g, b_g]$ and 1. This lift is a free homotopy

$$[\tilde{a}_1, \tilde{b}_1] \cdots [\tilde{a}_g, \tilde{b}_g] \sim h^n.$$

In other words, the two loops are homotopic (fixing based point) via conjugation by a power of h, i.e.,

$$[\tilde{a}_1, \tilde{b}_1] \cdots [\tilde{a}_g, \tilde{b}_g] = h^n$$

and it can be verified that n is the degree of the line bundle. Similarly, it can be deduced that every central extension of $\pi_1(\Sigma)$ by $\mathbb{Z}$ has such a presentation. $\square$

One can now describe the loop product and the BV operator on $L(T_1\Sigma)$ as follows and we only sketch an outline here. Let $G := \pi_1(T_1\Sigma)$ be the central extension corresponding to $n = 2 - 2g$ where g is the genus of Σ. Let us consider the group ring $\mathbb{Q}[G]$. The Hochschild homology of $\mathbb{Q}[G]$ with its cup product and the Connes b-operator provide a model (as shown by D. Vaintrob [37]) for the loop product and the BV operator. The case for $L\Sigma$ will be taken up in §2.3.1 and §2.3.2.

2.3 String Topology of Surfaces

We shall be dealing with oriented surfaces of non-zero genus. We treat the case of the torus separately, owing to the fact that it is a Lie group. The surfaces of positive genus follow a general pattern. For any space M we have a splitting

$$\pi_i(LM) \cong \pi_i(M) \oplus \pi_i(\Omega M) \cong \pi_i(M) \oplus \pi_{i+1}(M).$$

Now let M be any oriented surface, which is also a model for $K(\pi_1(M), 1)$. This means $\pi_1(LM) = \pi_1(M)$ and $\pi_2(LM) = 0$. Therefore, $\iota : M \to LM$ induces a homotopy equivalence between M and the component L^0M of LM, consisting of contractible loops. The free loop space LM decomposes into a disjoint union of $L^\alpha M$, components containing loops freely homotopic to α, i.e., the components of LM are indexed by the free homotopy classes of elements in $\pi_1(M)$. The circle acts on components of LM, mapping $L^\alpha M$ to itself.

Notation *We shall adopt the convention that* $L(0;\infty) = \mathbb{CP}^\infty$, $L(1;\infty) := S^\infty$ *and more generally,* $L(m;\infty)$ *is the infinite lens space with fundamental group* $\mathbb{Z}_m$*. We also assume that* $(m, 0) = m$ *and* $(0, 0) = 0$*.*

2.3.1 The torus

Since $G = S^1 \times S^1$ is a Lie group, the loop space fibration splits as a direct product $LG \cong \Omega G \times G$ and is also the disjoint union of components $L^\alpha G$ as α ranges over $\pi_1(G)$. It follows form the definitions that

$$\mathbb{H}_*(LG) = \mathbb{H}_*(LS^1) \otimes \mathbb{H}_*(LS^1).$$

Then using Proposition 2.14, we conclude that

$$\begin{aligned}
\mathbb{H}_{-2}(LG) &= H_0(LG) = \oplus_{\alpha\in\pi_1(G)}\mathbb{Z}1_\alpha \\
\mathbb{H}_{-1}(LG) &= H_1(LG) = \oplus_{\alpha\in\pi_1(G)}(\mathbb{Z}x_\alpha \oplus \mathbb{Z}y_\alpha) \\
\mathbb{H}_0(LG) &= H_2(LG) = \oplus_{\alpha\in\pi_1(G)}\mathbb{Z}z_\alpha,
\end{aligned}$$

where 1_α is the point $\alpha \in L^\alpha G$, x_α means the 1-dimensional family of loops traced out by changing the starting point of α along x, the meridian circle. Similarly, y_α means the 1-dimensional family got by tracing α along the longitudinal circle y and z_α is the 2-dimensional family that arises from changing the base point of α all over the torus.

For dimension reasons there are only four possible loop products up to commutativity. The map

$$\bullet : \mathbb{H}_0(LG) \otimes \mathbb{H}_0(LG) \longrightarrow \mathbb{H}_0(LG)$$

sends $z_\alpha \otimes z_\beta$ to $z_{\alpha\cdot\beta}$. The map (and its symmetrization)

$$\bullet : \mathbb{H}_{-1}(LG) \otimes \mathbb{H}_0(LG) \longrightarrow \mathbb{H}_{-1}(LG)$$

sends $x_\alpha \otimes z_\beta$ to $x_{\alpha\cdot\beta}$ and $y_\alpha \otimes z_\beta$ to $y_{\alpha\cdot\beta}$. The map

$$\bullet : \mathbb{H}_{-2}(LG) \otimes \mathbb{H}_0(LG) \longrightarrow \mathbb{H}_{-2}(LG)$$

sends $1_\alpha \otimes z_\beta$ to $1_{\alpha\cdot\beta}$. The map

$$\bullet : \mathbb{H}_{-1}(LG) \otimes \mathbb{H}_{-1}(LG) \longrightarrow \mathbb{H}_{-2}(LG)$$

sends $x_\alpha \otimes y_\beta$ to $1_{\alpha\cdot\beta}$, $x_\alpha \otimes x_\beta$ and $y_\alpha \otimes y_\beta$ to 0.

Since $\pi_1(G) = \mathbb{Z} \oplus \mathbb{Z}$ we may denote any loop α, up to homotopy, by $x^a y^b$, where x denotes the meridian circle and y denotes the longitudinal circle. Then it is easy to see geometrically or using (3.2.1) that

$$\Delta(1_\alpha) = bx_\alpha + ay_\alpha, \;\; \Delta(x_\alpha) = az_\alpha, \;\; \Delta(y_\alpha) = -bz_\alpha, \;\; \Delta(z_\alpha) = 0. \tag{2.3.1}$$

Let α, β denote the homotopy class of the loops $x^a y^b, x^c y^d$ respectively. Then one can easily compute the loop bracket to be the following :

$$\begin{aligned}
\{1_\alpha, 1_\beta\} &= 0 \\
\{1_\alpha, z_\beta\} &= dx_{\alpha\beta} + cy_{\alpha\beta} \\
\{1_\alpha, x_\beta\} &= (a-c)1_{\alpha\beta} \\
\{1_\alpha, y_\beta\} &= (d-b)1_{\alpha\beta} \\
\{x_\alpha, z_\beta\} &= -cz_{\alpha\beta} \\
\{y_\alpha, z_\beta\} &= dz_{\alpha\beta} \\
\{y_\alpha, y_\beta\} &= (d-b)y_{\alpha\beta} \\
\{x_\alpha, x_\beta\} &= (a-c)x_{\alpha\beta} \\
\{x_\alpha, y_\beta\} &= -bx_{\alpha\beta} - cy_{\alpha\beta} \\
\{z_\alpha, z_\beta\} &= 0.
\end{aligned}$$

The loop bracket has degree 1 and endows $\mathbb{H}_*(LG)$ with a Lie algebra structure. The operator Δ turns $\mathbb{H}_*(LG)$ into a BV algebra.

We shall denote the constant loop at $g \in G$ by γ_g. Define a map

$$\varphi_\alpha : L^0G \longrightarrow L^\alpha G, \;\; \varphi_\alpha(\gamma)(t) := \alpha(t)\gamma(t).$$

This map is a homeomorphism, whence $L^\alpha G$ is homotopy equivalent to G. The map $\iota : G \hookrightarrow L^0G$ is S^1-equivariant, inducing an isomorphism on equivariant homology. Therefore, we conclude that $\mathcal{H}_*(L^0G) = H_*^{S^1}(G)$. Since the action of S^1 on G is trivial, $G \times_{S^1} ES^1 = G \times \mathbb{CP}^\infty$ and

$$\mathcal{H}_*(L^0G) = H_*(G) \otimes H_*(\mathbb{CP}^\infty). \tag{2.3.2}$$

For $\alpha \in \pi_1(G) \setminus \{1\}$, let αG denote the space of loops $\alpha\gamma_g, g \in G$. This space is clearly homeomorphic to G. The natural inclusion

$$\alpha G \overset{\iota}{\hookrightarrow} L^\alpha G.$$

is a homotopy equivalence and ι is also S^1-equivariant. This can be seen by choosing $\alpha = x^m y^n$, where x is the meridian circle and y is the longitudinal circle. Since any element of the fundamental group can be homotoped to one of such α's as chosen, we may as well work with such representative elements. If α is irreducible (equivalently $(m,n) = 1$) then the circle acts freely on it. When α is not irreducible (equivalently $(m,n) > 1$) then the action of S^1 is typically not free. Let Fix(α) denote the elements of S^1 which fix α. It can be shown (refer to the discussion before Proposition 2.32) that Fix$(\alpha) = \{k/(m,n) \bmod 1 \mid k \in \mathbb{Z}\}$. Therefore, the action of S^1 on $\alpha G \times ES^1$ is given by

$$(\alpha\gamma_g, \underline{x}) \xrightarrow{\theta} (\alpha\gamma_{(e^{2\pi i m\theta}, e^{2\pi i n\theta})g}, e^{2\pi i\theta}\underline{x}).$$

If $\alpha \neq 0$ then at least one of the integers m or n is non-zero. We may assume that $n \neq 0$. Then we have the identifications

$$\begin{array}{rcl} (\alpha\gamma_{(w_1,w_2)}, \underline{x}) & \sim & (\alpha\gamma_{(w_1 e^{-2\pi i m\theta},1)}, e^{-2\pi i\theta}\underline{x}) \\ (\alpha\gamma_{(w,1)}, \underline{x}) & \sim & (\alpha\gamma_{(we^{2\pi i km/n},1)}, e^{2\pi i k/n}\underline{x}). \end{array}$$

where $k \in \mathbb{Z}$ and $w_2 = e^{2\pi i n\theta}$. If $m = 0$ then $\alpha \sim x^0 y^n$ and

$$\alpha G \times_{S^1} ES^1 = S^1 \times L(n; \ldots).$$

If $m \neq 0$ then write $m = m'(m,n), n = n'(m,n)$ and $1 = n'l + m'k$. We then have the identifications

$$(\alpha\gamma_{(w,1)}, \underline{x}) \sim (\alpha\gamma_{(we^{2\pi ij/n'},1)}, e^{2\pi ikj/n}\underline{x}),$$

where k is as chosen above by the Euclid's algorithm. Since the identifications of the circle is by the n' roots of unity and k is coprime to n', the action on ES^1 is actually by the group of (m,n)th roots of unity, whence

$$\alpha G \times_{S^1} ES^1 = S^1 \times L((m,n); \ldots).$$

Therefore, it follows that if $\alpha \sim x^m y^n \neq 0$ then

$$H_*^{S^1}(L^\alpha G) \cong H_*(S^1) \otimes H_*(L((m,n); \ldots)).$$

Recall that the rational homology of the lens spaces is concentrated in degree zero. Since the loop space LG is the union of components $L^\alpha G$ with α running over elements of $\pi_1(G)$, the rational equivariant homology can be written as :

(2.3.3) $$\mathcal{H}_*^{\mathbb{Q}}(LG) = H_*(G) \otimes H_*(\mathbb{CP}^\infty) \oplus \left(\oplus_{\mathbb{Z}^2 \setminus \{(0,0)\}} H_*(S^1)\right).$$

Even over the integers this is non-zero in every dimension since the homology of $L(m; \ldots)$ is just $\mathbb{Z}_m$ in every odd dimension.

The string bracket is possibly non-zero only between $\mathcal{H}_0^{\mathbb{Q}}$ and itself, $\mathcal{H}_1^{\mathbb{Q}}$ and itself and between $\mathcal{H}_1^{\mathbb{Q}}$ and $\mathcal{H}_0^{\mathbb{Q}}$. The bracket

$$[\ ,\ ] : \mathcal{H}_0^{\mathbb{Q}}(LG) \times \mathcal{H}_0^{\mathbb{Q}}(LG) \longrightarrow \mathcal{H}_0^{\mathbb{Q}}(LG)$$

is the famous *Goldman bracket*. It provides a Lie algebra structure on $\mathcal{H}_0^{\mathbb{Q}}(LG)$, the vector space of free homotopy classes of closed curves. More precisely, $\mathcal{H}_0(LG)$ is freely generated by elements of the form $x^m y^n$ where x is the meridian circle and y is the longitudinal circle. Observe that $\mathcal{M}(\alpha) = ax_\alpha + by_\alpha$, where $\alpha \sim x^a y^b$. Then let $\alpha = x^a y^b, \beta = x^c y^d$ and the Lie bracket is

(2.3.4) $$[\alpha, \beta] := \mathcal{E}((ax_\alpha + by_\alpha) \bullet (cx_\beta + dy_\beta)) = \mathcal{E}((ad - bc)1_{\alpha\cdot\beta}) = (ad - bc)\alpha\cdot\beta$$

and the bracket between any two elements is extended by bilinearity.

The bracket

$$[\ ,\ ] : \mathcal{H}_1^{\mathbb{Q}}(LG) \times \mathcal{H}_0^{\mathbb{Q}}(LG) \longrightarrow \mathcal{H}_1^{\mathbb{Q}}(LG)$$

can be calculated explicitly. For $\alpha = x^a y^b$, a non-constant loop, let $Z_\alpha \in \mathcal{H}_1^{\mathbb{Q}}(L^\alpha G) = \mathbb{Q}$ denote the generator given by $w \mapsto (wz^a, wz^b)$ and let $\beta = x^c y^d \in \mathcal{H}_0^{\mathbb{Q}}$. Since $\mathcal{M}(Z_\alpha) = z_\alpha$, $\mathcal{E}(x_\alpha) = bZ_\alpha$ and $\mathcal{E}(y_\alpha) = -aZ_\alpha$ we conclude that

$$[Z_\alpha, \beta] = -\mathcal{E}(z_\alpha \bullet (cx_\beta + dy_\beta)) = -\mathcal{E}(cx_{\alpha\beta} + dy_{\alpha\beta}) = (ad - bc)Z_{\alpha\beta}. \tag{2.3.5}$$

Since $\mathcal{M}(X) = 0$ for any element $X \in \mathcal{H}_1^{\mathbb{Q}}(L^0 G) = \mathbb{Q} \otimes \mathbb{Q}$ for dimension reasons, we have $[X, \beta] = 0$. It can be verified similarly that the bracket

$$[\ ,\ ] : \mathcal{H}_1^{\mathbb{Q}}(LG) \times \mathcal{H}_1^{\mathbb{Q}}(LG) \longrightarrow \mathcal{H}_2^{\mathbb{Q}}(LG) = \mathbb{Q}z_0$$

for elements $\alpha, \beta \neq 0$ is given by

$$[Z_\alpha, Z_\beta] = \begin{cases} 0 & \text{if } \alpha\beta \neq 0, \\ -z_0 & \text{if } \alpha\beta = 0. \end{cases} \tag{2.3.6}$$

And for $X \in \mathcal{H}_1^{\mathbb{Q}}(L^0 G)$ we have $[X,\ ] = 0$. The higher brackets $\overline{m}_k, k \geq 3$ can now be calculated easily and is left to the interested reader.

2.3.2 Surfaces of non-zero genus

We shall denote by M an oriented surface of genus at least 1 with possible boundary. If M has genus g and $k \geq 1$ boundary components then it is is homotopy equivalent to a bouquet of $2g + k - 1$ circles, whence its homology is torsion free. This also holds true when M has no boundary. As observed at the beginning of §2.3, M is homotopy equivalent to $L^0 M$ via the canonical inclusion map. Since this map is S^1-invariant, it induces an isomorphism

$$\mathcal{H}_*(L^0 M) \cong H_*^{S^1}(M).$$

The circle action on M is trivial, whence

$$H_*^{S^1}(M) = H_*(M \times \mathbb{CP}^\infty) = H_*(M) \otimes H_*(\mathbb{CP}^\infty)$$

by the Künneth formula and the fact that $H_*(M)$ is torsion free. We claim that each of the other components $L^\alpha M$ of LM are homotopy equivalent to S^1. Observe that elements of $\pi_1(L^\alpha M, \alpha)$ can be interpreted as maps of the form $\tilde{f} : S^1 \times S^1 \to M$ where $S^1 \times \{1\}$ maps to α. Such a map induces $\tilde{f}_* : \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(M)$, sending $(1, 0)$ to $[\alpha]$. To conclude that $\pi_1(M)$ has no torsion, we recall the following well known result :

Proposition 2.30. *Let M be a $K(\pi, 1)$ space. If M is a finite dimensional CW complex then $\pi = \pi_1(M)$ has no torsion.*

Proof Choose a subgroup $\mathbb{Z}_m$ of $\pi_1(M)$ if possible. The covering space of M corresponding to this subgroup has the homotopy type of the infinite dimensional lens space $L(m; \ldots)$, which has non-zero homology in every odd dimension. However, any cover of M is also finite dimensional, whence its homology vanishes beyond the dimension, a contradiction. $\square$

Using this we may conclude that the image under $\tilde{f}_*$ has rank 1 or 2. If it has rank 2 then $\tilde{f}_*$ is injective. This means the torus covers M, a contradiction. Therefore the image has rank 1. Writing $\tilde{f}_*((0,1)) = [\beta]$ we see that both $[\alpha]$ and $[\beta]$ are multiples of some irreducible loop $[\gamma]$, i.e., $[\gamma]$ cannot be written as a non-trivial power of some other loop. Now it is clear that $\pi_1(L^\alpha M) \cong \mathbb{Z}$ is generated by the twisting of α along γ. To prove our claim completely, we need to show that higher homotopy groups of $L^\alpha M$ vanish. Using the fibration $\Omega M \to LM \to M$ we see that ΩM is homotopy equivalent to a discrete space. This fibration splits into $(\Omega M)_\alpha \to L^\alpha M \to M$, which immediately implies that $L^\alpha M$ is homotopy equivalent to a covering space of M. Therefore, $L^\alpha M$ is a model for $K(\mathbb{Z}, 1)$ and homotopy equivalent to S^1.

Let us gather together the results on loop homology for convenience :

Proposition 2.31. *The loop homology of an oriented surface M of positive genus is supported only in dimensions $-2, -1, 0$. More precisely,*

$$\begin{aligned} \mathbb{H}_{-2}(LM) &= \mathbb{Z} \oplus (\oplus_{\alpha \in S} \mathbb{Z}) \\ \mathbb{H}_{-1}(LM) &= H_1(M) \oplus (\oplus_{\alpha \in S} \mathbb{Z} u_\alpha) \\ \mathbb{H}_0(LM) &= \mathbb{Z}. \end{aligned}$$

Here S denotes the set of free homotopy classes of non-trivial closed curves on M and u_α the generator of $H_1(L^\alpha M) \cong \mathbb{Z}$.

The loop product

$$\bullet : \mathbb{H}_0(LM) \times \mathbb{H}_0(LM) \longrightarrow \mathbb{H}_0(LM)$$

is just the map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ sending $(1,1) \to 1$ and extended by bimultiplicativity. The second loop product

$$\bullet : \mathbb{H}_{-1}(LM) \times \mathbb{H}_{-1}(LM) \longrightarrow \mathbb{H}_{-2}(LM)$$

is defined for elements $(u_\alpha \bullet u_\beta) = [\alpha, \beta]$, with the markings. Notice that erasing the marked points give us the Goldman bracket $[\,,\,] : \mathcal{H}_0 \otimes \mathcal{H}_0 \to \mathcal{H}_0$. Let $\gamma \in H_1(M)$, thought of as a loop in L^0M, i.e., $\gamma(t)$ is the constant loop $\gamma_{\gamma(t)}$. Define $\gamma \bullet u_\alpha = [\gamma, \alpha]$ with the marked points. The remaining two loop products

$$\bullet : \mathbb{H}_{-1}(LM) \times \mathbb{H}_0(LM) \longrightarrow \mathbb{H}_{-1}(LM)$$

maps $(u_\alpha, 1) \to u_\alpha$ and $(\gamma, 1) \to \gamma$ and

$$\bullet : \mathbb{H}_{-2}(LM) \times \mathbb{H}_0(LM) \longrightarrow \mathbb{H}_{-2}(LM)$$

maps $(1_\alpha, 1_0)$ to 1_α. All these maps are defined on the generators and extended by bimultiplicativity.

Towards computing the equivariant homology of LM, we need to find $L^\alpha M \times_{S^1} ES^1$. Denote by S^1_α, the image of α in $L^\alpha M$ when twisted along γ, where $\alpha = \gamma^n$. Then $\iota : S^1_\alpha \hookrightarrow L^\alpha M$ is a homotopy equivalence and a S^1-equivariant map, which induces an isomorphism of equivariant homology. Let

$$\mathrm{Fix}(\alpha) = \{\theta \in S^1 = [0,1]/0 \sim 1 \,|\, \alpha(t+\theta) = \alpha(t)\}$$

be the elements of S^1 which fix α. If this set contains an irrational number θ_0 then $\{m\theta_0\}$ is dense in S^1. This would imply, in conjunction with the continuity of α, that α is a constant. If this set doesn't have a minimum then there exists a sequence $\{x_n\} \subset \mathrm{Fix}(\alpha)$ of rational numbers converging towards 0. At this point, we choose α to be represented by a non-trivial closed curve with finitely many double points. Then $\alpha(t + x_n) = \alpha(t)$ would mean any neighbourhood of $\alpha(t)$ (for any t) is visited infinitely many often unless the curve is constant through $\alpha(t)$ to $\alpha(t + x_1)$ for any t. Again, by reasons of continuity, this invalidates the non-triviality of α. Therefore, the set $\mathrm{Fix}(\alpha)$ has a minimum and it equals $1/n$. As a consequence of Euclid's algorithm $\mathrm{Fix}(\alpha) = \{k/n \bmod 1 \,|\, k \in \mathbb{Z}\}$ whenever $n \neq 1$. If $\alpha = \gamma$ is irreducible then $\mathrm{Fix}(\alpha) = \{0\}$.

Recall that $ES^1 = S^\infty$ where S^1 acts naturally by multiplication. The action of S^1 on S^1_α is given by multiplication by z^n. Therefore, the diagonal action of S^1 on $S^1_\alpha \times S^\infty$ is given by $(w, \underline{x}) \to (z^n w, z\underline{x})$. In particular, we first identify $(w, \underline{x})$ with $(1, \underline{x'})$ and then identify $(1, \underline{x'})$ with $(1, e^{\frac{2\pi i}{n}} \underline{x'})$. Equivalently, we are taking the quotient of S^∞ by the action of $e^{\frac{2\pi i}{n}}$, which just produces the infinite dimensional lens space $L(n; \ldots)$ if $n \geq 2$. For $n = 1$ we get S^∞. Let us recall some known facts about such spaces. We refer the interested reader to pages 144-146 of [19] for a detailed discussion.

Proposition 2.32. *The infinite dimensional lens spaces*

$$L(n;\infty) := L(n; l_1, l_2, \ldots)$$

for $n \geq 2$ is defined as the union of an increasing sequence of finite dimensional lens spaces $L(n; l_1, \ldots, l_n)$ and assigned the direct limit topology. It can also be viewed as the quotient of S^∞ by $\mathbb{Z}_n$ acting via multiplication by the complex n^{th} roots of unity. It is a model for $K(\mathbb{Z}_n, 1)$ and its homotopy type doesn't depend on the l_i's. It can be given a CW complex structure with one cell in each dimension, i.e., the cellular chain complex consists of a $\mathbb{Z}$ in each dimension and the boundary maps alternate between 0 and n. Consequently, its homology is $\mathbb{Z}_n$ in each odd dimension and zero otherwise.

As a consequence of this result, when we are calculating rational equivariant homology then all the higher homology vanishes. Therefore, if we ignore the component L^0M then all the equivariant homology is concentrated in degree 0, i.e., on free homotopy classes of non-trivial loops in M. The string bracket is then just the Goldman bracket.

If we write the equivariant homology of LM, we can see that the Goldman bracket $[\,,\,]$ and the loop bracket $\{\,,\,\}$ are related.

$$\mathcal{H}_0 \otimes \mathcal{H}_0 \xrightarrow{\mathcal{M}^{\otimes 2}} H_1 \otimes H_1 \xrightarrow{\bullet} H_0 \xrightarrow{\mathcal{E}} \mathcal{H}_0 \xrightarrow{\mathcal{M}} H_1$$

The composition of the first three arrows give us $[\,,\,]$ while the last three arrows give us $\Delta(a \bullet b) = -\{a, b\}$, where $a, b \in H_1$. Moreover, the loop bracket $\{a, b\}$ for $a, b \in H_0 \oplus H_1$ is totally determined by the BV operator Δ on H_0 and the loop product $\bullet$.

2.3.3 Lie bialgebra of curves on surfaces

We briefly recall some basic definitions. Let V be a vector space with $s : V^{\otimes 2} \to V^{\otimes 2}$ denoting the swapping map and

$$\omega : V^{\otimes 3} \longrightarrow V^{\otimes 3},\ v_1 \otimes v_2 \otimes v_3 \mapsto v_1 \otimes v_3 \otimes v_2.$$

A *Lie algebra* structure on V is a skew symmetric map $[\,,\,] : V \otimes V \to V$ satisfying the Jacobi identity. Algebraically, $[\,,\,] \circ s = -[\,,\,]$ and

$$[\,,\,](\mathrm{Id} \otimes [\,,\,])(\mathrm{Id} + \omega + \omega^2) = 0.$$

A *Lie coalgebra* structure on V is given by a co-skew symmetric map $\nu : V \to V^{\otimes 2}$ satisfying the coJacobi identity. Again, this means $s \circ \nu = -\nu$ and

$$(\mathrm{Id} + \omega + \omega^2)(\mathrm{Id} \otimes \nu)\nu = 0.$$

$(V, [\, , \,], \nu)$ is called a *Lie bialgebra* if V is both, a Lie algebra and a Lie coalgebra, and the *compatibility condition*

$$\nu[a, b] = [\nu(a), b] + [a, \nu(b)]$$

holds for any $a, b \in V$. Here $[a, b \otimes c] = -[b \otimes c, a] := [a, b] \otimes c + b \otimes [a, c]$. A Lie bialgebra $(V, [\, , \,], \nu)$ is called *involutive* if $[\, , \,] \circ \nu = 0$ on V.

In what follows we recapitulate the construction of a Lie bialgebra, due to Goldman and Turaev, associated to surfaces. This bialgebra is involutive as well. If we work over a slightly larger space, the space of multicurves, then we may construct an operator D out of an extended version of the Lie bracket and the cobracket. The four structural identities of the Goldman-Turaev bialgebra - Jacobi, coJacobi, compatibility and involutive - can then be encoded into a single equation $D^2 = 0$.

The Lie bracket and the cobracket

Let us first recall the construction of Goldman. Let M be an oriented surface with a prescribed orientation. Set V_M to be the vector space generated by free homotopy classes of closed curves in M. Let $a, b \in V_M$ and we may assume, without loss of generality, that the curves are in general position, i.e., each of a and b intersect transversely, has no triple points or higher with only finitely many double points $p_1, \ldots, p_n$. To each point p_i we assign the free homotopy class $\{a \cdot_{p_i} b\}$ of the loop that starts at p_i, runs around a and then around b. We also associate a sign $\varepsilon_{p_i}(a, b)$ with this class : if the orientation given by the branches of a and b coming out at p_i coincide with that of M then we assign $+$; otherwise assign $-$. Then the *Lie bracket* is given by

$$[a, b] := \sum_{p \in a \cap b} \varepsilon_p(a, b)\{a \cdot_p b\}.$$

We briefly sketch why this definition is well defined by showing it to be invariant under the fundamental moves sketched below. First, this is sufficient since the space $\mathcal{S}_M$, consisting of curves in general position with only double points, is open and dense in LM. Any two such curves, if homotopic, can be

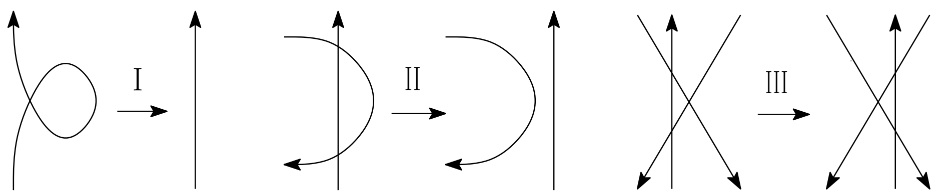

Figure 2.3: The three Reidemeister moves

homotoped via a finite sequence of the elementary moves. Apart from elements of $\mathcal{S}_M$, these moves only involve curves with cusps (move I), tangency (move II) or triple points (move III). Let γ be a path joining a and b in LM. The subspace of LM consisting of curves with tetra points has codimension 3, that of curves with points of valence n has codimension $n-1$. Since removing subspaces of codimension 2 or higher doesn't change connectivity, γ can be chosen to be as claimed.

As defined, $[\,,\,]$ is skew-symmetric. To verify Jacobi identity, take three curves a, b, c in general position. Now $[[a,b],c]$ is formal sum of closed curves written as

$$[[a,b],c] = \sum \varepsilon_{p'}(b \cdot_p a, c)\varepsilon_p(a,b)\{c \cdot_{p'} (b \cdot_p a)\},$$

where the sum runs over points $p \in a \cap b, p' \in c \cap (b \cdot_p a)$. Since no three curves intersect at a point, $c \cap (b \cdot_p a) = (c \cap b) \cup (c \cap a)$ and we conclude that

$$\begin{aligned}
[[a,b]\,,c] &= \underbrace{\sum_{p' \in c\cap a}\sum_{p\in b\cap a} \varepsilon_{p'}\varepsilon_p\{c \cdot_{p'} (b \cdot_p a)\}}_{I_1} + \underbrace{\sum_{p' \in c\cap b}\sum_{p\in b\cap a} \varepsilon_{p'}\varepsilon_p\{c \cdot_{p'} (b \cdot_p a)\}}_{J_1} \\
[[b,c]\,,a] &= \underbrace{\sum_{q' \in a\cap b}\sum_{q\in c\cap b} \varepsilon_{q'}\varepsilon_q\{a \cdot_{q'} (c \cdot_q b)\}}_{I_2} + \underbrace{\sum_{q' \in a\cap c}\sum_{q\in c\cap b} \varepsilon_{q'}\varepsilon_q\{a \cdot_{q'} (c \cdot_q b)\}}_{J_2} \\
[[c,a]\,,b] &= \underbrace{\sum_{r' \in b\cap c}\sum_{r\in a\cap c} \varepsilon_{r'}\varepsilon_r\{b \cdot_{r'} (a \cdot_r c)\}}_{I_3} + \underbrace{\sum_{r' \in b\cap a}\sum_{r\in a\cap c} \varepsilon_{r'}\varepsilon_r\{b \cdot_{r'} (a \cdot_r c)\}}_{J_3}.
\end{aligned}$$

Observe that for a typical curve in J_1, $\{c \cdot_{p'} (b \cdot_p a)\} = \{a \cdot_p (c \cdot_{p'} b)\}$, whence it also appears as a term in I_2. For such a pair of terms, the signs of the term in J_1 is $\varepsilon_{p'}(b \cdot_p a, c)\varepsilon_p(a,b)$ while the sign for the term in I_2 is $\varepsilon_{p'}(b,c)\varepsilon_p(c \cdot_{p'} b, a)$. We then have the following equalities due to skew symmetry and the relative positions of the loops :

$$\begin{aligned}\varepsilon_{p'}(b,c) &= \varepsilon_{p'}(b\cdot_p a,c)\\ \varepsilon_p(a,b) &= -\varepsilon_p(c\cdot_{p'} b,a).\end{aligned}$$

Therefore, J_1 cancels I_2 and in a similar fashion J_2 cancels I_3 and J_3 cancels I_1.

In the amazing paper of W. Goldman [17] it was proved that

Theorem 2.33. (Goldman)
If α is simple and β is any loop then α,β are freely homotopic to non-intersecting loops if and only if $[\{\alpha\},\{\beta\}]=0$.

The assumption that α is simple cannot be omitted. It is natural to ask to what extent the minimal intersection number measures the number of terms in the Goldman bracket. To that extent, M. Chas generalized (Main Theorem in [2]) Goldman's result.

Theorem 2.34. (Chas)
Let a and b be two free homotopy classes of closed curves on an orientable surface. If a can be represented by a simple closed curve then the number of terms, counted with multiplicity, of the Goldman Lie bracket is equal to the minimal number of intersection points of a and b.

Let V_0 denote the vector space generated by the class of the trivial loop $\{0\}$. For any element $\{\alpha\}\in V_M$, we can choose α in general position and choose the trivial loop representing $\{0\}$ to be disjoint from the image of α. Consequently, The Lie bracket of these two element vanishes, i.e., $[\{\alpha\},\{0\}]=0$. Since V_0 is in the center of the Lie algebra we may pass to the quotient space $\mathbb{V}(M):=V_M/V_0$, which becomes a Lie algebra. We denote the image of $\{\alpha\}$ in $\mathbb{V}$ by $\{\alpha\}_0$. We shall see that this is indeed the right space for the Lie coalgebra structure and hence for the Lie bialgebra structure.

Let us now recall the Lie cobracket construction of Turaev. Let α be a non-trivial loop in M in general position. Let $\#\alpha$ denote its (finite) set of double points. At each point $p\in\#\alpha$, there are two outgoing arcs of α. Label the corresponding loops α_p^1,α_p^2 so that the orientation induced by the ordered pair of the arcs of α_p^1 and α_p^2 (in that order) agree with the orientation of M. Up to a choice of parametrization $\alpha=\alpha_p^1\alpha_p^2$. We define

$$\nu(\{\alpha\})=\sum_{p\in\#\alpha}\{\alpha_p^1\}_0\otimes\{\alpha_p^2\}_0-\{\alpha_p^2\}_0\otimes\{\alpha_p^1\}_0. \tag{2.3.7}$$

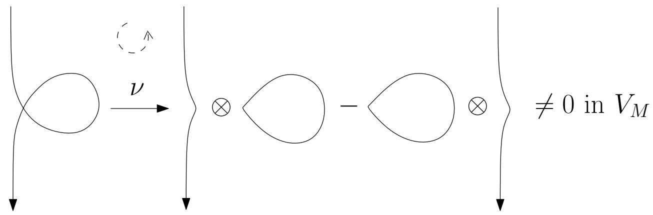

Figure 2.4: The effect of a type I move

This is not well defined on V_M since it's not invariant under a type I move (refer Figure 2.4). However, these two extra terms are zero if we work with $\mathbb{V}$. The following lemma, stated without proof, is useful towards proving the coJacobi identity for ν :

Lemma 2.35. *Let $\nu : V \to V \otimes V$ be a skew symmetric map. Since the image of ν lies in $\Lambda^2 V$, we extend ν, by Leibnitz rule, to a map $\nu : \Lambda^2 V \to \Lambda^3 V$. Then the coJacobi identity holds if and only if $\nu \circ \nu : V \to \Lambda^3 V$ is the zero map.*

We shall see in §2.3.3 (following Definition 2.39) that $\nu^2 = 0$, thereby proving coJacobi. It was proved (Theroem 8.3 in [36]) that :

Theorem 2.36. (Turaev)
The linear homomorphism $\nu : \mathbb{V} \to \mathbb{V} \otimes \mathbb{V}$ defined by (2.3.7) *on the generators is a Lie cobracket. The vector space $(\mathbb{V}, [\ ,\ ], \nu)$, equipped with the Goldman Lie bracket, is a Lie bialgebra.*

Observe that the compatibility condition can also be written as

$$\nu[a, b] = a \cdot \nu(b) - b \cdot \nu(a),$$

where $a \cdot (b \otimes c) = [a, b] \otimes c + b \otimes [a, c]$. This means ν can be thought of as a 1-cocycle of $\mathbb{V}$ with values in $\mathbb{V} \otimes \mathbb{V}$. For a simple loop α, $\nu\{\alpha\} = 0$. It follows that $\nu\{\alpha^n\} = 0$ as well since

$$\nu\{\alpha^n\} = \pm \left(\sum_{i=1}^{n-1} \{\alpha^i\}_0 \otimes \{\alpha^{n-i}\}_0 - \{\alpha^{n-i}\}_0 \otimes \{\alpha^i\}_0 \right).$$

This implies that ν is zero on any annulus. It was conjectured by Turaev that $\nu\{\alpha\} = 0$ only if $\{\alpha\} = \{\beta^n\}$ for some simple loop β. This was proved to false [3] in the case of any oriented surface with positive genus by M. Chas. On the other hand, the case for genus zero surfaces was answered in the positive by A. Le Donne [22].

