A twistor-sphere of generalized Kähler potentials on hyperkähler manifolds

A Thesis Presented

by

Malte Dyckmanns

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Master of Arts

in

Physics

Stony Brook University

August 2011

Stony Brook University

The Graduate School

Malte Dyckmanns

We, the thesis committee for the above candidate for the

Master of Arts degree, hereby recommend

acceptance of this thesis.

Martin Roček
Professor, Department of Physics and Astronomy,
Stony Brook University

Leonardo Rastelli
Associate Professor, Department of Physics and Astronomy,
Stony Brook University

Paul D. Grannis
Distinguished Research Professor, Department of Physics and Astronomy,
Stony Brook University

This thesis is accepted by the Graduate School

Lawrence Martin
Dean of the Graduate School

Abstract of the Thesis

A twistor-sphere of generalized Kähler potentials on hyperkähler manifolds

by

Malte Dyckmanns

Master of Arts

in

Physics

Stony Brook University

2011

We consider generalized Kähler structures (g,J_+,J_-) on a hyperkähler manifold (M,g,I,J,K), where we use the twistor space of M to choose J_+ and J_-. Relating semi-chiral to arctic superfields, we can determine the generalized Kähler potential for hyperkähler manifolds whose description in projective superspace is known. This is used to determine an S^2-family of generalized Kähler potentials for Euclidean space and for the Eguchi-Hanson geometry.

Contents

Chapter 1

Introduction

The main purpose of this thesis is to determine the generalized Kähler potential for hyperkähler manifolds.

First, we review aspects of hyperkähler geometry and its twistor space. We parametrize the twistor-sphere of complex structures by a complex coordinate ζ and introduce holomorphic Darboux coordinates $\Upsilon(\zeta)$, $\tilde{\Upsilon}(\zeta)$ for a certain holomorphic symplectic form. This construction is relevant for the projective superspace description of $\mathcal{N}=2$ sigma models, where Υ, $\tilde{\Upsilon}$ are arctic superfields. We consider four-dimensional hyperkähler manifolds and explicitly determine the partial differential equations that the coordinates describing those arctic superfields have to fulfill.

In chapter 3, we review the relevant features of generalized Kähler geometry in its bihermitian formulation. This geometry involves two complex structures J_+, J_- on a Riemannian manifold (M,g) and can locally be described by a generalized Kähler potential. In this thesis, we consider the case where the kernel of $[J_+,J_-]$ is trivial. Then the potential is defined as the generating function for a symplectomorphism between coordinates (x_L,y_L) and (x_R,y_R) that are holomorphic w.r.t. J_+ and J_- respectively. We also derive a condition that the generalized potential has to fulfill in order for M to be hyperkähler. Generalized Kähler geometry has initially been found as the target space geometry of $\mathcal{N}=(2,2)$ supersymmetric sigma models, where the potential turns out to be the superspace Lagrangian and the coordinates x_L, x_R describe semi-chiral superfields.

Using its twistor space, a hyperkähler manifold can be seen as a generalized Kähler manifold in various ways while keeping the metric fixed. In chapter 4, we consider a two-sphere of generalized Kähler structures on a hyperkähler manifold and express the coordinates $x_{L,R}$, $y_{L,R}$ in terms of Υ,$\tilde{\Upsilon}$. This enables us to determine the generalized Kähler potential on a hyperkähler manifold if we can find the decomposition of the arctic superfields Υ,$\tilde{\Upsilon}$ in terms of their $\mathcal{N}=1$ components.

We determine the generalized potential for Euclidean space, where the differential equations for Υ,$\tilde{\Upsilon}$ are easy to solve. As a nontrivial example, we look at the Eguchi-Hanson metric, where the relevant coordinates $\Upsilon,\tilde{\Upsilon}$ have been found previously. We give an explicit expression for the S^2-family of generalized Kähler potentials for this geometry, which belongs to the family of gravitational instantons and is thus of interest to physicists.

In this chapter we also extend our results from the first section to an $S^2 \times S^2$-family of generalized Kähler structures on a hyperkähler manifold, i.e. we let both J_+ and J_- be an arbitrary point on the twistor-sphere of complex structures. As an example we determine this $S^2 \times S^2$-family of generalized potentials for Euclidean space.

In chapter 4, we derive a method to determine the generalized Kähler potential for hyperkähler manifolds. The coordinates $\Upsilon(\zeta),\tilde{\Upsilon}(\zeta)$ are the starting point for this method. In projective superspace, these coordinates are described by arctic superfields. Since projective superspace provides powerful methods to determine the coordinates that we need, we briefly review this formalism in chapter 5. In the case of hyperkähler metrics on cotangent bundles of hermitian symmetric spaces, projective superspace methods have been successfully applied to the problem of finding $\Upsilon,\tilde{\Upsilon}$. We briefly show how the results used in chapter 4 for Euclidean space and the Eguchi-Hanson geometry arise in that context.

In the last chapter, we discuss possible applications and further research projects using our method for finding generalized Kähler potentials on hyperkähler manifolds.

Chapter 2

Hyperkähler manifolds and their twistor spaces

The main goal of this thesis is to consider generalized Kähler structures (g,J_+,J_-) on a hyperkähler manifold M and to investigate their generalized Kähler potentials. For the choice of the two complex structures J_+ and J_-, we will make use of the twistor space $\mathcal{Z} = M \times S^2$ of M.

Hyperkähler manifolds appear for instance as the target spaces for hypermultiplet scalars in four dimensional nonlinear σ-Models with $\mathcal{N} = 2$ supersymmetry on the base space [1]. In geometric terms, they are described by the data (M,g,I,J,K), where g is a Riemannian metric on M that is Kähler with respect to the three complex structures I,J,K, which fulfill $I \circ J = K$. Since complex structures square to $-id_{TM}$, $IJ = K$ implies that the complex structures fulfill the algebra of the quaternions:

$$IJ = K = -JI, \quad JK = I = -KJ, \quad KI = J = -IK. \tag{2.1}$$

The dimension of a hyperkähler manifold has to be a multiple of 4 and we define $\dim_{\mathbb{R}} M =: 4n$.

2.1 Twistor-sphere of complex structures

In fact, a hyperkähler manifold does not only admit the three complex structures I,J,K, but there exists a whole two-sphere of complex structures on M with respect to which g is a Kähler metric, namely $(M,g,\mathcal{J} = v_1 I + v_2 J + v_3 K)$ is Kähler for each $(v_1,v_2,v_3) \in S^2$. Using the stereographic projection,

$$S^2 \mapsto \mathbb{C}, \quad (v_1,v_2,v_3) \mapsto \zeta = \frac{v_2 + iv_3}{1 + v_1}, \tag{2.2}$$

we parametrize this family of complex structures on M in a chart of S^2 including the north-pole by a complex coordinate ζ:

$$\mathcal{J}(\zeta) := v_1(\zeta) I + v_2(\zeta) J + v_3(\zeta) K := \frac{1}{1+\zeta\bar{\zeta}} \left[(1-\zeta\bar{\zeta}) I + (\zeta+\bar{\zeta}) J + i\,(\bar{\zeta}-\zeta) K \right]. \tag{2.3}$$

We define the complex two-forms

$$\omega^{(2,0)} := \omega_2 + i\omega_3, \qquad \omega^{(0,2)} := \omega_2 - i\omega_3; \tag{2.4}$$

where $\omega_1 = gI$, $\omega_2 = gJ$, $\omega_3 = gK$ are the three Kähler forms. As the superscript indicates, $\omega^{(2,0)}$ and $\omega^{(0,2)} = \overline{\omega^{(0,2)}}$ are (2,0)- and (0,2)-forms with respect to I. (The $^-$ will always refer to the complex conjugate w.r.t. the first complex structure I.)

In fact, $\omega^{(2,0)}$ is a holomorphic symplectic form, i.e. $d\omega^{(2,0)} = 0$ and $\left(\omega^{(2,0)}\right)^n$ is nowhere vanishing. As a consequence of the famous Calabi-Yau theorem, Bochner's vanishing theorem and Berger's classification of irreducible holonomy groups, the existence of a holomorphic symplectic form on a compact Kähler manifold (M,g,I) implies that M is hyperkähler.

The main point of this chapter is the fact that we can construct a holomorphic symplectic form with respect to each $\mathcal{J}(\zeta)$ on the sphere of complex structures and introduce adapted coordinates that bring this symplectic form into its canonical form

$$\begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}. \tag{2.5}$$

Namely for each $\zeta \in \mathbb{C}$,

$$\Omega_H(\zeta) := \omega^{(2,0)} - 2\zeta\omega_1 - \zeta^2\omega^{(0,2)} \tag{2.6}$$

turns out to be a holomorphic symplectic form with respect to the complex structure $\mathcal{J}(\zeta)$ [2]. In particular, $\omega^{(2,0)} = \Omega_H(\zeta = 0)$ is a (2,0)-form w.r.t. $I = \mathcal{J}(\zeta = 0)$.

Starting from $\zeta = 0$, we can locally find holomorphic Darboux coordinates $\Upsilon^p(\zeta)$ and $\tilde{\Upsilon}_p(\zeta)$ ($p = 1,\ldots,n$) for $\Omega_H(\zeta)$ that are analytic in ζ such that [3]

$$\Omega_H(\zeta) = i\, d\Upsilon^p(\zeta) \wedge d\tilde{\Upsilon}_p(\zeta). \tag{2.7}$$

These canonical coordinates $\Upsilon,\tilde{\Upsilon}(\zeta)$ for Ω_H are crucial for the projective superspace formulation of $\mathcal{N} = 2$ σ-models[1], where they describe "arctic" superfields. They have been determined for instance in [3] for the Eguchi-Hanson metric and we will use them in chapter 4 to determine the generalized Kähler potential for hyperkähler manifolds. They can be found for a large class of hyperkähler metrics on cotangent bundles of Kähler manifolds using the projective superspace approach, which will be explained in chapter 5.

[1]If we define $\breve{\Upsilon}(\zeta) := \breve{\Upsilon}(-\frac{1}{\zeta})$, $\breve{\tilde{\Upsilon}}(\zeta) := \bar{\tilde{\Upsilon}}(-\frac{1}{\zeta})$, then $\Upsilon,\tilde{\Upsilon}$ and $\breve{\Upsilon},\breve{\tilde{\Upsilon}}$ are related by a ζ^2-twisted symplectomorphism whose generating function $f(\Upsilon,\breve{\Upsilon})$ can be interpreted as the $\mathcal{N} = 2$ projective superspace Lagrangian [3].

2.2 The four dimensional case

In this section, we consider the four dimensional case and explicitly determine the partial differential equations for $\Upsilon(\zeta)$ and $\tilde{\Upsilon}(\zeta)$ to be holomorphic with respect to $\mathcal{J}(\zeta)$ and to fulfill equation 2.7.

A four dimensional Kähler manifold (M,g,I) is hyperkähler if and only if there are holomorphic coordinates (z,u) on M such that the Kähler potential $K(z,u)$ fulfills the following Monge-Ampère equation [10]:

$$K_{z\bar{z}}K_{u\bar{u}} - K_{z\bar{u}}K_{u\bar{z}} = 1. \tag{2.8}$$

From a Kähler potential fulfilling this equation, we can construct the three Kähler forms and the metric:

$$\begin{aligned} \omega_1 &= -\frac{i}{2}\partial\bar{\partial}K, \\ \omega_2 &= \frac{i}{2}(dz \wedge du - d\bar{z} \wedge d\bar{u}), \\ \omega_3 &= \frac{1}{2}(dz \wedge du + d\bar{z} \wedge d\bar{u}), \\ ds^2 &= K_{z\bar{z}}\, dzd\bar{z} + K_{z\bar{u}}\, dzd\bar{u} + K_{u\bar{z}}\, dud\bar{z} + K_{u\bar{u}}\, dud\bar{u}. \end{aligned} \tag{2.9}$$

In their matrix representation with respect to the basis $\{dz,du,d\bar{z},d\bar{u}\}$, the Kähler forms and the metric are given by

$$(\omega_1) = -\frac{i}{2}\begin{pmatrix} 0 & 0 & K_{z\bar{z}} & K_{z\bar{u}} \\ 0 & 0 & K_{u\bar{z}} & K_{u\bar{u}} \\ -K_{z\bar{z}} & -K_{u\bar{z}} & 0 & 0 \\ -K_{z\bar{u}} & -K_{u\bar{u}} & 0 & 0 \end{pmatrix}, \quad (\omega_2) = \frac{i}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$(\omega_3) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad (g) = \frac{1}{2}\begin{pmatrix} 0 & 0 & K_{z\bar{z}} & K_{z\bar{u}} \\ 0 & 0 & K_{u\bar{z}} & K_{u\bar{u}} \\ K_{z\bar{z}} & K_{u\bar{z}} & 0 & 0 \\ K_{z\bar{u}} & K_{u\bar{u}} & 0 & 0 \end{pmatrix}. \tag{2.10}$$

Since (z,u) are holomorphic coordinates w.r.t. to the first complex structure I, I is in its canonical form with respect to those coordinates:

$$(I)_{(\partial_z,\partial_u,\partial_{\bar{z}},\partial_{\bar{u}})} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}. \tag{2.11}$$

Together with the metric's inverse

$$\left(g^{-1}\right) = \frac{2}{K_{z\bar{z}}K_{u\bar{u}} - K_{z\bar{u}}K_{u\bar{z}}} \begin{pmatrix} 0 & 0 & K_{u\bar{u}} & -K_{u\bar{z}} \\ 0 & 0 & -K_{z\bar{u}} & K_{z\bar{z}} \\ K_{u\bar{u}} & -K_{z\bar{u}} & 0 & 0 \\ -K_{u\bar{z}} & K_{z\bar{z}} & 0 & 0 \end{pmatrix}, \tag{2.12}$$

we get the other two complex structures (with respect to the basis $\{\partial_z,\partial_u,\partial_{\bar{z}},\partial_{\bar{u}}\}$):

$$(J) = g^{-1}\omega_2 = \frac{i}{K_{z\bar{z}}K_{u\bar{u}} - K_{z\bar{u}}K_{u\bar{z}}} \begin{pmatrix} 0 & 0 & -K_{u\bar{z}} & -K_{u\bar{u}} \\ 0 & 0 & K_{z\bar{z}} & K_{z\bar{u}} \\ K_{z\bar{u}} & K_{u\bar{u}} & 0 & 0 \\ -K_{z\bar{z}} & -K_{u\bar{z}} & 0 & 0 \end{pmatrix},$$

$$(K) = g^{-1}\omega_3 = \frac{1}{K_{z\bar{z}}K_{u\bar{u}} - K_{z\bar{u}}K_{u\bar{z}}} \begin{pmatrix} 0 & 0 & K_{u\bar{z}} & K_{u\bar{u}} \\ 0 & 0 & -K_{z\bar{z}} & -K_{z\bar{u}} \\ K_{z\bar{u}} & K_{u\bar{u}} & 0 & 0 \\ -K_{z\bar{z}} & -K_{u\bar{z}} & 0 & 0 \end{pmatrix}. \tag{2.13}$$

We see that equation 2.8 ensures that J and K indeed square to $-id_{TM}$. For the rest of the section, we will assume that equation 2.8 holds.

