Aspects of Superconformal Field Theories

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Abstract of the Dissertation

Aspects of Superconformal Field Theories

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Recently, a lot of progress has been made towards understanding the strongly coupled supersymmetric quantum gauge theories. The problem of strong coupling for $SU(N)$ gauge theories can be formulated in two separate regimes of interest, one at finite N and the other at large N in 't Hooft limit. In the first case electric/magnetic duality also called S-duality and in the second, AdS/CFT duality map the strongly coupled problem to a weakly coupled one. Both of the dualities have been well understood in the maximally supersymmetric $4d$ gauge theory, the $\mathcal{N} = 4$ super Yang-Mills. In this thesis, as a natural next step, we focus on the strong coupling behavior in $\mathcal{N} = 2$ supersymmetric gauge theories.

To my family.

Contents

List of Figures

List of Tables

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List of Publications

[1] *S-duality and 2d Topological QFT.*
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[2] *The Veneziano Limit of $\mathcal{N} = 2$ Superconformal QCD:*
Towards the String Dual of $\mathcal{N} = 2$ $SU(N_c)$ SYM with $N_f = 2N_c$.

[3] *The Superconformal Index of the E_6 SCFT.*
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[4] *Spin Chains in $\mathcal{N} = 2$ Superconformal Theories:*
From the $\mathbb{Z}_2$ Quiver to Superconformal QCD.
Abhijit Gadde, Elli Pomoni, Leonardo Rastelli.

[5] *On the Superconformal Index of $\mathcal{N} = 1$ IR Fixed Points:*
A Holographic Check
Abhijit Gadde, Leonardo Rastelli, Shlomo S. Razamat, Wenbin Yan

[6] *Twisted Magnons*
Abhijit Gadde, Leonardo Rastelli

[7] *Reducing the 4d Index to S^3 Partition Function*
Abhijit Gadde, Wenbin Yan

[8] *The 4d Superconformal Index from q-deformed 2d Yang-Mills*
Abhijit Gadde, Leonardo Rastelli, Shlomo Razamat, Wenbin Yan

Chapter 1

Introduction

Symmetries simplify the study of any dynamical system. Although, in *quantum* theories the conventional (bosonic) symmetries help but only a little. The constraints imposed by bosonic symmetries on quantum theories are weak. The quantum corrections rapidly become intractable with increasing loop order. However, the fermionic symmetry - Supersymmetry helps keep the quantum corrections under control. It is this tractability of quantum effects that has given supersymmetry a very special place in modern theoretical physics. Quantum field theories with more supersymmetry have larger number of fields in order to be consistent with all the supersymmetries. In spite of the increasing field content and hence the increasing apparent difficulty, a useful point of view is: the more supersymmetry the simpler the *quantum* field theory. The theory with the maximal supersymmetry in four dimensions, $\mathcal{N} = 4$ super Yang-Mills has even been termed as the harmonic oscillator of 21st century.

$\mathcal{N} = 4$ SYM has been studied in great detail. Large amount of supersymmetry allows it to enjoy a strong-weak coupling duality or electric-magnetic duality called S-duality [9]. Montonen and Olive conjectured $\mathcal{N} = 4$ SYM to be a self dual theory i.e. the massive monopoles of the weakly coupled theory become massless as coupling is increased and behave *exactly* as the quarks of a different weakly coupled or *S-dual* $\mathcal{N} = 4$ SYM. Rigorous checks of this duality have been performed [10, 11].

The strong-weak coupling duality gives us a handle on strong coupling behavior of $SU(N)$ $\mathcal{N} = 4$ SYM at finite N. On the other hand, AdS/CFT correspondence is proves to be a very useful tool in understanding the large N

behavior of $\mathcal{N}=4$ SYM at strong coupling [12–14]. The duality states that $SU(N)$ $\mathcal{N}=4$ SYM with coupling g_{YM} is dual to type IIB string theory in $AdS_5 \times S^5$ of radius $\sim l_s\lambda^{1/4}$. The string theory has the coupling constant λ/N. At large N with fixed λ, one can describe strongly interacting $\mathcal{N}=4$ in terms of free strings. If one further takes $\lambda \to \infty$ the dual description is in terms of the low energy limit of string theory, type IIB supergravity. Very strong checks of the large N AdS/CFT correspondence have been performed. Moreover, gauge theory computations in 't Hooft limit e.g. the computation of anomalous dimensions of single trace operators has led to the discovery of integrable structures in planar $\mathcal{N}=4$ SYM [15]. These integrable structures along with supersymmetry have played a very important role in verifying the duality at any 't Hooft coupling in the planar limit.

S-duality and *AdS/CFT correspondence* both are useful in understanding the strong coupling dynamics of $SU(N)$ $\mathcal{N}=4$ SYM, albeit in two different limits. The former is applicable to the gauge theory at finite N and later to the gauge theory at large N. We wish to take the next step in studying these two aspects of gauge theories. From our point of view the next step amounts to reducing the supersymmetry in half. Of $\mathcal{N}=2$ supersymmetric field theories, $\mathcal{N}=2$ superconformal theories are especially interesting. They do not have asymptotic freedom where perturbation theory is applicable but they do enjoy state-operator correspondence makes it relatively simpler to study their Hilbert space using radial quantization. In this thesis, we will mainly focus our attention to S-duality and large N dynamics of $\mathcal{N}=2$ superconformal field theories.

1.1 $\mathcal{N}=2$ S-duality

Recently, an S-duality for $\mathcal{N}=2$ supersymmetric gauge theories has been proposed by Gaiotto [16]. Most conveniently the duality is understood in terms of the geometry of Riemann surfaces. The appearance of a Riemann surface is manifest in M theory construction of $4d$ gauge theories [17]. A $4d$ gauge theory is obtained by compactifying N M5 branes on a Riemann surface (with a topological twist) and taking low energy limit. The complex structure moduli space of the Riemann surface turns out be the same as the space of exactly marginal deformations of the $\mathcal{N} = 2$ gauge theory. The action of S-duality on the gauge parameter space is the same as the action of the mapping class group of the Riemann surface on its complex moduli space. Riemann surface can be decomposed into pairs of pants in a variety of ways. Each of the decompositions represents a weakly coupled $4d$ gauge theory. In addition to being a strong/weak coupling duality, S-duality relates all these weakly coupled "corners" of the moduli space as well.

In the first part of the thesis, we mainly focus on this web of $\mathcal{N} = 2$ dualities. Our main tool to study and check the S-dualities will be the Witten index [18], a quantity that is invariant under the supersymmetric deformations of the theory. The Witten index of the superconformal field theories in radial quantization is called the superconformal index [19]. In addition to being invariant under all the exactly marginal supersymmetric deformations of the theory, the superconformal index also captures "cohomological" information about the protected states of the theory. It counts (with signs) protected states of the theory, upto an equivalence relation that sets to zero all sequences of short multiplets that may in principle recombine into a long multiplets. As the index is independent of the coupling, it can be easily computed in the weakly coupled limit whenever such limit is available. In due course, we will see that, conversely, using S-dualities one can even compute the superconformal index of nontrivial fixed points.

For the class of gauge theories obtained from M5 brane compactification on Riemann surface Σ, the gauge coupling independence also makes the index independent of the complex structure of Σ. That is, the index of the $4d$ theory abstractly defines a topological field theory on Σ. Having an independent microscopic description of this TQFT would prove very useful as it will enable use to compute the index of any $4d$ theory directly from the associated Σ. This is especially useful when the $4d$ theory doesn't admit a weakly coupled limit e.g. $4d$ theories obtained from compactifying M5 branes on a sphere with three punctures. Recall that the absence of complex structure moduli means absence of any exactly marginal deformations. We will show that a certain "reduced" superconformal index is computed by $2d$ q-YM on Σ in the zero area limit!

The superconformal index of a $4d$ gauge theory is the supersymmetric partition function of the theory on $S^3 \times S^1$. As the radius of S^1 goes to zero, the path integral is restricted to S^3. In this limit, we expect the superconformal index to yield the partition function of dimensionally reduced $3d$ gauge theory on S^3. Recently, the S^3 partition function of $3d$ gauge theories has been computed by Kapustin et. al. using localization methods [20, 21] and it indeed agrees with this limit of the superconformal index. Remarkably, the superconformal index of the "parent" $4d$ theory can be thought of as the q-deformation of the $3d$ partition function.

1.2 $\mathcal{N} = 2$ at large N

The gauge/gravity duality for $\mathcal{N} = 4$ SYM has been used as an approximation to the physics of strong interactions. The approximation is not the most ideal one due to the lack of quarks in fundamental representation. Most of the attempts to include the effect of fundamental matter have considered quarks in the probe approximation but in many physically relevant theories such as QCD one would need to account for the effect of unquenched flavor. It seems that the "simplest" theory that would incorporate this effect is the $\mathcal{N} = 2$ superconformal QCD, the $\mathcal{N} = 2$ super Yang Mills theory with gauge group $SU(N_c)$ and $N_f = 2N_c$ fundamental hyper multiplets. We attack the long-standing problem of finding its AdS dual. The theory admits a Veneziano expansion [22] of large N_c and large N_f, with N_f/N_c and $\lambda = g^2_{YM}N_c$ kept fixed. The topological structure of large N diagrams motivates a general conjecture: the *flavor-singlet sector* of a gauge theory in the Veneziano limit is dual to a closed string theory; single closed string states correspond to "generalized single-trace" operators, where adjoint letters and flavor-contracted fundamental/antifundamental pairs are stringed together in a closed chain. We look for the string dual of $\mathcal{N} = 2$ superconformal QCD from two fronts. From the bottom-up, we perform a systematic analysis of the protected spectrum using superconformal representation theory. We also evaluate the one-loop dilation operator in the scalar sector, finding a novel spin chain. From the top-down, we consider the decoupling limit of known brane constructions. In both approaches, more insight is gained by viewing the theory as the degenerate limit of the $\mathcal{N} = 2$ $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM, as one of the two gauge couplings is tuned to zero. A consistent picture emerges. We conclude that the string dual is a sub-critical background with seven "geometric" dimensions, containing both an AdS_5 and an S^1 factor. The supergravity approximation is never entirely valid, even for large λ, indeed the field theory has an exponential degeneracy of exactly protected states with higher spin, which must be dual to a sector of light string states.

Computing the spectrum of single trace operators in an interesting problem in itself. Very strong checks of AdS/CFT correspondence can be performed by matching the gauge theory operator spectrum with the string theory spectrum. In $\mathcal{N} = 4$ SYM a lot of progress was made on that front by thinking of single trace operators as spin chains and the dilatation operator as the Hamiltonian. The main reason for the success of this approach is the *integrability*[1] of the $\mathcal{N} = 4$ SYM spin chain. Integrability is the existence of infinitely many conserved charges. It leads to factorized scattering of fundamental excitations i.e. n-body scattering factorizes into sequence of two body scatterings. Hence complete spectrum of an integrable spin chain is encoded in just the two body the scattering matrix or the *S matrix* of fundamental excitations and can be found using *Bethe ansatz*.

We find preliminary evidence that $\mathcal{N} = 2$ superconformal QCD, the $SU(N_c)$ SYM theory with $N_f = 2N_c$ fundamental hypermultiplets, might be integrable in the large N Veneziano limit. We evaluate the one-loop dilation operator in the scalar sector of the $\mathcal{N} = 2$ superconformal quiver with $SU(N_c) \times SU(N_{\check{c}})$ gauge group, for $N_c \equiv N_{\check{c}}$. Both gauge couplings g and $\check{g}$ are exactly marginal. This theory interpolates between the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM, which corresponds to $\check{g} = g$, and $\mathcal{N} = 2$ superconformal QCD, which is obtained for $\check{g} \to 0$. The planar one-loop dilation operator takes the form of a nearest-neighbor spin-chain Hamiltonian. For superconformal QCD the spin chain is of novel form: besides the color-adjoint fields, which occupy individual sites of the chain, there are "dimers" of flavor-contracted fundamental fields, which occupy two neighboring sites. We solve the two-body scattering problem of magnon excitations and study the spectrum of bound states, for general $\check{g}/g$. The dimeric excitations of superconformal QCD are seen to arise smoothly for $\check{g} \to 0$ as the limit of bound wavefunctions of the interpolating theory. Finally

[1]See individual chapters for the references.

we check the *Yang-Baxter equation* for the two-magnon S-matrix, which is a necessary but not sufficient condition for integrability. It holds as expected at the orbifold point $\check{g} = g$. While violated for general $\check{g} \neq g$, it holds again in the limit $\check{g} \to 0$, hinting at one-loop integrability of planar $\mathcal{N} = 2$ superconformal QCD.

Although, integrability is broken for the theory interpolating between the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ and $\mathcal{N} = 2$ SCQCD, we use the centrally extended $SU(2|2)$ symmetry of the magnons to fix their dispersion relation and two body S-matrices as functions of exactly marginal couplings [23].

The rest of the thesis can be broadly divided into two parts: chapters 2,3,4,5 analyze S-duality aspects of $\mathcal{N} = 2$ superconformal theories while chapters 6,7,8 study the large N limit. In chapter 2, we study the superconformal index for the class of $\mathcal{N} = 2$ superconformal field theory defined by compactifying the (2,0) $6d$ theory on a Riemann surface with punctures. We interpret the index of $4d$ theory associated to n-punctured Riemann surface as the n-point correlation function of the topological QFT living on the surface. We focus on the A_1 case and calculate the 2 and 3 point function of the TQFT in terms of elliptic hypergeometric gamma functions and verify S-dulity. In chapter 3, we study the A_2 case which involves a strongly coupled SCFT with E_6 flavor symmetry. We compute its index using Argyres-Seiberg duality. In the next chapter, we give an independent identification of the $2d$ TQFT that computes a certain limit of the $4d$ superconformal index. We demonstrate the usefulness of this TQFT by computing the index of an infinite series of strongly coupled theories. Chapter 5 is dedicated to the S^1 reduction of the $4d$ index to the S^3 partition function of the $3d$ gauge theories. The second part begins with chapter 6 which is aimed at finding the holographic dual of $\mathcal{N} = 2$ superconformal QCD in the Veneziano limit. We approach the problem from gauge theory side first then from the string theory side starting from a Hanany-Witten type brane construction. In chapter 7, we compute the one loop anomalous dimension operator in the scalar sector and display signs of integrability in $\mathcal{N} = 2$ SCQCD. In the last chapter, we use the $SU(2|2)$ symmetries of the theory to determine the all-loop dispersion relation and the two body scattering matrix of magnons in a *non*-integrable theory that interpolates between $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SCQCD. We also comment on the dual description of such magnons. The thesis ends with a lot of appendices that supplement the bulk of the thesis.

Chapter 2

S-duality and 2d Topological QFT

Electric-magnetic duality (S-duality) in four-dimensional gauge theory has a deep connection with two-dimensional modular invariance. The canonical example is the $SL(2,\mathbb{Z})$ symmetry of $\mathcal{N}=4$ super-Yang-Mills, which can be interpreted as the modular group of a torus. A physical picture for this correspondence is provided by the existence of the six-dimensional $(2,0)$ superconformal field theory, whose compactification on a torus of modular parameter τ yields $\mathcal{N}=4$ SYM with holomorphic coupling τ (see [24] for a recent discussion).

Gaiotto [16] has recently discovered a beautiful generalization of this construction. A large class of $\mathcal{N}=2$ superconformal field theories in 4d is obtained by compactifying a twisted version of the $(2,0)$ theory on a Riemann surface Σ, of genus g and with n punctures. The complex structure moduli space $\mathcal{T}_{\mathsf{g},n}/\Gamma_{\mathsf{g},n}$ of Σ is identified with the space of exactly marginal couplings of the 4d theory. The mapping class group $\Gamma_{\mathsf{g},n}$ acts as the group of generalized S-duality transformations of the 4d theory. A striking correspondence between the Nekrasov's instanton partition function [25] of the 4d theory and Liouville field theory on Σ has been conjectured in [26] and further explored in [27–38]. Relations to string/M theory have been discussed in [39–42]. See also [43–45].

In this chapter we study the superconformal index [19] for this class of 4d SCFTs. The index captures "cohomological" information about the protected states of the theory. By construction, it counts (with signs) the protected states of the theory, up to equivalence relations that set to zero all sequences of short multiplets that may in principle recombine into long multiplets.

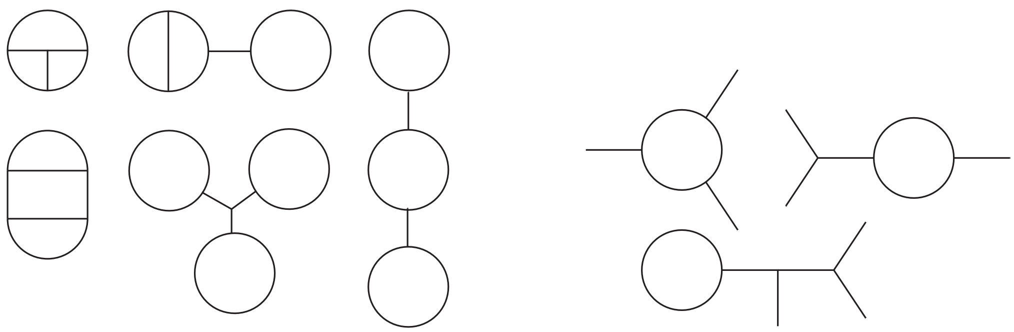
The index is invariant under continuous deformations of the theory, and is also expected to be invariant under the S-duality group $\Gamma_{\mathbf{g},n}$. *Assuming S-duality*, this implies that the index must be computed by a topological QFT living on Σ. The usual physical arguments involving the $(2,0)$ theory give a "proof" of this assertion, as follows. The index has a path integral representation [19] as the partition function of the 4d theory on $S^3 \times S^1$, twisted by various chemical potentials, which uplifts to a (suitably twisted) path integral of the $(2,0)$ theory on $S^3 \times S^1 \times \Sigma$. This path integral must be independent of the metric on Σ. In the limit of small Σ we recover the 4d definition; in the opposite limit of large Σ we expect a purely 2d description. Each puncture on Σ should be regarded as an operator insertion. By this logic, the index must be equal to the n-point correlation function of some TQFT on Σ. The question is whether one can describe this TQFT more directly, and in the process *check* the S-duality of the index.

It is likely that a "microscopic" Lagrangian formulation of the 2d TQFT may be derived from the dimensional reduction of the twisted $(2,0)$ theory that we have just described, but we will not pursue this here. Our approach will be to start with the 4d definition of the index [19] and write its concrete expression for Gaiotto's A_1 theories, which have a 4d Lagrangian description. We show in section 2.1 that the index does indeed take the form expected for a correlator in a 2d TQFT. We then evaluate explicitly the structure constants and metric of the TQFT operator algebra, and check its associativity, which is the 2d counterpart of S-duality (section 2.2). The metric and structure constants have elegant expressions in terms of elliptic Gamma functions and the index in terms of elliptic Beta integrals, a set of special functions which are a new and active branch of mathematical research, see *e.g.* [46–48] and references therein. For Gaiotto's A_1 theories associativity of the topological algebra (and thus S-duality) hinges on the invariance of a special case of the $E^{(5)}$ elliptic Beta integral under the Weyl group of F_4. A proof of this symmetry appeared on the math ArXiv just as the original paper was nearing completion [49].[1] In a related physical context, elliptic identities have been used in [50] (following

[1] We are grateful to Fokko J. van de Bult for sending us a draft of [49] prior to publication.

[51]) to prove equality of the superconformal index for Seiberg-dual pairs of $\mathcal{N} = 1$ gauge theories.

It is also natural to ask how things work for the original paradigm of a theory exhibiting S-duality, namely $\mathcal{N} = 4$ SYM. From the viewpoint of the superconformal index the only non-trivial $\mathcal{N} = 4$ dual pairs are the theories based on $SO(2n+1)/Sp(n)$ gauge groups. We study these cases in Appendix A. We write integral expressions for the index of two dual theories and check their equality "experimentally", for the first few orders in a series expansion in the chemical potentials. It would be nice to find an analytic proof.



(a) (b)

Figure 2.1: **(a)** Generalized quiver diagrams representing $\mathcal{N} = 2$ superconformal theories with gauge group $SU(2)^6$ and no flavor symmetries ($N_G = 6$, $N_F = 0$). There are five different theories of this kind. The internal lines of a diagram represent and $SU(2)$ gauge group and the trivalent vertices the trifundamental chiral matter. **(b)** Generalized quiver diagrams for $N_G = 3$, $N_F = 3$. Each external leg represents an $SU(2)$ flavor group. The upper left diagram corresponds the $\mathcal{N} = 2$ $\mathbb{Z}_3$ orbifold of $\mathcal{N} = 4$ SYM with gauge group $SU(2)$.

We end this introduction by recalling the basics of Gaiotto's analysis [16]. The main achievement of [16] is a purely four-dimensional construction of the SCFT implicitly defined by compactifying the A_{N-1} $(2,0)$ theory on Σ. In the A_1 case an explicit Lagrangian description is available, in terms of a generalized quiver with gauge group $SU(2)^{N_G}$, see Figure 2.1 for examples. The internal edges of a diagram correspond to the $SU(2)$ gauge groups, the ex-

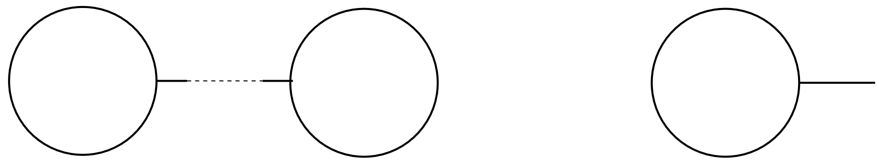


Figure 2.2: An example of a degeneration of a graph and appearance of flavour punctures. As one of the gauge coupling is taken to zero the corresponding edge becomes very long. Cutting the edge, each of the two resulting semi-infinite open legs will be associated to chiral matter in an $SU(2)$ flavor representation. In this picture setting the coupling of the middle legs in **(a)** to zero gives two copies of the theory represented in **(b)**, namely an $SU(2)$ gauge theory with a chiral field in the bifundamental representation of the gauge group and in the fundamental of a flavour $SU(2)$.

ternal legs to $SU(2)$ flavor groups and the the cubic vertices to chiral fields in the trifundamental representation (fundamental under each of the groups joining at the vertex). The corresponding Riemann surface is immediately pictured by thickening the lines of the graph into tubes – with the external tubes assumed to be infinitely long, so that they can be viewed as punctures. The plumbing parameters τ_i of the tubes are identified with the holomorphic gauge couplings; the degeneration limit when the surface develops a long tube corresponds to the weak coupling limit $\tau \to +i\infty$ of the corresponding gauge group (Figure 2.2). The different patterns of degenerations (pair-of-pants decompositions) of a surface Σ of genus g and N_F punctures give rise to the different connected diagrams with N_F external legs ($SU(2)$ flavor groups) and $N_G = N_F + 3(\mathsf{g} - 1)$ internal lines ($SU(2)$ gauge groups). Since the mapping class group permutes the diagrams, the associated field theories must be related by generalized S-duality transformations [16].

In the higher A_{N-1} cases the 4d theories are generically described by more complicated quivers that involve new exotic isolated SCFTs as elementary building blocks. While the correspondence between the index and 2d TQFT is general, in this chapter we will focus on the A_1 theories, where explicit calculations can be easily performed.

2.1 2d TQFT from the Superconformal Index

The superconformal index is defined as [19]

$$\mathcal{I} = \mathcal{I}^{WR} = \mathrm{Tr}(-1)^F t^{2(E+j_2)} y^{2j_1} v^{-(r+R)}\,, \tag{2.1}$$

where the trace is over the states of the theory on S^3 (in the usual radial quantization). For definiteness we are considering the "right-handed" Witten index $\mathcal{I}^{WR}$ of [19], which computes the cohomology of the supercharge $\bar{\mathcal{Q}}_{2+}$, in notations [52] where the supercharges are denoted as $\mathcal{Q}^I_\alpha$, $\bar{\mathcal{Q}}_{I\dot{\alpha}}$, $\mathcal{S}_{I\alpha}$, $\bar{\mathcal{S}}^I_{\dot{\alpha}}$, with $I = 1, 2$ $SU(2)_R$ indices and $\alpha = \pm$, $\dot{\alpha} = \pm$ Lorentz indices. (For the class of superconformal theories that we consider, the left-handed and right-handed Witten indices are equal.) The chemical potentials t, y, and v keep track of various combinations of quantum numbers associated to the supercorformal algebra $SU(2,2|2)$: E is the conformal dimension, (j_1, j_2) the $SU(2)_1 \times SU(2)_2$ Lorentz spins, and $(R\,, r)$ the quantum numbers under the $SU(2)_R \times U(1)_r$ R-symmetry.[2]

For a theory with a weakly-coupled description the index can be explicitly computed as a matrix integral,

$$\mathcal{I}(V, t, y, v) = \int [dU]\, \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_j f^{\mathcal{R}_j}(t^n, y^n, v^n) \cdot \chi_{\mathcal{R}_j}(U^n,\, V^n)\right)\,. \tag{2.2}$$

Here U is the matrix of the gauge group, V the matrix of the flavor group and $\mathcal{R}_j$ label representations of the fields under the flavor and gauge groups. The measure $[dU]$ is the invariant Haar measure, and it has the following property

$$\int [dU] \prod_{j=1}^{n} \chi_{\mathcal{R}_j}(U) = \#\text{of singlets in } \mathcal{R}_1 \otimes \cdots \otimes \mathcal{R}_n\,. \tag{2.3}$$

For the A_1 generalized quivers the index can be explicitly computed as a matrix integral,

$$\mathcal{I} = \int \prod_{\ell \in \mathcal{G}} [dU_\ell]\, \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\left[\sum_{i \in \mathcal{G}} f_n^{vect} \cdot \chi_{adj}(U_i^n) + \sum_{(i,j,k) \in \mathcal{V}} f_n^{chi} \cdot \chi_{3f}(U_i^n, U_j^n, U_k^n)\right]\right)\,. \tag{2.4}$$

[2] Our normalization for the R-symmetry charges is as in [52] and differs from [19]: $R_{here} = R_{there}/2$, $r_{here} = r_{there}/2$.

Here $f_n^{vect} = f^{vect}(t^n, y^n, v^n)$ and $f_n^{chi} = f^{chi}(t^n, y^n, v^n)$, with $f^{vect}(t, y, v)$ and $f^{chi}(t, y, v)$ the "single-letter partition functions" for respectively the vector and half-hyper degrees of freedom, multiplying the corresponding $SU(2)$ characters. The explicit expressions for f^{vect} and f^{chi} will be given in the next section. The $\{U_i\}$ are $SU(2)$ matrices. Their index i run over the $N_G + N_F$ edges of the diagram, both internal ("Gauge") and external ("Flavor"). The set $\mathcal{G}$ is the set of N_G internal edges while the set $\mathcal{V}$ is the set of trivalent vertices, each vertex being labelled by the triple (i, j, k) of incident edges. The integral over $\{U_\ell\,, \ell \in \mathcal{P}\}$, with $[dU]$ being the Haar measure, enforces the gauge-singlet condition. All in all, the index $\mathcal{I}$ depends on the chemical potentials t, y, v (through f^{vect} and f^{chi}) and on (the eigenvalues of) the N_F unintegrated flavor matrices.

The characters depend on a single angular variable α_i for each $SU(2)$ group U_i. Writing

$$U_i = V_i^\dagger \begin{pmatrix} e^{i\alpha_i} & 0 \\ 0 & e^{-i\alpha_i} \end{pmatrix} V_i\,, \tag{2.5}$$

we have

$$\chi_{adj}(U_i) = \mathrm{Tr}U_i\,\mathrm{Tr}U_i - 1 = e^{2i\alpha_i} + e^{-2i\alpha_i} + 1 \equiv \chi_{adj}(\alpha_i)\,, \tag{2.6}$$

$$\begin{aligned}\chi_{3f}(U_i, U_j, U_k) &= \mathrm{Tr}U_i\,\mathrm{Tr}U_j\,\mathrm{Tr}U_k = (e^{i\alpha_i} + e^{-i\alpha_i})(e^{i\alpha_j} + e^{-i\alpha_j})(e^{i\alpha_k} + e^{-i\alpha_k}) \\ &\equiv \chi_{3f}(\alpha_i, \alpha_j, \alpha_k)\,,\end{aligned} \tag{2.7}$$

where we have used the fact that $2 \sim \bar{2}$. Integrating over V_i, the Haar measure simplifies to

$$\int [dU_i] = \frac{1}{\pi}\int_0^{2\pi} d\alpha_i\ \sin^2\alpha_i \equiv \int d\alpha_i\,\Delta(\alpha_i)\,. \tag{2.8}$$

We now define

$$\begin{aligned} C_{\alpha_i\alpha_j\alpha_k} &\equiv \exp\left(\sum_{n=1}^{\infty}\frac{1}{n} f_n^{chi}\cdot\ \chi_{3f}(n\alpha_i, n\alpha_j, n\alpha_k)\right)\,, \\ \eta^{\alpha_i\alpha_j} &\equiv \exp\left(\sum_{n=1}^{\infty}\frac{1}{n} f_n^{vect}\cdot\ \chi_{adj}(n\alpha_i)\right)\hat{\delta}(\alpha_i, \alpha_j) \equiv \eta^{\alpha_i}\,\hat{\delta}(\alpha_i, \alpha_j), \end{aligned} \tag{2.9}$$

where $\hat{\delta}(\alpha, \beta) \equiv \Delta^{-1}(\alpha)\delta(\alpha - \beta)$ (with the understanding that α and β are defined modulo 2π) is the delta-function with respect to the measure (2.8). Further define the "contraction" of an upper and a lower α labels as

$$A^{\dots\alpha\dots}\, B_{\dots\alpha\dots} \equiv \int_0^{2\pi} d\alpha\, \Delta(\alpha)\, A^{\dots\alpha\dots}\, B_{\dots\alpha\dots}\,. \tag{2.10}$$

The superconformal index (4.2) can then be suggestively written as

$$\mathcal{I} = \prod_{\{i,j,k\}\in\mathcal{V}} C_{\alpha_i\alpha_j\alpha_k} \prod_{\{m,n\}\in\mathcal{G}} \eta^{\alpha_m\alpha_n}\,. \tag{2.11}$$

The internal labels $\{\alpha_i\ , i \in \mathcal{G}\}$ associate to the gauge groups are contracted, while the N_F external labels associated to the flavor groups are left open. The expression (2.11) is naturally interpreted as an N_F-point "correlation function" $\langle\alpha_1 \dots \alpha_{N_F}\rangle_{\mathtt{g}}$, evaluated by regarding the generalized quiver as a "Feynman diagram". The Feynman rules assign to each trivalent vertex the cubic coupling $C_{\alpha\beta\gamma}$, and to each internal propagator the inverse metric $\eta^{\alpha\beta}$. S-duality implies that the superconformal indices calculated from two diagrams with the same (N_F, N_G) must be equal. These properties can be summarized in the statement that the superconformal index is evaluated by a 2d Topological QFT (TQFT).

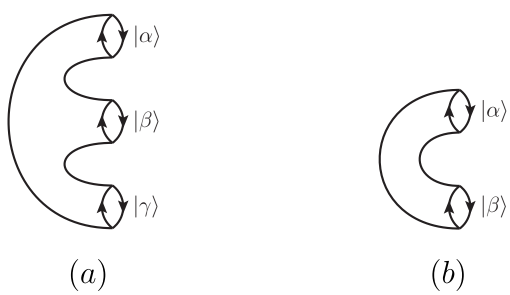


Figure 2.3: **(a)** Topological interpretation of the structure constants $C_{\alpha\beta\gamma} \equiv \langle C|\,|\alpha\rangle|\beta\rangle|\gamma\rangle$. The path integral over the sphere with three boundaries defines $\langle C| \in \mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{H}^*$. **(b)** Analogous interpretation of the metric $\eta_{\alpha\beta} \equiv \langle\eta||\alpha\rangle|\beta\rangle$, with $\langle\eta| \in \mathcal{H}^* \otimes \mathcal{H}^*$, in terms of the sphere with two boundaries.

At the informal level sufficient for our discussion, a 2d TQFT [53, 54] can be characterized in terms of the following data: a space of states $\mathcal{H}$; a non-degenerate, symmetric metric η: $\mathcal{H}\otimes\mathcal{H}\to\mathbb{C}$; and a completely symmetric triple product C: $\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}\to\mathbb{C}$. The states in $\mathcal{H}$ are understood physically as wavefunctionals of field configurations on the "spatial" manifold S^1. The metric and triple product are evaluated by the path integral over field configurations on the sphere with respectively two and three boundaries (Figure 2.3). The 2d surfaces where the TQFT is defined are assumed to be oriented, so the S^1 boundaries inherit a canonical orientation. To a boundary of inverse orientation (with respect to the canonical one) is associated the *dual* space $\mathcal{H}^*$. Choosing a basis for $\mathcal{H}$, we can specify the metric and triple product in terms of $\eta_{\alpha\beta}\equiv\eta(|\alpha\rangle,|\beta\rangle)$ and $C_{\alpha\beta\gamma}\equiv C(|\alpha\rangle,|\beta\rangle,|\gamma\rangle)$, or

$$\eta=\sum_{\alpha,\beta}\eta_{\alpha\beta}\langle\alpha|\langle\beta|\,,\quad C=\sum_{\alpha,\beta,\gamma}C_{\alpha\beta\gamma}\langle\alpha|\langle\beta|\langle\gamma|\,. \tag{2.12}$$

The inverse metric $\eta^{\alpha\beta}$ is associated to the sphere with two boundaries of inverse orientation, and as its name suggests it obeys $\eta^{\alpha\beta}\eta_{\beta\gamma}=\delta^{\alpha}_{\gamma}$, see Figure 2.4. Index contraction corresponds geometrically to gluing of S^1 boundary of compatible orientation.

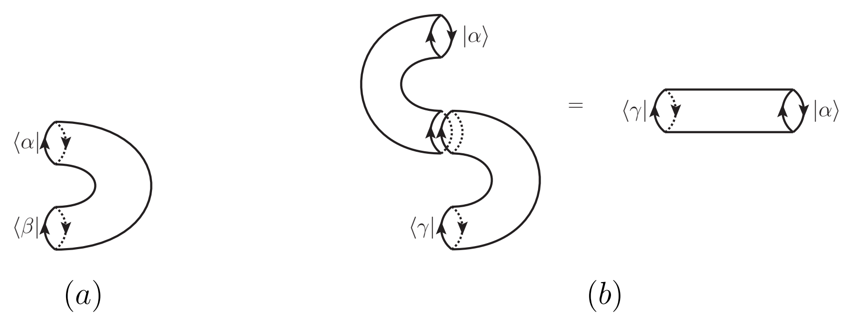


Figure 2.4: Topological interpretation of **(a)** the inverse metric $\eta^{\alpha\beta}$, **(b)** the relation $\eta_{\alpha\beta}\eta^{\beta\gamma}=\delta^{\gamma}_{\alpha}$. By convention, we draw the boundaries associated with upper indices facing left and the boundaries associated with the lower indices facing right.

The metric and triple product obey natural compatibility axioms which can be simply summarized by the statement that the metric and its inverse are used to lower and raise indices in the usual fashion. Finally the crucial requirement: the structure constants $C_{\alpha\beta}{}^{\gamma}\equiv C_{\alpha\beta\epsilon}\eta^{\epsilon\gamma}$ define an associative algebra

$$C_{\alpha\beta}{}^{\delta}\, C_{\delta\gamma}{}^{\epsilon} = C_{\beta\gamma}{}^{\delta}\, C_{\delta\alpha}{}^{\epsilon}\,, \tag{2.13}$$

as illustrated in Figure 2.5. From these data, arbitrary n-point correlators on a genus g surface can be evaluated by factorization (= pair-of-pants decomposition of the surface). The result is guaranteed to be independent of the specific decomposition.

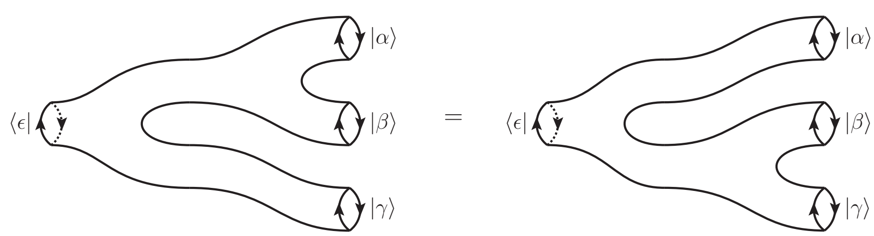


Figure 2.5: Pictorial rendering of the associativity of the algebra.

In our case the space $\mathcal{H}$ is spanned by the states $\{|\alpha\rangle\,, \alpha \in [0, 2\pi)\}$, where α parametrizes the $SU(2)$ eigenvalues, equ.(2.5). Alternatively we may "Fourier transform" to the basis of irreducible $SU(2)$ representations, $\{|R_K\rangle\,, K \in \mathbb{Z}_+\}$. We have concrete expressions (2.9, 2.10) for the cubic couplings $C_{\alpha\beta\gamma}$ and for the *inverse* metric $\eta^{\alpha\beta}$, which are manifestly symmetric under permutations of the indices. Formal inversion of (2.10) gives the metric $\eta_{\alpha\beta} \equiv (\eta^{\alpha})^{-1}\hat{\delta}(\alpha, \beta)$. Finally with the help of (2.10) we can raise, lower and contract indices at will. On physical grounds we expect these formal manipulations to make sense, since the superconformal index is well-defined as a series expansion in the chemical potential t, which should have a finite radius of convergence [19]. The explicit analysis of sections 3 and 4 will confirm these expectations. We will find expressions for the index as analytic functions of the chemical potentials. Our definitions satisfy the axioms of a 2d TQFT by construction, and independently of the specific form of the functions $f^{vect}(t, y, v)$ and $f^{chi}(t, y, v)$, except for the associativity axiom, which is completely non-trivial. Associativity of the 2d topological algebra is equivalent to 4d S-duality, and it can only hold for very special choices of field content, encoded in the single-letter partition functions f^{vect} and f^{chi}.

Letters	E	j_1	j_2	R	r	$\mathcal{I}$
ϕ	1	0	0	0	-1	$t^2 v$
$\lambda^1_\pm$	$\frac{3}{2}$	$\pm\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-t^3\, y,\ -t^3\, y^{-1}$
$\bar{\lambda}_{2+}$	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-t^4/v$
$\bar{F}_{++}$	2	0	1	0	0	t^6
$\partial_{-+}\lambda^1_+ + \partial_{++}\lambda^1_- = 0$	$\frac{5}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	t^6
q	1	0	0	$\frac{1}{2}$	0	$t^2/\sqrt{v}$
$\bar{\psi}_+$	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-t^4\sqrt{v}$
$\partial_{\pm+}$	1	$\pm\frac{1}{2}$	$\frac{1}{2}$	0	0	$t^3\, y,\ t^3\, y^{-1}$

Table 2.1: Contributions to the index from "single letters". We denote by $(\phi, \bar{\phi}, \lambda^I_\alpha, \lambda_{I\,\dot{\alpha}}, F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}})$ the components of the adjoint $\mathcal{N}=2$ vector multiplet, by $(q, \bar{q}, \psi_\alpha, \bar{\psi}_{\dot{\alpha}})$ the components of the trifundamental $\mathcal{N}=1$ chiral multiplet, and by $\partial_{\alpha\dot{\alpha}}$ the spacetime derivatives. Here $I=1,2$ are $SU(2)_R$ indices and $\alpha=\pm$, $\dot{\alpha}=\pm$ Lorentz indices.

2.2 Associativity of the Algebra

In this section we determine explicitly the structure constants and the metric of the TQFT and write them in terms of elliptic Beta integrals. With the help of a recent mathematical result [49] we prove analytically the associativity of the topological algebra.

2.2.1 Explicit Evaluation of the Index

The "single letters" contributing to the index, which must obey $\bar{\Delta} \equiv E - 2j_2 - 2R + r = 0$ [19], are enumerated in Table 2.1. The first block of the Table shows the contributing letters from the adjoint $\mathcal{N}=2$ vector multiplet (associated to each internal edge of a graph), including the equations of motion constraint. The second block shows the contributions from the $\mathcal{N}=1$ chiral multiples in the trifundamental representation, associated to each cubic vertex. Finally the last line of the Table shows the spacetime derivatives contributing to the index. Since each field can be hit by an arbitrary number of derivatives, the derivatives give a multiplicative contribution to the single-letter partition

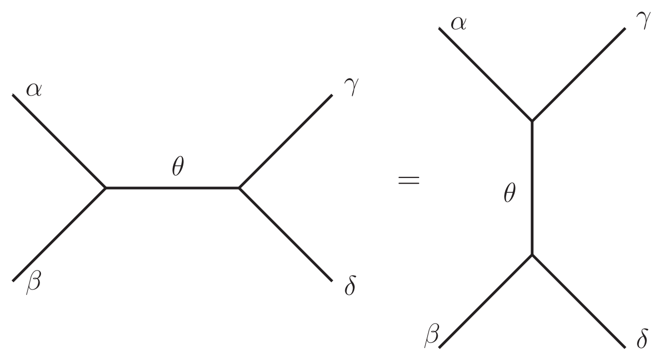


Figure 2.6: The basic S-duality channel-crossing. The two diagrams are two equivalent (S-dual) ways to represent the $\mathcal{N} = 2$ gauge theory with a single gauge group $SU(2)$ and four $SU(2)$ flavour groups, which is the basic building block of the A_1 generalized quiver theories. The indices on the edges label the eigenvalues of the corresponding $SU(2)$ groups.

functions of the form

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(t^3y)^m\,(t^3y^{-1})^n = \frac{1}{(1-t^3y)(1-t^3y^{-1})}\,. \tag{2.14}$$

All in all, the single letter partition function are given by

$$\text{adjoint} \quad : \quad f^{vect}(t,y,v) = \frac{t^2v - \frac{t^4}{v} - t^3(y+y^{-1}) + 2t^6}{(1-t^3\,y)(1-t^3y^{-1})}, \tag{2.15}$$

$$\text{trifundamental} \quad : \quad f^{chi}(t,y,v) = \frac{\frac{t^2}{\sqrt{v}} - t^4\sqrt{v}}{(1-t^3\,y)(1-t^3y^{-1})}\,. \tag{2.16}$$

We are now ready to check explicitly the basic S-duality move – S-duality with respect to one of the $SU(2)$ gauge groups, represented graphically as channel-crossing with respect to one of the edges of the graph (Figure 2.6). The full S-duality group of a graph is generated by repeated applications of the basic move to different edges. The contribution to the index from the left graph in Figure 2.6 is

$$\mathcal{I} = \int d\theta\,\Delta(\theta)\exp(\sum_{n=1}^{\infty}\frac{1}{n}\,\left[f_n^{vect}\cdot\chi_{adj}(n\theta) + f_n^{chi}\cdot\chi_{3f}(n\alpha,n\beta,n\theta) + f_n^{chi}\cdot\chi_{3f}(n\theta,n\gamma,n\delta)\right])$$

Substituting the expressions for the characters,

$$\begin{aligned}\mathcal{I} &= \frac{e^{\sum_{n=1}^{\infty}\frac{f_n^{vect}}{n}}}{\pi}\int_0^{2\pi} d\theta \sin^2\theta \\ &\quad e^{\sum_{n=1}^{\infty}\frac{2f_n^{vect}}{n}\cos 2n\theta} e^{\sum_{n=1}^{\infty}\frac{8f_n^{chi}}{n}[\cos n\alpha\cos n\beta+\cos n\gamma\cos n\delta]\cos n\theta}\,, \qquad (2.17)\end{aligned}$$

where $f_n^{vect} \equiv f^{vect}(t^n, y^n, v^n)$ and $f_n^{chi} \equiv f^{chi}(t^n, y^n, v^n)$. S-duality of the index is the statement this integral is invariant under permutations of the external labels $\alpha, \beta, \gamma, \delta$. Since symmetries under $\alpha \leftrightarrow \beta$ and (independently) under $\gamma \leftrightarrow \delta$ are manifest, the non-trivial requirement is symmetry under $\beta \leftrightarrow \gamma$, which gives the index associated to the crossed graph on the right of Figure 2.6.

The integrand of (2.17) is *not* invariant under $\beta \leftrightarrow \gamma$, but the integral is, as once can check order by order in a series expansion in the chemical potential t. Here is how things work to the first non-trivial order. We expand the integrand in t around $t = 0$, and set $y = v = 1$ for simplicity. The single-letter partition functions behave as

$$f^{vect}(t, y = 1, v = 1) \sim t^2 - 2\,t^3\,, \qquad f^{chi}(t, y = 1, v = 1) \sim t^2 - t^4\,. \quad (2.18)$$

The first non-trivial check is for the coefficient of $\mathcal{I}$ of order $O(t^4)$,

$$\begin{aligned}\mathcal{I} \sim t^4 \int_0^{2\pi} d\theta \sin^2\theta \Big(&\cos 4\theta + 2\cos^2 2\theta + 4A_2\, \cos 2\theta \qquad (2.19)\\ &+32A_1^2\, \cos^2\theta - 2\cos 2\theta + 16A_1\, \cos\theta\cos 2\theta - 8A_1\, \cos\theta\Big)\,,\end{aligned}$$

where $A_n \equiv \cos n\alpha \cos n\beta + \cos n\gamma \cos n\delta$. Performing the elementary integrals,

$$\mathcal{I} \sim t^4\,[6\pi + 2\pi\ \ (\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \cos 2\delta + 8\cos\alpha\cos\beta\cos\gamma\cos\delta)]\,,$$

which is indeed symmetric under $\alpha \leftrightarrow \beta \leftrightarrow \gamma \leftrightarrow \delta$. We stress that crossing symmetry depends crucially on the specific form of the single-letter partition functions (4.3) and thus on the specific field content. We have performed systematic checks by calculating the series expansion to several higher orders using *Mathematica*. Fortunately it is possible to give an analytic proof of crossing symmetry of the index, as we now describe.

