Extension of Lyapunov's Convexity Theorem to Subranges

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Abstract of the Dissertation

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Consider a measurable space with a finite vector measure. This measure defines a mapping of the σ-field into a Euclidean space. According to Lyapunov's convexity theorem, the range of this mapping is compact and, if the measure is atomless, this range is convex. Similar ranges are also defined for measurable subsets of the space. We show that the union of the ranges of all subsets having the same given vector measure is also compact and, if the measure is atomless, it is convex. We further provide a geometrically constructed convex compact set in the Euclidean space that contains this union. We show that, for two-dimensional measures, among all the subsets having the same given vector measure, there exists a set with the maximal range of the vector measure (maximal subset). Furthermore, for two-dimensional measures, the maximal subset, the above-mentioned union, and the above-mentioned convex compact set are equal sets. We also give counterexamples showing that, in three or higher dimensions, the maximal subset may not exist and these equalities may not hold. We use the existence of max-

imal subsets to strengthen the Dvoretzky-Wald-Wolfowitz purification theorem for the case of two measures. We show that there are no similar results for three or higher dimensions.

To Xinhui and Niuniu

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Chapter 1

Review of Lyapunov's Convexity Theorem

In this chapter, we give a review of Lyapunov's convexity theorem and the theory of purification developed on it. In Section 1.1, we give an introduction of Lyapunov's convexity theorem with some examples. Then we review the theorems on the purification of transition probabilities in Section 1.2. Finally, we discuss the application of these purification results to game theory in Section 1.3.

1.1 Lyapunov's Convexity Theorem

Let $(X, \mathcal{F})$ be a measurable space and $\mu = (\mu_1, ..., \mu_m)$, $m = 1, 2, \ldots$, be a finite vector measure on it. Consider the range $R_\mu(X) = \{\mu(Y) : Y \in \mathcal{F}\} \subset \mathbb{R}^m$ of the vector measures of all its measurable subsets Y. In [17], Lyapunov proved the following theorem, which is now known as Lyapunov's convexity theorem.

Theorem 1.1.1 (Lyapunov)**.** *The range $R_\mu(X)$ is compact and furthermore, if μ is atomless, this range is convex.*

We recall that a measure ν is called atomless if for each $Z \in \mathcal{F}$, such that $\nu(Z) > 0$, there exists $Z' \in \mathcal{F}$ such that $Z' \subset Z$ and $0 < \nu(Z') < \nu(Z)$. A vector measure $\mu = (\mu_1, ..., \mu_m)$, is called atomless if each measure μ_i, $i = 1 \ldots m$, is atomless. A measure is called atomic if it is not atomless. In some literature, the word nonatomic is used for atomless and the word nonatomless is used for atomic.

In the following, we consider some examples of ranges in various dimensions. The simplest (and trivial) cases are the ranges of one-dimensional measures. For instance, the range of any one-dimensional atomless probability measure on any measurable space, is $[0, 1]$, which is a convex and compact set. The example below shows the range of a one-dimensional atomic probability measure.

Example 1.1.2. Consider the probability space $(X, \mathcal{F}, \mu)$, where $X = \{a, b, c\}$, $\mathcal{F} = 2^X$, and

$$\mu(a) = 0.1, \quad \mu(b) = 0.2, \quad \mu(c) = 0.7. \tag{1.1.1}$$

Then the range $R_\mu(X) = \{0, 0.1, 0.2, 0.3, 0.7, 0.8, 0.9, 1\}$ (shown in Fig. 1.1) is a compact set, which simply follows from the fact that X is a finite set.

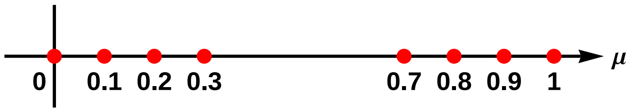

Figure 1.1: The range $R_\mu(X)$ in Example 1.1.2.

Note that, for a measure to be atomic, it is not necessary that the measurable space is countable. In the example above, if $X = [0, 1]$, $\mathcal{F}$ is the Borel σ-algebra on X, and the probability measure μ is still defined by (1.1.1), where $a, b, c \in X$ are three arbitrarily fixed elements of X. Then the range is the same as that of the above case shown in Fig. 1.1. Since the measure μ is atomic, the range $R_\mu(X)$ is not necessarily convex. $\square$

The following example shows that the range of an atomic measure may or may not be convex.

Example 1.1.3. Consider the probability space $(X, \mathcal{F}, \nu)$, where $X = \mathbb{N}$, $\mathcal{F} = 2^X$, and $\nu(i) = 2^{-i}$, for any $i \in \mathbb{N}$. Then the range $R_\nu(X) = [0, 1]$ is convex and compact. However, the convexity does not follow from Lyapunov's theorem.

Consider the same measurable space $(X, \mathcal{F})$ endowed with a different probability measure μ, where $\mu(1) = \frac{3}{4}$ and $\mu(i) = \frac{2^{-i}}{2}$, for $i = 2, 3, 4, \ldots$. The range $R_\mu(X)$ is shown in Fig. 1.2, which is compact but not convex. $\square$

Next, we give an example of the range of a two-dimensional atomless probability measure.

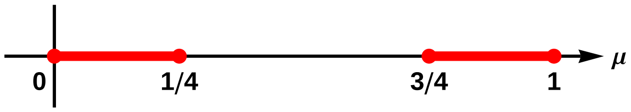

Figure 1.2: The range $R_{\mu}(X)$ in Example 1.1.3.

Example 1.1.4. Consider the probability space $(X, \mathcal{F}, \mu)$, where $X = [0,1]$, $\mathcal{F}$ is the Borel σ-algebra on X, and the measure μ is defined in terms of density functions

$$\mu\,(dx) = (\mu_1, \mu_2)\,(dx) = (1, 2x)\,dx.$$

Fig. 1.3 shows the range of the finite atomless probability measure μ. It is compact and convex. $\square$

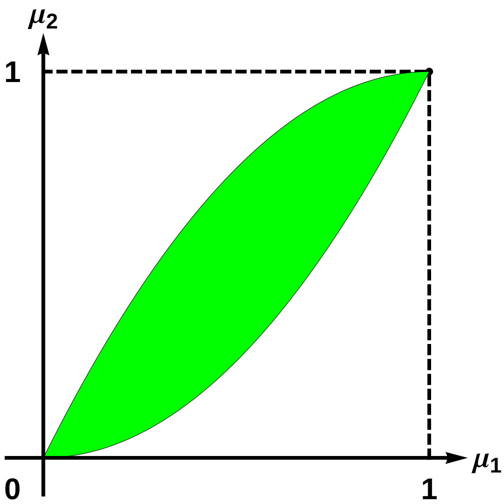

Figure 1.3: The range $R_{\mu}(X)$ in Example 1.1.4.

Finally, we consider the range of a three-dimensional finite atomless measure.

Example 1.1.5. Consider the measure space $(X, \mathcal{F}, \mu)$, where $X = [0,6]$, $\mathcal{F}$ is the Borel

σ-field on X, and $\mu(dx) = (\mu_1, \mu_2, \mu_3)(dx) = (f_1(x), f_2(x), f_3(x))\,dx$, where

$$f_1(x) = \begin{cases} 30 & x \in [0,1), \\ 40 & x \in [1,2), \\ 10 & x \in [2,4), \\ 15 & x \in [4,5), \\ 5 & x \in [5,6]; \end{cases} \quad f_2(x) = \begin{cases} 40 & x \in [0,1), \\ 10 & x \in [1,2), \\ 20 & x \in [2,4), \\ 10 & x \in [4,5), \\ 30 & x \in [5,6]; \end{cases} \quad f_3(x) = \begin{cases} 10 & x \in [0,1), \\ 20 & x \in [1,3), \\ 30 & x \in [3,4), \\ 20 & x \in [4,5), \\ 25 & x \in [5,6]. \end{cases}$$

These density functions are plotted in Fig. 1.4. The range $R_\mu(X)$ is plotted in Fig. 1.5 and it is a convex compact set in $\mathbb{R}^3$. $\square$

In addition to compactness and convexity, the range of a finite atomless measure is also centrally symmetric and contains the origin. Lyapunov [17] pointed out that a convex compact subset of $\mathbb{R}^2$, which is centrally symmetric and contains the origin, is the range of some two-dimensional vector measure. However, in $\mathbb{R}^3$ or higher dimensional Euclidean space, such a set may not necessarily be the range of a vector measure. In geometry, the range of a finite atomless measure is called a zonoid. For a review on Lyapunov's convexity theorem and its applications see [19]. For a review on zonoids see [3, 4].

1.2 Purification Theorems

Based on Lyapunov's convexity theorem, Dvoretzky, Wald, and Wolfowitz[7, 8] discovered the purification of transition probabilities. Let $(A, \mathcal{A})$ be a measurable space and π be a transition probability from $(X, \mathcal{F})$ to $(A, \mathcal{A})$; that is, $\pi(B|x)$ is a measurable function on $(X, \mathcal{F})$ for any $B \in \mathcal{A}$ and $\pi(\cdot|x)$ is a probability measure on $(A, \mathcal{A})$ for any $x \in X$. According to Dvoretzky, Wald, and Wolfowitz [7, 8], for measurable nonnegative functions f_i, $i = 1, \ldots, m$, on $X \times A$, two transition probabilities π_1 and π_2 are called equivalent if for each $i = 1, \ldots, m$,

$$\int_X \int_A f_i(x,a)\pi_1(da|x)\,\mu_i(dx) = \int_X \int_A f_i(x,a)\pi_2(da|x)\,\mu_i(dx).$$

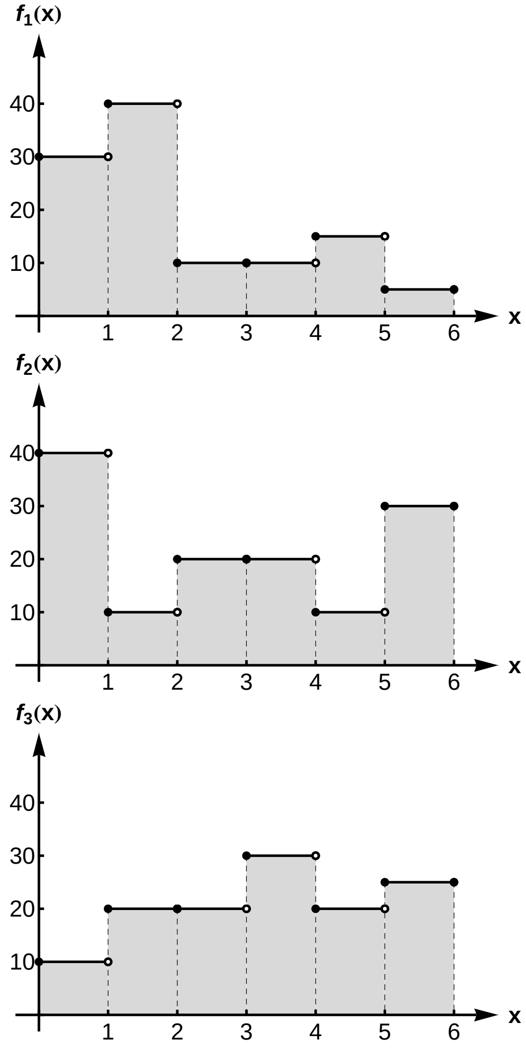

Figure 1.4: Density functions of the vector measure in Example 1.1.5.

They also defined strong equivalence. Two transition probabilities π_1 and π_2 are called strongly equivalent if for each $i = 1, \ldots, m$ and any $B \in \mathcal{A}$,

$$\int_X \pi_1 (B|x) \, \mu_i (dx) = \int_X \pi_2 (B|x) \, \mu_i (dx) . \tag{1.2.1}$$

A transition probability π is called pure if each probability measure $\pi(\cdot|x)$ is concentrated at one point. A pure transition probability π is defined by a measurable mapping $\varphi : X \to A$ such that $\pi(B|x) = I\{\varphi(x) \in B\}$ for all $B \in \mathcal{A}$. We say that a transition probability can be (strongly) purified if it is (strongly) equivalent to a pure transition probability. The

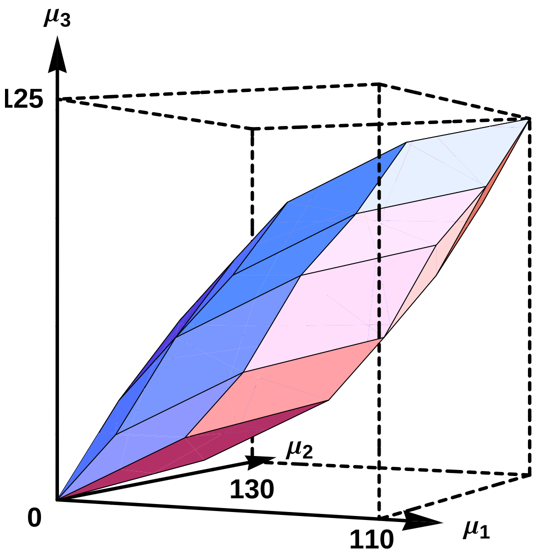

Figure 1.5: The range $R_\mu(X)$ in Example 1.1.5.

procedure to obtain the (strongly) equivalent pure transition probability is called (strong) purification.

We emphasize that, although the names seem to suggest that strong equivalence implies equivalence, neither of these two equivalence implies the other. Furthermore, a transition probability can be strongly purified does not imply that it can be purified, and vice versa. Finally, even if a transition probability can be both purified and strongly purified, one may still need to check if it can be purified in these two senses simultaneously. We will see that the purification of equilibriums in games will require simultaneous purification and strong purification.

Dvoretzky, Wald, and Wolfowitz [7, 8] proved the following theorem.

Theorem 1.2.1 (Dvoretzky, Wald, and Wolfowitz)**.** *For a finite measurable space* $(A, 2^A)$ *and a measurable space* $(X, \mathcal{F})$ *with an atomless vector measure* μ*, any transition probability can be purified and strongly purified.*

Edwards [10, Theorem 4.5] generalized this result to the case of a countable set A for strong purification. Khan and Rath [13, Theorem 2] gave another proof of this generalization.

Theorem 1.2.2 (Edwards)**.** *For a countable measurable space* $(A, 2^A)$ *and a measurable space* $(X, \mathcal{F})$ *with an atomless vector measure* μ*, any transition probability can be strongly* *purified.*

Feinberg and Piunovskiy [11] generalized Theorem 1.2.1 to the case of purification for $(A, \mathcal{A})$ being a Borel space. Khan and Rath [13, Corollary 1] also pointed out that purification exists when A is a countable set, which is a special case of the theorem by Feinberg and Piunovskiy below.

Theorem 1.2.3 (Feinberg and Piunovskiy)**.** *For $(A, \mathcal{A})$ being a Borel space and a measurable space $(X, \mathcal{F})$ with an atomless vector measure μ, any transition probability can be purified.*

However, strong purification may not be possible for $(A, \mathcal{A})$ being a Borel space. Loeb and Sun [16, Example 2.7] constructed an elegant example when a transition probability cannot be strongly purified for $m = 2$, $X = [0, 1]$, $A = [-1, 1]$, and atomless μ. We summarize these purification results in Table 1.1.

Table 1.1: Summary of purification theorems.

Action space	Finite	Countable	Borel
Purification	Exists	Exists	Exists
Strong purification	Exists	Exists	May not exist

It is also noteworthy that with additional assumptions on the measure space $(X, \mathcal{F}, \mu)$, strong purification exists for a more general action space A. Specifically, strong purification holds for a countable set of atomless, finite, signed Loeb measures, when A is a complete separable metric space [16, Corollary 2.6]. Podczeck [20] proved that strong purification holds for a countable set of finite signed measures μ_k absolutely continuous with respect to a measure μ, when $(X, \mathcal{F}, \mu)$ is a super-atomless probability space and A is a compact metric space.

We conclude this section by presenting an application of the purification theorems to the statistical decision problems, which was first mentioned in [7]. Without loss of generality, we limit ourselves to the case of finite action set A. Since the following application uses only theorems on purification (other than strong purification), the application can be extended to the cases of $(A, \mathcal{A})$ being countable or Borel spaces straightforwardly.

Example 1.2.4. Consider the outcome of an experiment described by a measurable space $(X, \mathcal{F})$ with m probability measures $\mu_1, \ldots, \mu_m$, where $m \in \mathbb{N}$ is a finite number. The true probability measure μ on $(X, \mathcal{F})$ is unknown, but known to be one of the above m measures.

On observing the outcome of the experiment, the statistician can make a decision a from a finite set of decisions A. A strategy $\pi(a|x)$ is a transition probability from $(X, \mathcal{F})$ to $(A, \mathcal{A})$ which specifies the probability that the statistician will adopt decision a on the condition that the outcome of the experiment is x. A strategy is called pure if it is a pure transition probability. Let $w_i(a, x)$ be the loss when the true measure is μ_i, $i = 1, \ldots, m$, the outcome of the experiment is x, and decision a is adopted. Then the risk r_i, $i = 1, \ldots, m$, or expected loss, when the true measure is μ_i and strategy π is adopted, can be expressed as

$$r_i(\pi) = \sum_{a \in A} \int_X w_i(a, x)\pi(a|x)\mu_i(dx).$$

According to Theorem 1.2.1, if all the measures $\mu_1, \ldots, \mu_m$ are atomless, the transition probability π can be purified. In other words, there exists a pure strategy $\pi^*(a|x)$ such that, $r_i(\pi) = r_i(\pi^*)$, for all $i = 1, \ldots, m$. □

1.3 Applications in Game Theory

One of fundamental problems in game theory is to prove the existence of an equilibrium with all players adopting the pure strategies (pure equilibrium). Such a proof generally requires convexifying effect of large numbers to appear in the model (see [1, 15, 18, 21, 23, 25] for such proofs for various games). In fact, this effect was the central point of Lyapunov's convexity theorem: when the measure of the atoms of the σ-algebra become smaller and smaller and eventually tend to zero (in other words, the measure becomes atomless), the range of the measure becomes more and more close to and eventually tends to be a convex set. The convexifying effect is also the reason why in many games it is easy to prove that an equilibrium with some of the players adopting mixed strategy (mixed equilibrium) exists, as mixed-strategies can provide convexity.