The coordinates $\Upsilon,\tilde{\Upsilon}(\zeta)$ in equation 2.7 must be holomorphic with respect to

$$(\mathcal{J}(\zeta)) = \frac{i}{1+\zeta\bar{\zeta}} \begin{pmatrix} 1-\zeta\bar{\zeta} & 0 & -2\zeta K_{u\bar{z}} & -2\zeta K_{u\bar{u}} \\ 0 & 1-\zeta\bar{\zeta} & 2\zeta K_{z\bar{z}} & 2\zeta K_{z\bar{u}} \\ 2\bar{\zeta}K_{z\bar{u}} & 2\bar{\zeta}K_{u\bar{u}} & \zeta\bar{\zeta}-1 & 0 \\ -2\bar{\zeta}K_{z\bar{z}} & -2\bar{\zeta}K_{u\bar{z}} & 0 & \zeta\bar{\zeta}-1 \end{pmatrix} \tag{2.14}$$

(see equation 2.3), so we first have to determine a basis for the forms that are (1,0) with respect to $\mathcal{J}(\zeta)$. We find the following eigenvector fields of $\mathcal{J}(\zeta)$ with eigenvalue $+i$:

$$(X_1) = \frac{1}{1+\zeta\bar{\zeta}} \begin{pmatrix} 1 \\ 0 \\ \bar{\zeta}K_{z\bar{u}} \\ -\bar{\zeta}K_{z\bar{z}} \end{pmatrix}, \qquad X_1 = \frac{1}{1+\zeta\bar{\zeta}}\left(\partial_z + \bar{\zeta}K_{z\bar{u}}\partial_{\bar{z}} - \bar{\zeta}K_{z\bar{z}}\partial_{\bar{u}}\right);$$

$$(X_2) = \frac{1}{1+\zeta\bar{\zeta}} \begin{pmatrix} 0 \\ 1 \\ \bar{\zeta}K_{u\bar{u}} \\ -\bar{\zeta}K_{u\bar{z}} \end{pmatrix}, \qquad X_2 = \frac{1}{1+\zeta\bar{\zeta}}\left(\partial_u + \bar{\zeta}K_{u\bar{u}}\partial_{\bar{z}} - \bar{\zeta}K_{u\bar{z}}\partial_{\bar{u}}\right). \tag{2.15}$$

The complex conjugates have eigenvalue $-i$:

$$\left(\bar{X}_1\right) = \frac{1}{1+\zeta\bar{\zeta}} \begin{pmatrix} \zeta K_{u\bar{z}} \\ -\zeta K_{z\bar{z}} \\ 1 \\ 0 \end{pmatrix}, \qquad \bar{X}_1 = \frac{1}{1+\zeta\bar{\zeta}} \left(\partial_{\bar{z}} + \zeta K_{u\bar{z}}\partial_z - \zeta K_{z\bar{z}}\partial_u\right);$$

$$\left(\bar{X}_2\right) = \frac{1}{1+\zeta\bar{\zeta}} \begin{pmatrix} \zeta K_{u\bar{u}} \\ -\zeta K_{z\bar{u}} \\ 0 \\ 1 \end{pmatrix}, \qquad \bar{X}_2 = \frac{1}{1+\zeta\bar{\zeta}} \left(\partial_{\bar{u}} + \zeta K_{u\bar{u}}\partial_z - \zeta K_{z\bar{u}}\partial_u\right). \tag{2.16}$$

From the rows of the matrix

$$\left(X_1\, X_2\, \bar{X}_1\, \bar{X}_2\right)^{-1} = \begin{pmatrix} 1 & 0 & -\zeta K_{u\bar{z}} & -\zeta K_{u\bar{u}} \\ 0 & 1 & \zeta K_{z\bar{z}} & \zeta K_{z\bar{u}} \\ -\bar{\zeta} K_{z\bar{u}} & -\bar{\zeta} K_{u\bar{u}} & 1 & 0 \\ \bar{\zeta} K_{z\bar{z}} & \bar{\zeta} K_{u\bar{z}} & 0 & 1 \end{pmatrix}, \tag{2.17}$$

we can read off the dual basis to $\{X_1,X_2,\bar{X}_1,\bar{X}_2\}$:

$$\begin{aligned} \left(\theta^1\right) &= (1,\, 0,\, -\zeta K_{u\bar{z}},\, -\zeta K_{u\bar{u}}), & \theta^1 &= dz - \zeta K_{u\bar{z}} d\bar{z} - \zeta K_{u\bar{u}} d\bar{u}, \\ \left(\theta^2\right) &= (0,\, 1,\, \zeta K_{z\bar{z}},\, \zeta K_{z\bar{u}}), & \theta^2 &= du + \zeta K_{z\bar{z}} d\bar{z} + \zeta K_{z\bar{u}} d\bar{u}, \\ \left(\bar{\theta}^1\right) &= (-\bar{\zeta} K_{z\bar{u}},\, -\bar{\zeta} K_{u\bar{u}},\, 1,\, 0), & \bar{\theta}^1 &= d\bar{z} - \bar{\zeta} K_{z\bar{u}} dz - \bar{\zeta} K_{u\bar{u}} du, \\ \left(\bar{\theta}^2\right) &= (\bar{\zeta} K_{z\bar{z}},\, \bar{\zeta} K_{u\bar{z}},\, 0,\, 1), & \bar{\theta}^2 &= d\bar{u} + \bar{\zeta} K_{z\bar{z}} dz + \bar{\zeta} K_{u\bar{z}} du. \end{aligned} \tag{2.18}$$

θ^1 and θ^2 form a basis for the (1,0)-forms with respect to $\mathcal{J}(\zeta)$.

For $\Upsilon,\tilde{\Upsilon}$ to be holomorphic with respect to $\mathcal{J}(\zeta)$, $d\Upsilon(\zeta)$ and $d\tilde{\Upsilon}(\zeta)$ must be linear combinations of θ^1 and θ^2 (here, the differential does not act on ζ):

$$d\Upsilon = \alpha\theta^1 + \beta\theta^2, \quad d\tilde{\Upsilon} = \eta\theta^1 + \lambda\theta^2; \tag{2.19}$$

where $\alpha,\beta,\eta,\lambda$ are complex functions in $(z,u;\zeta)$. Plugging this into 2.18 and expressing the differential on the left side in terms of $(dz,du,d\bar{z},d\bar{u})$, one can read off the coefficients

$$\alpha = \frac{\partial\Upsilon}{\partial z}, \quad \beta = \frac{\partial\Upsilon}{\partial u}, \quad \eta = \frac{\partial\tilde{\Upsilon}}{\partial z}, \quad \lambda = \frac{\partial\tilde{\Upsilon}}{\partial u}. \tag{2.20}$$

Plugging 2.18 and 2.20 into 2.19 and comparing the $d\bar{z}$- and $d\bar{u}$-terms on both sides, one obtains the following two partial differential equations:

$$\frac{\partial\Psi}{\partial\bar{z}} = \zeta\left(K_{z\bar{z}}\frac{\partial\Psi}{\partial u} - K_{u\bar{z}}\frac{\partial\Psi}{\partial z}\right), \quad \frac{\partial\Psi}{\partial\bar{u}} = \zeta\left(K_{z\bar{u}}\frac{\partial\Psi}{\partial u} - K_{u\bar{u}}\frac{\partial\Psi}{\partial z}\right) \quad (\Psi = \Upsilon,\tilde{\Upsilon}). \tag{2.21}$$

Υ and $\tilde{\Upsilon}$ have to fulfill the two equations in 2.21 in order to be holomorphic with respect to $\mathcal{J}(\zeta)$.

Apart from being holomorphic with respect to $\mathcal{J}(\zeta)$, we want Υ and $\tilde{\Upsilon}$ to be Darboux coordinates for the symplectic holomorphic form $\Omega_H(\zeta)$, i.e. to fulfill equation 2.7. We find that

$$\Omega_H(\zeta) = idz \wedge du + i\zeta\partial\bar{\partial}K + i\zeta^2 d\bar{z} \wedge d\bar{u} = i\theta^1 \wedge \theta^2, \tag{2.22}$$

so equation 2.7 corresponds to the requirement

$$d\Upsilon \wedge d\tilde{\Upsilon} = \theta^1 \wedge \theta^2. \tag{2.23}$$

2.19 implies

$$d\Upsilon \wedge d\tilde{\Upsilon} = (\alpha\lambda - \beta\eta)\theta^1 \wedge \theta^2. \tag{2.24}$$

Combining this with 2.20, we obtain the requirement

$$\frac{\partial\Upsilon}{\partial z}\frac{\partial\tilde{\Upsilon}}{\partial u} - \frac{\partial\Upsilon}{\partial u}\frac{\partial\tilde{\Upsilon}}{\partial z} = 1 \tag{2.25}$$

for Υ and $\tilde{\Upsilon}$ to fulfill equation 2.7.

2.3 Higher dimensional hyperkähler manifolds

In the last section, we have seen that given a Kähler manifold (M,g,I) with the Kähler potential K in terms of holomorphic coordinates (z,u), we can define the hyperkähler structure (g,I,J,K) on M with ω_2, ω_3 as in 2.9 if and only if K fulfills the Monge-Ampère equation 2.8. We want to generalize this to higher dimensions and determine what requirements $K(z^i,u^i)$ needs to fulfill in order for

$$\begin{aligned}
\omega_1 &= -\frac{i}{2}\partial\bar{\partial}K, \\
\omega_2 &= \frac{i}{2}\sum_{i=1}^{n}(dz^i \wedge du^i - d\bar{z}^i \wedge d\bar{u}^i), \\
\omega_3 &= \frac{1}{2}\sum_{i=1}^{n}(dz^i \wedge du^i + d\bar{z}^i \wedge d\bar{u}^i), \\
g &= K_{z^i\bar{z}^j}\, dz^i d\bar{z}^j + K_{z^i\bar{u}^j}\, dz^i d\bar{u}^j + K_{u^i\bar{z}^j}\, du^i d\bar{z}^j + K_{u^i\bar{u}^j}\, du^i d\bar{u}^j
\end{aligned} \tag{2.26}$$

to define a hyperkähler structure on M.

Here, we cannot invert g explicitly, but we know that $J^2 = (g^{-1}\omega_2)^2$ has to be $-\mathbb{1}$, so we get the requirement

$$(\omega_2^{-1}g)^2 = -\mathbb{1}. \tag{2.27}$$

In coordinates $(z^1,\ldots,z^n,u^1,\ldots,u^n,\bar{z}^1,\ldots,\bar{z}^n,\bar{u}^1,\ldots,\bar{u}^n)$, we have

$$(\omega_2)=\frac{i}{2}\begin{pmatrix}0 & \mathbb{1}_n & 0 & 0\\ -\mathbb{1}_n & 0 & 0 & 0\\ 0 & 0 & 0 & -\mathbb{1}_n\\ 0 & 0 & \mathbb{1}_n & 0\end{pmatrix},\quad \left(\omega_2^{-1}\right)=-2i\begin{pmatrix}0 & -\mathbb{1}_n & 0 & 0\\ \mathbb{1}_n & 0 & 0 & 0\\ 0 & 0 & 0 & \mathbb{1}_n\\ 0 & 0 & -\mathbb{1}_n & 0\end{pmatrix}$$

$$(g)=\frac{1}{2}\begin{pmatrix}0 & 0 & \mathbb{K}_{z\bar{z}} & \mathbb{K}_{z\bar{u}}\\ 0 & 0 & \mathbb{K}_{u\bar{z}} & \mathbb{K}_{u\bar{u}}\\ \mathbb{K}_{z\bar{z}} & \mathbb{K}_{u\bar{z}} & 0 & 0\\ \mathbb{K}_{z\bar{u}} & \mathbb{K}_{u\bar{u}} & 0 & 0\end{pmatrix}, \tag{2.28}$$

where $\mathbb{K}$ are now 4×4-matrices, e.g. $(\mathbb{K}_{z\bar{z}})_{ij}=K_{z^i\bar{z}^j}$. From this, we find $\left(\omega_2^{-1}g\right)^2$ to be

$$-\begin{pmatrix}\mathbb{K}_{u\bar{u}}\mathbb{K}_{z\bar{z}}-\mathbb{K}_{u\bar{z}}\mathbb{K}_{z\bar{u}} & \mathbb{K}_{u\bar{u}}\mathbb{K}_{u\bar{z}}-\mathbb{K}_{u\bar{z}}\mathbb{K}_{u\bar{u}} & 0 & 0\\ \mathbb{K}_{z\bar{z}}\mathbb{K}_{z\bar{u}}-\mathbb{K}_{z\bar{u}}\mathbb{K}_{z\bar{z}} & \mathbb{K}_{z\bar{z}}\mathbb{K}_{u\bar{u}}-\mathbb{K}_{z\bar{u}}\mathbb{K}_{u\bar{z}} & 0 & 0\\ 0 & 0 & \mathbb{K}_{u\bar{u}}\mathbb{K}_{z\bar{z}}-\mathbb{K}_{z\bar{u}}\mathbb{K}_{u\bar{z}} & \mathbb{K}_{u\bar{u}}\mathbb{K}_{z\bar{u}}-\mathbb{K}_{z\bar{u}}\mathbb{K}_{u\bar{u}}\\ 0 & 0 & \mathbb{K}_{z\bar{z}}\mathbb{K}_{u\bar{z}}-\mathbb{K}_{u\bar{z}}\mathbb{K}_{z\bar{z}} & \mathbb{K}_{z\bar{z}}\mathbb{K}_{u\bar{u}}-\mathbb{K}_{u\bar{z}}\mathbb{K}_{z\bar{u}}\end{pmatrix}.$$

The requirements one gets from setting this to $-\mathbb{1}_{4n}$ can be combined into

$$\begin{aligned} I)\quad & \mathbb{K}_{z\bar{z}}\mathbb{K}_{u\bar{u}}-\mathbb{K}_{z\bar{u}}\mathbb{K}_{u\bar{z}}=1\\ II)\quad & [\mathbb{K}_{\circ\bullet},\mathbb{K}_{\odot\otimes}]=0 \qquad (\circ,\bullet,\odot,\otimes=z,\bar{z},u,\bar{u}); \end{aligned} \tag{2.29}$$

i.e. K has to fulfill a higher dimensional analog of the Monge-Ampère equation and all $\mathbb{K}$-matrices have to commute.

2.4 Rotating the basis of complex structures $\{I,J,K\}$

In section 2.1, we have seen that a hyperkähler manifold (M,g,I,J,K) is naturally equipped not only with the three complex strctures I,J,K, but with a whole two-sphere of complex structures. All complex structures in the twistor-sphere of complex structures are given by a linear combination of I,J,K, so we view $\{I,J,K\}$ as an orthonormal basis for the space of complex structures. There is no reason for the basis $\{I,J,K\}$ to be special in any way, so we can rotate it to a new basis $\{I',J',K'\}$ and statements expressed in terms of $\{I,J,K\}$ should be equally valid after replacing $\{I,J,K\}$ by $\{I',J',K'\}$.

The choice of a basis for the complex structures leads to the choice of basis $\{\omega_1=gI,\ \omega_2=gJ,\ \omega_3=gK\}$ for the space of Kähler forms. Above, we have seen that $\omega^{(2,0)}=\omega_2+i\omega_3$ is a holomorphic symplectic form w.r.t. the complex structure I. From the point of view of twistor space, I does not play a special role, so we should be able to find such a decomposition of a holomorphic symplectic form w.r.t. an arbitrary complex structure $\mathcal{J}(\zeta)$ on the twistor sphere. We rotate the basis $\{I,J,K\}$ by the angle $\theta = \angle(I,\mathcal{J}(\zeta))$ around the axis perpendicular to I and $\mathcal{J}(\zeta)$ to get a new orthonormal basis $\{\mathcal{J}_1(\zeta),\mathcal{J}_2(\zeta),\mathcal{J}_3(\zeta)\}$ with $\mathcal{J}_1(\zeta) = \mathcal{J}(\zeta)$:

Let $\vec{\mathbf{I}} := (I,J,K)^T$. We express the complex structures w.r.t. that basis:

$$I = \vec{e}_1 \cdot \vec{\mathbf{I}},\ J = \vec{e}_2 \cdot \vec{\mathbf{I}},\ K = \vec{e}_3 \cdot \vec{\mathbf{I}};\ \mathcal{J}_1(\zeta) =: \vec{v}_1 \cdot \vec{\mathbf{I}},\ \mathcal{J}_2(\zeta) =: \vec{v}_2 \cdot \vec{\mathbf{I}},\ \mathcal{J}_3(\zeta) =: \vec{v}_3 \cdot \vec{\mathbf{I}}. \tag{2.30}$$

$\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ is the standard orthonormal basis of $\mathbb{R}^3$ and since $\mathcal{J}_1(\zeta) = \mathcal{J}(\zeta)$, we have

$$\vec{v}_1 = \frac{1}{1+\zeta\bar{\zeta}} \left(1-\zeta\bar{\zeta},\, \zeta+\bar{\zeta},\, i(\bar{\zeta}-\zeta)\right)^T. \tag{2.31}$$

The axis of rotation will be

$$\vec{n} = \frac{\vec{e}_1 \times \vec{v}_1}{|\vec{e}_1 \times \vec{v}_1|} = \frac{1}{\sqrt{4\zeta\bar{\zeta}}} \left(0,\, i(\zeta-\bar{\zeta}),\, \zeta+\bar{\zeta}\right) \tag{2.32}$$

and the angle θ is given by

$$\cos\theta = \vec{e}_1 \cdot \vec{v}_1 = \frac{1-\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}}, \quad \sin\theta = ||\vec{e}_1 \times \vec{v}_1|| = \frac{\sqrt{4\zeta\bar{\zeta}}}{1+\zeta\bar{\zeta}}. \tag{2.33}$$

We use 2.32 and 2.33 in the standard formula for the rotation around an axis $\vec{n}$ in $\mathbb{R}^3$:

$$\begin{aligned} R &= \begin{pmatrix} \cos\theta + n_x^2(1-\cos\theta) & n_x n_y(1-\cos\theta) - n_z\sin\theta & n_x n_z(1-\cos\theta) + n_y\sin\theta \\ n_y n_x(1-\cos\theta) + n_z\sin\theta & \cos\theta + n_y^2(1-\cos\theta) & n_y n_z(1-\cos\theta) - n_x\sin\theta \\ n_z n_x(1-\cos\theta) - n_y\sin\theta & n_z n_y(1-\cos\theta) + n_x\sin\theta & \cos\theta + n_z^2(1-\cos\theta) \end{pmatrix} \\ &= \frac{1}{1+\zeta\bar{\zeta}} \begin{pmatrix} 1-\zeta\bar{\zeta} & -\zeta-\bar{\zeta} & -i(\bar{\zeta}-\zeta) \\ \zeta+\bar{\zeta} & 1-\frac{1}{2}(\zeta^2+\bar{\zeta}^2) & \frac{i}{2}(\zeta^2-\bar{\zeta}^2) \\ i(\bar{\zeta}-\zeta) & \frac{i}{2}(\zeta^2-\bar{\zeta}^2) & 1+\frac{1}{2}(\zeta^2+\bar{\zeta}^2) \end{pmatrix}. \end{aligned} \tag{2.34}$$

From the columns, we can read off the rotated orthonormal basis: $R = (\vec{v}_1\, \vec{v}_2\, \vec{v}_3)$. This leads to the following rotated complex structures:

$$\begin{aligned} \mathcal{J}_1(\zeta) &= \vec{v}_1(\zeta) \cdot \vec{\mathbf{I}} = \frac{1}{1+\zeta\bar{\zeta}} \left((1-\zeta\bar{\zeta})I + (\zeta+\bar{\zeta})J + i(\bar{\zeta}-\zeta)K\right), \\ \mathcal{J}_2(\zeta) &= \vec{v}_2(\zeta) \cdot \vec{\mathbf{I}} = \frac{1}{1+\zeta\bar{\zeta}} \left(-(\zeta+\bar{\zeta})I + (1-\frac{1}{2}(\zeta^2+\bar{\zeta}^2))J + \frac{i}{2}(\zeta^2-\bar{\zeta}^2)K\right), \\ \mathcal{J}_3(\zeta) &= \vec{v}_3(\zeta) \cdot \vec{\mathbf{I}} = \frac{1}{1+\zeta\bar{\zeta}} \left(-i(\bar{\zeta}-\zeta)I + \frac{i}{2}(\zeta^2-\bar{\zeta}^2)J + (1+\frac{1}{2}(\zeta^2+\bar{\zeta}^2))K\right). \end{aligned} \tag{2.35}$$

Indeed, they fulfill the quaternion algebra $\mathcal{J}_1\mathcal{J}_2 = \mathcal{J}_3$.