2.2.2 Elliptic Beta Integrals and S-duality

The fundamental integral (2.17) can be recast in an elegant way in terms of special functions known as elliptic Beta integrals. We start by recalling the definition of the elliptic Gamma function, a two parameter generalization of the Gamma function,

$$\Gamma(z;p,q) \equiv \prod_{j,k\geq 0} \frac{1-z^{-1}\,p^{j+1}q^{k+1}}{1-z\,p^j q^k}\,. \tag{2.20}$$

For reviews of the elliptic Gamma function and of elliptic hypergeometric mathematics the reader can consult [46–48]. Throughout this thesis we will use the standard condensed notations

$$\begin{aligned} \Gamma(z_1,\dots,z_k;p,q) &\equiv \prod_{j=1}^{k}\Gamma(z_j;p,q), \\ \Gamma(z^{\pm1};p,q) &= \Gamma(z;p,q)\Gamma(1/z;p,q)\,. \end{aligned} \tag{2.21}$$

Two identities satisfied by the elliptic Gamma function that will be useful to us are

$$\Gamma(z^2;p,q) = \Gamma(\pm z, \pm\sqrt{q}\,z, \pm\sqrt{p}\,z, \pm\sqrt{pq}\,z;p,q)\,, \tag{2.22}$$

$$\Gamma\left(pq/z;p,q\right)\Gamma\left(z;p,q\right) = 1\,. \tag{2.23}$$

(As an illustration of the shorthand (2.21), the *rhs* of (2.22) is a product of eight Gamma functions.) Using the definition (2.20), it is straightforward to show [50]

$$\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\,\frac{t^{2n}z^n - t^{4n}z^{-n}}{(1-t^{3n}y^n)(1-t^{3n}y^{-n})}\right) = \Gamma(t^2\,z;p,q), \tag{2.24}$$

$$\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\,\frac{2t^{6n} - t^{3n}(y^n+y^{-n})}{(1-t^{3n}y^n)(1-t^{3n}y^{-n})}(z^n+z^{-n})\right) = -\frac{z}{(1-z)^2}\,\frac{1}{\Gamma(z^{\pm1};p,q)},$$

where

$$p = t^3 y, \qquad q = t^3 y^{-1} \,. \tag{2.25}$$

With these preparations, the building blocks (2.9) for the index can be written in the following compact form

$$\begin{aligned}
C_{\alpha_i\alpha_j\alpha_k} &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f_n^{chi}\; \chi_{3f}(n\alpha_i, n\alpha_j, n\alpha_k)\right) = \Gamma(\frac{t^2}{\sqrt{v}} a_i^{\pm 1} a_j^{\pm 1} a_k^{\pm 1}; p, q), \\
\eta^{\alpha_i} &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f_n^{vect}\; \chi_{adj}(n\alpha_i)\right) \\
&= \frac{1}{\Delta(\alpha_i)} \frac{(p;p)(q;q)}{4\pi} \Gamma(t^2\, v; p, q) \frac{\Gamma(t^2\, v\, a_i^{\pm 2}; p, q)}{\Gamma(a_i^{\pm 2}; p, q)} \,.
\end{aligned} \tag{2.26}$$

Here we have defined $a_i = \exp(i\alpha_i)$ and used

$$\exp(\sum_{n=1}^{\infty} \frac{1}{n} f_n^{vect}) = (p;p)(q;q)\, \Gamma(t^2\, v; p, q), \quad (a;b) \equiv \prod_{k=0}^{\infty} (1 - a\, b^k) \tag{2.27}$$

Again, the reader should keep in mind that the *rhs* of the first line in (2.26) is a product of eight elliptic Gamma functions according to the condensed notation (2.21).

Collecting all these definitions the fundamental integral (2.17) becomes

$$\kappa \Gamma\left(t^2 v; p, q\right) \oint \frac{dz}{z} \frac{\Gamma(t^2\, v\, z^{\pm 2}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \Gamma(\frac{t^2}{\sqrt{v}} a^{\pm 1} b^{\pm 1} z^{\pm 1}; p, q) \Gamma(\frac{t^2}{\sqrt{v}} c^{\pm 1} d^{\pm 1} z^{\pm 1}; p, q)$$

with $pq = t^6$, $\kappa \equiv (p;p)(q;q)/4\pi i$. As it turns out, this integral fits into a class of integrals which are an active subject of mathematical research, the elliptic Beta integrals

$$E^{(m)}(t_1, \ldots, t_{2m+6}) \sim \oint \frac{dz}{z} \frac{\Gamma(t_1 z, \ldots t_{2m+6} z; p, q)}{\Gamma(z^{\pm 2}; p, q)} \,, \qquad \prod_{k=1}^{2m+6} t_k = (pq)^{m+1} \tag{2.28}$$

Our integral is a special case of $E^{(5)}$. Elliptic Beta integrals have very interesting symmetry properties. For instance the symmetry of $E^{(2)}$ is related to the Weyl group of E_7. Very recently van de Bult proved [49] that special cases of

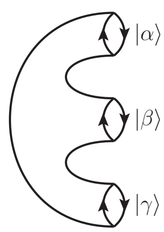
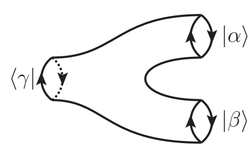
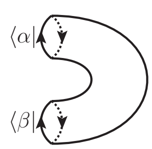
Symbol	Surface	Value
$C_{\alpha\beta\gamma}$		$\Gamma(\frac{t^2}{\sqrt{v}}a^{\pm1}b^{\pm1}c^{\pm1})$
$C_{\alpha\beta}{}^{\gamma}$		$\frac{i\kappa}{\Delta(\gamma)}\,\Gamma(t^2\,v)\,\frac{\Gamma(t^2\,v\,c^{\pm2})}{\Gamma(c^{\pm2})}\Gamma(\frac{t^2}{\sqrt{v}}a^{\pm1}b^{\pm1}c^{\pm1})$
$\eta^{\alpha\beta}$		$\frac{i\,\kappa}{\Delta(\alpha)}\,\Gamma(t^2\,v)\,\frac{\Gamma(t^2\,v\,a^{\pm2})}{\Gamma(a^{\pm2})}\,\hat{\delta}(\alpha,\beta)$

Table 2.2: The structure constants and the metric in terms of elliptic Gamma functions. For brevity we have left implicit the parameters of the Gamma functions, $p = t^3y$ and $q = t^3y^{-1}$. We have defined $a \equiv \exp(i\alpha)$, $b \equiv \exp(i\beta)$, and $c \equiv \exp(i\gamma)$. Recall also $\kappa \equiv (p;p)(q;q)/4\pi i$ and $\Delta(\alpha) \equiv (\sin^2\alpha)/\pi$.

the $E^{(5)}$ integral, which are equivalent to (2.28), are invariant under the Weyl group of F_4. In particular (2.28) is invariant under $b \leftrightarrow c$. This is theorem 3.2 in [49], with the parameters $\{t_{1,2,3,4},\, b\}$ of [49] related to the parameters $\{a,\, b,\, c,\, d,\, t^2 v\}$ in our equation (2.28) by the substitution

$$t_1 \to \frac{t^2}{\sqrt{v}}\, a\, b,\quad t_2 \to \frac{t^2}{\sqrt{v}}\, a/b,\quad t_3 \to \frac{t^2}{\sqrt{v}}\, c\, d,\; t_4 \to \frac{t^2}{\sqrt{v}}\, c/d,\quad b \to t^2\, v. \tag{2.29}$$

This completes the proof of crossing symmetry of the fundamental integral (2.17).

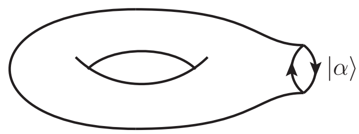


Figure 2.7: Handle-creating operator $\mathcal{J}_\alpha$

The expressions for the structure constants and metric of the topological algebra in terms of the elliptic Gamma functions are summarized in Table 2.2. These expressions are analytic functions of their arguments, except for for the metric $\eta^{\alpha\beta}$ which contains a delta-function. One can try and use the results of the theory of elliptic Beta integrals to represent the delta-function in a more elegant way, indeed such a representation is sometimes available in terms of a contour integral [55]. However, for generic choices of the parameters, the definition of [55] involves contour integrals not around the unit circle and thus using this representation one presumably should also change the prescription (2.10) for contracting indices. In the limit $v \to t$ the relevant contours do approach the unit circle and the formalism of [55] yields elegant expressions. This limit is however slightly singular. We discuss it in Appendix B.

As a simple illustration of the use of the expressions in Table 2.2 let us compute the superconformal index of the theory associated to diagram **(b)** in Figure 2.2. This is essentially the "handle-creating" vertex $\mathcal{J}_\alpha$ of the TQFT, Figure 2.7. We have

$$\mathcal{J}_\alpha = C_{\alpha\beta\gamma}\eta^{\beta\gamma} = \kappa\Gamma\left(t^2 v\right)\Gamma\left(\frac{t^2}{\sqrt{v}}a^{\pm1}\right)^2 \oint \frac{dz}{z}\frac{\Gamma(t^2 v\, z^{\pm2})}{\Gamma(z^{\pm2})}\Gamma\left(\frac{t^2}{\sqrt{v}}z^{\pm2}a^{\pm1}\right) \tag{2.30}$$

Multivariate extensions of elliptic Beta integrals have appeared in the calculation of the superconformal index for pairs of $\mathcal{N} = 1$ theories related by Seiberg duality [50]. Unlike our $\mathcal{N} = 2$ superconformal cases, there is no continuous deformation relating two Seiberg-dual theories, and it is not a priori obvious that their indices, evaluated at the free UV fixed points, should coincide – but it turns out that they do, thanks to identities satisfied by these multivariate integrals [56]. See also [57]. In Appendix A we tackle the $\mathcal{N} = 4$ case, evaluating the indices the S-dual pairs with gauge groups $Sp(n)$ and $SO(2n+1)$. Again S-duality predicts some new identities of elliptic Beta integrals, which we confirm to the first few orders in the t expansion. It appears that there is a general connection between elliptic hypergeometric mathematics and electric-magnetic duality of the index of $4d$ gauge theories.

2.3 Discussion

A rich class of 4d superconformal field theories arise by compactifying the 6d $(2,0)$ theory on a punctured Riemann surface Σ [16], and this has inspired a precise dictionary between 4d and 2d quantities [26–30]. In this chapter we have added a new entry to this dictionary. Previous work has focussed on the relation between the 4d theory on S^4 (or more generally on the theory in the Ω background) and Liouville theory on Σ. Here we have considered instead the superconformal index [19], which can be viewed as the partition function of the 4d theory on $S^3 \times S^1$, with twisted boundary conditions labelled by three chemical potentials. We have argued that the superconformal index is evaluated by a topological QFT on Σ. In the A_1 case we have computed explicitly the structure constants of the topological algebra and checked its associativity, using a rather non-trivial piece of contemporary mathematics [49]. Physically this result can be regarded as a precise check that the protected spectrum of operators is the same for the $SU(2)^{N_G}$ theories related by the generalized S-dualities of [16].

In the next chapter, we will compute the TQFT for the A_2 class of theories. S-dualities of the generalized A_2 quivers leads to a strongly coupled SCFT with E_6 flavor symmetry. S-duality invariance of the superconformal index will allow us to compute the index of this theory. This work will be extended to

A_n generalized quivers with an independent characterization of the TQFT in chapter 4. Finally, it would be illuminating to obtain a Lagrangian description of the $2d$ TQFT from a twisted compactification of the $(2, 0)$ theory on $S^3 \times S^1$, and reproduce by that route the structure constants evaluated in this chapter.

We suspect that we are just scratching the surface of a general connection between elliptic hypergeometric mathematics and S-duality. It is possible to generate new elliptic hypergeometric identities by calculating the superconformal index of S-dual theories. Already the simplest S-dualities (from a physical perspective), such as the $SO(2n + 1)/Sp(n)$ dualities in $\mathcal{N} = 4$ SYM, lead to identities that to the best of our knowledge have not appeared in the mathematical literature. One may wonder whether the logic can be reversed, and new S-dualities discovered from known elliptic identities. Elliptic Beta integrals are the most general known extensions of the classic Euler Beta integral, and as such they are the natural mathematical objects to appear in the calculation of "crossing-symmetric" physical quantities. It is perhaps not coincidental that the mathematics and the physics of the subject are being developed simultaneously, and we can look forward to a fruitful interplay between the two viewpoints.

Chapter 3

The Superconformal Index of the E_6 SCFT

In the last chapter we reviewed that the paradigmatic S-duality of $\mathcal{N} = 4$ super Yang-Mills is the simplest instance of a much more general web of duality connections relating $\mathcal{N} = 2$ $4d$ superconformal field theories. This viewpoint has been emphasized by Gaiotto [16], who introduced a large class of $\mathcal{N} = 2$ SCFTs by compactifying the $(2, 0)$ $6d$ theory on a Riemann surfaces Σ with punctures. Different ways of cutting Σ into pairs of pants correspond to different S-duality frames for the $4d$ theory. A remarkable dictionary relates $4d$ gauge theory quantities with calculations in $2d$ conformal field theory on Σ. For example, the partition function of the gauge theory on S^4, or more generally the Nekrasov instanton partition function [25], is reproduced exactly by a Liouville or Toda correlation function on Σ [26, 27].

This dictionary was extended in the previous chapter based on [1], by considering the superconformal index [19], which can be viewed as a twisted partition function of the $4d$ gauge theory $S^3 \times S^1$. The superconformal index counts the states of the $4d$ theory belonging to short multiplets, up to equivalent relations that set to to zero all sequences of short multiplets that may in principle recombine into long ones. By construction, the index is invariant under continuous deformations of the theory, and is also expected to be independent of the S-duality frame. *Assuming* S-duality, it follows that the index must be computed by a topological QFT living on Σ. In [1] this TQFT structure was discussed for the generalized quiver gauge theories with

$SU(2)^k$ gauge group, which arise from compactifications on Σ of the A_1 (2,0) theory. Invariance of the index under S-duality translates into associativity of the operator algebra of the $2d$ TQFT. In turn, associativity holds thanks to a beautiful mathematical identity for an elliptic hypergeometric integral [49].

What distinguishes the A_1 theories from their counterparts with $A_{n\geq 2}$ is that in all duality frames they have a Lagrangian description. This makes it easy to compute their superconformal index explicitly and to identify the structure constants of the $2d$ TQFT [1]. The situation for the generalized quiver theories with higher rank gauge groups is qualitatively different: in some duality frames the quivers contain intrinsically strongly-coupled blocks with no Lagrangian description. The prototypical example of this phenomenon was discussed by Argyres and Seiberg [58][1]: the SYM theory with $SU(3)$ gauge group and $N_f = 6$ fundamental hypermultiplets has a dual description involving the strongly-coupled SCFT with E_6 flavor symmetry [60]. In the absence of a Lagrangian description for the E_6 SCFT, it seems difficult to compute its superconformal index and to define the TQFT structure for generalized quivers with $SU(3)$ gauge groups.

We solve this problem in this chapter. By demanding consistency with Argyres-Seiberg duality, we are able to write down an explicit integral expression for the index of the E_6 SCFT (equation (3.24)). Technically, this is possible thanks to a remarkable inversion formula for a class of integral transforms [55]. By construction, the resulting expression for the index is guaranteed to be invariant under an $SU(6) \otimes SU(2)$ subgroup of the E_6 flavor symmetry. The index is seen *a posteriori* to be invariant under the full E_6 symmetry, providing an independent check of Argyres-Seiberg duality itself.[2] We proceed to define a TQFT structure for generalized quivers with $SU(3)$ gauge symmetries. We check associativity of the operator algebra, which is equivalent to a check of S-duality for Gaiotto's A_2 theories. Most of our checks are performed perturbatively, to several orders in an expansion in the chemical potentials that enter the definition of the index. Conversely, S-duality implies that associativity must hold exactly, so as a by-product of our analysis we conjecture new identities between integrals of elliptic Gamma functions.

The chapter is organized as follows. In section 3.1.1 we set up the stage by

[1]See also [59] for more examples.

[2]For earlier checks of Argyres-Seiberg duality see [61] and [62].

computing the superconformal index of $SU(N)$ gauge theories in terms of the elliptic Gamma functions. In section 3.1.2 the index of $N_f = 6$ $SU(3)$ theory is computed in the weakly-coupled frame and the usual S-duality invariance of this index is discussed. In section 3.1.3 we use Argyres-Seiberg duality to write down an explicit expression for the index of E_6 SCFT; we check perturbatively that the answer is E_6 covariant and that it is compatible with physical expectations about the Coulomb and Higgs branches of vacua. In section 3.2 we check invariance under S-duality of the superconformal index for the generalized $SU(3)$ quiver theories, and we present the TQFT interpretation of this index. In section 3.3 we briefly discuss our results. Four appendices complement the text with technical details.

3.1 Argyres-Seiberg duality and the index of E_6 SCFT

The S-duality group of the $\mathcal{N} = 2$ $SU(2)$ gauge theory with four flavors is $SL(2, \mathbb{Z})$. The action of this group on the gauge coupling is generated by $\tau \to \tau + 1$ and $\tau \to -1/\tau$. In Gaiotto's description [16] this theory is constructed by compactification of the $6d$ $(2,0)$ theory on a sphere with four punctures of the *same* kind. Then, the S-duality group could be understood as the mapping class group of this Riemann surface. The moduli space of the gauge coupling is shown in figure 3.1 (a). We can see that a fundamental domain can be chosen such that nowhere in the moduli space does the coupling take an infinite value.

For the case of $\mathcal{N} = 2$ $SU(3)$ gauge theory with 6 flavors, however, the S-duality group is $\Gamma^0(2)$. The action of the S-duality on the complex coupling is generated by the transformations $\tau \to \tau + 2$ and $\tau \to -1/\tau$. In Gaiotto's setup this theory is obtained by compactifying the $(2,0)$ theory on the sphere with two punctures of one type and two of another. The mapping class group of such a sphere is $\Gamma^0(2)$. The fundamental domain of this group is shown in the figure 3.1 (b) and, unlike the $SU(2)$ case, this does unavoidably contain a point with infinite coupling. In [58], it was shown that this infinitely coupled cusp could be described in terms of an $SU(2)$ gauge group weakly-coupled to a single hypermultiplet and a rank 1 interacting SCFT with E_6 flavor sym-

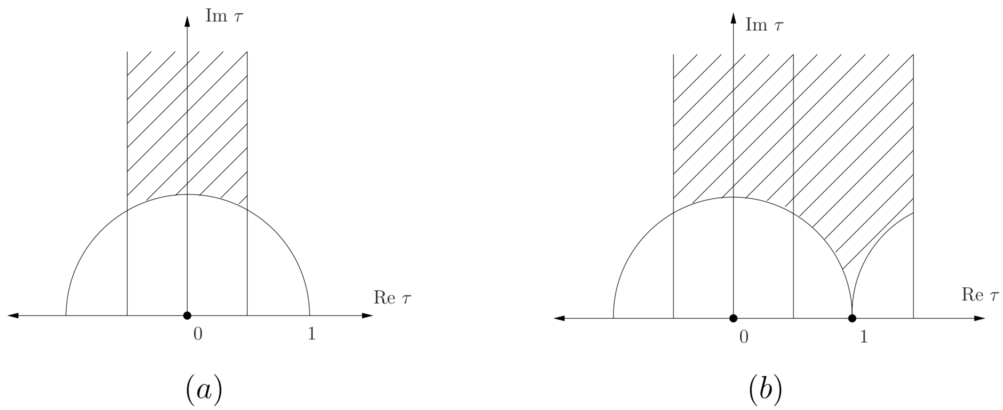


Figure 3.1: Moduli spaces for $\mathcal{N} = 2$ $SU(n)$ gauge theory with $2n$ flavors, (a) for $n = 2$ and (b) for $n = 3$ (in fact, for any $n > 2$). The shaded region in (a) is $H/SL(2, \mathbb{Z})$ while in (b) it is $H/\Gamma^0(2)$, where H is the upper half plane.

metry. Figure 3.3 describes this duality pictorially. The $SU(2)$ subgroup of the flavor symmetry of the SCFT that is gauged commutes with the $SU(6)$ subgroup of E_6. This $SU(6)$ combined with $SO(2)$ flavor symmetry of the single hypermultiplet generates the full $U(6)$ flavor symmetry of the original $SU(3)$ gauge theory. In other words, the $SO(2)$ flavor symmetry of the single hypermultiplet corresponds to the baryon number of the original $SU(3)$ gauge theory. The quarks of the $SU(3)$ theory are charged ± 1 under this $U(1)_B$ while the quarks of the $SU(2)$ theory are charged ± 3 under the same.

The E_6 SCFT has a Coulomb branch parametrized by the expectation value of a dimension 3 operator u which is identified with $\text{Tr}\phi^3$ of the dual $SU(3)$ theory, while the $\text{Tr}\phi^2$ of the $SU(3)$ theory corresponds to the Coulomb branch parameter of the $SU(2)$ gauge theory. The E_6 CFT also has a Higgs branch parametrized by the expectation value of dimension 2 operators $\mathbb{X}$, which transform in the adjoint representation of E_6 (**78**). As shown in [62] the Higgs branch operators obey a Joseph relation at quadratic order which leaves a 22 complex dimensional Higgs branch. When coupled to the $SU(2)$ gauge group, the resulting Higgs branch has complex dimension 20. The dual $SU(3)$ theory also has a Higgs branch of complex dimension 20 and its Higgs operators can be easily constructed by combination of squark fields. See appendix E for more details.

The moduli space might contain also other infinitely coupled cusps which however are S-dual to the weakly-coupled cusp $\tau = i\infty$. This is the usual S-dualty mapping the $N_f = 6$ $SU(3)$ gauge theory to itself with some of the $U(1)$ flavor factors interchanged. This duality is represented in figure 3.2.

We proceed to compute the superconformal index of the $SU(3)$ theory and, by using the Argyres-Seiberg duality, of the interacting E_6 SCFT.

3.1.1 Elliptic hypergeometric expressions for the index

The contribution to the integrand of (4.2) from hypers in a fundamental representation of an $SU(n)$ gauge group is

$$\exp\left(\sum_{k=1}^{\infty} \frac{1}{k} f^{chi}\left(t^k,\, v^k,\, y^k\right)\, \left[\chi_f(U^k) + \chi_{\bar{f}}(U^k)\right]\right) = \prod_{i=1}^{n} \Gamma\left(\frac{t^2}{\sqrt{v}} a_i^{\pm 1}; p, q\right). \tag{3.1}$$

The contribution to the integrand of (4.2) from the vector multiplet of $SU(n)$ is

$$\exp\left(\sum_{k=1}^{\infty} \frac{1}{k} f^{vect}\left(t^k,\, v^k,\, y^k\right)\, \chi_{adj}(U^k)\right) = \tag{3.2}$$

$$= \frac{\left[\Gamma(t^2\, v; p, q)\, (p; p)(q; q)\right]^{n-1}}{\Delta(\mathbf{a})\Delta(\mathbf{a}^{-1})} \prod_{i \neq j} \frac{\Gamma(t^2\, v\, a_i/a_j; p, q)}{\Gamma(a_i/a_j; p, q)}.$$

We have defined the characters of the fundamental representation to be

$$\chi_f = \sum_{i=1}^{n} a_i, \qquad \chi_{\bar{f}} = \sum_{i=1}^{n} \frac{1}{a_i}, \qquad \prod_{i=1}^{n} a_i = 1\,. \tag{3.3}$$

The character of the adjoint representation is

$$\chi_{adj} = \chi_f\, \chi_{\bar{f}} - 1 = \sum_{i \neq j} a_i/a_j + n - 1\,. \tag{3.4}$$

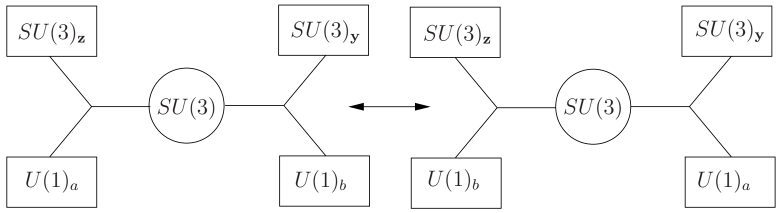


Figure 3.2: $SU(3)$ SYM with $N_f = 6$. The $U(6)$ flavor symmetry is decomposed as $SU(3)_{\mathbf{z}} \otimes U(1)_a \oplus SU(3)_{\mathbf{y}} \otimes U(1)_b$. S-duality $\tau \to -1/\tau$ interchanges the two $U(1)$ charges.

We have also defined

$$\Delta(\mathbf{a}) = \prod_{i \neq j} (a_i - a_j)\,. \tag{3.5}$$

The Haar measure is given by

$$\oint_{SU(n)} d\mu(\mathbf{a}) f(\mathbf{a}) = \frac{1}{n!} \oint_{\mathbb{T}^{n-1}} \prod_{i=1}^{n-1} \frac{da_i}{2\pi i\, a_i} \Delta(\mathbf{a})\Delta(\mathbf{a}^{-1}) f(\mathbf{a})\Bigg|_{\prod_{i=1}^{n} a_i = 1}\,, \tag{3.6}$$

where $\mathbb{T}$ is the unit circle. Whenever we gauge a symmetry we have a vector multiplet associated to the integrated group and thus we will use the following notation

$$\mathcal{F}_{\mathbf{a}}\, \mathcal{G}^{\mathbf{a}} \equiv \frac{\left[2\,\Gamma(t^2\, v; p, q)\, \kappa\right]^{n-1}}{n!} \oint_{\mathbb{T}_{n-1}} \prod_{i=1}^{n-1} \frac{da_i}{2\pi i\, a_i} \prod_{i \neq j} \frac{\Gamma(t^2\, v\, a_i/a_j; p, q)}{\Gamma(a_i/a_j; p, q)} \mathcal{F}\,(\mathbf{a})\; \mathcal{G}\,(\mathbf{a}^{-1})\Bigg|_{\prod_{i=1}^{n} a_i = 1}$$

where $\kappa \equiv (p;p)(q;q)/2$. In what follows for the sake of brevity we will omit the parameters p and q from the elliptic Gamma function, i.e. $\Gamma(x)$ should always be understood as $\Gamma(x;\, p, q)$.

3.1.2 Weakly-coupled frame

We take the chiral multiplets to be in the fundamental and antifundamental of the color and flavor. $U(1)_B$ rotates them into each other. The vector multiplet is in the adjoint of the color. The $SU(3)$ characters of the relevant representations are:

$$\chi_f = z_1 + z_2 + z_3 \quad \chi_{\bar{f}} = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \quad \text{and} \quad \chi_{adj} = \chi_f \chi_{\bar{f}} - 1 \tag{3.7}$$

while writing down these characters, we have to impose $z_1 z_2 z_3 = 1$.

Let z's stand for the eigenvalues of the flavor group and x's be the eigenvalues of the color group. The $U(1)_B$ charge is counted by the variable a. Let us write down the characters of the representation of the matter

$$\chi_{hyp} = \sum_{i=1}^{3}\sum_{j=1}^{3} a\, z_i\, x_j + \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{1}{a\, z_i\, x_j}\,. \tag{3.8}$$

Using (3.1) the index contributed by the matter can be written in a closed form as

$$C_{a,\mathbf{x},\mathbf{y}} = \prod_{i=1}^{3}\prod_{j=1}^{3} \Gamma\left(\frac{t^2}{\sqrt{v}}\,(a\, x_i\, y_j)^{\pm 1}\right)\,. \tag{3.9}$$

The index for the $SU(3)$ gauge theory with six hypermultiplets is then given by the following contour integral.

$$\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}} = C_{b,\mathbf{y},\mathbf{x}}\, C_{a,\mathbf{z}}{}^{\mathbf{x}} = \tag{3.10}$$

$$\frac{2}{3}\kappa^2 \Gamma(t^2 v)^2 \oint_{\mathbb{T}^2} \prod_{i=1}^{2} \frac{dx_i}{2\pi i\, x_i} \frac{\prod_{i=1}^{3}\prod_{j=1}^{3} \Gamma(\frac{t^2}{\sqrt{v}}(\frac{a z_i}{x_j})^{\pm 1})\Gamma(\frac{t^2}{\sqrt{v}}(b\, y_i\, x_j)^{\pm 1}) \prod_{i\neq j}\Gamma(t^2 v\,\frac{x_i}{x_j})}{\prod_{i\neq j}\Gamma(\frac{x_i}{x_j})}$$

By expanding this integral in t one can show that it is symmetric under interchanging the two $U(1)$ factors (see appendix C),

$$a \quad \leftrightarrow \quad b\,. \tag{3.11}$$

Interchanging the two $U(1)$s is equivalent to performing a usual S-duality between a weakly-coupled and infinitely-coupled points of the moduli space and thus we expect the index to be invariant under this operation.[3]

One can analytically prove this statement in a special case. Notice that if

[3]The integral (3.10) is an $SU(3)$ generalization of the $SU(2)$ integral in [1] for which the analogous statement to (3.11) has an analytic proof [49]. It is easy to generalize [3.10,3.11] for $SU(n)$ theories with arbitrary n, see appendix F.

$t = v$, the integral (3.10) is given by

$$\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}}\big|_{v=t} = I^{(2)}_{A_2}\left(1|\,t^{\frac{3}{2}}a^{-1}\mathbf{z}^{-1}, t^{\frac{3}{2}}b\mathbf{y}; t^{\frac{3}{2}}a\mathbf{z}, t^{\frac{3}{2}}b^{-1}\mathbf{y}^{-1}\right)\,, \tag{3.12}$$

where [56]

$$I^{(m)}_{A_n}(Z|t_0,\ldots,t_{n+m+1};u_0,\ldots,u_{n+m+1};\,p,q) = \tag{3.13}$$
$$\frac{2^n}{n!}\kappa^n\oint_{\mathbb{T}^{n-1}}\prod_{i=1}^{n-1}\frac{dx_i}{2\pi i\,x_i}\,\frac{\prod_{i=1}^{n}\prod_{j=0}^{m+n+1}\Gamma(t_j\,x_i,\,u_j/x_i;\,p,q)}{\prod_{i\neq j}\Gamma(x_i/x_j;\,p,q)}\Bigg|_{\prod_{i=1}^{n}x_i=Z}.$$

If the integral $I^{(m)}_{A_n}(Z|\ldots t_i\ldots;\ldots u_i\ldots)$ satisfies the condition that $\prod_{i=1}^{m+n+2}t_iu_i = (pq)^{m+1}$ then due to [56], the following theorem holds

$$I^{(m)}_{A_n}(Z|\ldots t_i\ldots;\ldots u_i\ldots) = I^{(n)}_{A_m}\left(Z|\ldots\frac{T^{\frac{1}{m+1}}}{t_i}\ldots;\ldots\frac{U^{\frac{1}{m+1}}}{u_i}\ldots\right)\prod_{r,s=1}^{m+n+2}\Gamma(t_ru_s)\,, \tag{3.14}$$

where $T \equiv \prod_{r=1}^{m+n+2}t_r$ and $U \equiv \prod_{r=1}^{m+n+2}u_r$.[4] Coincidently, our integral (3.10) satisfies the above requirement and applying the theorem we can transform it into

$$I^{(2)}_{A_2}\left(1|t^{\frac{3}{2}}b\mathbf{z}, t^{\frac{3}{2}}a^{-1}\mathbf{y}^{-1}; t^{\frac{3}{2}}b^{-1}\mathbf{z}^{-1}, t^{\frac{3}{2}}a\mathbf{y}\right) = I^{(2)}_{A_2}\left(1|t^{\frac{3}{2}}b^{-1}\mathbf{z}^{-1}, t^{\frac{3}{2}}a\mathbf{y}; t^{\frac{3}{2}}b\mathbf{z}, t^{\frac{3}{2}}a^{-1}\mathbf{y}^{-1}\right).\tag{3.15}$$

Note that the factor $\prod_{r,s=1}^{m+n+2}\Gamma(t_ru_s)$ in (3.14) reduces to 1 after pairwise cancelations using the property (2.23). What we have effectively achieved through this transformation is that we have exchanged the $U(1)$ quantum numbers of the matter charged under the $SU(3)^2$ flavor. This in particular implies that both the $SU(3)$ flavor groups are on the same footing and are not associated with separate $U(1)$'s.

3.1.3 Strongly-coupled frame and the index of E_6 SCFT

In the strongly-coupled S-duality frame, figure 3.3, we have a fundamental hypermultiplet coupled to an $SU(2)$ gauge theory. This gauge group is identified with an $SU(2)$ subgroup of the E_6 flavor symmetry of a strongly-coupled rank one SCFT. We do not know the field content of the strongly-coupled

[4]This identity was extensively used in [50] to show that certain theories related by Seiberg duality have equal superconformal indices [51]. In this context the authors of [57, 63] applied the elliptic hypergeometric techniques to a large class of Seiberg dualities.

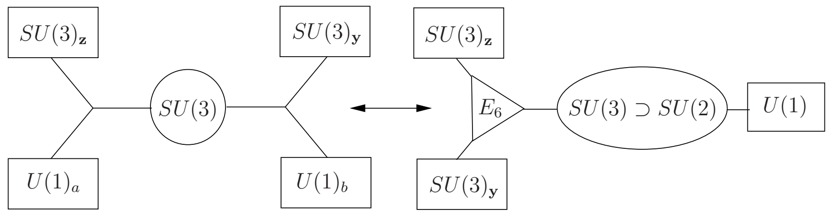


Figure 3.3: Argyres-Seiberg duality for $SU(3)$ SYM with $N_f = 6$.

rank 1 E_6 SCFT. This implies that we can not write down the "single letter" partition function for that theory and, a-priori, can not directly compute its index. In what follows we will use the index computed in the weakly-coupled frame (3.10) and the above statements about Argyres-Seiberg duality to infer the index of the E_6 SCFT.

Let $C^{(E_6)}$ denote the index of rank 1 E_6 SCFT [60]. The maximal subgroup of E_6 is $SU(3)^3$. Two among these three $SU(3)$'s are identified with the two $SU(3)$ factors in the flavor group of the weakly-coupled theory, see figure 3.3. Let the additional $SU(3)$ be denoted by $\mathbf{w}$. The fundamental representation of E_6 is decomposed under $SU(3)_{\mathbf{w}} \otimes SU(3)_{\mathbf{y}} \otimes SU(3)_{\mathbf{z}}$ as,

$$\mathbf{27}_{E_6} = (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) \oplus (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})\,. \tag{3.16}$$

Thus, the character of the E_6 fundamental fields is,

$$\chi_{\mathbf{27}} = \sum_{i,j=1}^{3} \left(\frac{w_i}{y_j} + \frac{z_i}{w_j} + \frac{y_i}{z_j} \right), \qquad \prod_{i=1}^{3} y_i = \prod_{i=1}^{3} z_i = \prod_{i=1}^{3} w_i = 1\,. \tag{3.17}$$

The index $C^{(E_6)}$ is thus a function of $\mathbf{w}$, $\mathbf{y}$, and $\mathbf{z}$. The S-duality picture suggests that we should decompose $SU(3)_{\mathbf{w}}$ as $SU(2)_e \otimes U(1)_r$. This amounts to the change of variables $\{w_1, w_2, w_2\} \to \{er, \frac{r}{e}, \frac{1}{r^2}\}$, for which the character of the fundamental of E_6 becomes

$$\chi_{\mathbf{27}} = (er + \frac{r}{e} + \frac{1}{r^2})(\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}) + (\frac{1}{er} + \frac{e}{r} + r^2)(z_1 + z_2 + z_3) + \sum_{i,j=1}^{3} \frac{y_i}{z_j} \tag{3.18}$$

Thus, the index of the E_6 SCFT can be denoted as $C^{(E_6)}\left((e,\, r), \mathbf{y}, \mathbf{z}\right)$. In the above notations the index of the additional hypermultiplet of the theory is

$$C_{s,\,e} = \Gamma\left(\frac{t^2}{\sqrt{v}}\, e^{\pm 1}\, s^{\pm 1}\right)\,. \tag{3.19}$$

Thus, one can write the superconformal index of the theory in the strongly-coupled frame as

$$\begin{aligned}\hat{\mathcal{I}}\,(s,r;\mathbf{y},\mathbf{z}) &= C_s{}^{e}\, C^{(E_6)}_{(e,r),\mathbf{y},\mathbf{z}} = \qquad (3.20)\\ &= \kappa\,\Gamma(t^2 v)\,\oint_{\mathbb{T}} \frac{de}{2\pi i\, e}\frac{\Gamma(t^2 v e^{\pm 2})}{\Gamma(e^{\pm 2})}\Gamma(\frac{t^2}{\sqrt{v}}e^{\pm 1}\, s^{\pm 1})\; C^{(E_6)}\,((e,\,r),\mathbf{y},\mathbf{z})\,.\end{aligned}$$

By Argyres-Seiberg duality we have to equate

$$\hat{\mathcal{I}}\,(s,r;\mathbf{y},\mathbf{z}) = \mathcal{I}_{a,\mathbf{z};b,\mathbf{y}}\,, \tag{3.21}$$

where $\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}}$ is given in (3.10), and we appropriately identify the $U(1)$ charges,

$$s = (a/b)^{3/2}, \qquad r = (a\,b)^{-1/2}\,. \tag{3.22}$$

It so happens that the integral of equation (3.20) has special properties which allow us to invert it (see appendix D and [55] for the details). One can write the following

$$\kappa \oint_{C_w} \frac{ds}{2\pi i\, s}\,\frac{\Gamma(\frac{\sqrt{v}}{t^2}w^{\pm 1}\, s^{\pm 1})}{\Gamma(\frac{v}{t^4},\; s^{\pm 2})}\,\hat{\mathcal{I}}\,(s,r;\mathbf{y},\mathbf{z}) \;=\; \Gamma(t^2 v\, w^{\pm 2})\; C^{(E_6)}\,((w,\,r),\mathbf{y},\mathbf{z}) \tag{3.23}$$

where the contour C_w is a deformation of the unit circle such that it encloses $s = \frac{\sqrt{v}}{t^2}\, w^{\pm 1}$ and excludes $s = \frac{t^2}{\sqrt{v}}\, w^{\pm 1}$ (for precise definition and details see appendix D and [55]). The above expression for the index $C^{(E_6)}$ does satisfy (3.20), but *a-priori* does not uniquely follow from it. However, as we will explicitly see below, (3.23) is consistent with what is expected from E_6 SCFT. We will comment on this issue in the end of this section. We can thus use the Argyres-Seiberg duality (3.21) to write a closed form expression for the E_6

index

$$
\boxed{\begin{gathered} C^{(E_6)}\left((w,\,r),\mathbf{y},\mathbf{z}\right) = \frac{2\,\kappa^3\Gamma(t^2v)^2}{3\,\Gamma(t^2v\,w^{\pm 2})}\oint_{C_w}\frac{ds}{2\pi i\,s}\,\frac{\Gamma(\frac{\sqrt{v}}{t^2}w^{\pm 1}\,s^{\pm 1})}{\Gamma(\frac{v}{t^4},\;s^{\pm 2})}\times \\ \times\oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i\,x_i}\frac{\prod_{i=1}^{3}\prod_{j=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{s^{\frac{1}{3}}\,z_i}{x_j\,r}\right)^{\pm 1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{s^{-\frac{1}{3}}\,y_i\,x_j}{r}\right)^{\pm 1}\right)\prod_{i\neq j}\Gamma\left(t^2v\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}\,. \end{gathered}}
\tag{3.24}
$$

One can rewrite the above expression without using the special integration contour. The integration contour C_w can be split into five pieces: a contour around the unit circle $\mathbb{T}$, two contours encircling the simple poles of $\Gamma(\frac{\sqrt{v}}{t^2}w^{\pm 1}\,s^{\pm 1})$ at $s=\frac{\sqrt{v}}{t^2}w^{\pm 1}$, and two contours encircling in the opposite direction the simple poles of $\Gamma(\frac{\sqrt{v}}{t^2}w^{\pm 1}\,s^{\pm 1})$ at $\frac{t^2}{\sqrt{v}}w^{\pm 1}$. Using the fact that elliptic Gamma function satisfies $\lim_{z\to 1}(1-z)\Gamma(z;p,q)=1/(p;p)(q;q)$ we have

$$
\begin{aligned}
C^{(E_6)}\left((w,\,r),\mathbf{y},\mathbf{z}\right) = {} & \frac{\kappa}{\Gamma(t^2vw^{\pm 2})}\oint_{\mathbb{T}}\frac{ds}{s}\,\frac{\Gamma(\frac{\sqrt{v}}{t^2}w^{\pm 1}\,s^{\pm 1})}{\Gamma(\frac{v}{t^4},\;s^{\pm 2})}\,\hat{\mathcal{I}}\left(s,r;\mathbf{y},\mathbf{z}\right) \\
&+\frac{1}{2}\frac{\Gamma(w^{-2})}{\Gamma(t^2vw^{-2})}\left[\hat{\mathcal{I}}\left(s=\frac{\sqrt{v}w}{t^2},r;\mathbf{y},\mathbf{z}\right)+\hat{\mathcal{I}}\left(s=\frac{t^2}{\sqrt{v}w},r;\mathbf{y},\mathbf{z}\right)\right] \\
&+\frac{1}{2}\frac{\Gamma(w^{2})}{\Gamma(t^2vw^{2})}\left[\hat{\mathcal{I}}\left(s=\frac{\sqrt{v}}{t^2w},r;\mathbf{y},\mathbf{z}\right)+\hat{\mathcal{I}}\left(s=\frac{t^2w}{\sqrt{v}},r;\mathbf{y},\mathbf{z}\right)\right]\,.
\end{aligned}
\tag{3.25}
$$

The index (3.24) encodes some information about the matter content of the E_6 theory. To extract this information it is useful to expand the index (3.24) in the chemical potentials. We define an expansion in t as

$$
C^{(E_6)}\equiv\sum_{k=0}^{\infty}a_k\,t^k\,. \tag{3.26}
$$

The first several orders in this expansion have the following form

$$\begin{aligned}
a_0 =& 1 \\
a_1 t =& a_2 t^2 = a_3 t^3 = 0 \\
a_4 t^4 =& \frac{t^4}{v} \chi^{E_6}_{\mathbf{78}} \\
a_5 t^5 =& 0 \\
a_6 t^6 =& -t^6 \chi^{E_6}_{\mathbf{78}} - t^6 + t^6 v^3 \\
a_7 t^7 =& \frac{t^7}{v}\left(y + \frac{1}{y}\right)\chi^{E_6}_{\mathbf{78}} + \frac{t^7}{v}\left(y + \frac{1}{y}\right) - t^7 v^2 \left(y + \frac{1}{y}\right) \\
a_8 t^8 =& \frac{t^8}{v^2}\left(\chi^{E_6}_{sym^2(\mathbf{78})} - \chi^{E_6}_{\mathbf{650}} - 1\right) + t^8 v + t^8 v \\
a_9 t^9 =& -t^9 \left(y + \frac{1}{y}\right)\chi^{E_6}_{\mathbf{78}} - 2t^9 \left(y + \frac{1}{y}\right) + t^9 v^3 \left(y + \frac{1}{y}\right) \\
a_{10} t^{10} =& -\frac{t^{10}}{v}(\chi^{E_6}_{\mathbf{78}}\chi^{E_6}_{\mathbf{78}} - \chi^{E_6}_{\mathbf{650}} - 1) + \frac{t^{10}}{v}\left(y^2 + 1 + \frac{1}{y^2}\right)\chi^{E_6}_{\mathbf{78}} + \\
& + \frac{t^{10}}{v}\left(y + \frac{1}{y}\right)^2 - t^{10} v^2 \left(y + \frac{1}{y}\right)^2 \\
a_{11} t^{11} =& \frac{t^{11}}{v^2}\left(y + \frac{1}{y}\right)(\chi^{E_6}_{\mathbf{78}}\chi^{E_6}_{\mathbf{78}} - \chi^{E_6}_{\mathbf{650}} - 1) + t^{11} v \left(y + \frac{1}{y}\right) + t^{11} v \left(y + \frac{1}{y}\right) .
\end{aligned} \tag{3.27}$$

The adjoint representation of E_6 , $\mathbf{78}$, decomposes in the following way in terms of its maximal $SU(3)^3$ subgroup

$$\mathbf{78} \quad = \quad (\mathbf{3},\mathbf{3},\mathbf{3}) + (\bar{\mathbf{3}},\bar{\mathbf{3}},\bar{\mathbf{3}}) + (\mathbf{8},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{8}) \,, \tag{3.28}$$

and $\mathbf{650}$ of E_6 is composed as

$$\mathbf{650} = \mathbf{27} \times \overline{\mathbf{27}} - \mathbf{78} - \mathbf{1} \,. \tag{3.29}$$

The Higgs branch operators $\mathbb{X}$ of E_6 theory are in the adjoint ($\mathbf{78}$) representation of E_6 flavor algebra. The terms of the index proportional to $\chi^{E_6}_{\mathbf{78}}$ are forming the following series,

$$\left[\frac{t^4}{v} - t^6 + \frac{t^7}{v}\left(y + \frac{1}{y}\right) - t^9 \left(y + \frac{1}{y}\right) + \cdots\right]\chi^{E_6}_{\mathbf{78}} \,, \tag{3.30}$$

which is the index of a multiplet with $\Delta = 2$, $j = \bar{j} = 0$ and $r = 0$ and of its derivatives (see appendix C.2 of [2]). Taken as a "letter" this multiplet has the following "single letter" partition function

$$\frac{t^4/v - t^6}{(1-t^3y)(1-t^3/y)}\,, \tag{3.31}$$

which matches the quantum numbers of the Higgs branch operators on the weakly-coupled side of the Argyres-Seiberg duality if we follow the identifications listed in [62].

The E_6 singlet part of the index contains yet another series,

$$t^6v^3 - t^7v^2\left(y+\frac{1}{y}\right) + t^8v + t^9v^3\left(y+\frac{1}{y}\right) + \cdots\,. \tag{3.32}$$

This series forms the index of a chiral multiplet with $\Delta = 3$, $j = \bar{j} = 0$ and $r = 3$ together with its derivatives (appendix C.1 of [2])

$$\frac{t^6v^3 - t^7v^2\left(y+\frac{1}{y}\right) + t^8v}{(1-t^3y)(1-t^3/y)}\,. \tag{3.33}$$

Since the Coulomb branch operator, u, of E_6 theory (which is identified as $\mathrm{Tr}\phi^3$ of the dual $SU(3)$ theory) has exactly the same quantum numbers, this multiplet is identified as the Coulomb branch operator.

The remaining singlet part of the index,

$$-t^6 + \frac{t^7}{v}\left(y+\frac{1}{y}\right) + t^8v - 2t^9\left(y+\frac{1}{y}\right) + \cdots\,, \tag{3.34}$$

is just the index of the stress tensor multiplet and its derivatives (appendix C.3 of [2])

$$\frac{-t^6 + \frac{t^7}{v}\left(y+\frac{1}{y}\right) + t^8v - t^9\left(y+\frac{1}{y}\right)}{(1-t^3y)(1-t^3/y)}\,. \tag{3.35}$$

Besides the matter content, the index also provides possible constraints among operators. For example, it was argued [62] that the Higgs branch operators of the E_6 theory should obey the Joseph relations,

$$(\mathbb{X}\otimes\mathbb{X})|_{\mathcal{I}_2} = 0\,, \tag{3.36}$$

where the representation $\mathcal{I}_2$ is defined as

$$sym^2(V(\mathbf{adj})) = V(2\mathbf{adj}) \oplus \mathcal{I}_2\,. \tag{3.37}$$

For E_6, $\mathbf{adj} = \mathbf{78}$, $2\mathbf{adj} = \mathbf{2430}$ and then $sym^2(\mathbf{78}) = \mathbf{2430} \oplus \mathbf{650} \oplus \mathbf{1}$. Thus, in our case

$$\mathcal{I}_2 = \mathbf{650} \oplus \mathbf{1}\,. \tag{3.38}$$

The Joseph relation in E_6 theory reads,

$$(\mathbb{X} \otimes \mathbb{X})|_{\mathbf{650}\oplus\mathbf{1}} = 0\,, \tag{3.39}$$

which means that these operators should not appear in the index. The index of $\mathbb{X}$ is t^4/v, then the index of $\mathbb{X} \otimes \mathbb{X}$ is t^8/v^2. (3.27) shows that our index is consistent with the Joseph relation.