Example 2.37. Let us calculate ν for $S^1 \times S^1$. Let x and y denote the meridian S^1 and longitude S^1 respectively. Orient the torus such that $\varepsilon_1(x, y) = 1$. Since these loops are simple,

$$\nu\{x^m\} = 0, \quad \nu\{y^n\} = 0.$$

It can be seen that $[\{x^m\}, \{y^n\}] = mn\{x^m \cdot y^n\}$, i.e., the bracket counts (with signs) the number of crossings. Therefore,

$$mn\, \nu\{x^m \cdot y^n\} = \nu\, [\{x^m\}, \{y^n\}] = [\nu\{x^m\}, \{y^n\}] + [\{x^m\}, \nu\{y^n\}] = 0.$$

Hence, ν is identically zero for the torus. This fact also follows from the observation that any closed loop γ has a representative in the free homotopy class of the form $x^m \cdot y^n$. Write $x^m \cdot y^n = (x^{m/k} \cdot y^{n/k})^k$ for $k = (m, n)$ and observe that $x^{m/k} \cdot y^{n/k}$ is a simple loop, whence ν kills any power of it. In particular, $\nu(\gamma) = 0$.

The space of multicurves

Let $\Lambda\mathbb{V}$ denote the (free) graded commutative associative algebra generated by the generators of $\mathbb{V}$, where the grading on $\Lambda\mathbb{V}$ is defined by setting the generators of $\mathbb{V}$ to be of degree 1. We will call the vector space $\Lambda\mathbb{V}$ the *space of multicurves*. A typical element of degree n in $\Lambda\mathbb{V}$ consists of finite linear combinations of elements of the form $\alpha_1 \wedge \ldots \wedge \alpha_n$, which we shall denote by $(\alpha_1, \ldots, \alpha_n)$. We shall use V_n to denote the elements of degree n and write $\Lambda\mathbb{V} = \oplus_{i \geq 0} V_i$.

Notation *For a homogeneous element $\alpha \in \Lambda\mathbb{V}$ we denote its degree by $|\alpha|$. For a general element x, its component of degree n is denoted by $|x|_n$.*

There is a canonical product on $\Lambda\mathbb{V}$ arising out of concatenation, i.e.,

$$(\alpha_1, \ldots, \alpha_n) \cdot (\alpha_{n+1}, \ldots, \alpha_{n+m}) := (\alpha_1, \ldots, \alpha_{m+n})$$

and then extended by bilinearity. It follows from the definition that this product is associative and (graded) commutative.

We have seen in §4.1 that the Goldman bracket on $\mathbb{V}$ that gave rise to a Lie algebra structure was of degree -1. We would like to extend the Lie bracket to $\Lambda\mathbb{V}$. For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in V_m$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in V_n$ we define

(2.3.8) $$[\boldsymbol{\alpha}, \boldsymbol{\beta}] := \sum_{i,j} (-1)^{i+j}([\alpha_i, \beta_j], \alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_m, \beta_1, \ldots, \widehat{\beta_j}, \ldots, \beta_n).$$

It is clear that this agrees on $V_1 = \mathbb{V}$. A simple computation shows that

$$[\boldsymbol{\alpha}, \boldsymbol{\beta}] = -(-1)^{(|\boldsymbol{\alpha}|-1)(|\boldsymbol{\beta}|-1)}[\boldsymbol{\beta}, \boldsymbol{\alpha}], \tag{2.3.9}$$

whence it is graded skew symmetric. Recall that

$$[a, b \otimes c] = [a, b] \otimes c + b \otimes [a, c]$$

by definition and its anti-symmetrization

$$[a, b \wedge c] = [a, b] \wedge c - [a, c] \wedge b$$

is a special case of (2.3.8) with $m = 1, n = 2$. One can see that

$$[\boldsymbol{\alpha}, \boldsymbol{\beta} \cdot \boldsymbol{\gamma}] = [\boldsymbol{\alpha}, \boldsymbol{\beta}] \cdot \boldsymbol{\gamma} + (-1)^{m(l+1)} \boldsymbol{\beta} \cdot [\boldsymbol{\alpha}, \boldsymbol{\gamma}]. \tag{2.3.10}$$

This implies that the bracket is a derivation of the product. We leave the proof of the Jacobi identity

$$[[\boldsymbol{\alpha}, \boldsymbol{\beta}], \boldsymbol{\gamma}] = [\boldsymbol{\alpha}, [\boldsymbol{\beta}, \boldsymbol{\gamma}]] - (-1)^{(|\boldsymbol{\alpha}|-1)(|\boldsymbol{\beta}|-1)}[\boldsymbol{\beta}, [\boldsymbol{\alpha}, \boldsymbol{\gamma}]] \tag{2.3.11}$$

as an exercise. We summarize what we have proved so far :

Proposition 2.38. *The space of multicurves* $(\Lambda\mathbb{V}, \cdot, [\,,\,])$ *is a Gerstanhaber algebra, i.e., it satisfies :*
(1) *The canonical product* $\cdot$ *defines a graded commutative, associative algebra.*
(2) $[\,,\,]$ *is a Lie bracket of degree* -1*, i.e.,*
  (i) *it is graded skew symmetric* (2.3.9),
  (ii) *it satisfies graded Jacobi identity* (2.3.11).
(3) *The Lie bracket is a graded derivation of the canonical product* (2.3.10).

We would like to carry out possible constructions using the cobracket too. We observe that

$$\nu(\alpha) := \sum_{p \in \#\alpha} \{\alpha_p^1\}_0 \otimes \{\alpha_p^2\}_0 - \{\alpha_p^2\}_0 \otimes \{\alpha_p^1\}_0$$

can be rewritten as

$$\nu(\alpha) = \sum_{p \in \#\alpha} (\alpha_p^1, \alpha_p^2). \tag{2.3.12}$$

Then one can extend this naturally as follows :

Definition 2.39. For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ define

$$\nu(\alpha_1, \ldots, \alpha_n) := \sum_{i=1}^{n} (-1)^i (\alpha_1, \ldots, \alpha_{i-1}, \nu(\alpha_i), \alpha_{i+1}, \ldots, \alpha_n) \tag{2.3.13}$$

and extend by multilinearity.

A couple of observations are in order. First,

$$\nu(\boldsymbol{\alpha} \cdot \boldsymbol{\beta}) = \nu(\boldsymbol{\alpha}) \cdot \boldsymbol{\beta} + (-1)^{|\boldsymbol{\alpha}|} \boldsymbol{\alpha} \cdot \nu(\boldsymbol{\beta}),$$

whence ν is a derivation. Secondly, observe that $\nu^2(\alpha) = 0$ for $\alpha \in V_1 = \mathbb{V}$:

$$\begin{aligned}
\nu^2(\alpha) &= \sum_{p \in \#\alpha} \nu(\alpha_p^1, \alpha_p^2) \\
&= \sum_{p \in \#\alpha} (\nu(\alpha_p^1), \alpha_p^2) - \sum_{p \in \#\alpha} (\alpha_p^1, \nu(\alpha_p^2)) \\
&= \sum_{q \in \#\alpha_p^1} \sum_{p \in \#\alpha} (\alpha_q^{11}, \alpha_q^{12}, \alpha_p^2) - \sum_{r \in \#\alpha_p^2} \sum_{p \in \#\alpha} (\alpha_p^1, \alpha_r^{21}, \alpha_r^{22}) \\
&= 0.
\end{aligned}$$

Here the last equality holds since the 3-tuples appearing on both sums, each listing all possible ways to split α into three curves, are identical. Armed with this observation, it is easy to verify that $\nu^2 = 0$ on $\Lambda\mathbb{V}$.

Remark 2.40. *The coJacobi identity is equivalent to $\nu^2 = 0$ (refer Lemma 2.35).*

One may ask how the differential ν interacts with the Lie bracket $[\, ,\,]$. To this end, we recall that

Proposition 2.41. *The Goldman-Turaev Lie bialgebra $\mathbb{V}$ is involutive.*

For a proof, we refer the reader to Appendix B of [3].

Definition 2.42. (The operator D) Define the operator D on a monomial $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ as follows :

$$\begin{aligned}
D(\boldsymbol{\alpha}) &= \sum_{i=1}^{n} (-1)^{i-1} (\alpha_1, \ldots, \alpha_{i-1}, \nu(\alpha_i), \alpha_{i+1}, \ldots, \alpha_n) \\
&\quad + \sum_{i<j} (-1)^{i+j} ([\alpha_i, \alpha_j], \alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_n).
\end{aligned} \tag{2.3.14}$$

It is then extended to all elements by multilinearity.

Pictorially, given a set of n closed curves in general position, D produces all possible curves that arise as a result of the operators ν and $[\,,\,]$ applied to it. Notice that D maps an element of degree n to elements of degree $n-1$ (corresponding to $[\,,\,]$) and elements of degree $n+1$ (corresponding to ν). We had seen earlier that ν, as used in this definition, is a derivation on $\Lambda\mathbb{V}$.

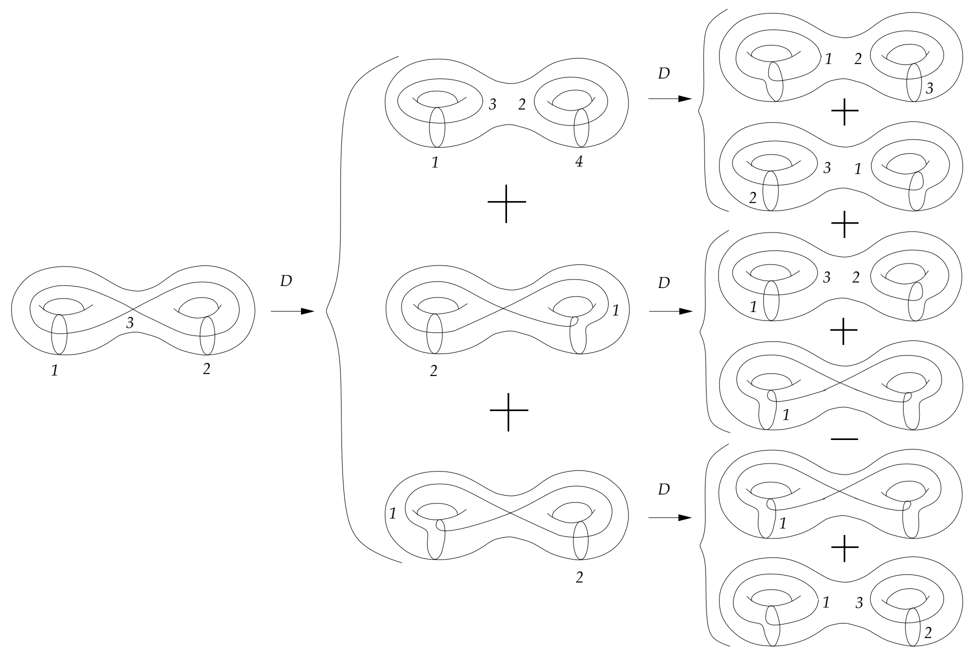

Figure 2.5: A pictorial description of D and why $D^2 = 0$

Theorem 2.43. *The operator D defined on $\Lambda\mathbb{V}$ satisfies $D^2 = 0$.*

Proof We only provide the briefest of proofs as the structure is conceptually clear and the details are, well, details! First observe that D is alternating. Since interchanging α_k and α_{k+1} differs by a negative sign,

$$D(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)}) = (-1)^{|\sigma|} D(\alpha_1, \ldots, \alpha_n).$$

We verify $D^2 = 0$ by starting with degree 1 elements. We have

$$D^2(\alpha) = D(\nu(\alpha)) = \nu^2(\alpha) + [\,,\,] \circ \nu(\alpha) = 0.$$

The last equality holds since $\nu^2 = 0$ and $\mathbb{V}$ is involutive. In general,

$$\left| D^2(\alpha_1, \ldots, \alpha_n) \right|_{n+2} = \nu^2(\alpha_1, \ldots, \alpha_n) = 0.$$

If we analyze the degree $n-2$ elements, we get $|D^2(\alpha_1,\ldots,\alpha_n)|_{n-2}$ as a sum which cancels out due to repeated application of the Jacobi identity (2.3.11). It remains to show that $|D^2(\alpha_1,\ldots,\alpha_n)|_n = 0$. A typical term in this arises out of the action of ν followed by $[\,,\,]$ or vice versa. After cancellations, these terms can be split into two sums. One of them cancels out due to involutivity and the other due to, no surprises here, compatibility. This completes the proof of $D^2 = 0$. □

Remark 2.44. *All these operations and identities can be visualized in a concise way via the following diagram, where V_n denotes the degree n elements of $\Lambda\mathbb{V}$:*

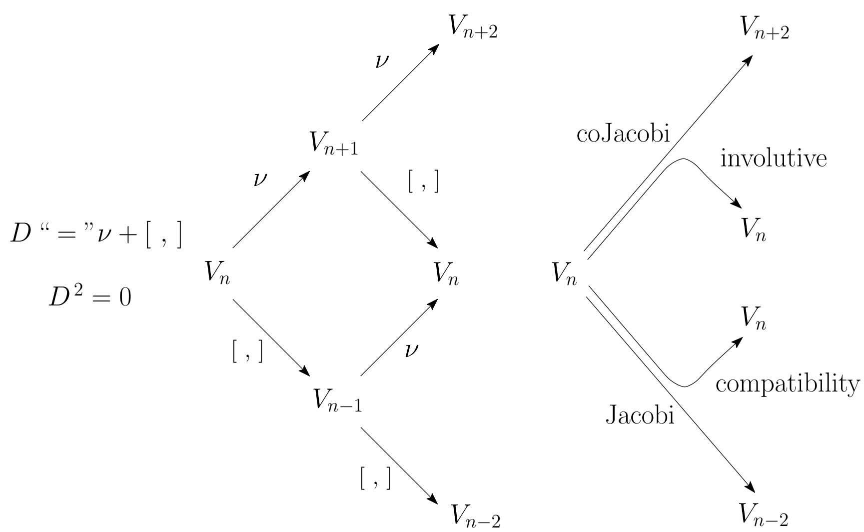

Figure 2.6: A conceptual proof of $D^2 = 0$

Recall that we had extended ν on $\Lambda\mathbb{V}$ to be a derivation. The operator D is not a derivation in general, unless $[\,,\,] = 0$. For particular examples like the torus, we had seen that $\nu = 0$. Therefore $D_{S^1\times S^1}$ is just the Goldman Lie bracket extended to all of $\Lambda\mathbb{V}$. For S^2, the bracket is zero, whence D_{S^2} is a derivation.

Chapter 3

Computations in String Topology

3.1 String Topology via Minimal Models

Spaces can be modelled algebraically in the sense of rational homotopy theory. Fibrations of spaces can be treated similarly - we model the base space and then model the twisting of the fibration appropriately. These algebraic models, often called *minimal models*, was originally introduced by D. Sullivan and is built out of piecewise linear differential forms.

We briefly review its existence for any simply connected manifold M. The interested reader can refer [34] for a detailed exposition. We also review the notion of formality in the context of minimal models. Since the free loop space LM appears as the total space of the fibration

$$\Omega M \hookrightarrow LM \longrightarrow M$$

we can use the model for M to construct one for LM. Moreover, one can construct an equivariant model for LM ([4], §9) by considering the circle action of rotating loops. It's possible to embellish these models with further appropriate structures in specific cases to model the loop product, loop bracket and the BV operator on LM. We show in subsequent computations that the loop product, loop bracket and the BV operator on $H_*(LM)$ are all non-trivial for formal manifolds with monogenic cohomology ring. However, the string bracket and the cobracket are both zero. In fact, we outline a sketch of the equivariant calculations for spheres and complex projective spaces in §3.1.2 and §3.1.3. In this section, unless mentioned otherwise, all subsequent models are rational models, i.e., the associated calculations are done with rational coefficients.

3.1.1 Formality and minimal models

Let ΛV be the free graded commutative algebra generated by a graded $\mathbb{R}$-vector space $V = \oplus_{i\geq 0} V_i$. When ΛV is equipped with a differential, we call $(\Lambda V, d)$ a *minimal algebra* if there is a set of generators $\{a_i\}$, indexed by an well ordered set I, such that $|a_i| \leq |a_j|$ if $i < j$ and da_i can be written in terms of a_j's for $j < i$. In other words, da_i doesn't have a linear term. A minimal algebra $(\Lambda V, d)$ is said to be a *minimal model* for a connected manifold M if there is a quasi-isomorphism[1] of algebras $\varphi : (\Lambda V, d) \to (\Omega_{dR} M, d)$, where $\Omega_{dR} M$ is the space of de Rham forms. These discussions can also be carried out with rational coefficients and replacing $\Omega_{dR} M$ with $C^*(M; \mathbb{Q})$, the cochain complex of M.

It is known that that minimal models exist for any connected manifold and is unique up to isomorphism ([13], Theorem 14.12). From D. Sullivan's work on rational homotopy theory, explicit models can be given for any *nilpotent space* M, i.e., $\pi_1(M)$ is nilpotent and its action on higher homotopy groups is nilpotent. For such manifolds, the dual of the real homotopy groups $\pi_i(M) \otimes \mathbb{R}$ can be taken to be V^i. For a detailed discussion on this refer [34]. We will need the following result ([34], page 637) which is also proved in [13] (Example 1, page 206).

Theorem 3.1. *If M is simply connected space with $\Lambda(x_1, x_2, \cdots, d)$ as its minimal model then the free loop space LM has the minimal model $(\Lambda(x_1, y_1, x_2, y_2, \cdots), \overline{d})$ where $|y_i| = |x_i| - 1$. The operator $\overline{d}$ is defined to be d on the x_i's and extended using $\overline{d}s + s\overline{d} = 0$, where s is the derivation of $\Lambda(x_1, x_2, \cdots)$ into $\Lambda(x_1, y_1, \cdots)$ defined by $s(x_j) = y_j$.*

Notice that $\Lambda(x_1, x_2, \cdots)$ is a subcomplex of $\Lambda(x_1, y_1, \cdots)$, and the image of d in $\Lambda(x_1, y_1, \cdots)$ is contained in the ideal $I = (x_1, x_2, \cdots)$. Thus, the induced d on $\Lambda(y_j) = \Lambda(x_1, y_1, \cdots)/I$ is zero. This algebraic picture corresponds to the natural fibration

$$\Omega M \hookrightarrow LM \xrightarrow{\text{ev}} M$$

since $(\Lambda(y_j), d = 0)$ can be taken to be a model of ΩM, being a H-space. The operator s corresponds to Δ in loop homology.

[1] In general, when one speaks of a quasi-isomorphism from F to W one always assumes it is given by a *zig-zag*

$$F \to W_1 \leftarrow F_1 \to \cdots \to F_n \to W$$

where each arrow induce isomorphism in (co)homology. In the case of minimal models, if F_i, W_i's and F are all minimal then all arrows are invertible and hence we make no mention of zig-zag when dealing with quasi-isomorphism.

To obtain a model for equivariant cohomology, we add to the existing model for LM a closed generator u of degree 2, i.e., $du = 0$ and $s(u) = 0$. It can be easily verified that $\bar{d}z := dz + s(z)u$ defines a differential. Note that that the equivariant cohomology arises from the circle action on the space $\mathcal{S}_M$ of general smooth mappings of the circle in M. This circle bundle has an associated characteristic class that corresponds to the closed generator u. The Gysin sequence for the circle bundle (dual to the sequence in homology)

$$S^1 \longrightarrow LM \times ES^1 \xrightarrow{\pi} LM \times_{S^1} ES^1$$

is the following long exact sequence :

$$(3.1.1) \quad \cdots \to H^{i-1}(LM) \xrightarrow{\mathcal{M}^*} \mathcal{H}^{i-2}(LM) \xrightarrow{\cup c} \mathcal{H}^i(LM) \xrightarrow{\pi^*} H^i(LM) \to \cdots$$

In terms of the Sullivan models described above, the map $\mathcal{M}^* = s, \cup c = \cup u$ and π^* maps u to 0.

A manifold M is called *formal* if there is a quasi-isomorphism between its minimal model Λ_M and its cohomology ring, i.e.,

$$\Phi : (\Lambda_M, d) \longrightarrow (H^*(M; \mathbb{Q}), 0)$$

is a quasi-isomorphism. In other words, the rational homotopy theory of the manifold is just a formal consequence of the rational cohomology algebra of M. Lie groups and symmetric spaces are known to be formal. It is also known that products and connected sums preserve formality. Moreover, if the Massey product is non-trivial then the manifold is not formal.

Example 3.2. (Spheres)

For S^{2k+1} we take the free (graded commutative) algebra $(\Lambda(x), d \equiv 0)$ generated by a generator x in degree $2k+1$. The map

$$\varphi : (\Lambda(x), d) \xrightarrow{\simeq} H^*(S^{2k+1}; \mathbb{Q})$$

is given by mapping x to the volume form. Similar models can be built for S^{2k}. Since the underlying vector spaces of the minimal model can also be built out of $\pi_i \otimes \mathbb{Q}$, we recover the beautiful result of J. P. Serre on homotopy groups of spheres, i.e.,

(i) $\pi_i(S^{2k+1}) \otimes \mathbb{Q} = \mathbb{Q}$ if $i = 0, 2k+1$ and zero otherwise,

(ii) $\pi_i(S^{2k}) \otimes \mathbb{Q} = \mathbb{Q}$ if $i = 0, 2k, 4k-1$ and zero otherwise.

Example 3.3. (Kähler manifolds)

The famous paper [12] by P. Deligne, P. Griffiths, J. Morgan and D. Sullivan used rational homotopy theory to prove that compact Kähler manifolds are formal. As a consequence of this rather deep theorem, the complex Grassmanians $G(k,n)$ are formal. In particular, $\mathbb{CP}^n$ is formal and this can be verified easily otherwise.

Actually, all the examples above (except $G(k,n), k \neq 1, n-1$) fit into the class of manifolds having a single generator for its cohomology ring. Moreover, any such manifold is known to be formal (refer the proof of the next proposition). Towards that end, we have :

Theorem 3.4. (String Topology of Spheres and Projective Spaces) *For a simply connected closed manifold M such that $H^*(M;\mathbb{Q})$ is a truncated algebra in one generator, the string bracket is trivial. The string cobracket on the reduced rational string homology is trivial.*

Proof Let $H^*(M;\mathbb{Q})$ be generated by α. Choose a representative cocycle c in $C^*(M;\mathbb{Q})$ for α. If $|\alpha|$ is odd then define a map of differential graded algebras $(\Lambda(x), 0) \to (C^*(M;\mathbb{Q}), \delta)$ which sends x to c. This is a quasi-isomorphism. Moreover, $\Lambda(x) \to H^*(M;\mathbb{Q})$ sending x to α is also a quasi-isomorphism. When $|\alpha|$ is even, similar considerations hold. As a consequence, M is formal and we can work with minimal models constructed from its cohomology ring. There are two mutually exclusive and exhaustive cases for $H^*(M;\mathbb{Q})$:

(i) $\Lambda(x), |x| = 2k+1$: These algebras model odd dimensional spheres. The associated string bracket and cobracket will be analyzed in §3.1.3 and will turn out to be zero (Proposition 3.10).

(ii) $\Lambda(x)/x^{k+1}, |x| = 2l$: This algebra and its equivariant version will be analyzed §3.1.2. The associated string bracket and cobracket again turns out to be zero (Proposition 3.7). ☐

3.1.2 A model for even spheres and projective spaces

We shall work with a specific differential graded algebra $\Lambda(k,l)$. This model is just the loop space model of a manifold with a monogenic cohomology ring generated by an element of degree $2l$ and of order $k-1$. Hence, it covers the case of even dimensional spheres and projective spaces. Moreover, the various string topology computations for these manifolds are obtained as a corollary of the analysis for $\Lambda(k,l)$ and its equivariant version $\Lambda^{S^1}(k,l)$.

Definition 3.5. For $k \geq 2, l \geq 1$ we define $\Lambda(k,l)$ to be the differential graded algebra generated by y_1, x_1, y_2, x_2 with the differential

$$dy_1 = 0 = dx_1,\ dx_2 = x_1^k,\ dy_2 = -kx_1^{k-1}y_1. \tag{3.1.2}$$

The element x_1 is of degree $2l$ while y_1 is of degree $2l-1$.

The *equivariant model* $\Lambda^{S^1}(k,l) := \Lambda(k,l)[u]$ with the differential defined by

$$dy_1 = 0,\ dx_1 = y_1u,\ dy_2 = -kx_1^{k-1}y_1,\ dx_2 = x_1^k + y_2u, \tag{3.1.3}$$

and extended as a derivation.

Before proceeding to analyze this model further, note that the rational loop cohomology model of S^{2n} is given by $\Lambda(2,n)$. The corresponding equivariant model is given by $\Lambda^{S^1}(2,n)$. The corresponding model for $L\mathbb{CP}^n$ is given by $\Lambda(n+1,1)$ and similar considerations hold for quarternionic projective spaces, i.e., $\Lambda(n+1,2)$ is a minimal model for $L\mathbb{HP}^n$. In all these cases, formality is essential in proving that these are minimal models.

Fix integers $k \geq 2, l \geq 1$. We shall be concerned with $\Lambda(k,l)$ and $\Lambda^{S^1}(k,l)$ henceforth. It follows from degree considerations that the equivariant cochain groups are

$$\begin{aligned} \mathcal{C}^{\text{even}} &= \mathbb{Q}_{\text{span}}\{y_1x_2y_2^ax_1^bu^c, y_2^px_2^qu^r\} \\ \mathcal{C}^{\text{odd}} &= \mathbb{Q}_{\text{span}}\{y_1y_2^px_1^qu^r, x_2y_2^ax_1^bu^c\} \end{aligned}$$

with the differential given by

$$\begin{aligned}
&(3.1.4) & d(y_1y_2^px_1^qu^r) &= 0 \\
&(3.1.5) & d(y_1x_2y_2^ax_1^bu^c) &= -y_1y_2^ax_1^{b+k}u^c - y_1y_2^{a+1}x_1^bu^{c+1} \\
&(3.1.6) & d(y_2^px_1^qu^r) &= qy_1y_2^px_1^{q-1}u^{r+1} - kpy_1y_2^{p-1}x_1^{k+q-1}u^r \\
&(3.1.7) & d(x_2y_2^ax_1^bu^c) &= y_2^ax_1^{b+k}u^c + y_2^{a+1}x_1^bu^{c+1} \\
&& &\quad +by_1x_2y_2^ax_1^{b-1}u^{c+1} - kay_1x_2y_2^{a-1}x_1^{k+b-1}u^c.
\end{aligned}$$

Notice that no linear combination of terms of the form $x_2y_2^ax_1^bu^c$ can be closed and can be seen by considering the term with the highest u exponent or the highest x_1 exponent. This implies that

$$\begin{aligned}
\ker^{\text{odd}}(d) &= \mathbb{Q}_{\text{span}}\{y_1 y_2^p x_1^q u^r\} \\
\text{image}^{\text{even}}(d) &= \mathbb{Q}_{\text{span}}\{-y_1 y_2^a x_1^{b+k} u^c - y_1 y_2^{a+1} x_1^b u^{c+1}\} \\
&\quad + \mathbb{Q}_{\text{span}}\{q y_1 y_2^p x_1^{q-1} u^{r+1} - kp y_1 y_2^{p-1} x_1^{k+q-1} u^r\} \\
&= \mathbb{Q}_{\text{span}}\{y_1 y_2^{a+1} x_1^b u^{c+1}, y_1 y_2^a x_1^{b+k} u^c, y_1 x_1^p u^{q+1}, y_1 y_2^r x_1^{k-1} u^s\} \\
&= \mathbb{Q}_{\text{span}}\{y_1 y_2^{a+1} x_1^b u^{c+1}, y_1 y_2^a x_1^{b+k-1} u^c, y_1 x_1^p u^{q+1}\},
\end{aligned}$$

whence the odd cohomology can be calculated to be

$$\begin{aligned}
\mathcal{H}^{\text{odd}} &= \frac{\mathbb{Q}_{\text{span}}\{y_1 y_2^p x_1^q u^r\}}{\mathbb{Q}_{\text{span}}\{y_1 y_2^{a+1} x_1^b u^{c+1}, y_1 y_2^a x_1^{b+k} u^c, y_1 x_1^i u^{j+1}, y_1 y_2^r x_1^{k-1} u^s\}} \\
&= \mathbb{Q}_{\text{span}}\{y_1 y_2^p x_1^q, y_1 x_1^m u^n \,|\, m, q < k-1\} / \mathbb{Q}_{\text{span}}\{y_1 x_1^p u^{q+1}\} \\
&= \mathbb{Q}_{\text{span}}\{y_1 y_2^p x_1^q, y_1 x_1^m \,|\, m, q < k-1\} \\
&= \mathbb{Q}_{\text{span}}\{y_1 y_2^p x_1^q \,|\, q < k-1\}.
\end{aligned}$$

Then it follows that

$$\operatorname{rank} \mathcal{H}^{2j+1} = |\{(p,q) \,|\, (lk-1)p + lq = j+1, p \geq 0, k-1 \geq q \geq 1\}|,$$

which is easily seen to be bounded by $k-1$ since q can take at most $k-1$ possible values.

To calculate the even cohomology, first observe that it follows from (3.1.5) that no linear combination of terms of the form $y_1 x_2 y_2^a x_1^b u^c$ is closed. Let us analyze when linear combinations of terms of the form $y_2^a x_1^b u^c$ is closed. We put an ordering on such terms by the exponent of u, followed by x_1 and then by y_2. Let σ, consisting of terms like $y_2^a x_1^b u^c$, be closed and irreducible, i.e., σ cannot be written as the sum of two closed forms. Then, in view of (3.1.6), the term in σ with the highest order necessarily has the form $y_2^p u^r$. Moreover, either $\sigma = u^r$ or $p \geq 1$ by irreducibility. Using (3.1.6) recursively, one can construct σ from the highest term while ensuring that σ is closed. In fact, a simple calculation shows that

$$\sigma = \sum_{i=0}^{\min\{p,r\}} \binom{p}{i} y_2^{p-i} x_1^{ik} u^{r-i}. \tag{3.1.8}$$

It is now easily seen that this is closed only when $r > p$ and when $r > p > 0$,

$$\sigma_p = \sum_{i=0}^{\min\{p,r\}} \binom{p}{i} y_2^{p-i} x_1^{ik} u^{r-i} = d(u^{r-p} (dx_2)^{p-1} x_2).$$

Now assume that σ is any closed irreducible element of even degree that consists of mixed terms. For $q \geq -1$ let $(q+1)y_1x_2y_2^px_1^qu^r$ be the term that is of the highest order among such terms present in σ. Using (3.1.6) and the fact that $d\sigma = 0$, we conclude that σ has the term $y_2^{p+1}x_1^{q+1}u^r$ and

$$\begin{aligned} d(y_2^{p+1}x_1^{q+1}u^r + (q+1)y_1x_2y_2^px_1^qu^r) &= -(kp+k+q+1)y_1y_2^px_1^{k+q}u^r \\ d(kpy_1x_2y_2^{p-1}x_1^{k+q}u^{r-1} - y_2^px_1^{k+q+1}u^{r-1}) &= -(kp+k+q+1)y_1y_2^px_1^{k+q}u^r, \end{aligned}$$

whence for $r \geq 1$

$$\sigma = y_2^{p+1}x_1^{q+1}u^r + (q+1)y_1x_2y_2^px_1^qu^r - kpy_1x_2y_2^{p-1}x_1^{k+q}u^{r-1} + y_2^px_1^{k+q+1}u^{r-1}.$$

Using (3.1.7), it follows that σ is exact and the even equivariant cohomology is $\mathcal{H}^{\text{even}} = \mathbb{Q}u^j$.

Towards calculating the cohomology of the algebra $\Lambda(k,l)$, notice that the cochain groups are

$$\begin{aligned} C^{\text{even}} &= \mathbb{Q}_{\text{span}}\{y_1x_2y_2^ax_1^b, y_2^px_2^q\} \\ C^{\text{odd}} &= \mathbb{Q}_{\text{span}}\{y_1y_2^px_1^q, x_2y_2^ax_1^b\}, \end{aligned}$$

with the differential given by

$$d(y_1y_2^px_1^q) = 0 \tag{3.1.9}$$

$$d(y_1x_2y_2^ax_1^b) = -y_1y_2^ax_1^{b+k} \tag{3.1.10}$$

$$d(y_2^px_1^q) = -kpy_1y_2^{p-1}x_1^{k+q-1} \tag{3.1.11}$$

$$d(x_2y_2^ax_1^b) = y_2^ax_1^{b+k} - kay_1x_2y_2^{a-1}x_1^{k+b-1}. \tag{3.1.12}$$

An element comprising entirely of terms of the form $x_2y_2^ax_1^b$ cannot be closed since the term with the highest x_1 exponent survives under the differential. Therefore, the closed odd degree elements are spanned by $y_1y_2^px_1^q$ while the image of the even elements only miss $y_1y_2^px_1^q$ where $q < k-1$. Hence,

$$H^{\text{odd}}(\Lambda(k,l)) = \mathbb{Q}_{\text{span}}\{y_1y_2^px_1^q \,|\, q < k-1\} \tag{3.1.13}$$

is exactly the same as the equivariant cohomology and has rank bounded by $k-1$. For the even degree elements,

$$d(y_2^px_1^{q+1} - kpy_1x_2y_2^{p-1}x_1^q) = 0$$

and (3.1.12) implies that

$$H^{\text{even}}(\Lambda(k,l)) = \mathbb{Q}_{\text{span}}\{1, y_2^px_1^{q+1} - kpy_1x_2y_2^{p-1}x_1^q \,|\, q < k-1\}, \tag{3.1.14}$$

whence the Betti numbers are again bounded by $k-1$. In conclusion, we have :

Theorem 3.6. *The cohomology of the algebra $\Lambda(k,l)$ is given by*

$$\begin{aligned} H^{2j} &= \mathbb{Q}_{\text{span}}\{\alpha_{p,q} \,|\, q < k-1, l(q+1) + (lk-1)p = j\} \oplus \delta_{j,0}\mathbb{Q}, \\ H^{2j+1} &= \mathbb{Q}_{\text{span}}\{y_1 y_2^p x_1^q \,|\, q < k-1, l(q+1) + (lk-1)p = j+1\}, \end{aligned}$$

where

$$\alpha_{p,q} := y_2^p x_1^{q+1} - kpy_1 x_2 y_2^{p-1} x_1^q.$$

The cohomology of the differential graded algebra $\Lambda^{S^1}(k,l)$ is given by

$$\begin{aligned} \mathcal{H}^{2j} &= \mathbb{Q}u^j, \\ \mathcal{H}^{2j+1} &= \mathbb{Q}_{\text{span}}\{y_1 y_2^p x_1^q \,|\, q < k-1, l(q+1) + (lk-1)p = j+1\}. \end{aligned}$$

Consider $\Lambda(k,l)$ as a model for LM for some manifold M of dimension $2l(k-1)$ and $\Lambda^{S^1}(k,l)$ its equivariant model. The associated Gysin sequence (refer (3.1.1)) is

$$\cdots \longrightarrow H^{j-1} \xrightarrow{s} \mathcal{H}^{j-2} \xrightarrow{\cup u} \mathcal{H}^j \xrightarrow{u \mapsto 0} H^j \longrightarrow \cdots .$$

The middle map given by cup product with u is an isomorphism whenever j is even. It is the zero map otherwise. Therefore, in terms of homology

$$\begin{array}{c} \cdots \longrightarrow H_{\text{even}} \xrightarrow{\mathcal{E}=0} \mathcal{H}_{\text{even}} \xrightarrow{\cong} \mathcal{H}_{\text{even}} \xrightarrow{\mathcal{M}=0} H_{\text{odd}} \longrightarrow \cdots \\ \cdots \longrightarrow H_{\text{odd}} \xrightarrow{\mathcal{E}} \mathcal{H}_{\text{odd}} \xrightarrow{0} \mathcal{H}_{\text{odd}} \xrightarrow{\mathcal{M}} H_{\text{even}} \longrightarrow \cdots \end{array}$$

and the string bracket $[\alpha, \beta]$ is seen to be zero just by a parity check. Moreover, the reduced equivariant homology exists only in odd degrees. Now suppose M is a manifold with $\Lambda(k,l)$ as a minimal model for LM. Since the manifold has even dimension and the Lie cobracket is of degree $2 - \dim M$, by degree considerations, it can be seen to be zero. Thus, we have :

Proposition 3.7. *If a manifold M has $\Lambda(k,l)$ as a minimal model for LM then the string bracket on the rational string homology is zero. The Lie cobracket on the reduced string homology is also trivial.*

3.1.3 A model for odd spheres

We shall be dealing with $S^{2k+1}, k \geq 1$. Recall that rational cochain models of the various spaces of relevance are given by the following :

$$\begin{array}{rcl} S^{2k+1} & : & \Lambda(x), |x| = 2k+1, d \equiv 0 \\ \Omega S^{2k+1} & : & \Lambda(y), |y| = 2k, d \equiv 0 \\ LS^{2k+1} & : & \Lambda(x, y), d \equiv 0 \\ LS^{2k+1} \times_{S^1} ES^1 & : & \Lambda(x, y, u), |u| = 2, dx = yu, dy = 0 = du. \end{array}$$

The equivariant cochain groups are then given by

$$\begin{array}{rcl} \mathcal{C}^{2j} & = & \mathbb{Q}_{\text{span}}\{y^c u^{j-ck}\} \\ \mathcal{C}^{2j+1} & = & \mathbb{Q}_{\text{span}}\{xy^a u^{j-(a+1)k}\} \end{array}$$

with the differential d from even to odd degrees being zero while

$$d(xy^a u^{j-(a+1)k}) = y^{a+1} u^{j+1-(a+1)k}$$

maps into $\mathcal{C}^{2j+2}$ and misses only u^{j+1}. Therefore, the equivariant cohomology reads

$$\mathcal{H}^{\text{even}}(LS^{2k+1}) = \mathbb{Q}_{\text{span}}\{y^a, u^j\} \tag{3.1.15}$$

$$\mathcal{H}^{\text{odd}}(LS^{2k+1}) = 0. \tag{3.1.16}$$

As a consequence, the string bracket and the Lie cobracket being both of degree $1-2k$, are forced to be trivial.