A natural way of proving the existence of the equilibrium in pure strategies consists of two steps. The first step is to prove the existence of a mixed equilibrium, and the second step (referred to as purification in games) is to prove that there exist pure strategies which are "equivalent" to the mixed strategies played at the equilibrium. In the second step, one generally needs Lyapunov's convexity theorem or purification results (reviewed in Section 1.2). In [14], based on the Dvoretzky-Wald-Wolfowitz purification theorem (Theorem 1.2.1), Khan, Rath, and Sun developed a unified framework for dealing with the purification in various finite action games considered in [18, 21, 23]. In [13], Khan and Rath extended this framework to countably infinite action games, based on the purification results for countable action sets (reviewed in Section 1.2).

In the example below, we show how purification works in a game introduced by Milgrom Weber [18], following Khan, Rath, and Sun's framework [14].

Example 1.3.1. Consider the situation in which l parties are evolved. After each party has fixed its strategy (which may be mixed, but only depends on the state they will be in), one of the m scenarios in the scenario set C will happen. In each scenario $c \in C$, each player i, $i = 1, \ldots, l$, will be randomly in a state from its state space $(X_i, \mathcal{F}_i)$, with respect to a probability measure μ_{ic}, which only depends on scenario c that has happened. The payoff of each party depends on: (1) scenario c that has happened, (2) the party's state and its strategy on such state, and (3) the strategies played by all others. Such situation can be described by the model of finite games with incomplete information introduced in [18]. Formally, this model consists of the following elements:

1. The game is played by l players.

2. The state space of the game $(X, \mathcal{F}) = (C \times \prod_{i=1}^{l} X_i, \mathcal{C} \times \prod_{i=1}^{l} \mathcal{X}_i)$ is the product of the common state space $(C, \mathcal{C})$ and the state space of each player $(X_i, \mathcal{F}_i)$, $i = 1, \ldots, l$. Assume that $C = \{1, 2, \ldots, m\}$ is a finite set and $\mathcal{C} = 2^C$. Let p be the probability measure on $(C, \mathcal{C})$, and μ_{ic}, $i = 1, \ldots, l$, $c = 1, \ldots, m$, be the probability measure on $(X_i, \mathcal{F}_i)$ on the condition that the common state is c. Then $\mu_i = (\mu_{i1}, \ldots, \mu_{im})$ is a m-dimensional vector measure on $(X_i, \mathcal{F}_i)$. We further assume that μ_i is an atomless probability vector measure for each $i = 1, \ldots, l$.

3. The action space of the game $(A, \mathcal{A}) = (\prod_{i=1}^{l} A_i, \prod_{i=1}^{l} \mathcal{A}_i)$ is the product of the action space of each player $(A_i, \mathcal{A}_i)$, $i = 1, \ldots, l$. Assume that $A_i = \{1, \ldots, n_i\}$ is a finite set and $\mathcal{A}_i = 2^{A_i}$.

4. The payoff of each player i, $i = 1, \ldots, l$, is denoted by a function $u_i : A \times C \times X_i \to \mathbb{R}$. In other words, the payoff of each player depends on the collection of actions $a = \{a_1, \ldots, a_l\} \in A$ taken by all players, the common state $c \in C$ of all the players, and its own state $x_i \in X_i$.

5. A strategy $\pi_i(a_i|x_i)$, $a_i \in A_i$, $x_i \in X_i$ of player i, $i = 1, \ldots, l$, is the transition probability from $(X_i, \mathcal{F}_i)$ to $(A_i, \mathcal{A}_i)$. In other words, it is the probability that the player will take action $a_i \in A_i$ on the condition that it is in state $x_i \in X_i$. π_i is called

a pure strategy if it is a pure transition probability. The collection $\pi = \{\pi_1, \dots, \pi_l\}$ is called a strategy profile. π is called a pure strategy profile if π_i is a pure transition probability for each $i = 1, \dots, l$.

Now we assume that a mixed equilibrium is known, where a mixed strategy profile π is adopted. Then at equilibrium, the expected payoff for player i is

$$U_i(\pi) = \int_X \int_A \left[p(dc) \prod_{j=1}^{l} \mu_{jc}(dx_j) \right] \left[\prod_{j=1}^{l} \pi_j(da_j|x_j) \right] u_i(a, x_i, c). \tag{1.3.1}$$

In (1.3.1) above, to make the expression concise, we have written all the sums as integrations and will restore the notation of sums where it is appropriate. Let $X_{-i} = \prod_{j \neq i} X_j$ and $A_{-i} = \prod_{j \neq i} A_j$, where $\prod_{j \neq i}$ is the abbreviation of $\prod_{1 \leq j \leq l, j \neq i}$. In addition, the notation a_{-i} is understood as the collection of variables a_j, where $1 \leq j \leq l$ and $j \neq i$. Then the expected payoff can be further written as

$$\begin{aligned} U_i(\pi) &= \sum_{c \in C} p(c) \int_{X_i} \mu_{ic}(dx_i) \sum_{a_i \in A_i} \pi_i(a_i|x_i) \\ &\quad \times \int_{X_{-i}} \left[\prod_{j \neq i} \mu_{jc}(dx_j) \right] \sum_{a_{-i} \in A_{-i}} \left[\prod_{j \neq i} \pi_j(a_j|x_j) \right] \\ &\quad \times u_i(a_i, a_{-i}, x_i, c) \\ &= \sum_{c \in C} p(c) \int_{X_i} \sum_{a_i \in A_i} \pi_i(a_i|x_i) \mu_{ic}(dx_i) \\ &\quad \times \sum_{a_{-i} \in A_{-i}} \left[\prod_{j \neq i} \int_{X_j} \pi_j(a_j|x_j) \mu_{jc}(dx_j) \right] \\ &\quad \times u_i(a_i, a_{-i}, x_i, c). \end{aligned} \tag{1.3.2}$$

For each $j = 1, \dots, l$, and any $B_j \in \mathcal{A}_j$, define

$$p_{jc}^{\pi}(B_j) = \int_{X_j} \pi_j(B_j|x_j) \mu_{jc}(dx_j). \tag{1.3.3}$$

Then, p^{π}_{jc} is a probability measure on $(A_j, \mathcal{A}_j)$, and (1.3.2) can be written as

$$\begin{aligned} U_i(\pi) &= \sum_{c\in C} p(c) \left[\int_{X_i} \sum_{a_i \in A_i} \pi_i(a_i|x_i)\mu_{ic}(dx_i) \right] \\ &\quad \times \sum_{a_{-i}\in A_{-i}} \left[u_i(a_i, a_{-i}, x_i, c) \prod_{j\neq i} p^{\pi}_{jc}(a_j) \right]. \end{aligned} \tag{1.3.4}$$

Furthermore, for each $i = 1, \ldots, l$, each $c = 1, \ldots, m$, and the strategy profile π, define

$$u^{\pi}_{ic}(a_i, t_i) = \sum_{a_{-i}\in A_{-i}} \left[u_i(a_i, a_{-i}, x_i, c) \prod_{j\neq i} p^{\pi}_{jc}(a_j) \right]. \tag{1.3.5}$$

Then the expected payoff for player i can be further expressed as

$$U_i(\pi) = \sum_{c=1}^{m} p(c) \left[\int_{X_i} \sum_{a_i\in A_i} u^{\pi}_{ic}(a_i, t_i)\pi_i(a_i|x_i)\mu_{ic}(dx_i) \right]. \tag{1.3.6}$$

Now we discuss whether there exists a pure strategy profile π^* under which the equilibrium still holds. This question is equivalent to the question whether there exists a pure strategy profile π^* under which, for each player i, $i = 1, \ldots, l$, the expected payoff is the same as that under π:

$$U_i(\pi) = U_i(\pi^*). \tag{1.3.7}$$

The answer is positive, and we can prove it by using the purification theorems in Section 1.2. According to an extension of Theorem 1.2.1 [14, Corollary 1], for each player i, $i = 1, \ldots, l$, there exists a pure strategy π^*_i such that: (1) for all $c \in C$,

$$\begin{aligned} &\int_{X_i} \sum_{a_i\in A_i} u^{\pi}_{ic}(a_i, t_i)\pi_i(a_i|x_i)\mu_{ic}(dx_i) \\ = &\int_{X_i} \sum_{a_i\in A_i} u^{\pi}_{ic}(a_i, t_i)\pi^*_i(a_i|x_i)\mu_{ic}(dx_i), \end{aligned} \tag{1.3.8}$$

and (2) for all $B_i \in \mathcal{A}_i$ and all $c \in C$,

$$\int_{X_i} \pi_i(B_i|x_i)\mu_{ic}(dx_i) = \int_{X_i} \pi^*_i(B_i|x_i)\mu_{ic}(dx_i). \tag{1.3.9}$$

We remark that the above result does not directly follow from Theorem 1.2.1, because we have claimed that there exists a pure strategy π_i^* that is simultaneously equivalent and strongly equivalent to π_i, where Theorem 1.2.1 does not address that π_i can be purified and strongly purified simultaneously.

Note that (1.3.8) alone does not imply (1.3.7). In order to show that the equality in (1.3.7) holds, we still need to show that, for any player i, and any $c \in C$,

$$u_{ic}^{\pi}(a_i, t_i) = u_{ic}^{\pi^*}(a_i, t_i). \tag{1.3.10}$$

Indeed, (1.3.9) implies that $p_{jc}^{\pi}(B_j) = p_{jc}^{\pi^*}(B_j)$, and according to the definition of u_{ic}^{π} in (1.3.5),

$$\begin{aligned} u_{ic}^{\pi}(a_i, t_i) &= \sum_{a_{-i} \in A_{-i}} \left[u_i(a_i, a_{-i}, x_i, c) \prod_{j \neq i} p_{jc}^{\pi}(a_j) \right] \\ &= \sum_{a_{-i} \in A_{-i}} \left[u_i(a_i, a_{-i}, x_i, c) \prod_{j \neq i} p_{jc}^{\pi^*}(a_j) \right] = u_{ic}^{\pi^*}(a_i, t_i). \end{aligned}$$

□

We conclude this section by three remarks. Firstly, there are no standard names for different kinds of equivalence and purification concepts in the current literature. In particular, the definitions of various types of equivalence in [14] are different from those of Dvoretzky, Wald, and Wolfowitz [7, 8]. Throughout this dissertation, we stick with the definitions of Dvoretzky, Wald, and Wolfowitz, and only use the concepts of (strong) equivalence and (strong) purification. For the readers' convenience, we compare the different notations in Table 1.2.

Table 1.2: Comparison of notations of [7] and those of [14].

Notations in [7]	Notations in [14]
Equivalence	Payoff Equivalence (PE)
Strong Equivalence	Distribution Equivalence (DE)
Equivalence & More Conditions	Strong PE (SPE)
Strong Equivalence & More Conditions	Strong DE (SDE)
Purification & Strong Purification	Strong Purification

Secondly, the idea of purification can be applied to more general settings of games. In particular, such purification framework is also suitable for other types of games described in [14], and can be generalized to the case of countable action set A [13].

Finally, Theorem 1 in [14], which considers not only the strategy profiles in equilibrium but also the general mixed strategy profiles, is a stronger result than that we have shown in Example 1.3.1, which is enough to serve the purpose of demonstration.

Chapter 2

Extension of Lyapunov's Convexity Theorem

In this chapter, we give an overview of our results about the extension of Lyapunov's convexity theorem to subranges [5, 6]. We first introduce the concept of subranges in Section 2.1. Then we present the results on maximal and minimal subranges in Section 2.2, and results on union and intersection of subranges in Section 2.3. Finally we give some examples to demonstrate our results for two-dimensional finite atomless measures in Section 2.4.

2.1 Subranges

Let $(X, \mathcal{F})$ be a measurable space and $\mu = (\mu_1, ..., \mu_m)$, $m = 1, 2, \ldots$, be a finite vector measure on it. For each $Y \in \mathcal{F}$ consider the range $R_\mu(Y) = \{\mu(Z) : Z \in \mathcal{F}, Z \subset Y\} \subset \mathbb{R}^m$ of the vector measures of all its measurable subsets Z. We call $R_\mu(Y)$ a subrange of X generated by its measurable subset Y. Lyapunov's convexity theorem [17] states that the range $R_\mu(X)$ is compact and furthermore, if μ is atomless, this range is convex. Of course, this is also true for any subrange $R_\mu(Y)$. Following is an example of a subrange.

Example 2.1.1. Let $(X, \mathcal{F})$ be a measurable space, where $X = [0, 1]$ and $\mathcal{F}$ is the Borel σ-algebra on X. Consider the vector measure defined in terms of density functions

$$\mu(dx) = (\mu_1, \mu_2)(dx) = (1, f(x))\, dx,$$

where

$$f(x) = \begin{cases} 4x, & \text{if } x \in \left[0, \frac{1}{2}\right); \\ 1, & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Let $Y = \left[0, \frac{1}{16}\right] \cup \left[\frac{7}{16}, \frac{7}{8}\right]$. Then $R_\mu(Y)$ is a subrange of X generated by Y. The range $R_\mu(X)$ and the subrange $R_\mu(Y)$ are plotted in Fig. 2.1. In particular, observe that the subrange $R_\mu(Y)$ is also convex and compact. □

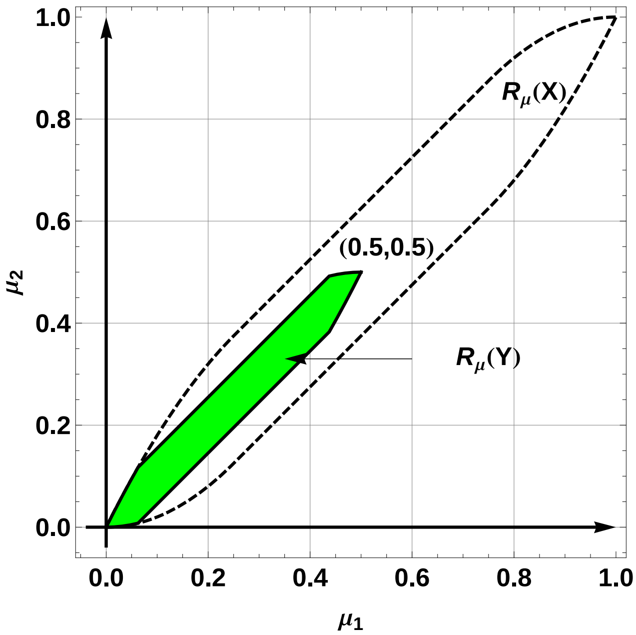

Figure 2.1: The range $R_\mu(X)$ and the subrange $R_\mu(Y)$ in Example 2.1.1.

Obviously, each measurable subset of X generates exactly one subrange. To study the properties of subranges, we will first divide $\mathcal{F}$ into classes and then study the properties of the subranges generated by the sets from each class. In fact, we can divide $\mathcal{F}$ into a three-level hierarchy of classes.

1. Note that $\mu(Y_1) = \mu(Y_2)$ is an equivalence relation on $\mathcal{F}$. Thus it partitions $\mathcal{F}$ into equivalence classes. Let $\mathcal{S}_\mu^p(X)$ be the set of all measurable subsets of X with the vector measure $p \in R_\mu(X)$,

$$\mathcal{S}_\mu^p(X) = \{Y \in \mathcal{F} : \mu(Y) = p\}.$$

Then, for each $p \in R_\mu(X)$, $\mathcal{S}_\mu^p(X)$ is one of the above-mentioned equivalence classes.

2. In addition, since the relation $Y_1 = Y_2$ (μ-everywhere) is an equivalence relation on $\mathcal{S}_\mu^p(X)$ for each $p \in R_\mu(X)$, it partitions $\mathcal{S}_\mu^p(X)$ further into equivalence subclasses.

3. One can look into more detailed classification on each of the subclasses above by using even stronger equivalence relation $Y_1 = Y_2$. However, this more detailed classification is not interesting in this dissertation, since $Y_1 = Y_2$ (μ-everywhere) is already a sufficient condition for $R_\mu(Y_1) = R_\mu(Y_2)$.

For an atomless μ, Lyapunov [17, Theorem III] proved that: (i) $\mathcal{S}_\mu^p(X)$ consists of one equivalence subclass if and only if p is an extreme point of $R_\mu(X)$, and (ii) if $p \in R_\mu(X)$ is not an extreme point of $R_\mu^p(X)$, then the set of equivalence subclasses in $\mathcal{S}_\mu^p(X)$ has cardinality of the continuum. Thus, in general, when $p \in R_\mu(X)$ is in the interior of $R_\mu^p(X)$, the elements of $\mathcal{S}_\mu^p(X)$, which are measurable subsets, may generate infinite number of different subranges of X. The following example shows that three different elements of $\mathcal{S}_\mu^p(X)$ generates three different subranges.