We denote the corresponding Kähler forms by $\Omega_1 := g\mathcal{J}_1$, $\Omega_2 := g\mathcal{J}_2$, $\Omega_3 := g\mathcal{J}_3$. In analogy to the fact that $\omega^{(2,0)} = \omega_2 + i\omega_3$ is holomorphic symplectic with respect to I, we expect that $\Omega^{(2,0)} := \Omega_2 + i\Omega_3$ is a holomorphic symplectic form with respect to $\mathcal{J}_1(\zeta)$. Indeed, we find that $\Omega^{(2,0)}$ is proportional to $\Omega_H(\zeta)$ which is known to be a holomorphic symplectic form with respect to $\mathcal{J}(\zeta)$:

$$\Omega^{(2,0)} = \Omega_2 + i\Omega_3 = \frac{1}{1+\zeta\bar{\zeta}}\left(-2\zeta\omega_1 + (1-\zeta^2)\omega_2 + i(1+\zeta^2)\omega_3\right) = \frac{1}{1+\zeta\bar{\zeta}}\Omega_H(\zeta). \quad (2.36)$$

For $\zeta = 0$, $\Omega^{(2,0)} = \Omega_2 + i\Omega_3$ becomes $\omega^{(2,0)} = \omega_2 + i\omega_3$. So here, we have explicitly shown that $\Omega_H(\zeta) = (1+\zeta\bar{\zeta})\Omega^{(2,0)}(\zeta)$ arises from $\omega^{(2,0)}$ by rotating the complex stucture I. Correspondingly $\Upsilon(\zeta)$ and $\tilde{\Upsilon}(\zeta)$ arise from $z = \Upsilon(\zeta = 0)$ and $u = \tilde{\Upsilon}(\zeta = 0)$ by rotating the complex structure.

Chapter 3

Generalized Kähler geometry

Generalized Kähler geometry first appeared in the study of 2D $\mathcal{N} = (2,2)$ nonlinear σ-Models [5] and was later rediscovered by mathematicians as a special case of generalized complex geometry, which is a generalization of the concepts of complex and symplectic geometry [4]. Generalized complex geometry is expressed in terms of generalized complex structures, which are maps $TM \oplus T^*M \to TM \oplus T^*M$. A generalized Kähler manifold admits two commuting generalized complex structures. Here we will, however, study generalized Kähler geometry in its original "bihermitian" formulation in terms of two ordinary complex structures $J_\pm : TM \to TM$.

3.1 Bihermitian Formulation

In its bihermitian formulation, generalized Kähler geometry consists of two (integrable) complex structures J_+, J_- on a Riemannian manifold (M,g), where the metric is hermitian w.r.t. J_+ and J_-. Furthermore, the forms $\omega_\pm := gJ_\pm$ have to fulfill [7]

$$d_+^c \omega_+ + d_-^c \omega_- = 0, \quad dd_+^c \omega_+ = 0, \tag{3.1}$$

where $d_\pm^c = i(\bar{\partial}_\pm - \partial_\pm)$. This allows us to define the closed three-form

$$H := d_+^c \omega_+ = -d_-^c \omega_-, \tag{3.2}$$

whose local two-form potential we denote by B ($H = dB$). Since $\omega_\pm = gJ_\pm$ and g is bihermitian, $\omega_\pm$ is a (1,1)-form with respect to $J_\pm$ respectively and thus, from equation 3.2, H has to be $(2,1) + (1,2)$, both with respect to J_+ and with respect to J_-.

In general, $\omega_\pm$ is not closed and thus $(M,g,J_\pm)$ is not Kähler. In this thesis however, we will mostly consider the case where $H = 0$. Then from equation 3.2, $\partial_\pm \omega_\pm$, $\bar{\partial}_\pm \omega_\pm$ have to vanish separately, so $d\omega_\pm = 0$, i.e. $(M,g,J_\pm)$ is Kähler.

A condition which is equivalent to 3.1 can be given in terms of the so called Bismut connections

$$\nabla^{\pm} = \nabla \pm \frac{1}{2} g^{-1} H, \tag{3.3}$$

where H is again no independent geometrical ingredient, but given by $H = d^c_+ \omega_+$. The integrability condition 3.1 is then equivalent to

$$\nabla^{\pm} J_{\pm} = 0. \tag{3.4}$$

Since $H = dB$ is a differential form, the torsion $T := g^{-1}H$ is asymmetric in its lower indices.

3.2 Poisson structures

On a generalized Kähler manifold, one can define the following three real tensor fields

$$\sigma := [J_+,J_-]g^{-1}, \quad \pi_{\pm} := (J_+ \pm J_-)g^{-1}; \tag{3.5}$$

which turn out to be Poisson structures on M [6]. They give rise to the following splitting of the tangent bundle TM into a direct sum[1]:

$$TM = \ker[J_+,J_-] \oplus (\ker[J_+,J_-])^{\perp} = \ker(J_+ - J_-) \oplus \ker(J_+ + J_-) \oplus (\ker[J_+,J_-])^{\perp}. \tag{3.6}$$

In sigma model language, these three subspaces are tangent to chiral, twisted chiral and semichiral coordinates respectively.

The Poisson structure σ plays a special role for the definition of the generalized Kähler potential. We state its important properties and prove some of them:

1.
$$\sigma(\theta,\eta) = -\sigma(\eta,\theta), \tag{3.7}$$
i.e. σ is a bivector field.

2.
$$J_{\pm} \sigma J_{\pm}^T = -\sigma. \tag{3.8}$$

[1]Let $X \in \Gamma(M;TM)$ s.t. $(J_+ + J_-)X = 0$, $(J_+ - J_-)X = 0$. Addition then gives $2J_+X = 0$ and thus $X = 0$, since J_+ is invertible. So $\ker(J_+ + J_-) \cap \ker(J_+ - J_-) = \{0\}$, i.e. the sum is direct.
From the identities $[J_+,J_-] = (J_+ - J_-)(J_+ + J_-) = -(J_+ + J_-)(J_+ - J_-)$, we immediately get $\ker(J_+ + J_-) \oplus \ker(J_+ - J_-) \subset \ker[J_+,J_-]$.
Let $X \in \Gamma(M;TM)$ s.t. $[J_+,J_-]X = 0$. Then $X = X_+ + X_-$, with $X_+ = -\frac{1}{2}J_+(J_+ - J_-)X$, $X_- = -\frac{1}{2}J_+(J_+ + J_-)X$. We find $(J_+ + J_-)X_+ = \frac{1}{2}J_-[J_+,J_-]X = 0$, $(J_+ - J_-)X_- = \frac{1}{2}J_-[J_+,J_-]X = 0$. Thus $\ker[J_+,J_-] \subset \ker(J_+ + J_-) \oplus \ker(J_+ - J_-)$.

This can be shown using $J_\pm g^{-1} J_\pm^T = g^{-1}$ (which follows from the fact that g is bihermitian) and $[J_+,J_-]J_\pm = -J_\pm[J_+,J_-]$. Thus, σ can be written as the sum of a (2,0)- and a (0,2)-bivector field both w.r.t. J_+ and w.r.t. J_-:

$$\sigma = \sigma_+^{(2,0)} + \sigma_+^{(0,2)} = \sigma_-^{(2,0)} + \sigma_-^{(0,2)}. \tag{3.9}$$

Explicitly, the decomposition is

$$\begin{aligned} \sigma_\pm^{(2,0)} &= \frac{1}{2}(\mathbb{1} - iJ_\pm)\sigma\frac{1}{2}(\mathbb{1} - iJ_\pm)^T = \frac{1}{2}(\mathbb{1} - iJ_\pm)\sigma, & (3.10)\\ \sigma_\pm^{(0,2)} &= \frac{1}{2}(\mathbb{1} + iJ_\pm)\sigma\frac{1}{2}(\mathbb{1} + iJ_\pm)^T = \frac{1}{2}(\mathbb{1} + iJ_\pm)\sigma, & (3.11)\end{aligned}$$

where the last equality uses the fact that σ does not contain a (1,1)-term with respect to $J_\pm$.

3.

$$\bar\partial_\pm \sigma_\pm^{(2,0)} = 0, \quad \partial_\pm \sigma_\pm^{(0,2)} = 0. \tag{3.12}$$

Together with 2., this makes σ a real holomorphic bivector with respect to $J_\pm$.

Proof. [8] Let $(z^\mu,\bar z^\mu \equiv z^{\bar\mu})_{\mu=1,\dots,n:=dim_{\mathbb{C}}M} : U \subset M \to \mathbb{C}^n$ be a holomorphic coordinate system w.r.t. to J_+. In these local coordinates, σ is given by

$$\sigma|_U = \sigma_+^{(2,0)}|_U + \sigma_+^{(0,2)}|_U = \frac{1}{2}\sigma^{\mu\nu}\frac{\partial}{\partial z^\mu}\wedge\frac{\partial}{\partial z^\nu} + \frac{1}{2}\sigma^{\bar\mu\bar\nu}\frac{\partial}{\partial z^{\bar\mu}}\wedge\frac{\partial}{\partial z^{\bar\nu}}, \tag{3.13}$$

with $\sigma^{\mu\nu} = \sigma(dz^\mu,dz^\nu)$ and $\sigma^{\bar\mu\bar\nu} = \sigma(dz^{\bar\mu},dz^{\bar\nu}) = \overline{\sigma^{\mu\nu}}$.

We will only proof $\bar\partial_+\sigma^{\mu\nu} = 0$. $\partial_+\sigma^{\bar\mu\bar\nu} = 0$ then follows from the fact that σ is real and the proof for the decompositon w.r.t. J_- is analogous to this one.

We use the fact that in our coordinate system J_+ is in its canonical form and that $g^{\mu\nu} = g^{\bar\mu\bar\nu} = 0$ to determine $\sigma^{\mu\nu}$ (here, Greek indices run over $\{1,\dots,n\}$ and Latin indices run over $\{1,\dots,n,\bar 1,\dots,\bar n\}$):

$$\begin{aligned} \sigma^{\mu\nu} &= \left([J_+,J_-]g^{-1}\right)^{\mu\nu} = [J_+,J_-]^\mu{}_{\bar\kappa} g^{\bar\kappa\nu} = \left({J_+}^\mu{}_j {J_-}^j{}_{\bar\kappa} - {J_-}^\mu{}_j {J_+}^j{}_{\bar\kappa}\right) g^{\bar\kappa\nu} \\ &= \left(i\delta^\mu{}_j {J_-}^j{}_{\bar\kappa} - {J_-}^\mu{}_j(-i\delta^j{}_{\bar\kappa})\right) g^{\bar\kappa\nu} = 2i{J_-}^\mu{}_{\bar\kappa} g^{\bar\kappa\nu}. \end{aligned} \tag{3.14}$$

Since J_+ is preserved by ∇^+, $\nabla_j^+ dz^\mu$ is a (1,0)-form w.r.t. J_+:

$$J_+^T(\nabla_j^+ dz^\mu) = \nabla_j^+(J_+^T dz^\mu) = i\nabla_j^+ dz^\mu \quad (\mu\in\{1,\dots,n\},\ j\in\{1,\dots,n,\bar 1,\dots,\bar n\}),$$

i.e. the (0,1) component of

$$\nabla_j^+ dz^\mu = -(\Gamma_j{}^\mu{}_k + \frac{1}{2}T_j{}^\mu{}_k)dz^k = -(\Gamma_j{}^\mu{}_\rho + \frac{1}{2}T_j{}^\mu{}_\rho)dz^\rho - (\Gamma_j{}^\mu{}_{\bar\rho} + \frac{1}{2}T_j{}^\mu{}_{\bar\rho})dz^{\bar\rho} \tag{3.15}$$

must vanish, leading to $\Gamma_j{}^{\mu}{}_{\bar\rho} = -\frac{1}{2}T_j{}^{\mu}{}_{\bar\rho}$. Using the symmetry of Γ (the Levi-Civita connection is torsionless) and the asymmetry of T in the lower indices, we get

$$\Gamma_{\bar\rho}{}^{\mu}{}_{j} = \frac{1}{2}T_{\bar\rho}{}^{\mu}{}_{j}. \tag{3.16}$$

In addition to this identity, we need $T_{\bar\rho}g^{-1} = -g^{-1}(T_{\bar\rho})^T$:

$$T_{\bar\rho}{}^{q}{}_{k}g^{kl} = T_{\bar\rho}{}^{ql} = -T_{\bar\rho}{}^{lq} = -g^{qj}T_{\bar\rho}{}^{l}{}_{j}. \tag{3.17}$$

Now we can finish the proof:

$$\begin{aligned}
\frac{1}{2i}\partial_{\bar\rho}\sigma^{\mu\nu} &= \partial_{\bar\rho}\left((J_-g^{-1})^{\mu\nu}\right) = \left(\nabla^+_{\bar\rho}(J_-g^{-1})\right)^{\mu\nu} - \left(\Gamma_{\bar\rho}{}^{\mu}{}_{j} + \frac{1}{2}T_{\bar\rho}{}^{\mu}{}_{j}\right)(J_-g^{-1})^{j\nu} \\
&- \left(\Gamma_{\bar\rho}{}^{\nu}{}_{j} + \frac{1}{2}T_{\bar\rho}{}^{\nu}{}_{j}\right)(J_-g^{-1})^{\mu j} \\
&= \left(\nabla^+_{\bar\rho}(J_-g^{-1})\right)^{\mu\nu} - T_{\bar\rho}{}^{\mu}{}_{j}J_-{}^{j}{}_{k}g^{k\nu} - J^{\mu}_{-q}g^{qj}T_{\bar\rho}{}^{\nu}{}_{j},
\end{aligned} \tag{3.18}$$

where for the last equality we used equation 3.16. The covariant derivative in equation 3.18 is

$$\begin{aligned}
\left(\nabla^+_{\bar\rho}(J_-g^{-1})\right)^{\mu\nu} &= \left((\nabla^+_{\bar\rho}J_-)g^{-1}\right)^{\mu\nu} = (T_{\bar\rho}(J_-))^{\mu}{}_{k}g^{k\nu} \\
&= \left(T_{\bar\rho}{}^{\mu}{}_{j}J_-{}^{j}{}_{k} - T_{\bar\rho}{}^{q}{}_{k}J_-{}^{\mu}{}_{q}\right)g^{k\nu} \\
&= T_{\bar\rho}{}^{\mu}{}_{j}J_-{}^{j}{}_{k}g^{k\nu} + J_-{}^{\mu}{}_{q}g^{qj}T_{\bar\rho}{}^{\nu}{}_{j},
\end{aligned} \tag{3.19}$$

where we have first used the fact that ∇^+ is preserving the metric, then $\nabla^+ = \nabla^- + T$ and $\nabla^- J_- = 0$; and finally equation 3.17. Plugging this into equation 3.18, we get $\partial_{\bar\rho}\sigma^{\mu\nu} = 0$, i.e. $\sigma_+^{(2,0)}$ is holomorphic. □

4.

$$\sigma^{il}\partial_l\sigma^{jk} + \sigma^{jl}\partial_l\sigma^{ki} + \sigma^{kl}\partial_l\sigma^{ij} = 0 \tag{3.20}$$

in any coordinate system or equivalently

$$\{f,g\} := \sigma(df,dg) \quad \forall f,g \in C^\infty(M) \tag{3.21}$$

fulfills the Jacobi identity [8]. Together with 1. this makes σ a Poisson structure.