Further constraints can also be derived from the higher order terms in (3.27). Let us consider the index at order t^{10}. The meaning of each term is clear. The first term corresponds to operators $\mathbb{X} \otimes (Q\mathbb{X})$ with the constraint $Q(\mathbb{X} \otimes \mathbb{X})_{\mathbf{650}+1} = 0$ which is a descendant of Joseph relation above (3.39). The last three terms are derivative descendants of $\frac{t^4}{v}\chi^{E_6}_{\mathbf{78}}$, $\frac{t^7}{v}\left(y + \frac{1}{y}\right)$ and $-t^7v^2\left(y + \frac{1}{y}\right)$ respectively. However, terms of the form

$$t^{10}v^2\chi^{E_6}_{\mathbf{78}}\,, \tag{3.40}$$

which would be corresponding to the $Higgs \otimes Coulomb$ operators are absent. This fact implies the constraint

$$\mathbb{X} \otimes u = 0\,. \tag{3.41}$$

This is consistent with the fact that the E_6 theory has rank 1. The absence of $-\frac{t^{10}}{v}\chi^{E_6}_{\mathbf{78}}$ also implies the constraint

$$\mathbb{X} \otimes T = 0\,, \tag{3.42}$$

where T is the stress tensor. The structure of the index at order t^{11} is consistent with these two constraints.

Finally, let us comment on the uniqueness of our proposal. In principle, the index (3.24) produced by the construction of this section might differ from the true index of the E_6 SCFT: $C^{(E_6)}_{true}((e,\,r),\mathbf{y},\mathbf{z}) = C^{(E_6)}((e,\,r),\mathbf{y},\mathbf{z}) + \delta C((e,\,r),\mathbf{y},\mathbf{z})$, with δC satisfying

$$\oint_{\mathbb{T}} \frac{de}{2\pi i\, e} \frac{\Gamma(\frac{t^2}{\sqrt{v}}\, e^{\pm 1} s^{\pm 1})\Gamma(t^2 v\, e^{\pm 2})}{\Gamma(e^{\pm 2})}\, \delta C((e,\,r),\mathbf{y},\mathbf{z}) = 0\,. \tag{3.43}$$

At this stage we are not able to rigorously rule out such a possibility. However, the E_6 covariance of our proposal, its consistency with physical expectations about protected operators and the further S-duality checks performed in the following section, make us confident that we have identified the correct index of the E_6 SCFT.

Note that the expression for the index (3.24) is not explicitly given in terms of E_6 characters. However, as one learns from the perturbative expansion (3.27), the characters of $SU(3)_{\mathbf{y}} \otimes SU(3)_{\mathbf{z}} \otimes SU(2)_w \otimes U(1)_r$ always combine into E_6 characters. Essentially, since the weakly-coupled frame has really $SU(6) \otimes U(1)$ flavor symmetry we can write an expression for the E_6 index which has a manifest $SU(6) \otimes SU(2)$ symmetry,[5] but not the full E_6. The fact that just by assuming Argyres-Seiberg duality we obtain an index for a theory with an E_6 flavor symmetry and with a consistent spectrum of operators is a non-trivial check of Argyres-Seiberg duality.

3.2 S-duality checks of the E_6 index

In the previous section we have discussed the superconformal index of the $N_f = 6$ $SU(3)$ theory and of its strongly-coupled dual. One can obtain this theory by compactifying a $(2,0)$ $6d$ theory on a sphere with four punctures, two $U(1)$ punctures and two $SU(3)$ punctures. The different S-duality frames are then given by the different degeneration limits of this Riemann surface. The weakly-coupled frames are obtained by bringing together one of the $U(1)$ punctures and one of the $SU(3)$ punctures, and the strongly-coupled frame is obtained by colliding the two $SU(3)$ ($U(1)$) punctures. The coupling constant of the theory is related to the cross ratio of the four punctured sphere.

[5] The fact that this symmetry can be manifestly seen in the expression for the index is very reminiscent of the construction of the E_6 symmetry using multi-pronged strings in [64]. It is very interesting to understand whether these facts are related.

In [16] Gaiotto suggested to generalize this picture by considering general Riemann surfaces with an arbitrary numbers of punctures of different types (two types in case of the $SU(3)$ theories). The claim is that all theories with the same number and type of punctures and same topology of the Riemann surface are related by S-dualities. The immediate consequence of this claim for the superconformal index is that all such theories have to have the same index as it is independent of the values of the coupling, i.e. the moduli of the Riemann surface. This implies that the superconformal index is a topological invariant of the punctured Riemann surface. It was claimed in [1] that the superconformal index can be actually interpreted as a correlator in a two dimensional topological quantum field theory. The structure constants of this TQFT are given by the index of the three punctured sphere and the contraction of indices (i.e. metric) is gauging of the flavor symmetries. The associativity of the algebra generated by the structure constants is equivalent to the invariance of the index of four punctured spheres under pair-of-pants decomposition into two three punctured spheres. The structure constants and the metric were constructed and the associativity was explicitly verified for the $SU(2)$ case.

In this section we will make the same analysis for the $SU(3)$ case. We have two types of punctures, associated to $U(1)$ and $SU(3)$ flavor symmetries. There are thus different three point functions one can construct. The index of the theory on a sphere with three $SU(3)$ punctures, i.e. the index of the E_6 theory, is a structure constant which we will denote by $C^{(333)}_{\mathbf{x},\mathbf{y},\mathbf{z}}$ and it is just given by (3.24),

$$C^{(333)}_{\mathbf{x},\mathbf{y},\mathbf{z}} = C^{(E_6)}\left(\left(\sqrt{\frac{x_1}{x_2}}, \sqrt{x_1 x_2}\right), \mathbf{y}, \mathbf{z}\right) . \tag{3.44}$$

This vertex corresponds to the E_6 theory which has rank one, and thus we will refer to it as a rank $\mathbf{1}$ vertex. We will denote by $C^{(133)}_{\mathbf{x},\mathbf{y},a}$ the index of the sphere with two $SU(3)$ punctures and one $U(1)$ puncture. This is a free theory consisting of a hypermultiplet in fundamental of two $SU(3)$ flavor groups and its value is given by (3.9),

$$C^{(133)}_{a,\mathbf{x},\mathbf{y}} = \prod_{i,j=1}^{3} \Gamma\left(\frac{t^2}{\sqrt{v}}\left(a x_i y_j\right)^{\pm}\right) . \tag{3.45}$$

This vertex corresponds to a free, rank $\mathbf{0}$, theory and we will refer to it as rank zero structure constant. Later on we will define yet another three point function, formally associated to a sphere with two $U(1)$ punctures and one $SU(3)$ puncture. This vertex will have effective rank $-\mathbf{1}$. The metric of the model, $\eta^{\mathbf{x},\mathbf{y}}$, is defined as

$$\eta^{\mathbf{x},\mathbf{y}} = \frac{2}{3}\,\kappa^2\,\Gamma^2(t^2 v) \prod_{1\leqslant i<j\leqslant 3} \frac{\Gamma\left(t^2 v\left(\frac{x_i}{x_j}\right)^{\pm}\right)}{\Gamma\left(\left(\frac{x_i}{x_j}\right)^{\pm}\right)}\hat{\Delta}(\mathbf{x}^{-1},\mathbf{y})\,, \tag{3.46}$$

where $\hat{\Delta}(\mathbf{x}^{-1},\mathbf{y})$ is a δ-function kernel defined by

$$\oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i\,x_i}\,\hat{\Delta}(\mathbf{x},\mathbf{w})\,f(\mathbf{x}) = f(\mathbf{w})\,, \qquad \mathbf{w}\in\mathbb{T}^2\,. \tag{3.47}$$

The indices are contracted as follows

$$A^{\cdots\mathbf{u}\cdots}B_{\cdots\mathbf{u}\cdots} \equiv \oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{du_i}{2\pi i u_i}\,A^{\cdots\mathbf{u}\cdots}B_{\cdots\mathbf{u}\cdots}\Big|_{\prod_{i=1}^{3}u_i=1}\,. \tag{3.48}$$

Following these definitions the superconformal indices of all the $SU(3)$ generalized quivers are obtained by contracting the structure constants in different ways.

For the S-duality to hold, and subsequently for the structure constants to have a TQFT interpretation, the algebra generated by these objects has to be associative. We proceed to verify this fact.

$(333)-(333)$ associativity

Let us consider the generalized quiver with genus zero and four $SU(3)$ punctures. The index should be invariant under the permutation of the four $SU(3)$ characters,

$$\mathcal{I}_{3333}(\mathbf{x},\mathbf{y};\mathbf{w},\mathbf{z}) \;=\; C^{(333)}_{\mathbf{x},\mathbf{y},\mathbf{u}}\eta^{\mathbf{u},\mathbf{v}}C^{(333)}_{\mathbf{v},\mathbf{z},\mathbf{w}} = C^{(333)}_{\mathbf{x},\mathbf{z},\mathbf{u}}\eta^{\mathbf{u},\mathbf{v}}C^{(333)}_{\mathbf{v},\mathbf{y},\mathbf{w}}\,. \tag{3.49}$$

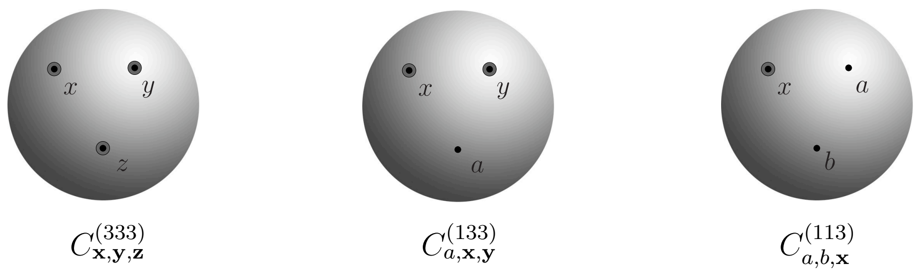


Figure 3.4: The three structure constants of the TQFT. The dots represent $U(1)$ punctures and the circled dots $SU(3)$ punctures.

At order $O(t^4)$ we find ,

$$\mathcal{I}_{3333} \sim t^4 \left[\frac{1}{v} (\chi_{\mathbf{8}}(\mathbf{x}) + \chi_{\mathbf{8}}(\mathbf{y}) + \chi_{\mathbf{8}}(\mathbf{z}) + \chi_{\mathbf{8}}(\mathbf{w})) + v^2 \right] , \tag{3.50}$$

and at order $O(t^6)$,

$$\mathcal{I}_{3333} \sim t^6 \left[-(\chi_{\mathbf{8}}(\mathbf{x}) + \chi_{\mathbf{8}}(\mathbf{y}) + \chi_{\mathbf{8}}(\mathbf{z}) + \chi_{\mathbf{8}}(\mathbf{w})) + 3v^3 \right] . \tag{3.51}$$

These axpressions are symmetric under the exchange $\mathbf{x} \leftrightarrow \mathbf{y} \leftrightarrow \mathbf{z} \leftrightarrow \mathbf{w}$. The associativity can be checked to hold to higher orders as well.

$(333) - (331)$ associativity

Let us consider the generalized quiver with genus zero, three $SU(3)$ punctures and one $U(1)$ puncture. The index should be invariant under permutations of the three $SU(3)$ characters

$$\mathcal{I}_{3331}(a, \mathbf{x}; \mathbf{y}, \mathbf{z}) = C^{(133)}_{a,\mathbf{x},\mathbf{u}} \eta^{\mathbf{uv}} C^{(333)}_{\mathbf{v},\mathbf{y},\mathbf{z}} = C^{(133)}_{a,\mathbf{y},\mathbf{u}} \eta^{\mathbf{uv}} C^{(333)}_{\mathbf{v},\mathbf{x},\mathbf{z}} . \tag{3.52}$$

We also expand the integrand in t around $t = 0$. The first non-trivial check is for the coefficient of $\mathcal{I}_{3331}$ at order $O(t^4)$,

$$\mathcal{I}_{3331} \sim t^4 \left[\frac{1}{v} (\chi_{\mathbf{8}}(\mathbf{x}) + \chi_{\mathbf{8}}(\mathbf{y}) + \chi_{\mathbf{8}}(\mathbf{z}) + 1) + v^2 \right] , \tag{3.53}$$

which is indeed symmetric under $\mathbf{x} \leftrightarrow \mathbf{y} \leftrightarrow \mathbf{z}$. At order $O(t^6)$,

$$\begin{aligned}\mathcal{I}_{3331} \;\sim\; & \frac{t^6}{v^{3/2}}\left(a^{-3}+a^{-1}\chi_{\overline{\mathbf{3}}}(\mathbf{x})\chi_{\overline{\mathbf{3}}}(\mathbf{y})\chi_{\overline{\mathbf{3}}}(\mathbf{z})+a\chi_{\mathbf{3}}(\mathbf{x})\chi_{\mathbf{3}}(\mathbf{y})\chi_{\mathbf{3}}(\mathbf{z})+a^3\right) \\ & -t^6\left(\chi_{\mathbf{8}}(\mathbf{x})+\chi_{\mathbf{8}}(\mathbf{y})+\chi_{\mathbf{8}}(\mathbf{z})+1\right)+2t^6v^3\,,\end{aligned} \tag{3.54}$$

which is also symmetric under $\mathbf{x} \leftrightarrow \mathbf{y} \leftrightarrow \mathbf{z}$. Again, we can perform systematic checks to arbitrary high order in t.

The (311) three point function and $(311)-(331)$ associativity

The index of the $N_f = 6$ $SU(3)$ theory in the strongly-coupled frame is given in terms of an integral over an $SU(2)$ character. Thus, we can not write it using the structure constants and the metric we defined in the beginning of this section. The strongly-coupled frame is obtained when two $U(1)$ punctures collide and thus in what follows we will formally define a structure constant with two $U(1)$ characters and an $SU(3)$ character such that when contracted with the E_6 structure constant using the metric above it will produce the index of the strongly-coupled frame.

Let us rewrite the index in the strongly-coupled frame,

$$\hat{\mathcal{I}}\,(s,r;\mathbf{y},\mathbf{z}) \;=\; \kappa\,\Gamma(t^2v)\oint_{\mathbb{T}}\frac{de}{2\pi i\,e}\frac{\Gamma(\frac{t^2}{\sqrt{v}}e^{\pm}\,s^{\pm})}{\Gamma(e^{\pm 2})}\,\Gamma(t^2v\,e^{\pm 2})\,C\left((e,\,r),\mathbf{y},\mathbf{z}\right) \tag{3.55}$$

as rank one (E_6) (333) and rank -1 (113) vertices contracted

$$\begin{aligned}&\hat{\mathcal{I}}\,(a,b;\mathbf{y},\mathbf{z}) = C^{(113)}_{a,b,\mathbf{x}}\,\eta^{\mathbf{x},\mathbf{x}'}\,C^{(333)}_{\mathbf{x}',\mathbf{y},\mathbf{z}} = \\ &\frac{2}{3}\,\kappa^2\,\Gamma(t^2v)^2\oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i\,x_i}\prod_{i\neq j}\frac{\Gamma(t^2v\,x_i/x_j)}{\Gamma(x_i/x_j)}\,C^{(113)}\left(a,b,\mathbf{x}^{-1}\right)\,C^{(333)}\left(\mathbf{x},\mathbf{y},\mathbf{z}\right)\end{aligned} \tag{3.56}$$

For this we define

$$C^{(113)}(a,b,\mathbf{x}^{-1}) \;=\; \frac{3}{2\kappa\Gamma(t^2v)}\oint_{\mathbb{T}}\frac{de}{2\pi i\,e}\frac{\Gamma(\frac{t^2}{\sqrt{v}}\,e^{\pm 1}\,s^{\pm 1})\,\Gamma(t^2v\,e^{\pm 2})}{\Gamma(e^{\pm 2})}\prod_{i\neq j}\frac{\Gamma(x_i/x_j)}{\Gamma(t^2v\,x_i/x_j)}\,\hat{\Delta}(\mathbf{x},\mathbf{w}) \tag{3.57}$$

Here, $\mathbf{w} = (e,\,r)$ with e an $SU(2)$ character and r a $U(1)$ character. The

$U(1)$ charges are related as in (3.22), $s=(a/b)^{3/2}$ and $r=(a\,b)^{-1/2}$. $\hat{\Delta}(\mathbf{x},\mathbf{w})$ is a δ-function kernel defined in (3.47). The (113) vertex has effective rank $-\mathbf{1}$. Using the above definition the TQFT algebra is well defined with all the contractions being $SU(3)$ integrals.

The associativity of (311) vertex contracted with a (333) vertex is achieved by construction: remember that we obtained the index of E_6 SCFT by requiring this property. Let us check the associativity of (331) contracted with (113)

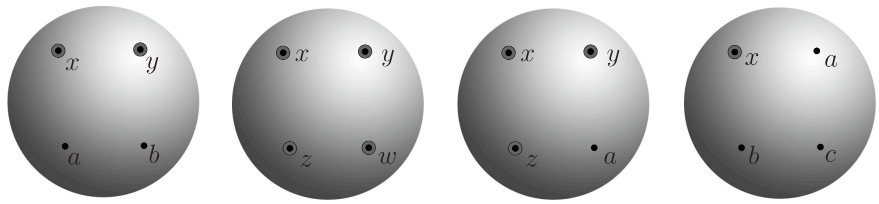
$$\begin{aligned}
&\mathcal{I}(a,b;\,c,\mathbf{y}) = C^{(113)}_{a,b,\mathbf{x}}\,\eta^{\mathbf{x},\mathbf{x}'}\,C^{(331)}_{\mathbf{x}',\mathbf{y},c} = \qquad (3.58)\\
&\frac{2}{3}\,\kappa^2\,\Gamma(t^2v)^2\,\oint\prod_{i=1}^{2}\frac{dx_i}{2\pi i\,x_i}\prod_{i\neq j}\frac{\Gamma(t^2v\,x_i/x_j)}{\Gamma(x_i/x_j)}\,C^{(113)}\left(a,b,\mathbf{x}^{-1}\right)\,\prod_{i,j}\Gamma(\frac{t^2}{\sqrt{v}}(c\,x_i\,y_j)^{\pm1})\\
&=\prod_{i=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{c\,y_i}{r^2}\right)^{\pm1}\right)\times\\
&\kappa\Gamma(t^2v)\oint\frac{de}{2\pi i\,e}\frac{\Gamma(t^2v\,e^{\pm2})}{\Gamma(e^{\pm2})}\Gamma\left(\frac{t^2}{\sqrt{v}}\,s^{\pm1}\,e^{\pm1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\,(c\,r\,y_i)^{\pm1}\,e^{\pm1}\right)
\end{aligned}$$

This is exactly the index of $SU(2)$ $N_f=4$ (the fourth line in (3.58)) with a decoupled hypermultiplet in the fundamental of an $SU(3)$ flavor (the third line in (3.58)). Remembering (3.22) and the results of [1, 49] it is easy to show that there is a permutation symmetry between the three $U(1)$ punctures a, b and c,

$$a\quad\leftrightarrow\quad b\quad\leftrightarrow\quad c\,. \qquad (3.59)$$

Using the definition (3.57) the index of a sphere with four $U(1)$ punctures is singular. However, we do not have a physical interpretation of this surface and it does not appear in any decoupling limit of a physical theory. Thus, making sense of this surface is not essential.

We have shown that the structure constants define an associative algebra and thus define a TQFT. In particular the superconformal index of theories with equal genus and equal number/type of punctures is the same in agreement with S-duality.



(a) (b) (c) (d)

Figure 3.5: The relevant four-punctured spheres for A_2 theories. The three different degeneration limits of a four-punctured sphere correspond to different S-duality frames. For example, in (a) two of the degeneration limits (when a $U(1)$ puncture collides with an $SU(3)$ puncture) correspond to the weakly-coupled $N_f = 6$ $SU(3)$ theory, the third limit (when two like punctures collide) corresponds to the Argyres-Seiberg theory. In (d) the degeneration limits correspond to the different duality frames of $SU(2)$ SYM with $N_f = 4$ theory plus a decoupled hypermultiplet.

3.3 Discussion

In this chapter we have obtained an explicit expression for the superconformal index of the strongly-coupled SCFT with an E_6 flavor symmetry [60]. The strategy is to use the Argyres-Seiberg duality, which relates a weakly-coupled theory, index of which can be easily obtained through the Lagrangian description of the theory, and E_6 SCFT with part of the global symmetry gauged. The index of the two theories should be the same. Thus, one obtains the index of the E_6 theory by "*inverting*" the gauging, see (3.24). Upon gauging a flavor symmetry one looses information about the theory by projecting on gauge invariant states. However, what allows us to "*invert*" the gauging in our case is the fact that additional matter is coupled to the $SU(2)$ gauge group along with the E_6 SCFT, and thus effectively preserves enough information to reconstruct the complete index of E_6 SCFT. We do not have a physical interpretation of the expression for the index (3.24) and it would be very interesting to find such an interpretation.

In principle one can try to use the same techniques to obtain the superconformal index for other strongly-coupled SCFTs of [16]. However, the generalization is not completely straightforward. Let us discuss the case of the

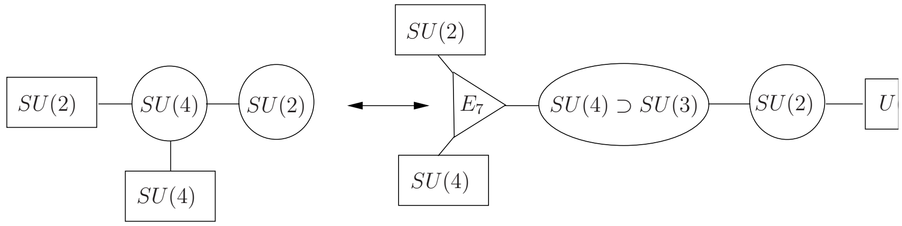


Figure 3.6: An Argyres-Seiberg duality relating a Lagrangian theory (left quiver) with a theory containing a strongly-coupled E_7 piece (right quiver).

E_7 theory [42, 58, 65] as an example. To obtain the E_7 SCFT we can apply Argyres-Seiberg duality to a Lagrangian theory with $SU(4) \otimes SU(2)$ gauge group, with a single hypermultiplet in the bi-fundamental representation and six hypermultiplets in the fundamental representation of $SU(4)$. The Argyres-Seiberg dual of this theory involves an E_7 strongly-coupled piece, with an $SU(3)$ subgroup of E_7 gauged. The theory has a second gauge group factor $SU(2)$ and two hypermultiplets: one in the fundamental of $SU(2)$ and the in bi-fundamental of the two gauge groups. See figure 3.6. The index of the weakly-coupled theory can be easily written down,

$$
\begin{aligned}
\mathcal{I}_{weak} \;=\;& \kappa\,\Gamma(t^2 v) \oint_{\mathbb{T}} \frac{de}{2\pi i\, e} \frac{\Gamma(t^2 v e^{\pm 2})}{\Gamma(e^{\pm 2})} \;\times \qquad (3.60)\\
& \frac{1}{3}\kappa^3\,\Gamma(t^2 v)^3 \oint_{\mathbb{T}^3} \prod_{i=1}^{3} \frac{du_i}{2\pi i\, u_i} \prod_{i\neq j} \frac{\Gamma(t^2 v \frac{u_i}{u_j})}{\Gamma(\frac{u_i}{u_j})} \Gamma(\frac{t^2}{\sqrt{v}}(e^{\pm 1}\, u_i\, a)^{\pm 1}) \;\times\\
& \prod_{i=1}^{4}\prod_{j=1}^{4} \Gamma(\frac{t^2}{\sqrt{v}}(y_j\, u_i\, b)^{\pm 1}) \prod_{i=1}^{4}\prod_{j=1}^{2} \Gamma(\frac{t^2}{\sqrt{v}}(z_j\, u_i\, c)^{\pm 1})\,.
\end{aligned}
$$

The index of the dual theory is given by

$$
\begin{aligned}
\mathcal{I}_{strong} \;=\;& \kappa\,\Gamma(t^2 v) \oint_{\mathbb{T}} \frac{de}{2\pi i\, e} \frac{\Gamma(t^2 v e^{\pm 2})}{\Gamma(e^{\pm 2})} \Gamma(\frac{t^2}{\sqrt{v}} e^{\pm 1}\, s^{\pm 1}) \;\times \qquad (3.61)\\
& \frac{2}{3}\kappa^2\,\Gamma(t^2 v)^2 \oint_{\mathbb{T}^2} \prod_{i=1}^{2} \frac{du_i}{2\pi i\, u_i} \prod_{i\neq j} \frac{\Gamma(t^2 v \frac{u_i}{u_j})}{\Gamma(\frac{u_i}{u_j})} \prod_{i=1}^{3} \Gamma(\frac{t^2}{\sqrt{v}}(e^{\pm 1}\, u_i\, m)^{\pm 1}) \;\times\\
& C^{(E_7)}\left((u_i,\, r)_{SU(4)}, \mathbf{y}_{SU(4)}, \mathbf{z}_{SU(2)}\right)\,.
\end{aligned}
$$

One can invert the $SU(2)$ integral by the same techniques we used for the E_6 index, but there is no simple inversion formula known to us for the $SU(3)$ integral. To obtain a closed form for the index of the strongly-coupled CFTs appearing in higher rank theories one has to learn how to "invert the superconformal tails". In the next chapter we bypass this problem and conjecture an independent microscopic description of this TQFT and generalize it to the case of A_n. Finally, from a pure mathematics viewpoint, we have seen that S-duality implies a number of identities that must be obeyed by integrals of elliptic Gamma functions and that we have checked perturbatively. We collect the exact identities in appendix F. It would be nice to find analytic proofs.

Chapter 4

The $4d$ Superconformal Index from q-deformed $2d$ Yang-Mills

In this chapter we describe a new powerful duality, relating physics in four and in two dimensions. We will argue that for a large class of *four-dimensional superconformal* gauge theories, non-trivial information about the operator spectrum is captured by correlators of a *two-dimensional non-supersymmetric* gauge theory. The $4d$ side of the duality is generically strongly-coupled, and difficult to analyze directly; on the other hand calculations on the $2d$ side will be explicit and algorithmic. Thus our conjecture gives new information about strongly-coupled $4d$ field theories.

Our proposal is in the same spirit as the Alday-Gaiotto-Tachikawa (AGT) relation between the partition function of a $4d\,\mathcal{N}=2$ gauge theory on S^4 and a correlator in $2d$ Liouville/Toda theory [26, 27]. In our case, the $4d$ observable is a (twisted) supersymmetric partition function of an $\mathcal{N}=2$ superconformal field theory on $S^3 \times S^1$, also known as the superconformal index. We will focus on a "reduced" index that depends on a single fugacity q. On the $2d$ side, instead of Liouville/Toda we have the *zero-area limit* of q-deformed Yang-Mills theory. The topological nature of this $2d$ theory dovetails with the independence of the $4d$ index on the gauge theory moduli.

We begin by reviewing the $4d$ side of the duality. The full $\mathcal{N}=2$ superconformal index is defined as [19, 66]

$$\mathcal{I} = \mathrm{Tr}(-1)^F p^{\frac{E-R}{2}+j_1} q^{\frac{E-R}{2}-j_1} u^{-(r+R)}\,, \tag{4.1}$$

where the trace is over the states of the theory on S^3 (in the usual radial quantization) and F the fermion number. The symbol E stands for the conformal dimension, (j_1, j_2) for the Cartan generators of the $SU(2)_1 \otimes SU(2)_2$ isometry group, and $(R\,, r)$ for the Cartan generators of the $SU(2)_R \otimes U(1)_r$ R-symmetry. The fugacities p, q, and u keep track of the maximal set of quantum numbers commuting with a single real supercharge, $\mathcal{Q} \equiv \tilde{\mathcal{Q}}_{1\dot{-}}$, which with no loss of generality has been chosen to have $R = \frac{1}{2}$, $r = -\frac{1}{2}$, $j_1 = 0$, $j_2 = -\frac{1}{2}$ and (of course) $E = \frac{1}{2}$. Only states that obey $2\{\mathcal{Q}, \mathcal{Q}^\dagger\} = E - 2j_2 - 2R + r = 0$ contribute to the index. Note that the variables p, q, and u are related to t, y, v of previous chapters as $p = t^3 y$, $q = \frac{t^3}{y}$ and $u = \frac{v}{t}$.

For a theory with a weakly-coupled Lagrangian description the index is computed explicitly by a matrix integral,

$$
\begin{aligned}
\mathcal{I}(p, q, u; V) = & \int [dU] \\
& \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_j f^{(j)}(p^n, q^n, u^n)\, \chi_{\mathcal{R}_j}(U^n,\, V^n) \right) .
\end{aligned}
\tag{4.2}
$$

Here U denotes an element of the gauge group, with $[dU]$ the invariant Haar measure, and V an element of the flavor group. The sum is over the different $\mathcal{N} = 2$ supermultiplets appearing in the Lagrangian, with $\mathcal{R}_j$ the representation of the j-th multiplet under the flavor and gauge groups and $\chi_{\mathcal{R}_j}$ the corresponding character. The functions $f^{(j)}$ are the "single-letter" partition functions, $f^{(j)} = f^{vect}$ or $f^{(j)} = f^{chi}$ according to whether the j-th multiplet is an $\mathcal{N} = 2$ vector or $\mathcal{N} = 2$ $\frac{1}{2}$-hypermultiplet. They are easily evaluated [19, 66]:

$$
f^{vect}(p, q, u) = \frac{(u - \frac{1}{u})\sqrt{pq} - (p + q) + 2pq}{(1 - p)(1 - q)}\,, \tag{4.3}
$$

$$
f^{chi}(p, q, u) = \frac{(pq)^{\frac{1}{4}} \frac{1}{\sqrt{u}} - (pq)^{\frac{3}{4}} \sqrt{u}}{(1 - p)(1 - q)}\,. \tag{4.4}
$$

We will focus on a *reduced* index, by setting

$$
u = 1, \qquad p = q\,, \tag{4.5}
$$

which leads to the significant simplification

$$f^{vect} = \frac{-2q}{1-q}, \qquad\qquad f^{chi} = \frac{q^{\frac{1}{2}}}{1-q}\,. \tag{4.6}$$

We consider a class of $\mathcal{N}=2$ $4d$ superconformal theories (SCFTs) constructed from a set of elementary building blocks [16]. The building blocks are isolated SCFTs with flavor symmetry $G_1 \otimes G_2 \otimes G_3$, $G_i \subseteq SU(N)$ for given N. In the simplest case of $N=2$, the only building block is the free $\frac{1}{2}$-hypermultiplet in the tri-fundamental representation of the $SU(2)^3$ flavor group. For $N>2$ most of the building blocks are intrinsically strongly-interacting theories with no Lagrangian description. One can "glue together" two building blocks by gauging a common $SU(N)$ flavor symmetry. Iterating this procedure one constructs a large class of $\mathcal{N}=2$ gauge theories, the $SU(N)$ "generalized quivers" [16]. There is a geometric interpretation of this construction, where one regards the building blocks as three-punctured spheres, with the punctures associated to the flavor symmetries; the gluing operation is performed by connecting the punctures with cylinders. The complex structure moduli of the resulting punctured Riemann surface correspond to the complexified gauge couplings. The same punctured Riemann surface can often be obtained by following several different gluing paths (different pairs-of-pants decompositions). The generalized quiver theories associated to different decompositions of the same surface are related by S-dualities [16].

The index of a generalized quiver can be written in terms of the index of its constituents. We parametrize the index of an elementary building block (3-punctured sphere) by "structure constants" $\mathcal{I}_N(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3})$ where $\mathbf{x}_i$ are fugacities dual to the Cartan subgroup of G_i: except in special cases these are *a priori* unknown functions. On the other hand we can easily write the index $\eta_N(\mathbf{x})$ of the $SU(N)$ vector multiplets used in the gluing (propagators),

$$\eta_N(\mathbf{x}) = \exp\left[-2\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1-q^n}\chi_{adj}(\mathbf{x}^n)\right]\,.$$

For example, gluing two 3-punctured spheres with one cylinder one obtains the following index

$$\int [dU(\mathbf{x})]\, \mathcal{I}_N(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x})\, \eta_N(\mathbf{x})\, \mathcal{I}_N(\mathbf{x}, \mathbf{x}_3, \mathbf{x}_4)\,. \tag{4.7}$$

By defining a metric

$$\eta_N(\mathbf{x}_1, \mathbf{x}_2) \equiv \eta_N(\mathbf{x}_1) \sum_{\mathcal{R}} \chi_{\mathcal{R}}(\mathbf{x}_1)\, \chi_{\mathcal{R}}(\mathbf{x}_2)\,, \tag{4.8}$$

with $\mathcal{R}$ running over irreducible and finite representations of $SU(N)$, we can re-write (4.7) as

$$\mathcal{I}_N(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x}) \cdot\, \eta_N(\mathbf{x}, \mathbf{x}') \cdot \mathcal{I}_N(\mathbf{x}', \mathbf{x_3}, \mathbf{x_4})\,, \tag{4.9}$$

where $\cdot$ multiplication means integration over the Haar measure. S-duality then implies that the metric and structure constants form an associative algebra and thus a $2d$ topological field theory (TQFT) [1]. (Strictly speaking, the state-space at each puncture, which is spanned by G_i representations, is infinite-dimensional, so one must slightly relax the standard mathematical axioms for a TQFT.) Associativity was directly verified for the $SU(2)$ and $SU(3)$ generalized quiver theories in [1, 3], for generic values of the fugacities p, q and u. In the following we will identify the $2d$ topological theory implicitly defined by the reduced index with an explicit model: q-deformed Yang-Mills (qYM) in the zero-area limit.

4.1 $SU(2)$ generalized quivers

Let us start with the simplest case, the $SU(2)$ quivers. Here the building blocks are free tri-fundamental $\frac{1}{2}$-hypermultiplets,

$$\mathcal{I}_{222}(a_1, a_2, a_3) = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^{\frac{1}{2}n}}{1-q^n} \chi_\square(a_1^n)\chi_\square(a_2^n)\chi_\square(a_3^n)\right].$$

Remarkably, one can prove (*e.g.* by comparing analytic properties) that $\mathcal{I}_{222}(a_1, a_2, a_3)$ admits the equivalent representation

$$\mathcal{I}_{222}(a_1, a_2, a_3) = \frac{(q;q)_\infty}{1-q} \prod_{i=1}^{3} \eta_2^{-\frac{1}{2}}(a_i) \sum_{\mathcal{R}} \frac{\chi_{\mathcal{R}}(a_1)\, \chi_{\mathcal{R}}(a_2)\, \chi_{\mathcal{R}}(a_3)}{[|\mathcal{R}|]_q} \,. \tag{4.10}$$

Here $(q;q)_\infty \equiv \prod_{i=1}^{\infty}(1-q^i)$. The sum is over irreducible $SU(2)$ representations $\mathcal{R}$, with $|\mathcal{R}|$ denoting the dimension of the representation. The $SU(2)$ characters are

$$\chi_{\mathcal{R}}(a) = \frac{a^{|\mathcal{R}|} - a^{-|\mathcal{R}|}}{a - a^{-1}} \,. \tag{4.11}$$

Finally the symbol $[x]_q$ denotes the *q-deformed* number,

$$[x]_q \equiv \frac{q^{-\frac{x}{2}} - q^{\frac{x}{2}}}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}} \,. \tag{4.12}$$

The structure constants contain the factors $\prod_i \eta_2^{-1/2}(a_i)$, which cancel with the metric $\eta_2(a_i)$ when two punctures are glued. It is then natural to define rescaled structure constants and metric,

$$\begin{aligned} \hat{\mathcal{I}}_{222}(a_1, a_2, a_3) &= \mathcal{N}_{222}(q) \sum_{\mathcal{R}} \frac{\chi_{\mathcal{R}}(a_1)\, \chi_{\mathcal{R}}(a_2)\, \chi_{\mathcal{R}}(a_3)}{[|\mathcal{R}|]_q} \,, \\ \hat{\eta}_2(a,\, b) &= \sum_{\mathcal{R}} \chi_{\mathcal{R}}(a)\, \chi_{\mathcal{R}}(b) \,, \end{aligned} \tag{4.13}$$

where $\mathcal{N}_{222}(q) = (q;q)_\infty/(1-q)$. Up to the overall normalization $\mathcal{N}_{222}$, these are precisely the structure constants and metric of $2d$ qYM in the zero area limit [67–69]!

Note that $[n]_q = \chi_n(q^{1/2})$. This implies that by setting one of the $SU(2)$ fugacities to $q^{1/2}$ we "close" a puncture,

$$\hat{\mathcal{I}}_{222}(a, b, q^{1/2}) = \mathcal{N}_{222}(q) \;\; \hat{\eta}_2(a,\, b) \,.$$

Applying this procedure again, we close another puncture and obtain the one-punctured sphere (the cap). For higher-rank groups we will encounter a similar procedure: setting some combination of the flavor fugacities to $q^{1/2}$ one obtains punctures with reduced flavor symmetry.

4.2 $SU(3)$ generalized quivers

Next let us consider the $SU(3)$ generalized quivers. Here two new generic features appear. First, the basic building block is an interacting theory with no Lagrangian description, the E_6 SCFT [16, 70]. Second, there is more than one type of puncture: in addition to the *maximal* $SU(3)$ flavor puncture there is a puncture with reduced flavor symmetry, $U(1)$ [16].

The representations of $SU(N)$ are parametrized by N integers $\lambda_1 \geq \lambda_2 ... \geq \lambda_{N-1} \geq \lambda_N = 0$, the row lengths of the corresponding Young diagram. The q-deformed dimension of the representation is

$$\dim_q \mathcal{R}_{\underline{\lambda}} = \prod_{i<j} \frac{[\lambda_i - \lambda_j + j - i]_q}{[j-i]_q}\,, \tag{4.14}$$

and the characters are given by Schur polynomials

$$\chi_{\underline{\lambda}}(\mathbf{x}) = \frac{\det\left({x_i}^{\lambda_j + k - j}\right)}{\det\left({x_i}^{k-j}\right)}\,. \tag{4.15}$$

Specializing to $SU(3)$ we can parametrize all the Young diagrams by (λ_1, λ_2). We observe again that the q-dimension of a representation is equal to the group character with a particular choice of fugacities,

$$\chi_{\lambda_1,\lambda_2}(q, 1, q^{-1}) = \dim_q \mathcal{R}_{\lambda_1,\lambda_2}\,. \tag{4.16}$$

4.2.1 Three Maximal Punctures

The sphere with three maximal punctures corresponds to the strongly coupled E_6 SCFT (the $SU(3)^3$ flavor symmetry is accidentally enhanced to E_6.) This theory has no Lagrangian description and thus we do not have a direct way to compute its index. However, this index was computed [3] indirectly by employing Argyres-Seiberg duality [70]. Inspired by the $SU(2)$ case, we conjecture that the index $\mathcal{I}_{E_6}(\{\mathbf{x}_i\}_{i=1}^3)$ of the E_6 SCFT is proportional to the structure constants $C_{SU(3)_q}$ of q-deformed $SU(3)$ Yang-Mills,

$$\mathcal{I}_{E_6}(\mathbf{x_i}) = \left[\prod_{i=1}^{3} \eta^{-\frac{1}{2}}(\mathbf{x}_i)\right] \mathcal{N}_{333}(q)\, C_{SU(3)_q}(\mathbf{x_i})\,,$$

where

$$C_{SU(3)_q}(\mathbf{x_i}) = \sum_{0\leq\lambda_2\leq\lambda_1}^{\infty} \frac{\chi_{\lambda_1,\lambda_2}(\mathbf{x}_1)\chi_{\lambda_1,\lambda_2}(\mathbf{x}_2)\chi_{\lambda_1,\lambda_2}(\mathbf{x}_3)}{\dim_q \mathcal{R}_{\lambda_1,\lambda_2}},$$

and $\mathcal{N}_{333}(q)$ a normalization factor. Using *Mathematica*, we have checked this proposal against the results of [3] to several orders in q, and in the process determined the normalization to be

$$\mathcal{N}_{333}(q) = \frac{(q;q)_\infty^2}{(1-q)^2(1-q^2)}\,. \tag{4.17}$$

4.2.2 Two Maximal and One $U(1)$ Puncture

Another building block is given by a sphere with two $SU(3)$ punctures and one $U(1)$ puncture. This corresponds to a free hypermultiplet in the bi-fundamental of $SU(3)^2$ and charged under the $U(1)$. The index of this theory is explicitly given by

$$\mathcal{I}_{331}(\mathbf{x_1},\mathbf{x_2};\ a) = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n}\frac{q^{\frac{1}{2}n}}{1-q^n}\chi_{hyp}(\mathbf{x_1}^n,\mathbf{x_2}^n;\ a^n)\right],$$

where the flavor character is

$$\chi_{hyp}(\mathbf{x_1},\mathbf{x_2};\ a) = \sum_{i,j}(x_1^i x_2^j a + \frac{1}{x_1^i x_2^j a})\,. \tag{4.18}$$

One can verify by series expansion in q that

$$\begin{aligned}\mathcal{I}_{331}(\mathbf{x_1},\mathbf{x_2};\ a) = C_{SU(3)_q}(\mathbf{x_1},\mathbf{x_2};\ a) \times \\ \frac{\prod_{i=1}^{2}\eta^{-\frac{1}{2}}(\mathbf{x}_i)}{\prod_{\ell=1}^{2}(1-q^\ell)}\ \exp\left[\sum_{n=1}^{\infty}\frac{q^{\frac{3}{2}n}}{1-q^n}\frac{a^{3n}+a^{-3n}}{n}\right],\end{aligned} \tag{4.19}$$

with

$$C_{SU(3)_q}(\mathbf{x_1}, \mathbf{x_2};\ a) = \sum_{0\le\lambda_2\le\lambda_1}^{\infty} \frac{\chi_{\lambda_1,\lambda_2}(\mathbf{x}_1)\,\chi_{\lambda_1,\lambda_2}(\mathbf{x}_2)\,\chi_{\lambda_1,\lambda_2}(a\,q^{1/2}, aq^{-1/2}, a^{-2})}{\dim_q \mathcal{R}_{\lambda_1,\lambda_2}}\,. \tag{4.20}$$

Note that this result can be recovered by starting from the structure constant with maximal punctures and "partially closing" one of the punctures by embedding $SU(2)$ fugacities $(q^{\frac{1}{2}}, q^{-\frac{1}{2}})$ into fugacities of $SU(3)$.

4.3 General statement

The generic building block of a higher-rank quiver is an interacting SCFT with no Lagrangian description. Unlike the case of $SU(2)$ and $SU(3)$ quivers it is very hard to calculate the index of these theories, either directly or indirectly. However, we can naturally extrapolate the relation to $2d$ qYM to higher-rank groups.

We conjecture that the reduced index of the theory corresponding to sphere with three maximal punctures (the T_N theory of [16]) is

$$\mathcal{I}_{T_N}(\mathbf{x_i}) = \frac{(q;q)_\infty^{N-1}\prod_{i=1}^{3}\eta^{-\frac{1}{2}}(\mathbf{x}_i)}{\prod_{\ell=1}^{N-1}(1-q^\ell)^{N-\ell}}\; C_{SU(N)_q}(\mathbf{x_i})$$

where

$$C_{SU(N)_q}(\mathbf{x_i}) = \sum_{\mathcal{R}} \frac{1}{\dim_q \mathcal{R}}\,\chi_{\mathcal{R}}(\mathbf{x}_1)\,\chi_{\mathcal{R}}(\mathbf{x}_2)\,\chi_{\mathcal{R}}(\mathbf{x}_3)$$

are the structure constant of $SU(N)$ qYM. The sum is over irreducible $SU(N)$ representations and $\{\mathbf{x}_i\}$ are the fugacities dual to the Cartan subgroup.

This conjecture can be tested against the numerous S-dualities of the generalized quivers [16]. For instance, a linear superconformal quiver theory with two $SU(4)$ nodes admits a dual description in terms of T_4 coupled to $SU(3)$ gauge theory which in turn is coupled to an $SU(2)$ gauge theory with a single hypermultiplet. We have checked, in the q expansion, that the indices on both sides of the duality indeed match if one uses our conjecture for the T_4 index.

Another test is to compare with physical expectations for the spectrum of protected operators. A class of protected operators in the T_N theories are the Higgs branch operators [39, 45]. These come in two families: $E = 2$, $R = 1$ in flavor representation $(adj, 1, 1) \oplus (1, adj, 1) \oplus (1, 1, adj)$ and $E = N-1, R = \frac{N-1}{2}$ in representation $(N, N, N) \oplus (\bar{N}, \bar{N}, \bar{N})$. It is straightforward to see that these operators appear in our conjecture for the index: the first family comes from the $\eta(\mathbf{x})^{-\frac{1}{2}}$ factors, and the second from the $\chi_{\square}(\mathbf{x}_1)\chi_{\square}(\mathbf{x}_2)\chi_{\square}(\mathbf{x}_3)$ and $\chi_{\overline{\square}}(\mathbf{x}_1)\chi_{\overline{\square}}(\mathbf{x}_2)\chi_{\overline{\square}}(\mathbf{x}_3)$ terms in $C_{SU(N)_q}$.