Example 2.1.2. Consider the measure space $(X, \mathcal{F}, \mu)$ described in Example 2.1.1. Let $Y^1 = \left[\frac{1}{2}, 1\right]$, $Y^2 = \left[0, \frac{1}{16}\right] \cup \left[\frac{7}{16}, \frac{7}{8}\right]$, and $Y^3 = \left[0, \frac{1}{2}\right]$, then $\mu(Y^1) = \mu(Y^2) = \mu(Y^*) = \left(\frac{1}{2}, \frac{1}{2}\right)$, and thus $Y^1, Y^2, Y^3 \in \mathcal{S}_\mu^p(X)$, where $p = \left(\frac{1}{2}, \frac{1}{2}\right)$. The subranges generated by Y^1, Y^2, and Y^3 are plotted in Fig. 2.2. Observe that these subranges are different. $\square$

2.2 Maximal and Minimal Subranges

For any $p \in R_\mu(X)$, we first answer the question whether the class $\mathcal{S}_\mu^p(X)$ contains an element that generates the maximal subrange. In other words, is it always true that for any $p \in R_\mu(X)$ there exists a measurable subset $Y^* \in \mathcal{S}_\mu^p(X)$ such that $R_\mu(Y^*) \supseteq R_\mu(Y)$ for any $Y \in \mathcal{S}_\mu^p(X)$? For a finite atomless μ, we show that the answer is positive when $m = 2$ (Theorem 3.2.1) and negative when $m > 2$ (Example 4.1.2).

Furthermore, for $m = 2$, this maximal subrange is equal to the set $Q_\mu^p(X) \subset \mathbb{R}^m$ constructed by a simple geometric transformation of $R_\mu(X)$. More specifically, $Q_\mu^p(X)$ is the intersection of the $R_\mu(X)$ with its shift by a vector $-(\mu(X) - p)$,

$$Q_\mu^p(X) = (R_\mu(X) - \{\mu(X) - p\}) \cap R_\mu(X),$$

where $S_1 - S_2 = \{q - r : q \in S_1, r \in S_2\}$ for $S_1, S_2 \subseteq \mathbb{R}^m$. In particular, $R_\mu(X) - \{r\}$ is a parallel shift of $R_\mu(X)$ by $-r$.

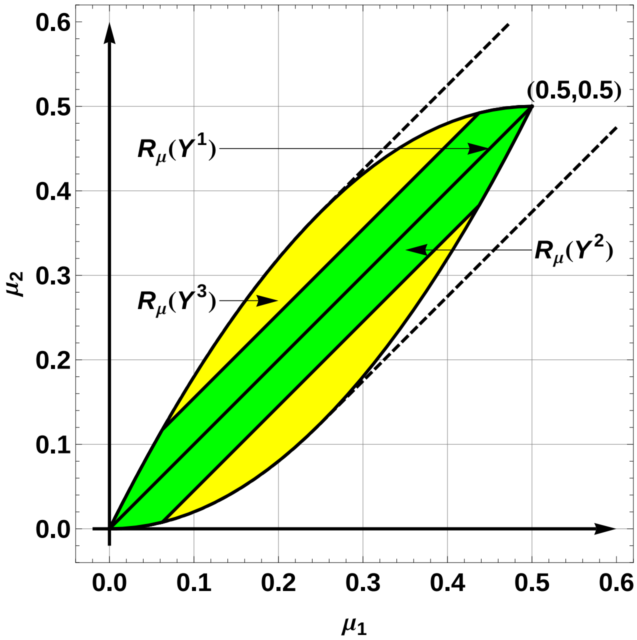

Figure 2.2: Three different subranges $R_\mu(Y^1)$, $R_\mu(Y^2)$, and $R_\mu(Y^3)$.

For an atomic μ, we show that a maximal subrange may not exist even for $m = 1$, and thus for any natural number m (Example 4.1.4).

In addition to the maximal subrange, it is possible to consider a minimal subrange. For $q \in R_\mu(X)$, the set $M^* \in \mathcal{S}^q_\mu(X)$ generates minimal subrange corresponding to q if $R_\mu(M^*) \subseteq R_\mu(M)$ for any $M \in \mathcal{S}^p_\mu(X)$. We show that a subset generates a maximal subrange corresponding to p if and only if its complement generates a minimal subrange corresponding to $\mu(X) - p$ (Theorem 3.2.2). Therefore, minimal subranges also exist for two-dimensional finite atomless measure μ and they may not exist for higher-dimensional or atomic measures.

The following example demonstrates minimal and maximal subranges.

Example 2.2.1. Consider the measure space $(X, \mathcal{F}, \mu)$ described in Example 2.1.1. Y^1 and Y^3 defined in Example 2.1.2 respectively generate the minimal and maximal subranges corresponding to $p = \left(\frac{1}{2}, \frac{1}{2}\right)$ (see Fig. 2.2). $\square$

2.3 Union and Intersection of Subranges

For any $p \in R_\mu(X)$, we further study the union of the subranges generated by all the elements of the class $\mathcal{S}_\mu^p(X)$. Let $R_\mu^p(X)$ denote such a union. In other words,

$$R_\mu^p(X) = \bigcup_{Y \in \mathcal{S}_\mu^p(X)} R_\mu(Y).$$

When $m = 2$ and μ is atomless, there exists a set $Y^* \in \mathcal{S}_\mu^p(X)$ that generates the maximal subrange $R_\mu(Y^*)$, and the maximal subrange is a convex compact set by Lyapunov's convexity theorem. Then, in this case, the set $R_\mu^p(X)$ is obviously a convex compact set, since $R_\mu^p(X) = R_\mu(Y^*)$. In addition, $R_\mu^p(X) = Q_\mu^p(X)$. However, a maximal subrange may not exist for an atomless vector measure μ when $m > 2$. In such cases, we raise two questions.

1. Is $R_\mu^p(X)$ still a convex compact set?
2. Does the equality $R_\mu^p(X) = Q_\mu^p(X)$ still hold?

We answer these two questions completely. In general, the union of an uncountably infinite number of convex compact sets may be neither closed nor convex. We prove (Theorem 3.3.1) that for any natural number m the set $R_\mu^p(X)$ is compact and, if μ is atomless, this set is convex. This is a generalization of Lyapunov's convexity theorem, which is a particular case of this statement for $p = \mu(X)$. We also prove that $R_\mu^p(X) \subseteq Q_\mu^p(X)$ (Theorem 3.3.2). Example 4.2.1 demonstrates that it is possible that the equality $R_\mu^p(X) = Q_\mu^p(X)$ does not hold when $m > 2$ and μ is atomless. Example 4.2.2 further demonstrates that such equality may not hold when μ is atomic even for $m = 1$ (and thus for any natural number m).

Naturally, one can also consider the intersection of the subranges generated by all the elements of the class $\mathcal{S}_\mu^p(X)$. Let $I_\mu^p(X)$ denote such an intersection. In other words,

$$I_\mu^p(X) = \bigcap_{Y \in \mathcal{S}_\mu^p(X)} R_\mu(Y).$$

Obviously, this intersection is compact and, if μ is atomless, it is also convex, because the intersection of compact or convex sets is still a compact or convex set. We further consider the relation between $R_\mu^p(X)$ and $I_\mu^p(X)$. We show (Theorem 3.3.3) that, for any natural number m, a m-dimensional finite atomless vector measure μ, and a vector $p \in R_\mu(X)$, $I_\mu^{\mu(X)-p}(X) = R_\mu(X) \ominus R_\mu^p(X)$.

2.4 Geometric construction

In [17], Lyapunov commented that a subset of the two-dimensional Euclidean space $\mathbb{R}^2$ is the range of some two-dimensional finite atomless vector measure on some measurable space if and only if it satisfies the following conditions: (1) it is convex; (2) it is closed; (3) it is centrally symmetric; (4) it contains the origin. Since the geometrically constructed set $Q_\mu^p(X)$ satisfies the conditions (1)-(4), it must be the range of some two-dimensional finite atomless vector measure on some measurable space. Theorem 3.2.1 immediately tells us that it is the range of the vector measure μ on the measurable space $(Z^*, \mathcal{F}_{Z^*})$, where $\mathcal{F}_{Z^*} = \{Z \in \mathcal{F} : Z \subseteq Z^*\}$. The second equality in Theorem 3.2.1 allows us to construct geometrically the set $R_\mu^p(x)$ by shifting the set $R_\mu(X)$ by $(p - \mu(X))$ and intersecting the shifted set with $R_\mu(X)$.

We consider three examples with the same set $X = [0,1]$, but with different probability vector measures. Let $p = (0.7, 0.8)$ in all these examples.

Example 2.4.1. Let the probability measures μ_1 and μ_2 be singular. Then the range $R_\mu(X)$ is the unit square enclosed by the dashed lines in Fig. 2.5. The area enclosed by the dotted lines is obtained by parallelly shifting $R_\mu(X)$ by $(-0.3, -0.2)$. The shaded area is the intersection of the above two areas and represents the identical sets $R_\mu(Z^*)$, $R_\mu^p(X)$, and $Q_\mu^p(X)$ with $p = (0.7, 0.8)$. □

Example 2.4.2. Consider the probability vector measure defined in terms of density functions

$$\mu(dx) = (\mu_1, \mu_2)(dx) = (1, f(x))\, dx,$$

where

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in \left[0, \frac{1}{2}\right); \\ \frac{3}{2}, & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then the range of $R_\mu(X)$ is the area enclosed by the dashed lines in Fig. 2.4. The area enclosed by the dotted lines is obtained by parallelly shifting $R_\mu(X)$ by $(-0.3, -0.2)$. The shaded area is the intersection of the above two areas and represents the identical sets $R_\mu(Z^*)$, $R_\mu^p(X)$, and $Q_\mu^p(X)$ with $p = (0.7, 0.8)$. □

Example 2.4.3. Consider the probability vector measure defined in terms of density functions

$$\mu(dx) = (\mu_1, \mu_2)(dx) = (1, 2x)\, dx.$$

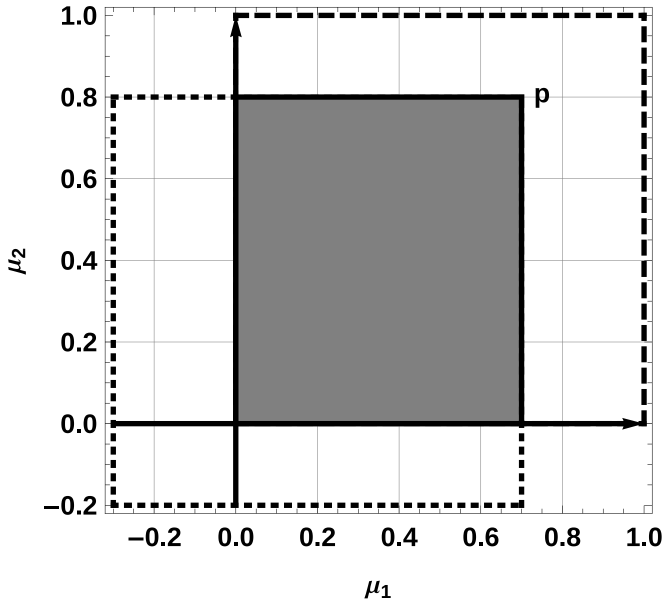

Figure 2.3: The sets $R_\mu(Z^*)$, $R^p_\mu(X)$, and $Q^p_\mu(X)$ in Example 2.4.1.

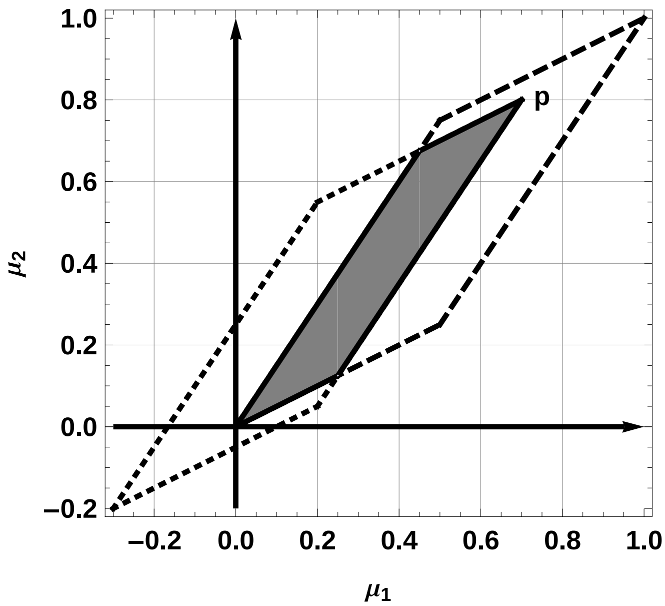

Figure 2.4: The sets $R_\mu(Z^*)$, $R^p_\mu(X)$, and $Q^p_\mu(X)$ in Example 2.4.2.

Then the range $R_\mu(X)$ is the area enclosed by the dashed lines in Fig. 2.5. The area enclosed by the dotted lines is obtained by parallelly shifting $R_\mu(X)$ by $(-0.3, -0.2)$. The shaded area is the intersection of the above two areas and represents the identical sets $R_\mu(Z^*)$, $R^p_\mu(X)$, and $Q^p_\mu(X)$ with $p = (0.7, 0.8)$. $\square$

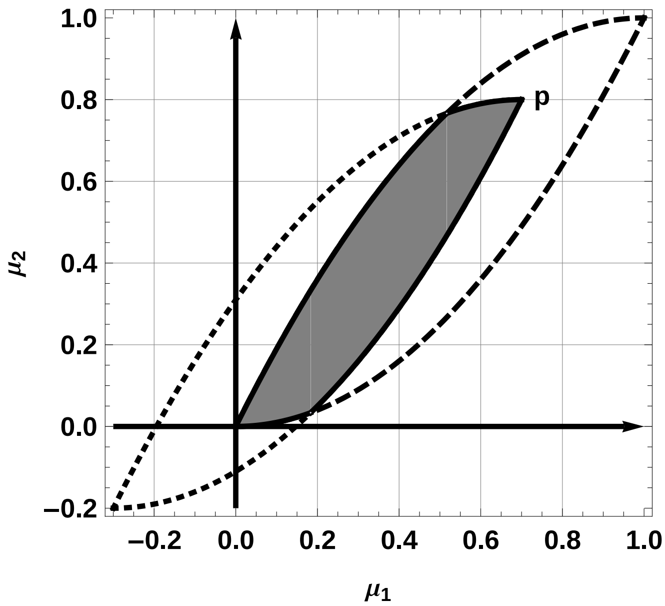

Figure 2.5: The sets $R_\mu(Z^*)$, $R^p_\mu(X)$, and $Q^p_\mu(X)$ in Example 2.4.3.

Chapter 3

Main Theorems

In this chapter, we present our results about subranges in a series of theorems as well as their proofs. In Section 3.1, we give formal definitions of some concepts. Then, we prove two theorems (Theorems 3.2.1 and 3.2.2) on maximal and minimal subranges in Section 3.2. Finally, we prove three more theorems (Theorems 3.3.1-3.3.3) on the union and the intersection of subranges in Section 3.3.

3.1 Definitions

In the following, we give the formal definitions of the sets $\mathcal{S}_{\mu}^{p}(X)$, $R_{\mu}^{p}(X)$, $I_{\mu}^{p}(X)$, and $Q_{\mu}^{p}(X)$, as well as those of the maximal and minimal subsets.

Definition 3.1.1. Given a measurable space $(X, \mathcal{F})$ with a vector measure μ and a vector $p \in R_{\mu}(X)$, we define

1. the set of all subsets of X with vector measure p,

$$\mathcal{S}_{\mu}^{p}(X) = \{Z \in \mathcal{F} : \mu(Z) = p\};$$

2. the union of the subranges generated by all measurable subsets of X with the vector measure p,

$$R_{\mu}^{p}(X) = \bigcup_{Z \in \mathcal{S}_{\mu}^{p}(X)} R_{\mu}(Z);$$

3. the intersection of the subranges generated by all measurable subsets of X with the

vector measure p,

$$I_{\mu}^{p}(X) = \bigcap_{Z \in \mathcal{S}_{\mu}^{p}(X)} R_{\mu}(Z);$$

4. the intersection of the $R_\mu(X)$ with its shift by a vector $-(\mu(X) - p)$,

$$Q_{\mu}^{p}(X) = (R_{\mu}(X) - \{\mu(X) - p\}) \cap R_{\mu}(X),$$

where $S_1 - S_2 = \{q - r : q \in S_1, r \in S_2\}$ for $S_1, S_2 \subseteq \mathbb{R}^m$.

Definition 3.1.2. Given a measurable space $(X, \mathcal{F})$ with a vector measure μ and vectors $p, q \in R_{\mu}(X)$, the set $Z^* \in \mathcal{S}_{\mu}^{p}(X)$, such that $R_{\mu}(Z^*) \supseteq R_{\mu}(Z)$ for any $Z \in \mathcal{S}_{\mu}^{p}(X)$, is called the maximal subset of X with the measure p. The set $M^* \in \mathcal{S}_{\mu}^{q}(X)$, such that $R_{\mu}(M^*) \subseteq R_{\mu}(M)$ for any $M \in \mathcal{S}_{\mu}^{q}(X)$, is called the minimal subset of X with the measure q.

In other words, maximal (minimal) subset of X with the measure p generates the maximal (minimal) subrange among the subranges generated by all elements of $\mathcal{S}_{\mu}^{p}(X)$.