Remark 3.8. *It follows from Serre spectral sequence applied to*

$$\Omega S^{2k+1} \hookrightarrow PS^{2k+1} \to S^{2k+1}$$

that

$$H_*(\Omega S^{2k+1}; \mathbb{Z}) \cong \mathbb{Z}[\alpha], \ |\alpha| = 2k.$$

Applying spectral sequence to the loop space fibration $\Omega S^{2k+1} \hookrightarrow LS^{2k+1} \to S^{2k+1}$ we gather that the homology of the free loop space has no torsion. We apply spectral sequence again to the fibration

$$S^1 \hookrightarrow LS^{2k+1} \times ES^1 \to LS^{2k+1} \times_{S^1} ES^1$$

we conclude that $\mathcal{H}_*(LS^{2k+1})$ is torsion free, whence (3.1.15),(3.1.16) *actually hold over $\mathbb{Z}$-coefficients.*

On the non-equivariant setting, the cochain groups $C^*(LS^{2k+1}) = \mathbb{Q}_{\text{span}}\{xy^i, y^j\}$ are precisely the cohomology groups since the differential is zero. This implies

$$H^{2k(j+1)+1} = \mathbb{Q}xy^j \tag{3.1.17}$$

$$H^{2jk} = \mathbb{Q}y^j. \tag{3.1.18}$$

Remark 3.9. *The example of S^3 is special as it is a Lie group. This means $LS^3 \cong \Omega S^3 \times S^3$ and*

$$\mathbb{H}_i(LS^3) = \big(H_{i+3}(\Omega S^3) \otimes \mathbb{H}_{-3}(S^3)\big) \oplus \big(H_i(\Omega S^3) \otimes \mathbb{H}_0(S^3)\big)$$

The loop product is given by the combination of the intersection product on S^3 and the Pontrjagin product on ΩS^3. In fact, combining the following isomorphisms

$$\begin{aligned} \big(\mathbb{H}_*(S^3;\mathbb{Z}), \cap\big) &\cong \Lambda_{\mathbb{Z}}(\alpha),\ |\alpha| = -3 \\ \big(H_*(\Omega S^3;\mathbb{Z}), \times\big) &\cong \mathbb{Z}[b],\ |b| = 2 \end{aligned}$$

we may conclude that

$$(\mathbb{H}_*(LS^3;\mathbb{Z}), \bullet) = \Lambda_{\mathbb{Z}}(\alpha) \otimes \mathbb{Z}[\beta],\ \alpha \in \mathbb{H}_{-3}, \beta = 1 \otimes b \in \mathbb{H}_2. \tag{3.1.19}$$

Quite magically, this formula remains true for any higher dimensional odd sphere (which has rational behaviour like Lie groups) as well [9].

One can construct the coalgebra structure on $\mathbb{H}^*(LS^{2k+1})$ by dualizing the algebra structure on the loop homology. Using this we calculate the equivariant operations. Since the based loop homology is just the polynomial algebra, the dual algebra is just the divided polynomial algebra. Therefore, we conclude that (refer (3.1.17), (3.1.18)) $xy^j/j!$ is dual to β^j while $y^j/j!$ is dual to $\alpha\beta^j$. We use the notation $y^{[j]} := y^j/j!, xy^{[j]} := xy^j/j!$. Let

$$\triangleleft : \mathbb{H}^* \longrightarrow \mathbb{H}^* \otimes \mathbb{H}^*$$

be the comultiplication dual to the loop product. Then simple calculations in the basis $xy^{[j]}, y^{[j]}$ tells us the following :

$$\triangleleft\,(xy^{[n]}) = \sum_{i=0}^{n} xy^{[i]} \otimes xy^{[n-i]} \tag{3.1.20}$$

$$\triangleleft(y^{[n]}) = \sum_{i=0}^{n} \big(xy^{[i]} \otimes y^{[n-i]} + y^{[i]} \otimes xy^{[n-i]}\big)\,. \tag{3.1.21}$$

In fact, since LS^{odd} is formal, the above formulae holds at the chain level. One can actually use this to define

$$\triangleleft_{l-1} : \mathbb{H}^* \to (\mathbb{H}^*)^{\otimes l},$$

which is dual to the loop product $\bullet^{\otimes(l-1)}$. Written explicitly,

$$\begin{aligned}
\triangleleft_{l-1}(xy^{[n]}) &= \sum_{i_1+\ldots+i_l=n} xy^{[i_1]}\otimes\cdots\otimes xy^{[i_l]} \\
\triangleleft_{l-1}(y^{[n]}) &= \sum_{j,i_1+\ldots+i_l=n} xy^{[i_1]}\otimes\cdots\otimes xy^{[i_{j-1}]}\otimes y^{[i_j]}\otimes xy^{[i_{j+1}]}\otimes\cdots\otimes xy^{[i_l]}.
\end{aligned}$$

It is now clear that

$$s^{\otimes l}\circ\triangleleft_{l-1}(y^{[n]}) = 0,$$

implying the following :

Proposition 3.10. *The higher operations $\overline{m}_k : \mathcal{H}_*^{\otimes k} \to \mathcal{H}_*$ are zero for $k \geq 2$. The higher coalgebra operations are also zero.*

Proof We have already proved that $\overline{m}_k$'s are zero. To figure out the coalgebra operations $c_k, k \geq 2$ first recall that the loop coproduct τ is dual to the product $\circledast$ on loop cohomology (Remark 2.9) and has even degree. Let

$$c^k : (\mathcal{H}^*)^{\otimes k} \longrightarrow \mathcal{H}^*, \ c^k = s\circ\circledast\circ\cdots\circ(\circledast\otimes\mathrm{id}^{\otimes(k-2)})\circ(u=0)^{\otimes k}$$

be the dual to c_k. It is now clear that $c^k \equiv 0$ since s kills any element $u^i, y^j \in \mathcal{H}^*$. Therefore, $c_k, k \geq 2$ are all zero. $\square$

It remains to calculate the BV operator. Observe that

$$s(xy^{[j]}) = (j+1)y^{[j+1]}$$

and s kills $y^{[j]}$, whence

$$\Delta(\beta^j) = 0, \ \ \Delta(\alpha\beta^j) = j\beta^{j-1}. \tag{3.1.22}$$

Then the loop bracket turns out to be

$$\{\beta^i,\beta^j\} = 0 \tag{3.1.23}$$
$$\{\beta^i,\alpha\beta^j\} = i\beta^{i+j-1} \tag{3.1.24}$$
$$\{\alpha\beta^i,\beta^j\} = -j\beta^{i+j-1} \tag{3.1.25}$$
$$\{\alpha\beta^i,\alpha\beta^j\} = (i-j)\alpha\beta^{i+j-1}. \tag{3.1.26}$$

Notice that all calculations done with rational coefficients hold over $\mathbb{Z}$ as well since the integral loop homology of odd spheres has no torsion. Hence, the loop bracket and Δ calculated above holds for $\mathbb{H}_*(LS^{2n+1};\mathbb{Z})$. Finally, the Massey triple products are all zero.

3.1.4 String topology as a Lie bialgebra

We briefly recall the occurrence of Lie bialgebras and how we interpret it in the setting of minimal models. Given a closed, oriented manifold M we have various algebraic structures associated to $H_*(LM)$, viz., the loop product $\bullet$ provides a commutative algebra structure on $H_*(LM)$. There is a BV operator Δ on $H_*(LM)$ given by rotating the loops. The associated bracket is the loop bracket $\{\,,\,\}$. There is also a loop coproduct on the reduced loop homology $H_*(LM, M)$. As observed in the discussion preceeding Remark 2.11, the Lie bialgebra structure arises when we look at the induced structure on the S^1-equivariant homology of LM. Moreover, we also invoke the discussion in §3.1.1 following (3.1.1). With these in mind, let's begin.

Assuming we have a minimal model $(\Lambda(x_1, \ldots, x_n), d)$ for M. Then we build a minimal model for LM as per Theorem 3.1, i.e.,

$$\Lambda_{LM} := (\Lambda(x_i, y_i), \overline{d}),$$

where $\overline{d} \equiv d$ on x_i's, $s(x_i) = y_i$ is of degree -1, both s and $\overline{d}$ are derivations satisfying

$$\overline{d}s + s\overline{d} = 0.$$

The equivariant model for LM is given by adding a variable u of degree 2 to Λ_{LM} and extending $\overline{d}$ suitably. The dual to the BV operator

$$\Delta : H_*(LM) \longrightarrow H_{*+1}(LM)$$

is the operator $\check{\Delta} := (u = 0) \circ s$. The loop product may not have a dual at the level of forms in general. Schematically we denote it by $\triangleleft$. Similarly, the coproduct τ may not have a dual at the level of forms but it is defined at the cohomology level; we use $\circledast$ to denote it. Then the various operations on the equivariant homology can be written down using minimal models. For example, the dual to the string bracket is given by

$$(s \otimes s) \circ \triangleleft \circ (u = 0)$$

while the string cobracket ν is given by

$$s \circ \circledast \circ (u = 0)^{\otimes 2}.$$

Of course, these equations are to be interpreted whenever they make sense! One can, by suitable iterations, write down the higher multiplications and comultiplications. We shall not go into the exact details here as we've already seen an example in §3.1.3; more examples will be computed in §3.2.

3.2 String Topology of Product Manifolds

Let M, N be closed, oriented manifolds. It is natural to ask how the string homology Lie algebras $\mathcal{H}_*(LM), \mathcal{H}_*(LN)$ relate to the string homology of the product manifold $M \times N$. The free loop space $L(M \times N)$ is just the product of the free loop space of the LM and LN. Moreover, we have a natural isomorphism of loop algebras

$$H_*(L(M \times N)) \xrightarrow{\cong} H_*(LM) \otimes H_*(LN).$$

Notation *We use $\mathcal{S}_M$ to denote the quotient of $LM \times ES^1$ by the diagonal action of S^1. The homology of $\mathcal{S}_M$ is denoted by $\mathcal{H}_*^M$. The homology of $LM \times ES^1$ is denoted by $\mathbb{H}_{*-m}^M$ where m is the dimension of M.*

However, $L(M \times N)$ admits an action of $S^1 \times S^1$. The reduced S^1-equivariant homology of $L(M \times N)$ is a Lie bialgebra and fibres over the tensor product over the Lie bialgebras given by $\mathcal{H}_*^M \otimes \mathcal{H}_*^N$. In general, the tensor product of Lie algebras is *not* a Lie algebra. Therefore, one expects $\mathcal{H}_*^{M\times N}$ to be slightly involved. Indeed, we show that even if the Lie bialgebras of M and N arising from string topology are trivial, i.e., have trivial bracket and cobracket, the Lie bialgebra of the product manifold is *non-trivial*. This seems to be quite interesting as it provides a potential geometric method to get a non-trivial Lie algebra starting with two trivial Lie algebras.

One can apply similar methods that we employ to calculate the string bracket of products of three or more manifolds. In fact, we explicitly calculate this for a product of three odd dimensional spheres. However, essentially formal computations do tend to be tedious with more than two factors. In what follows we compute $\mathcal{H}_*^M$ for M the product of two spheres. For simplicity of the algebraic computations, we also assume that these spheres are simply connected. We also outline an approach for including S^1 in these computations although no explicit computations are worked out in this case.

3.2.1 The equivariant minimal model

Let M, N be oriented compact manifolds of dimension m, n respectively. We shall be implicitly making natural identifications between $L(M \times N)$ and $LM \times LN$. Similar considerations hold for based loop spaces. In this section we work over rational coefficients throughout and all when we say homology it is understood to be over $\mathbb{Q}$ unless mentioned otherwise.

The circle action on LM by rotating the loops is not free. The Borel construction tells us to consider the diagonal action by the circle on $LM \times ES^1$. This action is free and the new space is homotopy equivalent to LM. Observe that the space $LM \times LN$ has a torus action. The diagonal circle inside the torus also acts on this space thereby resulting in a principal circle bundle

$$S^1 \hookrightarrow (LM \times ES^1) \times_{S^1} (LN \times ES^1) \xrightarrow{\pi} \mathcal{S}_M \times \mathcal{S}_N$$

which induces a long exact (Gysin) sequence :

$$\cdots \longrightarrow \mathcal{H}_i^{M\times N} \xrightarrow{\mathcal{E}'} \oplus_j(\mathcal{H}_j^M \otimes \mathcal{H}_{i-j}^N) \xrightarrow{c'} \oplus_j(\mathcal{H}_j^M \otimes \mathcal{H}_{i-2-j}^N) \xrightarrow{\mathcal{M}'} \mathcal{H}_{i-1}^{M\times N} \longrightarrow \cdots$$

This approach is not very useful as without the explicit knowledge of the maps it's hard to describe $\mathcal{H}_*^{M\times N}$. We shall outline a slightly different but rather tractable approach.

Notation *For a graded vector space $V = \oplus_{i\in\mathbb{Z}} V_i$, the graded vector space $\downarrow V$ is called the desuspension of V and $(\downarrow V)_i = V_{i+1}$.*

Recall that if $\mathcal{A}(M) := (\Lambda V_M, d_M)$ is a minimal model for a simply connected manifold M then the differential graded algebra

$$\mathcal{A}(LM) := (\Lambda(V_M \oplus \downarrow V_M), d_M)$$

is a model for computing the real (or rational) cohomology of LM. We should mention that d_M is defined on this bigger algebra by requiring $d_M s + s d_M = 0$ on ΛV, where s is the desuspension map extended to a derivation. The algebra $\mathcal{A}(\Omega M) := (\Lambda(\downarrow V_M), 0)$ is a model for ΩM. The loop space fibration $\Omega M \hookrightarrow LM \to M$ then corresponds to

$$\mathcal{A}(M) \hookrightarrow \mathcal{A}(LM) \xrightarrow{\pi_2} \mathcal{A}(\Omega M).$$

The algebra

$$\mathcal{A}(\mathcal{S}_M) := (\Lambda(V_M \oplus \downarrow V_M \oplus \mathbb{Q}u), \overline{d}_M)$$

is a model for $\mathcal{S}_M$, where $|u| = 2$, $d_M u = 0 = s(u)$ and $\overline{d}_M := d_M + us$. Then the (cohomology) Gysin sequence for $S^1 \hookrightarrow LM \times ES^1 \to \mathcal{S}_M$ is induced by the short exact sequence

$$\mathcal{A}(\mathcal{S}_M) \xrightarrow{\cup u} \mathcal{A}(\mathcal{S}_M) \xrightarrow{u=0} \mathcal{A}(LM),$$

where the connecting morphism is induced by $s : \mathcal{A}(LM) \to \mathcal{A}(\mathcal{S}_M)$.

Let M and N be two simply connected manifolds with minimal models $(\Lambda V_M, d_M)$ and $(\Lambda V_N, d_N)$. It can be easily verified that

$$(\Lambda V_{M\times N}, d\,) := (\Lambda(V_M \oplus V_N), d_M \otimes 1 + 1 \otimes d_N)$$

is a minimal model for $M \times N$. The Sullivan model for $\mathcal{S}_{M\times N}$ is given by

$$\mathcal{A}(\mathcal{S}_{M\times N}) := \Lambda(V_M \oplus \downarrow V_M \oplus V_N \oplus \downarrow V_N \oplus \mathbb{Q}u),$$

where the differential $\overline{d}_{M\times N}$ is $\overline{d}_M \otimes 1 + 1 \otimes \overline{d}_N$.

Let $\triangleleft_M$ denote the dual of the loop product, i.e.,

$$\triangleleft_M : \mathcal{A}(LM) \longrightarrow \mathcal{A}(LM) \otimes \mathcal{A}(LM)$$

is of degree m, where m is the dimension of M. One should note that this map is not always well defined (it's usually partially defined) at the level of forms or minimal models. Here we are supposing the existence of one which is fully defined. Then the dual to the string bracket is induced by

$$\begin{tikzcd}
\mathcal{A}(\mathcal{S}_{M\times N}) \arrow[r, "u=0"] & \mathcal{A}(LM) \otimes \mathcal{A}(LN) \arrow[d, "\triangleleft_M \otimes \triangleleft_N"] & \\
& \mathcal{A}(LM)^{\otimes 2} \otimes \mathcal{A}(LN)^{\otimes 2} \arrow[r, "(s\otimes s)\circ(23)"] & \mathcal{A}(\mathcal{S}_{M\times N})^{\otimes 2},
\end{tikzcd}$$

where (23) means interchanging the second and the third components following Koszul sign rule. We shall see explicit computations in the subsequent sections. In fact, the gravity algebra structure arising from the k-ary operations $\overline{m}_k, k \geq 2$ on $\mathcal{H}_*$ for $M \times N$ can also be written down in terms of that of M and N.

In the non-equivariant case, i.e., the loop homology $\mathbb{H}_*$, the BV operator and the loop bracket for $M \times N$ can be explicitly written down in terms of that of M and N. Let $\Delta_M, \{\ ,\ \}_M$ denote the BV operator and the loop bracket for M. Recall that the circle action on $L(M \times N) = LM \times LN$ is given by the action of the diagonal embedding of S^1 inside $S^1 \times S^1$. In homology, the class of this diagonal circle is the sum of the two generators of $H_1(S^1 \times S^1; \mathbb{Z})$. Therefore, for $a \otimes b \in \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LN)$,

$$\Delta_{M\times N}(a \otimes b) = \Delta_M(a) \otimes b + (-1)^{|a|+m} a \otimes \Delta_N(b). \tag{3.2.1}$$

This can also be seen by observing that the suspension map acts exactly the same way on the cohomology of free loop space. Using (3.2.1) we can calculate the loop bracket and conclude that

$$\{a_1 \otimes b_1, a_2 \otimes b_2\} = (-1)^{(|a_2|+1)|b_1|}\Big(\{a_1, a_2\} \otimes (b_1 \bullet b_2) + (-1)^{|a_2|+|b_1|+m}(a_1 \bullet a_2) \otimes \{b_1, b_2\}\Big). \tag{3.2.2}$$

Note that unlike the equivariant loop homology, where we restricted ourselves to simply connected manifolds, the formulae above for the BV operator and the loop bracket works in full generality.

3.2.2 The product of two odd spheres

Let S^m and S^n be two spheres of odd dimensions greater than one. Let $X = S^m \times S^n$ denote the product manifold. Then the minimal models are

$$\mathcal{A}(LX) = \Lambda(x_1, y_1, x_2, y_2)$$

with the trivial differential and

$$\mathcal{A}(\mathcal{S}_X) = \Lambda(x_1, y_1, x_2, y_2, u), |x_1| = m, |x_2| = n, |y_1| = m-1, |y_2| = n-1, |u| = 2,$$

where the differential is given by

$$\mathbf{d}x_1 = y_1 u,\ \mathbf{d}x_2 = y_2 u,\ \mathbf{d}y_1 = 0 = \mathbf{d}y_2,\ \mathbf{d}u = 0.$$

It therefore follows by a simple calculation that

$$\begin{aligned}
H^{\text{odd}}(LX) &= \mathbb{Q}_{\text{span}}\{x_1 y_1^a y_2^b, x_2 y_1^c y_2^d\} \\
H^{\text{even}}(LX) &= \mathbb{Q}_{\text{span}}\{y_1^a y_2^b, x_1 x_2 y_1^c y_2^d\} \\
\mathcal{H}^{\text{odd}}(LX) &= \mathbb{Q}_{\text{span}}\{(x_2 y_1 - x_1 y_2) y_1^c y_2^d\} \\
\mathcal{H}^{\text{even}}(LX) &= \mathbb{Q}_{\text{span}}\{y_1^c y_2^d, u^a\}.
\end{aligned}$$

The comultiplication map in loop cohomology

$$\triangleleft : \mathbb{H}^*(L(S^m \times S^n)) \longrightarrow \mathbb{H}^*(L(S^m \times S^n)) \otimes \mathbb{H}^*(L(S^m \times S^n))$$

is just the tensor product of the comultiplication of S^m and S^n respectively. We will need :

Lemma 3.11. *The comultiplication on $H^*(L(S^m \times S^n))$ is given by*

$$\begin{aligned}
\triangleleft(y_1^{[c]}y_2^{[d]}) &= (x_1x_2 \otimes 1 + 1 \otimes x_1x_2) \sum_{i=0,j=0}^{c,d} y_1^{[i]}y_2^{[j]} \otimes y_1^{[c-i]}y_2^{[d-j]} \\
&- \sum_{i=0,j=0}^{c,d} \Big(x_1 y_1^{[i]}y_2^{[j]} \otimes x_2 y_1^{[c-i]}y_2^{[d-j]} - x_2 y_1^{[c-i]}y_2^{[d-j]} \otimes x_1 y_1^{[i]}y_2^{[j]} \Big) \\
\triangleleft(x_1 y_1^{[c]}y_2^{[d]}) &= -(x_2 \otimes 1 + 1 \otimes x_2) \sum_{i=0,j=0}^{c,d} x_1 y_1^{[i]}y_2^{[j]} \otimes x_1 y_1^{[c-i]}y_2^{[d-j]} \\
\triangleleft(x_2 y_1^{[c]}y_2^{[d]}) &= (x_1 \otimes 1 + 1 \otimes x_1) \sum_{i=0,j=0}^{c,d} x_2 y_1^{[i]}y_2^{[j]} \otimes x_2 y_1^{[c-i]}y_2^{[d-j]} \\
\triangleleft(x_1x_2 y_1^{[c]}y_2^{[d]}) &= \sum_{i=0,j=0}^{c,d} x_1x_2 y_1^{[i]}y_2^{[j]} \otimes x_1x_2 y_1^{[c-i]}y_2^{[d-j]}.
\end{aligned}$$

Proof The comultiplication for an odd dimensional sphere was calculated before (refer (3.1.20) and (3.1.21)). Notice that by construction, y_1^j (resp. y_2^k) is an element of odd degree in $\mathbb{H}^*(LS^m)$ (resp. $\mathbb{H}^*(LS^n)$). Therefore,

$$(\alpha \otimes y_1^j) \cdot (y_2^k \otimes \beta) = -(\alpha y_2^k \otimes y_1^j \beta)$$

for arbitrary elements α, β. The identities now follow from easy but tedious calculations using the previous observation and the tensor product of the comultiplications on S^m and S^n respectively. $\square$

We can now shed light on the string bracket by calculating the cobracket and dualizing it.

Theorem 3.12. *Let $e_{c,d}, h_{c,d}, u_c$ denote the basis dual to $y_1^{[c]}y_2^{[d]}$, $(x_2y_1 - x_1y_2)y_1^{[c]}y_2^{[d]}$ and u^c respectively. The elements u_c are central and the string bracket in $\mathcal{H}_*(LX;\mathbb{Q})$ satisfies the following :*

(3.2.3) $$[h_{i,j}, h_{k,l}] = 0$$

(3.2.4) $$[e_{i,j}, e_{k,l}] = (il - jk)e_{i+k-1,j+l-1}$$

(3.2.5) $$[h_{i,j}, e_{k,l}] = (il - jk)h_{i+k-1,j+l-1}.$$

Proof Let η denote the dual of the string bracket. Recall that in our model

$$\eta = (\mathcal{M}^* \otimes \mathcal{M}^*) \circ \triangleleft \circ \mathcal{E}^* = (s \otimes s) \circ \triangleleft \circ (u = 0).$$

As observed before, we calculate the cobracket and then dualize to get the string bracket. Since $\mathcal{E}^*$ sends u to zero, $\eta(u^c) = 0$ and no u_c term arises as as term in the string bracket of any two elements. The map $\mathcal{E}^*$ is an isomorphism for $y_1^{[c]}y_2^{[d]}, (x_2y_1 - x_1y_2)y_1^{[c]}y_2^{[d]}$. Also notice that

$$\begin{aligned} s(x_1y_1^{[c]}y_2^{[d]}) &= (c+1)y_1^{[c+1]}y_2^{[d]} \\ s(x_2y_1^{[c]}y_2^{[d]}) &= (d+1)y_1^{[c]}y_2^{[d+1]}. \end{aligned}$$

Therefore, $\triangleleft \circ \mathcal{E}^*(y_1^{[c]}y_2^{[d]})$ equals

$$\Big(\sum_{i=0}^{c} y_1^{[i]} \otimes x_1y_1^{[c-i]} + x_1y_1^{[c-i]} \otimes y_1^{[i]}\Big) \cdot \Big(\sum_{j=0}^{d} y_2^{[j]} \otimes x_2y_2^{[d-j]} + x_2y_2^{[d-j]} \otimes y_2^{[j]}\Big)$$

and when we apply $s \otimes s$ to the above, the terms that have x_i's in only one component become zero. Recall that $(s \otimes s)(\alpha \otimes \beta) = (-1)^{|\alpha|}s(\alpha) \otimes s(\beta)$. Therefore, we have

$$\begin{aligned} \eta(y_1^{[c]}y_2^{[d]}) &= \sum_{i=0,j=0}^{c,d} s(x_2y_1^{[i]}y_2^{[j]}) \otimes s(x_1y_1^{[c-i]}y_2^{[d-j]}) \\ &\quad - \sum_{i=0,j=0}^{c,d} s(x_1y_1^{[i]}y_2^{[j]}) \otimes s(x_2y_1^{[c-i]}y_2^{[d-j]}) \\ &= - \sum_{i=0,j=0}^{c,d} \Big((j+1)(c+1-i)y_1^{[i]}y_2^{[j+1]} \otimes y_1^{[c+1-i]}y_2^{[d-j]} \\ &\quad -(i+1)(d+1-j)y_1^{[i+1]}y_2^{[j]} \otimes y_1^{[c-i]}y_2^{[d+1-j]}\Big) \\ &= \sum_{i+k=c+1,j+l=d+1} (il - jk)y_1^{[i]}y_2^{[j]} \otimes y_1^{[k]}y_2^{[l]}. \end{aligned}$$

The dual of this is exactly (3.2.4). Similarly, using the previous lemma and calculating (as before) we're led to

$$\begin{aligned} \eta((x_2y_1 - x_1y_2)y_1^{[c]}y_2^{[d]}) &= (il - jk)\Big(\sum_{i+k=c+1,j+l=d+1} (x_2y_1 - x_1y_2)y_1^{[i]}y_2^{[j]} \otimes y_1^{[k]}y_2^{[l]} \\ &\quad - y_1^{[i]}y_2^{[j]} \otimes (x_2y_1 - x_1y_2)y_1^{[k]}y_2^{[l]}\Big), \end{aligned}$$

which is dual to (3.2.5). Moreover, since the cobracket of no element has a term of the form $u^c \otimes \alpha, \alpha \otimes u^c$ or $(x_2y_1 - x_1y_2)y_1^{[k]}y_2^{[l]} \otimes (x_2y_1 - x_1y_2)y_1^{[i]}y_2^{[j]}$, (3.2.3) and the centrality of u_c follows. $\square$

One can verify, independently, that the bracket defined in the proposition does satisfy the Jacobi identity, whence is a Lie bracket.

Let $\widetilde{\mathcal{H}}_* = \mathcal{H}_*/\mathbb{Q}[u]$ denote the reduced equivariant loop homology. Then it splits as a vector space into a semisimple part generated by $h_{i,j}$'s and a non-nilpotent part generated by $e_{i,j}$'s. In $\widetilde{\mathcal{H}}_*$, $e_{0,0}$ becomes zero. In fact,

$$[e_{1,i+1}, e_{1,j+1}] = (i-j)e_{1,i+j+1}$$

is reminiscent of the Witt algebra, the Lie algebra of meromorphic vector fields on the circle. It was shown by Chas and Sullivan [5] that the reduced equivariant loop homology admits an involutive Lie bialgebra structure. It also admits a certain algebraic structure called the *gravity algebra*. This was first discovered in the mathematical realm and called so by E. Getzler. In particular, this generalizes the structure of a Lie algebra. The operations $\overline{m}_k, k \geq 2$ provide such a structure on $\mathcal{H}_*$, with shifted grading. We shall need :

Definition 3.13. Let M be a closed, oriented manifold of dimension m and let $\mathcal{H}_*$ denote the S^1-equivariant homology of LM. We define the *suspended string homology* to be

$$(\mathcal{H}[m-2])_* := \mathcal{H}_{*+m-2}. \tag{3.2.6}$$

With this new grading, the operations $\overline{m}_k$ are of degree $2-k$. In our case, since $M = S^m \times S^n$ is even dimensional, the elements $h_{i,j}$ are of odd degree and $e_{i,j}$ are of even degree with respect to this new grading. For $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l \in (\mathcal{H}[m+n-2])_*$ and $k > 2, l \geq 0$ we have the following equalities

$$\begin{aligned}\sum_{1 \leq i < j \leq k} (-1)^{\varepsilon_{ij}} \overline{m}_{k+l-1}(\overline{m}_2(\alpha_i, \alpha_j), \alpha_1, \ldots, \widehat{\alpha}_i, \ldots, \widehat{\alpha}_j, \ldots, \alpha_k, \beta_1, \ldots, \beta_l) \\ = \overline{m}_{l+1}(\overline{m}_k(\alpha_1, \ldots, \alpha_k), \beta_1, \ldots, \beta_l),\end{aligned} \tag{3.2.7}$$

where the right hand side is understood to be zero if $l = 0$ and

$$\varepsilon_{ij} = (|\alpha_1| + \cdots + |\alpha_{i-1}|)|\alpha_i| + (|\alpha_1| + \cdots + |\alpha_{j-1}|)|\alpha_j| + |\alpha_i||\alpha_j|.$$

A graded vector space with operations satisfying (3.2.7) is a working definition of *gravity algebra*. When $k = 3, l = 0$ we get precisely the Jacobi identity, whence such algebras are generalizations of Lie algebras. The gravity algebra structure for the case at hand can be written down explicitly.