We can generalize the conjecture to the structure constants with two maximal punctures and one $U(1)$ puncture,

$$\mathcal{I}_{NN1}(\mathbf{x_1}, \mathbf{x_2}, a) = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^{\frac{1}{2}n}}{1-q^n} \chi_{hyp}(\mathbf{x_1}^n, \mathbf{x_2}^n;\; a^n)\right] =$$
$$\frac{C_{SU(N)_q}(\mathbf{x_1}, \mathbf{x_2};\; a)}{\prod_{i=1}^{2} \eta^{\frac{1}{2}}(\mathbf{x}_i) \prod_{\ell=1}^{N-1}(1-q^\ell)} \exp\left[\sum_{n=1}^{\infty} \frac{q^{\frac{N}{2}n}}{1-q^n} \frac{a^{Nn} + a^{-Nn}}{n}\right],$$

where structure constants $C_{SU(N)_q}(\mathbf{x_1}, \mathbf{x_2}; a)$ are

$$C_{SU(N)_q}(\mathbf{x_1}, \mathbf{x_2}; a) = \sum_{\mathcal{R}} \frac{1}{\dim_q \mathcal{R}} \chi_{\mathcal{R}}(\mathbf{x}_1)\, \chi_{\mathcal{R}}(\mathbf{x}_2)\, \chi_{\mathcal{R}}(a q^{\frac{N-2}{2}}, .., a q^{-\frac{N-2}{2}}, a^{1-N})\,. \tag{4.21}$$

Again we have verified this conjecture in the q-expansion. Generic punctures

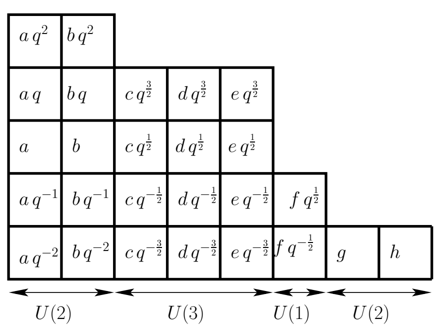


Figure 4.1: An example of the rule to associate flavor fugacities for a non-maximal puncture. Illustrated here is a puncture for $N = 26$ with flavor symmetry $S(U(3)U(2)^2U(1))$. The $S(\dots)$ constraint imposes $(ab)^5(cde)^4 f^2 gh = 1$.

are classified [16] by the embeddings $SU(2) \subset SU(N)$, which are specified by the decomposition of the fundamental of $SU(N)$ into $SU(2)$ representation. (In the terminology of [71], we focus on regular punctures). This information can be encoded into a Young diagram with N boxes, where the height of each column denotes the dimension of an $SU(2)$ representation. The commutant of this embedding is the flavor symmetry associated to the puncture. The maximal puncture corresponds to a single-row diagram, the closed puncture (*i.e.* no puncture) corresponds to a single-column diagram, and the $U(1)$ puncture to a two-column diagram with $N-1$ boxes in the first column and a single box in the second column. The Young diagram in Fig. 4.1 exemplifies a non-maximal puncture for $N = 26$ with $S(U(3)U(2)^2U(1))$ flavor symmetry. We are lead to the following conjecture for the index of a theory with three generic punctures corresponding to Young diagarms λ_i

$$\mathcal{I}(\Lambda_1, \Lambda_2, \Lambda_3) = \mathcal{N}_{\lambda_1,\lambda_2,\lambda_3}(q) \prod_{i=1}^{3} \mathcal{A}_{\lambda_i}(\Lambda_i) \times \\ \sum_{\mathcal{R}} \frac{1}{\dim_q \mathcal{R}} \chi_{\mathcal{R}}(\Lambda_1)\, \chi_{\mathcal{R}}(\Lambda_2)\, \chi_{\mathcal{R}}(\Lambda_3)\,,$$

with Λ_i labeling an association of flavor fugacities according to the Young diagram λ_i. The rule to associate the flavor fugacities to the $SU(N)$ fugacities is illustrated in Fig. 4.1. For all maximal punctures we have given the normalization factors ($\mathcal{N}$ and $\mathcal{A}$) above, while for generic punctures these factors can be in principle obtained by employing different S-dualities of the quivers [16]. As an example, consider the E_7 SCFT which is given by a sphere with two maximal punctures of $SU(4)$ and one square Young diagram with four boxes. Following the above procedure and fixing the normalization from the relevant Argyres-Seiberg duality [70], we are led to propose

$$\mathcal{I}_{E_7}(\mathbf{x}, \mathbf{y}; a) = \frac{\exp\left[\sum_{n=1}^{\infty} \frac{q^n(1+q^n)}{1-q^n} \frac{a^{2n}+a^{-2n}}{n}\right]}{\eta^{\frac{1}{2}}(\mathbf{x})\eta^{\frac{1}{2}}(\mathbf{y})(1-q)(1-q^2)^2(1-q^3)} \times \\ \sum_{\mathcal{R}} \frac{\chi_{\mathcal{R}}(\mathbf{x})\, \chi_{\mathcal{R}}(\mathbf{y})\, \chi_{\mathcal{R}}(q^{\frac{1}{2}}a, q^{-\frac{1}{2}}a, q^{\frac{1}{2}}/a, q^{-\frac{1}{2}}/a)}{\dim_q \mathcal{R}}\,,$$

Here $\mathbf{x}$, $\mathbf{y}$ label the two sets of $SU(4)$ fugacities and a the $SU(2)$ fugacity. The summation, as usual, is over finite irreducible representations of $SU(4)$. We have verified perturbatively in q that this expression is indeed E_7 covariant – a tight check of our logic.

4.4 Discussion

We have given compelling evidence that the reduced superconformal index of an $\mathcal{N}=2$ generalized $SU(N)$ quiver theory is exactly computed by a correlator in $2d$ $SU(N)_q$ Yang-Mills. This duality is new tool to investigate interacting field theories without a Lagrangian description. For example, it should be useful to study the constraints obeyed by the Higgs branch operators, generalizing to $N > 3$ the analysis of [62]. Two-dimensional qYM first appeared in a physical setting in the context of counting BPS states [67], and it would be interesting to find a relation with our work. An obvious question is whether our results can be generalized to the full index, with all fugacities turned on. It is already remarkable that the *known* structure constants of the $SU(2)$ quivers implicitly define a (q, p, u) deformation of $SU(2)$ Yang-Mills. Work is in progress in investigating the nature of this deformation, in order to extrapolate it to $N > 2$. The q and p fugacities appear on a symmetric footing, in a way which is strongly suggestive of an elliptic, or "dynamical", deformation of the quantum group structure $SU(N)_q$ that we have uncovered for $p = q$, $u = 1$. Indeed the full index is most elegantly expressed [50] in terms of elliptic Gamma functions [48]. Finally, a more conceptual understanding of the duality would be very desirable. As for the AGT correspondence [26], the existence, but not the details, of a $4d/2d$ relation can be traced to the definition of the $4d$ SCFT as the infrared limit of the $6d$ (2,0) theory on a Riemann surface. Whether this intuition can be turned into a microscopic derivation remains to be seen.

In the next chapter we focus our attention on an exactly computable observable in $3d$ gauge theories, their partition function on S^3. The index of the $4d$ gauge theories can be thought of as a twisted supersymmetric partition function on $S^3 \times S^1$. As the radius of the circle goes to zero, the partition function reduces to the path integral on S^3 of the dimensionally reduced $3d$ gauge theory. We study this reduction in the next chapter.

Chapter 5

Reducing the 4d Index to the S^3 Partition Function

String/M theory has led to a rich web of non-perturbative dualities between supersymmetric field theories. Checking/exploiting/extending these dualities requires exact computations in field theories. In recent years, using methods based on localization, several exact quantities in supersymmetric gauge theories have been computed. Two of such quantities, the superconformal index of $4d$ gauge theories [19, 66] and the partition function of supersymmetric gauge theories on S^3 [20, 21], are the main focus of this note.

The superconformal index of $\mathcal{N} = 1$ IR fixed points was computed in [50, 57, 63], there it served as a check of Seiberg duality. The indices of $\mathcal{N} = 4$ SYM and type IIB supergravity in AdS_5 were computed and matched in [19]. The superconformal index of $\mathcal{N} = 2$ supersymmetric gauge theories was used to check $\mathcal{N} = 2$ S-dualities conjectured by Gaiotto and to define a $2d$ topological field theory in the process [1, 3]. Recently the partition function of supersymmetric gauge theories on S^3 has been used to check a variety of $3d$ dualities including mirror symmetry [21] and Seiberg-like dualities [72]. Remarkably, the exact partition function has also allowed for a direct field theory computation of $N^{3/2}$ degrees of freedom of ABJM theory [73, 74]. The S^3 partition function of $\mathcal{N} = 2$ theories is extremized by the exact superconformal R-symmetry [75–77] so just like the a-maximization in $4d$, the $3d$ partition function can be used to determine the exact R-charges at interacting fixed points. The purpose of this note is to relate these two interesting and useful exactly calculable quantities in 3 and 4 dimensions.

The superconformal index of a $4d$ gauge theory can be computed as a path integral on $S^3 \times S^1$ with supersymmetric boundary conditions along S^1. All the modes on the S^1 contribute to this path integral. In a limit with the radius of the circle shrinking to zero the higher modes become very heavy and decouple. The index is then given by a path integral over just the constant modes on the circle. In other words, the superconformal index of the $4d$ theory reduces to a partition function of the dimensionally reduced $3d$ gauge theory on S^3. The $3d$ theory preserves all the supersymmetries of the "parent" $4d$ theory on $S^3 \times S^1$.

More generally, for any d dimensional manifold M^d, one would expect the index of a supersymmetric theory on $M^d \times S^1$ to reduce to the exact partition function of dimensionally reduced theory on M^d. This idea was applied by Nekrasov to obtain the partition function of $4d$ gauge theory on Ω-deformed background as a limit of the index of a $5d$ gauge theory [25].

A crucial property of the four dimensional index that facilitates its computation is the fact that it can be computed exactly by a saddle point integral. We show that in the limit of vanishing circle radius, this matrix integral reduces to the one that computes the partition function of $3d$ gauge theories on S^3 [20, 21]. It doesn't come as a surprise as the path integral of the $\mathcal{N} = 2$ supersymmetric gauge theory on S^3 was also shown to localize on saddle points of the action.

The note is organized a follows. In section 5.1 we write the superconformal index of $4d$ theory as a saddle point integral and describe the limit in which this integral reduces to the S^3 partition function. The limit is performed in section 5.2. In particular, we show that the building blocks of the matrix model that computes the superconformal index in $4d$ map separately to the building blocks of the $3d$ partition function matrix model. In section 5.3, we comment on the connections between $4d$ and $3d$ dualities. We conclude with an appendix that generalizes the Kapustin et. al. matrix model for $\mathcal{N} = 4$ gauge theories with two supersymmetric deformations. One such deformation involving squashed S^3 was studied in [78].

5.1 $4d$ Index as a path integral on $S^3 \times S^1$

The superconformal index is a Witten index with respect to one of the supercharges. For concreteness, let us restrict ourselves to the supercharge [1] $\mathcal{Q} \equiv \bar{\mathcal{Q}}_{2+} \in \mathcal{N} = 2$ superconformal algebra, although the index can be defined more generally. In radial quantization the superconformal index is defined as

$$\mathcal{I} = \mathrm{Tr}_{\mathcal{H}}(-1)^F t^{2(E+j_2)} y^{2j_1} v^{-(r+R)} \,. \tag{5.1}$$

The fugacities t, y and v couple to all possible $SU(2,2|2)$ charges that commute with $\mathcal{Q}$. E is the conformal dimension. (j_1, j_2) are the $SU(2)_1 \otimes SU(2)_2$ Lorentz spins and (R, r) are the charges of $SU(2)_R \times U(1)_r$ R-symmetry. The superconformal index doesn't depend on the couplings of the theory and hence it can be calculated in the weak coupling limit. The entire contribution to the supersymmetric partition function on $S^3 \times S^1$ thus comes from the saddle point approximation. One loop partition function of a $4d$ gauge theory on $S^3 \times S^1$ was computed in [79] in the presence of fugacities associated with various conserved charges. To compute the superconformal index, we only allow fugacities for charges which commute with $\mathcal{Q}$; i.e. t, y and v.

For the one loop computation in $SU(N)$ gauge theory, it is convenient to use the Coulomb gauge $\partial_i A^i = 0$ where i, j, k are S^3 coordinates and ∂_i are covariant derivatives. The residual gauge freedom is fixed by imposing $\partial_0 \alpha = 0$ where $\alpha = \frac{1}{V}\int_{S^3} A_0$ and V is the volume of S^3. The partition function is then written as

$$Z = \int d\alpha \Delta_2 \int \mathcal{D}A \Delta_1 e^{-S(A,\alpha)} \,, \tag{5.2}$$

where Δ_1 and Δ_2 are Fadeev-Popov determinants associated with the first and second gauge fixing conditions respectively. For a charge s that commutes with $\mathcal{Q}$, we can add a supersymmetric coupling with a constant background gauge field as

$$S \to S + \int d^4x \, s^\mu \chi_\mu, \tag{5.3}$$

[1] The supercharges of $\mathcal{N} = 2$ gauge theory are denoted as $\mathcal{Q}^I_\alpha$ and $\bar{\mathcal{Q}}_{I\dot{\alpha}}$ where $I = 1, 2$ is an $SU(2)_R$ index and $\alpha = \pm$, $\dot{\alpha} = \dot{\pm}$ are Lorentz indices.

where s^μ is associated conserved current. χ_μ is take to be a $(\chi, 0, 0, 0)$ and χ is identified with the chemical potential for charge s. The chemical potential is related to the fugacity, say x, of the Hamiltonian formalism as $x = e^{-\beta\chi}$. In our case, x can be any of the t, y and v.

After performing $\int \mathcal{D}A$, one gets an $SU(N)$ unitary matrix model

$$Z = \int [dU] e^{-S_{eff}[U]}\,, \tag{5.4}$$

where $U = e^{i\beta\alpha}$ and β is the circumference of the circle, $[dU]$ is the invariant Haar measure on the group $SU(N)$. We can write S_{eff} concisely as follows

$$S^{eff}[U] = \sum_{m=1}^{\infty} \frac{1}{m} \sum_j i_{\mathcal{R}_j}(t^m, y^m, v^m) \chi_{\mathcal{R}_j}(U^m, V^m)\,. \tag{5.5}$$

Here, V denotes the chemical potential that couples to the Cartan of the flavor group; $\mathcal{R}_j$ labels the representation of the fields under gauge and flavor groups and $i_{\mathcal{R}_j}$ is the single letter index of the fields in representation $\mathcal{R}_j$.

The circumference β of the circle is related to the fugacity t as $t = e^{-\beta/3}$. To produce the partition function of dimensionally reduced gauge theory on S^3 [20, 21] we also scale $v = e^{-\beta/3}$, $y = 1$, and take the limit $\beta \to 0$. In appendix G we restore the additional deformations by defining $v = e^{-\beta(1/3+u)}$ and set $y = e^{-\beta\eta}$ where u and η are chemical potentials for fugacities v and y respectively. The partition function of $3d$ gauge theories on squashed S^3 was computed in [78], the η deformation is related to the squashing parameter of S^3.

5.2 $4d$ Index to $3d$ Partition function on S^3

A matrix model for computing the partition function of $3d$ gauge theories on S^3 (S^3 matrix model) was obtained in [20, 21]. In this section, we will derive this matrix model as a $\beta \to 0$ limit of the matrix model that computes the superconformal index (5.5) (index matrix model) of the $4d$ gauge theories. Both matrix models involve integrals over gauge group parameters and their integrand contains one-loop contributions from vector- and hyper-multiplets. We will show that the gauge group integral together with the contribution from the vector multiplet map nicely from the index model to the S^3 model. The contributions of the hypermultiplets match up separately. We also show that the superconformal index is the q-deformation of the S^3 partition function of the daughter theory.

5.2.1 Building blocks of the matrix models

For concreteness, let us consider $4d$ $\mathcal{N}=2$ $SU(N)$ gauge theory. It is constructed using two basic building blocks: hyper-multiplets and vector multiplets.

Hyper-multiplet

As was first observed in [50], the index of the hypermultiplet can be written elegantly in terms of a special function [1]

$$\mathcal{I}^{hyp}=\prod_i \Gamma\left(\frac{t^2}{\sqrt{v}}a_i;t^3y,t^3y^{-1}\right), \tag{5.6}$$

where Γ is the elliptic gamma function [80] defined to be

$$\Gamma(z;r,s)=\prod_{j,k\geq 0}\frac{1-z^{-1}r^{j+1}s^{k+1}}{1-zr^js^k}, \tag{5.7}$$

and a_i are eigenvalues of the maximal torus of the gauge/flavor group satisfying $\prod_{i=1}^N a_i=1$. In this section, for the sake of simplicity, we set $v=t$ and $y=1$ and will discuss the general assignment of chemical potentials in appendix G. We choose a convenient variable $q\equiv e^{-\beta}$ to parametrize the chemical potentials of the Cartan of the flavor group as $a_i=q^{-i\alpha_i}$, and the chemical potential t as $t=q^{\frac{1}{3}}$. The index of the hyper-multiplet then becomes

$$\mathcal{I}^{hyp}=\prod_i\prod_{j,k\geq 0}\frac{1-q^{-\frac{1}{2}+i\alpha_i}q^{j+1}q^{k+1}}{1-q^{\frac{1}{2}-i\alpha_i}q^jq^k}=\prod_i\prod_{n\geq 1}\left(\frac{[n+\frac{1}{2}+i\alpha_i]_q}{[n-\frac{1}{2}-i\alpha_i]_q}\right)^n, \tag{5.8}$$

where $[n]_q\equiv\frac{1-q^n}{1-q}$ is the q-*number*. It has the property $[n]_q\xrightarrow{q\to 1}n$. So far we have fixed the chemical potentials v and y that couple to $-(R+r)$ and j_1 respectively. To recover $3d$ partition function on S^3 we should take the radius of S^1 to be very small, which corresponds to the limit $q \to 1$.

$$\mathcal{I}^{hyp} = \prod_i \prod_{n\geq 1} \left(\frac{n + \frac{1}{2} + i\alpha_i}{n - \frac{1}{2} - i\alpha_i} \right)^n = \prod_i (\cosh \pi\alpha_i)^{-\frac{1}{2}} \,. \tag{5.9}$$

One can find a proof of the second equality in [20]. From the limiting procedure, it is clear that the superconformal index of the hypermultiplet is the q-deformation of the $3d$ hypermultiplet partition function.

Vector multiplet

The index of an $\mathcal{N} = 2$ vector multiplet is given by

$$\mathcal{I}^{vector} = \prod_{i<j} \frac{1}{(1 - a_i/a_j)(1 - a_j/a_i)} \frac{\Gamma(t^2 v (a_i/a_j)^{\pm}; t^3 y, t^3 y^{-1})}{\Gamma((a_i/a_j)^{\pm}; t^3 y, t^3 y^{-1})} \,, \tag{5.10}$$

Here we have dropped an overall a_i-independent factor. We use the condensed notation, $\Gamma(z^{\pm 1}; r, s) = \Gamma(z^{-1}; r, s)\Gamma(z; r, s)$. With the same variable change as above we get

$$\begin{aligned}
\mathcal{I}^{vector} &= \prod_{i<j} \frac{1}{1 - q^{i(\alpha_i - \alpha_j)}} \frac{1}{1 - q^{-i(\alpha_i - \alpha_j)}} \frac{1}{\Gamma(q^{\pm i(\alpha_i - \alpha_j)}; q, q)} \\
&= \prod_{i<j} \frac{1}{1 - q^{i(\alpha_i - \alpha_j)}} \frac{1}{1 - q^{-i(\alpha_i - \alpha_j)}} \prod_{n\geq 1} \left(\frac{1 - q^{n + i(\alpha_i - \alpha_j) + 1}}{1 - q^{n - i(\alpha_i - \alpha_j) - 1}} \frac{1 - q^{n - i(\alpha_i - \alpha_j) + 1}}{1 - q^{n + i(\alpha_i - \alpha_j) - 1}} \right)^{-n} \\
&\stackrel{reg}{=} \prod_{i<j} \prod_{n\geq 1} \left(\frac{[n - i(\alpha_i - \alpha_j)]_q}{[n]_q} \frac{[n + i(\alpha_i - \alpha_j)]_q}{[n]_q} \right)^2 .
\end{aligned} \tag{5.11}$$

The last line involves regulating the infinite product in a way that doesn't depend on α. In the limit $q \to 1$, i.e. the radius of the circle goes to zero, we get

$$\mathcal{I}^{vector} = \prod_{i<j} \prod_{n\geq 1} \left(1 + \frac{(\alpha_i - \alpha_j)^2}{n^2} \right)^2 = \prod_{i<j} \left(\frac{\sinh \pi(\alpha_i - \alpha_j)}{\pi(\alpha_i - \alpha_j)} \right)^2 . \tag{5.12}$$

The last equality again is explained in [20]. Again, the we see that the index of the vector multiplet is the q-deformation of the $3d$ vector partition function. Most general expression for the one-loop contribution of the vector multiplet with u and η turned on is obtained in appendix G.

Gauge group integral

The gauge group integral in the $4d$ index matrix model is done with the invariant Haar measure

$$[dU] = \prod_i d\alpha_i \prod_{i<j} \sin^2\left(\frac{\beta(\alpha_i-\alpha_j)}{2}\right) \xrightarrow{\beta\to 0} \prod_i d\alpha_i \prod_{i<j}\left(\frac{\beta(\alpha_i-\alpha_j)}{2}\right)^2 \quad (5.13)$$

After appropriate regularization, the measure factor precisely cancels the weight factor in the denominator of the vector multiplet one-loop determinant. The unitary gauge group integral in the index matrix model can be done as a contour integral over a variables parametrizing the Cartan sub-group, i.e. $a \in \mathbb{T}$. After the change variables $a = q^{-i\alpha}$ the contour integral becomes a line integral as follows. We write $a = q^{-i\alpha} = e^{i\beta\alpha}$. The contour integral around the unit circle is then

$$\oint_{\mathbb{T}} \frac{da}{a} \cdots = \int_{-\pi/\beta}^{\pi/\beta} d\alpha \ldots \qquad : \qquad \beta \to 0, \qquad \oint_{\mathbb{T}} \frac{da}{a} \cdots = \int_{-\infty}^{\infty} d\alpha \ldots \quad (5.14)$$

5.3 $4d \leftrightarrow 3d$ dualities

S duality

Let us illustrate the reduction of a four dimensional index to three dimensional partition function with a simple example. Consider $\mathcal{N}=2$ $SU(2)$ gauge theory with four hypermultiplets in four dimensions. The index of this theory is given by the following expression (up to overall normalization constants)

$$\oint \frac{dz}{z} \frac{\Gamma(t^{3/2}a^{\pm 1}b^{\pm 1}z^{\pm 1};t^3,t^3)\,\Gamma(t^{3/2}c^{\pm 1}d^{\pm 1}z^{\pm 1};t^3,t^3)}{\Gamma(z^{\pm 2};t^3,t^3)}\,. \quad (5.15)$$

Here, a, b, c and d label the Cartans of $SU(2)^4 \subset SO(8)$ flavor group. The Gamma functions in the numerator come from the four hyper-multiplets; the Gamma functions in the denominator come from the $\mathcal{N}=2$ vector multiplet.

From the results of the previous section this expression for the index gives rise to the partition function of $\mathcal{N}=2$ $SU(2)$ gauge theory in three dimensions.

We scale $t \to 1$ and rewrite this as

$$\mathcal{Z}(\alpha, \beta, \gamma, \delta) = \int d\sigma \frac{\sinh^2 2\pi\sigma}{\cosh \pi(\sigma \pm \alpha \pm \beta) \cosh \pi(\sigma \pm \gamma \pm \delta)}, \tag{5.16}$$

where, each cosh is product of four factors with all sign combinations. The flavor (now mass) parameters α, β, γ and δ are related to the flavor parameters in $4d$ as before.

The superconformal index of the $\mathcal{N} = 2$ $SU(2)$ gauge theory with four hypermultiplets in four dimensions is expected to be invariant under the action of an S-duality group which permutes the four hypermultiplets. The expression above can be explicitly shown to exhibit this property [1]. The four dimensional S-duality implies that the three dimensional partition function is invariant under permuting α, β, γ, and δ. One can show (e.g. numerically or order by order expansion in α) that this is indeed true. Note that this implies a new kind of Seiberg-like duality in three dimensions. This computation can be generalized to any of the theories recently discussed by Gaiotto [16] in four dimensions. In particular the index of these theories was claimed to posses a TQFT structure [1]; and this structure is inherited by the three dimensional partition functions after doing the dimensional reduction. The reasoning in four dimensions and three dimensions is however different. In four dimensions one can associate a punctured Riemann surface to each of the superconfomal theories with the modular parameters of the surface related to the gauge coupling constants. The index does not depend on the coupling constants and thus is independent of the moduli giving rise to a topological quantity associated to the Riemann surface. After dimensionally reducing to three dimensions the theories cease to be conformal invariant and flow to a fixed point in the IR. The statement is then that at the IR fixed point the information about the original coupling constant is "washed away" and theories originally associated to punctured Riemann surfaces of the same topology flow to an equivalent fixed point in the IR.

Mirror symmetry

In principle one can try to use relations special to field theories in three dimensions to gain information about the four dimensional theories. Let us comment how this can come about. In three dimensions certain classes of theories are related by mirror symmetry. for example, in [81] it is claimed that mirror duals of T_N [16] theories have Lagrangian description and are certain star shaped quiver gauge theories. Let us see if the partition function of T_2 (free hypermultiplet in trifundamental of $SU(2)^3$) matches with the partition function of its mirror dual:

$$\begin{aligned} \mathcal{Z}_{T_2} &= \frac{1}{\cosh\pi(\alpha\pm\beta\pm\gamma)}, \\ \mathcal{Z}_{\tilde{T}_2} &= \int d\sigma d\mu d\nu d\rho \frac{\sinh^2 2\pi\sigma\, e^{2\pi i(\mu\alpha+\nu\beta+\rho\gamma)}}{\cosh\pi(\sigma\pm\mu)\cosh\pi(\sigma\pm\nu)\cosh\pi(\sigma\pm\rho)}. \end{aligned} \tag{5.17}$$

In $\mathcal{Z}_T$, the parameters α, β, γ appear as masses while in $\mathcal{Z}_{\tilde{T}}$ they appear as FI terms. Let us compute $\mathcal{Z}_{\tilde{T}_2}$. One can perform the $\mathcal{Z}_{\tilde{T}_2}$ integrations. First we work out

$$\int d\mu \frac{e^{2\pi i\alpha\mu}}{\cosh\pi(\mu\pm\sigma)} = \frac{2\sin 2\pi\alpha\sigma}{\sinh\pi\alpha\sinh 2\pi\sigma}\,.$$

Then we find that

$$\mathcal{Z}_{\tilde{T}_2} = \int d\sigma \frac{8\sin 2\pi\alpha\sigma\sin 2\pi\beta\sigma\sin 2\pi\gamma\sigma}{\sinh\pi\alpha\sinh\pi\beta\sinh\pi\gamma\sinh 2\pi\sigma} = \frac{1}{\cosh\pi(\alpha/2\pm\beta/2\pm\gamma/2)}\,. \tag{5.18}$$

$\mathcal{Z}_{\tilde{T}_2}$ is actually $\mathcal{Z}_{T_2}$ if we rescale α, β and γ in $\mathcal{Z}_{\tilde{T}_2}$ by a factor of 2. This fact can be in principle use to investigate the index of the strongly coupled SCFTs in four dimensions which do not have Lagrangian description. One can dimensionally reduce these theories to three dimensions, consider their mirror dual and compute its $3d$ partition function; finally, one can try to uplift this result to $4d$ and obtain thus the superconformal index of the original four dimensional theory. The feasibility of this approach is currently under investigation.

With this we end the first part of the thesis which studied the behavior of the $\mathcal{N}=2$ superconformal gauge theories of finite rank. In the second part, we will study their large N dynamics.

Chapter 6

Towards the string dual of $\mathcal{N} = 2$ Superconformal QCD

6.1 Motivation

How general is the gauge/string correspondence? 't Hooft's topological argument [82] suggests that any large N gauge theory should be dual to a closed string theory. However, the four-dimensional gauge theories for which an independent definition of the dual string theory is presently available are rather special. Even among conformal field theories, which are the best understood, an explicit dual string description is known only for a sparse subset of models. In some sense all examples are close relatives of the original paradigm of $\mathcal{N} = 4$ super Yang-Mills [12–14] and are found by considering stacks of branes at local singularities in critical string theory, or variations of this setup, *e.g.* [83–89].[1] Conformal field theories in this class can have lower or no supersymmetry, but are far from being "generic". Some of their special features are:

(i) The a and c conformal anomaly coefficients are equal at large N [91].

(ii) The fields are in the adjoint or in bifundamental representations of the gauge group. (Except possibly for a small number of fundamental flavors – "small" in the large N limit – as in [92]).

[1] We should perhaps emphasize from the outset that our focus is on *string* duals of gauge theories. There are strongly coupled field theories that admit *gravity* duals with no perturbative string limit, see *e.g.* [39, 90].

(iii) The dual geometry is ten dimensional.

(iv) The conformal field theory has an exactly marginal coupling λ, which corresponds to a geometric modulus on the dual string side. For large λ the string sigma model is weakly coupled and the supergravity approximation is valid.[2]

The situation certainly does not improve if one breaks conformal invariance – the field theories for which we can directly describe the string dual remain a very special set, which does not include some of the most relevant cases, such as pure Yang-Mills theory. Many more field theories, including pure Yang-Mills, can be described indirectly, as low-energy limits of deformations of $\mathcal{N} = 4$ SYM (as *e.g.* in [93] for $\mathcal{N} = 1$ SYM) or of other UV fixed points, not necessarily four-dimensional (as in [94] for $\mathcal{N} = 0$ YM or [95, 96] for $\mathcal{N} = 1$ SYM). These constructions count as physical "existence proofs" of the string duals, but if one wishes to focus just on the low-energy dynamics, one invariably encounters strong coupling on the dual string side. In the limit where the unwanted UV degrees of freedom decouple, the dual appears to be described (in the most favorable duality frame) by a closed-string sigma model with strongly curved target. This may well be only a technical problem, which would be overcome by an analytic or even a numerical solution of the worldsheet CFT. The more fundamental problem is that we lack a precise recipe to write, let alone solve, the limiting sigma model that describes only the low-energy degrees of freedom.

To break this impasse and enlarge the list of dual pairs outside the $\mathcal{N} = 4$ SYM universality class, we can try to attack the "next simplest case". A natural candidate for this role is $\mathcal{N} = 2$ SYM with gauge group $SU(N_c)$ and $N_f = 2N_c$ flavor hypermultiplets in the fundamental representation of $SU(N_c)$. The number of flavors is tuned to obtain a vanishing beta function. We refer to this model as $\mathcal{N} = 2$ superconformal QCD (SCQCD). The theory violates properties (i) and (ii) but it still has a large amount of symmetry (half the maximal superconformal symmetry) and it shares with $\mathcal{N} = 4$ SYM the crucial simplifying feature of a tunable, exactly marginal gauge coupling g_{YM}. (The theory also exhibits S-duality [16, 70, 97], though this will not be important

[2] In some cases, as in $\mathcal{N} = 4$ SYM, the opposite limit of small λ corresponds to a weakly coupled Lagrangian description on the field theory side. In other cases, like the Klebanov-Witten theory [86], the Lagrangian description is never weakly coupled.

for our considerations, since we will work in the large N limit, which does not commute with S-duality.)

The large N expansion of $\mathcal{N}=2$ SCQCD is the one defined by Veneziano [22]: the number of colors N_c and the number of fundamental flavors N_f are both sent to infinity, keeping fixed their ratio ($N_f/N_c \equiv 2$ in our case) and the combination $\lambda \equiv g_{YM}^2 N_c$. Which, if any, is the dual string theory? And what happens to it for large λ?

6.2 The Veneziano Limit and Dual Strings

6.2.1 A general conjecture

To understand in which sense we should expect a dual string description of a gauge theory in the Veneziano limit, we start by reviewing general elementary facts about large N counting, Feynman-diagrams topology, and operator mixing. At this stage we have in mind a generic field theory that contains both adjoint fields, which we collectively denote by ϕ^a_b, with $a, b = 1, \ldots, N_c$, and fundamental fields, denoted by q^a_i, with $i = 1, \ldots, N_f$. We can consider the theory both in the 't Hooft limit of large N_c with N_f fixed, and in the Veneziano limit of large $N_c \sim N_f$.

$\mathbf{N_c \to \infty}$, $\mathbf{N_f}$ fixed

Let us first recall the familiar analysis in the 't Hooft limit [82], where the number of colors N_c is sent to infinity, with $\lambda = g_{YM}^2 N_c$ *and* the number of flavors N_f kept fixed. In this limit it is useful to represent propagators for adjoint fields with *double* lines, and propagators for fundamental fields with *single* lines – the lines keep track of the flow of the a type (color) indices. Vacuum Feynman diagrams admit a topological classification as Riemann surfaces with boundaries: each flavor loop is interpreted as a boundary. The N dependence is $N_c^{2-2h-b} N_f^b$, for h the genus and b the number of boundaries.

The natural dual interpretation is then in terms of a string theory with coupling $g_s \sim 1/N_c$, containing both a closed and an open sector – the latter arising from the presence of N_f explicit "flavor" branes where open strings can end. Indeed this is the familiar way to introduce a small number of flavors in the AdS/CFT correspondence [98]: by adding explicit flavor branes to the bulk geometry (the simplest examples is adding D7 branes to the $AdS_5 \times S^5$ background). Since $N_f \ll N_c$, the backreaction of the flavor branes can be neglected (probe approximation).

According to the standard AdS/CFT dictionary, single-trace "glueball" composite operators, of the schematic form $\mathrm{Tr}\,\phi^\ell$ (where Tr is a color trace) are dual to closed string states, while "mesonic" composite operators, of the schematic form $\bar{q}^i \phi^\ell q_j$, are dual to open string states. At large N_c, these two classes of operators play a special role since they can be regarded as "elementary" building blocks: all other gauge-invariant composite operators of finite dimension can be built by taking products of the elementary (single-trace and mesonic) operators, and their correlation functions factorize into the correlation functions of the elementary constituents.[3] This factorization is dual to the fact for $g_s \to 0$ the string Hilbert space becomes the free multiparticle Fock space of open and closed strings.

Flavor-singlet mesons, of the form $\sum_{i=1}^{N_f} \bar{q}^i \phi^\ell q_i$, mix with glueballs in perturbation theory, but the mixing is suppressed by a factor of $N_f/N_c \ll 1$, so the distinction between the two classes of operators is meaningful in the 't Hooft limit. On the dual side, this translates into the statement that the mixing of open and closed strings in subleading since each boundary comes with a suppression factor of $g_s N_f \sim N_f/N_c$.

$\mathbf{N_c \sim N_f \to \infty}$

We can now repeat the analysis in the Veneziano limit of large N_c and large N_f with $\lambda = g_{YM}^2 N_c$ and N_f/N_c fixed. In this limit it is appropriate to use a double-line notation with two distinct types of lines [22]: color lines (joining a indices) and flavor lines (joining i indices). A ϕ propagator decomposes as two color lines with opposite orientations, while a q propagator is made of a color and a flavor line (Figure 6.1). Since $N_f \sim N_c \equiv N$, color and flavor lines are on the same footing in the counting of factors of N. It is natural to regard all vacuum Feynman diagrams as *closed* Riemann surfaces, whose N dependence is N^{2-2h}, for h the genus. At least at this topological level, by

[3]Note that in this discussion we are not considering baryonic operators, since they have infinite dimension in the strict large N_c limit. Baryons are interpreted as solitons of the large N_c theory; as familiar, in AdS/CFT they correspond to non-perturbative (D-brane) states on the string theory side [85].

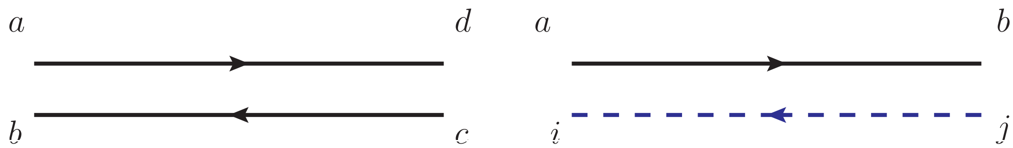


Figure 6.1: Double line propagators. The adjoint propagator $\langle \phi^a_{\ b}\, \phi^c_{\ d} \rangle$ on the left, represented by two color lines, and the fundamental propagator $\langle q^a_{\ i}\, \bar{q}^j_{\ b} \rangle$ on the right, represented by a color and a flavor line.

the same logic of [82], we should expect a gauge theory in the Veneziano limit to be described by the perturbative expansion of a closed string theory, with coupling $g_s \sim 1/N$. More precisely, there should be a dual *purely closed* string description of the *flavor-singlet* sector of the gauge theory.

This point can be sharpened looking at operator mixing. It is consistent to truncate the theory to flavor-singlets, since they close under operator product expansion. The new feature that arises in the Veneziano limit is the order-one mixing of "glueballs" and flavor-singlet "mesons". For large $N_c \sim N_f$, the basic "elementary" operators are what we may call *generalized single-trace* operators, of the form

$$\mathrm{Tr}\left(\phi^{k_1}\mathcal{M}^{\ell_1}\phi^{k_2}\dots\phi^{k_n}\mathcal{M}^{\ell_n}\right)\,, \qquad \mathcal{M}^a_{\ b} \equiv \sum_{i=1}^{N_f} q^a_{\ i}\, \bar{q}^i_{\ b}\,. \tag{6.1}$$

Here we have introduced a flavor-contracted combination of a fundamental and an antifundamental field, $\mathcal{M}^a_{\ b}$, which for the purpose of the large N expansion plays the role of just another adjoint field. The usual large N factorization theorems apply: correlators of generalized multi-traces factorize into correlators of generalized single-traces. In the conjectural duality with a closed string theory, generalized single-trace operators are dual to single-string states.

We can imagine to start with a dual closed string description of the field theory with $N_f = 0$, and first introduce a small number of flavors $N_f \ll N_c$ by adding flavor branes in the probe approximation. As we increase N_f to be $\sim N_c$, the probe approximation breaks down: boundaries are not suppressed and for fixed genus we must sum over worldsheets with arbitrarily many boundaries. The result of this resummation – we are saying – is a new closed string background dual to the flavor-singlet sector of the field theory. The large mixing of closed strings and flavor singlet open strings gives rise to *new effective closed-string degrees of freedom*, propagating in a backreacted geometry. This is the string theory interpretation of the generalized single-trace operators (7.1).

In stating the conjectured duality we have been careful to restrict ourselves to the flavor-singlet sector of the field theory. One may entertain the idea that "generalized mesonic operators" of the schematic form $\bar{q}^i\,\phi^{k_1}\mathcal{M}^{\ell_1}\phi^{k_2}\ldots\phi^{k_n}\mathcal{M}^{\ell_n}\,q_j$ (with open flavor indices i and j) would map to elementary open string states in the bulk. However this cannot be correct, because generalized mesons and generalized single-trace operators are not independent – already in free field theory they are constrained by algebraic relations – so adding an independent open string sector in the dual theory would amount to overcounting.

6.2.2 Outline of the chapter

In this chapter we focus on the concrete example of $\mathcal{N}=2$ SCQCD and look for a closed string theory description of its flavor-singlet sector. We work at the superconformal point (zero vev for all the scalars) and thus look for a string background with unbroken AdS_5 isometry. We attack the problem from two fronts: from the bottom-up, using the weakly-coupled Lagrangian description, and from the top-down, studying brane constructions in string theory. Correspondingly, the chapter is divided into two main parts. The field theory analysis occupies sections 6.3-6.5, the string theory analysis sections 6.7-6.8. Section 6.6 provides a bridge, a first attempt to put together the clues of the field theory analysis and guess features of the dual string theory. In the field theory sections we pose and answer in rigorous detail a well-defined question: what is the protected spectrum of $\mathcal{N}=2$ SCQCD in the generalized single-trace sector? The string theory analysis is more qualitative and our program not yet complete. We review brane constructions and argue that the decoupling limit leads to a *sub-critical* string background. We carry the analysis far enough to see that the string dual, which is largely constrained by symmetry, matches several field theory expectations, but we leave the determination of the precise non-critical background for future work.

In both the bottom-up and top-down approaches it is very useful to view $\mathcal{N} = 2$ SCQCD as part of an "interpolating" $\mathcal{N} = 2$ superconformal field theory (SCFT) that has product gauge group $SU(N_c) \times SU(N_{\check{c}})$ and correspondingly two exactly marginal couplings g and $\check{g}$. For $\check{g} \to 0$ one finds $\mathcal{N} = 2$ SCQCD *plus* a decoupled vector multiplet, while for $\check{g} = g$ one finds the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM. The orbifold theory has a well-known closed string dual, type IIB on $AdS_5 \times S^5/\mathbb{Z}_2$, and changing $\check{g}/g$ amounts to changing the period of the NSNS B-field through the blow-down cycle of the orbifold. As we are going to discuss in detail, the flavor-singlet operators of $\mathcal{N} = 2$ SCQCD are a subsector of the operators of the interpolating SCFT. So in a sense we are guaranteed success: we know a priori that the flavor-singlet sector of $\mathcal{N} = 2$ SCQCD must be described by the closed string theory obtained by following the limit $\check{g} \to 0$ in the bulk. This is however a rather subtle limit, and making sense of it will occupy us in the second part of the chapter.

In the next chapter (based on [99]) we have taken the next step of the bottom-up analysis. We have evaluated the planar one-loop dilation operator in the scalar sector of $\mathcal{N} = 2$ SCQCD, as well as of the interpolating SCFT, and written it as the Hamiltonian of a spin-chain system. The spin-chain for $\mathcal{N} = 2$ SCQCD is novel, since the chain is of the "generalized single-trace" form (7.1). The dynamics of magnon excitations is quite interesting. In particular it is amusing to see how the flavor-contracted fundamental/antifundamental pairs $\mathcal{M}^a{}_b$ arise as $\check{g} \to 0$ by a process of "dimerization" of the magnons of the interpolating SCFT. Some results of the next chapter will be an input in section 6.4 to the analysis of the protected spectrum of $\mathcal{N} = 2$ SCQCD.

A more detailed outline of the rest of chapter is as follows. We begin in section 6.3 with a review of the Lagrangian and symmetries of $\mathcal{N} = 2$ SCQCD and of the interpolating SCFT that connects it to the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM. In sections 6.4 and 6.5 we study the protected spectrum of short supermultiplets[4] of $\mathcal{N} = 2$ SCQCD and its relation with the spectrum of the interpolating SCFT. This turns out to be a rather intricate exercise in superconformal representation theory. A part of the protected spectrum

[4]We use the word "short" casually, to denote a multiplet that obeys any of type of shortening condition, unlike some authors who distinguish between "short" and "semi-short". We use the precise notation for multiplets reviewed in appendix H when we need to make such distinctions.

of $\mathcal{N} = 2$ SCQCD is easy to determine, namely the supermultiplets built on primaries made of scalar fields: (7.24) is the complete list of such primaries, as shown in [99] using the one-loop spin-chain. In section 6.4 we follow in detail the evolution of the protected states of the interpolating SCFT, starting at the orbifold point $\check{g} = g$ where the complete protected spectrum is easily determined. In the limit $\check{g} \to 0$ we recover (7.24) as the subsector of protected primaries of the interpolating SCFT that are flavor singlets. Now there are *many more* protected states in $\mathcal{N} = 2$ SCQCD than there are for generic $\check{g}$ in the interpolating SCFT: the extra protected states arise from long multiplets of the interpolating SCFT that split into short multiplets at $\check{g} = 0$. In section 6.5 we use the superconformal index to demonstrate the existence of these extra protected states. We show that the number of extra states grows exponentially with the conformal dimension. We also characterize the quantum numbers of the first few of them using a "sieve" algorithm; this characterization is up to a certain intrinsic ambiguity of the superconformal index, which can only determine "equivalence classes" of short multiplets, as we review in detail. Still, we have enough information to unambiguously demonstrate the existence of *higher-spin* protected states in the generalized single-trace sector, in sharp contrast with $\mathcal{N} = 4$ SYM.

In section 6.6 we use the clues offered by the protected spectrum to argue that the dual of $\mathcal{N} = 2$ SCQCD should be a sub-critical string background, with seven "geometric" dimensions, containing both an AdS_5 and an S^1 factor. There must be a sector of light string states, with mass of the order of the AdS scale for all λ, dual to the higher-spin protected states detected by the superconformal index – so even for large λ the supergravity approximation cannot be entirely valid. We suggest that there is also a separate sector of heavy string states, with $m^2 R_{AdS} \gg 1$ for $\lambda \to \infty$. We have in mind a scenario where in the interpolating SCFT there are two effective string lengths l_s and $\check{l}_s$, corresponding to the two 't Hooft couplings λ and $\check{\lambda}$: for $\check{\lambda} \to 0$ and fixed $\lambda \gg 1$, the string length $l_s \ll R_{AdS}$ is associated with the massive sector, while $\check{l}_s \sim R_{AdS}$ is associated with the light sector. In section 6.7 we review brane constructions of the interpolating SCFT and of $\mathcal{N} = 2$ SCQCD. The most useful construction is the Hanany-Witten setup with D4 branes suspended between NS5 branes. We argue that the relevant dynamics is captured by a sub-critical brane setup, with color D3 and flavor D5 boundary states in the exact IIB worldsheet CFT $\mathbb{R}^{5,1} \times SL(2)_2/U(1)/\mathbb{Z}_2$. We identify the dual of $\mathcal{N} = 2$ SCQCD with the backreacted background, where the D-branes are replaced by flux. We do not yet know the precise background, but it is largely constrained by symmetries. In section 6.8 we show that just assuming a solution exists, the results of the top-down approach are in nice qualitative agreement with the bottom-up expectations. A useful tool is the spacetime "effective action" of the non-critical theory, which we identify as the seven-dimensional maximal supergravity with the (non-standard) $SO(4)$ gauging. We conclude in section 6.9 with a brief discussion.

Several technical appendices supplement the text. In appendix A we review the shortening conditions of the $\mathcal{N} = 2$ superconformal algebra. In appendix B we review the $\mathcal{N} = 1$ chiral ring of $\mathcal{N} = 2$ SCQCD and of the interpolating SCFT. In appendix C we evaluate the superconformal index for various combinations of short multiplets. In appendix D we review the Kaluza-Klein reduction on $AdS_5 \times S^1$ of the $(2, 0)$ tensor multiplet, with a new detailed treatment of the zero modes. In appendix E we review the sub-critical IIB background $\mathbb{R}^{5,1} \times SL(2)_2/U(1)/\mathbb{Z}_2$ and its spectrum. We make a new claim about the $7d$ "effective action" describing the lowest plane-wave states, which we identify with maximally supersymmetric $SO(4)$-gauged supergravity.

6.2.3 Relation to previous work

The idea that sub-critical string theories play a role in the gauge/gravity correspondence is of course not new. Polyakov's conjecture that pure Yang-Mills theory should be dual to a $5d$ string theory, with the Liouville field playing the role of the fifth dimension, predates the AdS/CFT correspondence (see *e.g.* [100–102]). In fact one of the surprises of AdS/CFT was that some supersymmetric gauge theories are dual to simple backgrounds of *critical* string theory. General studies of AdS solutions of non-critical spacetime effective actions include [103, 104]. Non-critical holography has been mostly considered, starting with [105, 106], in the $\mathcal{N} = 1$ supersymmetric case, notably for $\mathcal{N} = 1$ super QCD in the Seiberg conformal window, which is argued to be dual to $6d$ non-critical backgrounds of the form $AdS_5 \times S^1$ with string-size curvature. There is an interesting literature on the RNS worldsheet description of these

6d non-critical backgrounds and their gauge-theory interpretation, see *e.g.* [107–110]. Non-critical RNS superstrings were formulated in [111, 112] and shown in [113–115, 115–117] to describe subsectors of critical string theory – the degrees of freedom localized near NS5 branes or (in the mirror description) Calabi-Yau singularities. Non-critical superstrings have been also considered in the Green-Schwarz and pure-spinor formalisms, see *e.g.* [118–122].