The combination of all these properties can be expressed as the statement that σ is a real holomorphic Poisson structure, i.e. the real part of a holomorphic Poisson structure.

3.3 The generalized Kähler potential

In [6] it was shown that like ordinary Kähler geometry, generalized Kähler geometry is locally described by a single function, the generalized Kähler potential. The proof uses the symplectic foliation of a generalized Kähler manifold induced by the Poisson structure σ:

A Poisson structure can be thought of as a map $T^*M \to TM$. The image $\mathcal{D}$ of this map constitutes (at least for regular Poisson manifolds) an integrable distribution, so there is a foliation of M by maximal integral manifolds of $\mathcal{D}$. These submanifolds are called symplectic leaves, since the restriction of the Poisson stucture to a symplectic leaf is invertible and the inverse of a Poisson structure is always a symplectic structure.

In this thesis, we restrict our attention to a symplectic leaf of σ or, equivalently, consider generalized Kähler manifolds where σ is invertible. In sigma model language this means that we consider Lagrangians that are exclusively described by semichiral superfields.

3.3.1 $Ker[J_+,J_-] = \{0\}$

Here we consider the case, where $[J_+,J_-]$ is invertible and recall how the generalized Kähler potential is defined in this case [13]:

Inverting σ gives

$$\Omega_G := \sigma^{-1} = g[J_+,J_-]^{-1}, \tag{3.22}$$

which is a real, closed and non-degenerate two-form that fulfills $J_\pm^T \Omega_G J_\pm = -\Omega_G$ [8], i.e. it is a real holomorphic symplectic form both w.r.t. J_+ and w.r.t. J_-. This means that Ω_G can be split into the sum of a (2,0)- and a (0,2)-form both w.r.t. J_+ and w.r.t. J_-:

$$\Omega_G = \Omega_+^{(2,0)} + \Omega_+^{(0,2)} = \Omega_-^{(2,0)} + \Omega_-^{(0,2)}, \tag{3.23}$$

where $\bar\partial_\pm \Omega_\pm^{(2,0)} = 0$ and $\partial_\pm \Omega_\pm^{(0,2)} = 0$.

One then introduces Darboux coordinates x_L^p and y_{Lp} (holomorphic w.r.t. J_+) for $\Omega_+^{(2,0)}$ and x_R^p and y_{Rp} (holomorphic w.r.t. J_-) for $\Omega_-^{(2,0)}$ [13].[2] Then

$$\begin{aligned}
\Omega_G &= \Omega_+^{(2,0)} + \Omega_+^{(0,2)} = dx_L^p \wedge dy_{Lp} + d\bar x_L^p \wedge d\bar y_{Lp}, \\
\Omega_G &= \Omega_-^{(2,0)} + \Omega_-^{(0,2)} = dx_R^p \wedge dy_{Rp} + d\bar x_R^p \wedge d\bar y_{Rp},
\end{aligned} \tag{3.24}$$

i.e. the coordinate transformation from $\{x_L,\bar x_L,y_L,\bar y_L\}$ to $\{x_R,\bar x_R,y_R,\bar y_R\}$ is a symplectomorphism (canonical transformation) preserving Ω_G. It is thus described by a generating function $P(x_L,\bar x_L,x_R,\bar x_R)$ such that

$$\frac{\partial P}{\partial x_L} = y_L, \quad \frac{\partial P}{\partial \bar x_L} = \bar y_L, \quad \frac{\partial P}{\partial x_R} = -y_R, \quad \frac{\partial P}{\partial \bar x_R} = -\bar y_R. \tag{3.25}$$

[2]The index p runs over $\{1,...,n\}$, where $dim_{\mathbb{R}} M = 4n$. We will omit it most of the time.

This generating function is the generalized Kähler potential and can be used to reconstruct all the geometric data of generalized Kähler geometry [13], i.e. the two complex structures J_+, J_- the metric g and the B-field. It also turns out to be the superspace Lagrangian for the $\mathcal{N}$ = (2,2) σ-models that led to the discovery of generalized Kähler geometry [6].

Note that we could as well choose to let the potential depend on other combinations of old and new coordinates. The potentials corresponding to the four different choices of variables are then related via Legendre transforms. In previous papers, the potential was chosen to depend on $\{x_L,\bar{x}_L,y_R,\bar{y}_R\}$ to avoid singularities for the case $J_+ = \pm J_-$. Formulas in previous papers for reconstructing g, J_+, J_- and B from the potential remain unchanged, however, in our basis $\{x_L,\bar{x}_L,x_R,\bar{x}_R\}$ (see their derivation below).

3.4 Reconstruction of geometric data

The local reconstruction of the geometric data from the generalized Kähler potential P is based on the fact that $J_\pm$ is in its canonical form in the coordinates $x_{L/R}$, $y_{L/R}$:

$$\begin{aligned} J_+ &= i\frac{\partial}{\partial x_L}\otimes dx_L - i\frac{\partial}{\partial \bar{x}_L}\otimes d\bar{x}_L + i\frac{\partial}{\partial y_L}\otimes dy_L - i\frac{\partial}{\partial \bar{y}_L}\otimes d\bar{y}_L, \\ J_- &= i\frac{\partial}{\partial x_R}\otimes dx_R - i\frac{\partial}{\partial \bar{x}_R}\otimes d\bar{x}_R + i\frac{\partial}{\partial y_R}\otimes dy_R - i\frac{\partial}{\partial \bar{y}_R}\otimes d\bar{y}_R. \end{aligned} \tag{3.26}$$

We denote the coordinate representation by surrounding brackets and use a subscript to indicate the set of coordinates used[3]:

$$(J_+)_L = \begin{pmatrix} \mathbb{J}_n & 0 \\ 0 & \mathbb{J}_n \end{pmatrix}, \quad (J_-)_R = \begin{pmatrix} \mathbb{J}_n & 0 \\ 0 & \mathbb{J}_n \end{pmatrix}; \tag{3.27}$$

where $\mathbb{J}_n$ is the $2n\times 2n$-matrix $\mathbb{J}_n = \mathrm{diag}(+i,-i,...)$. We will use the transformation matrices

$$\begin{aligned} \frac{\partial(x_L,y_L)}{\partial(x_L,x_R)} &= \begin{pmatrix} \partial_{x_L}x_L & \partial_{\bar{x}_L}x_L & \partial_{x_R}x_L & \partial_{\bar{x}_R}x_L \\ \partial_{x_L}\bar{x}_L & \partial_{\bar{x}_L}\bar{x}_L & \partial_{x_R}\bar{x}_L & \partial_{\bar{x}_R}\bar{x}_L \\ \partial_{x_L}y_L & \partial_{\bar{x}_L}y_L & \partial_{x_R}y_L & \partial_{\bar{x}_R}y_L \\ \partial_{x_L}\bar{y}_L & \partial_{\bar{x}_L}\bar{y}_L & \partial_{x_R}\bar{y}_L & \partial_{\bar{x}_R}\bar{y}_L \end{pmatrix} = \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 \\ P_{x_Lx_L} & P_{x_L\bar{x}_L} & P_{x_Lx_R} & P_{x_L\bar{x}_R} \\ P_{\bar{x}_Lx_L} & P_{\bar{x}_L\bar{x}_L} & P_{\bar{x}_Lx_R} & P_{\bar{x}_L\bar{x}_R} \end{pmatrix}, \\ \frac{\partial(x_R,y_R)}{\partial(x_L,x_R)} &= \begin{pmatrix} 0 & 0 & 1_n & 0 \\ 0 & 0 & 0 & 1_n \\ -P_{x_Rx_L} & -P_{x_R\bar{x}_L} & -P_{x_Rx_R} & -P_{x_R\bar{x}_R} \\ -P_{\bar{x}_Rx_L} & -P_{\bar{x}_R\bar{x}_L} & -P_{\bar{x}_Rx_R} & -P_{\bar{x}_R\bar{x}_R} \end{pmatrix}; \end{aligned} \tag{3.28}$$

[3] $x\widehat{=}(x_L,\bar{x}_L,x_R,\bar{x}_R)$, $L\widehat{=}(x_L,\bar{x}_L,y_L,\bar{y}_L)$, $R\widehat{=}(x_R,\bar{x}_R,y_R,\bar{y}_R)$.

to express all geometric data in terms of the coordinates $\{x_L, \bar{x}_L, x_R, \bar{x}_R\}$. Here, we used equation 3.25 to express the Jacobi matrix in terms of double derivatives of the generalized potential. This leads to the fact that all the expressions that we will derive for J_+, J_-, B and g in terms of the x-coordinates are nonlinear combinations of double derivatives of P. As always, we are omitting indices here, e.g. $P_{x_L x_L}$ stands for the $n \times n$-matrix $\left(P_{x_L^i x_L^j}\right)_{1 \leq i,j \leq n}$. From now on, we work with blocks of $2n \times 2n$-matrices and introduce the following abbreviations:

$$\frac{\partial(x_L, y_L)}{\partial(x_L, x_R)} =: \begin{pmatrix} \mathbb{1} & 0 \\ \mathbb{P}_{LL} & \mathbb{P}_{LR} \end{pmatrix}, \quad \frac{\partial(x_R, y_R)}{\partial(x_L, x_R)} =: \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{P}_{RL} & -\mathbb{P}_{RR} \end{pmatrix}. \tag{3.29}$$

In this blockform, inverting $\frac{\partial(x_L, y_L)}{\partial(x_L, x_R)}$ and $\frac{\partial(x_R, y_R)}{\partial(x_L, x_R)}$ is straightforward:

$$\begin{aligned} \frac{\partial(x_L, x_R)}{\partial(x_L, y_L)} &= \begin{pmatrix} \mathbb{1} & 0 \\ \mathbb{P}_{LL} & \mathbb{P}_{LR} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbb{1} & 0 \\ -\mathbb{P}_{RL}^{-1}\mathbb{P}_{LL} & \mathbb{P}_{RL}^{-1} \end{pmatrix}, \\ \frac{\partial(x_L, x_R)}{\partial(x_R, y_R)} &= \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{P}_{RL} & -\mathbb{P}_{RR} \end{pmatrix}^{-1} = \begin{pmatrix} -\mathbb{P}_{LR}^{-1}\mathbb{P}_{RR} & \mathbb{P}_{LR}^{-1} \\ \mathbb{1} & 0 \end{pmatrix}; \end{aligned} \tag{3.30}$$

where we use the notation $\mathbb{P}_{RL}^{-1} := (\mathbb{P}_{LR})^{-1}$, $\mathbb{P}_{LR}^{-1} := (\mathbb{P}_{RL})^{-1}$. Note that while the matrices in 3.29 and 3.30 are coordinate representations w.r.t. different sets of coordinates, their entries are all expressed in terms of the coordinates $\{x_L, \bar{x}_L, x_R, \bar{x}_R\}$.

Now we can express the complex structures with respect to the x-coordinates:

$$(J_+)_x = \frac{\partial(x_L, x_R)}{\partial(x_L, y_L)} (J_+)_L \frac{\partial(x_L, y_L)}{\partial(x_L, x_R)} = \begin{pmatrix} \mathbb{J}_n & 0 \\ \mathbb{P}_{RL}^{-1}\mathbb{C}_{LL} & \mathbb{P}_{RL}^{-1}\mathbb{J}_n\mathbb{P}_{LR} \end{pmatrix}, \tag{3.31}$$

$$(J_-)_x = \frac{\partial(x_L, x_R)}{\partial(x_R, y_R)} (J_-)_R \frac{\partial(x_R, y_R)}{\partial(x_L, x_R)} = \begin{pmatrix} \mathbb{P}_{LR}^{-1}\mathbb{J}_n\mathbb{P}_{RL} & \mathbb{P}_{LR}^{-1}\mathbb{C}_{RR} \\ 0 & \mathbb{J}_n \end{pmatrix}. \tag{3.32}$$

Here, we used the abbreviation $\mathbb{C}_{\circ\bullet} := \mathbb{J}_n\mathbb{P}_{\circ\bullet} - \mathbb{P}_{\circ\bullet}\mathbb{J}_n$.

Using the fact that in the L-coordinates the real holomorphic two-form Ω_G is the canonical symplectic structure, we can also easily express it in the x-basis:

$$\begin{aligned} \Omega_G &= dx_L \wedge dy_L + d\bar{x}_L \wedge d\bar{y}_L = dx_L \wedge \left(\frac{\partial y_L}{\partial \bar{x}_L} d\bar{x}_L + \frac{\partial y_L}{\partial x_R} dx_R + \frac{\partial y_L}{\partial \bar{x}_R} d\bar{x}_R\right) \\ &+ d\bar{x}_L \wedge \left(\frac{\partial \bar{y}_L}{\partial x_L} dx_L + \frac{\partial \bar{y}_L}{\partial x_R} dx_R + \frac{\partial \bar{y}_L}{\partial \bar{x}_R} d\bar{x}_R\right) \\ &= dx_L \wedge (P_{x_L x_R} dx_R + P_{x_L \bar{x}_R} d\bar{x}_R) + d\bar{x}_L \wedge (P_{\bar{x}_L x_R} dx_R + P_{\bar{x}_L \bar{x}_R} d\bar{x}_R), \end{aligned} \tag{3.33}$$

where we used the fact that $P_{x_L\bar{x}_L} = (P_{\bar{x}_L x_L})^T$. Written as a matrix, this gives

$$(\Omega_G)_x = \begin{pmatrix} 0 & 0 & P_{x_L x_R} & P_{x_L \bar{x}_R} \\ 0 & 0 & P_{\bar{x}_L x_R} & P_{\bar{x}_L \bar{x}_R} \\ -P_{x_R x_L} & -P_{x_R \bar{x}_L} & 0 & 0 \\ -P_{\bar{x}_R x_L} & -P_{\bar{x}_R \bar{x}_L} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{P}_{LR} \\ -\mathbb{P}_{RL} & 0 \end{pmatrix}. \tag{3.34}$$

The metric g and the B-field can then be determined via $g = \Omega_G[J_+,J_-]$, $B = \Omega_G\{J_+,J_-\}$ [11]. Since the expressions for g and B do not simplify, we just state the result for the individual parts $E := \frac{1}{2}(g+B) = \Omega_G J_+ J_-$ and $F := \frac{1}{2}(-g+B) = \Omega_G J_- J_+$:

$$\begin{aligned} (E)_x &= \begin{pmatrix} \mathbb{C}_{LL}\mathbb{P}_{LR}^{-1}\mathbb{J}_n\mathbb{P}_{RL} & \mathbb{C}_{LL}\mathbb{P}_{LR}^{-1}\mathbb{C}_{RR} + \mathbb{J}_n\mathbb{P}_{LR}\mathbb{J}_n \\ -\mathbb{P}_{RL}\mathbb{J}_n\mathbb{P}_{LR}^{-1}\mathbb{J}_n\mathbb{P}_{RL} & -\mathbb{P}_{RL}\mathbb{J}_n\mathbb{P}_{LR}^{-1}\mathbb{C}_{RR} \end{pmatrix}, \\ (F)_x &= \begin{pmatrix} \mathbb{P}_{LR}\mathbb{J}_n\mathbb{P}_{RL}^{-1}\mathbb{C}_{LL} & \mathbb{P}_{LR}\mathbb{J}_n\mathbb{P}_{RL}^{-1}\mathbb{J}_n\mathbb{P}_{LR} \\ -\mathbb{J}_n\mathbb{P}_{RL}\mathbb{J}_n - \mathbb{C}_{RR}\mathbb{P}_{RL}^{-1}\mathbb{C}_{LL} & -\mathbb{C}_{RR}\mathbb{P}_{RL}^{-1}\mathbb{J}_n\mathbb{P}_{LR} \end{pmatrix}. \end{aligned} \tag{3.35}$$