3.2 Maximal and Minimal Subranges

In this section, we prove two theorems on maximal and minimal subranges. The first theorem (Theorem 3.2.1) guarantees the existence of a maximal subrange in two-dimensional case and gives a way to geometrically construct it. The second theorem (Theorem 3.2.2) links the notions of maximal and minimal subsets.

Theorem 3.2.1. *For a measurable space $(X, \mathcal{F})$ with a two-dimensional finite atomless vector measure $\mu = (\mu_1, \mu_2)$ and for a vector $p \in R_{\mu}(X)$, there exists a maximal subset $Z^* \in \mathcal{S}_{\mu}^{p}(X)$ and, in addition, $R_{\mu}(Z^*) = Q_{\mu}^{p}(X)$.*

Theorem 3.2.2. *For a measurable space $(X, \mathcal{F})$ with a two-dimensional finite atomless vector measure $\mu = (\mu_1, \mu_2)$ and for a vector $p \in R_{\mu}(X)$, the set Z^* is the maximal subset of X with the measure p, if and only if $M^* = X \setminus Z^*$ is the minimal subset of X with the measure $\mu(X) - p$.*

With Theorem 3.2.2, the existence of the minimal subset $M^* \in \mathcal{S}_{\mu}^{q}(X)$ immediately follows from the existence of the maximal subset $Z^* \in \mathcal{S}_{\mu}^{\mu(X)-q}(X)$. Furthermore,

$$R_{\mu}(M^*) = (R_{\mu}(M^*) \oplus R_{\mu}(Z^*)) \ominus R_{\mu}(Z^*) = R_{\mu}(X) \ominus Q_{\mu}^{\mu(X)-q}(X).$$

Recall that, for sets $A, B \subseteq \mathbb{R}^m$, $A \oplus B = \bigcup_{b \in B} (A + b)$ is called the Minkowski addition (or sum), and $A \ominus B = \bigcap_{b \in B} (A - b)$ is called the Minkowski subtraction (or difference), where $A + b = \{a + b : a \in A\}$ and $A - b = A + (-b)$. These results are collected in the following corollary.

Corollary 3.2.3. *For a measurable space $(X, \mathcal{F})$ with a two-dimensional finite atomless vector measure $\mu = (\mu_1, \mu_2)$ and for a vector $q \in R_\mu(X)$, there exists a minimal subset $M^* \in \mathcal{S}_\mu^q(X)$ and, in addition, $R_\mu(M^*) = R_\mu(X) \ominus Q_\mu^{\mu(X)-q}(X)$.*

We first consider Theorem 3.2.1. Recall that for a set $S \subseteq \mathbb{R}^m$, its reflection across a point $c \in \mathbb{R}^m$ is $\mathrm{Ref}(S, c) = \{2c\} - S$. If $S = \{s\}$ is a singleton, we shall write $\mathrm{Ref}(s, c)$ instead of $\mathrm{Ref}(\{s\}, c)$. A set $S \subseteq \mathbb{R}^m$ is called centrally symmetric if $\mathrm{Ref}(S, c) = S$ for some point $c \in \mathbb{R}^m$ called the center of S. Any bounded centrally symmetric set has only one center.

Throughout this section, we let $Y \in \mathcal{F}$ be any measurable subset of X. Lemmas 3.2.4-3.2.6 hold for any finite atomless vector measure $\mu = (\mu_1, \ldots, \mu_m)$ on $(X, \mathcal{F})$, where $m = 1, 2, \ldots$.

Lemma 3.2.4. *The set $R_\mu(Y)$ is centrally symmetric with the center $\frac{1}{2}\mu(Y)$.*

Proof. The proof is straightforward, and this fact was observed by Lyapunov [17, p. 476]. □

Lemma 3.2.5. *The equality $R_\mu(Y) - \{\mu(Y) - p\} = \mathit{Ref}\big(R_\mu(Y), \frac{1}{2}p\big)$ holds for any $p \in R_\mu(Y)$.*

Proof. By Lemma 3.2.4, $R_\mu(Y) = \mathrm{Ref}\big(R_\mu(Y), \frac{1}{2}\mu(Y)\big) = \{\mu(Y)\} - R_\mu(Y)$. Therefore, $R_\mu(Y) - \{\mu(Y) - p\} = (\{\mu(Y)\} - R_\mu(Y)) - \{\mu(Y) - p\} = \{p\} - R_\mu(Y) = \mathrm{Ref}\big(R_\mu(Y), \frac{1}{2}p\big)$. □

Lemma 3.2.6. *Each of the sets $R_\mu^p(Y)$ and $Q_\mu^p(Y)$ is centrally symmetric with the center $\frac{1}{2}p$.*

Proof. According to Lemma 3.2.4, each set $Z \in \mathcal{S}_\mu^p(Y)$ is centrally symmetric with the center $\frac{1}{2}p$. Therefore, $R_\mu^p(Y)$, which is the union of all the sets in $Z \in \mathcal{S}_\mu^p(Y)$, is also centrally symmetric with the center $\frac{1}{2}p$.

In addition,

$$
\begin{aligned}
\operatorname{Ref}\left(Q_{\mu}^{p}(Y), \frac{1}{2} p\right) &= \operatorname{Ref}\left(\left(R_{\mu}(Y)-\{\mu(Y)-p\}\right) \cap R_{\mu}(Y), \frac{1}{2} p\right) \\
&= \operatorname{Ref}\left(\operatorname{Ref}\left(R_{\mu}(Y), \frac{1}{2} p\right) \cap R_{\mu}(Y), \frac{1}{2} p\right) \\
&= R_{\mu}(Y) \cap \operatorname{Ref}\left(R_{\mu}(Y), \frac{1}{2} p\right) \\
&= R_{\mu}(Y) \cap\left(R_{\mu}(Y)-\{\mu(Y)-p\}\right)=Q_{\mu}^{p}(Y),
\end{aligned}
$$

where the first and last equalities follow from the definition of Q_{μ}^{p}, the second and second to the last equalities follow from Lemma 3.2.5. The third equality holds because a reflection of intersections equals the intersection of reflections and, in addition, a reflection of a reflection across the same point is the original set. □

Here we present the major ideas of the proof of Theorem 3.2.1. First, as shown later, after Theorem 3.2.1 is proven for equivalent measures μ_1 and μ_2, this condition can be removed. So, we make the following assumption in Lemmas 3.2.8, 3.2.10-3.2.16.

Assumption 3.2.7. The measures μ_1 and μ_2 are finite, atomless, and equivalent.

Under Assumption 3.2.7, let $f(x)=\frac{d\mu_2}{d\mu_1}(x)$ be a Radon-Nikodym derivative of μ_2 with respect to μ_1. Since f is defined μ_1-*a.e.*, we fix any its version. We shall frequently use notations similar to

$$\{f(x)<l\}=\{x \in X : f(x)<l\}.$$

Second, under Assumption 3.2.7, for any $a \in [0, \mu_1(X)]$, we denote

$$l_a=\min\{l \geq 0 : \mu_1(\{f(x) \leq l\}) \geq a\}. \tag{3.2.1}$$

Observe that the minimum in (3.2.1) exists. Indeed, let

$$l_a=\inf\{l \geq 0 : \mu_1(\{f(x) \leq l\}) \geq a\}.$$

We need to show that $\mu_1(\{f(x) \leq l_a\}) \geq a$. If $l_a=\infty$ then $\mu_1(\{f(x) \leq \infty\})=\mu_1(X) \geq a$. Let $l_a<\infty$. Consider a sequence $l^k \searrow l_a$, $k=1,2,\ldots$. Then $\cap_{k=1}^{\infty}\{f(x) \leq l^k\}=\{f(x) \leq l_a\}$ and $\{f(x) \leq l^k\} \supseteq \{f(x) \leq l^{k+1}\}$. Therefore $\mu_1(\{f(x) \leq l_a\})=\lim_{k \to \infty} \mu_1(\{f(x) \leq l^k\}) \geq a$.

Third, it is possible to construct the maximal subset $Z^* = X \setminus M^*$, where M^* can be defined explicitly. Let $X^l = \{f(x) = l\}$. If $\mu_1(X^l) = 0$ for all $l \in [0, \infty)$, then the definition of M^* is easier and we explain it first. In this case, there exists $a^* \in [0, \mu_1(X)]$ such that $\mu_2(M^*) = \mu_2(X) - p_2$, and M^* can be defined as

$$M^* = \{l_{a^*} \le y < l_{a^*+(\mu_1(X)-p_1)}\}. \tag{3.2.2}$$

In the general situation, the number a^* can be chosen to satisfy

$$\mu_2(\{l_{a^*} < y < l_{a^*+(\mu_1(X)-p_1)}\}) \le \mu_2(X) - p_2 \le \mu_2(\{l_{a^*} \le y \le l_{a^*+(\mu_1(X)-p_1)}\}),$$

and

$$M^* = \{l_{a^*} < y < l_{a^*+(\mu_1(X)-p_1)}\} \cup Z^1 \cup Z^2, \tag{3.2.3}$$

for Z^i, $i = 1, 2$, being some measurable subsets of X^{l^i}, where $l^1 = l_{a^*}$ and $l^2 = l_{a^*+(\mu_1(X)-p_1)}$. In particular, if $\mu_1\left(X^{l^1}\right) = 0$, let $Z^1 = X^{l^1}$, and if $\mu_1\left(X^{l^2}\right) = 0$, let $Z^2 = \emptyset$. If $\mu_1\left(X^{l^1}\right) = \mu_1\left(X^{l^2}\right) = 0$, then (3.2.3) reduces to (3.2.2). It is easy to show that the number of l such that $\mu_1\left(X^l\right) = 0$ is countable, but we do not use this fact.

The proof of Theorem 3.2.1 is based on several lemmas.

Lemma 3.2.8. *Under Assumption 3.2.7, the numbers l_a, $a \in [0, \mu_1(X)]$ have the following properties: (a) $\mu_1(\{f(x) < l_a\}) \le a \le \mu_1(\{f(x) \le l_a\})$; (b) $l_a \le l_{a'}$ if $a \le a'$.*

Proof. For (a), by definition, $a \le \mu_1(\{f(x) \le l_a\})$ holds. To prove that $\mu_1(\{f(x) < l_a\}) \le a$, assume that $\mu_1(\{f(x) < l_a\}) > a$. If $l_a = 0$, then $\mu_1(\{f(x) < l_a\}) = 0 > a$ which contradicts the assumption that $a \ge 0$. If $l_a > 0$, let $\epsilon_k \searrow 0$, $k = 1, 2, \ldots$, be a sequence of positive numbers. Then, for $k = 1, 2, \ldots$,

$$\mu_1(\{f(x) < l_a\}) = \mu_1(\{f(x) \le l_a - \epsilon_k\}) + \mu_1(\{l_a - \epsilon_k < f(x) < l_a\}) > a.$$

Let $D_k = \{l_a - \epsilon_k < f(x) < l_a\}$. We observe that $D_{k+1} \subseteq D_k$ and $\cap_{u=1}^{\infty} D_k = \emptyset$. Therefore, $\lim_{k\to\infty} \mu_1(D_k) = 0$. Thus, $\mu_1(f(x) \le l_a - \epsilon) > a$ for some $\epsilon > 0$ and this contradicts (3.2.1). These contradictions imply the lemma.

For (b), assume $l_a > l_{a'}$, then $\mu_1(\{f(x) \le l_{a'}\}) \ge a' \ge a$, and this contradicts (3.2.1). ☐

Note that for each $l \in [0, \infty)$, there exists a subfamily

$$\left\{W_b\left(X^l\right) \in \mathcal{F}_{X^l} : b \in \left[0, \mu_1\left(X^l\right)\right]\right\}$$

such that (1) $\mu_1\left(W_b\left(X^l\right)\right) = b$ for each $b \in \left[0, \mu_1\left(X^l\right)\right]$ and (2) $W_b\left(X^l\right) \subset W_{b'}\left(X^l\right) \subseteq X^l$ whenever $b < b' \le \mu_1\left(X^l\right)$. This fact follows from Ross [22, Theorem 2(LT3)]. We set $W_0\left(X^l\right) = \emptyset$. From now on we fix a family of $W_b\left(X^l\right)$ for each $l \in [0, \infty)$.

Definition 3.2.9. Under Assumption 3.2.7, for each a, define the following set

$$L_a = \{f(x) < l_a\} \cup W_c\left(X^{l_a}\right), \tag{3.2.4}$$

where $c = a - \mu_1(\{f(x) < l_a\})$.

Note that property (a) in Lemma 3.2.8 guarantees that $c \in \left[0, \mu_1\left(X^l\right)\right]$.

Lemma 3.2.10. *Under Assumption 3.2.7, the sets $L_a \in \mathcal{F}$, $a \in [0, \mu_1(X)]$, have the following properties: (a) $\mu_1(L_a) = a$; (b) $\{f(x) < l_a\} \subseteq L_a \subseteq \{f(x) \le l_a\}$; (c) $L_a \subset L_{a'} \subseteq X$ if $a < a' \le \mu_1(X)$.*

Proof. For (a),

$$\begin{aligned}\mu_1(L_a) &= \mu_1(\{f(x) < l_a\}) + \mu_1\left(W_c\left(X^{l_a}\right)\right) \\ &= \mu_1(\{f(x) < l_a\}) + a - \mu_1(\{f(x) < l_a\}) = a.\end{aligned}$$

Property (b) follows from $W_c\left(X^{l_a}\right) \subseteq X^{l_a}$ and (3.2.4). For (c), if $l_a = l_{a'}$ then $c < c'$ where $c' = a' - \mu_1(\{f(x) < l_a\})$, and thus

$$L_a = \{f(x) < l_a\} \cup W_c\left(X^{l_a}\right) \subset \{f(x) < l_a\} \cup W_{c'}\left(X^{l_a}\right) = L_{a'}.$$

If $l_a < l_{a'}$ then $L_a \subseteq \{f(x) \le l_a\} \subset \{f(x) < l_{a'}\} \subseteq L_{a'}$. □

Let $M_{a,d} = L_{a+d} \setminus L_a$. For each $d \in [0, \mu_1(X)]$ and each $a \in [0, \mu_1(X) - d]$, denote $g_d(a) = \mu_2(M_{a,d}) = \int_{M_{a,d}} f(x)\mu_1(dx)$. The function $g_d(a)$ is non-decreasing and continuous in $a \in [0, \mu_1(X) - d]$ for each $d \in [0, \mu_1(X)]$. However, we will not use the fact that it is non-decreasing. So we only prove the continuity in the following lemma.

Lemma 3.2.11. *Under Assumption 3.2.7, for each $d \in [0, \mu_1(X)]$, the function $g_d(a)$ is continuous in $a \in [0, \mu_1(X) - d]$.*

Proof. We show that $\mu_2(L_{a+d})$ is continuous in $a \in ([0, \mu_1(X) - d]$ for any $d \in [0, \mu_1(X)]$. Since $g_d(a) = L_{a+d} - L_a$, this implies the lemma. Consider a sequence $\{a^k : k = 1, 2, \ldots\}$, where $a_k \in [0, \mu_1(X) - d]$. Let $a^k \nearrow a$. Then $L_{a^k+d} \subset L_{a^{k+1}+d} \subset B \subseteq L_{a+d}$, where

$B = \cup_{k=1}^{\infty} L_{a^k+d}$. Therefore, $\mu_i(L_{a^k+d}) \nearrow \mu_i(B)$ and $\mu_i(L_{a+d}) = \mu_i(B)+\mu_i(L_{a+d}\setminus B)$, $i = 1, 2$. Since $\mu_1(L_{a^k+d}) = a^k + d \nearrow a + d = \mu_1(L_{a+d})$, we have $\mu_1(B) = a + d$ and $\mu_1(L_{a+d} \setminus B) = 0$. Since μ_1 and μ_2 are equivalent measures, $\mu_2(L_{a+d} \setminus B) = 0$ and $\mu_2(L_{a^k+d}) \nearrow \mu_2(L_{a+d})$.

Now let $a^k \searrow a$. Then $L_{a^k+d} \supset L_{a^{k+1}+d} \supset D \supseteq L_{a+d}$, where $D = \cap_{k=1}^{\infty} L_{a^k+d}$, and $\mu_i(L_{a^k+d}) \searrow \mu_i(D)$, $\mu_i(L_{a+d}) = \mu_i(D)-\mu_i(D\setminus L_{a+d})$ for $i = 1, 2$. Similar to the previous case, $\mu_1(L_{a^k+d}) = a^k + d \searrow a + d = \mu_1(L_{a+d})$, so $\mu_1(D) = a + d$, $\mu_2(D \setminus L_{a+d}) = \mu_1(D \setminus L_{a+d}) = 0$, and $\mu_2(D) = \mu_2(L_{a+d})$. Thus, $\mu_2(L_{a^k+d}) \searrow \mu_2(L_{a+d})$. ☐

Observe that a point $q \in \mathbb{R}^2$ is on the upper (lower) boundary of $R_\mu(X)$, if and only if $q \in R_\mu(X)$ and $q'_2 \leq q_2$ ($q'_2 \geq q_2$) for any $q' \in R_\mu(X)$ with $q'_1 = q_1$.