Proposition 3.14. *Let $\alpha_1, \ldots, \alpha_k \in (\mathcal{H}[m+n-2])_*$ and $k \geq 2$. Then $\overline{m}_k \equiv 0$ if all the α_i's are of even degree or if at least three are of odd degree. Otherwise, for $i = i_1 + \cdots + i_k$, $j = j_1 + \cdots + j_k$ we have*

$$\begin{aligned}
\overline{m}_k(h_{i_1,j_1}, \ldots, e_{i_s,j_s}, \ldots, h_{i_k,j_k}) &= \begin{cases} (i_s j - i j_s) h_{i-1,j-1}, & \textit{if } s < k \\ (i j_s - i_s j) h_{i-1,j-1}, & \textit{if } s = k \end{cases} \\
\overline{m}_k(h_{i_1,j_1}, \ldots, e_{i_r,j_r}, \ldots, e_{i_s,j_s}, \ldots, h_{i_k,j_k}) &= \begin{cases} (i_s j_r - i_r j_s) e_{i-1,j-1}, & \textit{if } s < k \\ (i_r j_s - i_s j_r) e_{i-1,j-1}, & \textit{if } s = k. \end{cases}
\end{aligned}$$

Proof The result follows from simple but tedious computations in the spirit of what we have seen before. We spare the reader the details. □

The cobracket on the reduced equivariant homology of the free loop space is a map

$$\nu : \widetilde{\mathcal{H}}_* \to \widetilde{\mathcal{H}}_* \otimes \widetilde{\mathcal{H}}_*,$$

which is of degree $2-m-n$, skew-symmetric and satisfies coJacobi. Moreover, the Lie bialgebra $(\widetilde{\mathcal{H}}_*, [\,,\,], \nu)$ is involutive ($[\,,\,] \circ \nu = 0$) and the Drinfeld compatibility relation (ν is a derivation of $[\,,\,]$) holds. We use these constraints to get an ansatz for ν. We write the action of ν as

$$\begin{aligned}
\nu(e_{c,d}) &= \sum_{i=0,j=0}^{c-1,d-1} \mu_{i,j}^{c,d} e_{i,j} \wedge e_{c-1-i,d-1-j} \\
\nu(h_{c,d}) &= \sum_{i=0,j=0}^{c-1,d-1} \lambda_{i,j}^{c,d} h_{i,j} \wedge e_{c-1-i,d-1-j}
\end{aligned}$$

for some constants $\lambda_{i,j}^{c,d}, \mu_{i,j}^{c,d}$ that we are interested in. From degree considerations and the fact $e_{0,0} = 0$ we see that

$$\begin{aligned}
\nu(e_{1,1}) = \nu(e_{k,0}) = \nu(e_{0,k}) = 0, &\qquad k \geq 0 \\
\nu(h_{1,1}) = \nu(h_{k,0}) = \nu(h_{0,k}) = 0, &\qquad k \geq 0.
\end{aligned}$$

Same arguments also show that

$$\nu(e_{2,1}) = 0 = \nu(e_{1,2}).$$

Therefore, it follows that

$$-2c\nu(e_{1,c}) = \nu([e_{0,c}, e_{2,1}]) = [\nu(e_{0,c}), e_{2,1}] + [e_{0,c}, \nu(e_{2,1})] = 0, \;\; c > 1.$$

This means $\nu(e_{1,c}) = 0$ and interchanging the indices we get $\nu(e_{c,1}) = 0$. Therefore,

$$(cd-1)\nu(e_{c,d}) = \nu([e_{c,1}, e_{1,d}]) = 0$$

by Drinfeld compatibility. This compatibility relation again implies that

$$(c-1)\nu(h_{1,c}) = \nu([h_{1,1}, e_{1,c}]) = 0,$$

whence $\nu(h_{1,c}) = 0 = \nu(h_{c,1})$. Since

$$(cd-1)h_{c,d} = [h_{c,1}, e_{1,d}],$$

again by Drinfeld compatibility we conclude that ν kills $h_{c,d}$. Therefore,

Proposition 3.15. *The cobracket ν on the reduced string homology is trivial.*

3.2.3 The product of three odd spheres

Let $X = S^l \times S^m \times S^n$ denote the product of three odd dimensional spheres such that each is simply connected. The model for LX is

$$\mathcal{A}(LX) = \Lambda(x_1, x_2, x_3, y_1, y_2, y_3)$$

with the trivial differential and

$$|x_1| = l,\ |x_2| = m,\ |x_3| = n,\ |y_1| = l-1,\ |y_2| = m-1,\ |y_3| = n-1.$$

The model for $LX \times_{S^1} ES^1$ is

$$\mathcal{A}(\mathcal{S}_X) = \mathcal{A}(LX) \otimes \Lambda(u),\ |u| = 2$$

with the differential given by $\mathbf{d}x_i = y_i u$ and $\mathbf{d}y_i = 0 = \mathbf{d}u$. Calculating with these models we are lead to

$$\begin{aligned}
H^{\text{even}} &= \mathbb{Q}_{\text{span}}\{x_1x_2y_1^ay_2^by_3^c, x_2x_3y_1^ay_2^by_3^c, x_3x_1y_1^ay_2^by_3^c, y_1^ay_2^by_3^c\} \\
H^{\text{odd}} &= \mathbb{Q}_{\text{span}}\{x_1y_1^ay_2^by_3^c, x_2y_1^ay_2^by_3^c, x_3y_1^ay_2^by_3^c, x_1x_2x_3y_1^ay_2^by_3^c\} \\
\mathcal{H}^{\text{even}} &= \mathbb{Q}_{\text{span}}\{y_1^ay_2^by_3^c, u^r, (x_1x_2y_3 + x_2x_3y_1 + x_3x_1y_2)y_1^ay_2^by_3^c\} \\
\mathcal{H}^{\text{odd}} &= \mathbb{Q}_{\text{span}}\big\{(x_iy_{\sigma(i)} - x_{\sigma(i)}y_i)y_1^ay_2^by_3^c \,|\, \sigma \in \{(12),(23),(13)\}\big\}
\end{aligned}$$

Using (3.1.20),(3.1.21) we can determine the dual to the loop product for LX. Let $h_{i,a,b,c}$ denote the dual to $(x_ky_j - x_jy_k)y_1^{[a]}y_2^{[b]}y_3^{[c]}$, where i, j, k are elements of the cyclically ordered set $\{1,2,3\}$. Observe that the following relation holds as can be seen by dualizing :

$$(a+1)h_{1,a+1,b,c}+(b+1)h_{2,a,b+1,c}+(c+1)h_{3,a,b,c+1}=0. \tag{3.2.8}$$

Let $f_{a,b,c}$ be the dual to $(x_1x_2y_3+x_2x_3y_1+x_3x_1y_2)y_1^{[a]}y_2^{[b]}y_3^{[c]}$ while $e_{a,b,c}$ denote the dual to $y_1^{[a]}y_2^{[b]}y_3^{[c]}$. Therefore, the reduced equivariant homology is given by

$$\begin{aligned}\widetilde{\mathcal{H}}_{\text{even}} &= \mathbb{Q}_{\text{span}}\{e_{a,b,c}, f_{a,b,c}\}\\ \widetilde{\mathcal{H}}_{\text{odd}} &= \mathbb{Q}_{\text{span}}\{h_{1,a,b,c}, h_{2,a,b,c}, h_{3,a,b,0}\}\end{aligned}$$

As calculated in §3.2.2, we dualize the loop coproduct. Similar computations, long and tedious, and making use of (3.2.8) lead to the following :

Proposition 3.16. *The string bracket for $S^l\times S^m\times S^n$ is given by*

$$\begin{aligned}
[e_{i,j,k},e_{p,q,r}] &= 0\\
[f_{i,j,k},f_{p,q,r}] &= 0\\
[e_{i,j,k},h_{1,p,q,r}] &= (i-pk/(r+1))e_{i+p-1,j+q,k+r}\\
[e_{i,j,k},h_{2,p,q,r}] &= (j-qk/(r+1))e_{i+p,j+q-1,k+r}\\
[e_{i,j,k},h_{3,p,q,0}] &= ke_{i+p,j+q,k-1}\\
[f_{i,j,k},h_{1,p,q,r}] &= (i+p)f_{i+p-1,j+q,k+r}\\
[f_{i,j,k},h_{2,p,q,r}] &= -(j+q)f_{i+p,j+q-1,k+r}\\
[f_{i,j,k},h_{3,p,q,0}] &= kf_{i+p,j+q,k-1}\\
[h_{1,i,j,k},h_{2,p,q,r}] &= (i+p)h_{2,i+p-1,j+q,k+r}-(j+q)h_{1,i+p,j+q-1,k+r}\\
[h_{2,i,j,0},h_{3,p,q,0}] &= (j+q)h_{3,i+p,j+q-1,0}\\
[h_{2,i,j,k},h_{3,p,q,0}] &= -((i+p+1)(j+q)/k)h_{1,i+p+1,j+q-1,k-1}\\
&\quad -(k+(j+q)^2/k)h_{2,i+p,j+q,k-1},\quad k\geq 1\\
[h_{3,i,j,0},h_{1,p,q,0}] &= -(i+p)h_{3,i+p-1,j+q,0}\\
[h_{3,i,j,0},h_{1,p,q,r}] &= (r+(i+p)^2/r)h_{1,i+p,j+q,r-1}\\
&\quad +((i+p)(j+q+1)/r)h_{2,i+p-1,j+q+1,r-1},\quad r\geq 1\\
[f_{i,j,0},e_{p,q,0}] &= (iq-jp)h_{3,i+p-1,j+q-1,0}\\
[f_{i,j,k},e_{p,q,r}] &= \frac{(k+r)(jr-kq)-(iq-jp)(i+p)}{k+r}h_{1,i+p,j+q-1,k+r-1}\\
&\quad +\frac{(k+r)(kp-ir)-(iq-jp)(j+q)}{k+r}h_{2,i+p-1,j+q,k+r-1}.
\end{aligned}$$

Remark 3.17. (1) *Similar computations can be done for product of odd spheres allowing one or many of them to be circles. Although it's not included in this work, the author has computations to that extent.*
(2) *As is evident from the equations above, the string bracket complexifies quite a bit as the number of factors in the product grows. One hopes that there is a systematic procedure to study and classify such Lie algebras arising naturally from topology.*
(3) *It is possible that there is an understandable geometric reason behind the triviality of the string cobracket. However, this is more of thinking out loud on the author's part and may be unfounded!*

3.2.4 String Topology of Lie groups

We shall be dealing with compact Lie groups. It is known that any Lie group deformation retracts to any of its maximal compact subgroup, thereby inducing a homotopy equivalence. Compactness is essential for the fundamental class $[G] \in H^*(G;\mathbb{Z})$, which is the unit in loop homology algebra. We also assume, when necessary, that G is simply connected so that rational homotopy methods can be used easily.

The free loop space of G, denoted by LG, is an infinite dimensional Lie group. We will call it the *loop group* of G. The natural evaluation map ev : $LG \to G$, which evaluates a loop at $1 \in S^1$, gives rise to a fibration $\Omega G \hookrightarrow LG \to G$. It is easily seen that

$$\varphi : LG \to G \times \Omega G,\ \gamma \mapsto (\gamma(1), \gamma(1)^{-1}\gamma)$$

is a homeomorphism. It is a classical result of Bott that ΩG can be given a cellular structure using just even dimensional cells. This shows, using cellular homology for instance, that $H_*(\Omega G;\mathbb{Z})$ is free and is zero in odd dimensions. Therefore, the Kunneth theorem guarantees that

$$H_*(G \times \Omega G;\mathbb{Z}) \cong H_*(G;\mathbb{Z}) \otimes H_*(\Omega G;\mathbb{Z}).$$

Therefore, there is an induced isomorphism

$$\varphi_* : (\mathbb{H}_*(LG;\mathbb{Z}), \bullet) \xrightarrow{\cong} (\mathbb{H}_*(G;\mathbb{Z}), \cap) \otimes (H_*(\Omega G;\mathbb{Z}), \times),$$

where $\cap$ is the classical *intersection product* on homology and $\times$ is the concatenation of loops. The product $\times$ is classically known as the *Pontrjagin product*. However, $\times$ is homotopic to $*$, the pointwise multiplication of loops, via

$$\begin{array}{rcl} F_t & : & \Omega G \times \Omega G \times [0,1/2] \longrightarrow \Omega G, \\ F_t(\gamma_1,\gamma_2)(s) & := & \left\{ \begin{array}{rl} \gamma_1(2s(1-t))\gamma_2(2st), & s \in [0,1/2] \\ \gamma_1(2st-2t+1)\gamma_2(2s+2t-2st-1), & s \in [1/2,1]. \end{array} \right. \end{array}$$

As a consequence,

$$(H_*(\Omega G;\mathbb{Z}), \times) \cong (H_*(\Omega G;\mathbb{Z}), *).$$

Moreover, both $\mathbb{H}_*(G;\mathbb{Z})$ and $H_*(\Omega G;\mathbb{Z})$ are Hopf algebras with corresponding antipodes given by the inverse map in G. Therefore, $\mathbb{H}_*(G;\mathbb{Z}) \otimes H_*(\Omega G;\mathbb{Z})$ is a

Hopf algebra. Since LG is a topological group (we assume that LG is connected) it follows that $\mathbb{H}_*(LG;\mathbb{Z})$ is also a Hopf algebra. Although, at the level of chains/spaces this structure is different from that induced from the product of G and ΩG, it is the same when we pass to homology. In fact, we have an isomorphism of *involutive* Hopf algebras

$$\Phi : \mathbb{H}_*(LG;\mathbb{Z}) \xrightarrow{\cong} \mathbb{H}_*(G;\mathbb{Z}) \otimes H_*(\Omega G;\mathbb{Z}).$$

More precisely, at the level of spaces

$$\tilde{\iota}(g,\gamma) := (g^{-1}, g\gamma^{-1}g^{-1}) = (\iota(g), \mathrm{Ad}_g(\iota(\gamma))),\ \ (g,\gamma) \in G \times \Omega G.$$

We have the adjoint action of G on ΩG which induces a map

$$H_*(G;\mathbb{Z}) \otimes H_*(\Omega G;\mathbb{Z}) \xrightarrow{\mathrm{Ad}} H_*(\Omega G;\mathbb{Z}).$$

Since $H_*(\Omega G;\mathbb{Z})$ is only concentrated in even degrees, $\mathrm{Ad}(\alpha,\beta) = 0$ if α is of odd degree. Moreover, as we shall in the next section, $H_{\text{even}}(G;\mathbb{Z})$ is only torsion while $H_*(\Omega G;\mathbb{Z})$ has no torsion. Therefore, $\mathrm{Ad}(\alpha,\beta) = 0$ for α of even degree unless $\alpha \in H_0(G;\mathbb{Z})$. It is clear that $H_0(G;\mathbb{Z})$ acts as the identity on $H_*(\Omega G;\mathbb{Z})$. This means that Φ is an isomorphism as claimed.

Although $H_*(G;\mathbb{Z})$, with the intersection product, is a Frobenius algebra, $H_*(\Omega G;\mathbb{Z})$ doesn't admit a Frobenius algebra structure. Hence, it's not clear whether $H_*(LG;\mathbb{Z})$ has any Frobenius algebra structure. But, LG being an H-space in particular, is a universal enveloping algebra due to the main theorem of Milnor & Moore's famous paper [28]. For the remainder of this discussion, we shall mainly work over rational coefficients. The only information we'll be loosing is $\mathrm{Tor}(H_*(G;\mathbb{Z}))$. Before we proceed, a couple of remarks are in order.

Remark 3.18. (BV operator at the chain level)
It is possible to describe the loop product for LG at the chain level via chains on $G \times \Omega G$, i.e., on $C_*(G) \otimes C_*(\Omega G)$

$$\alpha \otimes \beta \xrightarrow{\Delta} \big((\alpha \circ \mathrm{pr}_k)\mathcal{T}(\beta \circ \mathrm{pr}^{i+j-k}), \mathcal{R}(\beta \circ \mathrm{pr}^{i+j-k})\big)$$

gives a model for $\bullet$ on $C_*(LG)$. If $\alpha : [0,1]^k \to G, \beta : [0,1]^{i+j-k} \to \Omega G$ then pr_l *denotes projection to the first l coordinates while* pr^l *denotes projection to the last l coordinates. The maps $\mathcal{T}, \mathcal{R}$ are given as follows : using the evaluation map $ev : \Omega G \times S^1 \to G$ one can define the transgression map*

$$\mathcal{T} : C_*(\Omega G) \to C_{*+1}(G),\ \ \mathcal{T}(\beta(q,\cdot)) := (\beta(q,s)).$$

This is chain map and induces a map on homology. There is a twisted rotation

$$\mathcal{R} : C_*(\Omega G) \to C_{*+1}(\Omega G),\ \mathcal{R}(\beta(q,\cdot), s) := \beta(q,s)^{-1}\beta(q, \cdot + s).$$

The induced map $\mathcal{R}_* : H_*(\Omega G;\mathbb{Z}) \to H_{*+1}(\Omega G;\mathbb{Z})$ *is zero due to the classical result by Bott that* $H_*(\Omega G;\mathbb{Z})$ *is free and exists only in even degrees. This approach has been taken up by Hepworth* [20] *where integral calculations were done for orthogonal groups.*

Remark 3.19. *A rational homotopy equivalence* $f : M \to N$ *of simply connected closed manifolds induces an isomorphism of vector spaces*

$$f_* : H_*^{S^1}(LM;\mathbb{Q}) \longrightarrow H_*^{S^1}(LN;\mathbb{Q}), \tag{3.2.9}$$

If it is known that (3.2.9) *is actually an isomorphism of Lie bialgebras then the said structure for a compact Lie group is characterized by the Lie bialgebra structure for products of odd spheres (already taken up in §3.2). This is very likely to be true although presently there aren't any proofs.*

Homology of loop groups

It follows from Serre's version (for example, this can be found in [13]) of the classical Whitehead theorem that if $\varphi : X \to Y$ is a rational homotopy equivalence of simply connected closed manifolds then $\varphi^* : H^*(X;\mathbb{Q}) \to H^*(Y;\mathbb{Q})$ is an isomorphism of rings. Moreover, as shown in [14], the rational loop homology algebra of X is isomorphic to the Hochschild cohomology of the cochains in X. If both manifolds X and Y are formal and φ is as above then there is an algebra isomorphism between the minimal models of X and Y, whence there is an algebra isomorphism between $H_*(LX;\mathbb{Q})$ and $H_*(LY;\mathbb{Q})$. This is useful since it is classically known, due to Serre, that a compact Lie group G is rational homotopy equivalent to a product of odd spheres. In fact, if r is the rank of G then there are odd spheres $S^{2i_j+1}, j = 1,\dots,r$ and a map

$$\varphi : S^{2i_1+1} \times \cdots \times S^{2i_r+1} \longrightarrow G \tag{3.2.10}$$

such that φ induces an isomorphism of rational homotopy groups. This also shows that the cohomology ring of G, being isomorphic to the product of the cohomology rings of the spheres, is of dimension 2^r.

It was shown in [9] that the following is an isomorphism of algebras

$$(\mathbb{H}_*(LS^{2k+1};\mathbb{Z}), \bullet) \cong \Lambda_{\mathbb{Z}}(\alpha) \otimes \mathbb{Z}[\beta], \tag{3.2.11}$$

where $\alpha = [x_0] \in H_0(LS^{2k+1};\mathbb{Z})$ is the class of the constant loop at a point, $1 \otimes 1 = [S^{2k+1}] \in H_{2k+1}(LS^{2k+1};\mathbb{Z})$ and $\beta \in H_{4k+1}(LS^{2k+1};\mathbb{Z})$. A few examples are in order, using the well known rational decomposition of classical Lie groups into spheres.

Example 3.20. (Spin groups)

Let $G = \text{Spin}(n)$ be the spin group. It is a double cover of the special orthogonal group $\text{SO}(n)$ of dimension $n(2n+1)$ and rank n. Using the fibrations $\text{SO}(n-1) \hookrightarrow \text{SO}(n) \to S^{n-1}$ one can calculate the rational homotopy groups of $\text{Spin}(n)$. We have a map

$$\varphi : S^3 \times S^7 \times \ldots \times S^{4n-1} \longrightarrow \text{Spin}(2n+1)$$

which is a rational homotopy equivalence. In this case,

$$\big(\mathbb{H}_*(L\text{Spin}(2n+1);\mathbb{Q}), \bullet\big) \cong \Lambda_{\mathbb{Q}}(\alpha_3, \alpha_7, \ldots, \alpha_{4n-1}) \otimes \mathbb{Q}[\beta_3, \ldots, \beta_{4n-1}],$$

where the $|\alpha_i| = -i$ and $|\beta_i| = i$. If we set $G = \text{Spin}(2n+2)$ then there is a rational homotopy equivalence

$$\varphi : S^3 \times S^7 \times \ldots \times S^{4n-1} \times S^{2n+1} \longrightarrow \text{Spin}(2n+2).$$

As a consequence

$$\big(\mathbb{H}_*^{\mathbb{Q}}(L\text{Spin}(2n+2)), \bullet\big) \cong \Lambda_{\mathbb{Q}}(\alpha_3, \ldots, \alpha_{4n-1}, \alpha'_{2n+1}) \otimes \mathbb{Q}[\beta_3, \ldots, \beta_{4n-1}, \beta'_{2n+1}].$$

In both cases, φ induces a map at the level of free loop spaces which commutes with the loop product and the action of the circle by rotation.

Proposition 3.21. (1) *There is an isomorphism of BV algebras*

$$\big(\mathbb{H}_*^{\mathbb{Q}}(L\text{Spin}(2n+1)), \bullet\big) \cong \Lambda_{\mathbb{Q}}(\alpha_3, \ldots, \alpha_{4n-1}) \otimes \mathbb{Q}[\beta_3, \beta_7 \ldots, \beta_{4n-1}],$$

where the BV operator $\mathbf{\Delta}$ *on the right hand side is given by*

$$\mathbf{\Delta}(\alpha \otimes \beta) := \sum_{i=1}^{n} \frac{\partial \alpha}{\partial \alpha_{4i-1}} \otimes \frac{\partial \beta}{\partial \beta_{4i-1}}. \tag{3.2.12}$$

Here $\partial/\partial\alpha_{4i-1}$ *is extended as an odd derivation while* $\partial/\partial\beta_{4i-1}$ *is considered an even derivation.*

(2) *There is an isomorphism of BV algebras*

$$\big(\mathbb{H}_*^{\mathbb{Q}}(L\text{Spin}(2n)), \bullet\big) \cong \Lambda_{\mathbb{Q}}(\alpha_3, \ldots, \alpha_{4n-5}, \alpha_{2n-3}) \otimes \mathbb{Q}[\beta_3, \ldots, \beta_{4n-5}, \beta_{2n-3}],$$

where $\mathbf{\Delta}$ denotes the BV operator on the right hand side; it's given by

$$\mathbf{\Delta}(\alpha \otimes \beta) = \frac{\partial \alpha}{\partial \alpha_{2n-3}} \otimes \frac{\partial \beta}{\partial \beta_{2n-3}} + \sum_{i=1}^{n-1} \frac{\partial \alpha}{\partial \alpha_{4i-1}} \otimes \frac{\partial \beta}{\partial \beta_{4i-1}}, \tag{3.2.13}$$

where $\partial/\partial\alpha_i$ is extended as an odd derivation while $\partial/\partial\beta_i$ is considered an even derivation.

Example 3.22. (Unitary groups)

Let $G = \mathsf{SU}(n)$ be the special unitary group. There is a tower of fibrations

$$\begin{tikzcd} S^3 = \mathsf{SU}(2) \arrow[r] & \mathsf{SU}(3) \arrow[r] \arrow[d] & \mathsf{SU}(4) \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathsf{SU}(n) \arrow[d] \\ & S^5 & S^7 & & S^{2n-1}. \end{tikzcd}$$

We conclude from the long exact sequence of homotopy groups and the fact that $\pi_i(S^{2k+1}) \otimes \mathbb{Q} = \mathbb{Q}$ only when $i = 2k+1$ that

$$\pi_{2i-1}(\mathsf{SU}(n)) \otimes \mathbb{Q} = \mathbb{Q},\ 2 \le i \le n.$$

Using the generators $f_{2i-1} : S^{2i-1} \to \mathsf{SU}(n)$ we can define the map

$$\varphi := f_3 \times \cdots \times f_{2n-1} : S^3 \times S^5 \times \ldots \times S^{2n-1} \longrightarrow \mathsf{SU}(n)$$

which is a rational homotopy equivalence. This is not a homotopy equivalence since $\pi_4(\mathsf{SU}(n)) = 0$ while $\pi_4(S^3) = \mathbb{Z}_2$. Therefore, there is an algebra isomorphism

$$\big(\mathbb{H}_*(L\mathsf{SU}(n);\mathbb{Q}), \bullet\big) \cong \Lambda_{\mathbb{Q}}(\alpha_3, \alpha_5, \ldots, \alpha_{2n-1}) \otimes \mathbb{Q}[\beta_3, \ldots, \beta_{2n-1}],$$

with generators α_i of degree $-i$ and β_i of degree $i-1$. Incidentally, this algebra structure above holds over $\mathbb{Z}$ although there is no map at the level of spaces, inducing it. Notice that $\varphi : LS^3 \times \cdots \times LS^{2n-1} \to L\mathsf{SU}(n)$ commutes with $\bullet$ and Δ. Moreover, since $\{\ ,\ \}$ is the deviation of Δ from being a derivation of $\bullet$, we conclude that φ commutes with the loop bracket as well. The precise statement follows.

Proposition 3.23. (1) *The map $\varphi : S^3 \times \cdots \times S^{2n-1} \to \mathsf{SU}(n)$, as defined previously, induces an isomorphism of BV algebras*

$$\big(\mathbb{H}_*(L\mathsf{SU}(n);\mathbb{Q}), \bullet, \Delta\big) \cong (\Lambda_{\mathbb{Q}}(\alpha_3, \alpha_5, \ldots, \alpha_{2n-1}) \otimes \mathbb{Q}[\beta_3, \ldots, \beta_{2n-1}], \mathbf{\Delta})$$

where $\mathbf{\Delta}$ denotes the BV operator for the product manifold $S^3 \times \cdots \times S^{2n-1}$. More precisely,

$$\mathbf{\Delta}(\alpha \otimes \beta) = \sum_{i=1}^{n-1} \frac{\partial \alpha}{\partial \alpha_{2i+1}} \otimes \frac{\partial \beta}{\partial \beta_{2i+1}}, \tag{3.2.14}$$

where $\partial / \partial \alpha_{2i+1}$ is extended as an odd derivation while $\partial / \partial \beta_{2i+1}$ is considered an even derivation.

(2) *The spaces $S^1 \times \mathrm{SU}(n)$ and $\mathrm{U}(n)$ are homeomorphic, via the map sending $(e^{i\theta}, A)$ to A with its first column multiplied by $e^{i\theta}$. Therefore, one has the following isomorphism of BV algebras*

$$\big(\mathbb{H}_*^{\mathbb{Q}}(L\mathrm{U}(n)), \bullet, \Delta\big) \cong \big(\Lambda_{\mathbb{Q}}(\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}) \otimes \mathbb{Q}[\beta_1, \beta_1^{-1}, \beta_3, \ldots, \beta_{2n-1}], \mathbf{\Delta}\big),$$

where the BV operator on the right is given by

$$\mathbf{\Delta}(\alpha \otimes \beta) = \frac{\partial \alpha}{\partial \alpha_1} \otimes \frac{\beta_1 \partial \beta}{\partial \beta_1} + \sum_{i=1}^{n-1} \frac{\partial \alpha}{\partial \alpha_{2i+1}} \otimes \frac{\partial \beta}{\partial \beta_{2i+1}}. \tag{3.2.15}$$

We may ask what is the induced map φ_* with integer coefficients. For starters, if $\varphi : S^3 \times S^5 \to \mathrm{SU}(3)$ then $\varphi_*([S^3])$ generates the 3-cycle in $\mathrm{SU}(3)$. However, $\varphi_*([S^5])$ is a positive multiple of the generating 5-cycle α_3 of $\mathrm{SU}(3)$. We see from the long exact sequence

$$\begin{array}{ccccccccc}
\pi_5(\mathrm{SU}(2)) & \longrightarrow & \pi_5(\mathrm{SU}(3)) & \longrightarrow & \pi_5(S^5) & \longrightarrow & \pi_4(\mathrm{SU}(2)) & \longrightarrow & \pi_4(\mathrm{SU}(3)) \\
\| & & \| & & \| & & \| & & \| \\
\mathbb{Z}_2 & \longrightarrow & \pi_5(\mathrm{SU}(3)) & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0
\end{array}$$

that the $\mathbb{Z}$-summand of $\pi_5(\mathrm{SU}(3))$ maps to $\pi_5(S^5)$ via multiplication by 2, i.e., $\varphi_*([S^5]) = 2\alpha_3$. Let

$$\mathbb{H}_*(LS^i; \mathbb{Z}) = \Lambda_Z(a_i) \otimes \mathbb{Z}[b_i], \ i = 3, 5, \ |a_i| = -i, |b_i| = i - 1.$$

Then we may write down a map

$$\widetilde{\varphi} : \mathbb{H}_*(LS^3; \mathbb{Z}) \otimes \mathbb{H}_*(LS^5; \mathbb{Z}) \xrightarrow{\cong} \mathbb{H}_*(L\mathrm{SU}(3); \mathbb{Z}) \tag{3.2.16}$$

$$\widetilde{\varphi}(a_3) = \frac{1}{2}\varphi_*(a_3) = \alpha_3, \ \widetilde{\varphi}(b_3) = \frac{1}{2}\varphi_*(b_3) = \beta_3, \ \widetilde{\varphi}(a_5) = \alpha_5, \ \widetilde{\varphi}(b_5) = \beta_5.$$

It can be checked that $\widetilde{\varphi}$ is an isomorphism of BV algebras.

Remark 3.24. *As pointed out before, there is no map* $f : S^3 \times S^5 \to \mathrm{SU}(3)$ *such that* $f_* = \widetilde{\varphi}$*. Any such* f *would induce an isomorphism on* H_5*. However, the Hurewicz map*

$$\mathcal{H} : \pi_5(\mathrm{SU}(3)) \longrightarrow H_5(\mathrm{SU}(3); \mathbb{Z})$$

is multiplication by 2*. Therefore, there cannot be a map from* S^5 *to* $\mathrm{SU}(3)$ *which induces isomorphism on* H_5*.*

More generally, one has $\mathrm{SU}(n-1) \hookrightarrow \mathrm{SU}(n) \to S^{2n-1}$ and the positive generator $f_{2n-1} : S^{2n-1} \to \mathrm{SU}(n)$ of $\pi_{2n-1}(\mathrm{SU}(n))$ maps $[S^{2n-1}]$ to a positive multiple of a chosen generator of $H^{\mathrm{free}}_{2n-1}(\mathrm{SU}(n); \mathbb{Z})$. Let us call this integer λ_n. We have seen that $\lambda_2 = 1$ and $\lambda_3 = 2$. Then one may define an isomorphism of BV algebras

$$\tilde{\varphi} : \mathbb{H}_*(L\mathrm{SU}(n-1); \mathbb{Z}) \otimes \mathbb{H}_*(LS^{2n-1}; \mathbb{Z}) \xrightarrow{\cong} H_*(L\mathrm{SU}(n); \mathbb{Z}). \tag{3.2.17}$$

Define $\tilde{\varphi}$ to be the inclusion on $L\mathrm{SU}(n-1)$ and send the generators of the BV algebra $\mathbb{H}_*(LS^{2n-1}; \mathbb{Z})$ to their natural corresponding generators but scaled inversely by λ_n. As for finding λ_n, one recalls the following part of the long exact sequence

$$\begin{tikzcd}
& \pi_{2n-1}(\mathrm{SU}(n)) \arrow[r] \arrow[d, equal] & \pi_{2n-1}(S^{2n-1}) \arrow[r] \arrow[d, equal] & \pi_{2n-2}(\mathrm{SU}(n-1)) \arrow[r] \arrow[d, equal] & \pi_{2n-2}(\mathrm{SU}(n)) \arrow[d, equal] \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\lambda_n"] & \mathbb{Z} \arrow[r] & \pi_{2n-2}(\mathrm{SU}(n-1)) \arrow[r] & 0
\end{tikzcd}$$

where the first and the last vertical equality follows from the celebrated Bott's periodicity theorem [1]. It is classically known, for instance it follows from Corollary 6.14 in [29], that $\pi_{2n}(\mathrm{SU}(n)) = \mathbb{Z}_{n!}$. Therefore, we arrive at the following result.

Theorem 3.25. *The map*

$$\Phi : \mathbb{H}_*(LS^3; \mathbb{Z}) \otimes \cdots \otimes \mathbb{H}_*(LS^{2n-1}; \mathbb{Z}) \longrightarrow \mathbb{H}_*(L\mathrm{SU}(n); \mathbb{Z}) \tag{3.2.18}$$

defined by scaling the generators of $\mathbb{H}_*(LS^{2i+1}; \mathbb{Z})$ *inversely by* $i!$ *is an isomorphism of BV algebras. Similar isomorphism holds for* $\mathrm{U}(n)$ *as well.*

Example 3.26. (Symplectic groups)

Let $G = \text{Sp}(n)$ be the symplectic group, i.e., we think of it as group of unitary $n \times n$ matrices with values in the quaternions. There is a map

$$\varphi : S^3 \times S^7 \times \ldots \times S^{4n-1} \longrightarrow \text{Sp}(n)$$

which is a rational homotopy equivalence. In fact, as has been the recurring motif, the integral cohomology ring of both spaces above are isomorphic ([19], Corollary 4D.3) but there isn't a map inducing it. However, as abstract algebras there is an isomorphism

$$(\mathbb{H}_*(L\text{Sp}(n);\mathbb{Z}), \bullet) \cong \Lambda_{\mathbb{Z}}(\alpha_3, \alpha_7, \ldots, \alpha_{4n-1}) \otimes \mathbb{Z}[\beta_3, \ldots, \beta_{4n-1}]$$

where $|\alpha_i| = -i$ and $|\beta_i| = i - 1$. As in the unitary case, we have a long exact sequence associated to the fibration $\text{Sp}(n-1) \hookrightarrow \text{Sp}(n) \to S^{4n-1}$ and we're interested in

$$\begin{tikzcd}
 & \pi_{4n-1}(\text{Sp}(n)) \arrow[r] \arrow[d, equal] & \pi_{4n-1}(S^{4n-1}) \arrow[r] \arrow[d, equal] & \pi_{4n-2}(\text{Sp}(n-1)) \arrow[r] \arrow[d, equal] & \pi_{4n-2}(\text{Sp}(n)) \arrow[d, equal] \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\mu_n"] & \mathbb{Z} \arrow[r] & \pi_{4n-2}(\text{Sp}(n-1)) \arrow[r] & 0
\end{tikzcd}$$

where the first and the last vertical equality follows from Bott's periodicity. More precisely, the leftmost and the rightmost groups in the first row are both in the stable range and the implication follows from

$$\pi_3(\mathbf{Sp}) = \pi_7(\mathbf{Sp}) = \mathbb{Z}, \;\; \pi_2(\mathbf{Sp}) = \pi_6(\mathbf{Sp}) = 0.$$

It was shown (Corollary[2] 6.14 in [29]) that

$$\pi_{4n+2}(\text{Sp}(n)) = \begin{cases} \mathbb{Z}_{2(2n-1)!} & \text{if } n \text{ is even} \\ \mathbb{Z}_{(2n-1)!} & \text{if } n \text{ is odd.} \end{cases} \tag{3.2.19}$$

Equivalently, we deduce what μ_n's, the sizes of $\pi_{4n-2}(\text{Sp}(n-1))$'s are. If one looks at the induced map

$$\varphi_* : \mathbb{H}_*(LS^3;\mathbb{Z}) \otimes \cdots \otimes \mathbb{H}_*(LS^{4n-1};\mathbb{Z}) \longrightarrow \mathbb{H}_*(L\text{Sp}(n);\mathbb{Z})$$

then the generators of $\mathbb{H}_*(LS^{4i-1};\mathbb{Z})$ get sent to the corresponding generators in $\mathbb{H}_*(L\text{Sp}(n);\mathbb{Z})$ but scaled by μ_i. Therefore, we have the following result.

[2]There is a typographical error here in an otherwise beautiful book where it reads π_{4n-2} as opposed to π_{4n+2}.

Theorem 3.27. *There is an isomorphism of BV algebras*

$$\Phi : \mathbb{H}_*(LS^3;\mathbb{Z}) \otimes \cdots \otimes \mathbb{H}_*(LS^{4n-1};\mathbb{Z}) \longrightarrow \mathbb{H}_*(L\mathrm{Sp}(n);\mathbb{Z}) \tag{3.2.20}$$

defined by scaling the generators of $\mathbb{H}_*(LS^{4i-1};\mathbb{Z})$ *inversely by*

$$\mu_i = |\pi_{4i-2}(\mathrm{Sp}(i-1))|.$$

Chapter 4

String Topology is Not a Homotopy Invariant

A loop in M can be thought of as an arc in $M \times M$ that start and end on the diagonal. More precisely, given

$$\gamma : [0, 2\pi] \longrightarrow M$$

such that $\gamma(0) = \gamma(2\pi)$, we define

$$\widetilde{\gamma} : [0, \pi] \longrightarrow M \times M, \;\; \widetilde{\gamma}(t) := (\gamma(t), \gamma(2\pi - t)).$$

Such arcs will be called *open strings*. With this change of viewpoint, structures

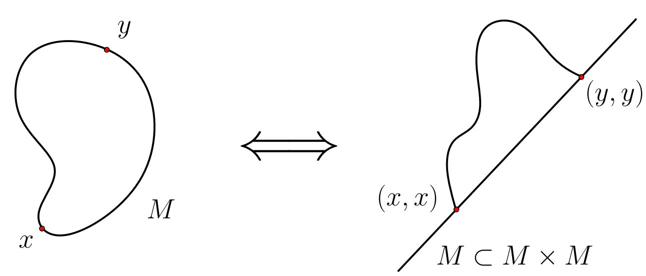

Figure 4.1: A loop is an open string

present in the free loop space can be interpreted as appropriate structures on the space of open strings in $M \times M$. This also motivates constructions where we study open strings in X with boundary conditions in Y, i.e., open strings in

X that start and end on an embedded submanifold Y. We take up this study in the subsequent sections and show that this leads to non-homotopy invariants of manifolds.

4.1 Transversal String Topology

4.1.1 Transversal open strings

We shall consider a smooth Riemannian manifold X which is closed and oriented. Due to a deep result of Grauert and Morrey, any (paracompact) smooth manifold admits a unique real analytic structure. We choose one such for our manifold X. Consequently, both T^*X and $\mathrm{Sym}^2(T^*X)$ become real analytic bundles. In the Whitney C^∞-topology, the analytic sections of an analytic bundle $E \to X$ are dense among smooth sections. Since a section of $\mathrm{Sym}^2(T^*X)$ is precisely a Riemannian metric, we conclude that there exists real analytic Riemannian metrics. We choose one such and denote the resulting C^ω-Riemannian manifold by (X, g). Notice that the Christoffel symbols only involve derivatives of g or its inverse and all such terms are real analytic. This implies that geodesics in (X, g) are real analytic solutions. In what follows, we shall assume (X, g) is a real analytic Riemannian manifold and geodesics will be assumed to be real analytic.

Geodesics on Riemannian manifolds are locally unique. By compactness of X, there is $\varepsilon_0 > 0$ such that if $x, y \in X$ have $d(x, y) \leq 2\varepsilon_0$ then there is a unique geodesic joining x and y.

Definition 4.1. (The tube) Let Y be a closed, oriented manifold embedded in X having codimension d. Let $\mathcal{N}_\varepsilon$ denote the ε-tubular neighbourhood of Y in X. For ε sufficiently small, $\mathcal{N}_\varepsilon$ is a d-dimensional disk bundle over Y. We shall set $\epsilon := \min\{\varepsilon, \varepsilon_0\}$ and $\mathscr{N} := \mathcal{N}_\epsilon$. The boundary of $\mathscr{N}$ will be denoted by T and called the *tube*.

The tube has a natural action of $\mathbb{Z}/2\mathbb{Z}$ by the pointwise anti-podal map on the normal sphere at $y \in Y$. We shall often denote by $\overline{p}$ the anti-pode of $p \in T$. This action will be denoted by $\mathfrak{a} : T \to T$.

Although we will not need the notion of order of tangencies for this discussion (we'll only be dealing with transversal intersections which will be tangencies of order zero) we shall nonetheless discuss it now. We'll have more to say about it later. Moreover, it gives a good idea of the setting of *transversal string topology* and it's possible future directions.