3.5 Hyperkähler condition

In order for M to be a hyperkähler manifold, the anticommutator of the two complex structures must be equal to a constant times the identity on TM. Setting $\{J_+,J_-\} = c\mathbb{1}$ gives the following four equations:

$$\begin{aligned} I) \quad & \mathbb{J}_n\mathbb{P}_{LR}^{-1}\mathbb{J}_n\mathbb{P}_{RL} + \mathbb{P}_{LR}^{-1}\left(\mathbb{J}_n\mathbb{P}_{RL}\mathbb{J}_n + \mathbb{C}_{RR}\mathbb{P}_{RL}^{-1}\mathbb{C}_{LL}\right) = c\mathbb{1}, \\ II) \quad & \mathbb{J}_n\mathbb{P}_{LR}^{-1}\mathbb{C}_{RR} + \mathbb{P}_{LR}^{-1}\mathbb{C}_{RR}\mathbb{P}_{RL}^{-1}\mathbb{J}_n\mathbb{P}_{LR} = 0, \\ III) \quad & \mathbb{P}_{RL}^{-1}\mathbb{C}_{LL}\mathbb{P}_{LR}^{-1}\mathbb{J}_n\mathbb{P}_{RL} + \mathbb{J}_n\mathbb{P}_{RL}^{-1}\mathbb{C}_{LL} = 0, \\ IV) \quad & \mathbb{P}_{RL}^{-1}\left(\mathbb{C}_{LL}\mathbb{P}_{LR}^{-1}\mathbb{C}_{RR} + \mathbb{J}_n\mathbb{P}_{LR}\mathbb{J}_n\right) + \mathbb{J}_n\mathbb{P}_{RL}^{-1}\mathbb{J}_n\mathbb{P}_{LR} = c\mathbb{1}. \end{aligned} \tag{3.36}$$

Using $(\mathbb{C}_{\circ\bullet})^T = -\mathbb{C}_{\bullet\circ}$ and $\left(\mathbb{P}_{LR}^{-1}\right)^T = \mathbb{P}_{RL}^{-1}$, we find

$$\mathbb{P}_{RL}(I)\mathbb{P}_{LR}^{-1} \widehat{=} (IV)^T. \tag{3.37}$$

Thus, requirement (I) implies (IV). Applying $\mathbb{P}_{RL}$ to (II) from the left yields

$$\mathbb{A} := \mathbb{P}_{RL}\mathbb{J}_n\mathbb{P}_{LR}^{-1}\mathbb{C}_{RR} = -\mathbb{C}_{RR}\mathbb{P}_{RL}^{-1}\mathbb{J}_n\mathbb{P}_{LR} = \left(\mathbb{P}_{RL}\mathbb{J}_n\mathbb{P}_{LR}^{-1}\mathbb{C}_{RR}\right)^T = \mathbb{A}^T. \tag{3.38}$$

$\mathbb{P}_{LR}(II)$ corresponds to

$$\mathbb{B} := \mathbb{P}_{LR}\mathbb{J}_n\mathbb{P}_{RL}^{-1}\mathbb{C}_{LL} = -\mathbb{C}_{LL}\mathbb{P}_{LR}^{-1}\mathbb{J}_n\mathbb{P}_{RL} = \left(\mathbb{P}_{LR}\mathbb{J}_n\mathbb{P}_{RL}^{-1}\mathbb{C}_{LL}\right)^T = \mathbb{B}^T. \tag{3.39}$$

So all in all, we find that the requirements for the manifold described by the generalized Kähler potential P to be hyperkähler are (I), $\mathbb{A} = \mathbb{A}^T$ and $\mathbb{B} = \mathbb{B}^T$.

The four dimensional case

In four dimensions, $\mathbb{P}_{RL}$ and $\mathbb{P}_{LR}$ are 2×2-matrices and thus easy to invert:

$$\mathbb{P}_{LR}^{-1} = \frac{1}{|P_{x_L x_R}|^2 - |P_{x_L \bar{x}_R}|^2} \begin{pmatrix} P_{\bar{x}_L \bar{x}_R} & -P_{\bar{x}_L x_R} \\ -P_{x_L \bar{x}_R} & P_{x_L x_R} \end{pmatrix}, \quad \mathbb{P}_{RL}^{-1} = \left(\mathbb{P}_{LR}^{-1}\right)^T. \tag{3.40}$$

The requirements $\mathbb{A} = \mathbb{A}^T$ and $\mathbb{B} = \mathbb{B}^T$ are then automatically satisfied and requirement (I) from equation 3.36 yields

$$2\frac{2P_{x_L \bar{x}_L} P_{x_R \bar{x}_R} - |P_{x_L x_R}|^2 + |P_{x_L \bar{x}_R}|^2}{|P_{x_L x_R}|^2 - |P_{x_L \bar{x}_R}|^2} = c. \tag{3.41}$$

The requirement for a four dimensional manifold that is described by the generalized potential P to be hyperkähler is thus that the left side of equation 3.41 is constant. This result has been first obtained in [11] for the special case $c = 0$ and then generalized in [12] for an arbitrary $c \in \mathbb{R}$.

Chapter 4

Generalized Kähler structures on hyperkähler manifolds

We want to transport the idea of a twistor space from hyperkähler to generalized Kähler geometry, namely we interpret a hyperkähler manifold (M,g,I,J,K) as a generalized Kähler manifold (M,g,J_+,J_-), where we fix the left complex structure $J_+ = I$ and let the right complex structure depend on ζ: $J_- = \mathcal{J}(\zeta)$ (see eq. 2.3). So for a given hyperkähler manifold, we consider an S^2-family of generalized Kähler structures whose generalized Kähler potentials we now try to determine.

4.1 Expressing semichiral superfields x_L,y_L, x_R,y_R in terms of arctic superfields $\Upsilon(\zeta)$,$\tilde{\Upsilon}(\zeta)$

The generalized Kähler potential is a generating function for the symplectomorphism from holomorphic coordinates $(x_L\bar{x}_L,y_L,\bar{y}_L)$ with respect to J_+ to holomorphic coordinates $(x_R,\bar{x}_R,$ $y_R,\bar{y}_R)$ with respect to J_- preserving the symplectic form Ω_G. To determine the generalized Kähler potential, we first need to find the coordinates x,y on a given hyperkähler manifold. In order to find them, we need an explicit expression for $\Omega_G = g[J_+,J_-]^{-1}$ (eq. 3.22), which now depends on ζ.

The anticommutator of two complex structures on an irreducible hyperkähler manifold is equal to a constant times the identity, $\{J_+,J_-\} = c\mathbb{1}$ [9]. If $J_+ \neq \pm J_-$, then $|c| < 2$ and $\frac{1}{\sqrt{4-c^2}}[J_+,J_-]$ is another complex structure, so in particular, it squares to $-\mathbb{1}$. Using this, we have

$$\Omega_G = g[J_+,J_-]^{-1} = -\frac{1}{4-c^2}g[J_+,J_-]. \tag{4.1}$$

The two complex structures for the generalized Kähler manifold are chosen to be

$$J_+ = I, \quad J_- = v_1 I + v_2 J + v_3 K = \frac{1}{1+\zeta\bar\zeta}\left[(1-\zeta\bar\zeta)I + (\zeta+\bar\zeta)J + i\,(\bar\zeta-\zeta)K\right], \tag{4.2}$$

where I,J,K pairwise anticommute. This gives $c = -2v_1 = -2\frac{1-\zeta\bar\zeta}{1+\zeta\bar\zeta}$ and $[J_+,J_-] = 2v_2K - 2v_3J$, so 4.1 becomes

$$\Omega_G(\zeta) = -\frac{1}{2-2v_1^2}\left(v_2\omega_3 - v_3\omega_2\right) = -\frac{1+\zeta\bar\zeta}{8\zeta\bar\zeta}\left[(\zeta+\bar\zeta)\omega_3 - i(\bar\zeta-\zeta)\omega_2\right]. \tag{4.3}$$

On the one hand, we need to split this into the sum of a (2,0)- and a (0,2)-form with respect to J_+. From the first chapter, we know that $\omega^{(2,0)} = \omega_2 + i\omega_3$ and $\omega^{(0,2)} = \omega_2 - i\omega_3$ are such forms. Indeed, we find a splitting into the holomorphic form $\Omega_+^{(2,0)} = i\bar\zeta\frac{1+\zeta\bar\zeta}{8\zeta\bar\zeta}\omega^{(2,0)}$ and the antiholomorphic form $\Omega_+^{(0,2)} = -i\zeta\frac{1+\zeta\bar\zeta}{8\zeta\bar\zeta}\omega^{(0,2)}$ with respect to J_+. Combining equations 2.6 and 2.7, we get[1]

$$\omega^{(2,0)} = id\Upsilon(\zeta=0)\wedge d\tilde\Upsilon(\zeta=0), \quad \omega^{(0,2)} = id\bar\Upsilon(\bar\zeta=0)\wedge d\bar{\tilde\Upsilon}(\bar\zeta=0). \tag{4.4}$$

Thus, we can choose the following Darboux coordinates for $\Omega_G(\zeta)$:

$$\begin{aligned} x_L^p &= \Upsilon^p(\zeta=0), \quad y_{L_p} = -\bar\zeta\frac{1+\zeta\bar\zeta}{8\zeta\bar\zeta}\tilde\Upsilon_p(\zeta=0), \\ \bar x_L^p &= \bar\Upsilon^p(\zeta=0), \quad \bar y_{L_p} = -\zeta\frac{1+\zeta\bar\zeta}{8\zeta\bar\zeta}\bar{\tilde\Upsilon}_p(\zeta=0); \end{aligned} \tag{4.5}$$

to ensure

$$\Omega_+^{(2,0)} = dx_L\wedge dy_L, \quad \Omega_+^{(0,2)} = d\bar x_L\wedge d\bar y_L. \tag{4.6}$$

On the other hand, we need to split Ω_G into a (2,0)- and a (0,2)-form with respect to $J_- = \mathcal{J}(\zeta)$. Again, we know from the first chapter that $\Omega_H(\zeta)$ and $\overline{\Omega_H(\zeta)}$ fulfill this property and indeed we find that $\Omega_-^{(2,0)}$ and $\Omega_-^{(0,2)}$ are proportional to $\Omega_H(\zeta)$ and $\overline{\Omega_H(\zeta)}$, respectively:

$$\Omega_-^{(2,0)} = i\bar\zeta\frac{1}{8\zeta\bar\zeta}\Omega_H(\zeta), \quad \Omega_-^{(0,2)} = -i\zeta\frac{1}{8\zeta\bar\zeta}\overline{\Omega_H(\zeta)}. \tag{4.7}$$

Knowing $\Omega_H(\zeta) = id\Upsilon(\zeta)\wedge d\tilde\Upsilon(\zeta)$ and imposing

$$\Omega_-^{(2,0)} \stackrel{!}{=} dx_R\wedge dy_R, \quad \Omega_-^{(0,2)} \stackrel{!}{=} d\bar x_R\wedge d\bar y_R, \tag{4.8}$$

[1] We denote the complex conjugate of $\Upsilon(\zeta)$ by $\bar\Upsilon \equiv \bar\Upsilon(\bar\zeta) \equiv \overline{\Upsilon(\zeta)}$ which is not to be confused with the notation in [3], where $\bar\Upsilon$ is short for $\breve\Upsilon(\zeta) = \bar\Upsilon(-\zeta^{-1})$.

we can choose

$$
\begin{aligned}
x_R^p &= \Upsilon^p(\zeta), \quad y_{R_p} = -\bar{\zeta}\frac{1}{8\zeta\bar{\zeta}}\tilde{\Upsilon}_p(\zeta);\\
\bar{x}_R^p &= \overline{\Upsilon^p(\zeta)}, \quad \bar{y}_{R_p} = -\zeta\frac{1}{8\zeta\bar{\zeta}}\overline{\tilde{\Upsilon}_p(\zeta)}.
\end{aligned}
\tag{4.9}
$$

The choice of Darboux coordinates in 4.5 and 4.9 is not unique. For instance, we could distribute factors differently and set

$$
x'_L = i\sqrt{\frac{1+\zeta\bar{\zeta}}{8\zeta}}\Upsilon(0),\ x'_R = i\sqrt{\frac{1}{8\zeta}}\Upsilon \text{ and } y'_L = i\sqrt{\frac{1+\zeta\bar{\zeta}}{8\zeta}}\tilde{\Upsilon}(0),\ y'_R = i\sqrt{\frac{1}{8\zeta}}\tilde{\Upsilon}. \tag{4.10}
$$

We could also use a more complicated symplectomorphism to make a different identification of x,y in terms of Υ and $\tilde{\Upsilon}$.

With the identifications 4.5 and 4.9, we are now able to express the coordinates $x_{L,R}$ and $y_{L,R}$ that describe semichiral superfields in 2D $\mathcal{N}=(2{,}2)$ models in terms of the coordinates $\Upsilon(\zeta)$, $\tilde{\Upsilon}(\zeta)$ describing arctic superfields in the projective superspace formulation of $\mathcal{N}=2$ supersymmetric sigma models. This will enable us to determine the ζ-dependent generalized Kähler potential for hyperkähler manifolds whose projective superspace description is already known.

4.2 Euclidean space

We now use the relation between x,y and Υ,$\tilde{\Upsilon}$ derived in the last section to determine the generalized Kähler potential for Euclidean space. Here the Kähler potential is given by

$$
K = z\bar{z} + u\bar{u}, \tag{4.11}
$$

which clearly fulfills equation 2.8. All the results in this section can be extended to higher dimensional Euclidean space by introducing additional indices.

Assuming that (z,u) are holomorphic coordinates w.r.t. I and setting $\omega^{(2,0)} = idz\wedge du$, we get the complex structures as described in section 2.2:

$$
\begin{aligned}
I &= i\frac{\partial}{\partial z}\otimes dz - i\frac{\partial}{\partial \bar{z}}\otimes d\bar{z} + i\frac{\partial}{\partial u}\otimes du - i\frac{\partial}{\partial \bar{u}}\otimes d\bar{u},\\
J &= -i\frac{\partial}{\partial z}\otimes d\bar{u} + i\frac{\partial}{\partial u}\otimes d\bar{z} + i\frac{\partial}{\partial \bar{z}}\otimes du - i\frac{\partial}{\partial \bar{u}}\otimes dz,\\
K &= \frac{\partial}{\partial z}\otimes d\bar{u} - \frac{\partial}{\partial u}\otimes d\bar{z} + \frac{\partial}{\partial \bar{z}}\otimes du - \frac{\partial}{\partial \bar{u}}\otimes dz.
\end{aligned}
\tag{4.12}
$$

They are the differentials of the imaginary basis quaternions[2] i,j,k acting on $\mathbb{H} \approx \mathbb{C}^2$ by left multiplication, where we make the identification $(z,u) = (x_0+ix_1,x_2+ix_3) \mapsto x_0+ix_1+jx_2+kx_3$.

$$\Upsilon(\zeta) = z - \zeta\bar{u}, \quad \tilde{\Upsilon}(\zeta) = u + \zeta\bar{z} \tag{4.13}$$

fulfill equations 2.21 and 2.25, i.e. they are holomorphic w.r.t. $\mathcal{J}(\zeta)$ and satisfy equation 2.7. Using equations 4.5 and 4.9, we make the identifications

$$x_L = \Upsilon(\zeta = 0) = z,\ x_R = \Upsilon(\zeta) \equiv \Upsilon \text{ and } y_L = -\frac{1+\zeta\bar{\zeta}}{8\zeta}u,\ y_R = -\frac{1}{8\zeta}\tilde{\Upsilon}. \tag{4.14}$$

Solving for y_L, y_R in terms of x_L, x_R, we get

$$y_L = -\frac{1+\zeta\bar{\zeta}}{8\zeta\bar{\zeta}}(\bar{x}_L - \bar{x}_R), \quad y_R = -\frac{1}{8\zeta\bar{\zeta}}\left((1+\zeta\bar{\zeta})\bar{x}_L - \bar{x}_R\right), \tag{4.15}$$

which leads (up to an additive constant) to the generating function (see equation 3.25)

$$P = -\frac{1}{8\zeta\bar{\zeta}}\left[x_R\bar{x}_R + (1+\zeta\bar{\zeta})\cdot(x_L\bar{x}_L - x_L\bar{x}_R - \bar{x}_Lx_R)\right]. \tag{4.16}$$

This is the generalized Kähler potential for Euclidean space, where $J_+ = I$ and J_- is an arbitrary point on the twistor-sphere of complex structures, $J_- \neq \pm I$. However, we notice that P only involves the combination $\zeta\bar{\zeta}$, i.e. it only depends on the angle between J_+ and J_- in the space spanned by the three complex structures (I,J,K). Also, P turns out to be asymmetric between left- and right-coordinates. This can be resolved however, as there are various ambiguities in the generalized Kähler potential. For instance, we could distribute factors differently in 4.14 or even perform a more complicated symplectomorphism, going to new coordinates $x'_{L/R}$, $y'_{L/R}$. Furthermore, we can perform Legendre transforms and express the potential in terms of a different set of variables.