Our analysis in sections 6.6 and 6.7 for $\mathcal{N} = 2$ SCQCD will be in the same spirit as the analysis of [107, 110] for $\mathcal{N} = 1$ super QCD. We will use the double-scaling limit defined in [116, 117] and further studied in *e.g.* [123–125]. One of our points is that the $\mathcal{N} = 2$ supersymmetric case should be the simplest for non-critical gauge/string duality. On the string side, more symmetry does not hurt, but the real advantage is on the field theory side. Little is known about the SCFTs in the Seiberg conformal window, since generically they are strongly coupled, isolated fixed points. By contrast $\mathcal{N} = 2$ SCQCD has an exactly marginal coupling λ, which takes arbitrary non-negative values. There is a weakly coupled Lagrangian description for $\lambda \to 0$, and we can bring to bear all the perturbative technology that has been so successful for $\mathcal{N} = 4$ SYM, for example in uncovering integrable structures.[5] At the same time we may hope, again in analogy with $\mathcal{N} = 4$ SYM, that the string dual will simplify in the strong coupling limit $\lambda \to \infty$.

There are also interesting approaches to holography for gauge theories with a large number of fundamental flavors in *critical* string theory/supergravity, see *e.g.* [126–134]. The critical setup inevitably implies that the boundary gauge theory will have UV completions with extra degrees of freedom (*e.g.* higher supersymmetry and/or higher dimensions).

6.3 Field Theory Lagrangian and Symmetries

In this section we briefly review the structure and symmetries of $\mathcal{N} = 2$ SCQCD, and its relation to the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM. Much insight is gained by viewing $\mathcal{N} = 2$ SCQCD, which has *one* exactly marginal parameter (the

[5] $\mathcal{N} = 1$ SQCD at the Seiberg self-dual point $N_f = 2N_c$ admits an exactly marginal coupling (the coefficient of a quartic superpotential), which however is bounded from below – the theory is never weakly coupled.

$SU(N_c)$ gauge coupling g_{YM}), as the limit of a *two*-parameter family of $\mathcal{N}=2$ superconformal field theories. This is the family of $\mathcal{N}=2$ theories with product gauge group[6] $SU(N_c)\times SU(N_{\check{c}})$ and two bifundamental hypermultiplets; its exactly marginal parameters are the two gauge-couplings g_{YM} and $\check{g}_{YM}$. For $\check{g}_{YM}\to 0$ one recovers $\mathcal{N}=2$ SCQCD *plus* a decoupled free vector multiplet in the adjoint of $SU(N_{\check{c}})$. At $\check{g}_{YM}=0$, the second gauge group is interpreted as a subgroup of the global flavor symmetry, $SU(N_{\check{c}})\subset U(N_f=2N_c)$. For $\check{g}_{YM}=g_{YM}$, we have instead the familiar $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM. Thus by tuning $\check{g}_{YM}$ we interpolate continuously between $\mathcal{N}=2$ SCQCD and the $\mathcal{N}=4$ universality class.

The a and c anomalies are constant, and equal to each other, along this exactly marginal line: at the end point $\check{g}_{YM}=0$, the $SU(N_{\check{c}})$ vector multiplets decouples, accounting for the "missing" $a-c$ in $\mathcal{N}=2$ SCQCD.

6.3.1 $\mathcal{N}=2$ SCQCD

Our main interest is $\mathcal{N}=2$ SYM theory with gauge group $SU(N_c)$ and $N_f=2N_c$ fundamental hypermultiplets. We refer to this theory as $\mathcal{N}=2$ SCQCD. Its global symmetry group is $U(N_f)\times SU(2)_R\times U(1)_r$, where $SU(2)_R\times U(1)_r$ is the R-symmetry subgroup of the superconformal group. We use indices $\mathcal{I},\mathcal{J}=\pm$ for $SU(2)_R$, $i,j=1,\dots N_f$ for the flavor group $U(N_f)$ and $a,b=1,\dots N_c$ for the color group $SU(N_c)$.

Table 6.1 summarizes the field content and quantum numbers of the model: The Poincaré supercharges $\mathcal{Q}^{\mathcal{I}}_{\alpha}$, $\bar{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}$ and the conformal supercharges $\mathcal{S}_{\mathcal{I}\alpha}$, $\bar{\mathcal{S}}^{\mathcal{I}}_{\dot{\alpha}}$ are $SU(2)_R$ doublets with charges $\pm 1/2$ under $U(1)_r$. The $\mathcal{N}=2$ vector multiplet consists of a gauge field A_m, two Weyl spinors $\lambda^{\mathcal{I}}_{\alpha}$, $\mathcal{I}=\pm$, which form a doublet under $SU(2)_R$, and one complex scalar ϕ, all in the adjoint representation of $SU(N_c)$. Each $\mathcal{N}=2$ hypermultiplet consists of an $SU(2)_R$ doublet $Q_{\mathcal{I}}$ of complex scalars and of two Weyl spinors ψ_α and $\tilde{\psi}_\alpha$, $SU(2)_R$ singlets. It is convenient to define the flavor contracted mesonic operators

$$\mathcal{M}_{\mathcal{J}\ b}^{\mathcal{I}a}\equiv\frac{1}{\sqrt{2}}Q_{\mathcal{J}\ i}^{\ a}\,\bar{Q}^{\mathcal{I}\ i}_{\ b}\,, \tag{6.2}$$

[6]The ranks of the two groups coincide, $N_c\equiv N_{\check{c}}$, but it will be useful to always distinguish graphically with a "check" all quantities pertaining to the second group $SU(N_{\check{c}})$.

	$SU(N_c)$	$U(N_f)$	$SU(2)_R$	$U(1)_r$
$\mathcal{Q}^{\mathcal{I}}_{\alpha}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$+1/2$
$\mathcal{S}_{\mathcal{I}\,\alpha}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$
A_m	Adj	$\mathbf{1}$	$\mathbf{1}$	0
ϕ	Adj	$\mathbf{1}$	$\mathbf{1}$	-1
$\lambda^{\mathcal{I}}_{\alpha}$	Adj	$\mathbf{1}$	$\mathbf{2}$	$-1/2$
$Q_{\mathcal{I}}$	$\square$	$\square$	$\mathbf{2}$	0
ψ_{α}	$\square$	$\square$	$\mathbf{1}$	$+1/2$
$\tilde{\psi}_{\alpha}$	$\bar{\square}$	$\bar{\square}$	$\mathbf{1}$	$+1/2$
$\mathcal{M}_{\mathbf{1}}$	Adj + $\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0
$\mathcal{M}_{\mathbf{3}}$	Adj + $\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	0

Table 6.1: Symmetries of $\mathcal{N}=2$ SCQCD. We show the quantum numbers of the supercharges $\mathcal{Q}^{\mathcal{I}}$, $\mathcal{S}_{\mathcal{I}}$, of the elementary components fields and of the mesonic operators $\mathcal{M}$. Conjugate objects (such as $\bar{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}$ and $\bar{\phi}$) are not written explicitly.

which may be decomposed into the $SU(2)_R$ singlet and triplet combinations

$$\mathcal{M}_{\mathbf{1}} \equiv \mathcal{M}_{\mathcal{I}}^{\ \mathcal{I}} \quad \text{and} \quad \mathcal{M}_{\mathbf{3}\,\mathcal{J}}^{\ \ \mathcal{I}} \equiv \mathcal{M}_{\mathcal{J}}^{\ \mathcal{I}} - \frac{1}{2}\mathcal{M}_{\mathcal{K}}^{\ \mathcal{K}}\,\delta_{\mathcal{J}}^{\mathcal{I}}\,. \tag{6.3}$$

The operators $\mathcal{M}$ decompose into adjoint plus singlet representations of the color group $SU(N_c)$; the singlet piece is however subleading in the large N_c limit.

The Lagrangian is

$$\mathcal{L} = \mathcal{L}_V + \mathcal{L}_H\,, \tag{6.4}$$

where $\mathcal{L}_V$ stands for the Lagrangian of the $\mathcal{N}=2$ vector multiplet and the $\mathcal{L}_H$ for the Lagrangian of $\mathcal{N}=2$ hypermultiplet. Explicitly[7]

$$\begin{aligned} \mathcal{L}_V &= -\mathrm{Tr}\Big[\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\,\bar{\lambda}_{\mathcal{I}}\bar{\sigma}^{\mu}D_{\mu}\lambda^{\mathcal{I}} + (D^{\mu}\phi)(D_{\mu}\phi)^{\dagger} \\ &\quad + i\sqrt{2}\,(g_{YM}\,\epsilon_{\mathcal{I}\mathcal{J}}\lambda^{\mathcal{I}}\lambda^{\mathcal{J}}\phi^{\dagger} - g_{YM}\,\epsilon^{\mathcal{I}\mathcal{J}}\bar{\lambda}_{\mathcal{I}}\bar{\lambda}_{\mathcal{J}}\phi) + \frac{g_{YM}^2}{2}\big[\phi\,,\,\phi^{\dagger}\big]^2\Big]\,. \end{aligned} \tag{6.5}$$

[7] In our conventions, $D_{\mu} \equiv \partial_{\mu} + i g_{YM} A_{\mu}$. We raise and lower $SU(2)_R$ indices with the antisymmetric symbols $\epsilon_{\mathcal{I}\mathcal{J}}$ and $\epsilon^{\mathcal{I}\mathcal{J}}$, which obey $\epsilon_{\mathcal{I}\mathcal{J}}\,\epsilon^{\mathcal{J}\mathcal{K}} = \delta_{\mathcal{I}}^{\mathcal{K}}$.

$$\begin{aligned}
\mathcal{L}_H &= -\Big[(D^\mu \bar{Q}^{\mathcal{I}})(D_\mu Q_{\mathcal{I}}) + i\,\bar{\psi}\bar{\sigma}^\mu D_\mu \psi + i\,\tilde{\psi}\bar{\sigma}^\mu D_\mu \bar{\tilde{\psi}} \\
&\quad + i\sqrt{2}\,(g_{YM}\,\epsilon^{\mathcal{I}\mathcal{J}}\,\bar{\psi}\bar{\lambda}_{\mathcal{I}} Q_{\mathcal{J}} - g_{YM}\,\epsilon_{\mathcal{I}\mathcal{J}}\bar{Q}^{\mathcal{I}}\lambda^{\mathcal{J}}\psi \\
&\quad + g_{YM}\,\tilde{\psi}\lambda^{\mathcal{I}} Q_{\mathcal{I}} - g_{YM}\,\bar{Q}^{\mathcal{I}}\bar{\lambda}_{\mathcal{I}}\bar{\tilde{\psi}} \\
&\quad + g_{YM}\,\tilde{\psi}\phi\psi - g_{YM}\,\bar{\psi}\bar{\phi}\bar{\tilde{\psi}}) \\
&\quad + g^2_{YM}\bar{Q}_{\mathcal{I}}(\phi^\dagger\phi + \phi\phi^\dagger)Q^{\mathcal{I}} + g^2_{YM}\mathcal{V}(Q)\Big]\,,
\end{aligned} \tag{6.6}$$

where the potential for the squarks is

$$\begin{aligned}
\mathcal{V}(Q) &= (\bar{Q}^{\mathcal{I}}{}_a{}^i Q_{\mathcal{I}}{}^a{}_j)(\bar{Q}^{\mathcal{J}}{}_b{}^j Q_{\mathcal{J}}{}^b{}_i) - \frac{1}{2}(\bar{Q}^{\mathcal{I}}{}_a{}^i Q_{\mathcal{J}}{}^a{}_j)(\bar{Q}^{\mathcal{J}}{}_b{}^j Q_{\mathcal{I}}{}^b{}_i) \\
&\quad + \frac{1}{N_c}(\frac{1}{2}(\bar{Q}^{\mathcal{I}}{}_a{}^i Q_{\mathcal{I}}{}^a{}_i)(\bar{Q}^{\mathcal{J}}{}_b{}^j Q_{\mathcal{J}}{}^b{}_j) - (\bar{Q}^{\mathcal{I}}{}_a{}^i Q_{\mathcal{J}}{}^a{}_i)(\bar{Q}^{\mathcal{J}}{}_b{}^j Q_{\mathcal{I}}{}^b{}_j))\,.
\end{aligned} \tag{6.7}$$

Using the flavor contracted mesonic operator (6.2), $\mathcal{V}$ can be written more compactly as

$$\begin{aligned}
\mathcal{V} &= \mathrm{Tr}[\mathcal{M}^{\mathcal{J}}{}_{\mathcal{I}}\mathcal{M}^{\mathcal{I}}{}_{\mathcal{J}}] - \frac{1}{2}\mathrm{Tr}[\mathcal{M}^{\mathcal{I}}{}_{\mathcal{I}}\mathcal{M}^{\mathcal{J}}{}_{\mathcal{J}}] \\
&\quad - \frac{1}{N_c}\mathrm{Tr}[\mathcal{M}^{\mathcal{J}}{}_{\mathcal{I}}]\mathrm{Tr}[\mathcal{M}^{\mathcal{I}}{}_{\mathcal{J}}] + \frac{1}{2}\frac{1}{N_c}\mathrm{Tr}[\mathcal{M}^{\mathcal{I}}{}_{\mathcal{I}}]\mathrm{Tr}[\mathcal{M}^{\mathcal{J}}{}_{\mathcal{J}}] \\
&= \mathrm{Tr}[\mathcal{M}_{\mathbf{3}}\mathcal{M}_{\mathbf{3}}] - \frac{1}{N_c}\mathrm{Tr}[\mathcal{M}_{\mathbf{3}}]\mathrm{Tr}[\mathcal{M}_{\mathbf{3}}]\,.
\end{aligned}$$

6.3.2 $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ and interpolating family of SCFTs

$\mathcal{N}=2$ SCQCD can be viewed as a limit of a family of superconformal theories; in the opposite limit the family reduces to a $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM. In this subsection we first describe the orbifold theory and then its connection to $\mathcal{N}=2$ SCQCD.

As familiar, the field content of $\mathcal{N}=4$ SYM comprises the gauge field A_m, four Weyl fermions λ^A_α and six real scalars X_{AB}, where $A, B = 1, \dots 4$ are indices of the $SU(4)_R$ R-symmetry group. Under $SU(4)_R$, the fermions are in the **4** representation, while the scalars are in **6** (antisymmetric self-dual) and obey the reality condition[8]

$$X^\dagger_{AB} = \frac{1}{2}\epsilon^{ABCD} X_{CD} \,. \tag{6.8}$$

We may parametrize X_{AB} in terms of six real scalars X_k, $k = 4, \dots 9$,

$$X_{AB} = \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 0 & X_4 + iX_5 & X_7 + iX_6 & X_8 + iX_9 \\ -X_4 - iX_5 & 0 & X_8 - iX_9 & -X_7 + iX_6 \\ \hline -X_7 - iX_6 & -X_8 + iX_9 & 0 & X_4 - iX_5 \\ -X_8 - iX_9 & X_7 - iX_6 & -X_4 + iX_5 & 0 \end{array} \right) \tag{6.9}$$

Next, we pick an $SU(2)_L \times SU(2)_R \times U(1)_r$ subgroup of $SU(4)_R$,

$$\begin{array}{cc} 1 & + \\ 2 & - \\ 3 & \hat{+} \\ 4 & \hat{-} \end{array} \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{SU(2)_R \times U(1)_r} & & \\ & & & \\ \hline & & & \\ & & \multicolumn{2}{c}{SU(2)_L \times U(1)_r^*} \end{array} \right) . \tag{6.10}$$

We use indices $\mathcal{I}, \mathcal{J} = \pm$ for $SU(2)_R$ (corresponding to $A, B = 1, 2$) and indices $\hat{\mathcal{I}}, \hat{\mathcal{J}} = \hat{\pm}$ for $SU(2)_L$ (corresponding to $A, B = 3, 4$). To make more manifest their transformation properties, the scalars are rewritten as the $SU(2)_L \times SU(2)_R$ singlet Z (with charge -1 under $U(1)_r$) and as the bifundamental $\mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}}$ (neutral under $U(1)_r$),

$$\mathcal{Z} \equiv \frac{X_4 + iX_5}{\sqrt{2}} \,, \qquad \mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} X_7 + iX_6 & X_8 + iX_9 \\ X_8 - iX_9 & -X_7 + iX_6 \end{pmatrix} . \tag{6.11}$$

Note the reality condition $\mathcal{X}^\dagger_{\mathcal{I}\hat{\mathcal{I}}} = -\epsilon_{\mathcal{I}\mathcal{J}}\epsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}\mathcal{X}_{\mathcal{J}\hat{\mathcal{J}}}$. Geometrically, $SU(2)_L \times SU(2)_R \cong SO(4)$ is the group of 6789 rotations and $U(1)_R \cong SO(2)$ the group of 45 rotations. Diagonal $SU(2)$ transformations $\mathcal{X} \to U\mathcal{X}U^{-1}$ ($U_R = U, U_L = U^*$) preserve the trace, $\mathrm{Tr}[\mathcal{X}] = 2iX_6$, and thus correspond to 789 rotations.

We are now ready to discuss the orbifold projection. In R-symmetry space,

[8] The $\dagger$ indicates hermitian conjugation of the matrix in color space. We choose hermitian generators for the color group.

the orbifold group is chosen to be $\mathbb{Z}_2 \subset SU(2)_L$ with elements $\pm\mathbb{I}_{2\times 2}$. This is the well-known quiver theory [135] obtained by placing N_c D3 branes at the A_1 singularity $\mathbb{R}^2\times\mathbb{R}^4/\mathbb{Z}_2$, with $(X_6, X_7, X_8, X_9) \to \pm(X_6, X_7, X_8, X_9)$ and X_4 and X_5 invariant. Supersymmetry is broken to $\mathcal{N} = 2$, since the supercharges with $SU(2)_L$ indices are projected out. The $SU(4)_R$ symmetry is broken to $SU(2)_L\times SU(2)_R\times U(1)_r$, or more precisely to $SO(3)_L\times SU(2)_R\times U(1)_r$ since only objects with integer $SU(2)_L$ spin survive. The $SU(2)_R \times U(1)_r$ factors are the R-symmetry of the unbroken $\mathcal{N} = 2$ superconformal group, while $SO(3)_L$ is an extra global symmetry under which the unbroken supercharges are neutral.

In color space, we start with gauge group $SU(2N_c)$, and declare the non-trivial element of the orbifold to be

$$\tau \equiv \begin{pmatrix} \mathbb{I}_{N_c\times N_c} & 0 \\ 0 & -\mathbb{I}_{N_c\times N_c} \end{pmatrix} . \tag{6.12}$$

All in all the $\mathbb{Z}_2$ action on the $\mathcal{N} = 4$ fields is

$$A_m \to \tau A_m \tau \,,\; Z_{\mathcal{I}\mathcal{J}} \to \tau Z_{\mathcal{I}\mathcal{J}}\tau \,,\; \lambda_{\mathcal{I}} \to \tau\lambda_{\mathcal{I}}\tau \,,\; \mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}} \to -\tau\mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}}\tau \,,\; \lambda_{\hat{\mathcal{I}}} \to -\tau\lambda_{\hat{\mathcal{I}}}\tau \,. \tag{6.13}$$

The components that survive the projection are

$$A_m = \begin{pmatrix} A^a_{\mu b} & 0 \\ 0 & \check{A}^{\check{a}}_{\mu\check{b}} \end{pmatrix} \quad Z = \begin{pmatrix} \phi^a{}_b & 0 \\ 0 & \check{\phi}^{\check{a}}{}_{\check{b}} \end{pmatrix} \tag{6.14}$$

$$\lambda_{\mathcal{I}} = \begin{pmatrix} \lambda^a_{\mathcal{I}b} & 0 \\ 0 & \check{\lambda}^{\check{a}}_{\mathcal{I}\check{b}} \end{pmatrix} \quad \lambda_{\hat{\mathcal{I}}} = \begin{pmatrix} 0 & \psi^a_{\hat{\mathcal{I}}\check{a}} \\ \tilde{\psi}^{\check{b}}_{\hat{\mathcal{I}}b} & 0 \end{pmatrix} \tag{6.15}$$

$$\mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}} = \begin{pmatrix} 0 & Q_{\mathcal{I}\hat{\mathcal{I}}\check{a}}^{\;a} \\ -\epsilon_{\mathcal{I}\mathcal{J}}\epsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}\bar{Q}^{\check{b}\hat{\mathcal{J}}\mathcal{J}}_{\;\;b} & 0 \end{pmatrix} . \tag{6.16}$$

The gauge group is broken to $SU(N_c)\times SU(N_{\check{c}})\times U(1)$, where the $U(1)$ factor is the *relative*[9] $U(1)$ generated by τ (equ.(8.17)): it must be removed by hand, since its beta function is non-vanishing. The process of removing the relative

[9]Had we started with $U(2N_c)$ group, we would also have an extra *diagonal* $U(1)$, which would completely decouple since no fields are charged under it.

	$SU(N_c)_1$	$SU(N_c)_2$	$SU(2)_R$	$SU(2)_L$	$U(1)_R$
$\mathcal{Q}^{\mathcal{I}}_{\alpha}$	**1**	**1**	**2**	**1**	+1/2
$\mathcal{S}_{\mathcal{I}\,\alpha}$	**1**	**1**	**2**	**1**	–1/2
A_m	Adj	**1**	**1**	**1**	0
$\check{A}_m$	**1**	Adj	**1**	**1**	0
ϕ	Adj	**1**	**1**	**1**	–1
$\check{\phi}$	**1**	Adj	**1**	**1**	–1
$\lambda^{\mathcal{I}}$	Adj	**1**	**2**	**1**	–1/2
$\check{\lambda}^{\mathcal{I}}$	**1**	Adj	**2**	**1**	–1/2
$Q_{\mathcal{I}\hat{\mathcal{I}}}$	$\square$	$\bar{\square}$	**2**	**2**	0
$\psi_{\hat{\mathcal{I}}}$	$\square$	$\bar{\square}$	**1**	**2**	+1/2
$\tilde{\psi}_{\hat{\mathcal{I}}}$	$\bar{\square}$	$\square$	**1**	**2**	+1/2

Table 6.2: Symmetries of the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM and of the interpolating family of $\mathcal{N}=2$ SCFTs.

$U(1)$ modifies the scalar potential by double-trace terms, which arise from the fact that the auxiliary fields (in $\mathcal{N}=1$ superspace) are now missing the $U(1)$ component. Equivalently we can evaluate the beta function for the double-trace couplings, and tune them to their fixed point [136].

Supersymmetry organizes the component fields into the $\mathcal{N}=2$ vector multiplets of each factor of the gauge group, $(\phi, \lambda_{\mathcal{I}}, A_m)$ and $(\check{\phi}, \check{\lambda}_{\mathcal{I}}, \check{A}_m)$, and into two bifundamental hypermultiplets, $(Q_{\mathcal{I},\hat{+}}, \psi_{\hat{+}}, \tilde{\psi}_{\hat{+}})$ and $(Q_{\mathcal{I},\hat{-}}, \psi_{\hat{-}}, \tilde{\psi}_{\hat{-}})$. Table 2 summarizes the field content and quantum numbers of the orbifold theory.

The two gauge-couplings g_{YM} and $\check{g}_{YM}$ can be independently varied while preserving $\mathcal{N}=2$ superconformal invariance, thus defining a two-parameter family of $\mathcal{N}=2$ SCFTs. Some care is needed in adjusting the Yukawa and scalar potential terms so that $\mathcal{N}=2$ supersymmetry is preserved. We find

$$
\begin{aligned}
\mathcal{L}_{Yukawa}(g_{YM}, \check{g}_{YM}) \; = \; & i\sqrt{2}\mathrm{Tr}\big[- g_{YM}\epsilon^{\mathcal{I}\mathcal{J}}\bar{\lambda}_{\mathcal{I}}\bar{\lambda}_{\mathcal{J}}\phi - \check{g}_{YM}\epsilon^{\mathcal{I}\mathcal{J}}\bar{\check{\lambda}}_{\mathcal{I}}\bar{\check{\lambda}}_{\mathcal{J}}\check{\phi} \\
& + \; g_{YM}\epsilon^{\hat{\mathcal{I}}\hat{\mathcal{J}}}\tilde{\psi}_{\hat{\mathcal{I}}}\phi\psi_{\hat{\mathcal{J}}} + \check{g}_{YM}\epsilon^{\hat{\mathcal{I}}\hat{\mathcal{J}}}\psi_{\hat{\mathcal{J}}}\check{\phi}\tilde{\psi}_{\hat{\mathcal{I}}} \\
& + \; g_{YM}\epsilon^{\hat{\mathcal{I}}\hat{\mathcal{J}}}\tilde{\psi}_{\hat{\mathcal{J}}}\lambda^{\mathcal{I}}Q_{\mathcal{I}\hat{\mathcal{I}}} + \check{g}_{YM}\epsilon^{\hat{\mathcal{I}}\hat{\mathcal{J}}}Q_{\mathcal{I}\hat{\mathcal{I}}}\check{\lambda}^{\mathcal{I}}\tilde{\psi}_{\hat{\mathcal{J}}} \\
& - \; g_{YM}\epsilon_{\mathcal{I}\mathcal{J}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{I}}\lambda^{\mathcal{J}}\psi_{\hat{\mathcal{J}}} - \check{g}_{YM}\epsilon_{\mathcal{I}\mathcal{J}}\psi_{\hat{\mathcal{J}}}\check{\lambda}^{\mathcal{I}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{J}}\big] + h.c.
\end{aligned} \tag{6.17}
$$

$$
\begin{aligned}
\mathcal{V}(g_{YM},\check{g}_{YM}) &= g_{YM}^2 \mathrm{Tr}\big[\frac{1}{2}[\bar{\phi},\phi]^2 + \mathcal{M}_{\mathcal{I}}^{\ \mathcal{I}}(\phi\bar{\phi}+\bar{\phi}\phi) + \mathcal{M}_{\mathcal{I}}^{\ \mathcal{J}}\mathcal{M}_{\mathcal{J}}^{\ \mathcal{I}} - \frac{1}{2}\mathcal{M}_{\mathcal{I}}^{\ \mathcal{I}}\mathcal{M}_{\mathcal{J}}^{\ \mathcal{J}}\big] \\
&+ \check{g}_{YM}^2 \mathrm{Tr}\big[\frac{1}{2}[\bar{\check{\phi}},\check{\phi}]^2 + \check{\mathcal{M}}^{\mathcal{I}}{}_{\mathcal{I}}(\check{\phi}\bar{\check{\phi}}+\bar{\check{\phi}}\check{\phi}) + \check{\mathcal{M}}^{\mathcal{I}}{}_{\mathcal{J}}\check{\mathcal{M}}^{\mathcal{J}}{}_{\mathcal{I}} - \frac{1}{2}\check{\mathcal{M}}^{\mathcal{I}}{}_{\mathcal{I}}\check{\mathcal{M}}^{\mathcal{J}}{}_{\mathcal{J}}\big] \\
&+ g_{YM}\check{g}_{YM}\mathrm{Tr}\big[-2Q_{\mathcal{I}\hat{\mathcal{I}}}\check{\phi}\bar{Q}^{\hat{\mathcal{I}}\mathcal{I}}\bar{\phi} + h.c.\big] - \frac{1}{N_c}\mathcal{V}_{d.t.}\,, \qquad (6.18)
\end{aligned}
$$

where the mesonic operators $\mathcal{M}$ are defined as[10]

$$
\mathcal{M}_{\mathcal{J}\ \ b}^{\ \mathcal{I}a} \equiv \frac{1}{\sqrt{2}} Q^{a}_{\ \mathcal{J}\hat{\mathcal{J}}\check{a}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{I}\check{a}}{}_{b}\,, \qquad \check{\mathcal{M}}^{\mathcal{I}\check{a}}_{\ \ \mathcal{J}\check{b}} \equiv \frac{1}{\sqrt{2}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{I}\check{a}}{}_{a}Q^{a}_{\ \mathcal{J}\hat{\mathcal{J}}\check{b}}\,, \qquad (6.19)
$$

and the double-trace terms in the potential are

$$
\begin{aligned}
\mathcal{V}_{d.t.} &= g_{YM}^2\big(\mathrm{Tr}[\mathcal{M}_{\mathcal{I}}^{\ \mathcal{J}}]\mathrm{Tr}[\mathcal{M}_{\mathcal{J}}^{\ \mathcal{I}}] - \frac{1}{2}\mathrm{Tr}[\mathcal{M}_{\mathcal{I}}^{\ \mathcal{I}}]\mathrm{Tr}[\mathcal{M}_{\mathcal{J}}^{\ \mathcal{J}}]\big) \qquad (6.20)\\
&\quad + \check{g}_{YM}^2\big(\mathrm{Tr}[\check{\mathcal{M}}^{\mathcal{I}}{}_{\mathcal{J}}]\mathrm{Tr}[\check{\mathcal{M}}^{\mathcal{J}}{}_{\mathcal{I}}] - \frac{1}{2}\mathrm{Tr}[\check{\mathcal{M}}^{\mathcal{I}}{}_{\mathcal{I}}]\mathrm{Tr}[\check{\mathcal{M}}^{\mathcal{J}}{}_{\mathcal{J}}]\big) \\
&= \big(g_{YM}^2 + \check{g}_{YM}^2\big)\big(\mathrm{Tr}[\mathcal{M}_{\mathcal{I}}^{\ \mathcal{J}}]\mathrm{Tr}[\mathcal{M}_{\mathcal{J}}^{\ \mathcal{I}}] - \frac{1}{2}\mathrm{Tr}[\mathcal{M}_{\mathcal{I}}^{\ \mathcal{I}}]\mathrm{Tr}[\mathcal{M}_{\mathcal{J}}^{\ \mathcal{J}}]\big)\,.
\end{aligned}
$$

The $SU(2)_L$ symmetry is present for all values of the couplings (and so is the $SU(2)_R \times U(1)_r$ R-symmetry, of course). At the orbifold point $g_{YM} = \check{g}_{YM}$ there is an extra $\mathbb{Z}_2$ symmetry (the quantum symmetry of the orbifold) acting as

$$
\phi \leftrightarrow \check{\phi}\,, \quad \lambda_{\mathcal{I}} \leftrightarrow \check{\lambda}_{\mathcal{I}}\,, \quad A_m \leftrightarrow \check{A}_m\,, \quad \psi_{\hat{\mathcal{I}}} \leftrightarrow \tilde{\psi}_{\hat{\mathcal{I}}}\,, \quad Q_{\mathcal{I}\hat{\mathcal{I}}} \leftrightarrow -\epsilon_{\mathcal{I}\mathcal{J}}\epsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}\bar{Q}^{\mathcal{J}\hat{\mathcal{J}}}\,. \qquad (6.21)
$$

Setting $\check{g}_{YM} = 0$, the second vector multiplet $(\check{\phi}, \check{\lambda}_{\mathcal{I}}, \check{A}_m)$ becomes free and completely decouples from the rest of theory, which happens to coincide with $\mathcal{N} = 2$ SCQCD (indeed the field content is the same and $\mathcal{N} = 2$ susy does the rest). The $SU(N_{\check{c}})$ symmetry can now be interpreted as a global flavor symmetry. In fact there is a symmetry enhancement $SU(N_{\check{c}}) \times SU(2)_L \to U(N_f = 2N_c)$: one sees in (6.17, 6.18) that for $\check{g}_{YM} = 0$ the $SU(N_{\check{c}})$ index $\check{a}$ and the $SU(2)_L$ index $\hat{\mathcal{I}}$ can be combined into a single flavor index $i \equiv (\check{a}, \hat{I}) = 1, \ldots 2N_c$.

In the rest of the chapter, unless otherwise stated, we will work in the large

[10] Note that $\mathrm{Tr}[\mathcal{M}_{\mathcal{I}}^{\ \mathcal{J}}] = \mathrm{Tr}[\check{\mathcal{M}}^{\mathcal{J}}{}_{\mathcal{I}}]$.

$N_c \equiv N_{\check{c}}$ limit, keeping fixed the ‘t Hooft couplings

$$\lambda \equiv g_{YM}^2 N_c \equiv 8\pi^2 g^2 \,, \qquad \check{\lambda} \equiv \check{g}_{YM}^2 N_{\check{c}} \equiv 8\pi^2 \check{g}^2 \,. \tag{6.22}$$

The normalizations of g and $\check{g}$ are convenient for the perturbative calculations of [99], in this chapter it is just important to keep in mind that they are (square roots of) the 't Hooft couplings. We will refer to the theory with arbitrary g and $\check{g}$ as the "interpolating SCFT", thinking of keeping g fixed as we vary $\check{g}$ from $\check{g} = g$ (orbifold theory) to $\check{g} = 0$ ($\mathcal{N} = 2$ SCQCD $\oplus$ extra $N_{\check{c}}^2 - 1$ free vector multiplets).

6.4 Protected Spectrum of the Interpolating Theory

In the present and in the following section we will study the protected spectrum of $\mathcal{N} = 2$ SCQCD at large N, in the flavor singlet sector, and its relation with the protected spectrum of the interpolating SCFT. We have argued that in the large N Veneziano limit, flavor singlets that diagonalize the dilation operator take the "generalized single-trace" form (7.1). We will look for the generalized single-trace operators belonging to short multiplets of the superconformal algebra. These are the operators expected to map to the Kaluza-Klein tower of massless single closed string states, so they are the first place to look in a "bottom-up" search for the string dual.

The determination of the complete list of short multiplets of $\mathcal{N} = 2$ SCQCD in this "generalized single-trace" sector turns out to be more subtle than expected. A class of short multiplets is relatively easy to isolate, namely the multiplets based on the following superconformal primaries:

$$\operatorname{Tr} \mathcal{M}_{\mathbf{3}} = (Q_i^a \bar{Q}_a^i)_{\mathbf{3}} \,, \qquad \operatorname{Tr} \phi^{\ell+2} \,, \qquad \operatorname{Tr}[\, T \phi^{\ell}\,] \,, \qquad \ell \geq 0 \,. \tag{6.23}$$

Here $T \equiv \phi\bar{\phi} - \mathcal{M}_{\mathbf{1}}$. We hasten to add that this will turn out to be only a small fraction of the complete set of protected operators. The set (7.24) is the complete list of one-loop protected primaries *in the scalar sector*, as we show in [99] by a systematic evaluation of the one-loop anomalous dimension of all operators that are made out of scalars and obey shortening conditions. The operators $\mathrm{Tr}\,\phi^\ell$ correspond to the vacuum of the spin-chain studied in [99], while the $\mathrm{Tr}\,T\phi^\ell$ correspond to the $p \to 0$ limit of a gapless magnon $T(p)$ of momentum p.

The operators $\mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}$ and $\mathrm{Tr}\,\phi^{\ell+2}$ obey the familiar BPS condition $\Delta = 2R+|r|$, where R is the $SU(2)_R$ spin and r the $U(1)_r$ charge, and they are generators of the chiral ring (with respect to an $\mathcal{N}=1$ subalgebra), see appendix B.[11] By contrast $\mathrm{Tr}[\,T\phi^\ell]$ obey a "semi-shortening" condition and it may be missed in a naive analysis; in these operators there is a large mixing of "glueballs" and "mesons" and the idea of considering "generalized single-traces" is essential. The $\mathrm{Tr}\,T$ multiplet plays a distinguished role since it contains the stress-energy tensor and R-symmetry currents.

Protection of the operators (7.24) can be understood from the viewpoint of the interpolating SCFT connecting $\mathcal{N}=2$ SCQCD with the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM, as follows. The complete spectrum of short multiplets at the orbifold point $g=\check{g}$ is well-known. We will argue, using superconformal representation theory [52], that the protected multiplets found at the orbifold point *cannot* recombine into long multiplets as we vary $\check{g}$, so in particular taking $\check{g} \to 0$ they must evolve into protected multiplets of the theory

$$\{\mathcal{N}=2 \text{ SCQCD } \oplus \text{ decoupled } SU(N_{\check{c}}) \text{ vector multiplet}\}\,. \tag{6.25}$$

The list (7.24) is precisely recovered by restricting to $U(N_f)$ singlets. Remarkably however, the superconformal index of $\mathcal{N}=2$ SCQCD, evaluated in the next section, will show the existence of *many more* protected states. The extra protected states arise from the splitting *long* multiplets of the interpolating theory into short multiplets as $\check{g} \to 0$.

We will make extensive use of the the list given by Dolan and Osborn[52] of all possible shortening conditions of the $\mathcal{N}=2$ superconformal algebra. We

[11] Incidentally, the analysis of the chiral ring extends immediately to flavor non-singlets. The only chiral ring generator which is not a flavor singlet is the $SU(2)_R$ triplet bilinear

$$\mathcal{O}^i_{\mathbf{3}\,j} \equiv (\bar{Q}^i{}_a Q^a{}_j)_{\mathbf{3}} = \bar{Q}^i_{a\,\{\mathcal{I}} Q^a_{\mathcal{J}\}\,j}\,, \tag{6.24}$$

in the adjoint of $SU(N_f)$. The conserved currents for the $SU(N_f) \subset U(N_f)$ flavor symmetry belong to the short multiplet with bottom component $\mathcal{O}^i_{\mathbf{3}\,j}$. Similarly the current for the $U(1) \subset U(N_f)$ baryon number belongs to the $\mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}$ multiplet.

summarize their results and establish notations in appendix H.

6.4.1 Protected Spectrum at the Orbifold Point

At the orbifold point ($g = \check{g}$) the state space of the field theory is the direct sum of an untwisted and a twisted sector, respectively even and odd under the "quantum" $\mathbb{Z}_2$ symmetry (6.21).

6.4.1.1 Untwisted sector

Operators in the untwisted sector of the orbifold descend from operators of $\mathcal{N} = 4$ SYM by projection onto the $\mathbb{Z}_2$ invariant subspace. Their correlators coincide at large N_c with $\mathcal{N} = 4$ correlators [137, 138]. In particular the complete list of untwisted protected states is obtained by projection of the protected states of $\mathcal{N} = 4$. We will be interested in single-trace operators; as is well-known, the only protected single-trace operators of $\mathcal{N} = 4$ belong to the $\frac{1}{2}$ BPS multiplets $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,p,0]}$, built on the chiral primaries $\mathrm{Tr}X^{\{i_1} \dots X^{i_p\}}$, with $p \geq 2$, in the $[0, p, 0]$ representation of $SU(4)_R$ (symmetric traceless of $SO(6)$) The decomposition of each $\frac{1}{2}$ BPS multiplet $\mathcal{N} = 4$ into $\mathcal{N} = 2$ multiplets reads [52]

$$\begin{aligned}
\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,p,0]} \simeq\ & (p+1)\hat{\mathcal{B}}_{\frac{1}{2}p} \oplus \mathcal{E}_{p(0,0)} \oplus \bar{\mathcal{E}}_{-p(0,0)} \\
& \oplus (p-1)\hat{\mathcal{C}}_{\frac{1}{2}p-1(0,0)} \oplus p(\mathcal{D}_{\frac{1}{2}(p-1)(0,0)} \oplus \bar{\mathcal{D}}_{\frac{1}{2}(p-1)(0,0)} \\
& \oplus \bigoplus_{k=1}^{p-2}(k+1)(\mathcal{B}_{\frac{1}{2}k,p-k(0,0)} \oplus \bar{\mathcal{B}}_{\frac{1}{2}k,k-p(0,0)}) \\
& \oplus \bigoplus_{k=0}^{p-3}(k+1)(\mathcal{C}_{\frac{1}{2}k,p-k-2(0,0)} \oplus \bar{\mathcal{C}}_{\frac{1}{2}k,k-p+2(0,0)}) \\
& \oplus \bigoplus_{k=0}^{p-4} \bigoplus_{l=0}^{p-k-4} (k+1)\mathcal{A}^{p}_{\frac{1}{2}k,p-k-4-2l(0,0)}\,, \qquad (6.26)
\end{aligned}$$

which can be understood by considering all possible ways to substitute $X^i \to \mathcal{Z}, \bar{\mathcal{Z}}, \mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}}$, *i.e.* $\mathbf{6} \to (0,0)_1 \oplus (0,0)_{-1} \oplus (\frac{1}{2},\frac{1}{2})_0$ in the branching $SU(4)_R \to SU(2)_L \times SU(2)_R \times U(1)_r$. The $\mathbb{Z}_2$ orbifold projection is then accomplished by the substitution (8.19); states with an even (odd) number of $\mathcal{X}$s are kept (projected out), or equivalently, states with integer (half-odd) $SU(2)_R$ spin are

Multiplet	Orbifold operator $(R, \ell \geq 0,\, n \geq 2)$
$\hat{\mathcal{B}}_{R+1}$	$\mathrm{Tr}[(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R+1}]$
$\bar{\mathcal{E}}_{-(\ell+2)(0,0)}$	$\mathrm{Tr}[\phi^{\ell+2} + \check{\phi}^{\ell+2}]$
$\hat{\mathcal{C}}_{R(0,0)}$	$\mathrm{Tr}[\sum \mathcal{T}(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R}]$
$\bar{\mathcal{D}}_{R+1(0,0)}$	$\mathrm{Tr}[\sum (Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R+1}(\phi + \check{\phi})]$
$\bar{\mathcal{B}}_{R+1,-(\ell+2)(0,0)}$	$\mathrm{Tr}[\sum_i (Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R+1}\phi^i\check{\phi}^{\ell+2-i}]$
$\bar{\mathcal{C}}_{R,-(\ell+1)(0,0)}$	$\mathrm{Tr}[\sum_i \mathcal{T}(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R}\phi^i\check{\phi}^{\ell+1-i}]$
$\mathcal{A}^{\Delta=2R+\ell+2n}_{R,-\ell(0,0)}$	$\mathrm{Tr}[\sum_i \mathcal{T}^n(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R}\phi^i\check{\phi}^{\ell-i}]$

Table 6.3: Superconformal primary operators in the untwisted sector of the orbifold theory. They descend from the $\frac{1}{2}$ BPS primaries of $\mathcal{N} = 4$ SYM. The symbol $\sum$ indicates summation over all "symmetric traceless" permutations of the component fields allowed by the index structure.

Multiplet	Orbifold operator $(\ell \geq 0)$
$\hat{\mathcal{B}}_1$	$\mathrm{Tr}[(Q^{+\hat{+}}\bar{Q}^{+\hat{-}} - Q^{+\hat{-}}\bar{Q}^{+\hat{+}})] = \mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}$
$\bar{\mathcal{E}}_{-\ell-2(0,0)}$	$\mathrm{Tr}[\phi^{\ell+2} - \check{\phi}^{\ell+2}]$

Table 6.4: Superconformal primary operators in the twisted sector of the orbifold theory.

kept (projected out). Table 6.3 lists all the superconformal primaries of the orbifold theory obtained by this procedure.

Let us explain the notation. The explicit expressions in terms of fields are schematic. The symbol $\sum$ indicates summation over all "symmetric traceless" permutations of the component fields allowed by the index structure. The symbol $\mathcal{T}$ stands for the appropriate combination of two scalar fields, neutral under the R symmetry. In the case of the multiplet $\hat{\mathcal{C}}_{0(0,0)}$, $\mathrm{Tr}\,\mathcal{T} = \mathrm{Tr}\,[T + \check{\phi}\bar{\check{\phi}}]$, the bottom component of the stress tensor multiplet of the orbifold theory. The $SU(2)_R \times U(1)_r$ quantum numbers are manifest as labels of the $\mathcal{N} = 2$ multiplets, while the $SU(2)_L$ quantum numbers can be seen from the multiplicity of each multiplet on the right hand side of (6.26) – the $SU(2)_L$ spin always equals the $SU(2)_R$ spin of the multiplet, because $SU(2)_R$ and $SU(2)_L$ indices always come in pairs $(\mathcal{I}\hat{\mathcal{I}})$ and are separately symmetrized.

6.4.1.2 Twisted sector

In the twisted sector, we claim that the complete list of single-trace superconformal primary operators obeying shortening conditions is

$$\begin{aligned}
\mathrm{Tr}[\tau Z^\ell] &= \mathrm{Tr}[\phi^\ell - \check{\phi}^\ell] \text{ for } \ell \geq 2 \\
\mathrm{Tr}[\tau \mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}} \mathcal{X}_{\mathcal{J}\hat{\mathcal{J}}} \epsilon^{\mathcal{I}\mathcal{J}}] &= -\mathrm{Tr}[Q_{\hat{\mathcal{I}}\{\mathcal{I}} \bar{Q}^{\hat{\mathcal{I}}}_{\mathcal{J}\}}] = -\mathrm{Tr}\mathcal{M}_{\mathbf{3}} \,.
\end{aligned} \tag{6.27}$$

That these operators are protected can be seen by the fact that they are the generators of the $\mathcal{N} = 1$ chiral ring in the twisted sector, as we show in appendix I. A priori there could be extra twisted states that do not belong to the chiral ring, as is the case for the untwisted sector. In the next section we will evaluate the superconformal index of the orbifold theory and find that it matches perfectly with the contribution of our claimed list of short multiplets.

The primary $\mathrm{Tr}[\phi^\ell - \check{\phi}^\ell]$ corresponds for each $\ell \geq 2$ to a second copy of the chiral multiplet $\bar{\mathcal{E}}_{-\ell(0,0)}$ – the first copy being the one in the untwisted sector built on $\mathrm{Tr}[\phi^\ell + \check{\phi}^\ell]$. The operator $\mathrm{Tr}[Q_{\hat{\mathcal{I}}\{\mathcal{I}} \bar{Q}^{\hat{\mathcal{I}}}_{\mathcal{J}\}}]$ is an $SU(2)_R$ triplet with vanishing $U(1)_r$ charge and $\Delta = 2$, and must be identified with the primary of a $\hat{\mathcal{B}}_1$ multiplet. This protected multiplet has been overlooked in previous discussions of the orbifold field theory. It is protected only in the theory where the relative $U(1)$ has been correctly subtracted (see section 6.3.2), as seen both in the chiral ring analysis of appendix B and in an explicit one-loop calculation.

6.4.2 From the orbifold point to $\mathcal{N} = 2$ SCQCD

As we move away from the orbifold point by changing $\check{g}$, the short multiplets that we have just enumerated may *a priori* recombine into long multiplets and acquire a non-zero anomalous dimension. The possible recombinations of short multiplets of the $\mathcal{N} = 2$ superconformal algebra were classified in [52]. For short multiplets with a Lorentz-*scalar* bottom component, the relevant rule is

$$\mathcal{A}^{2R+\ell+2}_{R,-\ell(0,0)} \simeq \bar{\mathcal{C}}_{R,-\ell(0,0)} \oplus \bar{\mathcal{B}}_{R+1,-(\ell+1)(0,0)} \,. \tag{6.28}$$

In the special case $\ell = 0$, the short multiplets on the right hand side further decompose into even shorter multiplets as

$$\mathcal{A}^{2R+2}_{R,0(0,0)} \simeq \hat{\mathcal{C}}_{R(0,0)} \oplus \mathcal{D}_{R+1(0,0)} \oplus \bar{\mathcal{D}}_{R+1(0,0)} \oplus \hat{\mathcal{B}}_{R+2(0,0)} \tag{6.29}$$

It follows that the short multiplets of the orbifold theory that that could in principle recombine are

$$\begin{aligned}
&\mathrm{Tr}[\sum_i T(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^R \phi^i \check{\phi}^{\ell-i}] \oplus \mathrm{Tr}[\sum_i (Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R+1} \phi^i \check{\phi}^{\ell-i}] \longrightarrow \mathcal{A}^{2R+\ell+2}_{R,-\ell(0,0)} \\
&\mathrm{Tr}[\sum T(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^R] \oplus \mathrm{Tr}[\sum_i (Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R+1} \bar{\phi}^i \bar{\check{\phi}}^{1-i}] \oplus \\
&\mathrm{Tr}[\sum_i (Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R+1} \phi^i \check{\phi}^{1-i}] \oplus \mathrm{Tr}[\sum (Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R+2}] \longrightarrow \mathcal{A}^{2R+2}_{R,0(0,0)}.
\end{aligned} \tag{6.30}$$

However we see that the proposed recombinations entail short multiplets with *different* $SU(2)_L$ quantum numbers, which is impossible since the supercharges are neutral under $SU(2)_L$. Thus $SU(2)_L$ selection rules forbid the recombination, and the protected multiplets of the orbifold theory remain short for all values of g and $\check{g}$. This conclusion was reached using superconformal representation theory, and it is a rigorous result valid at the full quantum level.[12]

In the limit $\check{g} \to 0$, we must be able to match the protected states of the interpolating SCFT with protected states of $\{\mathcal{N} = 2$ SCQCD $\oplus$ decoupled vector multiplet$\}$. In [99] we follow this evolution in detail using the one-loop spin chain Hamiltonian. The basic features of this evolution can be understood just from group theory. The protected states naturally splits into two sets: $SU(2)_L$ singlets and $SU(2)_L$ non-singlets. It is clear that all the (generalized) single-trace operators of $\mathcal{N} = 2$ SCQCD must arise from the $SU(2)_L$ singlets.