Definition 4.2. (Order of tangency) Fix a submanifold Y of X and an embedded real analytic curve $\gamma : (-\varepsilon, \varepsilon) \to X$ such that $\operatorname{Im} \gamma \cap Y = \gamma(0)$. Choose a local chart (U, φ) of $\gamma(0)$ in M such that $\varphi(\gamma(0)) = 0$ and

$$\varphi : U \to \mathbb{R}^{n+d}, \ \varphi(U \cap Y) = \mathbb{R}^n,$$

where $\mathbb{R}^n$ is thought of as the inclusion in the first n coordinates. Using Tayler's expansion around 0 we write γ as

$$\varphi(\gamma(t)) = \underbrace{\sum_{i=1}^{k} (\alpha_i; \beta_i) t^i}_{T_k} + \underbrace{(f_1(t); f_2(t))}_{\mathcal{O}(k+1)},$$

where $(\alpha_i; \beta_i) \in \mathbb{R}^n \oplus \mathbb{R}^d$. Let K be the smallest k for which at least one of the β_i's are non-zero in T_k. Then the *order of tangency* of γ with N is defined to be $K - 1$. If no such k exists, we say that the order is infinite.

It's easy to check that the order of tangency is well defined. Take another chart (V, ψ) around $\gamma(0)$ in X such that $\psi(U \cap Y) = \mathbb{R}^n$ and the transition map

$$g_{\psi\varphi} := \psi \circ \varphi^{-1} \in O(n) \times O(d).$$

Let $\beta_{i_0} \neq 0 \in \mathbb{R}^d$. Then

$$\psi(\gamma(t)) = \sum_{i=1}^{K} g_{\psi\varphi}(\alpha_i; \beta_i) t^i + g_{\psi\varphi}(f_1(t); f_2(t))$$

has $g_{\psi\varphi}(\alpha_{i_0}; \beta_{i_0})$ projects to a non-zero vector in $\mathbb{R}^d$. This means the order of tangency of γ defined using ψ is at most $K - 1$. Applying the argument with ψ and φ interchanged, we get an equality.

Remark 4.3. *A tangency of order zero is equivalent to being transversal while if* $\alpha_1 = 0$ *then this is actually perpendicular.*

Example 4.4. (Various orders of tangencies)
(i) If $\iota_1 : \mathbb{R} \hookrightarrow \mathbb{R}^2, n \geq 1$ and $\gamma_n(t) = (t, t^n)$, the order of tangency of γ to the x-axis is $n - 1$.
(ii) Let $\iota_1 : \mathbb{R} \hookrightarrow \mathbb{R}^3$ with $\gamma(t) = (0, t, t^2)$. In this case, γ is tangent to order zero.

(iii) With the same setup, if $\gamma(t) = (t, e^{-\frac{1}{t^2}})$ then the order of tangency is infinite although admittedly $e^{-\frac{1}{t^2}}$ is not real analytic.
(iv) For positive integers m, n, let $Y = \{(x, -x^m) \,|\, x \in \mathbb{R}\} \subset \mathbb{R}^2$ with $\gamma_n(t) = (t, t^n)$. Using a change of coordinate (for example $\varphi(x, y) = (x, y - x^m)$) one can take Y to be the x-axis and $\gamma_n(t) = (t, t^n + t^m)$. Therefore, the tangency is of order $\min\{m, n\} - 1$.

Definition 4.5. (Transversal open strings) We say that γ is an *open string* if $\gamma : [0, t_\gamma] \to X$ is piecewise real analytic and $\gamma(0), \gamma(t_\gamma) \in X$. We say such a γ is a *transversal open string* if near a point in $\gamma \cap Y$ the curve γ is an embedded C^ω-curve with zero order of tangency and this holds for $\gamma(0)$ and $\gamma(t_\gamma)$ as well. The

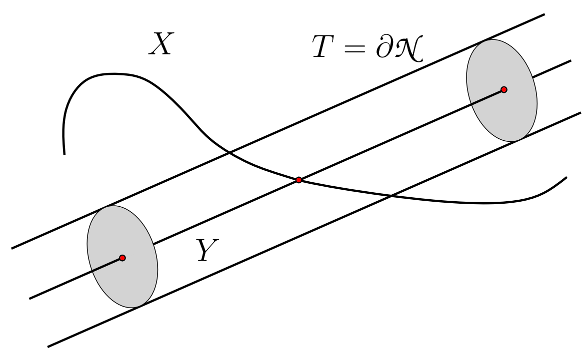

Figure 4.2: A transversal open string

space of transversal open strings will be denoted by $\mathcal{S}t$. This space is stratified; it's a disjoint union of $\{\mathcal{S}t_n\}_{n\geq 1}$ consisting of open string with $(n-1)$ interior intersections with Y.

Observe that we are using unparametrized open strings, i.e., we are no longer parametrizing arcs by $[0, 1]$. From a homotopical point of view this is fine. Indeed, the analogue in the free loop space picture will be to use the space of continuous maps from an unparametrized circle to M, i.e., the Moore space version of LM.

Definition 4.6. (Orthogonal open strings) The subspace of $\mathcal{S}t_n$ which consist of open string orthogonal to Y will be denoted by $\mathcal{S}t_n^\perp$ and the union will be denoted by $\mathcal{S}t^\perp$.

Definition 4.7. (Refined orthogonal open strings) For any $n \geq 1$, consider the subspace of $\mathcal{S}t_n^\perp$ which consist of open strings with the property that the intersection with $\mathcal{N}$ are just lines joining anti-podal points in $T := \partial\mathcal{N}$. This

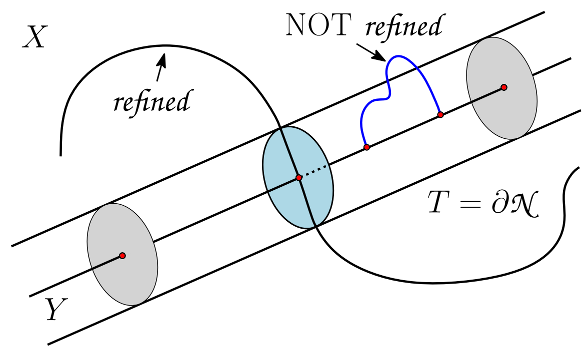

Figure 4.3: Various orthogonal open strings

space will be denoted by $\underline{\mathcal{S}t}$.

Lemma 4.8. *Each of the inclusions*

$$\underline{\mathcal{S}t}_n \hookrightarrow \mathcal{S}t_n^{\perp} \hookrightarrow \mathcal{S}t_n$$

is a homotopy equivalence.

Proof We first prove the statement for the rightmost inclusion. Given $\gamma \in \mathcal{S}t_n$ we construct $h(\gamma) \in \mathcal{S}t_n^{\perp}$ as follows. Let $\gamma : [0, T] \to M$ have transversal intersection points $\gamma(0), p_1, \ldots, p_{n-1}, \gamma(T) \in N$. Let

$$C_\gamma = \max_{t \in [0,T]} \|\gamma'(t)\|.$$

Choosing $\varepsilon > 0$ sufficiently small we can ensure that $\gamma^{-1}(\mathcal{N}_\varepsilon)$ is a disjoint union of closed intervals, i.e.,

$$\gamma^{-1}(\mathcal{N}_\varepsilon) = [0, \varepsilon_0] \cup [t_1 - \varepsilon_1, t_1 + \varepsilon_1] \cup \cdots \cup [t_{n-1} - \varepsilon_{n-1}, t_{n-1} + \varepsilon_{n-1}] \cup [1 - \varepsilon_n, 1]$$

with $\gamma(t_i) = p_i$ and $\varepsilon_i < \varepsilon/C$. Choose ε even smaller if necessary to ensure that $\mathcal{N}_\varepsilon \subset \mathcal{N}$. Since γ is transversal at p_i, there is a unit normal vector β_i at p_i along which γ points. It follows from the choices made that

$$\begin{aligned}
d(p_i + \varepsilon_i\beta_i, \gamma(t_i + \varepsilon_i)) &\leq d(p_i + \varepsilon_i\beta_i, p_i) + d(p_i, \gamma(t_i + \varepsilon_i) \\
&= \varepsilon_i + \int_0^{\varepsilon_i} \|\gamma'(t + t_i)\| \, dt \\
&\leq \varepsilon_i + C_\gamma \int_0^{\varepsilon_i} dt \\
&\leq 2\varepsilon_0.
\end{aligned}$$

Therefore, one can define $h(\gamma)$ to be the union of straight lines shooting across a normal diameter through p_i followed by the unique C^{ω}-geodesic joining $p_i+\varepsilon_i\beta_i$ with $\gamma(t_i+\varepsilon_i)$ and then followed by γ until we arrive at $\gamma(t_{i+1}-\varepsilon_{i+1})$ where we use the unique geodesic to go to $p_{i+1}-\varepsilon_{i+1}\beta_{i+1}$ and repeat this procedure. This is a continuous map since ε can be chosen in a continuous fashion.

It is clear that $h \circ \iota : \mathcal{S}t_n^{\perp} \to \mathcal{S}t_n^{\perp}$ is the identity map. Moreover, $\iota \circ h$ is homotopic to the identity by letting the ε used for γ to define h to get smaller, i.e., at time $t \in [0,1]$ use $\mathcal{N}_{t\varepsilon}$ to define h_t in a manner analogous to h. This constructs a homotopy between the identity map and $\iota \circ h$.

We now prove the homotopy equivalence of the first inclusion. $\square$

It follows that each of the three (stratified) spaces $\underline{\mathcal{S}t}$, $\mathcal{S}t^{\perp}$ and $\mathcal{S}t$ are homotopy equivalent.

Remark 4.9. (1) *Although we are using analytic open strings, we're only using the* 1*-jets in the definitions and related constructions. The higher jets associated to a transversal open string near the submanifold may have a crucial role in some of the future research directions we have in mind. For the purpose of the present discussion we wouldn't need it.*

(2) *We shall abuse notation and call elements (or chains) of* $\underline{\mathcal{S}t}$ *transversal open strings. The homotopy equivalence via Lemma* 4.8 *allows us to do so. Constructions described later on can be done on* $\mathcal{S}t$ *directly but to make the exposition simpler we use* $\underline{\mathcal{S}t}$.

Since we are interested in transversal strings, we may also consider the space of smooth curves or continuous curves that intersect N transversally. We shall be working with chains on $\mathcal{S}_n$ or on $\mathcal{S}_n^{\perp}$ without distinction since they are quasi-isomorphic. Given any i-chain α in $\mathcal{S}_n$, it follows from the compactness of α that for sufficiently small $\varepsilon > 0$ the intersection $\mathcal{N}_{\varepsilon} \cap \alpha$ consists of i-dimensional families of straight lines joining antipodal points. We call such an ε *admissable* and we set

$$\varepsilon_{\alpha} := \min\{\varepsilon_0, \max\{\varepsilon \,|\, \varepsilon \text{ is admissable}\}\}.$$

Definition 4.10. There is a natural diffeomorphism between $\partial\mathscr{N}$ and $\partial\mathcal{N}_{\varepsilon}$ for suitably small $\varepsilon > 0$ given by radial scaling. We shall denote this map by $r_{\varepsilon} : \partial\mathscr{N} \to \partial\mathcal{N}_{\varepsilon}$.

4.1.2 The category of bicomodules

We fix our base field $\mathbb{Q}$, the rational numbers. We shall reserve the notation $\otimes$ for $\otimes_{\mathbb{Q}}$.

Definition 4.11. (Bicomodule) Let $(\mathcal{C}, d)$ be a coassociative counital dg coalgebra over $\mathbb{Q}$, i.e., d is a coderivation of degree -1 and

$$\Delta : \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$$

is a chain map that is coassociative. The counit map will be denoted by $\varepsilon : \mathcal{C} \to \mathbb{Q}$.

A $\mathcal{C}$*-bicomodule* $(\mathcal{M}, d_{\mathcal{M}})$ is a dg vector space equipped with maps

$$\begin{aligned} \prec^{\mathcal{M}} &: \mathcal{M} \longrightarrow \mathcal{M} \otimes \mathcal{C} \\ \prec_{\mathcal{M}} &: \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M} \end{aligned}$$

which are coassociative, i.e.,

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\prec^{\mathcal{M}}"] \arrow[d, "\prec^{\mathcal{M}}"'] & \mathcal{M} \otimes \mathcal{C} \arrow[d, "\prec^{\mathcal{M}} \otimes \mathrm{id}"] \\ \mathcal{M} \otimes \mathcal{C} \arrow[r, "\mathrm{id} \otimes \Delta"] & \mathcal{M} \otimes \mathcal{C} \otimes \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M} \arrow[r, "\prec_{\mathcal{M}}"] \arrow[d, "\prec_{\mathcal{M}}"'] & \mathcal{C} \otimes \mathcal{M} \arrow[d, "\mathrm{id} \otimes \prec_{\mathcal{M}}"] \\ \mathcal{C} \otimes \mathcal{M} \arrow[r, "\Delta \otimes \mathrm{id}"] & \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{M} \end{tikzcd}$$

are commuting diagrams. Moreover, we also need the following commuting diagram

$$\begin{tikzcd} \mathcal{C} \otimes \mathcal{M} \arrow[dr, "\varepsilon \otimes \mathrm{id}"'] & \mathcal{M} \arrow[l, "\prec_{\mathcal{M}}"'] \arrow[r, "\prec^{\mathcal{M}}"] \arrow[d, "\mathrm{id}"] & \mathcal{M} \otimes \mathcal{C} \arrow[dl, "\mathrm{id} \otimes \varepsilon"] \\ & \mathbb{Q} \otimes \mathcal{M} \cong \mathcal{M} \otimes \mathbb{Q} & \end{tikzcd}$$

Finally, we require all the maps to be chain maps.

Notice that the requirement of being chain maps force $d_{\mathcal{M}}$ to be of degree -1.

Definition 4.12. (Cotensor product) Given two $\mathcal{C}$-bicomodules $\mathcal{M}_1$ and $\mathcal{M}_2$ we can form the *equalizer* of

$$\mathcal{M}_1 \otimes \mathcal{M}_2 \rightrightarrows \mathcal{M}_1 \otimes \mathcal{C} \otimes \mathcal{M}_2$$

given by $\prec^{\mathcal{M}_1} \otimes \mathrm{Id}$ and $\mathrm{Id} \otimes \prec_{\mathcal{M}_2}$. We shall denote the equalizer by $\mathcal{M}_1 \,\Box\, \mathcal{M}_2$ and call it the *cotensor product* of $\mathcal{M}_1$ and $\mathcal{M}_2$ over the coalgebra $\mathcal{C}$. More precisely,

$$\mathcal{M}_1 \otimes \mathcal{M}_2 \supseteq \mathcal{M}_1 \,\Box\, \mathcal{M}_2 := \Big\{ \sum_i m_i \otimes n_i \,\Big|\, \sum_i \prec^{\mathcal{M}_1}(m_i) \otimes n_i = \sum_i m_i \otimes \prec_{\mathcal{M}_2}(n_i) \Big\}.$$

Observe that $\mathcal{M}_1 \,\square\, \mathcal{M}_2$ is again a $\mathcal{C}$-bicomodule.

Example 4.13. (1) A dg coalgebra is a bicomodule over itself; the comultiplication map Δ serves as both $\prec^{\mathcal{C}}$ and $\prec_{\mathcal{C}}$.
(2) If $\mathcal{C}$ has a counit $\mathbb{1}$ then the maps

$$\prec^{\mathbb{Q}} : 1 \mapsto 1 \otimes \mathbb{1}, \quad \prec_{\mathbb{Q}} : 1 \mapsto \mathbb{1} \otimes 1$$

define a $\mathcal{C}$-bicomodule structure on $\mathbb{Q}$.

Example 4.14. (Fibre products) The usefulness of cotensor products will be clear if we consider maps $X \xrightarrow{f} Z \xleftarrow{g} Y$ of spaces. We can form the fibre product $X \times_Z Y$ and ask for a chain model built out of chains on X, Y and Z. We have maps

$$f := \mathrm{id} \times f : X \longrightarrow X \times Z, \quad g := g \times \mathrm{id} : Y \longrightarrow Z \times Y.$$

Recall the basic yet fundamental fact that the chains on any topological space is a unital coalgebra.

Theorem 4.15. (Eilenberg-Zilber)
There are chain maps

$$\begin{aligned} AW &: C_*(X \times Y) \longrightarrow C_*(X) \otimes C_*(Y) \\ EZ &: C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y) \end{aligned}$$

such that $(AW) \circ (EZ) = Id$ *and* $(EZ) \circ (AW)$ *is chain homotopic to the identity. Moreover,* AW *and* EZ *are natural and associative.*

Therefore, we may define $C_*(X)$ and $C_*(Y)$ as $C_*(Z)$-comodules via

$$\begin{aligned} AW \circ f_* &: C_*(X) \longrightarrow C_*(X) \otimes C_*(Z) \\ AW \circ g_* &: C_*(Y) \longrightarrow C_*(Z) \otimes C_*(Y). \end{aligned}$$

Proposition 4.16. *There is a quasi-isomorphism*

$$AW : C_*(X \times_Z Y) \xrightarrow{\simeq} C_*(X) \,\square\, C_*(Y)$$

where $\square$ *is the cotensor product over* $C_*(Z)$.

Proof There are natural maps

$$\mathcal{F} := (\mathrm{id} \times f) \times \mathrm{id}, \mathcal{G} := \mathrm{id} \times (g \times \mathrm{id}) : X \times Y \to X \times Z \times Y.$$

Notice that $C_*(X \times_Z Y)$ is the subcomplex of $C_*(X \times Y)$ which is the kernel of $\mathcal{F}_* - \mathcal{G}_*$. Since EZ is injective, one may identify $C_*(X) \,\square\, C_*(Y)$ with its image in $C_*(X \times Y)$. By definition

$$\begin{tikzcd}
C_*(X \times Y) \arrow[rr, "\mathcal{F}_*"] \arrow[dd, shift right, "AW"'] & & C_*(X \times Z \times Y) \arrow[d, shift left, "AW"] \\
& & C_*(X \times Z) \otimes C_*(Y) \arrow[u, shift left, "EZ"] \arrow[d, shift left, "AW"] \\
C_*(X) \otimes C_*(Y) \arrow[uu, shift right, "EZ"'] \arrow[r, "f_* \otimes \mathrm{id}"] & C_*(X \times Z) \otimes C_*(Y) \arrow[r, "AW"] & C_*(X) \otimes C_*(Z) \otimes C_*(Y). \arrow[u, shift left, "EZ"]
\end{tikzcd}$$

There is a similar diagram for $\mathcal{G}$. We conclude that elements of $C_*(X) \,\square\, C_*(Y)$ are precisely the kernel of the arrow composed of the leftmost bottom horizontal arrow (now thought of as $f_* - g_*$) followed by next horizontal arrow AW. But the commutativity of the diagram implies that

$$EZ\big(C_*(X) \,\square\, C_*(Y)\big) \subseteq \ker(\mathcal{F}_* - \mathcal{G}_*) = C_*(X \times_Z Y).$$

It also follows from the diagram that

$$AW : C_*(X \times_Z Y) \longrightarrow C_*(X) \,\square\, C_*(Y). \tag{4.1.1}$$

This is a quasi-isomorphism as AW and EZ are natural quasi-isomorphism. $\square$

Remark 4.17. *The notion of bicomodules is dual to the notion of bimodules. It may help to think of the duality of cotensor products and tensor products of bimodules as the duality between kernels and quotients or cokernels. More precisely, if all objects under consideration are of finite rank in each degree then one can dualize $M \,\square\, N$ over $\mathbb{Q}$. Let $L^\vee$ be the linear dual of L. We then have*

$$(M \,\square\, N)^\vee \cong M^\vee \otimes_{\mathcal{C}^\vee} N^\vee.$$

It's good to keep in mind that a cohomological model for a fibre product $X \times_Z Y$ is given by the tensor products of bimodules $\Lambda_X \otimes_{\Lambda_Z} \Lambda_Y$ where Λ_X models the forms on X.

Definition 4.18. (Category of bicomodules) Let $\mathcal{C}$ be a counital coassociative dg coalgebra over $\mathbb{Q}$. Let $\mathfrak{C}_{\mathcal{C}}$ be the monoidal category with objects $\mathcal{C}$-bicomodules and morphisms between objects given by maps that are

(i) chain maps and

(ii) are homomorphisms of left and right $\mathcal{C}$-comodules.

The monoidal structure is given by the cotensor product $\square$.

As is true in the category $\mathfrak{C}^R$ of R-bimodules that R plays the role of identity with respect to the tensor product $\otimes_R$, the analogous statement, stated without a proof, holds in $\mathfrak{C}_{\mathcal{C}}$.

Lemma 4.19. (1) *If $\mathcal{M}$ is a right $\mathcal{C}$-comodule then $\mathcal{M} \square \mathcal{C} \cong \mathcal{M}$ as right $\mathcal{C}$-comodules.* (2) *If $\mathcal{M}$ is a left $\mathcal{C}$-comodule then $\mathcal{C} \square \mathcal{M} \cong \mathcal{M}$ as left $\mathcal{C}$-comodules.*

Since we're working over $\mathbb{Q}$ and any $\mathbb{Q}$-module, i.e., a vector space, is naturally free (and hence flat), the monoidal structure given by $\square$ is associative. More formally, we invoke one of the properties of the cotensor product [35].

Proposition 4.20. (Associativity of $\square$) *Given $\mathcal{C}$-bicomodules $\mathcal{M}_i, i = 1, 2, 3$, there is a natural isomorphism of $\mathcal{C}$-bicomodules*

$$\mathscr{A} : (\mathcal{M}_1 \square \mathcal{M}_2) \square \mathcal{M}_3 \xrightarrow{\cong} \mathcal{M}_1 \square (\mathcal{M}_2 \square \mathcal{M}_3).$$

Example 4.21. insert a non-associative example

Definition 4.22. (Algebras and coalgebras in $\mathfrak{C}_{\mathcal{C}}$) An algebra $\mathcal{A}$ in the category $\mathfrak{C}_{\mathcal{C}}$ is an object, i.e., a $\mathcal{C}$-bicomodule, such that there is a chain map (of degree zero)

$$m : \mathcal{A} \square \mathcal{A} \longrightarrow \mathcal{A}$$

which is also a morphism in the category.

A coalgebra $\mathscr{C}$ in $\mathfrak{C}_{\mathcal{C}}$ is an object equipped with a map

$$\prec : \mathscr{C} \longrightarrow \mathscr{C} \square \mathscr{C}$$

which is a morphism in this category.

Remark 4.23. *If $\mathcal{A}$ is an algebra in $\mathfrak{C}_{\mathcal{C}}$ then so is $\mathcal{A} \square \mathcal{A}$. Similarly, if $\mathscr{C}$ is a coalgebra in this category then $\mathscr{C} \square \mathscr{C}$ naturally carries a coalgebra structure. However, the algebra objects are not closed under $\square$ and the main obstruction for this is the non-symmetric nature of cotensor products, i.e., there is no natural functor that identifies $M \square N$ and $N \square M$. In fact, there may be no such maps at all for specific examples.*

Example 4.24. (A Trivial Example) Consider $\mathbb{Q}$ as a $\mathcal{C}$-bicomodule (as described in (2) of Example 4.13). Then

$$\mathbb{Q}\,\square\,\mathbb{Q} = \mathbb{Q}\otimes\mathbb{Q} \cong \mathbb{Q}$$

The identification of $\mathbb{Q}$ with $\mathbb{Q}\otimes\mathbb{Q}$ in either direction provides an algebra or coalgebra structure on $\mathbb{Q}$ as an object in $\mathfrak{C}_{\mathcal{C}}$.

Example 4.25. (Transversal open strings) Consider an element $\gamma \in \mathcal{S}t_n$. Let $\gamma \in \mathcal{S}t_n$ with intersection points $p_0 = \gamma(0), p_1, \ldots, p_{n-1}, p_n = \gamma(t_\gamma)$. Assume that $\gamma(t_i) = p_i$ such that $[t_i - \varepsilon_i^{'}, t_i + \varepsilon_i^{''}]$ is mapped by γ to $\mathcal{N}$ with $\gamma(t_i + \varepsilon_i^{'}), \gamma(t_i - \varepsilon_i^{''}) \in T = \partial\mathcal{N}$. Define the *evaluation* maps

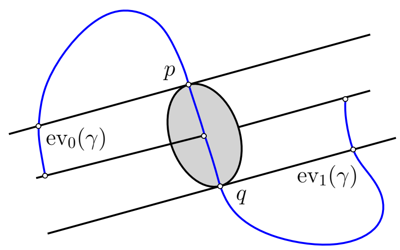

Figure 4.4: The evaluation maps from strings to the tube

$$\begin{aligned}
\mathrm{ev}_0 &: \mathcal{S}t_n \longrightarrow T, \;\; \gamma \mapsto \gamma(\varepsilon_0^{''}) \\
\mathrm{ev}_1 &: \mathcal{S}t_n \longrightarrow T, \;\; \gamma \mapsto \gamma(t_\gamma - \varepsilon_n^{'}).
\end{aligned}$$

These can be thought of as the first point of exit and the last point of entry of γ to the tube T.

Definition 4.26. ($\mathcal{S}t$ as a bicomodule over T) Chains on $\mathcal{S}t$ can be given a $C_*(T)$-bicomodule structure via the maps

(4.1.2) $$\prec^{s} : C_*(\mathcal{S}t_n) \xrightarrow{\mathrm{id}\times\overline{\mathrm{ev}_1}} C_*(\mathcal{S}t_n \times T) \xrightarrow{AW} C_*(\mathcal{S}t_n) \otimes C_*(T)$$

(4.1.3) $$\prec_{s} : C_*(\mathcal{S}t_n) \xrightarrow{\mathrm{ev}_0\times\mathrm{id}} C_*(T \times \mathcal{S}t_n) \xrightarrow{AW} C_*(T) \otimes C_*(\mathcal{S}t_n).$$

It's worth recalling that the chains on any space (and in particular $C_*(T)$) is a coalgebra via the diagonal map.

4.1.3 Structures on transversal open strings

We have already seen that the chain complex of transversal open strings has a natural structure of a bicomodule over the chains on the tube. However, we need a finer analysis. In general, given $\gamma \in \mathcal{S}t_n$ we have another evaluation map with more structure.

Definition 4.27. (Evaluation map $\mathcal{E}v$) Define the map

$$\mathcal{E}v : \mathcal{S}t_n \longrightarrow (T \times T)^{\times_T^n} \tag{4.1.4}$$

given by mapping an open string to the points of intersection with T.

A simple picture may be more helpful to the visual reader. While the definition

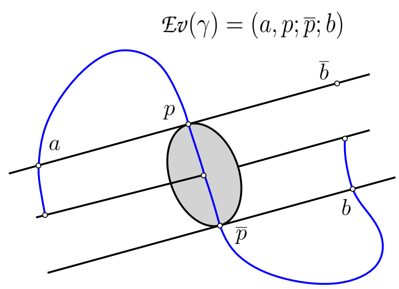

Figure 4.5: The evaluation map $\mathcal{E}v$

above may have been cryptic, let us now explain the terms involved and the map. We shall denote the fibre product

$$\begin{tikzcd}
(T \times T) \times_T (T \times T) \arrow[r] \arrow[d] & T \times T \arrow[d, "(a,b)\mapsto a"] \\
T \times T \arrow[r, "(x,y)\mapsto \overline{y}"] & T.
\end{tikzcd}$$

In our notation,

$$(T \times T)^{\times_T^n} := \underbrace{(T \times T) \times_T (T \times T) \times_T \cdots \times_T (T \times T)}_{n \text{ copies}}.$$

With $\gamma \in \mathcal{S}t_n$ as before we define

$$\mathcal{E}v(\gamma) := \Big(\gamma(\varepsilon_0''), \gamma(t_1 - \varepsilon_1'); \gamma(t_1 + \varepsilon_1''), \gamma(t_2 - \varepsilon_2'); \cdots ; \gamma(t_{n-1} + \varepsilon_{n-1}''), \gamma(t_\gamma - \varepsilon_n')\Big).$$

This map lands in the iterated fibre product due to

$$\overline{\gamma(t_i + \varepsilon_i^{''})} = \gamma(t_i - \varepsilon_i^{'}),$$

where the bar denotes the antipode. Using the notation

$$T^\infty := \bigcup_{n \geq 1} (T \times T)^{\times_T^n}.$$

one can write the (stratified) evaluation map

$$\mathcal{E}v : \underline{\mathfrak{St}} \longrightarrow T^\infty.$$

If one visualizes an element $\gamma \in \underline{\mathfrak{St}}_n$ then one can *split* it at each of the interior intersection points. We shall denote the *splitting map* at the ith intersection point by $\prec_i : \underline{\mathfrak{St}}_n \to \underline{\mathfrak{St}}_i \times_T \underline{\mathfrak{St}}_{n-i}$ where the target space is the fibre product

$$\begin{tikzcd}
\underline{\mathfrak{St}}_i \times_T \underline{\mathfrak{St}}_{n-i} \arrow[r] \arrow[d] & \underline{\mathfrak{St}}_{n-i} \arrow[d, "\mathrm{ev}_0"] \\
\underline{\mathfrak{St}}_i \arrow[r, "\overline{\mathrm{ev}_1}"] & T.
\end{tikzcd}$$

We have a commuting diagram

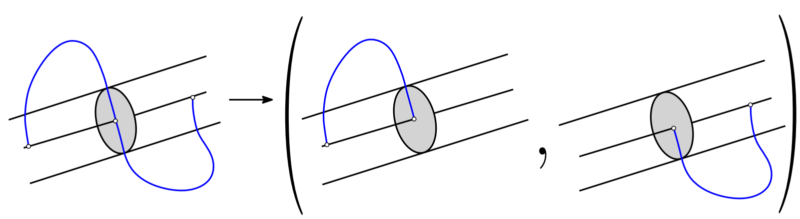

Figure 4.6: Splitting a transversal open string

$$\begin{tikzcd}
\underline{\mathfrak{St}}_n \arrow[r, "\prec_j"] \arrow[d, "\prec_i"'] & \underline{\mathfrak{St}}_j \times_T \underline{\mathfrak{St}}_{n-j} \arrow[d, "1 \times \prec_{i-j}"] \\
\underline{\mathfrak{St}}_i \times_T \underline{\mathfrak{St}}_{n-i} \arrow[r, "\prec_j \times 1"] & \underline{\mathfrak{St}}_i \times_T \underline{\mathfrak{St}}_{i-j} \times_T \underline{\mathfrak{St}}_{n-i}.
\end{tikzcd}$$

At the level of chains, we get a map

$$\prec := \oplus_{i=1}^{n-1} (\prec_i)_* : C_*(\underline{\mathfrak{St}}_n) \longrightarrow \bigoplus_{i=1}^{n-1} C_*(\underline{\mathfrak{St}}_i \times_T \underline{\mathfrak{St}}_{n-i}).$$

The commutative diagram above implies that $\prec$ is coassociative. In fact, one can explicitly describe (as seen is Example 4.25) $C_*(\mathcal{S}\underline{t}_* \times_T \mathcal{S}\underline{t}_*)$ (up to quasi-isomorphism) as the *cotensor product* of the bi-comodules $C_*(\mathcal{S}\underline{t}_*)$ over the coalgebra $C_*(T)$.

Definition 4.28. The *comultiplication* map is defined by post-composing $\prec$ by the Alexander-Whitney map AW, i.e.,

$$\Delta : C_*(\mathcal{S}\underline{t}_n) \longrightarrow \bigoplus_{i=1}^{n-1} \big(C_*(\mathcal{S}\underline{t}_i) \,\square\, C_*(\mathcal{S}\underline{t}_{n-i})\big). \tag{4.1.5}$$

Notice that ∂ commutes with $\prec$ whence Δ is a chain map.

The commutativity of the diagram

$$\begin{tikzcd}
C_*(\mathcal{S}\underline{t}) \arrow[r, "\prec"] \arrow[d, "\mathcal{E}v"] & C_*(\mathcal{S}\underline{t} \times_T \mathcal{S}\underline{t}) \arrow[r, "AW"] \arrow[d, "\mathcal{E}v \times \mathcal{E}v"] & C_*(\mathcal{S}\underline{t}) \,\square\, C_*(\mathcal{S}\underline{t}) \arrow[d, "\mathcal{E}v"] \\
C_*(T^\infty) \arrow[r, "\prec"] & C_*(T^\infty \times_T T^\infty) \arrow[r, "AW"] & C_*(T^\infty) \,\square\, C_*(T^\infty)
\end{tikzcd}$$

implies that the bottom row defines a comultiplication map Δ_T on $C_*(T^\infty)$, which is to be interpreted as a coalgebra in the category of bicomodules over $C_*(T)$. We set

$$\begin{aligned}
\mathfrak{C}_{i,j} &:= C_{i-j(d-1)}(\mathcal{S}\underline{t}_j) \\
\mathfrak{C}^T_{i,j} &:= C_{i-j(d-1)}\big((T \times T)^{\times_T^j}\big).
\end{aligned}$$

Therefore, we get bigraded complexes

$$\mathfrak{C}(X,Y) := \bigoplus\nolimits_{i,j} \mathfrak{C}_{i,j} = \bigoplus\nolimits_{i,j} C_{i-j(d-1)}(\mathcal{S}\underline{t}_j) \tag{4.1.6}$$

$$\mathfrak{C}(T) := \bigoplus\nolimits_{i,j} \mathfrak{C}^T_{i,j} = \bigoplus\nolimits_{i,j} C_{i-j(d-1)}\big((T \times T)^{\times_T^j}\big). \tag{4.1.7}$$

We may conclude the following :

Lemma 4.29. (1) *Δ is coassociative on $\mathfrak{C}(X,Y)$.*
(2) *Δ_T is coassociative on $\mathfrak{C}(T)$.*
(3) *$\mathcal{E}v$ is an intertwiner, i.e.,*

$$\mathcal{E}v \circ \Delta = \Delta_T \circ \mathcal{E}v.$$

Proof The map $\prec$ is coassociative and the Alexander-Whitney map AW is associative. This proves (1) and (2) while (3) is obvious from the definition. $\square$

It is conceivable that there is an operation on transversal open strings where we try to resolve the interior intersection points. This leads us to define the *resolve operator* $\mathscr{R}$.

Definition 4.30. (Resolve operation) Let $\gamma \in \mathcal{S}t_n$ with intersection points $p_0 = \gamma(0), p_1, \ldots, p_{n-1}, p_n = \gamma(t_n)$. Assume that $\gamma(t_i) = p_i$ such that

$$\gamma\big([t_i - \varepsilon_i^{'}, t_i + \varepsilon_i^{''}]\big) \subset \mathcal{N}$$

with $\gamma(t_i + \varepsilon_i^{'}), \gamma(t_i - \varepsilon_i^{''}) \in T = \partial\mathcal{N}$. Let S_i^{d-1} be the boundary of $\mathcal{N}$ at p_i. Consider the the *equator* e_i of S_i^{d-1} with $\gamma(t_i - \varepsilon_i^{'})$ as the north pole and $\gamma(t_i + \varepsilon_i^{''})$ as the south pole. For $\mathbf{x} \in e_i$ we can associate the *longitude* $\ell_{i,\mathbf{x}}$ that travels from the north pole to the south through $\mathbf{x}$ in time $\varepsilon_i^{'} + \varepsilon_i^{''}$. Now we associate to p_i the S^{d-2}-parametrized family $\mathscr{R}(\gamma, p_i)$ of elements from $\mathcal{S}t_{n-1}$ given by :

$$\gamma_{\mathbf{x}}(t) \quad = \quad \begin{cases} \gamma(t) & \text{if } t < t_i - \varepsilon_i^{'}, \\ \ell_{i,\mathbf{x}} & \text{if } t_i - \varepsilon_i^{'} \leq t \leq t_i + \varepsilon_i^{''}, \\ \gamma(t) & \text{if } t > t_i + \varepsilon_i^{''}. \end{cases}$$

We define the *resolve operator* $\mathscr{R}$ on γ by

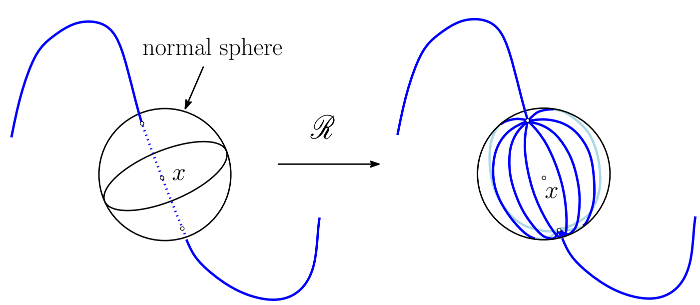

Figure 4.7: The resolve operator at a point

$$\mathscr{R}(\gamma) = \sum_{i=1}^{n-1} (-1)^i \mathscr{R}(\gamma, p_i).$$

For an i-chain α in $\mathcal{S}t_n$ we associate the $(i + d - 2)$-chain $\mathscr{R}(\alpha)$ in $\mathcal{S}t_{n-1}$.