If we make the identifications

$$x'_L = i\sqrt{\frac{1+\zeta\bar{\zeta}}{8\zeta}}z,\ x'_R = i\sqrt{\frac{1}{8\zeta}}\Upsilon \text{ and } y'_L = i\sqrt{\frac{1+\zeta\bar{\zeta}}{8\zeta}}u,\ y'_R = i\sqrt{\frac{1}{8\zeta}}\tilde{\Upsilon}, \tag{4.17}$$

the potential becomes left-right-symmetric. Performing an additional Legendre transform exchanging the roles of x'_R and y'_R for instance then leads the potential

$$P' = \sqrt{1+\zeta\bar{\zeta}}\cdot(x'_Ly'_R + \bar{x}'_L\bar{y}'_R) + \sqrt{\zeta\bar{\zeta}}\cdot(x'_L\bar{x}'_L + y'_R\bar{y}'_R). \tag{4.18}$$

[2]We stick to the convention from previous papers and include the i-factor in the choice of $\omega^{(2,0)}$. This interchanges the complex structures J and K, such that J acts like left multiplication by k and K acts like left multiplication by $-j$.

4.3 Example: Eguchi-Hanson geometry

The real function

$$K=\sqrt{1+4u\bar{u}(1+z\bar{z})^2}+\frac{1}{2}\ln\left[\frac{4u\bar{u}(1+z\bar{z})^2}{\left(1+\sqrt{1+4u\bar{u}(1+z\bar{z})^2}\right)^2}\right] \tag{4.19}$$

in the two complex variables z, u fulfills the Monge-Ampère equation 2.8. It thus defines a hyperkähler metric, where the Kähler forms are given by equation 2.9. The first Kähler form takes the form

$$\begin{aligned}\omega_1 &= -\frac{i}{2}\frac{1+z\bar{z}}{\sqrt{1+4u\bar{u}(1+z\bar{z})^2}}\Bigg[(1+z\bar{z})\,du\wedge d\bar{u}+2u\bar{z}\,dz\wedge d\bar{u} \\ &\qquad +2z\bar{u}\,du\wedge d\bar{z}+\left(\frac{1}{(1+z\bar{z})^3}+4u\bar{u}\right)dz\wedge d\bar{z}\Bigg],\end{aligned} \tag{4.20}$$

from which the metric can be read off. This is the well-known Eguchi-Hanson geometry: Setting $u=\frac{1}{2}u'^2$, $z=\frac{z'}{u'}$ and $r:=\sqrt{u'\bar{u}'+z'\bar{z}'}$ gives the familiar Kähler potential

$$K=\sqrt{1+r^4}+log\frac{r^2}{1+\sqrt{1+r^4}} \tag{4.21}$$

for the Eguchi-Hanson metric [3].

The holomorphic Darboux coordinates for $\Omega_H(\zeta)$ (fulfilling equations 2.21 and 2.25) can be chosen as [3]

$$\begin{aligned}\tilde{\Upsilon} &= u+\zeta^2\bar{z}^2\bar{u}+\frac{\bar{z}\zeta}{1+z\bar{z}}\sqrt{1+4u\bar{u}(1+z\bar{z})^2}, \\ \Upsilon &= z-\frac{2\bar{u}\zeta(1+z\bar{z})^2}{1+\sqrt{1+4u\bar{u}(1+z\bar{z})^2}+2\bar{u}\bar{z}\zeta(1+z\bar{z})}.\end{aligned} \tag{4.22}$$

We solve $\Upsilon,\bar{\Upsilon}(z,\bar{z},u,\bar{u})$ for u and $\bar{u}$ to get

$$u(z,\bar{z},\Upsilon,\bar{\Upsilon})=\frac{\zeta}{1+z\bar{z}}\cdot\frac{(\bar{z}-\bar{\Upsilon})(1+\Upsilon\bar{z})}{\zeta\bar{\zeta}(1+\bar{\Upsilon}z)(1+\Upsilon\bar{z})-(z-\Upsilon)(\bar{z}-\bar{\Upsilon})} \tag{4.23}$$

and its complex conjugate. Using this and the identifications derived in section 4 (equation 4.14), we get $y_L(x_L,x_R)$. We then integrate $y_L(x_L,x_R)$ w.r.t. x_L to get the generalized Kähler potential up to a possible additive term that is independent of x_L:

$$\begin{aligned}P &= \int y_L(x_L,x_R)\,dx_L=-\bar{\zeta}\frac{1+\zeta\bar{\zeta}}{8\zeta\bar{\zeta}}\int u(z,\Upsilon)\,dz \\ &= -\frac{1}{8}\cdot log\frac{1+x_L\bar{x}_L}{\zeta\bar{\zeta}(1+x_L\bar{x}_R)(1+\bar{x}_Lx_R)-(x_L-x_R)(\bar{x}_L-\bar{x}_R)}\end{aligned} \tag{4.24}$$

Plugging $u(z=x_L,\Upsilon=x_R)$ into $\tilde{\Upsilon}(z=x_L,u)$ (equation 4.22) gives

$$y_R(x_L,x_R) = -\bar{\zeta}\frac{1}{8\zeta\bar{\zeta}}\tilde{\Upsilon}(x_L,x_R) = -\frac{1}{8}\cdot\frac{(1+\zeta\bar{\zeta})\cdot\bar{x}_L - \left(1-\zeta\bar{\zeta}\cdot x_L\bar{x}_L\right)\bar{x}_R}{\zeta\bar{\zeta}(1+x_L\bar{x}_R)(1+\bar{x}_Lx_R) - (x_L-x_R)(\bar{x}_L-\bar{x}_R)}, \tag{4.25}$$

which is indeed equal to $-\frac{\partial P}{\partial x_R}$. P is real, so $\frac{\partial P}{\partial \bar{x}_L} = \bar{y}_L$ and $\frac{\partial P}{\partial \bar{x}_R} = -\bar{y}_R$ are also fulfilled and thus equation 4.24 gives indeed the ζ-dependent generalized Kähler potential for the Eguchi-Hanson geometry:

$$P(x_L,\bar{x}_L,x_R,\bar{x}_R) = -\frac{1}{8}\cdot log\frac{1+|x_L|^2}{\zeta\bar{\zeta}\cdot|1+x_L\bar{x}_R|^2 - |x_L-x_R|^2}. \tag{4.26}$$

Again, the generalized Kähler potential turns out to depend only on the combination $\zeta\bar{\zeta}$, i.e. on the angle between J_+ and J_-. Of course, there are again many ambiguities in the potential, but 4.26 seems to be already in its simplest form.

4.3.1 Legendre transform

Since the generalized Kähler potential was chosen to depend on y_R instead of x_R in all references, we originally determined it for this choice of variables. For the record, we state these results in this section and show that the potential 4.26 then arises from a Legendre transform exchanging y_R and x_R.

Inverting $\tilde{\Upsilon},\bar{\tilde{\Upsilon}}(z,u,\bar{z},\bar{u})$ from equation 4.22, we obtain

$$u = \frac{1}{(1+z\bar{z})(1-z\bar{z}\zeta\bar{\zeta})^2}\left((1+z\bar{z})(\tilde{\Upsilon}+\tilde{\Upsilon}^*\bar{z}^2\zeta^2) - \bar{z}\zeta\sqrt{(1-z\bar{z}\zeta\bar{\zeta})^2 + 4\tilde{\Upsilon}\tilde{\Upsilon}^*(1+z\bar{z})^2}\right). \tag{4.27}$$

Plugging this into $\Upsilon(z,u,\bar{z},\bar{u})$, one obtains

$$\Upsilon = z - 2\zeta\cdot\frac{1+z\bar{z}}{1-z\bar{z}\zeta\bar{\zeta}}\cdot\frac{(1+z\bar{z})(\tilde{\Upsilon}^*+\tilde{\Upsilon}z^2\bar{\zeta}^2) - z\bar{\zeta}\sqrt{(1-z\bar{z}\zeta\bar{\zeta})^2+4\tilde{\Upsilon}\tilde{\Upsilon}^*(1+z\bar{z})^2}}{1-z\bar{z}\zeta\bar{\zeta}+\sqrt{(1-z\bar{z}\zeta\bar{\zeta})^2+4\tilde{\Upsilon}\tilde{\Upsilon}^*(1+z\bar{z})^2} - 2z\bar{\zeta}\tilde{\Upsilon}(1+z\bar{z})}. \tag{4.28}$$

As in the last section, we make the replacements

$$x_L = z, \quad y_L = -\frac{1+\zeta\bar{\zeta}}{8\zeta}u; \quad x_R = \Upsilon(\zeta), \quad y_R = -\frac{1}{8\zeta}\tilde{\Upsilon}(\zeta). \tag{4.29}$$

Up to an additive constant in x_L, the generalized Kähler potential is

$$\begin{aligned}
P' &= \int y_L(x_L,y_R)\,dx_L = -\frac{1+\zeta\bar{\zeta}}{8\zeta}\int u(z,\tilde{\Upsilon})\,dz = \frac{1+\zeta\bar{\zeta}}{\zeta\bar{\zeta}}\frac{1}{1-\zeta\bar{\zeta}x_L\bar{x}_L}\left(\frac{y_R}{\bar{x}_L}+\zeta\bar{\zeta}\bar{x}_L\bar{y}_R\right) \\
&- \frac{1}{8}\log\left(\frac{1-\zeta\bar{\zeta}x_L\bar{x}_L+\sqrt{(1-\zeta\bar{\zeta}x_L\bar{x}_L)^2+256\zeta\bar{\zeta}y_R\bar{y}_R(1+x_L\bar{x}_L)^2}}{1+x_L\bar{x}_L}\right) \\
&+ \frac{1}{8}\frac{\sqrt{(1-\zeta\bar{\zeta}x_L\bar{x}_L)^2+256\zeta\bar{\zeta}y_R\bar{y}_R(1+x_L\bar{x}_L)^2}}{1-\zeta\bar{\zeta}x_L\bar{x}_L}+c_{x_L}(\bar{x}_L,y_L,\bar{y}_L).
\end{aligned} \tag{4.30}$$

Setting $c_{x_L}=-\frac{1+\zeta\bar{\zeta}}{\zeta\bar{\zeta}}\frac{y_R}{\bar{x}_L}$ makes P' real, so that it also fufills $\frac{\partial P'}{\partial \bar{x}_L}=\bar{y}_L$. P' then becomes

$$\begin{aligned}
P' &= \frac{1+\zeta\bar{\zeta}}{1-\zeta\bar{\zeta}x_L\bar{x}_L}(x_Ly_R+\bar{x}_L\bar{y}_R) \\
&- \frac{1}{8}\log\left(\frac{1-\zeta\bar{\zeta}x_L\bar{x}_L+\sqrt{(1-\zeta\bar{\zeta}x_L\bar{x}_L)^2+256\zeta\bar{\zeta}y_R\bar{y}_R(1+x_L\bar{x}_L)^2}}{1+x_L\bar{x}_L}\right) \\
&+ \frac{1}{8}\frac{\sqrt{(1-\zeta\bar{\zeta}x_L\bar{x}_L)^2+256\zeta\bar{\zeta}y_R\bar{y}_R(1+x_L\bar{x}_L)^2}}{1-\zeta\bar{\zeta}x_L\bar{x}_L}.
\end{aligned} \tag{4.31}$$

and fulfills $\frac{\partial P'}{\partial y_R}=x_R$ and $\frac{\partial P'}{\partial \bar{y}_R}=\bar{x}_R$ without any further corrections. It is thus the generalized Kähler potential.

Now we perform a Legendre transform replacing y_R,$\bar{y}_R$ by x_R,$\bar{x}_R$. The new coordinate in terms of the old one is given by $x_R=\frac{\partial P'}{\partial y_R}$:

$$x_R=\frac{1+\zeta\bar{\zeta}}{1-\zeta\bar{\zeta}x_L\bar{x}_L}x_L+\frac{1}{16y_R}\left(\frac{\sqrt{(1-\zeta\bar{\zeta}x_L\bar{x}_L)^2+256\zeta\bar{\zeta}y_R\bar{y}_R(1+x_L\bar{x}_L)^2}}{1-\zeta\bar{\zeta}x_L\bar{x}_L}-1\right). \tag{4.32}$$

Surprisingly, this causes the linear terms in the Legendre transform to cancel all terms in P' except for the logarithm (up to a an irrelevant additive constant):

$$\begin{aligned}
P &= P'-x_Ry_R-\bar{x}_R\bar{y}_R \qquad (4.33)\\
&= \frac{1}{8}-\frac{1}{8}\log\left(\frac{1-\zeta\bar{\zeta}x_L\bar{x}_L+\sqrt{(1-\zeta\bar{\zeta}x_L\bar{x}_L)^2+256\zeta\bar{\zeta}y_R\bar{y}_R(1+x_L\bar{x}_L)^2}}{1+x_L\bar{x}_L}\right).
\end{aligned}$$

Now it just remains to invert the relation 4.32 to get $y_R(x_R)$ and to plug this into 4.33. Indeed, after some calculation we find that (ignoring the additive constant) the resulting potential is exactly 4.26.

4.4 $S^2 \times S^2$-family of generalized complex structures

In this section, we generalize our results from section 4.1. Instead of fixing one complex structure, we let both J_+ and J_- depend on an individual complex coordinate and thus consider an $S^2 \times S^2$-family of generalized complex structures (M,g,J_+,J_-) on a given hyperkähler manifold (M,g,I,J,K). We parametrize vectors $\vec{u}$, $\vec{v} \in S^2\backslash(-1,0,0)$ by complex coordinates ζ_1, ζ_2 like in equation 2.3 and define

$$J_+ := \mathcal{J}(\zeta_1) = u_1 I + u_2 J + u_3 K, \quad J_- = \mathcal{J}(\zeta_2) = v_1 I + v_2 J + v_3 K. \tag{4.34}$$

The anticommutator depends only on the angle θ between $\vec{u}$ and $\vec{v}$:

$$\{J_+,J_-\} = -2(\vec{u}\cdot\vec{v})\mathbb{1} = -2\cos\theta\,\mathbb{1}. \tag{4.35}$$

The commutator turns out to be perpendicular to J_+ and J_- in the space spanned by (I,J,K):

$$[J_+,J_-] = 2(u_2v_3 - u_3v_2)I - 2(u_1v_3 - u_3v_1)J + 2(u_1v_2 - u_2v_1)K = 2(\vec{u}\times\vec{v})\cdot(I,J,K)^T. \tag{4.36}$$

In order to determine the coordinates x_R,y_R, we need to split Ω_G into a (2,0)- and a (0,2)-form w.r.t. J_-. Indeed, we find that

$$\begin{aligned} g[J_+,J_-] &= \frac{i}{(1+\zeta_2\bar{\zeta}_2)^2}\Big((\overline{\vec{a}(\zeta_2)}\cdot\vec{u})\,\Omega_H(\zeta_2) - (\vec{a}(\zeta_2)\cdot\vec{u})\,\overline{\Omega_H(\zeta_2)}\Big) \\ &= \frac{i}{(1+\zeta_2\bar{\zeta}_2)^2}\Big((\overline{\vec{a}(\zeta_2)}\cdot\vec{u})\,\vec{a}(\zeta_2) - (\vec{a}(\zeta_2)\cdot\vec{u})\,\overline{\vec{a}(\zeta_2)}\Big)\cdot\vec{\omega}, \end{aligned} \tag{4.37}$$

where $\vec{a}(\zeta) = (-2\zeta,1-\zeta^2,i(1+\zeta^2))^T$, i.e. $\Omega_H(\zeta) = \vec{a}(\zeta)\cdot\vec{\omega}$ (see eq. 2.6). So we find

$$\Omega_G = -\frac{1}{4-4(\vec{u}\cdot\vec{v})^2}g[J_+,J_-] = \Omega_-^{(2,0)} + \Omega_-^{(0,2)}, \tag{4.38}$$

where

$$\Omega_-^{(2,0)} = \frac{-i\,(\overline{\vec{a}(\zeta_2)}\cdot\vec{u})}{4\sin^2\theta\,(1+\zeta_2\bar{\zeta}_2)^2}\,\Omega_H(\zeta_2), \quad \Omega_-^{(0,2)} = \frac{i\,(\vec{a}(\zeta_2)\cdot\vec{u})}{4\sin^2\theta\,(1+\zeta_2\bar{\zeta}_2)^2}\,\overline{\Omega_H(\zeta_2)}. \tag{4.39}$$

Thus knowing that $\Omega_H(\zeta_2) = id\Upsilon(\zeta_2)\wedge d\tilde{\Upsilon}(\zeta_2)$, we can choose (omitting indices)

$$x_R = \Upsilon(\zeta_2) \equiv \Upsilon_2, \quad y_R = \frac{\overline{\vec{a}(\zeta_2)}\cdot\vec{u}}{4\sin^2\theta\,(1+\zeta_2\bar{\zeta}_2)^2}\tilde{\Upsilon}(\zeta_2) \equiv c_2\tilde{\Upsilon}_2. \tag{4.40}$$

to get $\Omega_G = dx_R\wedge dy_R + d\bar{x}_R\wedge d\bar{y}_R$.