The $SU(2)_L$ singlets are:

(i) One $\hat{\mathcal{B}}_1$ multiplet, corresponding to the primary $\mathrm{Tr}[Q_{\hat{\mathcal{I}}\{\mathcal{I}}\bar{Q}^{\hat{\mathcal{I}}}_{\mathcal{J}\}}] = \mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}$. Since this is the only operator with these quantum numbers, it cannot mix with anything and its form is independent of $\check{g}$.

(ii) Two $\bar{\mathcal{E}}_{-\ell(0,0)}$ multiplets for each $\ell \geq 2$, corresponding to the primaries

[12] We will rephrase the same result in the next section by computing a refined superconformal index that also keeps track of the $SU(2)_L$ quantum number.

$\mathrm{Tr}\,[\phi^\ell \pm \check{\phi}^\ell]$. For each ℓ, there is a two-dimensional space of protected operators, and we may choose whichever basis is more convenient. For $g = \check{g}$, the natural basis vectors are the untwisted and twisted combinations (respectively even and odd under $\phi \leftrightarrow \check{\phi}$), while for $\check{g} = 0$ the natural basis vectors are $\mathrm{Tr}\,\phi^\ell$ (which is an operator of $\mathcal{N} = 2$ SCQCD) and $\mathrm{Tr}\,\check{\phi}^\ell$ (which belongs to the decoupled sector).

(iii) One $\hat{\mathcal{C}}_{0(0,0)}$ multiplet (the stress-tensor multiplet), corresponding to the primary $\mathrm{Tr}\,\mathcal{T} = \mathrm{Tr}\,[T + \check{\phi}\bar{\check{\phi}}]$. We have checked that this combination is an eigenstate with zero eigenvalue for all $\check{g}$. For $\check{g} = 0$, we may trivially subtract out the decoupled piece $\mathrm{Tr}\,\check{\phi}\bar{\check{\phi}}$ and recover $\mathrm{Tr}\,T$, the stress-tensor multiplet of $\mathcal{N} = 2$ SCQCD.

(iv) One $\bar{\mathcal{C}}_{0,-\ell(0,0)}$ multiplet for each $\ell \geq 1$. In the limit $\check{g} \to 0$, we expect this multiplet to evolve to the $\mathrm{Tr}\,T\phi^\ell$ multiplet of $\mathcal{N} = 2$ SCQCD. We have checked this in detail in [99].

All in all, we see that this list reproduces the list (7.24) of one-loop protected scalar operators of $\mathcal{N} = 2$ SCQCD, *plus* the extra states $\mathrm{Tr}\check{\phi}^\ell$ that decouple for $\check{g} = 0$.

The basic protected primary of $\mathcal{N} = 2$ SCQCD which is charged under $SU(2)_L$ is the $SU(2)_L$ triplet contained in the mesonic operator $\mathcal{O}^i_{\mathbf{3_R}\,j} = (\bar{Q}^i_a Q^a_j)_{\mathbf{3_R}}$ (see footnote 11). Indeed writing the $U(N_f = 2N_c)$ flavor indices i as $i = (\check{a}, \hat{\mathcal{I}})$, with $\check{a} = 1, \ldots N_f/2 = N_c$ "half" flavor indices and $\mathcal{I} = \hat{\pm}$ $SU(2)_L$ indices, we can decompose

$$\mathcal{O}^i_{\mathbf{3_R}\,j} \to \mathcal{O}^{\check{a}}_{\mathbf{3_R 3_L}\,\check{b}}\,, \quad \mathcal{O}^{\check{a}}_{\mathbf{3_R 1_L}\,\check{b}}\,. \tag{6.31}$$

In particular we may consider the highest weight combination for both $SU(2)_L$ and $SU(2)_R$,

$$(\bar{Q}^{+\hat{+}} Q^{+\hat{+}})^{\check{a}}_{\check{b}}\,. \tag{6.32}$$

States with higher $SU(2)_L$ spin can be built by taking products of $\mathcal{O}_{\mathbf{3_R 3_L}}$ with $SU(2)_L$ and $SU(2)_R$ indices separately symmetrized – and this is the only way to obtain protected states of $\mathcal{N} = 2$ SCQCD charged under $SU(2)_L$ which have finite conformal dimension in the Veneziano limit. It is then a priori clear that a protected primary of the interpolating theory with $SU(2)_L$ spin L

must evolve as $\check{g} \to 0$ into a product of L copies of $(\bar{Q}^{+\hat{+}} Q^{+\hat{+}})$ and of as many additional decoupled scalars $\check{\phi}$ and $\bar{\check{\phi}}$ as needed to make up for the correct $U(1)_r$ charge and conformal dimension. Examples of this evolution are given in [99].

6.4.3 Summary

In summary all the short multiplets of the interpolating theory remain short as $\check{g} \to 0$, and have a natural interpretation in this limit. The $SU(2)_L$-singlet protected states evolve into the list (7.24) of protected states of SCQCD, plus some extra states made purely from the decoupled vector multiplet. The interpolating theory has also many single-trace protected states with non-trivial $SU(2)_L$ spin, which are flavor non-singlets from the point of view of $\mathcal{N} = 2$ SCQCD: we have seen that in the limit $\check{g} \to 0$, a state with $SU(2)_L$ spin L can be interpreted as a "multiparticle state", obtained by linking together L short "open" spin-chains with of SCQCD and decoupled fields $\check{\phi}$. This is also suggestive of a dual string theory interpretation: as $\check{g} \to 0$, single closed string states carrying $SU(2)_L$ quantum numbers disintegrate into multiple open strings.

Thus by embedding $\mathcal{N} = 2$ SCQCD into the interpolating SCFT we have confirmed that the operators (7.24) are protected at the full quantum level, since they arise as the limit of operators whose protection can be shown at the orbifold point and is preserved by the exactly marginal deformation. However this argument does *not* guarantee that (7.24) is the *complete* set of protected generalized single-trace primaries of $\mathcal{N} = 2$ SCQCD. Indeed we will exhibit many more such states in the next section: they arise from *long* multiplets of the interpolating theory splitting into short multiplets at $\check{g} = 0$.

6.5 Extra Protected Operators of $\mathcal{N} = 2$ SCQCD from the Index

The superconformal index [19] (see also [66]) computes "cohomological" information about the protected spectrum of a superconformal field theory. It counts (with signs) the multiplets obeying shortening conditions, up to equivalence relations that set to zero all sequences of short multiplets that may in principle recombine into long multiplets. The index is invariant under exactly marginal deformations and can thus be evaluated in the free field limit (if the theory admits a Lagrangian description). It should be kept in mind that the index does not completely fix the protected spectrum. A first issue is a certain ambiguity in the quantum numbers of the protected multiplets detected by the index. Short multiplets can be organized into "equivalence classes", such that each short multiplet in a class gives the same contribution to the index. For $\mathcal{N} = 2$ 4d superconformal theories these equivalence classes contain a finite number of short multiplets. This finite ambiguity could in principle be resolved by an explicit one-loop calculation, but in practice this is difficult since the diagonalization of the one-loop dilation operator becomes rapidly complicated as the conformal dimension increases. A second issue is that some sequences of short multiplets that are kinematically allowed to recombine into long multiplets may in fact remain protected for dynamical reasons. This dynamical protection is known to occur at large N_c in $\mathcal{N} = 4$ SYM for certain multi-trace operators, but not for single-trace operators.

Despite these caveats, the index is a very valuable tool. In this section, after reviewing the definition of the index [19], we explain exactly what kind of information can be extracted from it, by characterizing the "equivalence classes" of short multiplets that give the same contribution to the index. We then proceed to concrete calculations, evaluating the index for the interpolating SCFT and for $\mathcal{N} = 2$ SCQCD. The free field contents of the two theories, and thus their indices, are different: recall that the interpolating SCFT has an extra vector multiplet in the adjoint of $SU(N_{\check{c}})$. The index for the interpolating theory confirms the protected spectrum of single-trace operators discussed in the previous section. By contrast, the index for $\mathcal{N} = 2$ SCQCD reveals the existence of many more generalized single-trace operators obeying shortening conditions: their degeneracy grows exponentially with the conformal dimension. Interestingly, we find protected operators with arbitrarily high spin, though none of them is a higher-spin conserved current. We account for the origin of these extra protected states by identifying long multiplets of the interpolating theory that at $\check{g} = 0$ split into short multiplets: some of the resulting short multiplets belong purely to $\mathcal{N} = 2$ SCQCD (*i.e.* do not contain fields in the decoupled vector multiplet) and comprise the extra states.

6.5.1 Review of the Superconformal Index

The superconformal index [19] is just the Witten index with respect to one of the Poincaré supercharges, call it $\mathcal{Q}$, of the superconformal algebra. Let $\mathcal{S} = \mathcal{Q}^\dagger$ be the conformal supercharge conjugate to $\mathcal{Q}$, and $\delta \equiv 2\{\mathcal{S}, \mathcal{Q}\}$. Every state in a unitary representation of the superconformal algebra has $\delta \geq 0$. The index is defined as

$$\mathcal{I} = \mathrm{Tr}\,(-1)^F e^{-\alpha\delta + M}\,, \tag{6.33}$$

where the trace is over the Hilbert space of the theory on S^3, in the usual radial quantization, and M is any operator that commutes with $\mathcal{Q}$ and $\mathcal{S}$. The index receives contributions only from states with $\delta = 0$, which are in one-to-one correspondence with the cohomology classes of $\mathcal{Q}$. It is thus independent of α.

There are in fact two inequivalent possibilities for the choice of $\mathcal{Q}$, leading to a "left" index $\mathcal{I}^{\mathtt{L}}$ and a "right" index $\mathcal{I}^{\mathtt{R}}$. The choice $\mathcal{Q} = \mathcal{Q}^1_-$ leads to the "left" index $\mathcal{I}^{\mathtt{L}}$. In this case

$$\delta^{\mathtt{L}} = \Delta - 2j - 2R - r\,. \tag{6.34}$$

Introducing chemical potentials for all the operators that commute with $\mathcal{Q}$ and $\mathcal{S}$, one defines

$$\mathcal{I}^{\mathtt{L}}(t, y, v) \equiv \mathrm{Tr}\,(-1)^F\, t^{2(\Delta+j)}\, y^{2\bar{j}} v^{r-R}\,. \tag{6.35}$$

The choice $\mathcal{Q} = \bar{\mathcal{Q}}_{2+}$ gives instead the "right" index $\mathcal{I}^{\mathtt{R}}$. In this case

$$\begin{aligned} \delta^{\mathtt{R}} &\equiv \Delta - 2\bar{j} - 2R + r & (6.36)\\ \mathcal{I}^{\mathtt{R}}(t, y, v) &= \mathrm{Tr}\,(-1)^F\, t^{2(\Delta+\bar{j})}\, y^{2j} v^{-r-R}. & (6.37) \end{aligned}$$

The relation between the left and right index is simply $j \leftrightarrow \bar{j}$ and $r \leftrightarrow -r$. For an $\mathcal{N} = 2$ theory, which is necessarily non-chiral, the left and right indices are in fact equal as functions of the chemical potentials, $\mathcal{I}^{\mathtt{L}}(t, y, v) = \mathcal{I}^{\mathtt{R}}(t, y, v)$, but it will be useful to have introduced the definitions of both.

6.5.2 Equivalence Classes of Short Multiplets

We have mentioned that there is a certain finite ambiguity in extracting from the index which are the actual multiplets that remain short. Schematically, the issue is the following. Suppose that two short multiplets, S_1 and S_2, can recombine to form a long multiplet L_1,

$$S_1 \oplus S_2 = L_1 \,, \tag{6.38}$$

and similarly that S_2 can recombine with a third short multiplet S_3 to give another long multiplet L_2,

$$S_2 \oplus S_3 = L_2 \,. \tag{6.39}$$

By construction, the index evaluates to zero on long multiplets, so

$$\mathcal{I}(S_1) = -\mathcal{I}(S_2) = \mathcal{I}(S_3) \,. \tag{6.40}$$

We say that the two multiplets S_1 and S_3 belong to the same equivalence class, since their indices are the same. Note that S_2 *can* be distinguished from $S_1 \sim S_3$ by the overall sign of its index.

The recombination rules for $\mathcal{N} = 2$ superconformal algebra are [52]

$$\begin{aligned}
\mathcal{A}^{2R+r+2j+2}_{R,r(j,\bar{j})} &\simeq \mathcal{C}_{R,r(j,\bar{j})} \oplus \mathcal{C}_{R+\frac{1}{2},r+\frac{1}{2}(j-\frac{1}{2},\bar{j})} & (6.41)\\
\mathcal{A}^{2R-r+2\bar{j}+2}_{R,r(j,\bar{j})} &\simeq \bar{\mathcal{C}}_{R,r(j,\bar{j})} \oplus \bar{\mathcal{C}}_{R+\frac{1}{2},r-\frac{1}{2}(j,\bar{j}-\frac{1}{2})} & (6.42)\\
\mathcal{A}^{2R+j+\bar{j}+2}_{R,j-\bar{j}(j,\bar{j})} &\simeq \hat{\mathcal{C}}_{R(j,\bar{j})} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(j-\frac{1}{2},\bar{j})} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(j,\bar{j}-\frac{1}{2})} \oplus \hat{\mathcal{C}}_{R+1(j-\frac{1}{2},\bar{j}-\frac{1}{2})} & (6.43)
\end{aligned}$$

Notations are reviewed in appendix H. The $\mathcal{C}$, $\bar{\mathcal{C}}$ and $\hat{\mathcal{C}}$ multiplets obey certain "semi-shortening" conditions, see Table H.1, while $\mathcal{A}$ multiplets are generic long multiplets. A long multiplet whose conformal dimension is exactly at the unitarity threshold can be decomposed into shorter multiplets according to (6.41,6.42,6.43). We can formally regard any multiplet obeying some shortening condition (with the exception of the $\mathcal{E}$ and $\bar{\mathcal{E}}$ types) as a multiplet of type $\mathcal{C}$, $\bar{\mathcal{C}}$ or $\hat{\mathcal{C}}$ by allowing the spins j and $\bar{j}$, whose natural range is over the non-negative half-integers, to take the value $-1/2$ as well. The translation is as follows:

$$\mathcal{C}_{R,r(-\frac{1}{2},\bar{j})} \simeq \mathcal{B}_{R+\frac{1}{2},r+\frac{1}{2}(0,\bar{j})} . \tag{6.44}$$

$$\hat{\mathcal{C}}_{R(-\frac{1}{2},\bar{j})} \simeq \mathcal{D}_{R+\frac{1}{2}(0,\bar{j})}, \qquad \hat{\mathcal{C}}_{R(j,-\frac{1}{2})} \simeq \bar{\mathcal{D}}_{R+\frac{1}{2}(j,0)}\,. \tag{6.45}$$

$$\hat{\mathcal{C}}_{R(-\frac{1}{2},-\frac{1}{2})} \simeq \mathcal{D}_{R+\frac{1}{2}(0,-\frac{1}{2})} \simeq \bar{\mathcal{D}}_{R+\frac{1}{2}(-\frac{1}{2},0)} \simeq \hat{\mathcal{B}}_{R+1}. \tag{6.46}$$

Note how these rules flip statistics: a multiplet with bosonic primary ($j+\bar{j}$ integer) is turned into a multiplet with fermionic primary ($j+\bar{j}$ half-odd), and viceversa. With these conventions, the rules (6.41, 6.42, 6.43) are the most general recombination rules. The $\mathcal{E}$ and $\bar{\mathcal{E}}$ multiplets never recombine.

Let us start by characterizing the equivalent classes for $\mathcal{C}$-type multiplets. The right index vanishes identically on $\mathcal{C}$ multiplets. From (6.41), we have

$$\mathcal{I}^{\mathtt{L}}[\mathcal{C}_{R,r(j,\bar{j})}] + \mathcal{I}^{\mathtt{L}}[\mathcal{C}_{R+\frac{1}{2},r+\frac{1}{2}(j-\frac{1}{2},\bar{j})}] = 0\,. \tag{6.47}$$

Clearly $\tilde{R} \equiv R+j$, $\tilde{r} \equiv r+j$ and $\bar{j}$ and the overall sign are the invariant quantum numbers that label an equivalence class. We denote by $[\tilde{R},\tilde{r},\bar{j}]^{\mathtt{L}}_{+}$ the equivalence class of $\mathcal{C}$ multiplets with $\mathcal{I}^{\mathtt{L}} = \mathcal{I}^{\mathtt{L}}[\mathcal{C}_{\tilde{R},\tilde{r}(0,\bar{j})}]$, and by $[\tilde{R},\tilde{r},\bar{j}]^{\mathtt{L}}_{-}$ the class with $\mathcal{I}^{\mathtt{L}} = -\mathcal{I}^{\mathtt{L}}[\mathcal{C}_{\tilde{R},\tilde{r}(0,\bar{j})}]$,

$$\begin{aligned} [\tilde{R},\tilde{r},\bar{j}]^{\mathtt{L}}_{+} &= \{\mathcal{C}_{\tilde{R}-m,\tilde{r}-m\,(m,\bar{j})}\,|\,m=0,1,2\ldots,m\leq\tilde{R}\} && (6.48)\\ [\tilde{R},\tilde{r},\bar{j}]^{\mathtt{L}}_{-} &= \{\mathcal{C}_{\tilde{R}-m,\tilde{r}-m\,(m,\bar{j})}\,|\,m=-\frac{1}{2},\frac{1}{2},\frac{3}{2}\ldots,m\leq\tilde{R}\}\,. && (6.49)\end{aligned}$$

Explicitly, the left index of the class $[\tilde{R},\tilde{r},\bar{j}]^{\mathtt{L}}_{\pm}$ is:

$$\mathcal{I}^{\mathtt{L}}_{[\tilde{R},\tilde{r},\bar{j}]^{\mathtt{L}}_{\pm}} = \pm(-1)^{2\bar{j}+1}t^{6+4\tilde{R}+2\tilde{r}}v^{-2+\tilde{r}-\tilde{R}}\frac{(1-t^2v)(t-\frac{v}{y})(t-vy)}{(1-t^3y)(1-\frac{t^3}{y})}(y^{2\bar{j}}+\ldots+y^{-2\bar{j}}) \tag{6.50}$$

We have illustrated the equivalence classes $[1,1,0]^{\mathtt{L}}_{\pm}$ in Figure 6.2 by listing multiplets on the j axis. The allowed values of $\tilde{R}$ and $\bar{j}$ are $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, ...,

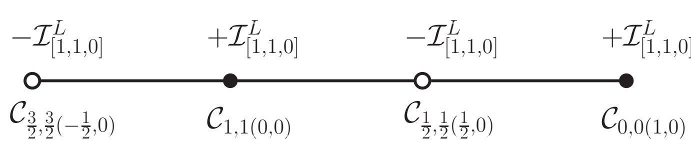
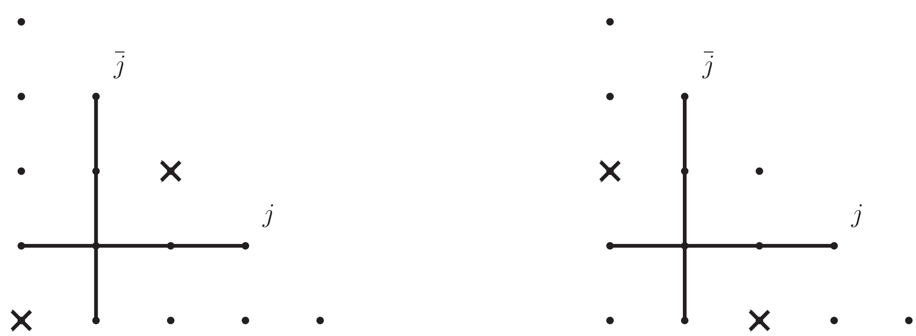


Figure 6.2: The equivalence classes $[1,1,0]^{\mathtt{L}}_{\pm}$. The multiplets belonging to $[1,1,0]^{\mathtt{L}}_{\pm}$ have index $\pm\mathcal{I}^{\mathtt{L}}_{[1,1,0]}$. The sum of the indices of adjacent multiplets is zero, as required by the recombination rule.

with the proviso that $j=-\frac{1}{2}$ or $\bar{j}=-\frac{1}{2}$ must be interpreted according to

(6.44). For the lowest value of $\tilde{R}$, $\tilde{R}=-\frac{1}{2}$, the class $[-\frac{1}{2},\tilde{r},\bar{j}]^{\mathtt{L}}_{+}$ is empty while the class $[-\frac{1}{2},\tilde{r},\bar{j}]^{\mathtt{L}}_{-}=\mathcal{B}_{\frac{1}{2},\tilde{r}+1(0,\bar{j})}$ consists of a single multiplet, which can then be determined without any ambiguity. For $\tilde{R}=0$, $[0,\tilde{r},\bar{j}]^{\mathtt{L}}_{+}=\mathcal{C}_{0,\tilde{r}(0,\bar{j})}$ and $[0,\tilde{r},\bar{j}]^{\mathtt{L}}_{-}=\mathcal{B}_{1,\tilde{r}+1(0,\bar{j})}$ both contain a single multiplet and again there is no ambiguity. Finally for $\tilde{R}=\frac{1}{2}$, $[\frac{1}{2},\tilde{r},\bar{j}]_{+}=\mathcal{C}_{\frac{1}{2},\tilde{r}(0,\bar{j})}$ contains a single multiplet, but $[\frac{1}{2},\tilde{r},\bar{j}]_{-}$ already has two and from the index alone cannot decide which of the two actually remains protected. Clearly the ambiguity grows linearly with $\tilde{R}$.

The analysis for the $\bar{\mathcal{C}}$ multiplets is entirely analogous, and follows from the previous discussion by the substitutions $j\leftrightarrow\bar{j}$, $r\leftrightarrow -r$. One needs to consider $\mathcal{I}^{\mathtt{R}}$, since now it is $\mathcal{I}^{\mathtt{L}}$ that evaluates to zero. The equivalence classes are defined to be the set of all the $\bar{\mathcal{C}}$ multiplets with same $\mathcal{I}^{\mathtt{R}}$ up to sign, and are denoted as $[\bar{\tilde{R}},\bar{\tilde{r}},j]^{\mathtt{R}}_{\pm}$, where $\bar{\tilde{R}}\equiv R+\bar{j}$, $\bar{\tilde{r}}\equiv -r+\bar{j}$.



(a) $\hat{\mathcal{C}}_{0(\frac{1}{2},\frac{1}{2})}$ and $\hat{\mathcal{C}}_{2(-\frac{1}{2},-\frac{1}{2})}\equiv\hat{\mathcal{B}}_{3(0,0)}$

(b) $\hat{\mathcal{C}}_{1(-\frac{1}{2},\frac{1}{2})}\equiv\mathcal{D}_{\frac{3}{2}(0,\frac{1}{2})}$ and $\hat{\mathcal{C}}_{1(\frac{1}{2},-\frac{1}{2})}\equiv\bar{\mathcal{D}}_{\frac{3}{2}(0,\frac{1}{2})}$

Figure 6.3: Example of two configurations of the $\hat{\mathcal{C}}$ multiplets with $R+j+\bar{j}=1$ contributing the same to both $\mathcal{I}^{\mathtt{L}}$ and $\mathcal{I}^{\mathtt{R}}$. The multiplets are denoted by crosses on the $(j,\bar{j})$ grid. The indices are the same for (a) and (b) because the *projections* on the j and $\bar{j}$ (*i.e.* the *sets* of j and $\bar{j}$ values) are the same.

The analysis for the $\hat{\mathcal{C}}$ multiplets is slightly more involved. Unlike $\mathcal{C}$ and $\bar{\mathcal{C}}$ multiplets, $\hat{\mathcal{C}}$ multiplets contribute to both $\mathcal{I}^{\mathtt{L}}$ and $\mathcal{I}^{\mathtt{R}}$. Moreover the quantum number r is fixed by the additional shortening condition $r=\bar{j}-j$. The left and right equivalence classes of $\hat{\mathcal{C}}_{R(j,\bar{j})}$ are $[R+j,\bar{j},\bar{j}]^{\mathtt{L}}_{\pm}$ and $[R+\bar{j},j,j]^{\mathtt{R}}_{\pm}$ respectively. The left index determines $\tilde{R}=R+j$ and the right index $\bar{\tilde{R}}=R+\bar{j}$, so all in all no two different $\hat{\mathcal{C}}$ multiplets give the same contribution to both $\mathcal{I}^{\mathtt{L}}$ and $\mathcal{I}^{\mathtt{R}}$. Nevertheless different *direct sums* of $\hat{\mathcal{C}}$ multiplets can have the same $\mathcal{I}^{\mathtt{L}}$ and

Multiplet	Equivalence class
$\mathcal{C}$	$[\tilde{R},\tilde{r},\bar{\jmath}]^{\mathtt{L}}_{\pm} \equiv [R+j, r+j, \bar{\jmath}]^{\mathtt{L}}_{\pm}$
$\bar{\mathcal{C}}$	$[\tilde{\bar{R}},\tilde{\bar{r}},j]^{\mathtt{R}}_{\pm} \equiv [R+\bar{\jmath}, -r+\bar{\jmath}, j]^{\mathtt{R}}_{\pm}$
$\hat{\mathcal{C}}$	$[\hat{R},\bar{\jmath}]^{\mathtt{L}}_{\pm} \equiv [R+j+\bar{\jmath}, \bar{\jmath}]^{\mathtt{L}}_{\pm}$ $[\hat{R},j]^{\mathtt{R}}_{\pm} \equiv [R+j+\bar{\jmath}, j]^{\mathtt{R}}_{\pm}$

Table 6.5: Summary of notation for equivalence classes of short multiplets.

$\mathcal{I}^{\mathtt{R}}$. It is convenient to introduce the quantum number $\hat{R} \equiv R + j + \bar{\jmath}$, which is an invariant for both the left and the right equivalence classes, and to label the equivalence classes for $\hat{\mathcal{C}}$ multiplets as $[\hat{R},\bar{\jmath}]^{\mathtt{L}}_{\pm}$ and $[\hat{R},j]^{\mathtt{R}}_{\pm}$. This new way to label the classes does not entail any loss of information, and makes it more convenient to analyze both the indices simultaneously. Explicitly, the left and right indices for these equivalence classes are:

$$\begin{aligned}\mathcal{I}^{\mathtt{L}}_{[\hat{R},\bar{\jmath}]^{\mathtt{L}}_{\pm}} &= \pm(-1)^{2\bar{\jmath}}\frac{t^{6-2\bar{\jmath}+4\hat{R}}v^{-1+2\bar{\jmath}-\hat{R}}(1-t^2v)}{(1-t^3y)(1-t^3/y)} \\ &\quad (t(y^{2\bar{\jmath}+1}+\ldots+y^{-(2\bar{\jmath}+1)})-v(y^{2\bar{\jmath}}+\ldots+y^{-2\bar{\jmath}})) \qquad (6.51)\\ \mathcal{I}^{\mathtt{R}}_{[\hat{R},j]^{\mathtt{R}}_{\pm}} &= \pm(-1)^{2j}\frac{t^{6-2j+4\hat{R}}v^{-1+2j-\hat{R}}(1-t^2v)}{(1-t^3y)(1-t^3/y)} \\ &\quad (t(y^{2j+1}+\ldots+y^{-(2j+1)})-v(y^{2j}+\ldots+y^{-2j}))\,. \qquad (6.52)\end{aligned}$$

Now the point is that given a collection of $\hat{\mathcal{C}}$ multiplets with the same value of $\hat{R}$, the left index determines the *set* of $\bar{\jmath}$ values while the right index determines the set of j values, but in general there is not enough information to fix uniquely all quantum numbers. Figure 6.3 illustrates the ambiguity in a simple example: two different configurations, each consisting of two $\hat{\mathcal{C}}$ multiplets, give the same contribution to both $\mathcal{I}^{\mathtt{L}}$ and $\mathcal{I}^{\mathtt{R}}$.

Letters	Δ	j	$\bar{j}$	R	r	$\mathcal{I}^{\mathtt{R}}$
ϕ	1	0	0	0	-1	t^2v
λ^1_+	3/2	1/2	0	1/2	-1/2	$-t^3y$
λ^1_-	3/2	-1/2	0	1/2	-1/2	$-t^3y^{-1}$
$\bar{\lambda}_{2+}$	3/2	0	1/2	1/2	1/2	$-t^4v^{-1}$
$\bar{F}_{++}$	2	0	1	0	0	t^6
∂_{++}	1	1/2	1/2	0	0	t^3y
∂_{-+}	1	-1/2	1/2	0	0	t^3y^{-1}
$\partial_{-+}\lambda^1_+ + \partial_{++}\lambda^1_- = 0$	5/2	0	1/2	1/2	1/2	t^6

Table 6.6: Letters with $\delta^{\mathtt{R}} = 0$ from the $\mathcal{N} = 2$ vector multiplet

6.5.3 The Index of the Interpolating Theory

We now review the calculation of the index for the orbifold theory [19, 139].[13] The index is invariant under exactly marginal deformation and is thus the same for the whole family of interpolating SCFTs. The procedure is well-established. One enumerates the "letters" of the theory with $\delta = 0$ and then counts all possible gauge-invariants words. This is done efficiently by a matrix model, which for large N can be evaluated by saddle point. Tables 6.6 and 6.7 list the $\delta^{\mathtt{R}} = 0$ letters from the $\mathcal{N} = 2$ vector and hyper multiplets.[14] Equations of motion are accounted for by introducing words with "wrong" statistics. One finds the single-letter indices for the vector multiplet and the "half" hyper multiplet

$$f_V(t,y,v) = \frac{t^2v - t^3\left(y+y^{-1}\right) - t^4v^{-1} + 2t^6}{\left(1-t^3y\right)\left(1-t^3y^{-1}\right)} \tag{6.53}$$

$$f_H(t,y,v) = \frac{t^2}{v^{1/2}}\frac{\left(1-t^2v\right)}{\left(1-t^3y\right)\left(1-t^3y^{-1}\right)}\,. \tag{6.54}$$

[13] While we agree with the general procedure followed in [139], we disagree with the final result, equ.(3.5) of [139]. The discrepancy can be traced to an incorrect subtraction of the $U(1)$ factors in [139], they are apparently taken to be $\mathcal{N} = 1$ rather than $\mathcal{N} = 2$ vector multiplets (equ.(2.12) of [139]). For the same reason we disagree with the expression ((3.7) of [139]) for the contribution to the index of the $6d$ $(2,0)$ massless tensor multiplet, which we evaluate in appendix J.

[14] For definiteness we evaluate $\mathcal{I}^{\mathtt{R}}$, but recall that $\mathcal{I}^{\mathtt{L}}(t,y,v) = \mathcal{I}^{\mathtt{R}}(t,y,v)$. The concrete letters with $\delta^{\mathtt{L}} = 0$ are different but the left and right single-letter indices coincide.

Letters	Δ	j	$\bar{j}$	R	r	$\mathcal{I}^{\mathtt{R}}$
q	1	0	0	1/2	0	$t^2v^{-1/2}$
$\bar{\psi}_+$	3/2	0	1/2	0	-1/2	$-t^4v^{1/2}$
$\tilde{q}$	1	0	0	1/2	0	$t^2v^{-1/2}$
$\bar{\tilde{\psi}}_+$	3/2	0	1/2	0	-1/2	$-t^4v^{1/2}$

Table 6.7: Letters with $\delta^{\mathtt{R}} = 0$ from the hyper multiplet

The single-letter index then reads

$$\begin{aligned} i_{orb}(t,y,v;U,\check{U}) &= f_V(t,y,v)(\mathrm{Tr}U\,\mathrm{Tr}U^\dagger - 1) + f_V(t,y,v)(\mathrm{Tr}\check{U}\,\mathrm{Tr}\check{U}^\dagger - 1) \\ &\quad + \left(w + \frac{1}{w}\right) f_H(t,y,v)(\mathrm{Tr}U\,\mathrm{Tr}\check{U}^\dagger + \mathrm{Tr}U^\dagger\,\mathrm{Tr}\check{U})\,. \end{aligned} \tag{6.55}$$

Here U and $\check{U}$ are is an $N_c \times N_c$ unitary matrices out of which we construct the relevant characters of $SU(N_c)$ and $SU(N_{\check{c}})$. We have also introduced a potential w that keeps track of $SU(2)_L$ quantum numbers: $w + \frac{1}{w}$ is the character of the fundamental representation of $SU(2)_L$. The index is obtained by enumerating all gauge-invariant operators in terms of the matrix integral

$$\mathcal{I}_{orb} = \int [dU][d\check{U}] \exp\left(\sum_n \frac{1}{n} i_{orb}(t^n, y^n, v^n; U^n \check{U}^n)\right), \tag{6.56}$$

which for large N_c can be carried out explicitly,

$$\mathcal{I}_{orb} \cong \prod_{n=1}^{\infty} \frac{e^{-\frac{2}{n} f_V(t^n,y^n,v^n)}}{(1 - f_V(t^n,y^n,v^n))^2 - (w^{2n} + w^{-2n} + 2) f_H^2(t^n,y^n,v^n)} \equiv \mathcal{I}_{orb}^{m.t.}\,. \tag{6.57}$$

This expression contains the contribution from all the gauge-invariant operators of the theory, which at large N_c are multi-traces, hence the superscript in $\mathcal{I}_{orb}^{m.t.}$. To extract the contribution from single-traces we evaluate the plethystic logarithm (see *e.g.* [140])

$$\begin{aligned}
\mathcal{I}^{s.t.}_{orb} &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log[\mathcal{I}^{m.t}_{orb}(t^n, y^n, v^n)] && (6.58)\\
&= -\sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \log[(1 - f_V(t^n, y^n, v^n))^2 - (w^{2n} + w^{-2n} + 2) f_H^2(t^n, y^n, v^n)] \\
&\quad -2 f_V(t, y, v) && (6.59)\\
&= 2\left[\frac{t^2 v}{1 - t^2 v} - \frac{t^3 y}{1 - t^3 y} - \frac{t^3 y^{-1}}{1 - t^3 y^{-1}}\right] + \frac{\frac{t^4 w^2}{v}}{1 - \frac{t^4 w^2}{v}} + \frac{\frac{t^4}{v w^2}}{1 - \frac{t^4}{v w^2}} \\
&\quad -2 f_V(t, y, v)\,. && (6.60)
\end{aligned}$$

Here $\mu(n)$ is the Moebius function ($\mu(1) \equiv 1$, $\mu(n) \equiv 0$ if n has repeated prime factors, and $\mu(n) = (-1)^k$ if n is the product of k distinct primes), and $\varphi(r)$ is the Euler Phi function, defined as the number of positive integers less than or equal to r that are coprime with respect to r. We have used the properties

$$\sum_{d|n} d\, \mu\left(\frac{n}{d}\right) = \varphi(n)\,, \qquad \sum_r \frac{\varphi(r)}{r} \log(1 - x^r) = \frac{-x}{1-x}\,. \qquad (6.61)$$

The index is of course independent of g and $\check{g}$. At the orbifold point $g = \check{g}$ it makes sense organize the spectrum into a twisted and an untwisted sector. Protected operators in the untwisted sectors are known from inheritance from $\mathcal{N} = 4$ SYM. To evaluate the contribution to the index from the untwisted sector we start with the single-trace index for $SU(N_c)$ $\mathcal{N} = 4$ SYM and project onto the $\mathbb{Z}_2$ invariant subspace. The single-trace index for $\mathcal{N} = 4$ is found by regarding $\mathcal{N} = 4$ as an $\mathcal{N} = 2$ theory with one adjoint vector and one adjoint hyper. A short calculation gives [19][15]

$$\begin{aligned}
\mathcal{I}_{\mathcal{N}=4} &= \frac{t^2 v}{1 - t^2 v} + \frac{\frac{t^2 w}{\sqrt{v}}}{1 - \frac{t^2 w}{\sqrt{v}}} + \frac{\frac{t^2}{w\sqrt{v}}}{1 - \frac{t^2}{w\sqrt{v}}} - \frac{t^3 y}{1 - t^3 y} - \frac{t^3 y^{-1}}{1 - t^3 y^{-1}} \\
&\quad - f_V(t, y, v) - (w + \frac{1}{w}) f_H(t, y, v)\,. && (6.62)
\end{aligned}$$

The $\mathbb{Z}_2$ acts as $w \to -w$ leaving invariant the under potentials, so the index

[15] Our notations for the chemical potentials are slightly different from [19].

of the untwisted sector of the $\mathbb{Z}_2$ orbifold theory is

$$\begin{aligned}\mathcal{I}_{untwist} &= \frac{1}{2}(\mathcal{I}_{\mathcal{N}=4}(t,y,v,w)+\mathcal{I}_{\mathcal{N}=4}(t,y,v,-w)) \\ &= \frac{t^2v}{1-t^2v}-\frac{t^3y}{1-t^3y}-\frac{t^3y^{-1}}{1-t^3y^{-1}}+\frac{\frac{t^4w^2}{v}}{1-\frac{t^4w^2}{v}}+\frac{\frac{t^4}{vw^2}}{1-\frac{t^4}{vw^2}}-f_V(t,y,v)\,.\end{aligned} \tag{6.63}$$

Subtracting the contribution of the untwisted sector from the total index (6.60), we finally find

$$\mathcal{I}_{twist} = \frac{t^2v}{1-t^2v}-\frac{t^3y}{1-t^3y}-\frac{t^3y^{-1}}{1-t^3y^{-1}}-f_V(t,y,v)\,. \tag{6.64}$$

In appendix J we confirm that this precisely matches with the contribution from the twisted multiplets $\{\mathcal{M}_{\mathbf{3}}, \mathrm{Tr}(\phi^{2+\ell}-\check{\phi}^{2+\ell})\,, \ell\geq 0\}$, which are the generators of the $\mathcal{N}=1$ chiral ring in the twisted sector.

6.5.4 The Index of $\mathcal{N}=2$ SCQCD and the Extra States

The single-letter index for $\mathcal{N}=2$ SCQCD is

$$i_{QCD}(t,y,v;U,V) = f_V(t,y,v)(\mathrm{Tr}U\,\mathrm{Tr}U^\dagger - 1)+f_H(t,y,v)(\mathrm{Tr}U\,\mathrm{Tr}V^\dagger+\mathrm{Tr}U^\dagger\,\mathrm{Tr}V) \tag{6.65}$$

where U an $N_c\times N_c$ matrix and V an $N_f\times N_f$ matrix, with $N_f=2N_c$. We are interested in gauge *and* flavor-singlets, so we integrate over both U and V,

$$\mathcal{I}_{QCD} = \int [dU][dV]\exp\left(\sum_n \frac{1}{n} i_{QCD}(t^n,y^n,v^n;U^nV^n)\right)\,. \tag{6.66}$$

For large N_c and N_f with N_f/N_c fixed we can again use saddle point,

$$\mathcal{I}_{QCD} \cong \prod_{n=1}^{\infty}\frac{e^{-\frac{1}{n}f_V(t^n,y^n,v^n)}}{(1-f_V(t^n,y^n,v^n))-f_H^2(t^n,y^n,v^n)} \equiv \mathcal{I}_{QCD}^{m.t.}\,. \tag{6.67}$$

The index that enumerates (generalized) single-trace operators is then

$$\mathcal{I}_{QCD}^{s.t.} = -\sum_{n=1}^{\infty}\frac{\varphi(n)}{n}\log[(1-f_V(t^n,y^n,v^n))\,-f_H^2(t^n,y^n,v^n)]-f_V(t,y,v)\,. \tag{6.68}$$

Unlike the orbifold theory, there is no nice factorization of the single-letter index and we cannot extract the plethystic log explicitly. This is already an indication of a more complicated structure than expected. The naive expectation is that all protected generalized single-trace multiplets of $\mathcal{N}=2$ SCQCD are exhausted by the list $\{\mathcal{M}_{\mathbf{3}}\,, \mathrm{Tr}\,\phi^{2+\ell}\,, \mathrm{Tr}\,T\phi^{\ell}\,, \ell\geq 0\}$, obtained by projecting the protected single-trace spectrum of the interpolating theory onto $U(N_f)$ singlets. We evaluate the corresponding index in appendix J,

$$\mathcal{I}_{naive}=\frac{1}{(1-t^3y)(1-\frac{t^3}{y})}[-t^6(1-\frac{t}{v}(y+\frac{1}{y}))-\frac{t^{10}}{v}+\frac{t^4v^2(1-\frac{t}{vy})(1-\frac{ty}{v})}{1-t^2v}+\frac{t^4}{v}(1-t^2v)]$$

which is different from the correct index (6.68). Expanding in powers of t, the first discrepancy appears at $O(t^{13})$.

To get some insight, let us rewrite the single-trace index of the orbifold theory as

$$\begin{aligned}\mathcal{I}^{s.t.}(h,k) \quad &= \quad -\sum_{n=1}^{\infty}\Big[\frac{\varphi(n)}{n}\log[(1-f_V(t^n,y^n,v^n))(1-hf_V(t^n,y^n,v^n)) \\ &\qquad -(k(w^{2n}+1+w^{-2n})+1)f_H^2(t^n,y^n,v^n)\Big]-f_V(t,y,v)\end{aligned} \quad (6.69)$$

We have introduced a variable h that keeps track of the number of $SU(N_{\check{c}})$ vector multiplets, and a variable k associated with the *triplet* combination of two neighboring $SU(2)_L$ indices. The index (6.68) for $\mathcal{N}=2$ SCQCD is recovered in the limit $(h,k)\to(0,0)$. Indeed setting $(h,k)=(0,0)$: this amounts to omitting the "second" vector multiplet and to project onto $U(N_f)$ singlets, which is equivalent to first projecting onto $SU(N_{\check{c}})$ singlets (automatically done in the interpolating theory) and then contracting all neighboring $SU(2)_L$ indices into the singlet combination. The grading of gauge-invariant words by powers of h (number of letters in the $SU(N_{\check{c}})$ vector multiplet) makes sense only for $\check{g}=0$. Similarly, for $\check{g}\neq 0$ only the overall $SU(2)_L$ spin of a state is a meaningful quantum number, not the specific way neighboring $SU(2)_L$ indices are contracted. (For example it is clearly possible to construct $SU(2)_L$ singlets which are not $U(N_f)$ singlets.) At $\check{g}\neq 0$ words with different h or k grading will generically mix.

The origin of the extra protected states is then clear. As $\check{g}\to 0$, a long multiplets of the interpolating theory, which obviously does not contribute to $\mathcal{I}_{orb}$, may hit the unitarity bound and decompose into a sum of short multiplets, some of which are $U(N_f)$ singlets and thus belong to $\mathcal{N}=2$ SCQCD, but some of which have instead non-trivial h or k grading. Schematically

$$\lim_{\check{g}\to 0} L = \oplus S_{(h,k)=(0,0)} \ \oplus S_{(h,k)\neq(0,0)} \,. \tag{6.70}$$

The operators $\{S_{(h,k)=(0,0)}\}$ are the extra states. They are protected in $\mathcal{N}=2$ SCQCD because they have no partners to recombine with.

Remarkably the extra protected states are vastly more numerous than the naive list. The asymptotic growth of states in the naive list is clearly linear in the conformal dimension – the number of states with $\Delta < N$ grows as $\sim 2N$, in other terms the density of states $\rho(\Delta)$ is constant. This modest growth is consistent with the fact that the naive single-trace index does not "deconfine", *i.e.* it does not diverge as a function of $t = e^{-1/T}$ for any finite temperature T. The same behavior holds for the orbifold theory or for $\mathcal{N}=4$ SYM. By contrast, the single-trace index of $\mathcal{N}=2$ SCQCD exhibits Hagedorn behavior. Setting for simplicity all other potentials to 1, we encounter a divergence at $t = t_H$ such that

$$1 - f_V(t_H,1,1) - f_H^2(t_H,1,1) = 0 \longrightarrow t_H \cong 0.897769\,. \tag{6.71}$$

This implies an exponential growth in the density of states contributing to the index,

$$\rho(E') \sim e^{\beta_H E'}\,, \quad E' \equiv \Delta + j\,, \quad \beta_H = -\ln t_H \cong 0.107842\,. \tag{6.72}$$

It is interesting to compare this behavior with the density of *generic* generalized single-trace operators of $\mathcal{N}=2$ SCQCD. The density of generic states, unlike the density of protected states, is of course a function of the coupling. For $g=0$, it is obtained by calculating the phase transition temperature of the complete generalized single-trace partition function (with no $(-1)^F$). We find $\sim e^{\beta'_H(\Delta+j)}$ with $\beta'_H = 1.34254$. Not surprisingly, $\beta_H < \beta'_H$. The density of protected states, while exponential, grows at a much slower rate than the density of the generic states, or at least this is the behavior for small g.

6.5.5 Sieve Algorithm

We would like to list the quantum numbers of the extra protected states, up to the finite equivalence class ambiguity intrinsic to the index. There is no closed-form expression for $\mathcal{I}^{s.t.}_{QCD}$ but we can identity the equivalence classes contributing to it in a systematic expansion in powers of t, by implementing a "sieve" algorithm similar in spirit to the one of [141].