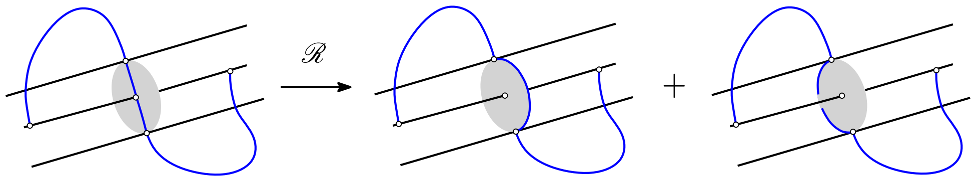

Figure 4.8: The resolve operator for codimension 2

We have the homotopy equivalence

$$\iota : \underline{\mathcal{S}t}_n \hookrightarrow \mathcal{S}t_n, \quad \mathcal{S}t_n \xrightarrow{r} \underline{\mathcal{S}t}_n$$

where $r \circ \iota = \mathrm{id}$ and $\iota \circ r$ is homotopic to id. We use this to define the resolve operator on $\underline{\mathcal{S}t}_n$ and we denote it by $\underline{\mathscr{R}}$, i.e., the composition

$$C_*(\mathcal{S}t_n) \xrightarrow{r} C_*(\underline{\mathcal{S}t}_n) \xrightarrow{\mathscr{R}} C_*(\underline{\mathcal{S}t}_{n-1}) \xrightarrow{\iota} C_*(\mathcal{S}t_{n-1}).$$

Moreover, one can define $\mathscr{R}$ on $\underline{\mathcal{S}t}_1 \times_T \underline{\mathcal{S}t}_1 \times_T \cdots \times_T \underline{\mathcal{S}t}_1$ in exactly the same way. Towards that end the following result will be very useful later on.

Lemma 4.31. *The inclusion*

$$\iota : \underline{\mathcal{S}t}_k \hookrightarrow \underline{\mathcal{S}t}_1 \times_T \underline{\mathcal{S}t}_1 \times_T \cdots \times_T \underline{\mathcal{S}t}_1$$

is a quasi-isomorphism with homotopy inverse given by the smoothing map κ^1.

Proof We shall only define the *smoothing map* κ for $\underline{\mathcal{S}t}_1 \times_T \underline{\mathcal{S}t}_1$ and prove it to be the homotopy inverse to ι for $k = 2$. The same definition and proof works in general. Given $(\gamma, \eta) \in \underline{\mathcal{S}t}_1 \times_T \underline{\mathcal{S}t}_1$ let

$$p = \mathrm{ev}_1(\gamma) = \overline{\mathrm{ev}_0}(\eta)$$

lie in the normal sphere of $x \in X$. We assume that it takes ε_γ (resp. ε_η) to reach from p to x via γ (resp. η). We define $\kappa(\gamma, \eta)$ to be a real analytic map that joins p to $\overline{p}$ through the axis passing through x. We also require $\kappa(\gamma, \eta)$ to agree with the jet of γ at p and the jet of η at $\overline{p}$. If we write (in a local chart around x, which is now set to 0)

$$\gamma(t) = \sum_{i \geq 1} \gamma_i \, t^i, \quad \eta(t) = \sum_{i \geq 1} \eta_i \, t^i$$

then we define, for $\varepsilon \in [0,1]$,

$$\kappa^\varepsilon(\gamma,\eta) := \begin{cases} \gamma(t) & \text{if } t \leq -\varepsilon\varepsilon_\gamma \\ \sum_{i\geq 1} \xi_{\gamma_i,\eta_i}^{-\varepsilon\varepsilon_\gamma,\varepsilon\varepsilon_\eta} t^i & \text{if } t \in (-\varepsilon\varepsilon_\gamma, \varepsilon\varepsilon_\eta) \\ \eta(t) & \text{if } t \geq \varepsilon\varepsilon_\eta. \end{cases}$$

The map ξ, embellished with indices, is defined to be a smooth map

$$\xi_{\gamma_i,\eta_i}^{-\varepsilon\varepsilon_\gamma,\varepsilon\varepsilon_\eta} : \mathbb{R} \longrightarrow [\gamma_i, \eta_i]$$

with the following properties :
(i) it is γ_i to the left of $-\varepsilon\varepsilon_\gamma$,
(ii) it is η_i to the right of $\varepsilon\varepsilon_\eta$,
(iii) it is monotonic, and,
(iv) all its derivatives are zero at $-\varepsilon\varepsilon_\gamma$ and $\varepsilon\varepsilon_\eta$.
With this definition of

$$\kappa^1 : \underline{\mathcal{S}t}_1 \times_T \underline{\mathcal{S}t}_1 \longrightarrow \underline{\mathcal{S}t}_2$$

we see that $\kappa^1 \circ \iota = \mathrm{id}$ and $\iota \circ \kappa^1$ is homotopic to the identity via $\{\kappa^t\}_{t\in[0,1]}$. $\square$

It follows that $\mathscr{R}$ on $\underline{\mathcal{S}t}_1^{\times_T^k}$ can also be thought of as $\kappa^1 \circ \mathscr{R} \circ \iota$.

The following is a nice and almost tautological feature of the resolve operator.

Lemma 4.32. *The resolve operator $\mathscr{R}$ commutes with the boundary of the chains.*

Proof Since spheres don't have boundary, the image of a chain under $\mathscr{R}$ has boundary components labelled by the resolve operator applied to the boundary components of the chain. $\square$

We also have the following :

Lemma 4.33. *The map $\underline{\mathscr{R}} : C_*(\mathcal{S}t_n) \longrightarrow C_{*+d-2}(\mathcal{S}t_{n-1})$ satisfies $\underline{\mathscr{R}}^2 = 0$.*

Proof Observe that when $n = 1$ the map $\mathscr{R} \equiv 0$ and when $n = 2$ we have $\mathscr{R}^2 \equiv 0$. So, we assume that $n \geq 3$ and let $\gamma \in \mathcal{S}_n$. For $i = 1, \ldots, n-1$ let p_i be its interior intersection points at time t_i. For $\mathbf{x} \in e_i$ and $\mathbf{y} \in e_{i+1}$ consider the elements $\gamma_{\mathbf{x}} \in \mathscr{R}(\gamma)$ and $\gamma_{\mathbf{y}} \in \mathscr{R}(\gamma)$. Notice that

$$(\gamma_{\mathbf{x}})_{\mathbf{y}} \equiv (\gamma_{\mathbf{y}})_{\mathbf{x}}$$

while these two terms appear with different signs in $\mathscr{R}^2(\gamma)$. The same argument applied to families prove the result for $\underline{\mathcal{S}t}_n$. To complete the proof notice that $\underline{\mathscr{R}}^2$ on $C_*(\underline{\mathcal{S}t})$ is the composition

$$C_*(\mathcal{S}t_n) \xrightarrow{r} C_*(\underline{\mathcal{S}t}_n) \xrightarrow{\mathscr{R}^2} C_*(\underline{\mathcal{S}t}_{n-2}) \xrightarrow{\iota} C_*(\mathcal{S}t_{n-2})$$

which is zero. $\square$

Recall the grading on $C_*(\underline{\mathcal{S}t})$ introduced before. It easily follows that $\mathscr{R} : \mathfrak{C}_{i,j} \to \mathfrak{C}_{i,j-1}$. Therefore, both $\mathscr{R}$ and ∂ lower the anti-diagonal degree (given by $i+j$) by 1. Moreover, $(\partial + \mathscr{R})^2 = 0$ and the comultiplication Δ still has degree zero. Gathering all the previous results we get :

Theorem 4.34. *The chain complex $(\mathfrak{C}(X,Y), \partial + \mathscr{R}, \Delta)$ is a differential graded coalgebra in the category of bicomodules over the coalgebra $C_*(T)$.*

It's natural to explore relations between the operations $\mathscr{R}$ and $\prec$. These operations can be thought of as analogues to $\bullet$ and τ. In the usual setting of string topology (with $X = M \times M$ and $Y = M$)

$$\tau(x \bullet y) = \tau(x) \bullet y + x \bullet \tau(y).$$

Notice that by definition a transversal open string in $M \times M$ when considered as a closed loop in M can never be the constant loop. The role played by τ in $H_*(LM, M)$ is now played by $\prec$ in $\mathfrak{C}(M \times M, M)$. The loop product $\bullet$ is replaced by $\mathscr{R}$ applied in a specific way. We have the following commutative diagrams :

$$\begin{tikzcd}
 & \underline{\mathcal{S}t}_m \times_T \underline{\mathcal{S}t}_k \times_T \underline{\mathcal{S}t}_{n-k} \arrow[r, "\mathscr{R}\times 1"] & \left(\underline{\mathcal{S}t}_{m+k-1} \times_T \underline{\mathcal{S}t}_{n-k}\right)^{S^{d-2}} \arrow[d, equal] \\
\underline{\mathcal{S}t}_m \times_T \underline{\mathcal{S}t}_n \arrow[ur, "1\times\prec_k"] \arrow[r, "\mathscr{R}"] & \left(\underline{\mathcal{S}t}_{m+n-1}\right)^{S^{d-2}} \arrow[r, "\prec_{m+k-1}"] & \left(\underline{\mathcal{S}t}_{m+k-1} \times_T \underline{\mathcal{S}t}_{n-k}\right)^{S^{d-2}}
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\mathcal{S}t}_m \times_T \underline{\mathcal{S}t}_n \arrow[r, "\mathscr{R}"] \arrow[dr, "\prec_j\times 1"'] & \left(\underline{\mathcal{S}t}_{m+n-1}\right)^{S^{d-2}} \arrow[r, "\prec_j"] & \left(\underline{\mathcal{S}t}_j \times_T \underline{\mathcal{S}t}_{m+n-j-1}\right)^{S^{d-2}} \arrow[d, equal] \\
 & \underline{\mathcal{S}t}_j \times_T \underline{\mathcal{S}t}_{m-j} \times_T \underline{\mathcal{S}t}_n \arrow[r, "1\times\mathscr{R}"] & \left(\underline{\mathcal{S}t}_j \times_T \underline{\mathcal{S}t}_{m+n-j-1}\right)^{S^{d-2}}
\end{tikzcd}$$

We have used the notation X^{S^k} to denote the space of continuous maps from S^k to X. In the example above, this mapping space arises due to the resolve operator which, at a point, is parametrized by the equatorial S^{d-2}.

The diagram above is equivalent to

$$\prec\left(\mathscr{R}(\alpha_1, \alpha_2)\right) = (\mathbb{1} \times \mathscr{R})(\prec(\alpha_1), \alpha_2) \pm (\mathscr{R} \times \mathbb{1})(\alpha_1, \prec(\alpha_2)) \tag{4.1.8}$$

where α_1, α_2 are appropriate chains. The sign $\pm$ depends on the stratification index of α_1, i.e., it's $(-1)^{k-1}$ if $\alpha_1 \in C_*(\mathcal{S}t_k)$. Finally, $\mathbb{1}(\beta)$ is the chain β thickened trivially by S^{d-2}, i.e., it is the composition of $\Delta \times S^{d-2} \xrightarrow{\mathrm{pr}_1} \Delta$ with β. In other words, one may think of $\prec$ as a derivation of $\mathscr{R}$ in this setting.

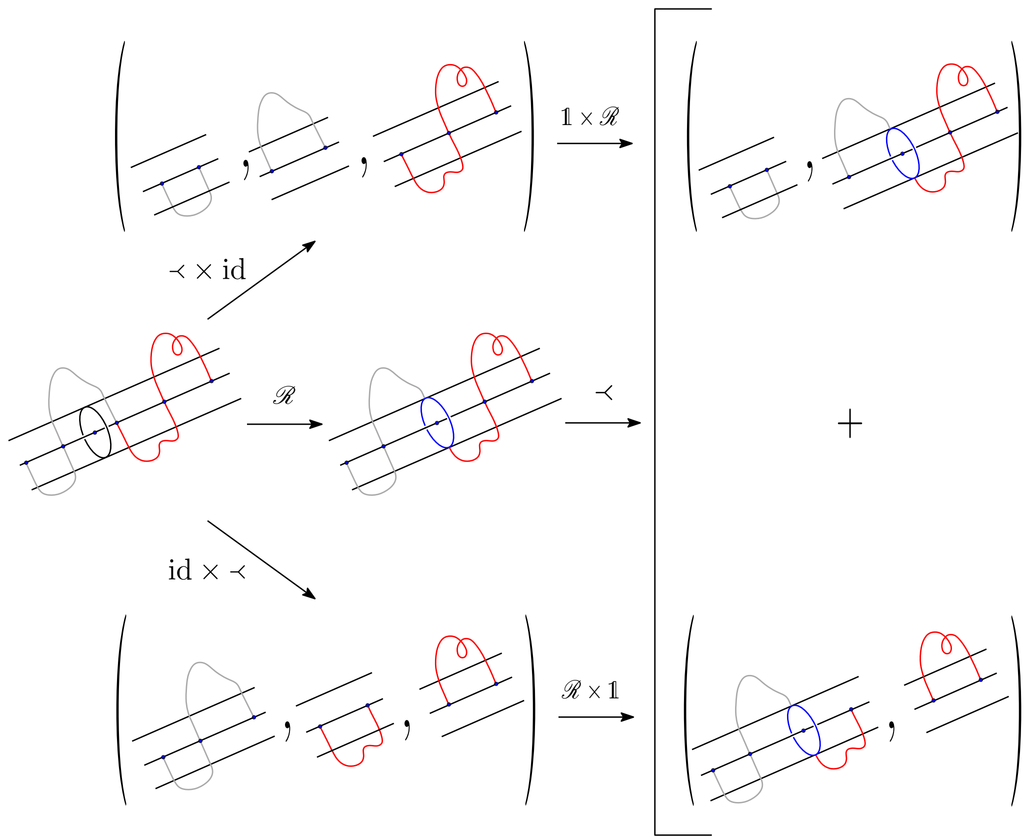

Figure 4.9: Splitting is a "derivation" of resolve

4.2 Bar and Cobar Construction

4.2.1 Simplicial objects and their realizations

We assume that the reader is familiar with the notions of simplicial sets and Delta sets. We refer the reader to [25] for review. Briefly, a *simplicial set* is a contravariant functor

$$\mathcal{F} : \Delta_{\text{simp}} \longrightarrow \mathbf{Set}$$

from the *simplicial category* Δ_{simp} to the category **Set** of sets. Similarly, a *Delta set* is a contravariant functor $\mathcal{F} : \Delta \to \mathbf{Set}$ from the *delta category* to **Set**. More specifically, a simplicial set is a collection of finite sets $\mathcal{K} := \{K_n\}_{n\geq 0}$ equipped with maps

$$d_i : K_n \longrightarrow K_{n-1},\ \ i = 0, \dots, n$$
$$s_i : K_n \longrightarrow K_{n+1},\ \ i = 0, \dots, n.$$

These maps satisfy the relations

$$d_i d_j = d_{j-1} d_i,\ \ i < j \tag{4.2.1}$$
$$s_i s_j = s_{j+1} s_i,\ \ i \leq j \tag{4.2.2}$$
$$d_i s_j = \begin{cases} s_{j-1} d_i & i < j \\ \mathrm{Id} & i = j, j+1 \\ s_j d_{i-1} & i > j+1. \end{cases} \tag{4.2.3}$$

These equations come from the boundary/face maps of simplices and degeneracies. A *Delta set* is a collection of sets $\mathcal{K} := \{K_n\}_{n\geq 1}$ such that we have operators d_i as above satisfying (4.2.1). We do not require the degeneracy maps s_i's. Note that the categories $\Delta_{\mathbf{simp}}$ and Δ have the same objects but there are more morphisms in the simplicial category and there are more operators, viz., the degeneracy maps, in $\Delta_{\mathbf{simp}}$.

Consider the standard n-simplex

$$\Delta_n := \{(t_0, t_1, \dots, t_n, t_{n+1}) \in \mathbb{R}^{n+2} \,|\, 0 = t_0 \leq t_1 \leq \cdots \leq t_n \leq t_{n+1} = 1\}.$$

The face and degeneracy maps

$$\partial_i : \Delta_{n-1} \longrightarrow \Delta_n,\ \ s_i : \Delta_{n+1} \longrightarrow \Delta_n$$

for $i = 0, \dots, n$ are given by

$$\partial_i(t_0, t_1, \dots, t_n) = (t_0, t_1, \dots, t_i, t_i, \dots, t_n) \tag{4.2.4}$$
$$s_i(t_0, t_1, \dots, t_{n+2}) = (t_0, t_1, \dots, t_i, t_{i+2}, \dots, t_{n+2}). \tag{4.2.5}$$

For the 3-simplex we have the following picture : These maps satisfy relations

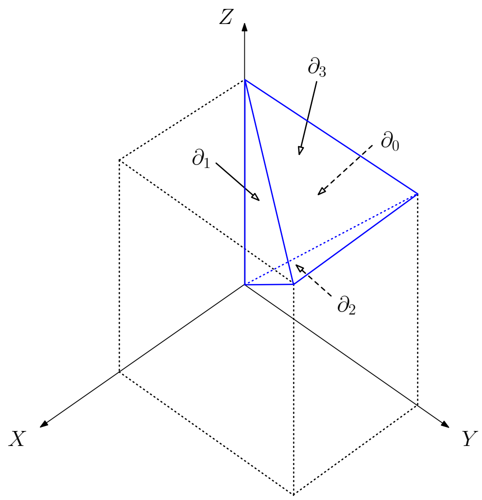

Figure 4.10: Boundary maps on the 3-simplex

adjoint to the relations above, viz.,

$$\partial_i \partial_j = \partial_j \partial_{i-1}, \quad i > j \tag{4.2.6}$$

$$s_i s_j = s_j s_{i+1}, \quad i \geq j \tag{4.2.7}$$

$$s_i \partial_j = \begin{cases} \partial_j s_{i-1} & i > j \\ \mathrm{Id} & i = j, j-1 \\ \partial_{j-1} s_i & i < j-1. \end{cases} \tag{4.2.8}$$

The significance of this construction arises from the idea of *geometric realization* of a simplicial set. Given $\mathcal{K}$ with the discrete topology form the topological sum

$$\overline{\mathcal{K}} := (K_0 \times \Delta_0) + (K_1 \times \Delta_1) + \cdots + (K_n \times \Delta_n) + \cdots$$

We generate the equivalence relation

$$\begin{aligned} (d_i k, x) &\sim (k, \partial_i x) \\ (s_i k, x') &\sim (k, s_i x') \end{aligned}$$

where $k \in K_n, x \in \Delta_{n-1}, x' \in \Delta_{n+1}$. The identification space $|\mathcal{K}| := \overline{\mathcal{K}}/\sim$ will be called the *geometric realization* of $\mathcal{K}$. This a CW-complex with exactly one n-cell for each non-degenerate n-simplex in $\mathcal{K}$. Conversely, to any CW-complex

X one can associate a simplicial set $\mathcal{S}(X) = \{\mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X)\}_{n\geq 0}$. This is precisely the singular complex of X, originally due to Eilenberg. For all of this and more, Milnor [27] is, as usual, a great read!

One can study simplicial objects or Delta objects in other categories as well, i.e., contravariant functors $\mathcal{F} : \Delta_{\mathbf{simp}} \to \mathscr{C}$ or $\mathcal{F} : \Delta \to \mathscr{C}$. We shall focus our attention $\mathscr{C} = \mathbf{Ch}_{\mathbb{k}}$, the category of chain complexes over a ring $\mathbb{k}$. A simplicial object $\mathcal{K}$ in $\mathbf{Ch}_{\mathbb{k}}$ is a collection of chain complexes $\mathcal{K} := \{(K(n)_*, \partial)\}_{n\geq 0}$ such that the usual commutativity of d_i, s_i's with the internal boundary map ∂ holds. The maps d_i, s_i have degree zero. At the cost of causing mild confusion, we shall also denote the internal boundary maps of chain complexes coming from spaces by ∂. Define a new chain complex

$$\overline{\mathcal{K}} := \oplus_{i=0}^{\infty}\big(K(n)_* \otimes_{\mathbb{k}} C_*(\Delta_n; \mathbb{k})\big). \tag{4.2.9}$$

The differential D is given by the usual Leibnitz rule, i.e.,

$$D(\alpha \otimes \beta) := (\partial\alpha \otimes \beta) + (-1)^{|\alpha|}(\alpha \otimes \partial\beta). \tag{4.2.10}$$

Let $\mathcal{L}$ be the $\mathbb{k}$-module generated by the elements

$$(d_i\alpha) \otimes \beta - \alpha \otimes (\partial_i\beta),\ (s_i\alpha) \otimes \beta' - \alpha \otimes (s_i\beta'),$$

where $\alpha \in K(n)_*, \beta \in C_*(\Delta_{n-1}), \beta' \in C_*(\Delta_{n+1})$. Let $\mathcal{L}_\Delta \subset \mathcal{L}$ be the $\mathbb{k}$-submodule generated by $(d_i\alpha) \otimes \beta - \alpha \otimes (\partial_i\beta)$. Notice that

$$\begin{aligned} D\big(d_i\alpha) \otimes \beta - \alpha \otimes (\partial_i\beta)\big) &= \big(d_i(\partial\alpha) \otimes \beta - \partial\alpha \otimes \partial_i\beta\big) \\ &\quad + (-1)^{|\alpha|}\big(d_i\alpha \otimes \partial\beta - \alpha \otimes d_i(\partial\beta)\big) \in \mathcal{L}_\Delta \\ D\big(s_i\alpha) \otimes \beta' - \alpha \otimes (s_i\beta')\big) &= \big(s_i(\partial\alpha) \otimes \beta' - \partial\alpha \otimes s_i\beta'\big) \\ &\quad + (-1)^{|\alpha|}\big(s_i\alpha \otimes \partial\beta' - \alpha \otimes s_i(\partial\beta')\big) \in \mathcal{L}. \end{aligned}$$

Therefore, $\mathcal{L}$ and $\mathcal{L}_\Delta$ are both closed under D whence $(\overline{\mathcal{K}}/\mathcal{L}, D)$ and $(\overline{\mathcal{K}}/\mathcal{L}_\Delta, D)$ are chain complexes.

Definition 4.35. Given a simplicial object $\mathcal{K}$ in $\mathbf{Ch}_{\mathbb{k}}$ we call $(\overline{\mathcal{K}}/\mathcal{L}, D)$ the *realization* of $\mathcal{K}$ and denote it by $|\mathcal{K}|$. The chain complex $(\overline{\mathcal{K}}/\mathcal{L}_\Delta, D)$ is called the *Delta realization* and denoted by $|\mathcal{K}|_\Delta$.

Observe that for a Delta object $\mathcal{K}$ in $\mathbf{Ch}_{\mathbb{k}}$ we only have one realization.

Proposition 4.36. *Let $\mathcal{K}$ be a simplicial (or Delta) object in $\mathbf{Ch}_{\Bbbk}$, equipped with chain maps*

$$K(n)_* \xrightarrow{\prec_j} K(j)_* \otimes K(n-j)_*$$

for $j = 0, \dots, n$ which are coassociative. Then the realization $|\mathcal{K}|$ carries a natural structure of a differential graded coalgebra.

Proof The standard n-simplex

$$\Delta_n = \{(t_0, t_1, \dots, t_n, t_{n+1}) \in \mathbb{R}^{n+2} \,|\, 0 = t_0 \leq t_1 \leq \cdots \leq t_n \leq t_{n+1} = 1\}$$

splits naturally via

$$\prec_k : (0, t_1, \dots, t_n, 1) \mapsto ((0, t_1, \dots, t_k, 1), (0, t_{k+1}, \dots, t_n, 1))\,.$$

which induces a map

$$C_*(\Delta_n) \xrightarrow{\prec_k} C_*(\Delta_k \times \Delta_{n-k}) \xrightarrow{AW} C_*(\Delta_k) \otimes C_*(\Delta_{n-k})$$

via the Alexander-Whitney (AW) map. Then we have a natural splitting

$$K(n)_* \otimes C_*(\Delta_n) \longrightarrow \bigoplus_{j=0}^{n} \left\{ \big(K(j)_* \otimes C_*(\Delta_j)\big) \otimes \big(K(n-j)_* \otimes C_*(\Delta_{n-j})\big) \right\}$$

induced by $\prec_j$ and $\prec_j$ and using Koszul sign convention when we switch terms. The coassociativity follows from the coassociativity of $\prec_j$ and AW. $\square$

Let us look at a few examples, the first two being classical and well known, which fit in this framework.

Example 4.37. (Geometric realization) Let $\mathcal{C} = C_*(\mathcal{K}; \Bbbk)$ be a simplicial object in $\mathbf{Ch}_{\Bbbk}$ arising from taking the chains on a simplicial set $\mathcal{K}$. Since K_n is given the discrete topology,

$$(C_*(K_n; \Bbbk), \partial) \xrightarrow{\simeq} \bigoplus_{p \in K_n} \Bbbk$$

where the right hand side has no internal boundary. Then

$$\overline{\mathcal{C}} \simeq \bigoplus_{i=0}^{\infty} \big(\bigoplus_{p \in K_n} C_*(\Delta_n; \Bbbk) \big).$$

If follows from the definition of $|\mathcal{K}|$ that $C_*(|\mathcal{K}|; \Bbbk)$ has identifications that correspond to the relations on chains that generate $\mathcal{L}$. Therefore, we get

$$C_*(|\mathcal{K}|; \Bbbk) \xrightarrow{\simeq} |C_*(\mathcal{K}; \Bbbk)|.$$

The coalgebra structure is the natural coalgebra on the chains arising from the diagonal map $|\mathcal{K}| \to |\mathcal{K}| \times |\mathcal{K}|$.

Example 4.38. (Bar construction of an algebra) Let $(\mathcal{A} := \oplus_{n\in\mathbb{Z}} A_n, d)$ be a differential graded algebra[1] with $d : A_n \to A_{n-1}$. We can form a Delta object $\mathfrak{A} := \{C_*(n)\}_{n\geq 0}$ in $\mathbf{Ch}_{\Bbbk}$ by setting

$$C_*(n) := \mathcal{A}^{\otimes n}.$$

The boundary maps $d_i : \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes(n-1)}$ are given by

$$\begin{aligned} d_0(a_1 \otimes \cdots \otimes a_n) &= 0 \\ d_i(a_1 \otimes \cdots \otimes a_n) &= (-1)^{|a_1|+\cdots+|a_i|}(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_n), \ \ 0 < i < n \\ d_n(a_1 \otimes \cdots \otimes a_n) &= 0. \end{aligned}$$

The associativity is equivalent to (4.2.1) while $d_i s_i = \mathrm{Id}$, in the existence of a possible degeneracy map, hints towards a unit in $\mathcal{A}$. In fact, if $\mathcal{A}$ has one then we define

$$s_i(a_1 \otimes \cdots \otimes a_n) = (a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n).$$

In any case, $|\mathfrak{A}|_\Delta$ exists. Using the underlying simplicial structure of Δ_n we can replace $C_*(\Delta_n; \Bbbk)$ by a direct sum of copies of $\Bbbk$, one each for each subsimplex at the appropriate degree.

Notation *Let $\Bbbk[n]$ denote the vector space $\Bbbk$ placed at degree n. We shall use $1[n]$ to denote the generator of the vector space. If there are more than one such vector spaces of the same grading present simultaneously then we shall distinguish them by using a subscript, for e.g., $\Bbbk_i[n]$ and $1_i[n]$.*

If we write $C_*(\Delta_1) = \Bbbk[1] \oplus (\oplus_{i=0}^{1} \Bbbk_i[0])$ then

$$C_*(1) \otimes C_*(\Delta_1) \simeq \big(\mathcal{A} \otimes \Bbbk[1]\big) \oplus \big(\oplus_{i=0}^{1} \mathcal{A}\big).$$

Notice that $\mathcal{A} \otimes \Bbbk[1]$ is the *suspension* of $\mathcal{A}$ and will be henceforth be denoted by $s\mathcal{A}$. More generally, let $\Bbbk[n]$ denote the generator of the top cell in $C_*(\Delta_n)$. Then

$$C_*(n) \otimes C_*(\Delta_n) \simeq \big(\mathcal{A}^{\otimes n} \otimes \Bbbk[n]\big) \oplus \cdots \oplus \big(\oplus_{i=0}^{n} \mathcal{A}^{\otimes n} \otimes \Bbbk_i[0]\big).$$

[1] It is not assumed that a dga automatically has a unit.

It is clear that the leading term in $C_*(n) \otimes C_*(\Delta_n)$ is $(s\mathcal{A})^{\otimes n}$. Moreover,

$$
\begin{aligned}
D(a_1 \otimes \cdots \otimes a_n \otimes 1[n]) - d(a_1 \otimes \cdots \otimes a_n) \otimes 1[n] \\
&= (-1)^{|a_1|+\cdots+|a_n|}(a_1 \otimes \cdots \otimes a_n) \otimes \partial(1[n]) \\
&= (-1)^{i+\sum_{j=1}^{n}|a_j|}(a_1 \otimes \cdots \otimes a_n) \otimes 1_i[n-1] \\
&= (-1)^{i+\sum_{j=1}^{n}|a_j|}(a_1 \otimes \cdots \otimes a_n) \otimes \partial_i(1[n-1]) \\
&= (-1)^{i+\sum_{j=1}^{n}|a_j|}d_i(a_1 \otimes \cdots \otimes a_n) \otimes 1[n-1]
\end{aligned}
$$

where the last equality holds modulo $\mathcal{L}_\Delta$. The differential D on $\overline{\mathfrak{A}}/\mathcal{L}_\Delta$ looks very much like the differential of the well known *bar construction* of an augmented unital dga. Moreover, we emphasize that $\Bbbk_i[n]$ is the top cell of the ith face of Δ_n and therefore is identified with $\partial_i\Bbbk[n-1]$. The equivalence relation implies that ∂_i transfers to d_i on $\mathcal{A}^{\otimes n}$. In summary, $\mathcal{A}^{\otimes n} \otimes \Bbbk_i[n-1]$ gets identified with parts of $\mathcal{A}^{\otimes(n-1)} \otimes \Bbbk[n-1]$. Similarly, $\mathcal{A}^{\otimes n} \otimes \Bbbk_\sigma[n-j]$ gets identified with parts of $\mathcal{A}^{\otimes(n-j)} \otimes \Bbbk[n-j]$ by repeated application of the equivalence relations and the gluing data. In conclusion, we have the well known

Proposition 4.39. *There is an isomorphism of* $\Bbbk$*-modules*

$$
|\mathfrak{A}|_\Delta \cong \oplus_{n\geq 0}\big(\mathcal{A}^{\otimes n} \otimes \Bbbk[n]\big) = \Bbbk \oplus s\mathcal{A} \oplus (s\mathcal{A})^{\otimes 2} \oplus \cdots . \tag{4.2.11}
$$

Moreover, the differential D can be transported to $T(s\mathcal{A})$ to get a derivation.

It can be verified that this construction agrees with the usual[2] bar construction of an augmented unital dga $\mathscr{A}$ by applying this to $\overline{\mathscr{A}}$, the kernel of the augmentation map.

The coalgebra structure $\prec$ is very clear when written as $T(s\mathcal{A})$. It is essentially due to the fact that an n-simplex given by $[0, 1, \ldots, n]$ splits into $[0, 1, \ldots, k] \times [k, \ldots, n]$ for $k = 1, \ldots, n-1$.

Example 4.40. (Bar construction of a dg category) What we take home from the previous example is that an algebra could be regarded as a category with one object and the algebra itself as the morphisms. Since morphisms are composable using the multiplication we get n-composable morphisms as our nth level chain complex which fit together into a simplicial object[3] in $\mathbf{Ch}_\Bbbk$.

[2]Sometimes it is referred to as the *reduced* bar construction.

[3]It is tedious to write *a simplicial object in the category* $\mathscr{C}$ all the time and usually one writes *a simplicial* $\mathscr{C}$. However, when $\mathscr{C} = \mathbf{Ch}_\Bbbk$ we are consciously avoiding writing it since a *simplicial chain complex* usually means something completely different.

Let $\mathscr{C}$ be a category with objects $\mathcal{Ob}$ and morphism $\mathcal{Mor}(A, B)$[4]. Assume that $\mathcal{Mor}(A, B)$ is a chain complex over $\Bbbk$ for any $A, B \in \mathcal{Ob}$ and that there is a *composition law*

$$\mathcal{Mor}(A, B) \otimes_{\Bbbk} \mathcal{Mor}(B, C) \xrightarrow{m} \mathcal{Mor}(A, C)$$

which is a map of chain complexes of degree zero. We assume that m is associative. We form a Delta (or simplicial as the case may be) object $\mathcal{N}\mathscr{C}$ in $\mathbf{Ch}_{\Bbbk}$ called the *nerve of the category* $\mathscr{C}$. Let

$$\mathcal{Mor} \otimes_{\mathcal{Ob}} \mathcal{Mor} := \oplus_{A,B,C \in \mathcal{Ob}} \mathcal{Mor}(A, B) \otimes \mathcal{Mor}(B, C)$$

be the collection 2*-composable morphisms*. This is a chain complex. One can similarly define

$$\mathcal{Mor}_{\mathcal{Ob}}^{\otimes n} := \oplus_{A_i \in \mathcal{Ob}} \mathcal{Mor}(A_1, A_2) \otimes \cdots \otimes \mathcal{Mor}(A_n, A_{n+1}) \tag{4.2.12}$$

the chain complex of n-composable morphisms and we follow the convention that

$$\mathcal{Mor}_{\mathcal{Ob}}^{\otimes 0} = \mathcal{Ob}.$$

Define the operators

$$d_i : \mathcal{Mor}_{\mathcal{Ob}}^{\otimes n} \longrightarrow \mathcal{Mor}_{\mathcal{Ob}}^{\otimes (n-1)}$$

by setting $d_0 = d_n = 0$ and d_i composes the ith and the $(i+1)$th morphism using m. Therefore,

$$\mathcal{N}\mathscr{C} := \{\mathcal{Mor}_{\mathcal{Ob}}^{\otimes n}\}_{n \geq 0}$$

is a Delta object in $\mathbf{Ch}_{\Bbbk}$. If $1_A \in \mathcal{Mor}(A, A)$ then we can write the degeneracy maps s_i's as well and $\mathcal{N}\mathscr{C}$ can be enriched to be a simplicial object. We define the *bar* of $\mathscr{C}$ to be the realization $|\mathcal{N}\mathscr{C}|_{\Delta}$ of the nerve.

Notice that $|\mathcal{N}\mathscr{C}|_{\Delta}$ is the quotient of $\mathcal{Mor}_{\mathcal{Ob}}^{\otimes n} \otimes C_*(\Delta_n)$ and as in the previous example after taking the quotient we get

$$|\mathcal{N}\mathscr{C}|_{\Delta} \cong \oplus_{n \geq 0} \big(\mathcal{Mor}_{\mathcal{Ob}}^{\otimes n} \otimes \Bbbk[n]\big) = \oplus_{n \geq 0} (s\mathcal{Mor})_{\mathcal{Ob}}^{\otimes n},$$

where $s(x)$ shifts the degree of the element x by $+1$. The coalgebra structure $\prec$ is given by splitting at all possible intermediate junctions.

[4]We are not requiring the existence of $1_A \in \mathcal{Mor}(A, A)$ although this is usually assumed to be part of the definition of any category.

4.2.2 Bar and cobar - construction and adjunction

Let's fix the field $\mathbb{Q}$ for what we will be discussing from the outset. Let's start off on a little flexible note and assume that $\mathcal{A}$ is a positively graded differential graded *algebra* with the differential d of degree -1. The underlying space of $\mathcal{A}$ is a graded vector space over $\mathbb{Q}$ and by an *algebra* we mean any structure of the form

$$m : \mathcal{A} \boxtimes \mathcal{A} \longrightarrow \mathcal{A}$$

which commutes with d (defined suitably on the left hand side). Examples include

(i) $\otimes_{\mathbb{Q}}$ - the usual algebra over $\mathbb{Q}$,
(ii) $\otimes_R$ - algebra in the category of bimodules over a ring R (over $\mathbb{Q}$),
(iii) $\square_{\mathcal{C}}$ - algebra in the category of bicomodules over a coalgebra $\mathcal{C}$.