Exchanging the roles of $\vec{u}$,$\vec{v}$ and ζ_1, ζ_2, respectively, and considering the antisymmetry of $[J_+,J_-]$, we get the following splitting w.r.t. J_-:

$$\Omega_+^{(2,0)} = \frac{i\,(\overline{\vec{a}(\zeta_1)}\cdot\vec{v})}{4\sin^2\theta\,(1+\zeta_1\bar{\zeta}_1)^2}\,\Omega_H(\zeta_1), \quad \Omega_+^{(0,2)} = \frac{-i\,(\vec{a}(\zeta_1)\cdot\vec{v})}{4\sin^2\theta\,(1+\zeta_1\bar{\zeta}_1)^2}\,\overline{\Omega_H(\zeta_1)}, \tag{4.41}$$

which allows us to choose

$$x_L = \Upsilon(\zeta_1) \equiv \Upsilon_1, \quad y_L = \frac{-\overline{\vec{a}(\zeta_1)}\cdot\vec{v}}{4\sin^2\theta\,(1+\zeta_1\bar{\zeta}_1)^2}\tilde{\Upsilon}(\zeta_1) \equiv c_1\tilde{\Upsilon}_1. \tag{4.42}$$

The constants c_1,$c_2(\zeta_1,\zeta_2)$ can be written as

$$\begin{aligned} c_1 &= \frac{-(\bar{\zeta}_2-\bar{\zeta}_1)}{2(1+\zeta_1\bar{\zeta}_1)^2(1+\zeta_2\bar{\zeta}_2)^2\sin^2\theta}(1+\bar{\zeta}_1\zeta_2)(1+\zeta_2\bar{\zeta}_2), \\ c_2 &= \frac{-(\bar{\zeta}_2-\bar{\zeta}_1)}{2(1+\zeta_1\bar{\zeta}_1)^2(1+\zeta_2\bar{\zeta}_2)^2\sin^2\theta}(1+\zeta_1\bar{\zeta}_2)(1+\zeta_1\bar{\zeta}_1). \end{aligned} \tag{4.43}$$

We see that by exchanging ζ_1 with ζ_2, we exchange x_L with x_R and y_L with $-y_R$.

In the special case where $\zeta_1 = 0$ (i.e. $\vec{u} = (1,0,0)$) and $\zeta_2 = \zeta$, we have

$$\sin^2\theta = \frac{4\zeta\bar{\zeta}}{(1+\zeta\bar{\zeta})^2} \tag{4.44}$$

and 4.40, 4.42 reduce to the results 4.5, 4.9 from section 4.1.

Using 4.40 and 4.42, we can now determine an $S^2\times S^2$-family of generalized Kähler potentials P_{ζ_1,ζ_2} for hyperkähler manifolds.

4.4.1 Example: Euclidean space

We extend the results from section 4.2 and determine the generalized Kähler potential for Euclidean space depending on two complex variables ζ_1 and ζ_2. We set

$$x_L = \Upsilon(\zeta_1) = z - \zeta_1\bar{u}, \quad x_R = \Upsilon(\zeta_2) = z - \zeta_2\bar{u}. \tag{4.45}$$

Inverting this leads to

$$z = \frac{\zeta_2 x_L - \zeta_1 x_R}{\zeta_2 - \zeta_1}, \quad u = \frac{\bar{x}_L - \bar{x}_R}{\bar{\zeta}_2 - \bar{\zeta}_1}. \tag{4.46}$$

The y-coordinates are then given in terms of the x-coordinates by

$$\begin{aligned} y_L &= c_1\tilde{\Upsilon}(\zeta_1) = c_1(u+\zeta_1\bar{z}) = \frac{c_1}{\bar{\zeta}_2-\bar{\zeta}_1}\left((1+\zeta_1\bar{\zeta}_2)\bar{x}_L - (1+\zeta_1\bar{\zeta}_1)\bar{x}_R\right), \\ y_R &= c_2\tilde{\Upsilon}(\zeta_2) = c_2(u+\zeta_2\bar{z}) = \frac{c_2}{\bar{\zeta}_2-\bar{\zeta}_1}\left((1+\zeta_2\bar{\zeta}_2)\bar{x}_L - (1+\bar{\zeta}_1\zeta_2)\bar{x}_R\right). \end{aligned} \tag{4.47}$$

Checking a few identities involving expressions of c_1,c_2,ζ_1,ζ_2, we see that

$$P = \frac{1}{\bar{\zeta}_2 - \bar{\zeta}_1}\Big(c_1(1+\zeta_1\bar{\zeta}_2)x_L\bar{x}_L - c_1(1+\zeta_1\bar{\zeta}_1)x_L\bar{x}_R$$
$$+c_2(1+\bar{\zeta}_1\zeta_2)x_R\bar{x}_R - c_2(1+\zeta_2\bar{\zeta}_2)\bar{x}_L x_R\Big) \tag{4.48}$$

is the generalized Kähler potential, i.e. fulfills 3.25. Plugging in c_1, c_2 from 4.43, this becomes

$$P = \frac{-1}{2(1+\zeta_1\bar{\zeta}_1)^2(1+\zeta_2\bar{\zeta}_2)^2\sin^2\theta}\Big((1+\zeta_1\bar{\zeta}_2)(1+\bar{\zeta}_1\zeta_2)\left((1+\zeta_2\bar{\zeta}_2)x_L\bar{x}_L + (1+\zeta_1\bar{\zeta}_1)x_R\bar{x}_R\right)$$
$$-(1+\zeta_1\bar{\zeta}_1)(1+\zeta_2\bar{\zeta}_2)\left((1+\bar{\zeta}_1\zeta_2)x_L\bar{x}_R + (1+\zeta_1\bar{\zeta}_2)\bar{x}_L x_R\right)\Big),$$

which is perfectly left-right symmetric.

For the special case $\zeta_1 = 0$, $\zeta_2 = \zeta$, this becomes exactly 4.16.

Chapter 5

Projective Superspace

The target space of 4D $\mathcal{N} = 2$ sigma models is constrained to be a hyperkähler manifold [1]. Projective superspace provides methods to construct such models and thus can be used to find new hyperkähler metrics [17]. We will consider models that describe self-couplings of the so called polar multiplet consisting of arctic and antarctic superfields [3]. For a large class of examples, the projective superspace formalism can be used to extract from these models the coordinates Υ and $\tilde{\Upsilon}$ that we need to determine the generalized Kähler potential of the hyperkähler target space using the method derived in chapter 4 [23].

5.1 Review of projective superspace

In ordinary 4D $\mathcal{N} = 2$ superspace, we have eight fermionic derivatives [14]

$$D_{a\alpha} = \partial_{a\alpha} + \frac{1}{2}\bar{\theta}^{\dot{\beta}}_{a} i\partial_{\alpha\dot{\beta}}, \quad \bar{D}^{a}_{\dot{\alpha}} = \bar{\partial}^{a}_{\dot{\alpha}} + \frac{1}{2}\theta^{a\beta} i\partial_{\beta\dot{\alpha}}, \tag{5.1}$$

where $a = 1{,}2$ are coordinates in the fundamental representation of $SU(2)$ and $\alpha{,}\dot{\alpha} = \pm$ are left and right handed spinor indices respectively. They fulfill the algebra

$$\{D_{a\alpha}{,}D_{b\beta}\} = \{\bar{D}^{a}_{\dot{\alpha}}{,}\bar{D}^{b}_{\dot{\beta}}\} = 0, \quad \{D_{a\alpha}{,}\bar{D}^{b}_{\dot{\beta}}\} = i\delta^{b}_{a}\partial_{\alpha\dot{\beta}}. \tag{5.2}$$

In projective superspace we now parametrize a two-sphere of $\mathcal{N} = 1$ subalgebras by a complex coordinate ζ (in a chart around the north pole):

$$\nabla_{\alpha}(\zeta) := D_{2\alpha} + \zeta D_{1\alpha}, \quad \bar{\nabla}_{\dot{\alpha}}(\zeta) := \bar{D}^{1}_{\dot{\alpha}} - \zeta\bar{D}^{2}_{\dot{\alpha}}. \tag{5.3}$$

Using 5.2 one can check that the new derivatives anticommute, e.g. $\{\nabla_{\alpha}{,}\bar{\nabla}_{\dot{\beta}}\} = 0$. Thus we can impose the constraints

$$\nabla_{\alpha}\Upsilon = \bar{\nabla}_{\dot{\alpha}}\Upsilon = 0 \tag{5.4}$$

without causing Υ to be constant in the bosonic coordinates.

Projective superfields are expansions in ζ,

$$\Upsilon = \sum_{j=p}^{q} \Upsilon_j \zeta^j, \tag{5.5}$$

where the coefficients Υ_j are ordinary $\mathcal{N} = 2$ superfields and where we can choose $p < q$ in $\mathbb{Z} \cup \{-\infty,\infty\}$. Projective superfields are constrained as in 5.4 and in addition to the choice of p and q, one sometimes imposes a reality condition in terms of the real structure $\Upsilon(\zeta) \mapsto \breve{\Upsilon}(\zeta) := \overline{\bar{\Upsilon}(-\frac{1}{\bar{\zeta}})}$ [3]. Applying the constraints 5.4 to the expansion of Υ in powers of ζ (5.5), one obtains the following constraints for the $\mathcal{N} = 2$ components Υ_j:

$$D_{1\alpha}\Upsilon_{j-1} + D_{2\alpha}\Upsilon_j = \bar{D}^2_{\dot{\alpha}}\Upsilon_{j-1} - \bar{D}^1_{\dot{\alpha}}\Upsilon_j = 0. \tag{5.6}$$

We will exclusively consider the polar multiplet in this thesis. It is described by an arctic superfield $\Upsilon(\zeta) = \sum_{j=0}^{\infty} \Upsilon_j \zeta^j$ and its conjugate antarctic superfield

$$\breve{\Upsilon}(\zeta) := \overline{\Upsilon\left(-\frac{1}{\bar{\zeta}}\right)} \equiv \bar{\Upsilon}\left(-\frac{1}{\zeta}\right) = \sum_{j=0}^{\infty} \bar{\Upsilon}_j (-\zeta)^{-j}. \tag{5.7}$$

So arctic superfields are determined by the choice $p = 0$, $q = \infty$ and as the name indicates they are analytic around the north-pole, while antarctic superfields have $p = -\infty$, $q = 0$ and are analytic around the south-pole.

Since the constraint 5.6 completely determines the θ^2-dependence of projective superfields in terms of their θ^1-dependence, we can go to $\mathcal{N} = 1$ components $\Upsilon_j|_{\theta^2,\bar{\theta}_2=0}$ without losing any information. From now on, we are always looking at $\mathcal{N} = 1$ components and omit the subscript: $\Upsilon_j \equiv \Upsilon_j|_{\theta^2,\bar{\theta}_2=0}$. In combination with 5.2, the constraints 5.6 imply the following $\mathcal{N} = 1$ constraints for $\Phi := \Upsilon_0$ and $\Sigma := \Upsilon_1$:

$$\bar{D}^1_{\dot{\alpha}}\Phi = 0, \quad \bar{D}^1_{\dot{\alpha}}\bar{D}^{1\dot{\alpha}}\Sigma = 0, \tag{5.8}$$

i.e. Φ is a chiral and Σ is a complex linear (or nonminimal) $\mathcal{N} = 1$ superfield. $X_j := \Upsilon_j$ ($j \geq 2$) is unconstrained as a $\mathcal{N} = 1$ superfield [23]. The decomposition of $\Upsilon(\zeta)$, $\breve{\Upsilon}(\zeta)$ into ordinary $\mathcal{N} = 1$ superfields is thus given by

$$\Upsilon(\zeta) = \sum_{j=0}^{\infty} \zeta^j \Upsilon_j = \Phi + \zeta\Sigma + \sum_{j=2}^{\infty} \zeta^j X_j, \quad \breve{\Upsilon}(\zeta) = \sum_{j=0}^{\infty} (-\zeta)^{-j}\bar{\Upsilon}_j. \tag{5.9}$$

The polar multiplet can be used to define the action

$$S(\Upsilon^i,\breve{\Upsilon}^i) = \frac{1}{2\pi i}\oint_C \frac{d\zeta}{\zeta}\int d^4x D^\alpha_1 D_{1\alpha}\bar{D}^{1\dot{\beta}}\bar{D}^1_{\dot{\beta}} f(\Upsilon^i(\zeta),\breve{\Upsilon}^i(\zeta);\zeta), \tag{5.10}$$

which admits off-shell $\mathcal{N}=2$ supersymmetry [3]. f is a real-valued function and is called the projective superspace Lagrangian. C is a contour in the ζ-plane that orbits the origin once in counterclockwise direction. The unconstrained $\mathcal{N}=1$ superfields X_j, $j \geq 2$, are auxiliary and have to be integrated out using their equations of motion given by

$$\begin{aligned} \frac{1}{2\pi i}\oint_C \frac{d\zeta}{\zeta}\zeta^j \frac{\partial f}{\partial \Upsilon^j} &= 0 \quad \forall j \geq 2, \\ \frac{1}{2\pi i}\oint_C \frac{d\zeta}{\zeta}(-\zeta)^{-j} \frac{\partial f}{\partial \breve{\Upsilon}^j} &= 0 \quad \forall j \geq 2. \end{aligned} \tag{5.11}$$

If the projective superspace Lagrangian f in the sigma model 5.10 does not explicitly depend on ζ, it can be interpreted as the Kähler potential of a Kähler manifold M. The target space of such a model turns out to be part of the tangent bundle TM of M and the target space of the dualized model is the cotangent bundle T^*M [19]. Since the projective sigma model 5.10 admits $\mathcal{N}=2$ supersymmetry, its target space and the target space of the dual model are hyperkähler manifolds [1]. This agrees with the theorem that (part of) the cotangent bundle of a Kähler manifold admits a hyperkähler structure ([15],[16]). The problem in finding the $\mathcal{N}=1$ component decomposition of the Lagrangian, which after dualization yields the Kähler potential on T^*M, is to eliminate the occuring infinite set of auxiliary superfields. This problem has been solved for $M = \mathbb{C}P^n = SU(n+1)/[SU(n)\times U(1)]$ ([18],[19]) and $Q^n = SO(n+2)/[SO(n)\times U(1)]$ [20] and in fact for many more Hermitian symmetric spaces ([21],[22]).

5.2 Hyperkähler structures on cotangent bundles of Kähler manifolds

If we have a chart $(\phi^i,\bar{\phi}^i)_{i=1,\ldots,n}$ on a Kähler manifold M with Kähler potential $K(\phi^i,\bar{\phi}^i)$ and if we take $f(\Upsilon^i,\breve{\Upsilon}^i) := K(\Upsilon^i,\breve{\Upsilon}^i)$ to be the projective superspace Lagrangian, then we obtain a sigma model whose target space is part of the tangent bundle of M ([17],[18],[20]). The constrained $\mathcal{N}=1$ components

$$\Phi^i = \Upsilon^i(\zeta=0), \quad \Sigma^i = \frac{\partial \Upsilon^i}{\partial \zeta}\Big|_{\zeta=0} \tag{5.12}$$

of Υ^i can be interpreted as coordinates on M and on the fibers of TM respectively. Namely a holomorphic reparametrization

$$\Upsilon^i(\zeta) \to F^i(\Upsilon(\zeta)) \tag{5.13}$$

leads to

$$\Phi^i \to F^i(\Phi), \quad \Sigma^i \to \left(\frac{F^i(\Upsilon(\zeta))}{\partial \Upsilon^k}\frac{\partial \Upsilon^k}{\partial \zeta}\right)\Bigg|_{\zeta=0} = \frac{\partial F^i(\Phi)}{\partial \Phi^k}\Sigma^k. \tag{5.14}$$

Thus the fields Φ^i transform like coordinates on M and the Σ^i transform like components of a tangent vector at the point described by Φ^i. If we dualize the model and exchange the complex linear superfield Σ with a chiral field ψ and integrate out ζ and the unconstrained fields X_j, then we can read off the Kähler potential of the cotangent space from the Lagrangian [19]. The Σ^i independend part then reduces to the original Kähler potential on M.