The first discrepancy between $\mathcal{I}^{s.t.}_{QCD}$ is the $O(t^{13})$ term

$$\mathcal{I}_{QCD} - \mathcal{I}_{naive} = -\frac{t^{13}}{v}(y + \frac{1}{y}) + \dots \tag{6.73}$$

On the other hand, expanding (6.50) in powers of t, the lowest term is

$$- t^{6+4\tilde{R}+2\tilde{r}} v^{\tilde{r}-\tilde{R}} (y^{2\bar{j}} + \dots + y^{-2\bar{j}}) \,. \tag{6.74}$$

Matching with (6.73) we determine the equivalence class of the first new protected multiplet to be $[\tilde{R}, \tilde{r}, \bar{j}]^{\mathtt{L}}_{+} = [\frac{3}{2}, \frac{1}{2}, \frac{1}{2}]^{\mathtt{L}}_{+}$. Since $\tilde{r} = \bar{j}$, this is actually a $\hat{\mathcal{C}}$ multiplet so we rewrite its equivalence class as $[\hat{R}, \bar{j}]^{\mathtt{L}} = [2, \frac{1}{2}]^{\mathtt{L}}_{+}$. Subtracting the whole index of the class from the discrepancy we proceed to the next mismatch in the t expansion, and so on. In this way, we can systematically construct the equivalence classes of all the extra protected multiplets of the SCQCD. The results from $\mathcal{I}^{\mathtt{L}}$ for first few multiplets are:

- $\mathcal{C}$ multiplets: $[2, 2, 0]^{\mathtt{L}}_{+}$, $[2, 3, 0]^{\mathtt{L}}_{+}$, $[2, 4, 0]^{\mathtt{L}}_{+}$, $[3, 2, 0]^{\mathtt{L}}_{-}$, $[3, 2, 1]^{\mathtt{L}}_{-}$, $\dots$
- $\hat{\mathcal{C}}$ multiplets: $[2, \frac{1}{2}]^{\mathtt{L}}_{+}$, $[4, 1]^{\mathtt{L}}_{+}$, $[4, \frac{3}{2}]^{\mathtt{L}}_{+}$, $\dots$

From the analysis of $\mathcal{I}^{\mathtt{R}}$ we can write down the *right* equivalence classes of the protected multiplets. Since $\mathcal{I}^{\mathtt{R}} = \mathcal{I}^{\mathtt{L}}$, the list of *right* equivalence classes is obtained immediately from the list of *left* equivalence classes by the substitutions $\mathcal{C} \to \bar{\mathcal{C}}$ and $\mathtt{L} \to \mathtt{R}$.

Protected $\bar{\mathcal{C}}$ multiplets are just conjugates of protected $\mathcal{C}$ multiplets. The $\hat{\mathcal{C}}$ multiplets, however, appear in both left and right classes, and as we discussed this gives more information. For example the $\hat{\mathcal{C}}$ multiplet in $[2, \frac{1}{2}]^{\mathtt{L}}_{+}$ also belongs to $[2, \frac{1}{2}]^{\mathtt{R}}_{+}$ and furthermore it is the only multiplet with $\hat{R} = R + j + \bar{j} = 2$. The left equivalence class determines $\bar{j} = \frac{1}{2}$, the right equivalence class $j = \frac{1}{2}$ and both also imply $R = \hat{R} - j - \bar{j} = 1$. This determines the lowest-lying extra protected $\hat{\mathcal{C}}$ multiplet to be $\hat{\mathcal{C}}_{1(\frac{1}{2},\frac{1}{2})}$. For $\hat{R}=4$, there are two multiplets with $\bar{j}=1,\frac{3}{2}$ and with same values of j. Two possible $(j,\bar{j})$ Lorentz spins are $(1,1)$, $(\frac{3}{2},\frac{3}{2})$ or $(1,\frac{3}{2})$, $(\frac{3}{2},1)$ but we also know that it is a bosonic multiplet from the subcript $+$. This picks out the pair $(1,1)$, $(\frac{3}{2},\frac{3}{2})$ with $R=4-1-1=2$ and $R=4-\frac{3}{2}-\frac{3}{2}=1$ respectively. This determines the next protected $\hat{\mathcal{C}}$ multiplets to be $\hat{\mathcal{C}}_{1(\frac{3}{2},\frac{3}{2})}$ and $\hat{\mathcal{C}}_{2(1,1)}$. To summarize, the first three protected $\hat{\mathcal{C}}$ multiplets are:

- $\hat{\mathcal{C}}$ multiplets: $\hat{\mathcal{C}}_{1(\frac{1}{2},\frac{1}{2})}$, $\hat{\mathcal{C}}_{1(\frac{3}{2},\frac{3}{2})}$, $\hat{\mathcal{C}}_{2(1,1)}$, ...

A striking feature of the extra protected multiplets is that they contain states with higher spin, in fact we believe that the sieve will produce arbitrarily high spin. To the best of our knowledge this is the first time that higher-spin protected multiplets are found in an *interacting* 4d superconformal field theory. Note that *none* of the protected states we find are higher spin *conserved currents*, which correspond to the multiplets $\hat{\mathcal{C}}_{0(j,\bar{j})}$. This is not surprising: higher spin conserved currents are the hallmark of a free theory, but $\mathcal{N}=2$ SCQCD is most definitely an interacting quantum field theory. As in $\mathcal{N}=4$ SYM [142], higher spin conserved currents exist at strictly zero coupling, but they are anomalous and recombine into long multiplets at non-zero coupling.

6.6 Dual Interpretation of the Protected Spectrum

As we have repeatedly emphasized, $\mathcal{N}=2$ SCQCD can be obtained as the $\check{g}_{YM}\to 0$ limit of a family of $\mathcal{N}=2$ superconformal field theories, which reduces for $g_{YM}=\check{g}_{YM}$ to the $\mathcal{N}=2$ $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM. This latter theory has a familiar dual description has IIB string theory on $AdS_5\times S^5/\mathbb{Z}_2$ [83], so it would seem that to find the dual of $\mathcal{N}=2$ SCQCD we simply need to follow the fate of the bulk string theory under the exactly marginal deformation. Recall that at the orbifold point the NSNS B-field has half-unit period through the blown-down S^2 of the orbifold singularity, $\int_{S^2} B_{NS}=1/2$ [143]. Taking $\check{g}_{YM}\neq g_{YM}$ is dual to changing the period of B-field, according to the dictionary [84, 144]

$$\frac{1}{g_{YM}^2} + \frac{1}{\check{g}_{YM}^2} = \frac{1}{2\pi g_s} \tag{6.75}$$

$$\frac{\check{g}_{YM}^2}{g_{YM}^2} = \frac{\beta}{1-\beta}\,, \qquad \beta \equiv \int_{S^2} B_{NS}\,. \tag{6.76}$$

The catch is that the limit $\check{g}_{YM} \to 0$ translates on the dual side to the singular limit of vanishing B_{NS} and vanishing string coupling g_s, and the IIB background $AdS_5 \times S^5/\mathbb{Z}_2$ becomes ill-defined. We will study in the next section how to handle this subtle limit. In this section we will try to learn about the string dual of $\mathcal{N} = 2$ SCQCD from the "bottom-up", collecting the clues offered by the spectrum of protected operators. We start by reviewing the well-known bulk-boundary dictionary for the protected states of the orbifold theory.

6.6.1 KK interpretation of the orbifold protected specrum

The untwisted spectrum of the orbifold field theory (summarized in Table 6.3), has a transparent dual interpretation as the Kaluza-Klein spectrum of IIB supergravity on $AdS_5 \times S^5/\mathbb{Z}_2$. It is appropriate to write the metric of $S^5/\mathbb{Z}_2$ as [92]

$$ds^2_{S^5/\mathbb{Z}_2} = d\alpha^2 + \sin^2\alpha\, d\varphi^2 + \cos^2\alpha\, ds^2_{S^3/\mathbb{Z}_2}\,, \qquad 0 \le \varphi \le 2\pi\,, \quad 0 \le \alpha \le \frac{\pi}{2}\,. \tag{6.77}$$

Momentum on S^1 corresponds to the $U(1)_r$ charge r. The $SO(4) \cong SU(2)_L \otimes SU(2)_R$ isometry of the 3-sphere is broken to $SO(3)_L \otimes SU(2)_R$ by the $\mathbb{Z}_2$ orbifold, which projects out harmonics with j_L half-odd. Needless to say, $SU(2)_R$ and $SO(3)_L$ are interpreted as the field theory symmetry groups of the same name, so in particular the right spin j_R is identified with the quantum number R. Finally the harmonics on the α interval are parametrized by an integer n, dual to the power of neutral scalar $\mathcal{T}$ (with $\Delta = 2$) in the schematic expressions of the operators in Table 6.3. It is not difficult to carry an explicit KK expansion and confirm that $\Delta = |r| + 2R + 2n$. A nice shortcut is to consider the KK expansion of the ten dimensional dilaton-axion [92], since only *scalar* harmonics on $S^5/\mathbb{Z}_2$ are required. Scalar harmonics on $S^3/\mathbb{Z}_2$

have $(j_L, j_R) = (2R, 2R)$ with $2R$ a non-negative integer. One finds $\Delta = |r| + 2R + 2n + 4$ [92], as expected from the fact that the KK modes of the dilaton-axion are dual to the descendants obtained by acting with $\mathcal{Q}^4\bar{\mathcal{Q}}^4$ on the superconformal primaries of Table 6.3.

The twisted states of the orbifold field theory (shown in Table 7.3), must map on the dual side to twisted closed string states localized at the fixed locus of the orbifold, which is $AdS_5 \times S^1$, corresponding to $\alpha = \pi/2$ in the parametrization (6.77). The massless twisted states of IIB on the A_1 singularity comprise one massless six-dimensional tensor multiplet, so the KK reduction of the tensor multiplet on $AdS_5 \times S^1$ must reproduce the protected twisted states of the orbifold field theory. It does, as we review in appendix K following the analysis of [145], to which we add a detailed treatment of the zero modes. We find that the zero modes of the tensor multiplet correspond to the multiplet build on the "exceptional state" $\mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}$.

6.6.2 Interpretation for $\mathcal{N} = 2$ SCQCD?

The protected spectrum of $\mathcal{N} = 2$ SCQCD (restricting as usual to flavor singlets, and in the large N Veneziano limit) consists of two sectors: the "naive" list of protected primaries (7.24) easily found by a one-loop calculation in the scalar sector [99]; and the many more extra "exotic" states found in the analysis of the superconformal index.

The "naive" spectrum arises from a truncation of the protected spectrum of the interpolating theory (as $\check{g} \to 0$) to $U(N_f)$ singlets. We have already discussed the reason to focus on the flavor-singlet sector: flavor-singlet operators, which necessarily are of "generalized single-trace type" in the Veneziano limit, are expected to map to single closed string states. The restriction to $U(N_f)$ singlets has an interesting geometric interpretation: flavor singlets are in particular $SU(2)_L$ singlets, and thus they are dual to supergravity states with *no* angular momentum on $S^3/\mathbb{Z}_2$ in the parametrization (6.77). So in performing this restriction we are "losing" three spatial dimensions. As explained around (6.32), the protected primaries of the interpolating theory that are *not* flavor-singlets can be decomposed in the limit $\check{g} \to 0$ as products of "mesonic" operators $(\bar{Q}^{+\hat{+}}Q^{+\hat{+}})^{\check{a}}_{\check{b}}$ and decoupled scalars of the "second" vector multiplet. The dual interpretation in the bulk is that as $\check{g} \to 0$ KK modes on $S^3/\mathbb{Z}_2$ become multi-particle states of open strings. The flavor singlet sector of $\mathcal{N}=2$ SCQCD does not "see" the $S^3/\mathbb{Z}_2$ portion of the geometry. We regard the "loss" of $S^3/\mathbb{Z}_2$ as a first hint that the string dual to the singlet sector of $\mathcal{N}=2$ SCQCD should be a *sub-critical* string background. The S^1 factor on the other hand is preserved.

We may also ignore the relation of $\mathcal{N}=2$ SCQCD with the orbifold theory, and consider the protected states (7.24) at face value: they are immediately suggestive of Kaluza-Klein reduction on a circle. The dual geometry must contain an AdS_5 factor to implement the conformal symmetry, and an S^1 factor to generate the two KK towers dual to $\{\mathrm{Tr}\, T\, \phi^\ell\}$ and $\{\mathrm{Tr}\, \phi^{\ell+2}\}$. Moreover the radii of the AdS_5 and S^1 factor must be equal. Indeed Kaluza-Klein reduction on S^1 gives a mass spectrum $m^2 \sim \ell^2/R^2_{S^1}$ (for ℓ large), and correspondingly a conformal dimension $\Delta \cong m R_{AdS} \cong \ell \frac{R_{AdS}}{R_{S^1}}$. Inspection of (7.24) gives $R_{AdS} = R_{S^1}$. The isometry of S^1 is interpreted as the $U(1)_r$ R-symmetry. On the other hand, there is no hint in the protected spectrum (7.24) of a "geometrically" realized $SU(2)_R$. The relation with the interpolating theory makes it clear that indeed the geometric factor $S^3/\mathbb{Z}_2$, with isometry $SU(2)_R \otimes SO(3)_L$, is lost in the limit $\check{g} \to 0$.

We can further split the "naive" spectrum (7.24) into the primaries $\{\mathrm{Tr}\, \mathcal{M}_{\mathbf{3}}, \mathrm{Tr}\phi^\ell\}$ and the primaries $\{\mathrm{Tr} T\phi^\ell\}$. The first set, of course, is isomorphic to the twisted states of the orbifold, and can be precisely matched with the KK reduction on $AdS_5 \times S^1$ of one tensor multiplet of $(2,0)$ chiral supergravity. A first guess is that the primaries $\{\mathrm{Tr}\, T\phi^\ell\}$ correspond to the KK reduction of the $6d$ $(2,0)$ *gravity* multiplet on $AdS_5 \times S^1$, but this is incorrect. The zero modes of the $6d$ gravity multiplet correctly match the stress-energy tensor multiplet (whose bottom component is the primary $\mathrm{Tr}\, T$), but there are not enough states in the higher KK modes to match the states in the $\mathrm{Tr}\, T\phi^\ell$ for $\ell > 0$. This could have been anticipated by tracing the origin of the states $\{\mathrm{Tr}\, T\phi^\ell\}$ in the orbifold theory: the dual supergravity states have no angular momentum on $S^3/\mathbb{Z}_2$ in the parametrization (6.77), but they are extended in the remaining *seven* dimensions. So a better guess is that the states $\{\mathrm{Tr}\, T\phi^\ell\}$ should have an interpretation in seven-dimensional supergravity.

In summary, with some hindsight, the "naive" spectrum appears to indicate a sub-critical string background, with seven "geometric" dimensions, and containing both an AdS_5 and an S^1 factor, with $R_{AdS} = R_{S^1}$.

The extra exotic protected states teach another important lesson. They arise in the limit $\check{g} \to 0$ from long multiplets on the interpolating theory that hit the unitarity bound and split into short multiplets. In the dual string theory, this means that a fraction of the massive closed string states become massless in the limit $\check{g} \to 0$. It is a substantial enough fraction to give rise to a Hagedorn degeneracy, as we saw in section 6.5.4. This has the crucial implication that *the dual description of $\mathcal{N} = 2$ SCQCD is never in terms of supergravity*, since even in the limit $\lambda \equiv g_{YM}^2 N_c \to \infty$ there is an infinite tower of "light" closed string states, with a mass of the order of the AdS scale. However it seems plausible to conjecture that there is also a second sector of "heavy" string states that decouple for $\lambda \to \infty$.

The picture that we have in mind is the following. There are really two 't Hooft couplings in the interpolating theory, $\lambda \equiv g_{YM}^2 N_c$ and $\check{\lambda} \equiv \check{g}_{YM}^2 N_c$, and correspondingly *two* effective string tensions $T_s \sim 1/l_s^2$ and $\check{T}_s \sim 1/\check{l}_s^2$. The idea of two effective string tensions is intuitive from the spin chain viewpoint, since the bifundamental fields separate different regions of the chain, occupied by adjoint fields of the two different groups $SU(N_c)$ and $SU(N_{\check{c}})$ and thus governed by the two different gauge couplings. At the orbifold point, of course, $\lambda = \check{\lambda}$. In the limit in which the unique 't Hooft coupling of the orbifold theory is sent to infinity the string length goes to zero in AdS units according to the usual AdS/CFT dictionary $R_{AdS_5}/l_s \sim \lambda^{1/4}$, leading to the decoupling of all massive string states. To approach $\mathcal{N} = 2$ SCQCD we are interested in what happens as λ is kept large, but $\check{\lambda}$ is sent to zero. At present we do not know how to modify the AdS/CFT dictionary in this limit. The most naive extrapolation would suggest a hierarchy between two different scales: there should be one sector of closed string states governed by $l_s \sim \lambda^{-1/4} R_{AdS}$ and thus very massive, and another governed by $\check{l}_s \sim R_{AdS}$ and thus light. The latter would correspond to the exotic protected states revealed by the index.

6.7 Brane Constructions and Non-Critical Strings

The interpolating SCFT has a dual description as IIB on $AdS_5 \times S^5/\mathbb{Z}_2$, but this description breaks down in the $\check{g} \to 0$ limit that we wish to study. We must describe the theory in a different duality frame. We will argue that the correct description is in terms of a *non-critical* superstring background. In this section we reconsider the IIB brane setup leading to the interpolating SCFT, and review how it can be T-dualized to a IIA Hanany-Witten setup (see *e.g.* [146] for a review). The T-dual frame allows for a more transparent understanding of the limit $\check{g} \to 0$, as a double-scaling limit in which two brane NS5 collide while the string coupling is sent to zero. In this limit the near-horizon dynamics is described a non-critical string background, which (before the backreaction of the D-branes) admits an exact worldsheet description as $\mathbb{R}^{5,1}$ times $SL(2)_2/U(1)$, the supersymmetric cigar CFT. We are led to identify the near-horizon backreacted background, where D-branes are replaced by flux, with the dual of $\mathcal{N} = 2$ SCQCD.

6.7.1 Brane Constructions

The interpolating SCFT arises at the low-energy limit on N_c D3 branes sitting at the orbifold singularity $\mathbb{R}^2 \times \mathbb{R}^4/\mathbb{Z}_2$. The blow-up modes of the orbifold are set to zero, since they correspond to massive deformations of the 4d field theory. The NSNS period β is related to g_{YM} and $\check{g}_{YM}$ by the dictionary (6.75). As $\beta \to 0$ the D-strings obtained by wrapping $D3$ branes on the blow-down cycle of the orbifold become tensionless and string perturbation theory breaks down. It is useful to T-dualize to a IIA Hanany-Witten description, where the deformation β can be pictured more easily. To perform the T-duality we should first replace the A_1 singularity $\mathbb{R}^4/\mathbb{Z}_2$ with its S^1 compactification, a two-center Taub-NUT space of radius $\tilde{R}$. The local singularity is recovered for $\tilde{R} \to \infty$.

Recall, more generally, that the S^1 compactification of the resolved A_{k-1} singularity is a k-center Taub-NUT, a hyperkäler manifold which can be concretely described as an S^1 fibration of $\mathbb{R}^3$. Let $\tilde{\tau}$ be the coordinate of the S^1 fiber and $\vec{y}$ the coordinates of the $\mathbb{R}^3$ base. The S^1 fiber degenerates to zero size at k points on the base, $\vec{y} = \vec{y}^{(a)}$, $a = 1, \dots k$, and goes to a finite radius $\tilde{R}$

at the infinity of $\mathbb{R}^3$. (Topologically the S^1 is non-trivially fibered over the S^2 boundary of $\mathbb{R}^3$, with monopole charge k.) Rotations of the $\vec{y}$ coordinates are interpreted as the $SU(2)$ symmetry that rotates the complex structures. From the viewpoint of the worldvolume theory of D3 branes probing the singularity, this is the $SU(2)_R$ R-symmetry. The geometry has also an extra $U(1)_L$ symmetry acting as angular rotation in the S^1 fiber.[16] (Finally the $U(1)_r$ of the $4d$ gauge theory corresponds to an isometry outside the Taub-NUT, namely rotations in the $\mathbb{R}^2$ factor of $\mathbb{R}^2 \times \mathbb{R}^4/\mathbb{Z}_2$.)

The metric of a k-center Taub-NUT space has $3(k-1)$ non-trivial hyperkähler moduli (after setting say $\vec{y}^{(1)} \equiv 0$ by an overall translation), which correspond to the blow-up modes of the $(k-1)$ cycles – one $SU(2)_R$ triplet for each cycle. In the string sigma model one needs to further specify the periods of B_{NSNS} and B_{RR} on each cycle,which gives two extra real moduli for each cycle, singlets under $SU(2)_R$. Altogether the $5 = 3+1+1$ moduli for each cycle are the scalar components of a tensor multiplet living in the six transverse directions to the Taub-NUT (or ALE) space. T-duality along the $\tilde{\tau}$ direction yields a string background with non-zero NSNS H flux and non-trivial dilaton, which is interpreted as the background produced by k NS5 branes [113, 147]. The NS5 branes sit at $\vec{y}^a$ in the $\mathbb{R}^3$ directions, and are localized on the dual circle.[17] The NSNS periods map to the relative angles of the NS5 branes on the dual circle.

Let us apply these rules to our case. We start on the IIB side with the configuration

IIB	x_0	x_1	x_2	x_3	x_4	x_5	$\tilde{\tau}$	y_1	y_2	y_3
TN_2							$\times$	$\times$	$\times$	$\times$
D3	$\times$	$\times$	$\times$	$\times$						

The two-center Taub-NUT TN_2 has radius $\tilde{R}$, vanishing blow-up modes ($\vec{y}^{(1)} = \vec{y}^{(2)} = 0$) and $\int_{S^2} B_{NSNS} = \beta$. T-duality gives the IIA configuration

[16] The A_1 singularity ($k = 2$, $\vec{y}_a = 0$, $\tilde{R} = \infty$) has a symmetry enhancement $U(1)_L \to SO(3)_L$, whose field theory manifestation is the $SO(3)_L$ global symmetry of the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM, discussed in section 6.3.2. The symmetry is broken to $U(1)_L$ for finite $\tilde{R}$; the full $SO(3)_L$ is recovered in the infrared.

[17] Naive application of the T-duality rules gives NS5 branes smeared on the dual circle. The localized solution arises after taking into account worldsheet instanton corrections [148].

IIA	x_0	x_1	x_2	x_3	x_4	x_5	τ	y_1	y_2	y_3
2 NS5	×	×	×	×	×	×				
D4	×	×	×	×			×			

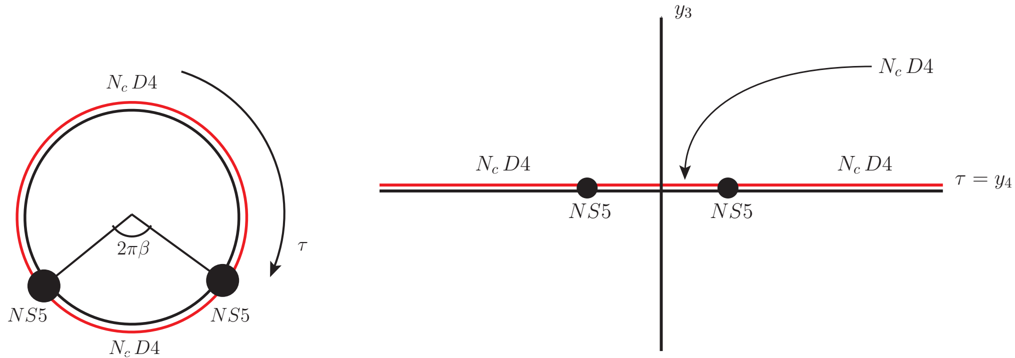


Figure 6.4: Hanany-Witten setup for the interpolating SCFT (on the left) and for $\mathcal{N} = 2$ SCQCD (on the right).

The two NS5 branes, at the origin of $\mathbb{R}^3$ are localized on the dual circle of radius $R = \alpha'/\tilde{R}$ and at an angle $2\pi\beta$ from each other. The string couplings are related as

$$g_s^A = \frac{R}{l_s} g_s^B = \frac{l_s}{\tilde{R}} g_s^B \,. \tag{6.78}$$

T-duality maps the N_c $D3$ branes on the IIB side (which can also be thought as *two stacks* of fractional branes [149]) to two stacks of N_c D4 branes on the IIA side, each stack ending on the two NS5 branes and extended along either arc segment of the τ circle (see Figure 6.4). This is the familiar Hanany-Witten setup for the $\mathbb{Z}_2$ orbifold field theory. The four-dimensional field theory living on the non-compact directions 0123 decouples from the higher dimensional and stringy degrees of freedom in the limit

$$\begin{aligned} & g_s^A \to 0 \quad l_s \to 0\,, \quad R \to 0\,, \\ & \text{with } \frac{\beta R}{2\pi g_s^A l_s} \equiv \frac{1}{g_{YM}^2} \text{ and } \frac{(1-\beta)R}{2\pi g_s^A l_s} \equiv \frac{1}{\check{g}_{YM}^2} \text{ fixed}\,. \end{aligned} \tag{6.79}$$

At this stage we are still keeping both gauge couplings g_{YM} and $\check{g}_{YM}$ finite. If L is the 4d length scale above which the field theory is a good description, we have the hierarchy of scales

$$L \gg l_s \gg R \cong g_s^A l_s \,. \tag{6.80}$$

Again, rotations in the y_i directions correspond to the $SU(2)_R$ R-symmetry of the $\mathcal{N}=2$ $4d$ field theory, while rotations in the 45 plane correspond to the $4d$ $U(1)_r$ symmetry. Finally the $U(1)_L$ symmetry, which was related to momentum conservation along the S^1 fiber in the IIB setup, is T-dualized to *winding* symmetry in the Hanany-Witten IIA setup. It gets enhanced in the infrared to the $SO(3)_L$ symmetry of the $4d$ field theory.

6.7.2 From Hanany-Witten to a Non-Critical Background

The limit $\breve{g}_{YM} \to 0$ (with g_{YM} fixed) can now be understood more geometrically: it corresponds to $\beta \to 0$, the limit of coincident NS5 branes. In this limit we can ignore the periodicity of the τ direction and think of two NS5 branes located in $\mathbb{R}^4$ at a distance $\tau_0 \equiv 2\pi\beta R$ from each other, with $\tau_0 \to 0$. There is a stack of N_c D4 branes suspended between the two NS5s and two stacks of N_c semi-infinite D4s, ending on either NS5 brane. As is well-known, $k \geq 2$ *coincident* NS5 branes generate a string frame background with a strongly coupled near horizon region – the string coupling blows up down the infinite throat towards the location of the branes. The throat region is the CHS background [150]

$$\mathbb{R}^{5,1} \times SU(2)_k \times \mathbb{R}_\rho \,, \quad \text{with dilaton } \Phi = -\frac{\rho}{\sqrt{2k}} \,, \tag{6.81}$$

where ρ is the radial direction (the NS5 branes are located at $\rho = -\infty$). The supersymmetric $SU(2)_k$ WZW model describes the angular S^3; it arises by combining the bosonic $SU(2)_{k-2}$ and three free fermions ψ_i, $i = 1, 2, 3$, which make up an $SU(2)_2$. This description breaks down for large negative ρ where the string coupling e^Φ is large. In Type IIA (our case), we must uplift to M-theory to obtain the correct description of the near horizon region strictly coincident NS5 branes. However, what we are really interested in is bringing the branes together in a controlled fashion, simultaneously turning off the string coupling g_s^A. We can break the limit (6.79) into two steps:

(i) We first take the double scaling limit [116, 117]

$$\tau_o \to 0\,, \quad g_s^A \to 0\,, \quad \frac{\tau_0}{l_s g_s^A} \equiv \frac{1}{g_{eff}} \sim \frac{1}{g_{YM}^2} \text{ fixed}\,, \quad l_s \text{ fixed.} \tag{6.82}$$

(ii) We then send $l_s \to 0$.

Let us first consider the purely closed background without the D4 branes. The double-scaling limit (i) has been studied in detail in [116, 117], precisely with the motivation of avoiding strong coupling. In this limit the region near the location of the NS5 branes decouples from the rest of the geometry and is described by a perfectly regular background of *non-critical* superstring theory [116, 117]. To describe the background as a worldsheet CFT it is useful to perform a further T-duality, in an angular direction around the branes. If $\tau \equiv y_4$ is the direction along which the branes are separated, we pick say the $y_3 y_4$ plane and perform a T-duality around $\chi = \arctan y_3/y_4$. The result is the exact IIB background

$$\mathbb{R}^{5,1} \times SL(2)_2/U(1)/\mathbb{Z}_2\,. \tag{6.83}$$

The $\mathbb{Z}_2$ orbifold implements the GSO projection. The Kazama-Susuki coset $SL(2)_2/U(1)$ is the supersymmetric Euclidean 2d black hole, or supersymmetric cigar, at level $k = 2$. The corresponding sigma-model background is

$$ds^2 = d\rho^2 + \tanh^2(\frac{Q\rho}{2})d\theta^2 + dX^\mu dX_\mu \qquad \theta \sim \theta + \frac{4\pi}{Q} \tag{6.84}$$

$$\Phi = -\ln\cosh(\frac{Q\rho}{2}), \qquad B_{ab} = 0\,. \tag{6.85}$$

In appendix E we review several properties of this background. An equivalent (mirror) description of $SL(2)/U(1)$ is as the $\mathcal{N} = 2$ superLiouville theory [151]. The two descriptions are manifestly equal in the asymptotic region $\rho \to \infty$, where they reduce to ($S^1\times$ linear dilaton). At large ρ, the leading perturbation away from the linear dilaton takes a different form in the semiclassical cigar and Liouville descriptions, but in the complete quantum description both the cigar and Liouville perturbations are present. The cigar description is more appropriate for $k \to \infty$, since in this limit the cigar perturbation dominates at large ρ over the Liouville perturbation, while the Liouville description is more appropriate for $k \to 0$, where the opposite is true. For $k = 2$ both descriptions are precisely on the same footing – the cigar and Liouville perturbations are present with equal strength and are in fact rotated into each another by the $SU(2)_R$ symmetry [125]. For $k = 2$ the asymptotic radius of the cigar is $\sqrt{2\alpha'}$, which is the free fermion radius, implying that for large ρ the angular coordinate θ and its superpartner ψ_θ can then be replaced by three free fermions ψ_i, or equivalently by $SU(2)_2$. The cigar background is thus a smoothed out version of the CHS background (6.81) – the negative ρ region of CHS has been cut-off and the string coupling is now bounded from above by its value g_{eff} at the tip of the cigar.[18]

To summarize, we started from a IIA configuration of two separated NS5 branes in flat space, and took the double-scaling limit (6.82). In this limit the near-horizon region decouples from the asymptotic flat space region, and is described by the exact non-critical IIB background (6.83). (The switch from IIA and IIB is due to the angular T-duality along χ.) The reduction of degrees of freedom from critical to non-critical strings happens because we are focusing on a subsector of the full theory, namely the degrees of freedom near the singularity produced by the colliding NS5 branes. The transverse direction ρ can be thought of as a worldsheet RG scale, with the asymptotically flat region at large ρ playing the role of the UV and the cigar geometry playing the role of the IR – in focusing to the near horizon region we lose the asymptotic flat space degrees of freedom. In particular, what remains of the transverse S^3 is just the "stringy" $SU(2)_2$ associated with the free fermions ψ_i, $i = 1, 2, 3$.

We can easily follow the fate of the D-branes through the double scaling limit and T_χ-duality: the D4 branes suspended between the two NS5s become D3 branes localized at the tip of the cigar, while the semi-infinite D4 branes

[18] As an aside, it is worth recalling the generalization of this discussion to k NS5 branes, equally spaced on a contractible circle in the $y_3 y_4$ plane. T-duality around the angular coordinate χ produces the background [116]

$$\mathbb{R}^{5,1} \times (SL(2)_k/U(1) \times SU(2)_k/U(1))/\mathbb{Z}_k\,. \tag{6.86}$$

The central charges are of the Kazama-Susuki cosets are

$$c(SL(2)_k/U(1)) = 3 + \frac{6}{k}\,, \quad c(SU(2)_k/U(1)) = 3 - \frac{6}{k}\,. \tag{6.87}$$

The CFT (6.86) In the semiclassical limit $k \to \infty$ we have a weakly curved "geometric" $10d$ background, while in the opposite limit $k = 2$ the curvature is string scale, the $SU(2)/U(1)$ piece disappears and we have the "non-critical" string background (6.83).

become D5 branes extended on the cigar. This at least is the intuitive geometric picture. Since the cigar background has string-size curvature near the tip, a more appropriate description of the D-branes is in terms of the exact boundary states. Boundary states for the Kazama-Susuki coset $SL(2)/U(1)$ (equivalently, for the superLiouville CFT) have been studied in several papers [152–156], following the construction of boundary states in bosonic Liouville theory, and used in $\mathcal{N} = 1$ non-critical holography in [107, 108, 110]. There are indeed natural candidates for the two types of cigar D-branes that we need. The branes localized near the tip of the cigar are the analog of Liouville ZZ [157] branes, while the branes extended along the cigar are the analog of the Liouville FZZT [158, 159] branes. The non-critical string setup can be summarized by the following diagram:

IIB	x_0	x_1	x_2	x_3	x_4	x_5	ρ	θ
D3	$\times$	$\times$	$\times$	$\times$				
D5	$\times$	$\times$	$\times$	$\times$			$\times$	$\times$

We could have taken this as our starting point. The theory on the worldvolume of the N_c D3 branes (the "color" branes) reduces for energies much smaller than the string scale to $\mathcal{N} = 2$ $SU(N_c)$ SYM, coupled to $N_f = 2N_c$ hypermultiplets arising from the open strings stretched between the D3s and the "flavor" D5s. This is true by construction, since we obtained this non-critical setup as a limit of a well-known brane realization of the same field theory, and it could also be checked directly, by examining the open string spectrum and preserved supersymmetries.

To decouple the field theory we need to take $l_s \to 0$ (step (ii) in our previous discussion of the field theory limit). This amounts on the gravity side to the near-horizon limit of the geometry produced by the D-branes. By the usual arguments [12], we are led to conjecture that the resulting non-critical string background is dual to $\mathcal{N} = 2$ SCQCD.

6.8 Towards the String Dual of $\mathcal{N} = 2$ SCQCD

The explicit construction of the background after the backreaction of the D-branes is left for future work. In this section we outline a line of attack, based on a $7d$ "effective action" which we identify as maximal supergravity with $SO(4)$ gauging. In fact several features of the background can be determined from symmetry considerations alone, and just assuming that a solution exists we will find a nice qualitative agreement with the bottom-up field theory analysis, notably in the protected spectrum of operators.

6.8.1 Symmetries

Let us start by recapitulating the symmetries. The obvious bosonic symmetries of the closed string background (6.83) (the background before introducing D-branes, henceforth the "cigar background") are the Poincaré group in $\mathbb{R}^{5,1}$ and the $U(1)$ isometry of the θ circle. In fact since as $\rho \to \infty$ the θ circle is at the free fermion radius, there is an asymptotic "stringy" enhancement of the $U(1)$ symmetry to $SU(2)_{\psi_i} \times SU(2)_{\tilde{\psi}_i} \cong SO(4)$. At finite ρ the cigar and super-Liouville interactions break this symmetry to the *diagonal* $SU(2)$. This has a clear geometric interpretation in the HW picture (before the angular T_χ-duality) of the two colliding NS5 branes: the $SO(4)$ symmetry is the isometry of the transverse four directions to two *coincident* NS5 branes; separating the branes along one direction ($\tau = y_4$ in the picture on the right of Figure 4) breaks the symmetry to $SO(3) \cong SU(2)$ (rotations of y_i, $i = 1, 2, 3$). This surviving diagonal $SU(2)$ is interpreted as the $SU(2)_R$ R-symmetry of the $\mathcal{N} = 2$ $4d$ gauge theory. Adding the color D3 branes and the flavor D5 branes breaks the $6d$ Poincaré symmetry to $4d$ Poincaré symmetry in the directions x_m, $m = 0, 1, 2, 3$, times the rotational symmetry in the 45 plane. The latter is interpreted as the $U(1)_r$ R-symmetry of the gauge theory. Note that the branes preserve the same (diagonal) $SU(2)$ as the cigar and super-Liouliville interactions. This is again transparent in the picture of colliding NS5 branes, since both the "compact" D4 branes and the "non-compact" D4 branes, which become respectively the color D3s and the flavor D5s after T_χ-duality, are oriented along the same $\tau = y_4$ direction in which the two NS5s are separated. Finally we should mention the fermionic symmetries. As we review in appendix E, the background (6.83) has 16 real supercharges, corresponding to the $(2, 0)$ Poincaré superalgebra in $\mathbb{R}^{5,1}$. Adding the D-branes breaks the supersymmetry in half, so that 8 Poincaré supercharges survive (that D3s and D5s break the *same* half is again obvious in the T-dual frame where they are both (parallel)

D4 branes). Taking the near-horizon geometry is expected to give the usual supersymmetry enhancement, restoring a total of 16 supercharges that form the $\mathcal{N} = 4$ AdS_5 superalgebra (isomorphic to the $\mathcal{N} = 2$ $4d$ superconformal algebra).

6.8.2 The cigar background and $7d$ maximal $SO(4)$-gauged supergravity

The cigar background (6.83) is analyzed in some detail in appendix E, which the reader is invited to read at this point. Let us summarize some of the relevant points. The physical spectrum of the cigar background consists of: (i) normalizable states localized at the tip of the cigar $\rho \sim 0$, living in $\mathbb{R}^{5,1}$: they fill a tensor multiplet of $(2,0)$ $6d$ supersymmetry; (ii) delta-function normalizable states, corresponding to plane waves in the radial ρ direction; (iii) non-normalizable vertex operators, supported in the large ρ region.

We are only interested in the cigar background as an intermediate step towards the background dual to $\mathcal{N} = 2$ SCQCD, obtained in the near-horizon limit of the D3/D5 brane configuration. A possible strategy is to use the cigar background, which admits an exact CFT description, to derive a spacetime "effective action". The spacetime action is expected to be background independent and should admit as classical solutions *both* the cigar background *and* the background dual to $\mathcal{N} = 2$ SCQCD. (In this respect, the cigar background is analogous to the $10d$ flat background of IIB string theory, which is described at low energies by $10d$ IIB supergravity; *another* solution of IIB supergravity is the $AdS_5 \times S^5$ background dual to $\mathcal{N} = 4$ SYM.) For the purpose of deriving an "effective action" the relevant part of the spectrum is (ii), the continuum of plane-wave states. Performing a KK reduction on the θ circle, the plane-wave states are naturally organized in a tower of increasing $7d$ mass (which gets contribution both from the θ momentum and from string oscillators). There is is no real separation of scales between the lowest mass level and the higher ones, because the linear dilaton has string-size gradient. Nevertheless the states belonging to lowest level are special: although they obey "massive" $7d$ wave-equations, this is an artifact of the linear dilaton; the counting of degrees of freedom is that of massless $7d$ states because of gauge invariances.

Remarkably, we find that for large ρ the lowest-mass level of the continuum spectrum is described by seven dimensional *maximally supersymmetric* supergravity (32 supercharges), but with a non-standard gauging: only an $SO(4)$ of the full $SO(5)$ R-symmetry is gauged. This supergravity has been constructed only quite recently [160, 161]. The maximal supersymmetry (which, as we shall see momentarily, is spontaneously broken to half-maximal, consistently with our previous counting) can be understood as follows. After fermionizing the angular coordinate θ, we have a total of ten left-moving fermions, ψ_μ, $\mu = 0 \dots 5$ along $\mathbb{R}^{5,1}$, ψ_ρ and ψ_i, $i = 1, 2, 3$ (the last three corresponding to $\partial\theta$, ψ_θ), and similarly ten right-moving fermions. So the construction of the lowest-level physical states of our sub-critical theory is entirely isomorphic to the construction of the massless states of the standard critical IIB string theory, except of course that the momenta are now seven dimensional. The $SO(4)$ that is being gauged is the asymptotic $SU(2)_{\psi_i} \times SU(2)_{\tilde{\psi}_i} \cong SO(4)$ that we have mentioned. It turns out that unlike the standard $SO(5)$-gauged $7d$ sugra, which admits the maximally supersymmetric AdS_7 vacuum, the $SO(4)$-gauged theory breaks half of the supersymmetry spontaneously. The scalar potential of the $SO(4)$-gauged theory does not admit a stationary solution but only a domain wall solution [160, 161], which is nothing but the linear dilaton background, with 16 unbroken supercharges – the $6d$ $(2, 0)$ super-Poincaré invariance discussed earlier.

Incidentally, we believe that this is a general phenomenon: non-critical superstrings in various dimensions must admit (non-standard) gauged supergravities as their spacetime "effective actions", in the sense that we have discussed. It may be worth to explore this connection systematically.

6.8.3 An Ansatz

We expect the $SO(4)$-gauged $7d$ sugra that describes the "massless" fields to be a useful tool, though not a perfect one because we know that the higher levels are not truly decoupled. The next step is to look for a solution of this supergravity with all the expected symmetries. In the seven dimensional theory the $SU(2)_R$ symmetry is not realized geometrically – its last remnant was the (string-size) θ circle, over which we have KK reduced to get down to $7d$. On the other hand, the $U(1)_r$ symmetry *is* geometric, and conformal symmetry is expected to arise in the near-horizon geometry, which must then contain both an S^1 and an AdS_5 factor. The most general ansatz for the $7d$ metric with the expected isometries is

$$ds^2 = f(y)ds^2_{AdS_5} + g(y)d\varphi^2 + C(y)dy^2\,. \tag{6.88}$$

Here φ is the angular coordinate of the S^1 associated to $U(1)_r$ isometry, while the y has range in a finite interval, say $y \in [0,1]$. Restoring the θ coordinate, the non-critical background would have the form

$$ds^2 = f(y)ds^2_{AdS_5} + g(y)d\varphi^2 + h(y)d\theta^2 + C(y)dy^2\,. \tag{6.89}$$

Comparing with the brane setup, which is again

IIB	x_0	x_1	x_2	x_3	x_4	x_5	ρ	θ
D3	$\times$	$\times$	$\times$	$\times$				
D5	$\times$	$\times$	$\times$	$\times$			$\times$	$\times$

we identify φ is angular coordinate in the 45 plane, while y could be taken to be a relative angle between the radial distance in the 45 plane and the radial distance ρ along the cigar, $y = \frac{2}{\pi}\arctan(\rho/\sqrt{x_4^2+x_5^2})$. The D5 branes sit at $y = 1$.

The program is then to look for a solution (6.88) of the $SO(4)$-gauged $7d$ supergravity, possibly allowing for singular behavior at the original location $y = 1$ of the flavor branes. For fixed N_c and $N_f(= 2N_c)$, we expect a one-parameter family of solutions, because the 't Hooft coupling λ is exactly marginal – the AdS scale should be a modulus, as in the familiar $AdS_5 \times S^5$ case. The color (D3) branes are magnetically charged under the RR one-form $C^{(2,2)}_{\hat{\mu}}$ (see Table 18) and the flavor branes (which are actually D4 branes from the viewpoint in the $7d$ theory) are magnetically charged under the RR zero-form $C^{(2,2)}$. The corresponding fluxes will be turned on in the solution. As usual the color branes will be completely replaced by flux. Our analysis of the large N Veneziano limit suggests that new effective closed string degrees of freedom, dual to "generalized single-trace" operators, arise from the resummation of open string perturbation theory. This favors the scenario in which also the flavor branes are completely replaced by flux. This fundamental issue would be illuminated by an explicit solution.

The program of finding a supergravity background for $\mathcal{N}=2$ SCQCD was also discussed in critical IIB supergravity [162] and in $11d$ supergravity [39], but no explicit solutions are yet known. It would be interesting to understand the relation of these approaches with our sub-critical setup. In particular a somewhat singular limit of solutions found in [39] should correspond to $\mathcal{N}=2$ SCQCD, and it would be nice to understand this in detail.

6.8.4 Spectrum

Already at this stage we can recognize that the top-down (string theory) and bottom-up (field theory) analyses are in qualitative agreement. Both suggest that the string dual of $\mathcal{N}=2$ SCQCD is a sub-critical background with an AdS_5 and an S^1 factor. In the field theory protected spectrum we found a sharp difference between the $U(1)_r$ and $SU(2)_R$ factors of the R-symmetry group: there are towers of states with increasing $U(1)_r$, but no analogous towers for $SU(2)_R$. The brane construction confirms the natural interpretation of this fact: while the $U(1)_r$ is realized geometrically as the isometry of a "large" S^1_φ, with its towers of KK modes, the $SU(2)_R$ is associated to the string-sized S^1_θ of the cigar (and in fact the very enhancement from the θ isometry $U(1) \subset SU(2)_R$ to the full $SU(2)_R$ is a stringy phenomenon). The "naive" part of the protected spectrum nicely matches:

(i) The multiplets built on the primaries $\{\mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}\,, \mathrm{Tr}\,\phi^{2+\ell}\}$ correspond to the KK modes on S^1_φ of the $6d$ tensor multiplet (see appendix D): these are the truly normalizable states of the cigar background, localized at the tip of the cigar ($y=0$ in the parametrization (6.88)).

(ii) The multiplets built on $\{\mathrm{Tr}\,T\phi^\ell\}$ correspond to the KK modes on S^1_φ of the bulk $7d$ $SO(4)$-gauged supergravity: this is the lowest level of the plane-wave spectrum of the cigar background. While we have not performed a detailed KK reduction, for which the precise geometry is required, it is clear that the bulk graviton maps to the stress tensor, which is part of the $\mathrm{Tr}\,T$ multiplet, and that the ℓ-th KK mode of the graviton maps to the unique spin 2 state in the $\mathrm{Tr}\,T\phi^\ell$ multiplet. Supersymmetry should do the rest.

The "extra" protected states of the field theory must correspond to light string states in the bulk, with mass of order of the AdS scale, but we do not know how to establish a more precise dictionary at this point. We have suggested in section 6.6 that the string theory dual to $\mathcal{N} = 2$ SCQCD may contain two sectors of string states, in correspondence with the two effective string scales l_s and $\check{l}_s$ of the interpolating theory: a light sector, controlled by $\check{l}_s \sim R_{AdS}$ for all λ, and a heavy sector, controlled by $l_s \ll R_{AdS}$ for $\lambda \gg 1$. The string length of the cigar background should be identified with l_s, so the massive string states of the cigar background would correspond to the heavy sector and decouple for large λ. The light sector is more mysterious. A tantalizing speculation is that the light states correspond to cohomology classes with non-normalizable $\mathcal{N} = 2$ Liouville dressing, *i.e.* supported at large ρ (operators of type (iii) in the list of section L.4). It is clearly possible to tune the ρ-momentum to achieve "massless" six-dimensional states, at the expense of making them non-normalizable in the ρ direction. Perhaps the extra protected states of $\mathcal{N} = 2$ SCQCD are somewhat analogous to the discrete states of the $c = 1$ matrix model, which are indeed dual to vertex operators with non-normalizable Liouville dressing.[19]

If indeed $l_s \ll R_{AdS}$ for large λ, the $7d$ supergravity, while not capturing the whole theory even in this limit (as we know from the existence of the extra protected states), may still offer a useful description of a subsector.