We initially assume that $\square$ has degree zero and is associative. Let us assume that we do not have a unit for $\mathcal{A}$. We form the geometric realization of the Delta complex

$$\mathfrak{A} := \{\mathcal{A}^{\boxtimes n}\}$$

and call this the *bar construction* of $\mathcal{A}$.

Remark 4.41. *If $\mathcal{A}$ has a unit then we have the appropriate degeneracy maps using the unit. In such a case $\mathfrak{A}$ is a simplicial complex and we take its realization. In the usual examples, presence of a unit is often equivalent to an augmentation and we may work with $\overline{\mathcal{A}}$, the augmentation coideal.*

Let us review the the classical case where $\boxtimes$ is the usual tensor product.

Definition 4.42. (Bar construction)
Let $(\mathcal{A} = \oplus_{n\in\mathbb{Z}} A_n, m, d, \varepsilon)$ be an augmented differential $\mathbb{Z}$-graded algebra with $d(A_n) \subseteq A_{n-1}$ and $\overline{\mathcal{A}} := \ker \varepsilon$. The *bar* of $(\mathcal{A}, m, d, \varepsilon)$ is given by

$$B\mathcal{A} := (T(s\overline{\mathcal{A}}), D)$$

where

(i) $T(s\overline{\mathcal{A}})$ is the tensor algebra of $s\overline{\mathcal{A}} := \overline{\mathcal{A}}[1]$, s being the *suspension operator*.
(ii) The differential D, written as $d + m$, is given by

$$\begin{aligned} d(sa_1 \otimes \cdots \otimes sa_n) &= -\sum_{i=1}^{n} \varepsilon_i (sa_1 \otimes \cdots \otimes sa_{i-1} \otimes s(da_i) \otimes \cdots \otimes sa_n) \\ m(sa_1 \otimes \cdots \otimes sa_n) &= \sum_{i=2}^{n} \varepsilon_i (sa_1 \otimes \cdots \otimes sa_{i-2} \otimes s(a_{i-1}a_i) \otimes \cdots \otimes sa_n) \end{aligned}$$

where the signs are given by

$$\varepsilon_i := (-1)^{|sa_1|+\cdots+|sa_{i-1}|}.$$

The coalgebra structure on $B(\mathcal{A}, d, \varepsilon)$ is given by the reduced diagonal

$$\Delta(sa_1 \otimes \cdots \otimes sa_n) = \sum_{i=1}^{n-1} (sa_1 \otimes \cdots \otimes sa_i) \otimes (sa_{i+1} \otimes \cdots \otimes sa_n)$$

while the algebra structure is given by the *shuffle product*

$$sh(x_1 \otimes \cdots \otimes x_p; x_{p+1} \otimes \cdots \otimes x_{p+q}) := \sum_{\sigma} \pm x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(p+q)}$$

where σ runs through the set of permutations of $\{1, \ldots, p+q\}$ which satisfy $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. The sign is given by the Koszul rule.

Remark 4.43. *The shuffle product is associative and commutative and together with the reduced diagonal, it turns $B\mathcal{A}$ into a Hopf algebra. However, D satisfies Leibnitz with sh if and only if $\mathcal{A}$ is commutative. In that case, $B\mathcal{A}$ is a commutative dga and hence a commutative differential graded Hopf algebra.*

Definition 4.44. (Cobar construction)
Given $(\mathcal{C} = \oplus_{n\in\mathbb{Z}} C_n, \Delta, \partial, \eta)$, a cocommutative differential $\mathbb{Z}$-graded coaugmented coalgebra with $\partial(C_n) \subseteq C_{n-1}$, $\overline{\mathcal{C}} := \mathcal{C}/\eta(\Bbbk)$ and $\overline{\Delta} : \overline{\mathcal{C}} \to \overline{\mathcal{C}} \otimes \overline{\mathcal{C}}$ we define the *cobar* to be

$$\Omega\mathcal{C} := (T(s^{-1}\overline{\mathcal{C}}), \delta)$$

where

(i) $T(s^{-1}\overline{\mathcal{C}})$ is the tensor algebra of $s^{-1}\overline{\mathcal{C}} := \overline{\mathcal{C}}[-1]$, s^{-1} being the *desuspension operator*.

(ii) The differential δ, written as $\partial + \Delta$, acts on $c := s^{-1}c_1 \otimes \cdots \otimes s^{-1}c_n$ as

$$\begin{aligned} \partial(c) &= -\sum_{i=1}^{n} \varepsilon_i (s^{-1}c_1 \otimes \cdots \otimes s^{-1}(\partial c_i) \otimes \cdots \otimes s^{-1}c_n) \\ \Delta(c) &= \sum_{i=1}^{n} \sum_{j} \varepsilon_i (-1)^{|c'_{ij}|} (s^{-1}c_1 \otimes \cdots \otimes s^{-1}c'_{ij} \otimes s^{-1}c''_{ij} \otimes \cdots \otimes s^{-1}c_n), \end{aligned}$$

where

$$\begin{aligned} \varepsilon_i &:= (-1)^{\sum_{k=1}^{i-1} |s^{-1}c_k|} \\ \overline{\Delta}(c_i) &= \sum_j c_{ij}^{'} \otimes c_{ij}^{''}. \end{aligned}$$

Note that δ, being a differential on an algebra, is determined by its values on $s^{-1}c$ for $c \in \overline{\mathcal{C}}$.

The canonical isomorphism

$$(s^{-1}\overline{\mathcal{C}})^{\otimes i} \otimes (s^{-1}\overline{\mathcal{C}})^{\otimes j} \to (s^{-1}\overline{\mathcal{C}})^{\otimes (i+j)}$$

induces a product $\mu : T(s^{-1}\overline{\mathcal{C}}) \otimes T(s^{-1}\overline{\mathcal{C}}) \to T(s^{-1}\overline{\mathcal{C}})$. The coalgebra structure is given by the *shuffle coproduct* given by splitting an element of $(s^{-1}\overline{\mathcal{C}})^{\otimes n}$ into elements of $(s^{-1}\overline{\mathcal{C}})^{\otimes i} \otimes (s^{-1}\overline{\mathcal{C}})^{\otimes (n-i)}$ along with all possible signed permutations.

Remark 4.45. *The shuffle coproduct is coassociative and cocommutative and alongwith μ this turns $\Omega\mathcal{C}$ into a Hopf algebra. However, δ satisfies Leibnitz with shuffle coproduct if and only if $\mathcal{C}$ is cocommutative. In that case, $\Omega\mathcal{C}$ is a differential graded cocommutative Hopf algebra.*

The bar and cobar constructions are applicable in the context of augmented dga's and connected dg colagebras. Recall that a coalgebra is *connected* if the coagumentation coideal is $C_{\geq 1}$ or equivalently $C_0 = \mathbb{k}$. Thus, one can think of B (resp. Ω) as functors from algebras (resp. coalgebras) to coalgebras (resp. algebras) respectively.

Theorem 4.46. (Husemoller-Moore-Stasheff)
For any augmented dga $\mathcal{A}$ and any connected dg coalgebra $\mathcal{C}$ (both of which are non-negatively graded), there are natural adjunction morphisms

$$\alpha : \Omega B\mathcal{A} \longrightarrow \mathcal{A}, \ \beta : \mathcal{C} \longrightarrow B\Omega\mathcal{C}$$

which are quasi-isomorphisms of algebras and coalgebras respectively. Here,

$$\alpha : \Omega B\mathcal{A} \longrightarrow \mathcal{A}$$

is defined to be zero on $s^{-1}T^{\geq 2}(s\overline{\mathcal{A}})$ and the canonical isomorphism $s^{-1}(s\mathcal{A}) \to \mathcal{A}$ and then extended naturally. The map β is the unique lifting of

$$\mathcal{C} \to s^{-1}\overline{\mathcal{C}} \to \overline{T(s^{-1}\overline{\mathcal{C}})} \to s\overline{T(s^{-1}\overline{\mathcal{C}})}.$$

We shall give a proof of the fact that α is a quasi-isomorphism in what follows by constructing an explicit deformation retraction of $\Omega B\mathcal{A}$ onto $\mathcal{A}$.

Theorem 4.47. (Bar-Cobar adjunction) *For the maps $\alpha : \Omega B\mathcal{A} \to \mathcal{A}$ and $\iota : \mathcal{A} \hookrightarrow \Omega B\mathcal{A}$ we have $\alpha \circ \iota = 1$ and there is a chain homotopy $\hbar : (\Omega B\mathcal{A}, \delta) \to (\Omega B\mathcal{A}, \delta)$ of degree 1 such that*

$$1 - \iota \circ \alpha = \delta\hbar + \hbar\delta.$$

We fix notations for the elements of $\Omega B\mathcal{A}$ and write down its differential δ, consisting of d_Ω (arising from d), d_Δ (arising from the tensor coproduct) and d_m (arising from the multiplication on $\mathcal{A}$). Using the augmentation $\varepsilon : \mathcal{A} \to \Bbbk$ we get a splitting $\mathcal{A} = \Bbbk \oplus \overline{\mathcal{A}}$. Note that

$$\Omega B\mathcal{A} = T(s^{-1}(\overline{T(s\overline{\mathcal{A}})})) = T(\mathfrak{s}_0 \oplus \mathfrak{s}_1 \oplus \cdots)$$

where

$$\mathfrak{s}_r := s^{-1}((s\overline{\mathcal{A}})^{\otimes(r+1)}).$$

In other words,

$$\Omega B\mathcal{A} = \Bbbk \oplus \left(\oplus_{j=0}^{\infty} \mathfrak{s}_j\right) \oplus \left(\oplus_{i,j=0}^{\infty} \mathfrak{s}_{i,j}\right) \oplus \cdots ,$$

where $\mathfrak{s}_{i_1,\dots,i_n} = \mathfrak{s}_{i_1+1} \otimes \cdots \otimes \mathfrak{s}_{i_n+1}$. Notice that $\mathcal{A} = \Bbbk \oplus \mathfrak{s}_0 \xhookrightarrow{\iota} \Omega B\mathcal{A}$.[5] Therefore, $\mathcal{A} \hookrightarrow \Omega B\mathcal{A} \xrightarrow{\alpha} \mathcal{A}$ is the identity map.

Notation *An element $y = s^{-1}(sa_1 \otimes sa_2 \otimes \cdots \otimes sa_k) \in \mathfrak{s}_{k-1}$ will be denoted by $[a_1|a_2|\cdots|a_k]$.*

By unravelling the definitions one sees that the differential δ on $\Omega B\mathcal{A}$ is given by $d_\Omega + d_\Delta + d_m$ where

$$\begin{aligned}
d_\Omega([a_1|\cdots|a_k]) &= \sum_{i=1}^{k} (-1)^{|sa_1|+\cdots+|sa_{i-1}|}[a_1|\cdots|a_{i-1}|da_i|\cdots|a_k] \\
d_\Delta([a_1|\cdots|a_k]) &= \sum_{i=1}^{k-1} (-1)^{|sa_1|+\cdots+|sa_i|}[a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_k] \\
d_m([a_1|\cdots|a_k]) &= -\sum_{i=2}^{k} (-1)^{|sa_1|+\cdots+|sa_{i-1}|}[a_1|\cdots|a_{i-2}|a_{i-1}a_i|a_{i+1}|\cdots|a_k].
\end{aligned}$$

[5] This map is *not* a map of algebras thereby explaining the direction of the map $\Omega B\mathcal{A} \to \mathcal{A}$.

In other words,

$$\delta : s_n \to s_n \oplus s_{n-1} \oplus \left(\oplus_{i=0}^{n-1} s_i \otimes s_{n-1-i}\right)$$

and then it is extended by Leibnitz. Notice that if $(\mathcal{A}, d)$ was a dga with trivial multiplication then $d_m \equiv 0$.

Lemma 4.48. *Let $\mathcal{A}$ be a differential graded algebra equipped with the trivial multiplication, i.e. $d_m \equiv 0$. Then bar-cobar adjunction holds.*

Proof It suffices to find chain maps $\alpha' : \Omega B\mathcal{A} \to \mathcal{A}$, $\iota : \mathcal{A} \to \Omega B\mathcal{A}$ and $h : (\Omega B\mathcal{A}, \delta) \to (\Omega B\mathcal{A}, \delta)$ such that $\alpha \circ \iota = \text{id}$ and $\delta h + h\delta = 1 - \iota\alpha'$. Define $\alpha' : \Omega B\mathcal{A} \to \mathcal{A}$ which maps $\Bbbk \oplus s_0$ canonically to $\mathcal{A}$ and maps every other $s_{i_1,\ldots,i_n}$ to zero. Define a *homotopy* $h : (\Omega B\mathcal{A}, d_\Omega) \to (\Omega B\mathcal{A}, d_\Omega)$ by declaring

$$\begin{aligned} h(s_{i_1,\ldots,i_k}) &= 0 \ \text{ if } \ i_1 > 0 \\ h\big([a_1] \otimes [a_{2,1}|\cdots|a_{2,i_2}] \otimes \cdots\big) &= (-1)^{|sa_1|}[a_1|a_{2,1}|\cdots|a_{2,i_2}] \otimes \cdots . \end{aligned}$$

Then one can check that

$$hd_\Omega + d_\Omega h = \text{id} - \iota \circ \alpha'$$

by checking that if $i_1 > 0$ then $hd_\Omega(s_{i_1,\ldots,i_k}) \equiv s_{i_1,\ldots,i_k}$. On the other hand if $i_1 = 0$ then there are cancellations between terms of hd_Ω and $d_\Omega h$ that leaves the correct term from $d_\Omega h$ remaining. More precisely, let

$$a = [a_1] \otimes \underbrace{[a_{2,1}|\cdots|a_{2,i_2}]}_{a_2} \otimes \cdots \otimes \underbrace{[a_{k,1}|\cdots|a_{k,i_k}]}_{a_k}$$

and observe that

$$d_\Omega h + hd_\Omega \equiv 0 \ \text{ on } \Omega B\mathcal{A}. \tag{4.2.13}$$

As an useful notation, let $\mathbf{a}_{12} := [a_1|a_{2,1}|\cdots|a_{2,i_2}]$. One can verify that

$$\begin{aligned}
d_\Delta h(\mathbf{a}) &= (-1)^{|sa_1|} d_\Omega\big(\underbrace{[a_1|a_{2,1}|\cdots|a_{2,i_2}]}_{\mathbf{a}_{12}} \otimes \mathbf{a}_2 \otimes \cdots \otimes \mathbf{a}_k\big) \\
&= \mathbf{a} + \sum_{\lambda=1}^{i_2-1} \pm [a_1|a_{2,1}|\cdots|a_{2,\lambda}] \otimes [a_{2,\lambda+1}|\cdots|a_{2,i_2}] \otimes \cdots \otimes \mathbf{a}_k \\
&\quad + \sum_{j=3,\lambda=1}^{k,i_j-1} \pm \mathbf{a}_{12} \otimes \cdots \otimes [a_{j,1}|\cdots|a_{j,\lambda}] \otimes [a_{j,\lambda+1}|\cdots|a_{j,i_j}] \otimes \cdots \otimes \mathbf{a}_k \\
h d_\Delta(\mathbf{a}) &= \sum_{\lambda=1}^{i_2-1} \mp [a_1|a_{2,1}|\cdots|a_{2,\lambda}] \otimes [a_{2,\lambda+1}|\cdots|a_{2,i_2}] \otimes \cdots \otimes \mathbf{a}_k \\
&\quad + \sum_{j=3,\lambda=1}^{k,i_j-1} \mp \mathbf{a}_{12} \otimes \cdots \otimes [a_{j,1}|\cdots|a_{j,\lambda}] \otimes [a_{j,\lambda+1}|\cdots|a_{j,i_j}] \otimes \cdots \otimes \mathbf{a}_k.
\end{aligned}$$

The summations above appear with opposite signs (it's tedious to write down and has been checked by the author). Therefore, we conclude that

$$d_\Delta h(\mathbf{a}) + h d_\Delta(\mathbf{a}) = \mathbf{a}. \tag{4.2.14}$$

Combining (4.2.13) and (4.2.14) we have our result. $\square$

Proof of theorem First observe from the construction of h (in Lemma 4.48) that hd_m (and hence $d_m h$) are pointwise nilpotent, i.e., given $x \in \Omega B\mathcal{A}$ there exists $n > 0$ such that $(d_m h)^n x = 0$. This is easiest proved by induction but is clear otherwise. We shall construct a homotopy

$$\mathbf{h} : (\Omega B\mathcal{A}, d_\Omega + d_m) \longrightarrow (\Omega B\mathcal{A}, d_\Omega + d_m)$$

for a suitable algebra morphism $\alpha : (\Omega B\mathcal{A}, d_\Omega + d_m) \to (\mathcal{A}, d)$. We do this by deforming h and α' (defined in Lemma 4.48) suitably. First notice that we have the following identities :

$$\alpha' h = 0,\ \ h\iota = 0,\ \ h^2 = 0.$$

Define

$$\Lambda = \sum_{i \geq 0} (-1)^i (h d_m)^i.$$

This is a well defined degree 0 operator due to the nilpotency of hd_m. We define

$$\begin{aligned}
\alpha &= \alpha'(1 - d_m \Lambda h) \\
\hbar &= \Lambda \circ h.
\end{aligned}$$

Traditionally one should define a new ι by setting it to $\Lambda \circ \iota$. In this case, this equals ι. Also notice that the way hd_m acts on $\Omega B\mathcal{A}$ precisely reflects the fact that α as defined here agrees with the definition given in the proposition. Therefore, we have perturbed α' to α, a morphism of dga's.

There is another way to think about $\hbar$. Define

$$\begin{aligned}
\mathcal{S}_0 &= \Bbbk \oplus \big(\oplus_{i \geq 0}\, s_i \big) \oplus \big(\oplus_{i_1 > 0, n > 1}\, s_{i_1, i_2, \ldots, i_n} \big) \\
\mathcal{S}_1 &= \oplus_{i_2 > 0}\, s_{0, i_2, \ldots, i_n} \\
\mathcal{S}_n &= \oplus\, s_{0,0,i_3,\ldots,i_n},\ n \geq 2.
\end{aligned}$$

We know that $\hbar \equiv 0$ on $\mathcal{S}_0$. If $y \in \mathcal{S}_i$ then one can check that

$$y - \delta h(y) \in \oplus_{j=0}^{i-1} \mathcal{S}_j.$$

Define $\hat{H}$ of degree 1 inductively on $\mathcal{S}_i$ by declaring $\hat{H} \equiv 0$ on $\mathcal{S}_0$ and setting

$$\hat{H}(y) = h(y) + \hat{H}(y - \delta h(y)).$$

One sees that

$$\begin{aligned}
h(y) + \Lambda h(y - \delta h(y)) &= h(y) + \Lambda h(h\delta - hd_m - d_m h + \iota\alpha')(y) \\
&= (h - \Lambda h d_m h)(y) \\
&= \Lambda h(y)
\end{aligned}$$

and therefore $\hbar$ satisfies the same recursive relation as $\hat{H}$. Since they agree on $\mathcal{S}_0$, they agree everywhere.

It remains to check that $\delta\hbar + \hbar\delta = 1 - \iota\alpha$. We prove this for $\mathcal{S}_i$ by induction on i. First observe that the image of h and $\Bbbk \oplus s_0$ are all of $\mathcal{S}_0$. Now

$$\begin{aligned}
(\hbar\delta + \delta\hbar)h &= \Lambda h \delta h + \delta \Lambda h^2 \\
&= \Lambda h d_m h + \Lambda h d_\Omega h \\
&= (1 - \Lambda)h + \Lambda(1 - \iota'\alpha' - d_\Omega h)h \\
&= (1 - \Lambda\iota'\alpha')h \\
&= (1 - \iota\alpha)h.
\end{aligned}$$

This implies that $\hbar$ is a chain homotopy for $\mathcal{S}_0$. Now assume that $\delta\hbar + \hbar\delta = 1-\iota\alpha$ is true for $\mathcal{S}_i$. Then we conclude that for $y \in \mathcal{S}_{i+1}$ (whence $y-\delta h(y) \in \mathcal{S}_i$)

$$\begin{aligned}
(\hbar\delta + \delta\hbar)(y) &= (\hbar\delta + \delta\hbar)(\delta h(y)) + (\hbar\delta + \delta\hbar)(y - \delta h(y)) \\
&= (\delta\hbar\delta h)(y) + (1-\iota\alpha)(y - \delta h(y)) \\
&= \delta(1-\iota\alpha)h(y) + (1-\iota\alpha)(y) - (1-\iota\alpha)\delta h(y) \\
&= (1-\iota\alpha)(y) - \delta\iota\alpha h(y) + \iota\alpha\delta h(y) \\
&= (1-\iota\alpha)(y).
\end{aligned}$$

This concludes the proof. □

Remark 4.49. *The proof of Theorem 4.47 is canonical in the sense that given the existence of maps α', ι, h and an object $\Omega B\mathcal{A}$ (as in Lemma 4.48) where $\mathcal{A}$ is an algebra in some category, the proof goes through. To have a general bar-cobar adjunction for an algebra $\mathcal{A}$ with an appropriate multiplication $m : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$, we just need to prove Lemma 4.48.*

4.2.3 Application to open strings

Let $Y \subset X$ be an embedded submanifold of codimension d. Let $\mathcal{S}\underline{t}_k$ be as discussed in §4.1.1. We shall denote by $V[k]$ a graded vector space V shifted up by k. Applying this consideration to the chain complex $C_*(\mathcal{S}\underline{t}_k)[k(d-2)]$, a geometric i-chain is placed in degree in $i + k(d-2)$.

Definition 4.50. (Cord space) The space $\mathcal{S}\underline{t}_1$ of transversal open strings in M that start and end in N with no interior intersections will be called the *cord space*. The chains on the cord space (appropriately shifted)

$$\mathscr{A} = C_*(\mathcal{S}\underline{t}_1)[d-2] \tag{4.2.15}$$

have a evaluation map

$$ev : \mathscr{A} \longrightarrow C_*(T \times T)[d-2] \tag{4.2.16}$$

induced by $\mathrm{ev}_0, \overline{\mathrm{ev}_1}$ (defined in §4.25).

Recall that the resolve operator $\mathscr{R}$ resolves a family in $\mathcal{S}\underline{t}_2$ and produces a family in $\mathcal{S}\underline{t}_1$. If one thinks of $\mathcal{S}\underline{t}_2$ as $\mathcal{S}\underline{t}_1 \times_T \mathcal{S}\underline{t}_1$ then one can imagine a map

$$\overline{\mathscr{R}} : \mathscr{A} \,\square\, \mathscr{A} \longrightarrow \mathscr{A}.$$

To make this precise, we need the Alexnader-Whitney and Eilenberg-Zilber maps. Recall that

$$EZ : \mathscr{A} \,\Box\, \mathscr{A} \longrightarrow C_{*-2d+4}(\mathcal{S}t_1 \times_T \mathcal{S}t_1)$$

and we can apply the resolve operator (of degree $d-2$) on the right hand side above to land in $\mathscr{A} := C_{*-d+2}(\mathcal{S}t_1)$.

Definition 4.51. (Resolve operator as a multiplication) The resolve operator can be interpreted as a multiplication giving rise to an algebra structure on $\mathscr{A}$ in the category of bicomodules over $C_*(T)$. Define

$$\overline{\mathscr{R}} : \mathscr{A} \,\Box\, \mathscr{A} \longrightarrow \mathscr{A} \tag{4.2.17}$$

as the composition $\mathscr{R} \circ EZ$. More generally, define

$$\overline{\mathscr{R}} : \mathscr{A}^{\Box k} \longrightarrow \mathscr{A}^{\Box (k-1)}$$

as the composition $AW \circ \mathscr{R} \circ EZ$.

With this necessary discussion of cord space (as an algebra) in place, we now consider the simplicial object

$$\left\{ C_*(\mathcal{S}t_k)[k(d-2)] \right\}_{k \geq 0}$$

where $C_*(\mathcal{S}t_0) := C_*(T)$ plays the role of the ground object. The role of the d_i's are played by resolving an element of $\mathcal{S}t_k$ at the ith transversal intersection (refer §4.1.3 for the operator $\mathscr{R}(\gamma, p_i)$). It also follows from the proof of the Proposition 4.16 that there are natural maps

$$AW \circ \iota : C_*(\mathcal{S}t_k)[k(d-2)] \longrightarrow \mathscr{A}^{\Box k}$$

where AW is the Alexander-Whitney map and

$$\iota : \mathcal{S}t_k \hookrightarrow \mathcal{S}t_1 \times_T \mathcal{S}t_1 \times_T \cdots \times_T \mathcal{S}t_1$$

is a quasi-isomorphism by Lemma 4.31. By a shifting, it follows that

$$AW \circ \iota : \bigoplus_{k \geq 0} \left(C_*(\mathcal{S}t_k)[k(d-1)] \right) \longrightarrow \bigoplus_{k \geq 0} \left(\mathscr{A}^{\Box k}[k] \right) \tag{4.2.18}$$

is a quasi-isomorphism.

Remark 4.52. *This shifting can also be realized as arising from*

$$AW \circ \iota : \bigoplus_{k\geq 0} \big(C_*(\mathcal{S}t_k)[k(d-2)] \otimes C_*(\Delta_k)\big) \longrightarrow \bigoplus_{k\geq 0} \big(\mathscr{A}^{\square k} \otimes C_*(\Delta_k)\big)$$

and via the bar construction. Recall from Example 4.38 that the bar construction (or geometric realization of simplicial objects in the category of chain complexes) is the quotient of the complexes above by a suitable ideal.

The left hand side of (4.2.18) is precisely $\mathfrak{C}(X,Y) \oplus C_*(T)$ while the right hand side is $B\mathscr{A}$, the bar construction of the algebra $\mathscr{A}$ in the category of bicomodules.

Theorem 4.53. *There is a natural quasi-isomorphism*

$$\mathcal{AW} : \mathfrak{C}(X,Y) \oplus C_*(T) \longrightarrow B\mathscr{A} \tag{4.2.19}$$

of differential graded coalgebras in the category of bicomodules over $C_*(T)$. Moreover, this quasi-isomorphism holds with the evaluation maps $\mathcal{E}v$ and ev.

Corollary 4.54. *There is a quasi-isomorphism*

$$\Psi : \Omega\big(\mathfrak{C}(X,Y)\big) \longrightarrow \mathscr{A} \oplus C_*(T) \tag{4.2.20}$$

of differential graded algebras in the category of bicomodules over $C_*(T)$. This quasi-isomorphism holds with the evaluation maps $\mathcal{E}v$ and ev.

Proof of corollary The map Ψ is the composition

$$\Psi := \alpha \circ \Omega \circ \mathcal{AW}$$

where α is the map from Theorem 4.47. Applying Theorem 4.47 to the cobar construction applied to the quasi-isomorphism $\mathcal{AW}$ of Theorem 4.53 we have the required conclusion. It is clear from the definition of the evaluation maps and the naturality of these constructions that quasi-isomorphism holds including these maps. $\square$

Proof The map $\mathcal{AW}$ is the collection of maps $AW \circ \iota$ stratified as usual. Since AW is already a quasi-isomorphism, to prove that $\mathcal{AW}$ is a quasi-isomorphism, it suffices to prove that ι is. This follows from Lemma 4.31. It remains to check that $\mathcal{AW}$ commutes, up to homotopy, with the differential. It clearly commutes with ∂. On the other hand, the diagram

$$\begin{tikzcd} C_*(\mathcal{S}\underline{t}_k)[k(d-1)] \arrow[r, "AW\circ\iota"] \arrow[d, "\mathscr{R}=\kappa^1\circ\mathscr{R}\circ\iota"'] & (\mathscr{A}[1])^{\square k} \arrow[d, "\overline{\mathscr{R}}:=AW\circ\mathscr{R}\circ EZ"] \\ C_*(\mathcal{S}\underline{t}_{k-1})[(k-1)(d-1)] \arrow[r, "AW\circ\iota"] & (\mathscr{A}[1])^{\square (k-1)} \end{tikzcd}$$

commutes up to homotopy due to $\iota\circ\kappa^1 \sim \mathrm{id}$ and $EZ\circ AW \sim \mathrm{id}$. The stratified map

$$\mathcal{EZ} := \kappa^1 \circ EZ : \bigoplus\nolimits_{k\geq 0}\big(\mathscr{A}^{\square k}[k]\big) \longrightarrow \bigoplus\nolimits_{k\geq 0}\big(C_*(\mathcal{S}\underline{t}_k)[k(d-1)]\big) \tag{4.2.21}$$

is a (homotopy) inverse to $\mathcal{AW}$. $\square$

We observe that there are two possible choice of augmentation (as coalgebras) for

$$T \longrightarrow T^\infty$$

given by the simultaneous maps (for all $n\geq 1$)

$$\nu : x \mapsto (x,\overline{x},\cdots,x,\overline{x}) \in (T\times T)^{\times_T^n}$$

or $\overline{\nu}(x) := \nu(\overline{x})$. The induced map at the chain level gives a map of coalgebras. Let us recall some notation :

$$\begin{aligned} \mathfrak{C}(X,Y) &= \bigoplus_{i,j} C_{i-j(d-1)}(\mathcal{S}\underline{t}_j) \\ \mathfrak{C}(T) &= \bigoplus_{i,j} C_{i-j(d-1)}\big((T\times T)^{\times_T^j}\big) \end{aligned}$$

Definition 4.55. We define the *homotopy fibre* of

$$\Omega\mathcal{E}v : \Omega\big(\mathfrak{C}(X,Y)\big) \longrightarrow \Omega\big(\mathfrak{C}(T)\big)$$

to be the cobar on the chains on the *homotopy fibre* of the actual evaluation

$$\mathcal{E}v : \mathcal{S}\underline{t} \longrightarrow T^\infty,$$

i.e., the fibre over a point in T when we pull back $\mathcal{E}v$ by ν or $\overline{\nu}$.

This definition is analogous to the case when we have free chain models of a fibration as classical coalgebras. We map a point (or contractible space) into the base and the chains on the pullback bundle is the homotopy fibre. The chains on this space is model for our fibre. The chains on a point, the ground field, provide an augmentation of the coalgebra. Similarly, here ν or $\overline{\nu}$ provide an augmentation of the coalgebra $\mathfrak{C}(T)$.

Unraveling the definitions, we see that the homotopy fibre of $\mathcal{S}\underline{t} \to T^\infty$ consists of transversal strings $\mathcal{S}$ that start from $x \in T$ and end in $\overline{x} \in T$ and whenever it passes through T it does along the axis joining x and $\overline{x}$. Therefore, the fibre of $\Omega\mathcal{E}v$ is just the classical cobar construction applied to the chains on the (stratified) space $\mathcal{S}$ described above. We keep in mind that the boundary operator on $C_*(\mathcal{S})$ is the sum of the resolve operator and the geometric boundary map. Let ℓ be a longitude joining x to $\overline{x}$ along the normal sphere on which these two points lie. We can identify $C_*(\mathcal{S})$ with the the bar construction of the chains on the based loop space of the complement of Y in X, equipped with a twisted Pontrjagin product arising from viewing the resolve product (using ℓ) as an appropriate element of the homotopy groups of $\Omega(X \setminus Y)$. Therefore, the classical bar-cobar adjunction implies that the homotopy fibre of $\Omega\mathcal{E}v$ is quasi-isomorphic to the chains on the *naive* fibre over $\mathcal{S}\underline{t}_1 \to T \times T$. Spatially, this is precisely the based loop space with the twisted Pontrjagin product. We shall see in §4.3 that this twisted Pontrjagin product on $\Omega(X \setminus Y)$ is not a homotopy invariant.

4.3 Detecting Non-homotopy Invariants

We discuss the configuration space of two points for lens spaces and present the arguments in [23] in the setting of minimal models introduced by Sullivan. Not only is this setting helpful for us but the arguments presented in §4.3.1 can now be applied in the context of configuration spaces of 4-manifolds or fibrations. This will be taken up elsewhere. For now, we use this to show that the (twisted) Pontrjagin product of the based loop space of the 2-point configuration space is not a homotopy invariant. In §4.3.3 we use this and connect it with the theory of transversal open strings developed in §4.1 and §4.2.3, concluding with the proof of Theorem 4.74, our main result from Chapter 4.

4.3.1 Configuration spaces of two points

The configuration space of points in any space is interesting in its own right. It's a well studied object, interesting in its own right and also appear in numerous contexts. Personally, it's interesting even more in the light of the fact that it contains non-homotopy information, i.e., there are homotopy equivalent spaces X_1 and X_2 such that the associated configuration spaces are not. The example we'll look at, due to Longoni and Salvatore [23], involves lens spaces. We shall briefly review it and present a minimal model approach to study such configuration spaces.

We shall think of the 3-sphere

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$$

as the unit quarternions. We denote by $L_{p,q}$ the lens space arising as the quotient of S^3 by the pth roots of unity as follows :

$$e^{i2\pi k/p}(z_1, z_2) = (e^{i2\pi k/p} z_1, e^{i2\pi kq/p} z_2).$$

Since S^3 is the universal cover of $L_{p,q}$, $\pi_1(L_{p,q}) = \mathbb{Z}_p$. The following is well known.

Theorem 4.56. (Reidemeister, Brody)
Consider the lens spaces L_{p,q_1} and L_{p,q_2}.
(i) They are homotopy equivalent if and only if $q_1 q_2 \equiv \pm n^2 \bmod p$.
(ii) They are homeomorphic if and only if $q_1 \equiv \pm q_2^{\pm 1} \bmod p$.

Remark 4.57. *Since $H_2(L_{p,q};\mathbb{Z}) = 0$, instead of Poincaré duality one uses the torsion linking form $H_1(L_{p,q};\mathbb{Z})\times H_1(L_{p,q};\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$, which classifies these spaces up to homotopy equivalence. Using Reidemeister torsion, defined as*

$$\Delta(L_{p,q}) := (t-1)(t^{q^{-1}}-1) \in \mathbb{Q}[t]/(t^p-1),$$

a classification up to PL homeomorphism was given by Reidemeister. Later on, Brody showed that this is a homeomorphism classification. There is a similar story for more general lens spaces denoted by $L(m;l_1,\ldots,l_d)$ of dimension $(2d-1)$. The homotopy classification was given by Olum in 1953 and the homeomorphism classification was done by Franz in 1935.

It follows that $L_{7,1}$ and $L_{7,2}$ are homotopy equivalent but not homeomorphic. It is a good point to mention a recent and rather surprising result, which we'll be using in a reinterpreted form.

Theorem 4.58. (Longoni-Salvatore)
The configuration spaces of $L_{7,1}$ and $L_{7,2}$ are not homotopy equivalent.

Remark 4.59. *Their proof starts by passing to the 49-sheeted universal covers of the configuration space of two distinct points in $L_{7,j}$. Then these resulting spaces can be distinguished by Massey products. In particular, for $j=1$ the space is formal, being homotopic to $(\vee_6 S^2)\times S^3$ while for $j=2$ there are non-trivial Massey products. This can then be used to distinguish higher configuration spaces.*

We shall denote the 2-point configuration space of M by $F_2(M)$[6]. Let M be the lens space $L_{p,q}$. The universal cover of $M-\{x_0\}$ is $S^3-\{\mathbb{Z}_p x_0\}$, which is homotopy equivalent to a bouquet of $(p-1)$-copies of 2-spheres. A specific p-fold cover of $F_2(M)$ fibres over M with fibre this bouquet, viz., using a cross section of the fibration $F_2(M)\to M$ (this always exists since M has a non-vanishing vector field) and replace the fibre $M-\{x_0\}$ by its universal cover $\vee_{p-1}S^2$. It's necessary to have a cross section for this construction to work since a section gives a consistent choice of base points in the fibre and we replace this with the paths based at the base point. Observe that the Hopf fibration $S^3\to S^2$ has no such replacement.