5.2.1 Euclidean space

To see that the results for Υ, $\tilde{\Upsilon}$ that one obtains from the projective superspace formalism agree with the examples in chapter 4, we consider the projective sigma model on the cotangent bundle of Euclidean space $M = \mathbb{R}^{2n} = \mathbb{C}^n$, i.e. we set the projective superspace Lagrangian to

$$f(\Upsilon(\zeta),\breve{\Upsilon}(\zeta)) = \sum_{j=1}^{n} \Upsilon^j \breve{\Upsilon}^{\bar{j}}. \tag{5.15}$$

With C being a contour in the ζ-plane that orbits the origin once in counterclockwise direction, the equations of motion 5.11 for the auxiliary superfields X_n^i, $n \geq 2$ become

$$X_n^i = \bar{X}_n^i = 0, \tag{5.16}$$

i.e. Υ and $\breve{\Upsilon}$ are of the form

$$\Upsilon^i(\zeta) = \Phi^i + \zeta\Sigma^i, \quad \breve{\Upsilon}^{\bar{i}}(\zeta) = \bar{\Phi}^{\bar{i}} - \frac{1}{\zeta}\bar{\Sigma}^{\bar{i}}. \tag{5.17}$$

In terms of $\mathcal{N} = 1$ superfields, the projective superspace Lagrangian then becomes

$$f(\Upsilon^i(\Phi^i,\Sigma^i;\zeta),\breve{\Upsilon}^{\bar{i}}(\bar{\Phi}^{\bar{i}},\bar{\Sigma}^{\bar{i}};\zeta)) = \sum_{j=1}^{n} \left(\phi^j + \zeta\Sigma^j\right) \left(\bar{\Psi}^{\bar{j}} - \frac{1}{\zeta}\bar{\Sigma}^{\bar{j}}\right). \tag{5.18}$$

After integrating out ζ, only the ζ-independent part of f remains and we get an $\mathcal{N} = 1$ model in ordinary superspace with chiral and complex linear superfields:

$$S = \int d^8z(\Phi^j\bar{\Phi}^{\bar{j}} - \Sigma^j\bar{\Sigma}^{\bar{j}}). \tag{5.19}$$

To dualize the action and express it purely in terms of chiral superfields, we replace it by

$$S = \int d^8z(\Phi^j\bar{\Phi}^{\bar{j}} - U^j\bar{U}^{\bar{j}} - U^j\psi_j - \bar{U}^{\bar{j}}\bar{\psi}_{\bar{j}}), \tag{5.20}$$

with an unconstrained auxiliary superfield U, i.e. we perform a Legendre transform interchanging Σ and Ψ. Integrating out U gives

$$U^i = -\bar{\psi}_{\bar{i}}, \quad \bar{U}^{\bar{i}} = -\psi_i \tag{5.21}$$

and thus the action becomes

$$S = \int d^8 z (\Phi^j \bar{\Phi}^{\bar{j}} + \psi_j \bar{\psi}_{\bar{j}}), \tag{5.22}$$

from which the unsurprising result for the hyperkähler potential of the cotangent bundle $T^*\mathbb{R}^{2n} \approx \mathbb{R}^{4n}$ can be read off.

However, we are more interested in the transformation $\Sigma^i = -\bar{\psi}_{\bar{i}}$ (see eq. 5.21) that we need to determine

$$\Upsilon^i(\Phi,\psi;\zeta) = \Phi^i - \zeta\bar{\psi}_{\bar{i}}, \quad \breve{\Upsilon}(\Phi,\psi;\zeta) = \bar{\Phi}^{\bar{i}} + \frac{1}{\zeta}\psi_i; \tag{5.23}$$

and

$$\tilde{\Upsilon}^i(\zeta) = \zeta \frac{\partial f}{\partial \Upsilon^i} = \zeta \breve{\Upsilon}^i = \psi_i + \zeta \bar{\Phi}^{\bar{i}}. \tag{5.24}$$

With $\Phi \equiv z$, $\Psi \equiv u$ this matches the results from chapter 4.

5.2.2 $M = \mathbb{C}P^n$

We consider the projective sigma model on the cotangent bundle of $\mathbb{C}P^n$, i.e. we take the projective superspace Lagrangian f to be the Kähler potential of $\mathbb{C}P^n$, where the arctic superfields Υ replaces the coordinates on $\mathbb{C}P^n$ and $\breve{\Upsilon}$ their complex conjugates:

$$f(\Upsilon^i(\zeta),\breve{\Upsilon}^{\bar{i}}(\zeta)) = a^2 ln \left(1 + \frac{\sum_{j=1}^n \Upsilon^j \breve{\Upsilon}^{\bar{j}}}{a^2}\right). \tag{5.25}$$

Here a is a real parameter. Solving the equations of motion for the auxiliary superfields yields Υ in terms of chiral and complex linear $\mathcal{N} = 1$ superfields [20]:

$$\Upsilon^i = \Phi^i + \zeta \frac{\Sigma^i}{1 - \zeta \frac{\bar{\Phi}^{\bar{k}} \Sigma^k}{a^2 + \Phi^l \bar{\Phi}^{\bar{l}}}}. \tag{5.26}$$

This result has been obtained in [20] by writing down a solution for Υ at $\Phi = 0$ and rotating this solution to an arbitrary point of $\mathbb{C}P^n = SU(n+1)/U(n)$ using its $SU(n+1)$ isometry. This is the generalization of the method used in [3] to find Υ for the Eguchi-Hanson geometry, which is defined on $T^*\mathbb{C}P^1$; and it also generalizes to arbitrary Hermitian symmetric spaces $M = G/H$ [23].

Dualizing the $M = \mathbb{C}P^n$ model to go from the complex linear coordinate Σ to the chiral coordinate ψ gives the equations

$$\psi_i = -\frac{g_{i\bar{j}}\bar{\Sigma}^{\bar{j}}}{1 - \frac{g_{k\bar{l}}\Sigma^k\bar{\Sigma}^{\bar{l}}}{a^2}}, \quad \bar{\psi}_{\bar{i}} = -\frac{g_{\bar{i}j}\Sigma^j}{1 - \frac{g_{k\bar{l}}\Sigma^k\bar{\Sigma}^{\bar{l}}}{a^2}}; \tag{5.27}$$

which have to be solved for the old coordinates Σ, $\bar{\Sigma}$ in terms of ψ, $\bar{\psi}$. Here $g_{i\bar{j}}$ is the Fubini-Study metric on $\mathbb{C}P^n$

$$g_{i\bar{j}} = \frac{a^2\delta_{ij}}{a^2 + \Phi^k\bar{\Phi}^{\bar{k}}} - \frac{a^2\bar{\Phi}^{\bar{i}}\Phi^j}{(a^2 + \Phi^l\bar{\Phi}^{\bar{l}})^2} \tag{5.28}$$

and $g^{i\bar{j}}$ its inverse. We find the following solution:

$$\Sigma^i = -\frac{2g^{i\bar{j}}\bar{\psi}_{\bar{j}}}{1 + \sqrt{1 + 4\frac{g^{k\bar{l}}\psi_k\bar{\psi}_{\bar{l}}}{a^2}}}, \quad \bar{\Sigma}^{\bar{i}} = -\frac{2g^{\bar{i}j}\psi_j}{1 + \sqrt{1 + 4\frac{g^{k\bar{l}}\psi_k\bar{\psi}_{\bar{l}}}{a^2}}}. \tag{5.29}$$

Plugging this into 5.26 gives the arctic superfields Υ in terms of chiral $\mathcal{N} = 1$ superfields Φ and ψ which are coordinates on the base space $M = \mathbb{C}P^n$ and on the fibers of T^*M respectively:

$$\Upsilon^i = \Phi^i - \zeta\frac{2\bar{\psi}_{\bar{j}}g^{i\bar{j}}}{1 + \sqrt{1 + 4\frac{g^{k\bar{l}}\psi_k\bar{\psi}_{\bar{l}}}{a^2}} + 2\zeta\frac{g^{p\bar{q}}\bar{\Phi}^p\bar{\psi}_{\bar{q}}}{a^2 + \Phi^m\bar{\Phi}^{\bar{m}}}}. \tag{5.30}$$

Together with

$$\tilde{\Upsilon}^i = \zeta\frac{\partial f}{\partial\Upsilon^i} = \frac{\zeta\breve{\Upsilon}^i}{1 + \frac{\Upsilon^j\breve{\Upsilon}^{\bar{j}}}{a^2}}, \tag{5.31}$$

this is all the information we need to use the methods from chapter 4 to determine the generalized Kähler potential for $T^*\mathbb{C}P^n$.

For $n = 1$, $g_{\Phi\bar{\Phi}} = \frac{a^4}{(a^2+\Phi\bar{\Phi})^2}$ and if we set $a = 1$, then 5.30 reduces to the result that we obtained earlier from [3] for the Eguchi-Hanson metric (see equation 4.22), where $z \equiv \Phi$ and $u \equiv \psi$.

Chapter 6

Discussion

The Eguchi-Hanson metric lives on the cotangent bundle of $\mathbb{C}P^1$. The method used in [3] to determine the arctic superfields $\Upsilon(z,u;\zeta)$, $\tilde{\Upsilon}(z,u;\zeta)$ using the $SU(2)$-isometry of the total space generalizes to the hyperkähler structure defined on $T^*\mathbb{C}P^{n-1}$ with its $SU(n)$-isometry ([18],[19]). Generally, projective superspace can be used to find hyperkähler metrics on the cotangent bundles of Kähler manifolds. In order to use this in practice, however, the equations of motion for an infinite tower of unconstrained auxiliary $\mathcal{N}=1$ superfields have to be solved. There is an increasing number of examples, most notably among Hermitian symmetric spaces, where this problem has been solved ([20],[21]). For these manifolds, we thus have the decomposition of the $\mathcal{N}=2$ arctic superfields Υ, $\tilde{\Upsilon}$ in terms of their $\mathcal{N}=1$ components (z,u) and are able to apply the methods developed in this paper to find their generalized Kähler potentials. Having whole classes of manifolds available for our analysis, one could try to find more general statements about the generalized Kähler potential in the case of hyperkähler manifolds.

The Eguchi-Hanson geometry is one of the hyperkähler manifolds that can be obtained from the generalized Legendre transform construction in [2] (generalized T-duality). The manifolds stemming from that construction are $4n$-dimensional hyperkähler manifolds admitting n commuting tri-holomorphic killing vectors. They are called toric hyperkähler manifolds and have been classified in [24]. It should be possible to determine the relevant coordinates $\Upsilon(z,u;\zeta)$ and $\tilde{\Upsilon}(z,u;\zeta)$ for toric hyperkähler manifolds. For four-dimensional toric hyperkähler manifolds, [11] gives a formula for the generalized Kähler potential as a certain threefold Legendre transform in the special case $\zeta\bar{\zeta}=1$. One could compare this construction with our results at least for the examples given in this paper or try to relate the two methods in general for four-dimensional toric hyperkähler manifolds. As a further explicit example, one could for instance consider the Taub-NUT geometry and determine its generalized Kähler potential.

The generalized Kähler potential for the Eguchi-Hanson geometry can also be obtained from a generalized quotient of Euclidean 8-dimensional space by a $U(1)$-isometry and in this setting turns out to be exactly 4.26 as well [12].

The relation between the coordinates $x_{L/R}$, $y_{L/R}$ and Υ, $\tilde{\Upsilon}$ has been obtained in this paper from a purely differential geometric approach. $x_{L/R}$, $y_{L/R}$ describe left- and right-semichiral superfields in $2D$ $\mathcal{N}=(2{,}2)$ sigma models. For a target space that is hyperkähler, these models admit $\mathcal{N}=(4{,}4)$ supersymmetry. The coordinates $\Upsilon(\zeta)$, $\tilde{\Upsilon}(\zeta)$ however describe arctic superfields in $4D$ $\mathcal{N}=2$ sigma models in projective superspace. The field theoretical interpretation and understanding of this relation between $4D$ $\mathcal{N}=2$ models and the $2D$ $\mathcal{N}=(4{,}4)$ models remains an open problem.

Bibliography

[1] L. Alvarez-Gaume, D. Z. Freedman, "Geometrical Structure and Ultraviolet Finiteness in the Supersymmetric Sigma Model," Commun. Math. Phys. **80**, 443 (1981).

[2] N. J. Hitchin, A. Karlhede, U. Lindstrom, M. Rocek, "Hyperkahler Metrics and Supersymmetry," Commun. Math. Phys. **108**, 535 (1987).

[3] U. Lindstrom, M. Rocek, "Properties of hyperkahler manifolds and their twistor spaces," Commun. Math. Phys. **293**, 257-278 (2010). [arXiv:0807.1366 [hep-th]].

[4] M. Gualtieri, "Generalized complex geometry," Oxford University DPhil thesis, [arXiv:math.DG/0401221].

[5] S. J. Gates, Jr., C. M. Hull, M. Rocek, "Twisted Multiplets and New Supersymmetric Nonlinear Sigma Models," Nucl. Phys. **B248**, 157 (1984).

[6] U. Lindstrom, M. Rocek, R. von Unge, M. Zabzine, "Generalized Kahler manifolds and off-shell supersymmetry," Commun. Math. Phys. **269**, 833-849 (2007). [hep-th/0512164].

[7] U. Lindstrom, M. Rocek, R. von Unge, M. Zabzine, "Linearizing Generalized Kahler Geometry," JHEP **0704**, 061 (2007). [hep-th/0702126].

[8] N. Hitchin, "Instantons, Poisson structures and generalized Kahler geometry," Commun. Math. Phys. **265**, 131-164 (2006). [math/0503432 [math-dg]].

[9] A. Moroianu, "Lectures on Kähler geometry," London Mathematical Society Student Texts **69**, Cambridge University Press, 2007.

[10] S. Chakravarty, L. J. Mason, E. T. Newman, "Canonical structures on antiselfdual four manifolds and the diffeomorphism group," J. Math. Phys. **32**, 1458-1464 (1991).

[11] J. Bogaerts, A. Sevrin, S. van der Loo, S. Van Gils, "Properties of semichiral superfields," Nucl. Phys. **B562**, 277-290 (1999). [hep-th/9905141].

[12] P. M. Crichigno, private communication, in preparation.

[13] U. Lindstrom, M. Rocek, R. von Unge, M. Zabzine, "A potential for Generalized Kahler Geometry," IRMA Lect. Math. Theor. Phys. **16**, 263-273 (2010). [hep-th/0703111].

[14] S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel, "Superspace Or One Thousand and One Lessons in Supersymmetry," Front. Phys. **58**, 1 (1983) [arXiv:hep-th/0108200].

[15] B. Feix, "Hyperkähler metrics on cotangent bundles," J. Reine Angew. Math. **532** (2001), 33-46

[16] D. Kaledin, "Hyperkähler structures on total spaces of holomorphic cotangent bundles" [arxiv:alg-geom/9710026]
D. Kaledin, "A canonical hyperkähler metric on the total space of a cotangent bundle" [arXiv:math/0011256]

[17] U. Lindstrom, "Hyperkahler metrics from projective superspace," arXiv:hep-th/0703181.

[18] S. J. J. Gates and S. M. Kuzenko, "The CNM hypermultiplet nexus," Nucl. Phys. B **543**, 122 (1999) [arXiv:hep-th/9810137].

[19] S. J. J. Gates and S. M. Kuzenko, "4-D, N=2 supersymmetric off-shell sigma models on the cotangent bundles of Kahler manifolds," Fortsch. Phys. **48**, 115 (2000) [arXiv:hep-th/9903013].

[20] M. Arai and M. Nitta, "Hyper-Kahler sigma models on (co)tangent bundles with SO(n) isometry," Nucl. Phys. B **745**, 208 (2006) [arXiv:hep-th/0602277].

[21] M. Arai, S. M. Kuzenko and U. Lindstrom, "Hyperkahler sigma models on cotangent bundles of Hermitian symmetric spaces using projective superspace," JHEP **0702**, 100 (2007) [arXiv:hep-th/0612174].

[22] M. Arai, S. M. Kuzenko and U. Lindstrom, "Polar supermultiplets, Hermitian symmetric spaces and hyperkahler metrics," JHEP **0712**, 008 (2007) [arXiv:0709.2633 [hep-th]].

[23] M. Arai, S. M. Kuzenko and U. Lindstrom, "Projective superspace and hyperkahler sigma models on cotangent bundles of Hermitian symmetric spaces," AIP Conf. Proc. **957**, 293 (2007) [arXiv:0709.1257 [hep-th]].

[24] R. Bielawski, “Complete hyperKahler 4n manifolds with n commuting triHamiltonian vector fields,” [math/9808134].