Lemma 3.2.12. *Under Assumption 3.2.7, a point $q \in \mathbb{R}^2$ is on the lower boundary of $R_\mu(X)$ if and only if $0 \leq q_1 \leq \mu_1(X)$ and $q_2 = \mu_2(L_{q_1})$, and it is on the upper boundary of $R_\mu(X)$ if and only if $0 \leq q_1 \leq \mu_1(X)$ and $q_2 = \mu_2\left(X \setminus L_{\mu_1(X)-q_1}\right)$.*

Proof. For the lower boundary, let $q_2 = \mu_2(L_{q_1})$. Since $q_1 = \mu_1(L_{q_1})$, we have $q = \mu(L_q) \in R_\mu(X)$. For any set $Z \in \mathcal{F}$ with $\mu_1(Z) = q_1$, define disjoint sets $Z_1 = Z \setminus L_{q_1}$, $Z_2 = L_{q_1} \setminus Z$, and $M = Z \cap L_{q_1}$. Then $Z = Z_1 \cup M$, $L_{q_1} = Z_2 \cup M$, and $\mu_1(Z_1) = \mu_1(Z_2)$, since $\mu_1(Z) = q_1 = \mu_1(L_{q_1})$. Furthermore, $Z_1 \subseteq \{f(x) \geq l_{q_1}\}$ and $Z_2 \subseteq \{f(x) \leq l_{q_1}\}$. Therefore,

$$\begin{aligned}
\mu_2(Z_1) &= \int_{Z_1} f(x)\,\mu_1(dx) \geq l_{q_1} \int_{Z_1} \mu_1(dx) \\
&= l_{q_1} \int_{Z_2} \mu_1(dx) \geq \int_{Z_2} f(x)\,\mu_1(dx) = \mu_2(Z_2).
\end{aligned}$$

So $\mu_2(Z) = \mu_2(Z_1) + \mu_2(M) \geq \mu_2(Z_2) + \mu_2(M) = \mu_2(L_{q_1})$, and thus q is on the lower boundary of $R_\mu(X)$.

If q is on the lower boundary of $R_\mu(X)$, then $q_2 \leq \mu_2(L_{q_1})$. Since $q \in R_\mu(X)$, there exists $Z \in \mathcal{F}$ with $\mu(Z) = q$. But, as proved above, $\mu_2(Z) \geq \mu_2(L_{q_1})$ for any $Z \in \mathcal{F}$ with $\mu_1(Z) = q_1$. Thus $q_2 \geq \mu_2(L_{q_1})$. Therefore, $q_2 = \mu_2(L_{q_1})$.

The statement on the upper boundary follows from the symmetry of the range $R_\mu(X)$. ☐

Lemma 3.2.13. *Under Assumption 3.2.7, given $u = (u_1, u_2) \in R_\mu(X)$, there exists $a^* \in [0, \mu_1(X) - u_1]$ such that $\mu(M_{a^*,u_1}) = u$.*

Proof. Since $\mu_1(L_0) = 0$ and μ_1 and μ_2 are equivalent, $\mu_2(L_0) = 0$. Therefore, $g_{u_1}(0) = \mu_2(L_{u_1} \setminus L_0) = \mu_2(L_{u_1}) - \mu_2(L_0) = \mu_2(L_{u_1})$. Similarly, since $\mu_1\left(X \setminus L_{\mu_1(X)}\right) = \mu_1(X) -$

$\mu_1\left(L_{\mu_1(X)}\right)=0$ and μ_1 and μ_2 are equivalent, $\mu_2\left(X\setminus L_{\mu_1(X)}\right)=0$. Thus

$$\begin{aligned} g_{u_1}(\mu_1(X)-u_1) &= \mu_2\left(L_{\mu_1(X)}\setminus L_{\mu_1(X)-u_1}\right)=\mu_2\left(L_{\mu_1(X)}\right)-\mu_2\left(L_{\mu_1(X)-u_1}\right) \\ &= \mu_2(X)-\mu_2\left(X\setminus L_{\mu_1(X)}\right)-\mu_2\left(L_{\mu_1(X)-u_1}\right) \\ &= \mu_2\left(X\setminus L_{\mu_1(X)-u_1}\right). \end{aligned}$$

According to Lemma 3.2.12, the point $(u_1, g_{u_1}(0))$ is on the lower boundary of the range $R_\mu(X)$ and the point $(u_1, g_{u_1}(\mu_1(X)-u_1))$ is on the upper boundary of the range $R_\mu(X)$. So $u_2\in[g_{u_1}(0), g_{u_1}(\mu_1(X)-u_1)]$. Since $g_{u_1}(a)$ is continuous in $a\in[0,\mu_1(X)-u_1]$, there exists a^*, such that $g_{u_1}(a^*)=u_2$. That is, $\mu_2(M_{a^*,u_1})=u_2$. By definition, $\mu_1(M_{a^*,u_1})=u_1$. Therefore, $\mu(M_{a^*,u_1})=u$. □

Note that Lemmas 3.2.8, and 3.2.10-3.2.13 hold if one replaces everywhere the set X with any measurable subset $Z\in\mathcal{F}$. In particular, expressions such as $\{f(x)<l\}$ should be replaced with $\{x\in Z: f(x)<l\}$. We define explicitly

$$l_a(Z)=\min\{l\geq 0:\mu_1(\{x\in Z: f(x)\leq l\})\geq a\}. \tag{3.2.5}$$

Let $Z^l=\{x\in Z: f(x)=l\}$. As follows from Ross [22, Theorem 2(LT3)], for each $l\in[0,\infty)$, there exists a family

$$\left\{W_b\left(Z^l\right)\in\mathcal{F}_{Z^l}: b\in\left[0,\mu_1\left(Z^l\right)\right]\right\}$$

such that (1) $\mu_1\left(W_b\left(Z^l\right)\right)=b$ for each $b\in\left[0,\mu_1\left(Z^l\right)\right]$ and (2) $W_b\left(Z^l\right)\subset W_{b'}\left(Z^l\right)\subseteq Z^l$ whenever $b<b'\leq\mu_1\left(Z^l\right)$. Again, we fix a family of $W_b\left(Z^l\right)$ for each $l\in[0,\infty)$ and each Z, and define

$$L_a(Z)=\{x\in Z: f(x)<l_a\}\cup W_c\left(Z^{l_a}\right),$$

where $c=a-\mu_1(\{x\in Z: f(x)<a\})$. Note that $l_a(X)=l_a$ and $L_a(X)=L_a$, for each $a\in[0,\mu_1(X)]$. In the following two lemmas and their proofs, for a given $u\in R_\mu(X)$, we consider a point $a^*\in[0,\mu_1(X)-u_1]$ with $\mu(M_{a^*,u_1})=u$ and the set $Z=X\setminus M_{a^*,u_1}$. The existence of a^* is stated in Lemma 3.2.13. Later it will become clear that that Z is the maximal subset with the vector measure $p=\mu(X)-u$ and M_{a^*,u_1} is the the minimal subset with the vector measure $p=u$.

Lemma 3.2.14. *Let Assumption 3.2.7 hold. For a given $u=(u_1,u_2)\in R_\mu(X)$, consider*

$a^* \in [0, \mu_1(X) - u_1]$ *such that* $\mu\left(M_{a^*,u_1}\right) = u$. *Then*

$$\mu_2\left(L_a(Z)\right) = \begin{cases} \mu_2\left(L_a\right), & \text{if } a \in [0, a^*]\,; \\ \mu_2\left(L_{a+u_1} \setminus M_{a^*,u_1}\right), & \text{if } a \in (a^*, \mu_1\left(X\right) - u_1]\,. \end{cases} \tag{3.2.6}$$

Proof. First, consider the case $a \in [0, a^*]$. We have $Z = X \setminus M_{a^*,u_1} \supseteq L_{a^*} \supseteq L_a = \{f(x) < l_a\} \cup W_c\left(X^{l_a}\right)$, where $c = a - \mu_1\left(\{f(x) < l_a\}\right)$. In addition, $\{f(x) < l_a\} \cup W_c\left(X^{l_a}\right) \subseteq \{f(x) \le l_a\}$. Therefore

$$\begin{aligned} &\mu_1\left(\{x \in Z : f(x) \le l_a\}\right) \\ &\quad = \mu_1(Z \cap \{f(x) \le l_a\}) \ge \mu_1\left(Z \cap (\{f(x) < l_a\} \cup W_c(X^{l_a}))\right) \\ &\quad = \mu_1\left(\{f(x) < l_a\} \cup W_c(X^{l_a})\right) = a. \end{aligned}$$

Thus, (3.2.5) implies that $l_a(Z) \le l_a$. On the other hand, take an arbitrary $l < l_a$. Since $Z \subseteq X$,

$$\mu_1\left(\{x \in Z : f(x) \le l\}\right) \le \mu_1\left(\{f(x) \le l\}\right) < a.$$

Therefore, $l_a(Z) > l$ for all $l < l_a$. Thus, $l_a(Z) \ge l_a$. We conclude that $l_a(Z) = l_a$.

Denote $A = \{f(x) < l_a\}$. Since $Z \supseteq L_a \supseteq A$ and $l_a(Z) = l_a$, then $\{x \in Z : f(x) < l_a(Z)\} = A$. By definition, each of the sets L_a and $L_a(Z)$ is the union of two disjoint subsets: $L_a = A \cup W_c(X^{l_a})$ and $L_a(Z) = A \cup W_b(Z^{l_a})$ with $c = a - \mu_1(A) = b$. Thus, since $X^{l_a} \supseteq Z^{l_a}$ and $f(x) = l_a$ when $x \in X^{l_a}$, we have $\mu_2(W_c(X^{l_a})) = \mu_2(W_c(Z^{l_a})) = l_a c$. So, $\mu_2(L_a(Z)) = \mu_2(A) + \mu_2(W_c(Z^{l_a})) = \mu_2(A) + \mu_2(W_c(X^{l_a})) = \mu_2(L_a)$.

Second, consider the case $a \in (a^*, \mu_1(X) - u_1]$. Observe that

$$M_{a^*,u_1} \subseteq L_{a^*+u_1} \subset L_{a+u_1} = \{f(x) < l_{a+u_1}\} \cup W_c\left(X^{l_{a+u_1}}\right),$$

where $c = a + u_1 - \mu_1\left(\{f(x) < l_{a+u_1}\}\right)$. In addition,

$$\{f(x) < l_{a+u_1}\} \cup W_c\left(X^{l_{a+u_1}}\right) \subseteq \{f(x) \le l_{a+u_1}\}.$$

Therefore,

$$
\begin{aligned}
&\mu_1\left(\{x \in Z : f(x) \le l_{a+u_1}\}\right) \\
&\quad = \mu_1\left(\{f(x) \le l_{a+u_1}\} \cap Z\right) = \mu_1\left(\{f(x) \le l_{a+u_1}\} \setminus M_{a^*,u_1}\right) \\
&\quad \ge \mu_1\left(\{f(x) < l_{a+u_1}\} \cup W_c\left(X^{l_{a+u_1}}\right) \setminus M_{a^*,u_1}\right) \\
&\quad = a + u_1 - u_1 = a.
\end{aligned}
$$

Thus, (3.2.5) implies that $l_a(Z) \le l_{a+u_1}$. On the other hand, observe that $M_{a^*,u_1} \subseteq \{f(x) \le l_a(Z)\}$. Indeed, since $a > a^*$, we have $l_a(Z) \ge l_{a^*}(Z) = l_{a^*}$. Assume $l_{a^*} \le l_a(Z) < l_{a^*+u_1}$, then $\{x \in Z : f(x) \le l_a(Z)\} = \{f(x) \le l_a(Z)\} \setminus M_{a^*,u_1} = L_{a^*}$, and $a = \mu_1(\{x \in Z : f(x) \le l_a(Z)\}) = \mu_1(L_{a^*}) = a^*$, which is a contradiction. Therefore, $l_a(Z) \ge l_{a^*+u_1}$ and $M_{a^*,u_1} \subseteq \{f(x) \le l_{a^*+u_1}\} \subseteq \{f(x) \le l_a(Z)\}$. Thus, $\{x \notin Z : f(x) \le l_a(Z)\} = \{x \in M_{a^*,u_1} : f(x) \le l_a(Z)\} = M_{a^*,u_1}$ and

$$
\mu_1\left(\{f(x) \le l_a(Z)\}\right) = \mu_1\left(\{x \in Z : f(x) \le l_a(Z)\}\right) + \mu_1\left(M_{a^*,u_1}\right) \ge a + u_1,
$$

where the last step follows from property (b) in Lemma 3.2.10. Formula (3.2.1) implies that $l_a(Z) \ge l_{a+u_1}$. Therefore, $l_a(Z) = l_{a+u_1}$.

Consider again the identity $L_{a+u_1} = \{f(x) < l_{a+u_1}\} \cup W_c(X^{l_a+u_1})$, where the sets in the union are disjoint and $c = (a+u_1) - \mu_1(\{f(x) < l_{a+u_1}\})$. Similarly, $L_a(Z) = \{x \in Z : f(x) < l_{a+u_1}\} \cup W_b(Z^{l_a+u_1})$, where $b = a - \mu_1(\{x \in Z : f(x) < l_a(Z)\})$. Since $l_a(Z) = l_{a+u_1}$ and $\{f(x) < l_{a+u_1}\} \supset M_{a^*,u_1}$, we have $b = a - \mu_1(\{x \in Z : f(x) < l_{a+u_1}\}) = a - \mu_1(\{f(x) < l_{a+u_1}\} \setminus M_{a^*,u_1}) = (a+u_1) - \mu_1(\{f(x) < l_{a+u_1}\}) = c$. Thus,

$$
\begin{aligned}
\mu_2(L_a(Z)) &= \mu_2(\{x \in Z : f(x) < l_a(Z)\}) + \mu_2(W_b(Z^{l_a(Z)})) \\
&= \mu_2(\{x \in Z : f(x) < l_{a+u_1}\}) + l_{a+u_1}\mu_1(W_b(Z^{l_{a+u_1}})) \\
&= \mu_2(\{f(x) < l_{a+u_1}\}) - \mu_2(M_{a^*,u_1}) + l_{a+u_1}\mu_1(W_c(X^{l_{a+u_1}})) \\
&= \mu_2(L_{a+u_1}) - \mu_2(M_{a^*,u_1}) = \mu_2(L_{a+u_1} \setminus M_{a^*,u_1}),
\end{aligned}
$$

where the second equality holds because $l_a(Z) = l_{a+u_1}$, $f(x) = l_{a+u_1}$ for $x \in X^{l_{a+u_1}}$, and $Z^{l_{a+u_1}} \subseteq X^{l_{a+u_1}}$ (in fact $Z^{l_{a+u_1}} = X^{l_{a+u_1}}$, but we do not use this). The third equality holds because of $\{x \in Z : f(x) < l_{a+u_1}\} = \{f(x) < l_{a+u_1}\} \setminus M_{a^*,u_1}$, $\{f(x) < l_{a+u_1}\} \supset M_{a^*,u_1}$, and $b = c$. The fourth equality follows from $l_{a+u_1}\mu_1(W_c(X^{l_{a+u_1}})) = \mu_2(W_c(X^{l_{a+u_1}}))$. $\square$

Lemma 3.2.15. *Let Assumption 3.2.7 hold. For a given $u = (u_1, u_2) \in R_\mu(X)$, consider*

$a^* \in [0, \mu_1(X) - u_1]$ such that $\mu(M_{a^*,u_1}) = u$. Let $q = (q_1, q_2)$ be on the lower (upper) boundary of $R_\mu(Z)$. If $q_1 \in [0, a^*]$ ($q_1 \in [0, \mu_1(X) - u_1 - a^*)$), then q is on the lower (upper) boundary of $R_\mu(X)$ and, if $q_1 \in (a^*, \mu_1(X) - u_1]$ ($q_1 \in [\mu_1(X) - u_1 - a^*, \mu_1(X) - u_1]$), then $r = \mu(X) - u - q$ is on the upper (lower) boundary of $R_\mu(X)$.

Proof. When q is on the lower boundary of $R_\mu(Z)$, according to Lemma 3.2.12, $\mu_2(L_{q_1}(Z)) = q_2$. If $q_1 \in [0, a^*]$, then by Lemma 3.2.14, $\mu_2(L_{q_1}) = \mu_2(L_{q_1}(Z)) = q_2$, and Lemma 3.2.12 implies that q is on the lower boundary of $R_\mu(X)$.

If $q_1 \in (a^*, \mu(X) - u_1]$, then for $r = (r_1, r_2)$

$$\begin{aligned} r_2 &= \mu_2(X) - u_2 - q_2 = \mu_2(X) - \mu_2(M_{a^*,u_1}) - \mu_2(L_{q_1}(Z)) \\ &= \mu_2(X) - (\mu_2(M_{a^*,u_1}) + \mu_2(L_{q_1+u_1} \setminus M_{a^*,u_1})) \\ &= \mu_2(X) - \mu_2(L_{q_1+u_1}) = \mu_2(X \setminus L_{q_1+u_1}) = \mu_2\left(X \setminus L_{\mu_1(X)-r_1}\right), \end{aligned}$$

where the first and last equalities follow from the definition of r, the second equality follows from Lemma 3.2.13, the third equality follows from Lemma 3.2.14, and the fourth equality follows from $q_1 > a^*$. According to Lemma 3.2.12, r is on the upper boundary of $R_\mu(X)$.