We can pull back the universal covering space of M to this total space to make a second fibration - the total space is the universal cover $\overline{F_2(M)}$ of $F_2(M)$,

[6]We shall not use the standard notation $C_2(M)$ for the configuration space of 2 points since in a discussion of chain complexes a notation like $C_2(M)$ is bound to cause a confusion.

the base is S^3 and the fibre is still $\vee_{p-1}S^2$. This fibration is determined up to homotopy by how the bouquet is mapped to itself along the equator S^2 of the base S^3, i.e., $\overline{F_2(M)}$ is classified by $\pi_2(\mathrm{Aut}_0(\vee_{p-1}S^2))$. Also note that

$$\overline{F_2(M)} = \{(x,y) \in S^3 \times S^3 \,|\, x \neq e^{i2\pi k/p}y\}.$$

Therefore, for any non-integer t the map $s : x \mapsto (x, e^{i2\pi t/p}x)$ is a bona-fide section. This means that the cohomology of $\overline{F_2(M)}$ ring splits as a tensor product of the base and the fibre.

Example 4.60. (The lens space $L_{7,1}$)

We think of S^3 as the unit quarternions and $\mathbb{Z}_7 = \langle\zeta\rangle$, the group generated by $\zeta = e^{\frac{2\pi i}{7}}$, acts on S^3 by left translations. Let $S^3 \xrightarrow{\pi} L_{7,1} := S^3/\mathbb{Z}_7$ with $\mathbf{1} := \pi(1)$. For $x, y \in L_{7,1}$ we choose lifts $\overline{x}, \overline{y}$ in S^3 and define a map

$$\begin{aligned} \varphi : F_2(L_{7,1}) &\longrightarrow L_{7,1} \times (L_{7,1} \setminus \{\mathbf{1}\}) \\ \varphi(x,y) &:= \big(x, \pi((\overline{x})^{-1}\overline{y})\big)\,. \end{aligned}$$

This maps into the right target space and is well defined because if $\chi = \zeta^k\overline{x}$, $y = \zeta^l\overline{y}$ are two different lifts then

$$\pi(\chi^{-1}y) = \pi((\overline{x})^{-1}\zeta^{l-k}\overline{y}) = \pi(\zeta^{k-l}(\overline{x})^{-1}\overline{y}).$$

Moreover, φ is an injective continuous map. It is also surjective (hence a homeomorphism) since

$$\varphi(x, \pi(\overline{x}\,\overline{y})) = (x,y) \in L_{7,1} \times (L_{7,1} \setminus \{\mathbf{1}\})\,.$$

Therefore, we have a homeomorphism

$$\varphi : \overline{F_2(L_{7,1})} \xrightarrow{\cong} S^3 \times \big(S^3 - \mathbb{Z}_7\big)$$

which implies that $\overline{F_2(L_{7,1})}$, being the product of formal spaces, has no Massey products.

We apply the based loop space functor

$$\Omega\varphi : \Omega\overline{F_2(L_{7,1})} \xrightarrow{\cong} \Omega S^3 \times \Omega\left(S^3 - \mathbb{Z}_7\right),$$

to get a homeomorphism of H-spaces thereby inducing an isomorphism of Hopf algebras

$$\Phi : H_*^{\mathbb{Q}}\big(\Omega\overline{F_2(L_{7,1})}\big) \xrightarrow{\cong} H_*^{\mathbb{Q}}(\Omega S^3) \otimes H_*^{\mathbb{Q}}\big(\Omega(S^3 - \mathbb{Z}_7)\big). \tag{4.3.1}$$

Notice that by Milnor-Moore's theorem [28], the Hurewicz map for a simply connected space X induces an isomorphism of Hopf algebras

$$\mathcal{H} : U(\pi_*^{\mathbb{Q}}(\Omega X)) \xrightarrow{\cong} (H_*(\Omega X; \mathbb{Q}), \times) . \tag{4.3.2}$$

Combining (4.3.1) and (4.3.2) we conclude

$$U\big(\pi_*^{\mathbb{Q}}\big(\Omega\overline{F_2(L_{7,1})}\big)\big) \xrightarrow{\cong} U\big(\pi_*^{\mathbb{Q}}(\Omega S^3)\big) \otimes U\big(\pi_*^{\mathbb{Q}}(\Omega(\vee_6 S^2))\big). \tag{4.3.3}$$

We would like to find the center of the big universal algebra above. Towards that we need :

Lemma 4.61. *The Lie algebra $\pi_*^{\mathbb{Q}}(\Omega(\vee_6 S^2))$ is the free Lie algebra generated by $V = \mathbb{Q}^6$ in degree 1.*

Before we embark on the proof, it is useful to recall the following schematic. Let X be a simply connected space with a minimal model (Λ_X, d). One such model is given by taking the dual of $\pi_*^{\mathbb{Q}}(X)$ and the Whitehead product corresponds to the quadratic part of d. The Whitehead product is also equivalent to the Samelson product on $\pi_*^{\mathbb{Q}}(\Omega X)$. The dual of this corresponds to a model of the forms on ΩX equipped with a coalgebra, dual to the Pontrjagin product. Finally, a model for ΩX is given by taking the generators of Λ_X and shifting them down by 1 and setting $d \equiv 0$.

Proof We observe that $\vee_k S^2$ is a formal space being the wedge of formal spaces. A minimal model is given by taking k closed generators x_i in degree 2 and then adding $\binom{k+1}{2}$ generators $\{y_{ij}\}_{1\leq i\leq j\leq k}$ with the relation $dy_{ij} = x_i x_j$. Any higher relations that appear thus far need to be killed appropriately. The terms that need to be killed look like graded commutativity of the Lie bracket dual to the product in this model. Shifting everything down by 1, the based loop space of the bouquet with the Samelson product is the free lie algebra on k generators in degree one. □

Corollary 4.62. *The center of $U\big(\pi_*^{\mathbb{Q}}\big(\Omega\overline{F_2(L_{7,1})}\big)\big)$ is*

$$U(\pi_*^{\mathbb{Q}}(\Omega S^3)) \cong \mathbb{Q}[\overline{\alpha}],\ |\overline{\alpha}| = 2.$$

Proof Recall that the universal enveloping algebra of a free Lie algebra on V is the free associative algebra on V. In conjunction with Lemma 4.61, the universal enveloping algebra of $\pi_*^{\mathbb{Q}}(\Omega(\vee_6 S^2))$ doesn't have central elements. The claim now follows from (4.3.3). □

Example 4.63. (The lens space $L_{7,2}$)
This is a very brief overview of [23] and we just recall the relevant Massey products since the actual details will be relevant for us in the next section.

Let $\zeta = e^{\frac{2\pi i}{7}}$ be a primitive root of unity. The space $\overline{F_2(L_{7,2})}$ is the complement of the union of the "diagonals" in $S^3 \times S^3$, i.e.,

$$\overline{F_2(L_{7,2})} = S^3 \times S^3 - \left(\cup_{k=0}^{6} \Delta_k\right),$$

where $\Delta_k := \{(x, \zeta^k x) \,|\, x \in S^3\}$. By the Alexander-Lefschetz duality we have an isomorphism

$$H^p\big(\overline{F_2(L_{7,2})}\big) \cong H_{6-p}\big(S^3 \times S^3, \big(\cup_{k=0}^{6} \Delta_k\big)\big).$$

This identifies the cup product in cohomology with the intersection product in homology. Let $A_k \cong S^3 \times [0,1]$ be the submanifold defined by elements of the form

$$((x_1, x_2), (\zeta^{k-1+t} x_1, \zeta^{2(k-1+t)} x_2)),\ \ t \in [0,1], (x_1, x_2) \in S^3.$$

Define an action of $\zeta : A_k \to A_{k+1}$ via the map

$$\zeta : ((x_1, x_2), (\zeta^{k-1+t} x_1, \zeta^{2(k-1+t)} x_2)) \mapsto ((x_1, x_2), (\zeta^{k+t} x_1, \zeta^{2(k+t)} x_2)).$$

This action preserves transversality, i.e., since $A_1 \pitchfork A_4$ (as proved in Lemma 4.2 [23]) we conclude that $A_k \pitchfork A_{k+3}$ where we are considering modulo 7. Define

$$\mathbb{D}_{k,k+3} := \left\{ \big((r, x), (\zeta^{4t+k-1} r, \zeta^{t+2k-2} x)\big) \,|\, (r, x) \in S^3, r^2 + |x|^2 = 1, t \in [0,1] \right\}.$$

Notice that $\mathbb{D}_{k,k+3}$ has three boundary components, one each for $r = 0, t = 0$ and $t = 1$. It can be shown that

$$\partial_{r=0} \mathbb{D}_{k,k+3} = \left\{ \big((0, x), (0, \zeta^{2k-2+t} x)\big) \,|\, |x| = 1, t \in [0,1] \right\} = A_k \cap A_{k+3}$$

while

$$\partial_{t=0} \mathbb{D}_{k,k+3} \subset \Delta_{k-1},\ \ \partial_{t=1} \mathbb{D}_{k,k+3} \subset \Delta_{k+3}.$$

Similarly, one concludes $\mathbb{D}_{k,k+3} \cap A_{k+5} = \phi$ and

$$A_{k+1} \cap ((1,0) \times S^3) = \mathbb{D}_{k,k+3} \cap A_{k+1}.$$

Let us denote the dual (in cohomology) of A_k by a_k and the dual of $(1,0) \times S^3$ by α. We can then conclude the following theorem of [23] :

Theorem 4.64. (Longoni-Salvatore)
The Massey product $\langle a_{k+3}, a_k, a_{k+1} + a_{k+5}\rangle$ *contains the class* $a_{k+1} \cup \alpha$.

The non-triviality of the Massey product is clear and is essentially due to the addition of a_{k+5} to a_{k+1}. Observe that $a_0 + \cdots a_6 = 0$. This is most transparent when thinking of $\overline{F_2(L_{7,2})}$ as a fibration over S^3 with fibre S^3 with seven punctures. In other words, the fibre is homotopy equivalanet to a wedge of six 2-spheres and the top cohomology classes of these generate $H^2\big(\overline{F_2(L_{7,2})}\big)$. In fact, with suitable orientation, one can argue that a_k is the fibrewise volume form on the normal sphere bundle of A_k inside $S^3 \times S^3$.

We shall take up the calculation of the centre of the universal enveloping algebra associated to $\Omega\overline{F_2(L_{7,2})}$ in the next section. The required twisted version will also be analyzed.

4.3.2 Computations in universal enveloping algebra

Recall that $\vee_6 S^2 \hookrightarrow \overline{F_2(L_{7,i})} \to S^3$ is a fibration of simply connected spaces we may apply the based loop functor to get another fibration

$$\Omega(\vee_6 S^2) \hookrightarrow \Omega\overline{F_2(L_{7,i})} \longrightarrow \Omega S^3.$$

Each of the arrows above induce a map of Lie algebras of rational homotopy groups with the Samelson product as the Lie bracket. Further applying the universal enveloping algebra functor we have

$$U\big(\pi_*^{\mathbb{Q}}(\Omega(\vee_6 S^2))\big) \longrightarrow U\big(\pi_*^{\mathbb{Q}}\big(\Omega\overline{F_2(L_{7,i})}\big)\big) \longrightarrow U\left(\pi_*^{\mathbb{Q}}(\Omega S^3)\right).$$

By Milnor-Moore's theorem, the Hurewicz map induces an isomorphism of Hopf algebras

$$\mathcal{H} : U(\pi_*^{\mathbb{Q}}(\Omega X)) \xrightarrow{\cong} (H_*(\Omega X; \mathbb{Q}), \times).$$

Using this (or via minimal model description of the original fibration) one checks surjectivity at $U(\pi_*^{\mathbb{Q}}(\Omega S^3))$ and injectivity at the left hand side. Thus, the objects of interest appear as the middle object in short exact sequences of Hopf algebras

$$1 \to V \to U_i \to W \to 1.$$

However, we have already seen that

$$U\left(\pi_*^{\mathbb{Q}}(\Omega\overline{F_2(L_{7,1})})\right) \cong U\left(\pi_*^{\mathbb{Q}}(\Omega S^3)\right) \otimes U\left(\pi_*^{\mathbb{Q}}(\Omega(\vee_6 S^2))\right).$$

Moreover, it has one central element (up to scaling) in every even degree. On the other hand, the existence of non-trivial Massey products in $F_2(L_{7,2})$ may imply the non-existence of central elements. Our purpose at hand is precisely this.

A minimal model for $\overline{F_2(L_{7,2})}$ is given as follows. Let Λ_F be the minimal model for $\vee_6 S^2$ and $\Lambda(\alpha), d\alpha = 0, |\alpha| = 3$ be the minimal model for S^3. Notice that Λ_F starts off with 6 generators x_i in degree 2 and then it is formal thereafter. More precisely one needs to *kill* $x_i x_j$ by setting $dy_{ij} = x_i x_j$ for $i \leq j$. Similarly, we need z_{ijk} such that

$$dz_{ijk} = x_i y_{jk} - y_{ij} x_k.$$

This last equation is very reminiscent to the triple Massey product. Indeed, without further terms this says that the Massey products vanish. But our model for $\overline{F_2(L_{7,2})}$ is given by $\Lambda(\alpha) \otimes \Lambda_F$ with the differential D twisted by the Massey product. For example, using Theorem 4.64, we conclude that

$$D(z_{412} + z_{416}) = x_2\alpha + x_4(y_{12} + y_{16}) - y_{41}(x_2 + x_6).$$

Just as a quick fact, there are 21 y_{ij}'s and 70 z_{ijk}'s.

Lemma 4.65. *For any simply connected space X let Λ_X be a quadratic minimal model for X. Then a minimal model for Ω_X is given by shifting the generators of Λ_X down by 1 and setting the differential to be zero. However, the dual to the Pontrjagin product gives rise to a dg coalgebra structure on $\Lambda_{\Omega X}$.*

This is possibly fairly well known but we sketch a proof for completeness.

Proof The underlying vector space of a minimal model for such a space can be taken to be the rational homotopy groups. Then the quadratic term of the differential in this model is dual to the Whitehead product which is equivalent to the Samelson product on the based loop space. But the Samelson product is the commutator induced by the Pontrjagin product. □

A minimal model (as a coalgebra) for $\Omega\overline{F_2(L_{7,2})}$ is given by

$$\Lambda(\overline{\alpha}) \otimes \Lambda(\overline{x}_i; \overline{y}_{ij}; \overline{z}_{ijk}; \cdots), d \equiv 0$$

with the coalgebra structure arising from the differential D. Here we have used the convention that $\overline{x}$ denotes the element x shifted down in degree by 1.

Proposition 4.66. *There is no central element of degree* 2 *in* $U\big(\pi_*^{\mathbb{Q}}(\Omega\overline{F_2(L_{7,2})})\big)$.

Proof It follows from Theorem 4.64 that

$$a_2 \cup \alpha \in \langle a_4, a_1, a_2 + a_6\rangle\,.$$

The elements $\overline{x}_i$'s are the same as $\overline{a}_i$'s. Therefore, $\overline{\alpha}$ is not central because

$$\overline{x}_2 \otimes \overline{\alpha} - \overline{\alpha} \otimes \overline{x}_2 = \overline{z}_{412} + \overline{z}_{416}.$$

Let $\xi = \lambda\overline{\alpha} + \sum_{i\leq j} \lambda_{ij}\overline{y}_{ij}$ denote a generic element of degree 2. If ξ is central then $[\overline{x}_2, \xi] = 0$. However, using the minimal model we have

$$[\overline{x}_2, \xi] \;\; = \;\; \lambda\overline{z}_{412} + \lambda\overline{z}_{416} + \sum_{i\leq j} \lambda_{ij}(\overline{z}_{2ij} - \overline{z}_{ij2}).$$

The term $\lambda\overline{z}_{416}$ cannot be cancelled by anti-symmetry or by Jacobi identity unless $\lambda = 0$. In that case, ξ is a central element and is in $U\big(\pi_*^{\mathbb{Q}}(\Omega(\vee_6 S^2))\big)$. This is the universal enveloping algebra of a free Lie algebra and is isomorphic as graded algebras to the tensor algebra on the graded vector space generated by $\pi_*^{\mathbb{Q}}(\Omega(\vee_6 S^2))$. Therefore, it doesn't have a centre whence $\lambda_{ij} \equiv 0$. $\square$

Corollary 4.67. *The Pontrjagin ring* $\big(H_*(\Omega\overline{F_2(L_{7,2})};\mathbb{Q}), \times\big)$ *detects Massey products.*

We would like to prove an analogous statement about the centre of the universal enveloping algebra but now with respect to a twisted multiplication. The necessity of this was motivated before but will certainly be transparent a little later.

Definition 4.68. (Twisted multiplication) Let $\mathcal{A}$ be an algebra. For $\beta \in \mathcal{A}$ we can *twist* the multiplication and define

$$x \cdot_\beta y := x\beta y, \quad x, y \in \mathcal{A}.$$

An element $\xi \in \mathcal{A}$ is in the *centre* if $\xi\beta x = x\beta\xi$ for any $x \in \mathcal{A}$.

If β has an inverse (and necessarily $\mathcal{A}$ has a unit) then

$$xy = \beta(\beta^{-1}x \cdot_\beta \beta^{-1}y).$$

In other words, one *translates* x and y by β and then uses the twisted multiplication and then *translates* in reverse by β.

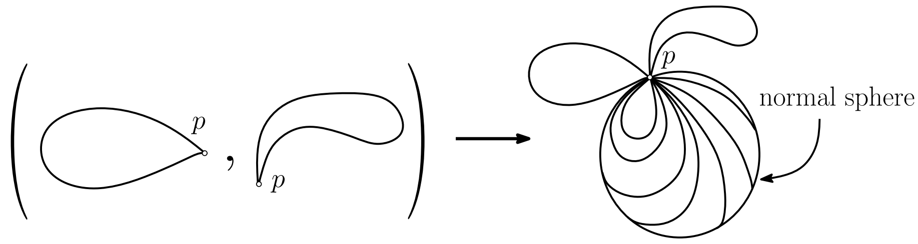

Figure 4.11: The twisted Pontrjagin product in $\Omega F_2(X)$

Definition 4.69. Let $U_{\mathscr{R},i} := U\big(\pi_*^{\mathbb{Q}}(\Omega\overline{F_2(L_{7,i})})\big)$ be the algebra induced by twisting the universal enveloping algebra by the element $\overline{a}_7$. This multiplication will also be denoted by $\times_{\mathscr{R}}$.

Notice that $\overline{\alpha}$ and its powers are in the centre of $U_{\mathscr{R},1}$ since it commutes with everything. On the other hand we have :

Proposition 4.70. *There is no element of degree* 2 *in the centre of* $U_{\mathscr{R},2}$.

Proof First we show that $\overline{\alpha}$ is not in the centre. We rewrite the Massey products of Theorem 4.64 in terms of $x_1, \ldots, x_6$ and replacing x_7 by $-x_1 - \cdots - x_6$. We keep in mind that the notation used in Theorem 4.64 uses a_i and these the same as x_i's used in our model. We have

$$
\begin{array}{rcl}
a_2 \cup \alpha & \in & \langle a_4, a_1, a_2 + a_6 \rangle \\
a_4 \cup \alpha & \in & \langle a_6, a_3, a_4 + a_1 \rangle \\
a_6 \cup \alpha & \in & \langle a_1, a_5, a_6 + a_3 \rangle \\
a_3 \cup \alpha & \in & \langle a_5, a_2, -a_1 - a_2 - a_4 - a_5 - a_6 \rangle \\
a_5 \cup \alpha & \in & \langle -a_1 - \cdots - a_6, a_4, a_5 + a_2 \rangle \\
a_1 \cup \alpha & \in & \langle a_3, -a_1 - \cdots - a_6, a_1 + a_5 \rangle \, .
\end{array}
$$

This implies that

$$\text{(4.3.4)}\quad \begin{aligned}\overline{\alpha}\otimes\overline{a}_7\otimes 1 - 1\otimes\overline{a}_7\otimes\overline{\alpha} &= \overline{z}_{311}+\overline{z}_{315}+\overline{z}_{321}+\overline{z}_{325}+\overline{z}_{331}\\ &\quad +\overline{z}_{335}+\overline{z}_{341}+\overline{z}_{345}+\overline{z}_{351}+\overline{z}_{355}\\ &\quad +\overline{z}_{361}+\overline{z}_{365}-\overline{z}_{412}-\overline{z}_{416}-\overline{z}_{634}\\ &\quad -\overline{z}_{631}+\overline{z}_{521}+\overline{z}_{522}+\overline{z}_{524}+\overline{z}_{525}\\ &\quad +\overline{z}_{526}-\overline{z}_{156}-\overline{z}_{153}+\overline{z}_{145}+\overline{z}_{142}\\ &\quad +\overline{z}_{245}+\overline{z}_{242}+\overline{z}_{345}+\overline{z}_{342}+\overline{z}_{445}\\ &\quad +\overline{z}_{442}+\overline{z}_{545}+\overline{z}_{542}+\overline{z}_{645}+\overline{z}_{642}.\end{aligned}$$

which is non-zero. The easiest way to see this is to observe that $\overline{z}_{416}$ doesn't cancel with any other terms. We shall call the right hand side above $\mathcal{Z}$. Let $\xi = \lambda\overline{\alpha} + \sum_{i\leq j}\lambda_{ij}\overline{y}_{ij}$ denote a generic central element (with respect to this twisted multiplication) of degree 2.

$$\xi\otimes\overline{a}_7\otimes 1 - 1\otimes\overline{a}_7\otimes\xi = \lambda\mathcal{Z} + \sum_{k=1}^{6}\sum_{1\leq i\leq j\leq 6}\lambda_{ij}(\overline{z}_{kij}-\overline{z}_{ijk}).$$

We shall be making use of the Jacobi identity

$$\overline{z}_{ijk}+\overline{z}_{jki}+\overline{z}_{kij}=0,\quad i\neq j\neq k.$$

We will also use $\overline{z}_{ijk} = -\overline{z}_{kji}$ and $\overline{z}_{iji}=0$ in what follows.

$$\begin{aligned}\sum_{k=1}^{6}\sum_{1\leq i\leq j\leq 6}\lambda_{ij}(\overline{z}_{ijk}-\overline{z}_{kij}) &= \sum_{i<j,k\neq i,k\neq j}\lambda_{ij}(\overline{z}_{ijk}-\overline{z}_{kij}) + \sum_{i\neq k}\lambda_{ii}(\overline{z}_{kii}-\overline{z}_{iik})\\ &\quad +\sum_{i<j}\lambda_{ij}(\overline{z}_{iij}-\overline{z}_{ijj})\\ &= \sum_{i<j,k\neq i,k\neq j}\lambda_{ij}(\overline{z}_{ijk}-\overline{z}_{kij}) + \sum_{i<j}\lambda_{ij}(\overline{z}_{iij}-\overline{z}_{ijj})\\ &\quad -\sum_{i<k}2\lambda_{ii}\overline{z}_{iik}+\sum_{i>k}2\lambda_{ii}\overline{z}_{kii}\\ \text{(4.3.5)}\qquad &= \sum_{i<j,k\neq i,k\neq j}\lambda_{ij}(\overline{z}_{ijk}-\overline{z}_{kij}) - \sum_{i<j}(2\lambda_{ii}-\lambda_{ij})\overline{z}_{iij}\\ &\quad +\sum_{k<l}(2\lambda_{ll}-\lambda_{kl})\overline{z}_{kll}.\end{aligned}$$

If ξ is central then

$$\lambda\mathcal{Z} + \sum_{k=1}^{6} \sum_{1\leq i\leq j\leq 6} \lambda_{ij}(\overline{z}_{kij} - \overline{z}_{ijk}) = 0.$$

Comparing coefficients of terms of the form $\overline{z}_{ijj}$ from (4.3.4) and (4.3.5) we see that

$$\begin{aligned}
\lambda_{12} &= 2\lambda_{11} && \text{coefficient of } \overline{z}_{112}\\
\lambda_{12} &= 2\lambda_{22} && \text{coefficient of } \overline{z}_{122}\\
\lambda_{14} &= 2\lambda_{11} && \text{coefficient of } \overline{z}_{114}\\
\lambda_{14} &= 2\lambda_{44} && \text{coefficient of } \overline{z}_{144}\\
\lambda_{24} &= 2\lambda_{22} && \text{coefficient of } \overline{z}_{224}\\
\lambda_{24} + \lambda &= 2\lambda_{44} && \text{coefficient of } \overline{z}_{244}.
\end{aligned}$$

The first four equations imply that $\lambda_{22} = \lambda_{44}$ (they both equal λ_{11}) which in conjunction with the last two equations imply that $\lambda = 0$. This implies that $\lambda_{ii} = \kappa$ for any i and $\lambda_{ij} = 2\kappa$ if $i < j$. Notice that

$$\xi = \kappa\left(\sum_{i=1}^{6} \overline{y}_{ii} + \sum_{i<j} \overline{y}_{ij}\right)$$

now commutes with $\overline{a}_7$.

It is enough to show that $\kappa = 0$. Consider the twisted commutator[7] of ξ with $\overline{\alpha}$:

$$\begin{aligned}
\xi \otimes \overline{a}_7 \otimes \overline{\alpha} - \overline{\alpha} \otimes \overline{a}_7 \otimes \xi &= \overline{a}_7 \otimes \xi \otimes \overline{\alpha} - \overline{\alpha} \otimes \overline{a}_7 \otimes \xi\\
&= \overline{a}_7 \otimes \overline{\alpha} \otimes \xi - \overline{\alpha} \otimes \overline{a}_7 \otimes \xi\\
&= -\mathcal{Z} \otimes \xi.
\end{aligned}$$

The result is an element in $U\left(\pi_*^{\mathbb{Q}}(\Omega(\vee_6 S^2))\right)$ which is a free tensor algebra. Since $\mathcal{Z} \neq 0$, $\mathcal{Z} \otimes \xi = 0$ if and only if $\kappa = 0$. This implies that $\xi = 0$. □

In the case of $U_{\mathscr{R},1}$ the element $\overline{\alpha}$ is clearly in the centre with respect to the twisted multiplication.

Corollary 4.71. *The twisted Pontrjagin ring* $\left(H_*(\Omega\overline{F_2(L_{7,2})};\mathbb{Q}), \times_{\mathscr{R}}\right)$ *detects Massey products.*

[7] We are using the Koszul rule of sign where $[\alpha,\beta]_{\mathscr{R}} := \alpha \otimes \overline{a}_7 \otimes \beta + (-1)^{(|\alpha|+1)(|\beta|+1)}\beta \otimes \overline{a}_7 \otimes \alpha$.

4.3.3 Towards the twisted Pontrjagin ring

In our aim to show that transversal string topology distinguishes manifolds which are not homeomorphic (or diffeomorphic in the case of three manifolds) we have seen the following :

(i) From the detailed discussions in §4.2.3 one knows how to derive a free model of $C_*(\Omega(X \setminus Y))$ with the twisted Pontrjagin product starting from transversal open strings.

(ii) For a suitable lens space M, set $X = M \times M$ and $Y = M$. In §4.3.2 we have seen that the twisted Pontrjagin ring

$$\left(H_*(\Omega(\overline{X \setminus Y})), \times_{\mathscr{R}}\right)$$

detects Massey products, where $\overline{X \setminus Y}$ denotes the universal cover of $X \setminus Y$.

To complete the proof we need to show that the twisted Pontrjagin ring associated to the based loop space $\Omega(X \setminus Y))$ detects Massey products. Combined with (i) this would mean that one can detect non-homotopy invariants starting from $\mathfrak{C}(X, Y)$.

We begin by analyzing the relationship between the based loop space of a space and its universal cover. These results are well known and is included for completeness of the text.

Lemma 4.72. *Let A be a topological space with $\widetilde{A}$ as its universal cover. Fix $a_0 \in A$ with a lift $\tilde{a}_0 \in \widetilde{A}$. Then the natural projection map $p : (\Omega\widetilde{A}, \tilde{a}_0) \to (\Omega_0 A, a_0)$ to the connected component of the trivial loop based at a_0 is a homotopy equivalence.*

Proof The natural covering map $(\widetilde{A}, \tilde{a}_0) \xrightarrow{p} (A, a_0)$ induces a map of based loop spaces which is a fibration in the sense of Serre. Since any loop $\gamma \in \Omega\widetilde{A}$ is connected to the trivial loop at $\tilde{a}_0$, it follows that the image of p is in $(\Omega_0 A, a_0)$. Moreover, since contractible loops in A can be lifted to loops in $\widetilde{A}$, we conclude that

$$p : (\Omega\widetilde{A}, \tilde{a}_0) \longrightarrow (\Omega_0 A, a_0)$$

is a surjective map and a Serre fibration. Finally, notice that $p^{-1}(a_0)$ consists of all loops in $\widetilde{A}$ based at $\tilde{a}_0$ and lying in the (discrete) fibre over a_0, i.e., $p^{-1}(a_0) = \tilde{a}_0$. Therefore, the fibre is contractible and p is a homotopy equivalence. $\square$

Corollary 4.73. *Let (A, a_0) be a pointed connected topological space. There is a homotopy equivalence*

$$\varphi : \Omega A \sim \bigcup_{\pi_1(A)} \Omega\widetilde{A}$$

of $\pi_1(A; a_0)$-spaces. Moreover, there is an isomorphism of H-spaces

$$\Phi : H_*(\Omega A) \xrightarrow{\cong} \bigoplus_{\pi_1(A)} H_*(\Omega \widetilde{A}).$$

Proof Notice that ΩA is an H-space and for any $\alpha \in \pi_1(A; a_0)$, the component $\Omega_\alpha A$ containing α can be identified (up to homotopy equivalence) with $\Omega_0 A$ which can be further identified with $\Omega \widetilde{A}$. $\square$

Recall that we had the twisted Pontrjagin product on $\tilde{A}$ for $A = X \setminus Y$. If Y is of codimension d in X then the twisting element α belongs to $\pi_{d-2}(\Omega A)$. In fact, it is generated by the normal $d-1$-sphere around Y that contains the base point. This sphere can be thought of as a family of based loops parametrized by the equator. Since each loop in this family is contractible, $\alpha \in \pi_{d-2}(\Omega_0 A)$. By Lemma 4.72

$$\pi_{d-2}(\Omega_0 A) \cong \pi_{d-2}(\Omega \tilde{A}),$$

whence the twisted Pontrjagin product on $H_*(\Omega(X \setminus Y))$ can be described as the original twisted Pontrjagin product on $\pi_1(A)$-copies of $H_*(\Omega \widetilde{A})$ along with the $\pi_1(A)$-action permuting these copies. The resulting structure, denoted by

$$\big(H_*(\Omega(X \setminus Y)), \times_{\mathscr{R}}\big),$$

is still not a homotopy invariant. In conclusion, we have our main result :

Theorem 4.74. *Let $\mathfrak{C}(X, Y)$ be the dg coalgebra associated to the space of transversal open strings in X with end points in Y. The map*

$$M \longrightarrow \mathfrak{C}(M \times M, M)$$

does not preserve homotopy equivalence. In particular, there are homotopy equivalent manifolds M and N but the associated differential graded coalgebras are not quasi-isomorphic.

Bibliography

[1] R. Bott, *The stable homotopy of the classical groups*, Ann. Math., 70 (1959), pp. 313–337.

[2] M. Chas, *Minimal intersection of curves on surfaces*. Geometriae Dedicata (to appear), available at `http://arxiv.org/abs/0706.2439`.

[3] ——, *Combinatorial Lie bialgebras of curves on surfaces*, Topology, 43 (2004), pp. 543–568.

[4] M. Chas and D. Sullivan, *String topology*. available at `http://arxiv.org/abs/math/9911159`.

[5] ——, *Closed string operators in topology leading to Lie bialgebras and higher string algebra*, in The Legacy of Niels Henrik Abel, Springer-Verlag, Berlin, 2004, pp. 771–788.

[6] D. Chataur and J.-F. Le Borgne, *Homology of regular loops in the sphere*, Algebr. Geom. Topol., 9 (2009), pp. 935–977.

[7] R. L. Cohen and V. Godin, *A polarized view of string topology*, in Topology, Geometry and Quantum Field Theory, vol. 308 of London Mathematical Society LNS, Cambridge, 2004, Cambridge Univ. Press, pp. 127–154.

[8] R. L. Cohen and J. D. S. Jones, *A homotopy theoretic realization of string topology*, Math. Ann., 324 (2002), pp. 773–798.

[9] R. L. Cohen, J. D. S. Jones, and J. Yan, *The loop homology algebra of spheres and projective spaces*, in Progress in Mathematics, Birkhäuser, Basel, 2004, pp. 77–92.

[10] R. L. Cohen, J. R. Klein, and D. Sullivan, *The homotopy invariance of string topology loop product and string bracket*, J. Topol., 1 (2008), pp. 391–408.

[11] K. Costello, *Topological conformal field theories and Calabi-Yau categories*, Adv. Math., 210 (2007), pp. 165–214.

[12] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math., 29 (1975), pp. 245–274.

[13] Y. FÉlix, S. Halerpin, and J.-C. Thomas, *Rational Homotopy Theory* (GTM 205), Springer-Verlag, Berlin, 1st ed., 2001.

[14] Y. FÉlix, J.-C. Thomas, and M. Vigué-Poirier, *Rational string topology*, J. Eur. Math. Soc., 9 (2005), pp. 123–156.

[15] V. Godin, *Higher string topology operations*. preprint (2008), available at `http://arxiv.org/abs/0711.4859`.

[16] ——, *The unstable integral homology of the mapping class groups of a surface with boundary*, Math. Ann., 337 (2007), pp. 15–60.

[17] W. M. Goldman, *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*, Invent. Math., 85 (1986), pp. 263–302.

[18] M. Goresky and N. Hingston, *Loop products and closed geodesics*, Duke Math J., 150 (2009), pp. 117–210.

[19] A. Hatcher, *Algebraic Topology*, Cambridge Univ. Press, Cambridge, 1st ed., 2001.

[20] R. Hepworth, *String Topology for Lie Groups*, J. Topol., 3 (2010), pp. 424–442.

[21] W. Jaco, *Lectures on Three-Manifold Topology*, CBMS Regional Conference Series in Mathematics.

[22] A. Le Donne, *On Lie bialgebras of loops on orientable surfaces*, J. Knot Theory Ramifications, 17 (2008), pp. 351–359.

[23] R. Longoni and P. Salvatore, *Configuration spaces are not homotopy invariant*, Topology, 44 (2005), pp. 375–380.

[24] J. Lurie, *On the classification of topological field theories*. preprint (2009), available at `http://arxiv.org/abs/0905.0465`.

[25] J. P. May, *Simplicial objects in algebraic topology*, Univ. of Chic. Pr., Chicago, 1982.

[26] C. McCrory, *A characterization of homology manifolds*, J. London Math. Soc., 16 (1977), pp. 149–159.

[27] J. Milnor, *The geometric realization of a semi-simplicial complex*, Ann. Math., 65 (1957), pp. 357–362.

[28] J. Milnor and G. Moore, *On the structure of Hopf algebras*, Ann. Math., 81 (1965), pp. 211–264.

[29] M. Mimura and H. Toda, *Topology of Lie Groups, I and II*, Translations of Mathematical Monographs.

[30] K. Poirier, *String topology and compactified moduli spaces*, PhD thesis, 2010. available at `http://math.berkeley.edu/~poirier/thesis.pdf`.

[31] S. Smale, *Regular curves on Riemannian manifolds*, Trans. Amer. Math. Soc., 87 (1958), pp. 492–512.

[32] D. Sullivan, *String topology : background and present state*, in Current Developments in Mathematics 2005, Boston, MA, International Press.

[33] ——, *Open and closed string field theory interpreted in classical algebraic topology*, in Topology, Geometry and Quantum Field Theory, Proceedings of the 2002 Oxford Symposium, vol. 308 of London Mathematical Society LNS, Cambridge, 2001, Cambridge Univ. Press, pp. 344–357.

[34] D. Sullivan and M. Vigué-Poirier, *The homology theory of closed geodesic problem*, J. Differential Geom., 11 (1976), pp. 633–644.

[35] M. Takeuchi, *Morita theorems for categories of comodules*, J. Fac. Sci. Uni. Tokyo, 24 (1977), pp. 629–644.

[36] V. Turaev, *Skein quantization of Poisson algebras of loops on surfaces*, Ann. Sci. École Norm. Sup., 24 (1991), pp. 635–704.

[37] D. Vaintrob, *The string topology BV algebra, Hochschild cohomology and the Goldman bracket on surfaces*. available at `http://arxiv.org/abs/0702859`.