If q is on the upper boundary of $R_\mu(Z)$, then, because of symmetry of $R_\mu(Z)$, $r = \mu(X) - u - q$ is on the lower boundary of $R_\mu(Z)$. If $q_1 \in [\mu_1(X) - u_1 - a^*, \mu_1(X) - u_1]$, then $\mu_1(X) - u_1 - q_1 \in [0, a^*]$. From the first part of the proof, $r = \mu(X) - u - q$ is on the lower boundary of $R_\mu(X)$. If $q_1 \in [0, \mu_1(X) - u_1 - a^*)$, then $\mu_1(X) - u_1 - q_1 \in (a^*, \mu_1(X) - u_1]$. Again, from the first part of the proof, $\mu(X) - u - (\mu(X) - u - q_1) = q_1$ is on the upper boundary of $R_\mu(X)$. ◻

Lemma 3.2.16. *Under Assumption 3.2.7, for any vector $p \in R_\mu(X)$, there exists a maximal subset $Z^* \in \mathcal{S}_p(X)$ and, in addition, $R^p_\mu(X) = Q^p_\mu(X)$.*

Proof. For $u = \mu(X) - p$, consider a^* defined in Lemma 3.2.13. For $Z^* = X \setminus M_{a^*,\mu_1(X)-p_1}$, the following three statements are true: (1) $R_\mu(Z^*) \subseteq R^p_\mu(X)$; (2) $R^p_\mu(X) \subseteq Q^p_\mu(X)$; (3) $Q^p_\mu(X) \subseteq R_\mu(Z^*)$.

For (1), $\mu(Z^*) = \mu\left(X \setminus M_{a^*,\mu_1(X)-p_1}\right) = \mu(X) - \mu\left(M_{a^*,\mu_1(X)-p_1}\right) = \mu(X) - (\mu(X) - p) = p$, where the second to the last equality follows from Lemma 3.2.13. Thus, $R_\mu(Z^*) = R^p_\mu(X)$.

For (2), assume that there exists a vector $q \in R^p_\mu(X)$ such that $q \notin Q^p_\mu(X)$. Then Definition 3.1.1 implies that either $q \notin R_\mu(X) - \{\mu(X) - p\}$ or $q \notin R_\mu(X)$. However $q \in R^p_\mu(X) \subseteq R_\mu(X)$. Therefore, $q \notin R_\mu(X) - \{\mu(X) - p\}$, which is equivalent to $p - q \notin \{\mu(X)\} - R_\mu(X) = R_\mu(X)$, where the equality follows from Lemma 3.2.4. Since $R^p_\mu(X) \subseteq$

$R_\mu(X)$, we have $p - q \notin R_\mu^p(X)$. By Lemma 3.2.6, $R_\mu^p(X)$ is centrally symmetric with the center $\frac{p}{2}$. Therefore $q \notin R_\mu^p(X)$. The above contradiction implies (2).

For (3), assume $q \in Q_\mu^p(X)$, but $q \notin R_\mu(Z^*)$. By Lyapunov's theorem $R_\mu(Z^*)$ is a convex compact set. Let $q^u = (q_1, q_2^u)$ and $q^l = \left(q_1, q_2^l\right)$ be the intersection points of the vertical line $\mu_1 = q_1$ and the upper and lower boundaries of $R_\mu(Z^*)$ respectively. Then one of the following must be true: $q_2 > q_2^u$ or $q_2 < q_2^l$. Without loss of generosity, we consider the former case. Since q_u is on the upper boundary of $R_\mu(Z^*)$, according to Lemma 3.2.15, one of the following is true: (a) q^u is on the upper boundary of $R_\mu(X)$ or (b) $r = p - q^u$ is on the lower boundary of $R_\mu(X)$. For (a), $q_2 > q_2^u$ implies $q \notin R_\mu(X)$. Thus $q \notin Q_\mu^p(X)$. This contradicts our assumption. For (b), we let $r' = p - q$. Obviously, $r_1' = r_1$ and $r_2' < r_2$. This implies that r' is below the lower boundary point r. Thus, $r' \notin R_\mu(X)$ and $r' \notin Q_\mu^p(X)$. But according to Lemma 3.2.6, this means $q \notin Q_\mu^p(X)$, which contradicts to our assumption. Statement (1)-(3) imply the lemma. □

Let D be a two-by-two invertible matrix with positive entries, and $A \subseteq \mathbb{R}^2$. We denote by AD the set $\{pD : p \in A\}$. For a vector measure $\mu = (\mu_1, \mu_2)$, let $\nu = \mu D$ be the vector measure $(\nu_1, \nu_2) = (D_{11}\mu_1 + D_{21}\mu_2, D_{12}\mu_1 + D_{22}\mu_2)$. Then the measure ν_1 and ν_2 are equivalent.

Lemma 3.2.17. *(a) $R_\mu(Y)D = R_\nu(Y)$ for all $Y \in \mathfrak{F}$; (b) $R_\mu^p(X)D = R_\nu^{pD}(X)$ for all $p \in R_\mu(X)$; (c) $Q_\mu^p(X)D = Q_\nu^{pD}(X)$ for all $p \in R_\mu(X)$.*

Proof. (a) For any point $q \in R_\nu(Y)$, there exists a set $Z \in \mathfrak{F}_Y$ such that $\nu(Z) = q$. Since $\mu(Z) = qD^{-1}$ and $qD^{-1} \in R_\mu(Y)$, we have $q \in R_\mu(Y)D$. For any point $q \in R_\mu(Y)D$, we have $qD^{-1} \in R_\mu(Y)$. Thus there exists a set $Z \in \mathfrak{F}_Y$ such that $\mu(Z) = qD^{-1}$, and $\nu(Z) = q$. Therefore, $\nu(Z) \in R_\nu(Y)$.

(b) For any point $q \in R_\nu^{pD}(X)$, there exist sets $Y \in \mathfrak{F}$ and $Z \in \mathfrak{F}_Y$ such that $\nu(Y) = pD$ and $\nu(Z) = q$. So $\mu(Y) = p$ and $\mu(Z) = qD^{-1}$. Thus, $qD^{-1} \in R_\mu^p(X)$ and therefore, $q \in R_\mu^p(X)D$. For any point $q \in R_\mu^p(X)D$, we have $qD^{-1} \in R_\mu^p(X)$. So there exist sets $Y \in \mathfrak{F}$ and $Z \in \mathfrak{F}_Y$ such that $\mu(Y) = p$ and $\mu(Z) = qD^{-1}$, and consequently $\nu(Y) = pD$ and $\nu(Z) = q$. Thus $q \in R_\mu^{pD}(X)$.

(c) According to Definition 3.1.1, $Q_\mu^p(X)D = (R_\mu(X)D - \{\mu(X)D - pD\}) \cap R_\mu(X)D = (R_\nu(X) - \{\nu(X) - pD\}) \cap R_\nu(X) = Q_\nu^{pD}(X)$. □

Proof of Theorem 3.2.1. According to Lemma 3.2.16, Theorem 3.2.1 holds under Assumption 3.2.7 that states that μ_1 and μ_2 are equivalent. If μ_1 and μ_2 are not equivalent, consider

$\nu = \mu D$. Since ν_1 and ν_2 are equivalent, $Q^p_\mu(X) = Q^{pD}_\nu(X)D^{-1} = R^{pD}_\nu(X)D^{-1} = R^p_\mu(X)$, where the first equality and the last equality is by Lemma 3.2.17, and the second equality is due to Lemma 3.2.16. Furthermore, according to Lemma 3.2.16, there exists a maximal subset Z^*, such that $R_\nu(Z^*) = R^{pD}_\nu(X)$. Therefore, $R_\mu(Z^*) = R_\nu(Z^*)\,D^{-1} = R^{pD}_\nu(X)D^{-1} = R^p_\mu(X)$. □

Now we consider Theorem 3.2.2. The proof of this theorem uses the following lemma.

Lemma 3.2.18. *Let $A_1, A_2, B_1, B_2 \subseteq \mathbb{R}^2$ be convex and compact sets such that $A_1 \oplus B_1 = A_2 \oplus B_2$ and $B_1 \subseteq B_2$. Then $A_2 \subseteq A_1$.*

Proof. According to [24, Lemma 3.1.8], if $A, B \subseteq \mathbb{R}^2$ are convex and compact sets then $(A \oplus B) \ominus B = A$. Thus if $a \in A_2$, then $a \in (A_2 \oplus B_2) \ominus B_2$, and consequently $a \in (A_1 \oplus B_1) \ominus B_2$. So $a \in (A_1 \oplus B_1) - b$, for any $b \in B_2$. Since $B_1 \subseteq B_2$, we have $a \in (A_1 \oplus B_1) - b$, for any $b \in B_1$, and thus $a \in (A_1 \oplus B_1) \ominus B_1 = A_1$. □

Proof of Theorem 3.2.2. Now let Z^* be the maximal subset with the measure p. Then, $\mu(X \setminus Z^*) = \mu(X) - p$. Consider any set M, such that $\mu(M) = \mu(X) - p$. Obviously, $R_\mu(M) \oplus R_\mu(X \setminus M) = R_\mu(X \setminus Z^*) \oplus R_\mu(Z^*) = R_\mu(X)$. In addition, $R_\mu(X \setminus M) \subseteq R_\mu(Z^*)$ by definition. Thus according to Lemma 3.2.18, $R_\mu(X \setminus Z^*) \subseteq R_\mu(M)$.

Similarly, let $M^* = X \setminus Z^*$ be the minimal subset with the measure $\mu(X) - p$. Then, $\mu(Z^*) = p$. Consider any set Z, such that $\mu(Z) = p$. Obviously, $R_\mu(Z) \oplus R_\mu(X \setminus Z) = R_\mu(Z^*) \oplus R_\mu(X \setminus Z^*) = R_\mu(X)$. In addition, $R_\mu(X \setminus Z^*) = R_\mu(M^*) \subseteq R_\mu(X \setminus Z)$ by definition. Thus according to Lemma 3.2.18, $R_\mu(Z) \subseteq R_\mu(Z^*)$. □

3.3 Union and Intersection of Subranges

Theorem 3.2.1 immediately implies that, when μ is a two-dimensional finite atomless measure on $(X, \mathcal{F})$, the set $R^p_\mu(X)$, which is the union of the ranges of μ on Y, for all $Y \in \mathcal{S}^p_\mu(X)$, is a convex compact set. Furthermore, if $R_\mu(X)$ and p are given, the set $R^p_\mu(X)$ is defined by two simple geometric operations, a shift and an intersection, since $Q^p_\mu(X)$ is defined by these operations.

In this section, we remove the assumption that μ is a two-dimensional measure and prove a theorem (Theorem 3.3.1) on the compactness and convexity of the union $R^p_\mu(X)$. In addition, for any natural number m, we prove that $R^p_\mu(X) \subseteq Q^p_\mu(X)$ (Theorem 3.3.2).

Theorem 3.3.1. *For a measurable space $(X, \mathcal{F})$ with a finite vector measure μ and for any vector $p \in R_\mu(X)$, the set $R^p_\mu(X)$ is compact and, in addition, if the vector measure μ is atomless, this set is convex.*

Theorem 3.3.2. *For a measurable space $(X, \mathcal{F})$ with a finite vector measure μ and for any vector $p \in R_\mu(X)$, $R^p_\mu(X) \subseteq Q^p_\mu(X)$.*

We also give a theorem (Theorem 3.3.3) that links the union $R^p_\mu(X)$ and the intersection $I^{\mu(X)-p}_\mu(X)$.

Theorem 3.3.3. *For a measurable space $(X, \mathcal{F})$ with a finite atomless vector measure μ and for any vector $p \in R_\mu(X)$, $I^{\mu(X)-p}_\mu(X) = R_\mu(X) \ominus R^p_\mu(X)$.*

We first prove Theorem 3.3.1.

Proof of Theorem 3.3.1. We say that a partition is measurable, if all its elements are measurable sets. Consider the set

$$\begin{aligned} V_{\mu,3}(X) \quad = \quad & \{(\mu(S_1), \mu(S_2), \mu(S_3)) : \\ & \{S_1, S_2, S_3\} \text{ is a measurable partition of } X\}. \end{aligned}$$

According to Dvoretzky, Wald, and Wolfowitz [9, Theorems 1 and 4], $V_{\mu,3}(X)$ is compact and, if μ is atomless, this set is convex. Now let

$$\begin{aligned} W^p_\mu(X) \quad = \quad & \{(s_1, s_2, s_3) : (s_1, s_2, s_3) \in \\ & V_{\mu,3}(X),\ s_3 = \mu(X) - p,\ s_1 + s_2 = p\}. \end{aligned}$$

This set is compact and, if μ is atomless, it is convex. This is true, because $W^p_\mu(X)$ is an intersection of $V_{\mu,3}(X)$ and two planes in $\mathbb{R}^{3m}$. These planes are defined by the equations $s_3 = \mu(X) - p$ and $s_1 + s_2 = p$ respectively. We further define

$$S^p_\mu(X) = \left\{s_1 : (s_1, s_2, s_3) \in W^p_\mu(X)\right\}.$$

Since $S^p_\mu(X)$ is a projection of $W^p_\mu(X)$, the set $S^p_\mu(X)$ is compact and, if μ is atomless, it is convex.

The last step of the proof is to show that $S^p_\mu(X) = R^p_\mu(X)$ by establishing that (i) $S^p_\mu(X) \subseteq R^p_\mu(X)$, and (ii) $S^p_\mu(X) \supseteq R^p_\mu(X)$. Indeed, for (i), for any $s_1 \in S^p_\mu(X)$, there exists

$(s_1, s_2, s_3) \in W^p_\mu(X)$ or equivalently there exists a measurable partition $\{S_1, S_2, S_3\}$ of X such that $\mu(S_3) = \mu(X) - p$ and $\mu(S_1) + \mu(S_2) = p$. Let $Z = S_1 \cup S_2$. Then $\mu(Z) = p$, $s_1 \in R_\mu(Z)$, and thus $s_1 \in R^p_\mu(X)$. For (ii), for any $s_1 \in R^p_\mu(X)$, there exists a set $Z \in \mathcal{F}$, such that $\mu(Z) = p$ and $s_1 \in R_\mu(Z)$, which further implies that there exists a measurable subset S_1 of Z such that $\mu(S_1) = s_1$. Let $S_2 = Z \setminus S_1$ and $S_3 = X \setminus Z$. Then $\mu(S_1)+\mu(S_2) = p$ and $\mu(S_3) = \mu(X) - p$, which further implies that $(s_1, \mu(S_2), \mu(S_3)) \in W^p_\mu(X)$. Thus $s_1 \in S^p_\mu(X)$. ☐

Now we consider Theorem 3.3.2. The proof of Theorem 3.3.2 uses the following lemma, which was proved in [5] and presented in Section 3.2 above (Lemma 3.2.6).

Lemma 3.3.4. ([5, Lemma 3.3]) *For any vector $p \in R(X)$, each of the sets $R^p_\mu(X)$ and $Q^p_\mu(X)$ is centrally symmetric with the center $\frac{1}{2}p$.*

Though it is assumed in [5] that the measure μ is atomless, this assumption is not used in the proofs of Lemmas 3.1-3.3 therein.

Proof of Theorem 3.3.2. Let $q \in R^p_\mu(X)$. Since $R^p_\mu(X) \subseteq R_\mu(X)$, then $q \in R_\mu(X)$. Furthermore, in view of Lemma 3.3.4, $p - q \in R^p_\mu(X)$. Therefore, $p - q \in R_\mu(X)$. Since $R_\mu(X)$ is centrally symmetric with the center $\frac{1}{2}\mu(X)$, then $R_\mu(X) = \{\mu(X)\} - R_\mu(X)$ and $p - q \in R_\mu(X) = \{\mu(X)\} - R_\mu(X)$. Therefore, $q \in R_\mu(X) - \{\mu(X) - p\}$. As follows from the definition of $Q^p_\mu(X)$ in Definition 3.1.1, $q \in Q^p_\mu(X)$. ☐

Finally, to prove Theorem 3.3.3, observe that the Minkowski difference of two sets $A, B \subseteq \mathbb{R}^m$ (defined in Section 3.2) can also be written as $A \ominus B = \{r \in \mathbb{R}^m : B + r \subseteq A\}$.

Proof of Theorem 3.3.3. Consider any $q \in I^{\mu(X)-p}_\mu(X)$. Then, for all $Z \in \mathcal{S}^p_\mu$, $R_\mu(Z) + q \subseteq R_\mu(X)$, since $q \in R_\mu(X \setminus Z)$ and $R_\mu(Z) \oplus R_\mu(X \setminus Z) = R(X)$. Thus, $R^p_\mu(X) + q \subseteq R_\mu(X)$, and $q \in R_\mu(X) \ominus R^p_\mu(X)$. Therefore $I^{\mu(X)-p}_\mu(X) \subseteq R_\mu(X) \ominus R^p_\mu(X)$.

On the other hand, consider any $q \in R_\mu(X) \ominus R^p_\mu(X)$. Then $R^p_\mu(X) + q \subseteq R_\mu(X)$. This implies that, for all $Z \in \mathcal{S}^p_\mu$, $R_\mu(Z)+q \subseteq R_\mu(X)$. Thus, $q \in R_\mu(X) \ominus R_\mu(Z) = R_\mu(X \setminus Z)$, for all $Z \in \mathcal{S}^p_\mu$. In other words, $q \in R_\mu(Y)$, for all $Y \in \mathcal{S}^{\mu(X)-p}_\mu$. It follows that $q \in I^{\mu(X)-p}_\mu(X)$. Therefore $I^{\mu(X)-p}_\mu(X) \supseteq R_\mu(X) \ominus R^p_\mu(X)$. ☐

Chapter 4

Counterexamples

In this chapter, we give some counterexamples to demonstrate that the theorems we have presented in Chapter 3 do not hold any more if various assumptions are removed from these theorems. In Section 4.1, we show that the maximal and minimal subranges may not exist when the dimension of the vector measure $m > 2$ or the vector measure μ is atomic. In Section 4.2, we show that the union of subranges $R_\mu^p(X)$ may not be equal to $Q_\mu^p(X)$ when $m > 2$ or μ is atomic.

4.1 Maximal and Minimal Subranges

In this section, we present an example of a measurable space $(X, \mathcal{F})$ endowed with a three-dimensional atomless finite measure $\nu = (\nu_1, \nu_2, \nu_3)$ and a vector $p \in R_\nu(X)$ such that a maximal subset of X with the measure p does not exist. Theorem 3.2.2 implies that the minimal subset does not exist either in this example.

Recall that, with respect to a measure μ, set A and B are said to be equal up to null sets (denoted by $A \simeq B$) if $\mu(A \setminus B) = \mu(B \setminus A) = 0$. Also recall that $X^l = \{f(x) = l\}$.

Proposition 4.1.1. *Let $\mu = (\mu_1, \mu_2)$ satisfy Assumption 3.2.7 and $Y \in \mathcal{F}$. If $\mu_1 \left\{X^{l_{\mu_1(Y)}}\right\} = 0$ and $\mu_2(Y) = \mu_2\left(L_{\mu_1(Y)}\right)$, then $Y \simeq L_{\mu_1(Y)}$.*

Proof. Assume that $Y \simeq L_{\mu_1(Y)}$ does not hold. We define three disjoint sets $Z_1 = Y \setminus L_{\mu_1(Y)}$, $Z_2 = L_{\mu_1(Y)} \setminus Y$, and $M = Y \cap L_{\mu_1(Y)}$. Observe that $Y = Z_1 \cup M$ and $L_{\mu_1(Y)} = Z_2 \cup M$. These equalities and $\mu_1(Y) = \mu_1(L_{\mu_1(Y)})$ imply $\mu_1(Z_1) = \mu_1(Z_2)$. Furthermore, $Z_1 \subseteq \left\{f(x) \geq l_{\mu_1(Y)}\right\}$ and $Z_2 \subseteq \left\{f(x) < l_{\mu_1(Y)}\right\}$, because according to (3.2.4), $L_{\mu_1(Y)} =$

$\{f(x) < l_{\mu_1(Y)}\}$ when $\mu_1\{X^{l_{\mu_1(Y)}}\} = 0$. Therefore,

$$\begin{aligned}\mu_2(Z_1) &= \int_{Z_1} f(x)\,\mu_1(dx) \geq l_{\mu_1(Y)} \int_{Z_1} \mu_1(dx) \\ &= l_{\mu_1(Y)} \int_{Z_2} \mu_1(dx) > \int_{Z_2} f(x)\,\mu_1(dx) = \mu_2(Z_2).\end{aligned}$$

So $\mu_2(Y) = \mu_2(Z_1) + \mu_2(M) > \mu_2(Z_2) + \mu_2(M) = \mu_2(L_{\mu_1(Y)})$. This contradiction implies the proposition. $\square$

Example 4.1.2. Let $X = [0,1]$ and $\mathcal{F}$ is the Borel σ-field. Consider the three-dimensional vector measure $\nu(dx) = (\nu_1, \nu_2, \nu_3)(dx) = (1, 2x, \rho(x))\,dx$, where

$$\rho(x) = \begin{cases} 4x, & \text{if } x \in \left[0, \frac{1}{2}\right); \\ 4x - 2, & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Consider the points $p = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $q^1 = \left(\frac{1}{4}, \frac{1}{16}, \frac{1}{8}\right)$, and $q^2 = \left(\frac{1}{4}, \frac{5}{32}, \frac{1}{16}\right)$. It is easy to show that $q^1, q^2 \in R_\nu^p(X)$. Indeed let $Z^1 = \left[0, \frac{1}{4}\right) \cup \left[\frac{3}{4}, 1\right]$, $Z^2 = \left[0, \frac{1}{8}\right) \cup \left[\frac{3}{8}, \frac{5}{8}\right) \cup \left[\frac{7}{8}, 1\right]$, $W^1 = \left[0, \frac{1}{4}\right) \subseteq Z^1$, and $W^2 = \left[0, \frac{1}{8}\right) \cup \left[\frac{1}{2}, \frac{5}{8}\right) \subseteq Z^2$, and we have $\nu(Z^1) = \nu(Z^2) = p$, $\nu(W^1) = q^1$, and $\nu(W^2) = q^2$. Since Z^1 and Z^2 are not equal up to a null set, Proposition 4.1.3 implies that there does not exist a set Z such that $\nu(Z) = p$ and $q^1, q^2 \in R_\mu(Z)$.

Proposition 4.1.3. *Consider the sets X, Z^1, Z^2, the measure ν and vectors p, q^1, q^2 from Example 4.1.2. Let $Z \in \mathcal{S}_\nu^p(X)$. For each $i = 1, 2$, if $q^i \in R_\nu(Z)$, then $Z \simeq Z^i$.*

Proof. Let $i = 1$. Since $q^1 \in R_\nu(Z)$, there exists a set $W^1 \in \mathcal{F}_Z$ such that $\nu(W^1) = q^1$. Define two-dimensional vector measure $\mu = (\mu_1, \mu_2) = (\nu_1, \nu_2)$. Then $\mu(W^1) = \left(\frac{1}{4}, \frac{1}{16}\right)$. Observe that, according to (3.2.1) and (3.2.4), $l_{\mu_1(W^1)} = l_{\frac{1}{4}} = \frac{1}{2}$ and $L_{\mu_1(W^1)} = L_{\frac{1}{4}} = \left[0, \frac{1}{4}\right)$. In addition, $\mu_1\left(X^{l_{\mu_1(W^1)}}\right) = 0$ and $\mu_2(W^1) = \frac{1}{16} = \mu_2\left(L_{\mu_1(W^1)}\right)$. Therefore, according to Proposition 4.1.1, $W^1 \simeq L_{\mu_1(W^1)} = \left[0, \frac{1}{4}\right)$. On the other hand, let $Y = W^1 \cup (X \setminus Z)$. Since $W^1 \subseteq Z$, $\nu(Y) = \nu(W^1) + (\nu(X) - \nu(Z)) = q^1 + (\nu(X) - p) = \left(\frac{3}{4}, \frac{9}{16}, \frac{5}{8}\right)$, and thus, $\mu(Y) = \left(\frac{3}{4}, \frac{9}{16}\right)$. Observe that, according to (3.2.1) and (3.2.4), $l_{\mu_1(Y)} = l_{\frac{3}{4}} = \frac{3}{2}$ and $L_{\mu_1(Y)} = L_{\frac{3}{4}} = \left[0, \frac{3}{4}\right)$. In addition, $\mu_1\left(X^{l_{\mu_1(Y)}}\right) = 0$ and $\mu_2(Y) = \frac{9}{16} = \mu_2\left(L_{\mu_1(Y)}\right)$. Therefore, according to Proposition 4.1.1, $Y \simeq L_{\mu_1(Y)} = L_{\frac{3}{4}}$. Above observations imply that $Z = W^1 \cup (X \setminus Y) \simeq L_{\frac{1}{4}} \cup \left(X \setminus L_{\frac{3}{4}}\right) = \left[0, \frac{1}{4}\right) \cup \left([0,1] \setminus \left[0, \frac{3}{4}\right)\right) = \left[0, \frac{1}{4}\right) \cup \left[\frac{3}{4}, 1\right] = Z^1$.

Let $i = 2$. Since $q^2 \in R_\nu(Z)$, there exists a set $W^2 \in \mathcal{F}_Z$ such that $\nu(W^2) = q^2$. Define two-dimensional vector measure $\mu = (\mu_1, \mu_2) = (\nu_1, \nu_3)$. Then $\mu(W^2) = \left(\frac{1}{4}, \frac{1}{16}\right)$. Observe that, according to (3.2.1) and (3.2.4), $l_{\mu_1(W^2)} = l_{\frac{1}{4}} = \frac{1}{2}$ and $L_{\mu_1(W^2)} = L_{\frac{1}{4}} = \left[0, \frac{1}{8}\right) \cup \left[\frac{1}{2}, \frac{5}{8}\right)$. In addition, $\mu_1\left(X^{l_{\mu_1(W^2)}}\right) = 0$ and $\mu_2(W^2) = \frac{1}{16} = \mu_2\left(L_{\mu_1(W^2)}\right)$. Therefore, according to Proposition 4.1.1, $W^2 \simeq L_{\mu_1(W^2)} = \left[0, \frac{1}{8}\right) \cup \left[\frac{1}{2}, \frac{5}{8}\right)$. On the other hand, let $Y = W^2 \cup (X \setminus Z)$. Since $W^2 \subseteq Z$, $\nu(Y) = \nu(W^2) + (\nu(X) - \nu(Z)) = q^2 + (\nu(X) - p) = \left(\frac{3}{4}, \frac{21}{32}, \frac{9}{16}\right)$, and thus, $\mu(Y) = \left(\frac{3}{4}, \frac{9}{16}\right)$. Observe that, according to (3.2.1) and (3.2.4), $l_{\mu_1(Y)} = l_{\frac{3}{4}} = \frac{3}{2}$ and $L_{\mu_1(Y)} = L_{\frac{3}{4}} = \left[0, \frac{3}{8}\right) \cup \left[\frac{1}{2}, \frac{7}{8}\right)$. In addition, $\mu_1\left(X^{l_{\mu_1(Y)}}\right) = 0$ and $\mu_2(Y) = \frac{9}{16} = \mu_2\left(L_{\mu_1(Y)}\right)$. Therefore, according to Proposition 4.1.1, $Y \simeq L_{\mu_1(Y)} = L_{\frac{3}{4}}$. Above observations imply that $Z = W^2 \cup (X \setminus Y) \simeq L_{\frac{1}{4}} \cup \left(X \setminus L_{\frac{3}{4}}\right) = \left(\left[0, \frac{1}{8}\right) \cup \left[\frac{1}{2}, \frac{5}{8}\right)\right) \cup \left([0,1] \setminus \left(\left[0, \frac{3}{8}\right) \cup \left[\frac{1}{2}, \frac{7}{8}\right)\right)\right) = \left[0, \frac{1}{8}\right) \cup \left[\frac{3}{8}, \frac{5}{8}\right) \cup \left[\frac{7}{8}, 1\right] = Z^2$. □

The following counterexample shows that, if μ is atomic, then even for $m = 1$ (and, therefore, for any natural number m) a maximal subset Z^*, defined in Definition 3.1.2, may not exist.

Example 4.1.4. Consider the probability space $(X, 2^X, \mu)$, where

$$X = \{1, 2, 3, 4\},$$

and

$$\mu(\{1\}) = 0.1,\ \mu(\{2\}) = 0.4,\ \mu(\{3\}) = 0.2,\ \mu(\{3\}) = 0.3.$$

Let $p = 0.5$. Then $\mathcal{S}_\mu^p = \{\{1, 2\}, \{3, 4\}\}$. In other words, the only subsets that have the measure 0.5 are $Z^1 = \{1, 2\}$ and $Z^2 = \{3, 4\}$. However, $R_\mu(Z^1)$ is not a subset of $R_\mu(Z^2)$ and vice versa. Therefore, a maximal subset does not exist for $p = 0.5$. □

4.2 Union and Intersection of Subranges

The following example shows that the equality $R_\mu^p(X) = Q_\mu^p(X)$ may not hold when μ is atomless and $m > 2$. In particular, the inclusion in Theorem 3.3.2 cannot be substituted with the equality.

Example 4.2.1. Consider the measure space $(X, \mathcal{F}, \mu)$ described in Example 1.1.5. Note that $\mu(X) = (110, 130, 125)$ and

$$R_\mu(X) = \left\{ \sum_{i=1}^{6} \alpha_i p^i : \alpha_i \in [0, 1],\ i = 1, \ldots, 6 \right\}$$

is a zonotope, where $p^1 = \mu([0,1)) = (30,40,10)$, $p^2 = \mu([1,2)) = (40,10,20)$, $p^3 = \mu([2,3)) = (10,20,20)$, $p^4 = \mu([3,4)) = (10,20,30)$, $p^5 = \mu([4,5)) = (15,10,20)$, $p^6 = \mu([5,6)) = (5,30,25)$.

Let $p = p^1 + p^2 + p^3 = (80,70,50)$. Observe that p is an extreme point of $R_\mu(X)$. Indeed, consider the vector $d = \left(\frac{7}{5}, 1, -\frac{8}{5}\right)$ and the linear function $l_d(\alpha)$ defined for all $\alpha \in \mathbb{R}^6$ by the scalar product

$$\begin{aligned} l_d(\alpha) &= d \cdot \left(\sum_{i=1}^{6} \alpha_i p^i\right) = \sum_{i=1}^{6} \alpha_i (d \cdot p^i) \\ &= 66\alpha_1 + 34\alpha_2 + 2\alpha_3 - 14\alpha_4 - \alpha_5 - 3\alpha_6. \end{aligned}$$

For $\alpha \in [0,1]^6$, this function achieves maximum at the unique point $\alpha^* = (1,1,1,0,0,0)$, and $l_d(\alpha^*) = 66 + 34 + 2 = 102$. In addition, $\sum_{i=1}^{6} \alpha_i^* p^i = p$. So, $d \cdot r - 102 \leq 0$ for all $r \in R_\mu(X)$ and the equality holds if and only if $r = p$. Thus, $d \cdot r - 102 = 0$ is a supporting hyperplane of the convex polytope $R_\mu(X)$, and the intersection of the polytope and hyperplane consists of the single point p. This implies that p is an extreme point of $R_\mu(X)$.

According to the definition of $R_\mu(X)$, for $p \in R_\mu(X)$ there exists a measurable subset $Z \in \mathfrak{F}$ such that $\mu(Z) = p$ and, according to [17, Theorem III] described in Section 2.1, since p is extreme, such Z is unique up to null sets. In particular, $p = \mu(Z)$ for $Z = [0,3]$. Thus,

$$R_\mu^p(X) = R_\mu(Z) = \left\{ \sum_{i=1}^{3} \alpha_i p^i : \alpha_i \in [0,1],\ i = 1,2,3 \right\}.$$

Choose $q = (56,29,31)$ and observe that $q \notin R_\mu^p(X)$. Indeed, $q \in R_\mu^p(X)$ if and only if there exist $\alpha_1, \alpha_2, \alpha_3 \in [0,1]$, such that $\sum_{i=1}^{3} \alpha_i p^i = q$, which is equivalent to

$$\alpha_1(30,40,10) + \alpha_2(40,10,20) + \alpha_3(10,20,20) = (56,29,31), \tag{4.2.1}$$

but the only solution to the linear system of equations (4.2.1) is

$$\alpha_1 = \frac{3}{10},\ \alpha_2 = \frac{11}{10},\ \alpha_3 = \frac{3}{10},$$

where $\alpha_2 \notin [0,1]$.

On the other hand, $q \in Q_\mu^p(X)$, because: (i) $q \in R_\mu(X)$, and (ii) $q \in R_\mu(X) - \{\mu(X) - p\}$.

Indeed, (i) holds since, for $Z_1 = \left[0, \frac{42}{115}\right) \cup \left[1, 1\frac{229}{230}\right) \cup \left[2, 2\frac{33}{460}\right) \cup \left[4, 4\frac{3}{10}\right)$,

$$\begin{aligned}
\mu\left(Z_1\right) &= \frac{42}{115} \times (30, 40, 10) + \frac{229}{230} \times (40, 10, 20) \\
&+ \frac{33}{460} \times (10, 20, 20) + \frac{3}{10} \times (15, 10, 20) \\
&= (56, 29, 31) = q.
\end{aligned}$$

Notice that (ii) is equivalent to $q + \mu(X) - p \in R_\mu(X)$, where $q + \mu(X) - p = (56, 29, 31) + (110, 130, 125) - (80, 70, 50) = (86, 89, 106)$. Let $Z_2 = \left[0, \frac{15}{46}\right) \cup \left[1, 1\frac{45}{46}\right) \cup \left[2, 2\frac{209}{230}\right) \cup [3, 5) \cup \left[5, 5\frac{3}{5}\right)$. Then

$$\begin{aligned}
\mu\left(Z_2\right) &= \frac{15}{46} \times (30, 40, 10) + \frac{45}{46} \times (40, 10, 20) + \frac{209}{230} \times (10, 20, 20) \\
&+ 1 \times (10, 20, 30) + 1 \times (15, 10, 20) + \frac{3}{5} \times (5, 30, 25) \\
&= (86, 89, 106) = q + \mu(X) - p.
\end{aligned}$$

Thus (ii) holds too, and $R^p_\mu(X) \neq Q^p_\mu(X)$. $\square$

In conclusion of this section, we provide a simple example showing that, if μ is not atomless, then even for $m = 1$ (and, therefore, for any natural number m), the equality $R^p_\mu(X) = Q^p_\mu(X)$ may not hold.

Example 4.2.2. Consider the probability space $(X, 2^X, \mu)$, where

$$X = \{1, 2, 3\}$$

and

$$\mu(\{1\}) = 0.1,\ \mu(\{2\}) = 0.55,\ \mu(\{3\}) = 0.35.$$

The range of μ on X is $R_\mu(X) = \{0, 0.1, 0.35, 0.45, 0.55, 0.65, 0.9, 1\}$. Let $p = 0.55$. Then $Q^p_\mu = \{0, 0.1, 0.45, 0.55\}$ and $R^p_\mu = \{0.55\}$. Thus $R^p_\mu \subset Q^p_\mu$, but $R^p_\mu \neq Q^p_\mu$. $\square$

Chapter 5

Application to Purification Theorems

In this chapter we present a theorem (Theorem 5.1.1) that strengthens Theorem 1.2.2 for two-dimensional vector measures. We prove this theorem in Section 5.1, and give a counterexample to show that this stronger result does not hold when $m > 2$ in Section 5.2.

5.1 Partition of the State Space

We will use Theorem 3.2.1 to prove the following theorem.

Theorem 5.1.1. *Consider a measurable space $(X, \mathcal{F})$ with a two-dimensional finite atomless vector measure μ, a countable set A, and a countable set of two-dimensional vectors $\{p^a : a \in A\}$. A partition $\{Z^a \in \mathcal{F} : a \in A\}$ of X, with $p^a = \mu(Z^a)$ for all $a \in A$, exists if and only if (i) $\sum_{a \in A} p^a = \mu(X)$ and (ii) $\sum_{a \in B} p^a \in R_\mu(X)$ for any finite subset $B \subset A$.*

For any $B \subseteq A$, denote $p(B) = \sum_{a \in B} p^a$, where either $A = \{1, 2, \ldots\}$ or $A = \{1, \ldots, n\}$ for some $n = 1, 2, \ldots$.

Lemma 5.1.2. *Let $\mu = (\mu_1, \mu_2)$ be a two-dimensional finite atomless measure. If $p(B) \in R_\mu(X)$ for all $B \subset A$ and $\sum_{a \in A} p^a = \mu(X)$, then there exists a partition $\{Z^a \in \mathcal{F} : a \in A\}$ of X, such that $p^a = \mu(Z^a)$ for each $a \in A$.*

Proof. Consider $p = \mu(X) - p^1$. According to Theorem 3.2.1, there exists a maximal subset $Z^* \in \mathcal{S}^p_\mu(X)$ and $R_\mu(Z^*) = Q^p_\mu(X)$. Let $Z^1 = X \setminus Z^*$, $X^1 = Z^*$, and $A^1 = A \setminus \{1\}$. Note that $p^1 = \mu(Z^1)$ and $p(B) \in R_\mu(X^1)$ for all $B \subseteq A^1$. Indeed, $p(B) + p^1 = p(B \cup \{1\}) \in R_\mu(X)$. Thus, $p(B) \in R_\mu(X) - \{(\mu(X) - p)\}$, and in addition $p(B) \in R_\mu(X)$. Therefore, $p(B) \in Q^p_\mu(X) = R_\mu(X^1)$.

Now for $p^2 \in \{p^a : a \in A^1\}$ there exists a maximal subset $Z^* \in \mathcal{S}_\mu^p(X^1)$, where $p = \mu(X^1) - p^2$. Let $Z^2 = X^1 \setminus Z^*$, $X^2 = Z^*$, and $A^2 = A^1 \setminus \{2\}$, then $p^2 = \mu(Z^2)$ and $p(B) \in R_\mu(X^2)$ for all $B \subseteq A^2$. The repetition of this procedure generates the desired partition $\{Z^a \in \mathcal{F} : a \in A\}$. □

Proof of Theorem 2.5. The necessity is obvious. For the sufficiency, in view of Lemma 5.1.2, it is enough to prove that condition (ii) implies $p(B) \in R_\mu(X)$ for all $B \subseteq A$. If B is finite, condition (ii) implies $p(B) \in R_\mu(X)$. If B is infinite, let $B = \{a^1, a^2, \dots\}$ and $B_n = \{a^1, a^2, \dots a^n\}$, $n = 1, 2, \dots$. Then $p(B) = \lim_{n\to\infty} p(B_n)$ and $p(B_n) \in R_\mu(X)$ for $n = 1, 2, \dots$, according to condition (ii). Since $R_\mu(X)$ is closed, $p(B) \in R_\mu(X)$. □

Finally we show that, when $m = 2$, the Dvoretzky-Wald-Wolfowitz purification theorem for a countable image set A [10, 13] is a particular case of Theorem 5.1.1. Let $p^a = \int_X \pi(a|x)\,\mu(dx)$, $a \in A$. If these vectors p^a satisfy conditions (i) and (ii) of Theorem 5.1.1, then Theorem 5.1.1 implies that transition probability can be purified in the case of countable A and $m = 2$. Indeed, for (i), obviously $\sum_{a\in A} p_a = \mu(X)$. For (ii), if $B \subseteq A$ then

$$\sum_{a\in B} p^a = \sum_{a\in B} \int_X \pi(a|x)\,\mu(dx) = \int_X \pi(B|x)\,\mu(dx) \in R_\mu(X),$$

where the inclusion follows from a version of Lyapunov's theorem [2, p. 218].

5.2 A Counterexample

The following example demonstrates that the necessary conditions in Theorem 5.1.1 for the existence of a measurable partition $\{X^a : a \in A\}$ with $\mu(X^a) = p^a$, $a \in A$, is not sufficient for an atomless measure μ when $m > 2$. In this example, A consists of three points. According to Theorem 5.1.1, this condition is necessary and sufficient when $m = 2$, A is countable, and μ is atomless. If A consists of two points, say a and b, and $p^a \in R_\mu(X)$, $p^b = \mu(X) - p^a$, then the partition $\{X^a, X^b\}$ always exists with X^a selected as any $X^a \in \mathcal{F}$ satisfying $\mu(X^a) = p^a$ and with $X^b = X \setminus X^a$.

Example 5.2.1. Consider the measure space $(X, \mathcal{F}, \mu)$ described in Example 1.1.5. Let $p^1 = (56, 29, 31)$, $p^2 = (24, 41, 19)$, $p^3 = (30, 60, 75)$, and $A = \{1, 2, 3\}$. Then $p^1 + p^2 + p^3 = \mu(X)$. We further observe that: (1) p^1 is the vector q from Example 4.2.1, so $p^1 \in R_\mu(X)$ and therefore $p^2 + p^3 = \mu(X) - p^1 \in R_\mu(X)$; (2) $p^1 + p^3$ is the vector $q + \mu(X) - p$ from Example 4.2.1, so $p^1 + p^3 \in R_\mu(X)$ and therefore $p^2 = \mu(X) - p^1 - p^3 \in R_\mu(X)$; (3) $p^1 + p^2$ is the vector p from Example 4.2.1, so $p^1+p^2 \in R_\mu(X)$ and therefore $p^3 = \mu(X)-p^1-p^2 \in R_\mu(X)$. Thus, the vectors p^a, $a \in A$, satisfy conditions (i) and (ii) in Theorem 5.1.1.

If there exists a partition $\{X^a \in \mathcal{B} : a \in A\}$ of X with $\mu(X^a) = p^a$ for all $a \in A$, let $Y = X^1 \cup X^2$. Since $X^1 \cap X^2 = \emptyset$, $\mu(X^1) = p^1 = q$, and $\mu(Y) = p^1+p^2 = p$, then $q \in R^p_\mu(X)$. However, according to Example 4.2.1, $q \notin R^p_\mu(X)$. This contradiction implies that a partition $\{X^a \in \mathcal{B} : a \in A\}$ of X, with $\mu(X^a) = p^a$ for all $a \in A$, does not exist. $\square$

Chapter 6

Outlook

In this chapter we mention three directions of future developments on the topic discussed in this dissertation. In Section 6.1, we discuss whether similar results hold for signed measures. In Section 6.2, we ask the question whether the Minkowski sum of the union $R_\mu^p(X)$ and the intersection $I_\mu^{\mu(X)-p}(X)$ is equal to the range $R_\mu(X)$. In Section 6.3, we consider a potential extension of the purification theorems.

6.1 Extension to Signed Measures

Up to this point, we have only considered nonnegative vector measures. A measure ν on the measurable space $(X, \mathcal{F})$ is called nonnegative if $\nu(Y) \geq 0$ for any $Y \in \mathcal{F}$. A vector measure is called nonnegative if all its components are nonnegative. In this section, we discuss what will happen when this assumption is removed. A measure that is not nonnegative is called a signed measure.

Note that the proofs of Theorems 3.3.1-3.3.3 did not use the nonnegativity of the vector measure, so we do not actually need this assumption in them and they can be generalized to the following theorems.

Theorem 6.1.1. *For a measurable space $(X, \mathcal{F})$ with a finite signed vector measure $\mu = (\mu_1, \mu_2)$ and for any vector $p \in R_\mu(X)$, the set $R_\mu^p(X)$ is compact and, in addition, if the vector measure μ is atomless, this set is convex.*

Theorem 6.1.2. *For a measurable space $(X, \mathcal{F})$ with a finite signed vector measure $\mu = (\mu_1, \mu_2)$ and for any vector $p \in R_\mu(X)$, $R_\mu^p(X) \subseteq Q_\mu^p(X)$.*

Theorem 6.1.3. *For a measurable space $(X, \mathcal{F})$ with a finite atomless signed vector measure μ and for any vector $p \in R_\mu(X)$, $I_\mu^{\mu(X)-p}(X) = R_\mu(X) \ominus R_\mu^p(X)$.*

Concerning the maximal and minimal subranges when μ is two-dimensional and signed, we raise the following question.

Question 6.1.4. *For a measurable space $(X, \mathcal{F})$ with a two-dimensional finite atomless signed vector measure $\mu = (\mu_1, \mu_2)$ and for a vector $p \in R_\mu(X)$, does there exist a maximal subset $Z^* \in \mathcal{S}_\mu^p(X)$?*

Note that, if the maximal subset above does exist, Theorem 3.2.2 and Corollary 3.2.3 can be straightforwardly extended to the case of signed measures. In addition, even if the answer to Question 6.1.4 is positive, the equality $R_\mu(Z^*) = Q_\mu^p(X)$ does not hold as for nonnegative measures. The following example demonstrates this fact.

Example 6.1.5. For the measurable space $(X, \mathcal{F})$, where $X = [-1, 1]$ and $\mathcal{F}$ is the Borel σ-field on X, consider the measure $\mu(dx) = (\mu_1, \mu_2)(dx) = (f(x), 2x)\, dx$, where

$$f(x) = \begin{cases} -1, & \text{if } x \in [-1, 0)\,; \\ 1, & \text{if } x \in [0, 1]\,. \end{cases}$$

Let $p = (1, 1)$ and $Z^* = [0, 1]$. Then obviously, $\mu(Z^*) = p$. Furthermore $\mathcal{S}_\mu^p(X)$ consists of one equivalence subclass. In other words, $Z = Z^*$ (μ-everywhere) for any $Z \in \mathcal{S}_\mu^p(X)$, and thus, $R_\mu(Z) = R_\mu(Z^*)$. Therefore Z^* is the maximal subset with measure p.

However, $R_\mu(Z^*) \neq Q_\mu^p(X)$. To show this, let $q = \left(\frac{1}{2}, \frac{1}{8}\right)$. Then obviously $q \notin R_\mu(Z^*)$, but $q \in Q_\mu^p(X)$. Indeed, observe that (i) $q \in R^\mu(X)$, since $\mu\left(\left[-1, -\frac{3}{4}\right] \cup \left[0, \frac{3}{4}\right]\right) = \left(\frac{1}{2}, \frac{1}{8}\right) = q$, and (ii) $q \in R^\mu(X) - \{\mu(X) - p\}$, since $\mu\left(\left[-1, -\frac{1}{4}\right] \cup \left[0, \frac{1}{4}\right]\right) = \left(-\frac{1}{2}, -\frac{7}{8}\right) = q + (\mu(X) - p)$. $\square$

6.2 Minkowski Sum of the Union and the Intersection

Consider the measurable space $(X, \mathcal{F})$ with a m-dimensional finite atomless vector measure μ. Recall that the range $R_\mu(X)$ can be written as Minkowski sums of various pairs of subranges in $\mathbb{R}^m$. In particular, for any $Y \in \mathcal{F}$, $R_\mu(X) = R_\mu(Y) \oplus R_\mu(X \setminus Y)$. Naturally, we ask the question whether the Minkowski sum of $R_\mu^p(X)$ and $I_\mu^{\mu(X)-p}(X)$ equals to the range $R_\mu(X)$.

Question 6.2.1. *Given a measurable space $(X, \mathcal{F})$ with a m-dimensional finite atomless vector measure μ, and a vector $p \in R_\mu(X)$, Does the following equality hold?*

$$R_\mu(X) = R_\mu^p(X) \oplus I_\mu^{\mu(X)-p}(X) \tag{6.2.1}$$

We remark that it is straightforward to prove that $R_\mu(X) \supseteq R_\mu^p(X) \oplus I_\mu^{\mu(X)-p}(X)$. Indeed, consider any $q \in R_\mu^p(X) \oplus I_\mu^{\mu(X)-p}(X)$. Then there exist $q_1 \in R_\mu^p(X)$ and $q_2 \in I_\mu^{\mu(X)-p}(X)$, such that $q = q_1 + q_2$. According to Definition 3.1.1, there exists $Y \in \mathcal{S}_\mu^p$, such that $q_1 \in R_\mu(Y)$, and thus there exists $Z_1 \in \mathcal{F}$ such that $Z_1 \subset Y$ and $q_1 = \mu(Z_1)$. In addition, since $q_2 \in R_\mu(X \setminus Y)$ according to Definition 3.1.1, there exists $Z_2 \in \mathcal{F}$ such that $Z_2 \subset X \setminus Y$ and $q_2 = \mu(Z_2)$. Therefore, $q = \mu(Z_1) + \mu(Z_2) = \mu(Z_1 \cup Z_2)$, and thus $q \in R_\mu(X)$.

In Section 3.3, we proved that the union $R_\mu^p(X)$ and the intersection $I_\mu^{\mu(X)-p}$ have the following relation (see Theorem 3.3.3),

$$I_\mu^{\mu(X)-p}(X) = R_\mu(X) \ominus R_\mu^p(X). \tag{6.2.2}$$

It is noteworthy that (6.2.2) does not always imply (6.2.1). To show that (6.2.1) holds, one has to show that $R_\mu^p(X)$ is a summand of $R_\mu(X)$. Recall that, for convex compact sets $A, B \subset \mathbb{R}^m$, A is called a summand of B, if there exists a convex compact set $C \subset \mathbb{R}^m$ such that $A \oplus C = B$.

6.3 Extension of the Purification Theorems

The results on strong purification presented in Section 1.2 (see Theorems 1.2.1 and 1.2.2) can be rephrased into the following theorem.

Theorem 6.3.1. *Consider a measurable space $(X, \mathcal{F})$ with a m-dimensional finite atomless vector measure μ, a countable set A, and a countable set of m-dimensional vectors $\{p^a : a \in A\}$. A partition $\{Z^a \in \mathcal{F} : a \in A\}$ of X, with $p^a = \mu(Z^a)$ for all $a \in A$, exists if and only if there exists a transition probability $\pi(a|x)$ from $(X, \mathcal{F})$ to A, such that*

$$p^a = \int_X \pi(a|x)\mu(dx), \quad \textit{for all } a \in A.$$

For $m = 2$, Theorem 5.1.1 provides a different necessary and sufficient condition for the partition described in Theorem 6.3.1 to exist. This condition is that: (i) $\sum_{a \in A} p^a = \mu(X)$, and (ii) $\sum_{a \in B} p^a \in R_\mu(X)$ for any finite subset $B \subset A$. Obviously, this is a necessary condition for any natural number m. However Example 5.2.1 implies that it is not sufficient for an atomless μ, when $m > 2$ and A consists of more than two points.

For any $B \subseteq A$, denote $p(B) = \sum_{a \in B} p^a$, where either $A = \{1, 2, \ldots\}$ or $A = \{1, \ldots, n\}$ for some $n = 1, 2, \ldots$. Then $p(\cdot)$ is a measure on the space $(A, 2^A)$. In addition, the existence of the partition described in Theorem 6.3.1 can be restated as the existence of a pure transition probability $\pi^*(B|x) = I\{\varphi(x) \in B\}$ from $(X, \mathcal{F})$ to $(A, 2^A)$, such that

$$p(B) = \int_X I\{\varphi(x) \in B\}\mu(dx) \quad \text{for all } B \in 2^A.$$

Now consider a general measurable space $(A, \mathcal{A})$. We raise the following question.

Question 6.3.2. *Given measurable spaces $(A, \mathcal{A})$ and $(X, \mathcal{F})$, endowed with m-dimensional vector measures η and μ respectively. What is the necessary and sufficient condition for the existence of a pure transition probability from $(X, \mathcal{F})$ to $(A, \mathcal{A})$,*

$$\pi^*(B|x) = I\{\varphi(x) \in B\},$$

such that

$$\eta(B) = \int_X I\{\varphi(x) \in B\}\mu(dx) \quad \textit{for all } B \in \mathcal{A}. \tag{6.3.1}$$

Obviously, the following condition is necessary: (i) $\eta(A) = \mu(X)$ and (ii) $R_\eta(A) \subseteq R_\mu(X)$. In addition, from Theorems 6.3.1 and 5.1.1, we know that for (6.3.1) to hold, either of the following conditions is sufficient:

1. (i) m is finite, (ii) A is countable, (iii) μ is finite and atomless, and (iv) there exists a transition probability $\pi(a|x)$ from $(X, \mathcal{F})$ to $(A, 2^A)$, such that

$$\eta(a) = \int_X \pi(a|x)\mu(dx), \quad \text{for all } a \in A;$$

2. (i) $m = 2$, (ii) A is countable, (iii) μ is finite and atomless, (iv) $\eta(A) = \mu(X)$, and (v) $R_\eta(A) \subseteq R_\mu(X)$.

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