6.9 Discussion

We may now look back to section 6.1, at the list of special features shared by all $4d$ CFTs for which an explicit string dual is presently known. We have studied in some detail perhaps the most symmetric theory that violates property (i) (since $a \neq c$ at large N) and property (ii) (since it has a large number of fields in the fundamental representation), while still satisfying the nice simplifying feature (iv) of an exactly marginal coupling λ. We have argued that the dual string theory is not ten dimensional, thus violating (iii), and proposed a

[19] Alternatively, our idea of two effective string scales may be wrong, and the unique scale l_s may be of the order of R_{AdS} for all λ. In this case all anomalous dimensions would remain small for large λ. The extra protected states would be special only in that their anomalous dimension is exactly zero for all λ. This is certainly a logical possibility.

sub-critical string dual in eight dimensions (including the string-size θ). The theory emerges as a limit of a family of superconformal field theories that have $a = c$ and admit ten dimensional string duals. In this singular limit some fields decouple on the field theory side, leading to $a \neq c$, while on the string side two dimensions are lost (counting θ as a dimension). It is tempting to link the two phenomena. The natural speculation is that the $4d$ gauge theories in the "$\mathcal{N} = 4$ universality class" (which among other things are characterized by $a = c$) have $10d$ string dual, while theories with "genuinely" fewer supersymmetries have sub-critical duals. A plausible pattern for (susy —dimension) is ($\mathcal{N}$ —d) = (4—10), (2—8), (1—6), (0—5). We have given evidence for the $\mathcal{N} = 2 \leftrightarrow d = 8$ connection, while [106, 107, 110] focused on $\mathcal{N} = 1 \leftrightarrow d = 6$.

Our example is in harmony with the no-go theorem that $a = c$ for all field theories with an AdS_5 gravity dual, since we argued that even for large λ the supergravity approximation to the dual of $\mathcal{N} = 2$ SCQCD cannot be entirely valid. The imbalance between a and c must arise from higher-curvature terms in the AdS_5 gravity theory [163]. We believe that the stringy origin of these higher curvature terms is the Wess-Zumino action of the flavor branes, as in the example studied in [164, 165]: the flavor Wess-Zumino terms were shown to generate $\mathcal{R}^2$ corrections to the $5d$ Einstein-Hillbert action, contributing at order $O(N_f/N_c)$ to $a - c$. In the example of [164, 165] $N_f \ll N_c$, while in our case $N_f \sim N_c$ and $a - c = O(1)$, but the mechanism must be the same. It is important to keep in mind that the higher-curvature terms from the WZ action are topological in nature and are on a different footing from the higher-curvature corrections due to the closed string sigma-model loops, which are instead suppressed by powers of l_s/R_{AdS}. So there is no contradiction in principle between our suggestion that for large λ the non-critical background has a string length $l_s \ll R_{AdS}$, and the fact that $a - c = O(1)$, since $a - c$ arises from the higher-curvature terms coming from the WZ action, since they are not suppressed.

It is worth pointing out a simple relation between our $\mathcal{N} = 2$ story and the $\mathcal{N} = 1$ story of [106, 107, 110], if we specialize their setup to $\mathcal{N} = 1$ super QCD with $N_f = 2N_c$, the Seiberg self-dual theory. This theory can be viewed as the $\check{g} \to 0$ limit of a family of $\mathcal{N} = 1$ SCFTs with product gauge-group

$SU(N_c) \times SU(N_{\check{c}})$; when the couplings are equal the family reduces to the Klebanov-Witten theory [86], which is dual to $AdS_5 \times T^{1,1}$. This is entirely analogous to the relation between $\mathcal{N} = 2$ SCQCD and the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM, and of course this is not a coincidence: the two-parameter family of $\mathcal{N} = 1$ theories is obtained from the two-parameter family of $\mathcal{N} = 2$ theories flowing in the IR by a *relevant* deformation. For $g = \check{g}$, this is the well-known RG flow from the $\mathbb{Z}_2$ orbifold to the KW theory triggered by $\mathrm{Tr}(\phi^2 - \check{\phi}^2)$ [86]. Unlike the $\mathcal{N} = 2$ family, for $\mathcal{N} = 1$ the couplings are bounded from below and the family of $\mathcal{N} = 1$ SCFTs is never weakly coupled. The exactly marginal coupling of the self-dual $\mathcal{N} = 1$ super QCD is the coefficient of a quartic superpotential – it cannot be taken arbitrarily small but it can be taken arbitrarily large. Our analysis of appendix E should easily generalize to this case, to find the gauged supergravity describing the lightest modes of the continuum spectrum. Only an isolated supergravity solution exists [106] (for arbitrary $N_f \sim N_c$), but in the special case $N_f = 2N_c$ a one-parameter family of solutions is expected. This is also confirmed by the vanishing of the dilaton tadpole when $N_f = 2N_c$ [110]. It would be nice to understand this point better.

Clearly there are many open questions. The bottom-up analysis would be greatly enhanced if we could determine the large λ behavior of generic non-protected operators. This may eventually be possible if $\mathcal{N} = 2$ SCQCD exhibits an all-loop integrable structure. In the next spin-chain chapter [99] we find a preliminary hint of one-loop integrability. In the top-down approach, work is in progress to verify whether the ansatz (6.88) is indeed a solution of the $SO(4)$-gauged supergravity. It will be interesting to understand its physical implications, especially the role of the warping factors and their possible singularity at $y = 1$.

Ultimately an accurate description of the string dual will require the full non-critical sigma-model in RR background. It would be very interesting to start with the sigma-model for $AdS_5 \times S^5/\mathbb{Z}_2$, which can be quantized either in the generalized light-cone gauge or in the pure-spinor formalism, and understand the transition to a non-critical sigma-model in the $\check{g} \to 0$ limit. This may well be the simplest instance of such a transition – we should learn the rules of the game in this highly symmetric example.

Chapter 7

Spin Chains in $\mathcal{N} = 2$ Superconformal Theories

In the previous chapter, we made some progress towards the AdS dual of $\mathcal{N} = 2$ SCQCD. We attacked the problem from two fronts: from the bottom-up, we performed a systematic analysis of the protected spectrum using superconformal representation theory; from the top-down, we considered the decoupling limit of known brane constructions in string theory. We concluded that the string dual is a sub-critical string background with seven geometric dimensions, containing both and AdS_5 and an S^1 factor. In this chapter we take the next step of the bottom-up (=field theory) analysis, by evaluating the one-loop dilation operator in the scalar sector of the theory.

Perturbative calculations of anomalous dimensions have given important clues into the nature of $\mathcal{N} = 4$ SYM. They gave the first hint for integrability of the planar theory: the one-loop dilation operator in the scalar sector is the Hamiltonian of the integrable $SO(6)$ spin chain [15] – a result later generalized to the full theory and to higher loops, using the formalism of the asymptotic Bethe ansatz (see *e.g.* [166–170] for a very incomplete list of references.) Remarkably, the asymptotic S-matrix of magnon excitations in the field theory spin chain can be exactly matched with the analogous S-matrix for the dual string sigma-model. Thus perturbative calculations open a window into the structure of the dual string theory.[1] It is natural to attempt the same strategy

[1]The calculation of the circular Wilson loop by localization techniques [171] is another interesting probe of the dual theory.

for $\mathcal{N}=2$ SCQCD. As explained in the previous chapter, the theory admits a large N expansion in the Veneziano sense [22]: the number of colors N_c and the number of fundamental flavors N_f are both sent to infinity keeping fixed their ratio ($N_f/N_c \equiv 2$ in our case) and the combination $\lambda = g_{YM}^2 N_c$. We focus on the flavor-singlet sector of the theory as before. Let us recall few essentials from the last chapter. A generic color-adjoint field by ϕ^a_b, with $a, b = 1, \dots N_c$, and a generic color-fundamental and flavor-fundamental field by $Q^a_{\ i}$, where $i = 1, \dots N_f$; we are suppressing all other quantum numbers. In the Veneziano limit, single-trace "glueball" operators, of the schematic form $\mathrm{Tr}\,\phi^\ell$, are *not* closed under the action of the dilation operator – this is a major difference with respect to the the standard 't Hooft limit of large N_c with N_f fixed [82]. Rather, glueball operators mix at order one (in the large N counting) with flavor-singlet meson operators of the form $\sum_i \bar{Q}^i \phi^k Q_i$. The simplest example is the mixing of $\mathrm{Tr}(\phi\bar{\phi})$ with the singlet meson $\sum_i \bar{Q}^i Q_i$, which occurs at one-loop in planar perturbation theory (order $O(\lambda)$). The basic "elementary" operators are thus what we call *generalized single-trace* operators, of the schematic form

$$\mathrm{Tr}\left(\phi^{k_1}\mathcal{M}^{\ell_1}\phi^{k_2}\dots\phi^{k_n}\mathcal{M}^{\ell_n}\right)\,, \qquad \mathcal{M}^a_{\ b} \equiv \sum_{i=1}^{N_f} Q^a_{\ i}\,\bar{Q}^i_{\ b}\,, \tag{7.1}$$

where Tr is a color trace. We have introduced a flavor-contracted combination of a fundamental and an antifundamental field, $\mathcal{M}^a_{\ b}$, which for the purpose of the large N expansion plays the role of just another color-adjoint field. The usual large N factorization theorems apply: correlators of generalized multi-traces factorize into correlators of generalized single-traces. In particular, acting with the dilation operator on a generalized single-trace operator yields (at leading order in N) another generalized single-trace operator, so we may consistently diagonalize the dilation operator in the space of generalized single-traces. The dilation operator acting on generalized single-traces can then be interpreted, in the usual fashion, as the Hamiltonian of a closed spin chain. Just as in the 't Hooft limit, planarity of the perturbative diagrams translates into locality of the spin chain: at one-loop the spin chain has only nearest neighbor interactions, at two two-loops there are next-to-nearest neighbors interactions, and each higher loop spreads the range interaction one site further.

More insight is gained by viewing $\mathcal{N} = 2$ SCQCD as part of an "interpolating" $\mathcal{N} = 2$ superconformal field theory (SCFT) that has a product gauge group $SU(N_c) \times SU(N_{\check{c}})$, with $N_{\check{c}} \equiv N_c$, and correspondingly two exactly marginal couplings g and $\check{g}$. For $\check{g} \to 0$ one recovers $\mathcal{N} = 2$ SCQCD *plus* a decoupled free vector multiplet, while for $\check{g} = g$ one finds the familiar $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM. We have evaluated the one-loop dilation operator for the whole interpolating theory, in the sector of operators made out of scalar fields. The magnon excitations of the spin chain and their bound states undergo an interesting evolution as a function of $\kappa = \check{g}/g$. For $\kappa = 0$ (that is, for $\mathcal{N} = 2$ SCQCD itself), the basic asymptotic excitations of the spin chain are linear combinations of the the adjoint impurity $\bar{\phi}$ and of "dimer" impurities $\mathcal{M}^a{}_b$ (we refer to them as dimers since they occupy two sites of the chain). From the point of view of the interpolating theory with $\kappa > 0$, these dimeric asymptotic states of $\mathcal{N} = 2$ SCQCD are *bound states* of two elementary magnons; the bound-state wavefunction localizes in the limit $\kappa \to 0$, giving an impurity that occupies two sites.

Armed with the one-loop Hamiltonian in the scalar sector, we can easily determine the complete spectrum of one-loop protected composite operators made of scalar fields. It is instructive to follow the evolution of the protected eigenstates as a function of κ, from the orbifold point to $\mathcal{N} = 2$ SCQCD. Some of these results were quoted with no derivation in the previous chapter, where they served as input to the analysis of the full protected spectrum, carried out with the help of the superconformal index [19].

An important question is whether the one-loop spin chain of $\mathcal{N} = 2$ SCQCD is integrable. The spin chain for the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM (which by definition has $\check{g} = g$) is known to be integrable [172, 173]. We find that as we move away from the orbifold point integrability is broken, indeed for general $\kappa = \check{g}/g$ the Yang-Baxter equation for the two-magnon S-matrix does not hold. Remarkably however the Yang-Baxter equation is satisfied again in the $\mathcal{N} = 2$ SCQCD limit $\kappa \to 0$. Ordinarily a check of the Yang-Baxter equation is strong evidence in favor of integrability. In our case things are more subtle: the elementary Q excitations freeze in the limit $\kappa \to 0$ (their dispersion relation becomes constant), while some (but not all) of their dimeric bound states retain non-trivial dynamics. Nevertheless, for infinitesimal κ the elementary Qs *are* propagating excitations, and the Yang-Baxter equation fails only infinitesimally, so it seems plausible that one can define consistent Bethe equations by taking small κ as a regulator, to be removed at the end of the calculation. A systematic analysis of this approach is in progress.

In section 7.1.1 we evaluate the one-loop dilation operator of SCQCD (in the scalar sector), and write it as a spin-chain Hamiltonian. In section 7.1.2 we find the spectrum of magnon excitations of this spin chain. These calculations are repeated in sections 7.1.3 and 7.1.4 for the the interpolating SCFT. A simplified derivation of the Hamiltonians is presented in appendix M, while appendix N contains an equivalent way to write the Hamiltonian for $\mathcal{N} = 2$ SCQCD in terms of composite (dimeric) impurities. In section 7.2 we study the spectrum of protected operators of the interpolating theory, and follow its evolution in the limit $\kappa \to 0$. In section 7.3 we solve the two-magnon scattering problem and analyze the spectrum of bound states in the spin chain of the interpolating SCFT, with particular attention to the $\kappa \to 0$ limit. In section 7.3 we check the Yang-Baxter equation for the two-body S-matrix of the interpolating theory, finding that it holds for $\kappa = 1$ and $\kappa \to 0$. We conclude in section 7.4 with a brief discussion of integrability and of future directions of research.

7.1 One-loop Dilation Operator in the Scalar Sector

At large $N_c \sim N_f$, the dilation operator of $\mathcal{N} = 2$ SCQCD can be diagonalized in the sector of generalized single-trace operators, of the form (7.1), indeed the mixing with generalized multi-traces is subleading. Motivated by the success of the analogous calculation in $\mathcal{N} = 4$ SYM [15], we have evaluated the one-loop dilation operator on generalized single-trace operators made out of scalar fields. An example of such an operator is

$$\mathrm{Tr}[\bar{\phi}\phi\phi Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}}\bar{\phi}] = \bar{\phi}^a{}_b\phi^b{}_c\phi^c{}_d Q^d_{\mathcal{I}\,i}\bar{Q}^{\mathcal{J}i}{}_e\bar{\phi}^e{}_a, \quad a,\ldots,e = 1,\ldots N_c\,, \quad i = 1,\ldots N_f \tag{7.2}$$

Since the color or flavor indices of consecutive elementary fields are contracted, we can assign each field to a definite "lattice site"[2] and think of a generalized single-trace operator as a state in a periodic spin chain. In the scalar sector, the state space V_l at each lattice site is six-dimensional, spanned by $\{\phi, \bar{\phi}, Q_{\mathcal{I}}, \bar{Q}^{\mathcal{J}}\}$. However the index structure of the fields imposes restrictions on the total space $\otimes_{l=1}^{L} V_l$: not all states in the tensor product are allowed. Indeed a Q at site l must always be followed by a $\bar{Q}$ at site $l+1$, and viceversa a $\bar{Q}$ must always be preceded by a Q. Equivalently, as in appendix N, we may use instead the color-adjoint objects ϕ, $\bar{\phi}$, $\mathcal{M}_{\mathbf{1}}$ and $\mathcal{M}_{\mathbf{3}}$ (recall the definitions (6.3)), where the $\mathcal{M}$'s are viewed as "dimers" occupying two sites of the chain.

As usual, we may interpret the perturbative dilation operator as the Hamiltonian of the spin chain. It is convenient to factor out the overall coupling from the definition of the Hamiltonian H,

$$\Gamma^{(1)} \equiv g^2 H \,, \qquad g^2 \equiv \frac{\lambda}{8\pi^2} \,, \quad \lambda \equiv g_{YM}^2 N_c \,, \tag{7.3}$$

where $\Gamma^{(1)}$ is the one-loop anomalous dimension matrix. By a simple extension of the usual arguments, the Veneziano double-line notation (see figure M.1 for an example) makes it clear that for large $N_c \sim N_f$ (with λ fixed) the dominant contribution comes from planar diagrams. Planarity implies that the one-loop Hamiltonian is of nearest-neighbor type, $H = \sum_{l=1}^{L} H_{kk+1}$ (with $k \equiv k + L$), where $H_{k,k+1} : V_k \otimes V_{k+1} \to V_k \otimes V_{k+1}$. The two-loop correction is next-to-nearest-neighbor and so on. In section 7.1.1 we present our results for the one-loop Hamiltonian of the spin chain for SCQCD. We then derive (section 7.1.2) the one-particle "magnon" excitations of the infinite chain above the BPS vacuum $\ldots \phi\phi\phi \ldots$. The one-particle eigenstates are interesting admixtures of the adjoint $\bar{\phi}$ impurity and of the "dimeric" $Q\bar{Q}$ impurities.

The generalization to the full interpolating SCFT is straightforward and is carried out in sections 7.1.3 and 7.1.4. The structure of this more general spin chain is in a sense more conventional, and it is somewhat reminiscent of the spin chain [174–177] for the ABJM [178] and ABJ [179] theories. There are two types of color indices, for the two gauge groups $SU(N_c)$ and $SU(N_{\check{c}})$, with adjoint fields $\phi^a_{\ b}$ and $\check{\phi}^{\check{a}}_{\ \check{b}}$ carrying two indices of the same type, and

[2] Up to cyclic re-ordering of course, under which the trace is invariant.

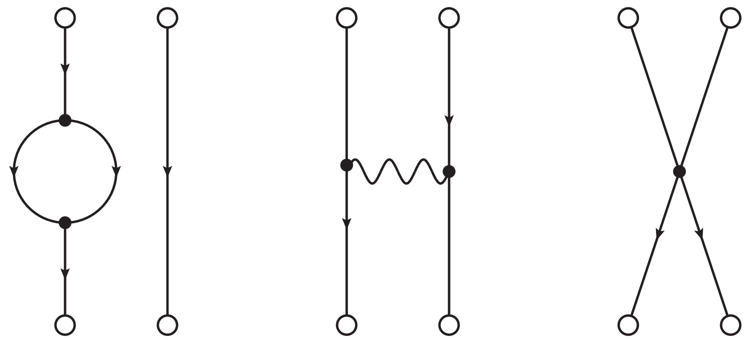


Figure 7.1: Various types of Feynman diagrams that contribute, at one loop, to anomalous dimension. The first diagram is the self-energy contribution. The second diagram represents the gluon exchange contribution whereas the third one stands for the quartic interaction between the fields. The first and the second diagrams are proportional to the identity in the R symmetry space while the third one carries a nontrivial R symmetry index structure.

bifundamental fields $Q^{a}{}_{\check{b}}$ and $\bar{Q}^{\check{a}}{}_{b}$ carrying two indices of opposite type. Of course one must contract neighboring indices of the same type. Now a Q and a $\bar{Q}$ need not be adjacent since they can be separated for $\check{\phi}$ fields. The infinite chain admits two BPS vacua, the state with all ϕs and the state with all $\check{\phi}$s. The magnons are momentum eigenstates containing a single Q or $\bar{Q}$ impurity, separating one BPS vacuum on the left from the other vacuum on the right. We will see in section 7.3 how the "dimeric" $Q\bar{Q}$ impurities of the SCQCD chain arise in the limit $\check{g} \to 0$ from the localization of the bound state wavefunctions of the interpolating chain.

The spin-chain Hamiltonian of the interpolating SCFT violates parity for $\check{g} \neq g$. This is expected from the dual picture, where the difference of the 't Hooft couplings maps to a worldsheet θ angle [84, 144].[3] Parity is restored in the SCQCD spin chain.

7.1.1 Hamiltonian for $\mathcal{N} = 2$ super QCD

We have determined the one-loop dilation operator in the scalar sector by explicit evaluation of the divergent part of all the relevant Feynman diagrams,

[3]A similar parity violation is expected in the spin chain of the ABJ theory, which somewhat surprisingly however appears to be parity invariant to the first non-trivial perturbative order (two loops) [177, 180].

which can be classified as self energy diagrams, gluon interaction diagrams and quartic vertex diagrams and are schematically shown in figure 7.1. The calculation is straightforward and its details will not be reproduced here. In appendix M we present a shortcut derivation that bypasses the explicit evaluation of the self-energy and gluon exchange diagrams, whose contribution can be fixed by requiring the vanishing of the anomalous dimension of certain protected operators.

As we are at it, we may as well consider the case of arbitrary N_f, though we are ultimately interested in the conformal case $N_f = 2N_c$. In the non-conformal case, it is more useful to normalize the fields so that the Lagrangian has an overall factor of $1/g_{YM}^2$ in front [181]. This different normalization affects the anomalous dimension of composite operators for $N_f \neq 2N_c$, which acquire an extra contribution due to the beta function, but it is of course immaterial for $N_f = 2N_c$. It is in this normalization that the chiral operator $\mathrm{Tr}\phi^\ell$ has vanishing anomalous dimension for all N_f.

We find[4]

$$H_{k,k+1} = \tag{7.4}$$

$$\begin{array}{cc}
 & \begin{array}{ccccc} \phi^{\mathfrak{p}}\phi^{\mathfrak{q}} & Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}} & \bar{Q}^{\mathcal{K}}Q_{\mathcal{L}} & \bar{Q}^{\mathcal{I}}\phi^{\mathfrak{p}} & \phi^{\mathfrak{p}}Q_{\mathcal{I}} \end{array} \\
\begin{array}{c} \phi_{\mathfrak{p}'}\phi_{\mathfrak{q}'} \\ \bar{Q}^{\mathcal{I}'}Q_{\mathcal{J}'} \\ \\ Q_{\mathcal{K}'}\bar{Q}^{\mathcal{L}'} \\ \\ Q_{\mathcal{I}'}\phi_{\mathfrak{p}'} \\ \phi_{\mathfrak{p}'}\bar{Q}^{\mathcal{I}'} \end{array} &
\left(\begin{array}{ccccc}
2\delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathfrak{q}}_{\mathfrak{q}'} + g^{\mathfrak{p}\mathfrak{q}}g_{\mathfrak{p}'\mathfrak{q}'} - 2\delta^{\mathfrak{p}}_{\mathfrak{q}'}\delta^{\mathfrak{q}}_{\mathfrak{p}'} & \sqrt{\frac{N_f}{N_c}}g_{\mathfrak{p}'\mathfrak{q}'}\delta^{\mathcal{J}}_{\mathcal{I}} & 0 & 0 & 0 \\
\sqrt{\frac{N_f}{N_c}}g^{\mathfrak{p}\mathfrak{q}}\delta^{\mathcal{I}'}_{\mathcal{J}'} & \begin{array}{c}(2\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'} - \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'})\frac{N_f}{N_c} \\ +\frac{1}{2}(1+\xi)\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'}\end{array} & 0 & 0 & 0 \\
0 & 0 & \begin{array}{c}2\delta^{\mathcal{K}}_{\mathcal{L}}\delta^{\mathcal{L}'}_{\mathcal{K}'} \\ -\frac{1}{2}(1+\xi)\delta^{\mathcal{K}}_{\mathcal{K}'}\delta^{\mathcal{L}'}_{\mathcal{L}}\end{array} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4}(7-\xi)\delta^{\mathcal{I}}_{\mathcal{I}'}\delta^{\mathfrak{p}}_{\mathfrak{p}'} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{4}(7-\xi)\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathfrak{p}}_{\mathfrak{p}'}
\end{array}\right)
\end{array}$$

The indices $\mathfrak{p}, \mathfrak{q} = \pm$ label the $U(1)_r$ charges of ϕ and $\bar{\phi}$, in other terms we have defined $\phi^- \equiv \phi$, $\phi^+ \equiv \bar{\phi}$, and $g_{\mathfrak{pq}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The parameter ξ is the gauge parameter that appears in the gluon propagator as $\frac{1}{k^2}(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2})$. Although the form of nearest-neighbor Hamiltonian depends on gauge choice

[4] The spin chain with this nearest-neighbor Hamiltonian reproduces the one-loop anomalous dimension of all operators with $L > 2$, where L is the number of sites. The $L = 2$ case is special: the double-trace terms in the scalar potential, which give subleading contributions (at large N) for $L > 2$, become important for $L = 2$ and must be added separately. This special case plays a role in the protection of $\mathrm{Tr}\boldsymbol{\mathcal{M}_3}$, see section 7.2.

ξ, it is easy to check that ξ dependence drops when H acts on a closed chain. In the following we will set $\xi = -1$.[5]

We may rewrite H_{kk+1} more concisely (we have set $\xi = -1$) as

$$H_{k,k+1} = \begin{array}{c} \\ \phi\phi \\ \bar{Q}Q \\ Q\bar{Q} \\ Q\phi \\ \phi\bar{Q} \end{array} \!\!\! \begin{array}{c} \begin{array}{ccccc} \phi\phi & Q\bar{Q} & \bar{Q}Q & \bar{Q}\phi & \phi Q \end{array} \\ \begin{pmatrix} 2\mathbb{I} + \mathbb{K} - 2\mathbb{P} & \sqrt{\frac{N_f}{N}} & 0 & 0 & 0 \\ \sqrt{\frac{N_f}{N}} & (2\mathbb{I} - \mathbb{K})\frac{N_f}{N_c} & 0 & 0 & 0 \\ 0 & 0 & 2\mathbb{K} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \end{array} \tag{7.5}$$

The symbols $\mathbb{I}, \mathbb{P}$ and $\mathbb{K}$ for identity, permutation and trace operators respectively. Their position in the matrix specifies the space in which they act. For example, the operator $\mathbb{P}$ that appears in the matrix element of $\langle \phi_{\mathtt{p}'}\phi_{\mathtt{q}'}|\phi^{\mathtt{p}}\phi^{\mathtt{q}}\rangle$ is $\delta^{\mathtt{p}}_{\mathtt{q}'}\delta^{\mathtt{q}}_{\mathtt{p}'}$, the operator $\mathbb{K}$ that appears in the matrix element $\langle \bar{Q}^{\mathcal{I}'}Q_{\mathcal{J}'}|Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}}\rangle$ stands for the operator $\delta^{\mathcal{I}'}_{\mathcal{J}'}\delta^{\mathcal{J}}_{\mathcal{I}}$ and so on. The entries where no symbols appear have an unambiguous index structure.

Although not obvious from the form (7.5), the Hamiltonian of the SCQCD spin chain preserves parity, once the constraints on the states allowed by the index structure are taken into account. In appendix B we rewrite the Hamiltonian in terms of composite (dimeric) impurities, and parity is then manifest.

For $N_f = 0$, the Hamiltonian can be consistently truncated to the space of ϕ (and $\bar{\phi}$): it reduces $2\mathbb{I}_{\phi\phi} + \mathbb{K}_{\phi\phi} - 2\mathbb{P}_{\phi\phi}$, which is Hamiltonian of the XXZ spin chain, confirming the result found in [181] for pure $\mathcal{N} = 2$ SYM. The $N_f \neq 0$ the ϕ sector is not closed in our case due to the leading order glueball-meson mixing.

[5]This choice corresponds to setting to zero the self-energy of Q and $\bar{Q}$. Then our Hamiltonian can also be used as is to calculate the anomalous dimension of operators with open *flavor* indices, of the schematic form $\bar{Q}^i \dots Q_j$. For $\xi \neq -1$ there are extra contributions form the self-energy of the Q^i and $\bar{Q}_j$ at the edge of the chain.

7.1.2 Magnons in the SCQCD spin chain

The chiral operator Tr ϕ^ℓ and the antichiral operator Tr $\bar{\phi}^\ell$ are zero-energy eigenstates (in particular the mixing element that is responsible for $\phi\phi \to QQ$ is proportional to $\mathbb{K}$ in ϕ space, and thus vanishes when two neighboring ϕ fields have the same $U(1)_r$ index). They correspond to the two ferromagnetic ground states of the spin chain (all spins up or all down). We choose for definiteness the chiral vacuum Tr ϕ^ℓ. Recall that in our conventions the $U(1)_r$ charge of ϕ is $r = -1$, so the ground state obeys $\Delta + r = 0$, where Δ is the total conformal dimension. Both Q and $\bar{Q}$ have $\Delta + r = 1$, but the index structure forbids the insertion of only one of them. The simplest impurities that can be excited on the ground state are $\bar{\phi}$, $\mathcal{M}_{\mathbf{1}}$ and $\mathcal{M}_{\mathbf{3}}$, where the last two are "dimeric" impurities which occupy two sites (recall (6.19)). All of them have $\Delta + r = 2$, and should be viewed in this sense as double excitations, though they are the most elementary we can find in the spin chain for $\mathcal{N} = 2$ SCQCD. We will see that they can be viewed as bound states of the elementary impurities of the interpolating theory with $\check{g} \neq 0$. This hidden compositeness makes the scattering problem somewhat harder than usual.

In the map from the (generalized) single-trace operators to the states of the spin chain, cyclycity of the trace gives periodic boundary conditions on the chain, along with the constraint that the total momentum of all the impurities in the spin be zero. As usual, it is convenient to first consider the chain to be infinite, and impose later the zero-momentum constraint on multi-impurity states. We now proceed to diagonalize the Hamiltonian on the space of states containing a single impurity (which in the present context means a single $\bar{\phi}$ or $\mathcal{M}_{\mathbf{1}}$ or $\mathcal{M}_{\mathbf{3}}$). The action of H on single impurities in position space is

$$\begin{aligned}
H[\bar{\phi}(x)] &= 6\bar{\phi}(x) - \bar{\phi}(x+1) - \bar{\phi}(x-1) &(7.6)\\
&\quad + \sqrt{\frac{2N_f}{N_c}}\mathcal{M}_{\mathbf{1}}(x) + \sqrt{\frac{2N_f}{N_c}}\mathcal{M}_{\mathbf{1}}(x-1) &(7.7)\\
H[\mathcal{M}_{\mathbf{1}}(x)] &= 4\mathcal{M}_{\mathbf{1}}(x) + \sqrt{\frac{2N_f}{N_c}}\bar{\phi}(x) + \sqrt{\frac{2N_f}{N_c}}\bar{\phi}(x+1) &\\
H[\mathcal{M}_{\mathbf{3}}(x)] &= 8\mathcal{M}_{\mathbf{3}}(x)\,, &(7.8)
\end{aligned}$$

where the coordinate x denotes the site of the impurity on the chain; for the dimeric impurities $\mathcal{M}_{\mathbf{1}}$ and $\mathcal{M}_{\mathbf{3}}$ we use the coordinate of the first site. To diagonalize the Hamiltonian on the $\bar{\phi}/\ \mathcal{M}_{\mathbf{1}}$ sector, we go to momentum space,

$$\begin{aligned}
\bar{\phi}(p) &\equiv \sum_x \bar{\phi}(x)e^{ipx}\,, \quad \mathcal{M}_{\mathbf{1}}(p) \equiv \sum_x \mathcal{M}_{\mathbf{1}}(x)e^{ipx} && (7.9)\\
H\begin{pmatrix} \bar{\phi}(p) \\ \mathcal{M}_{\mathbf{1}} \end{pmatrix} &= \begin{pmatrix} 6-e^{ip}-e^{-ip} & (1+e^{-ip})\sqrt{\frac{2N_f}{N_c}} \\ (1+e^{ip})\sqrt{\frac{2N_f}{N_c}} & 4 \end{pmatrix}\begin{pmatrix} \bar{\phi}(p) \\ \mathcal{M}_{\mathbf{1}} \end{pmatrix} && (7.10)
\end{aligned}$$

The expressions for the eigenvalues and eigenvectors are not very illuminating for generic values of the ratio N_f/N_c. For the conformal case of $N_f = 2N_c$, however, they simplify. The eigenstates for $N_f = 2N_c$ are

$$\begin{aligned}
T(p) &\equiv -\frac{1}{2}(1+e^{-ip})\bar{\phi}(p) + \mathcal{M}_{\mathbf{1}}(p) = \sum_x e^{ipx}[-\frac{1}{2}(\bar{\phi}(x)+\bar{\phi}(x+1)) + \mathcal{M}_{\mathbf{1}}(x)] && (7.11)\\
\widetilde{T}(p) &\equiv \bar{\phi}(p) + \frac{1}{2}(1+e^{ip})\mathcal{M}_{\mathbf{1}}(p) = \sum_x e^{ipx}[\bar{\phi}(x) + \frac{1}{2}(\mathcal{M}_{\mathbf{1}}(x)+\mathcal{M}_{\mathbf{1}}(x-1))]\,, && (7.12)
\end{aligned}$$

with eigenvalues

$$\begin{aligned}
HT(p) &= 4\sin^2(\frac{p}{2})\,T(p) && (7.13)\\
H\widetilde{T}(p) &= 8\,\widetilde{T}(p)\,. && (7.14)
\end{aligned}$$

Interestingly, precisely at the conformal point $N_f = 2N_c$ the magnon excitation $T(p)$ becomes gapless: in general the gap of $T(p)$ is $4-2\sqrt{2N_f/N_c}$. From now on we will only consider the superconformal case and set $N_f \equiv 2N_c$. Besides $T(p)$ and $\widetilde{T}(p)$, we have of course also the $\mathcal{M}_{\mathbf{3}}$ momentum eigenstate,

$$\mathcal{M}_{\mathbf{3}}(p) \equiv \sum_x \mathcal{M}_{\mathbf{3}}(x)e^{ipx}\,, \tag{7.15}$$

which has the same momentum-independent energy as $\widetilde{T}(p)$,

$$H\mathcal{M}_{\mathbf{3}}(p) = 8\,\mathcal{M}_{\mathbf{3}}(p)\,. \tag{7.16}$$

7.1.3 Hamiltonian for the interpolating SCFT

We have generalized the calculation of the one-loop dilation operator to the full interpolating family of $\mathcal{N}=2$ SCFTs, in the scalar sector. We find

$$
\begin{array}{rcl}
H & = & \begin{array}{c} \\ \phi_{\mathfrak{p}'}\phi_{\mathfrak{q}'} \\ \bar{Q}^{\hat{\mathcal{I}}'\mathcal{I}'}Q_{\mathcal{J}'\hat{\mathcal{J}}'} \end{array}
\begin{array}{c}
\begin{array}{cc} \phi^{\mathfrak{p}}\phi^{\mathfrak{q}} & Q_{\mathcal{I}\hat{\mathcal{I}}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{J}} \end{array} \\
\begin{pmatrix}
(2\delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathfrak{q}}_{\mathfrak{q}'} + g^{\mathfrak{p}\mathfrak{q}}g_{\mathfrak{p}'\mathfrak{q}'} - 2\delta^{\mathfrak{p}}_{\mathfrak{q}'}\delta^{\mathfrak{q}}_{\mathfrak{p}'}) & \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\, g_{\mathfrak{p}'\mathfrak{q}'} \\
\delta^{\mathcal{I}'}_{\mathcal{J}'}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{J}}'}g^{\mathfrak{p}\mathfrak{q}} & (2\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'} - \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'})\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{J}}'} + 2\kappa^2\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{I}}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{J}}'}
\end{pmatrix}
\end{array} \\[2em]
& \oplus & \begin{array}{c} \\ \check{\phi}_{\mathfrak{p}'}\check{\phi}_{\mathfrak{q}'} \\ Q_{\mathcal{J}'\hat{\mathcal{J}}'}\bar{Q}^{\hat{\mathcal{I}}'\mathcal{I}'} \end{array}
\begin{array}{c}
\begin{array}{cc} \check{\phi}^{\mathfrak{p}}\check{\phi}^{\mathfrak{q}} & \bar{Q}^{\hat{\mathcal{J}}\mathcal{J}}Q_{\mathcal{I}\hat{\mathcal{I}}} \end{array} \\
\begin{pmatrix}
\kappa^2(2\delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathfrak{q}}_{\mathfrak{q}'} + g^{\mathfrak{p}\mathfrak{q}}g_{\mathfrak{p}'\mathfrak{q}'} - 2\delta^{\mathfrak{p}}_{\mathfrak{q}'}\delta^{\mathfrak{q}}_{\mathfrak{p}'}) & \kappa^2\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\, g_{\mathfrak{p}'\mathfrak{q}'} \\
\kappa^2\delta^{\mathcal{I}'}_{\mathcal{J}'}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{J}}'}g^{\mathfrak{p}\mathfrak{q}} & \kappa^2(2\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'} - \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'})\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{J}}'} + 2\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{I}}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{J}}'}
\end{pmatrix}
\end{array} \\[2em]
& \oplus & \begin{array}{c} \\ \phi_{\mathfrak{p}'}\bar{Q}^{\hat{\mathcal{I}}'\mathcal{I}'} \\ \bar{Q}^{\hat{\mathcal{I}}'\mathcal{I}'}\check{\phi}_{\mathfrak{p}'} \end{array}
\begin{array}{c}
\begin{array}{cc} \phi^{\mathfrak{p}}Q_{\mathcal{I}\hat{\mathcal{I}}} & Q_{\mathcal{I}\hat{\mathcal{I}}}\check{\phi}^{\mathfrak{p}} \end{array} \\
\begin{pmatrix}
2\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{I}}}\delta^{\mathfrak{p}}_{\mathfrak{p}'} & -2\kappa\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{I}}}\delta^{\mathfrak{p}}_{\mathfrak{p}'} \\
-2\kappa\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{I}}}\delta^{\mathfrak{p}}_{\mathfrak{p}'} & 2\kappa^2\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{I}}}\delta^{\mathfrak{p}}_{\mathfrak{p}'}
\end{pmatrix}
\end{array} \\[2em]
& \oplus & \begin{array}{c} \\ \check{\phi}_{\mathfrak{p}'}Q_{\mathcal{J}'\hat{\mathcal{J}}'} \\ Q_{\mathcal{J}'\hat{\mathcal{J}}'}\phi_{\mathfrak{p}'} \end{array}
\begin{array}{c}
\begin{array}{cc} \check{\phi}^{\mathfrak{p}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{J}} & \bar{Q}^{\hat{\mathcal{J}}\mathcal{J}}\phi^{\mathfrak{p}} \end{array} \\
\begin{pmatrix}
2\kappa^2\delta^{\mathcal{J}}_{\mathcal{J}'}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{J}}'}\delta^{\mathfrak{p}}_{\mathfrak{p}'} & -2\kappa\delta^{\mathcal{J}}_{\mathcal{J}'}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{J}}'}\delta^{\mathfrak{p}}_{\mathfrak{p}'} \\
-2\kappa\delta^{\mathcal{J}}_{\mathcal{J}'}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{J}}'}\delta^{\mathfrak{p}}_{\mathfrak{p}'} & 2\delta^{\mathcal{J}}_{\mathcal{J}'}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{J}}'}\delta^{\mathfrak{p}}_{\mathfrak{p}'}
\end{pmatrix}
\end{array}
\end{array}
\tag{7.17}
$$

In concise form,[6]

$$
H_{k,k+1} =
$$

$$
\begin{array}{c}
\\ \phi\phi \\ Q\bar{Q} \\ \check{\phi}\check{\phi} \\ \bar{Q}Q \\ \phi Q \\ Q\check{\phi} \\ \check{\phi}\bar{Q} \\ \bar{Q}\phi
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} \phi\phi & Q\bar{Q} & \check{\phi}\check{\phi} & \bar{Q}Q & \phi Q & Q\check{\phi} & \check{\phi}\bar{Q} & \bar{Q}\phi \end{array} \\
\begin{pmatrix}
(2+\mathbb{K}-2\mathbb{P}) & \mathbb{K} & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{K} & (2-\mathbb{K})\hat{\mathbb{K}}+2\kappa^2\mathbb{K} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \kappa^2(2+\mathbb{K}-2\mathbb{P}) & \kappa^2\mathbb{K} & 0 & 0 & 0 & 0 \\
0 & 0 & \kappa^2\mathbb{K} & \kappa^2(2-\mathbb{K})\hat{\mathbb{K}}+2\mathbb{K} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -2\kappa & 0 & 0 \\
0 & 0 & 0 & 0 & -2\kappa & 2\kappa^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2\kappa^2 & -2\kappa \\
0 & 0 & 0 & 0 & 0 & 0 & -2\kappa & 2
\end{pmatrix}
\end{array}
$$

where

$$
\kappa \equiv \frac{\check{g}}{g}\,, \quad g^2 \equiv \frac{g_{YM}^2 N}{8\pi^2}\,, \quad \check{g}^2 \equiv \frac{\check{g}_{YM}^2 N}{8\pi^2}\,. \tag{7.18}
$$

It is easy to check that in the limit $\kappa \to 0$ this Hamiltonian reduces to that of the SCQCD spin chain, as it should.[7]

---

[6]The meaning of the different operators can be read off by comparing with the explicit form above. Note in particular that to avoid cluttering we have dropped the identity symbol $\mathbb{I}$. Also in the subspaces $Q\bar{Q}$, $\bar{Q}Q$ we use the notation $\mathbb{K}$ for the trace operator acting on $SU(2)_R$ indices and $\hat{\mathbb{K}}$ that acts on the $SU(2)_L$ indices.

[7]In the comparison, it is important to take into account the factors that arise by normalizing to one the tree-level two-point function. Recall that in SCQCD $\bar{Q}_i Q^i$ is contracted

The Hamiltonian can also be compactly written in terms of the $\mathbb{Z}_2$-projected $SU(2N_c)$ adjoint fields Z and $\mathcal{X}$,

$$Z = \begin{pmatrix} \phi & 0 \\ 0 & \check{\phi} \end{pmatrix}, \qquad \mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}} = \begin{pmatrix} 0 & Q_{\mathcal{I}\hat{\mathcal{I}}} \\ -\epsilon_{\mathcal{I}\mathcal{J}}\epsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{J}} & 0 \end{pmatrix}. \tag{7.19}$$

In this notation,

$$g^2 H = \begin{array}{c} \begin{array}{cccc} ZZ & \mathcal{X}\mathcal{X} & Z\mathcal{X} & \mathcal{X}Z \end{array} \\ \begin{pmatrix} (g_+ + \gamma g_-)^2(2+\mathbb{K}-2\mathbb{P}) & (g_+ + \gamma g_-)^2\mathbb{K}\hat{\mathbb{K}} & 0 & 0 \\ (g_+ + \gamma g_-)^2\mathbb{K}\hat{\mathbb{K}} & \begin{array}{c}(g_+ + \gamma g_-)^2(2\hat{\mathbb{K}} - \mathbb{K}\hat{\mathbb{K}}) \\ +2(g_+ - \gamma g_-)^2\mathbb{K}\end{array} & 0 & 0 \\ 0 & 0 & 2(g_+ + \gamma g_-)^2 & -2({g_+}^2 - {g_-}^2) \\ 0 & 0 & -2({g_+}^2 - {g_-}^2) & 2(g_+ - \gamma g_-)^2 \end{pmatrix} \end{array} \tag{7.20}$$

where γ is the twist operator (8.17), and we have defined $g_\pm \equiv (g \pm \check{g})/2$. Parity is broken for $g \neq \check{g}$ by the terms odd in the twist operator. Although not obvious from this form of the Hamiltonian, parity is actually restored for SCQCD ($\check{g} = 0$), as seen most clearly in the dimer picture of appendix B.

7.1.4 Magnons in the interpolating spin chain

The spin chain of the interpolating SCFT admits two degenerate chiral vacua with $\Delta + r = 0$, namely Tr ϕ^ℓ and Tr $\check{\phi}^\ell$. The elementary impurities are Q and $\bar{Q}$, which have $\Delta + r = 1$. In the infinite chain it makes sense to consider states with a single impurity. A single Q impurity separates the ϕ vacuum to its left from the $\check{\phi}$ vacuum on its right; viceversa for a $\bar{Q}$ impurity.

The action of the Hamiltonian on a single Q impurity in position space is

$$g^2 H Q_{\mathcal{I}\hat{\mathcal{I}}}(x) = 2(g^2 + \check{g}^2)Q_{\mathcal{I}\hat{\mathcal{I}}}(x) - 2g\check{g}[Q_{\mathcal{I}\hat{\mathcal{I}}}(x-1) + Q_{\mathcal{I}\hat{\mathcal{I}}}(x+1)] \tag{7.21}$$

summing over the $N_f = 2N_c$ flavors, while in the interpolating SCFT $\bar{Q}_{\check{a}}Q^{\check{a}}$ is contracted summing over the N_c colors (leaving open the $SU(2)_L$ indices).

Fourier transforming as $Q(p) = \sum_x e^{ipx} Q(x)$ we have

$$\begin{aligned} g^2 H Q_{\mathcal{I}\hat{\mathcal{I}}}(p) &= 2(g^2 + \check{g}^2 - 2g\check{g}\cos p) Q_{\mathcal{I}\hat{\mathcal{I}}}(p) \\ &= [2(g-\check{g})^2 + 4g\check{g}(1-\cos p)] Q_{\mathcal{I}\hat{\mathcal{I}}}(p) \\ &= [2(g-\check{g})^2 + 8g\check{g}\sin^2(\frac{p}{2})] Q_{\mathcal{I}\hat{\mathcal{I}}}(p) \end{aligned} \tag{7.22}$$

Hence the dispersion relation for $Q_{\mathcal{I}\hat{\mathcal{I}}}(p)$ is,

$$E(p;\kappa) = 2(1-\kappa)^2 + 8\kappa \left(\sin^2 \frac{p}{2}\right) . \tag{7.23}$$

The magnon is gapless at the orbifold point $\kappa = 1$, and it develops a gap as we move towards SCQCD. Precisely at the SCQCD point, the single impurity state ceases to be meaningful and its dispersion relation trivializes. An identical analysis holds for the $\bar{Q}$ impurity, leading to the same dispersion relation.

7.2 Protected Spectrum

In this section we put to use the one-loop Hamiltonian to study the protected spectrum of $\mathcal{N} = 2$ SCQCD and of the interpolating SCFT. The results presented here were quoted without proof and used in the previous chapter. The remainder of the present chapter is independent of this section, and readers mainly interested in dynamics and integrability of the spin chain may proceed directly to section 7.3.

We are going to determine all the generalized single-trace operators in the scalar sector of SCQCD having vanishing one-loop anomalous dimension. We find the complete list of such operators to be:[8]

$$\mathrm{Tr}\,\phi^{k+2}, \qquad \mathrm{Tr}[T\phi^k], \qquad \mathrm{Tr}\mathcal{M}_{\mathbf{3}}. \tag{7.24}$$

Here, $T \equiv \phi\bar{\phi} - \mathcal{M}_{\mathbf{1}}$ and $k \geq 0$. We are first led to (7.24) by an educated guess. In section H we list all operators in the scalar sector that obey any of the the shortening or semi-shortening conditions of the $\mathcal{N} = 2$ supercon-

[8]As explained in [2], $\mathcal{N} = 2$ SCQCD has a second class of protected operators, which are outside the scalar sector.

formal algebra, which have been completely classified [52, 182–185]. Using the spin-chain Hamiltonian, we compute the one-loop anomalous dimension of these candidate protected states, and find that only (7.24) have zero anomalous dimension. Even though here we only perform a one-loop analysis, the operators (7.24) can be seen to be protected at full quantum level using the superconformal index [2].

In section 7.2.2, we list the protected operators of the orbifold theory (they can be exhaustively enumerated by a variety of methods [2]) and follow their evolution along the exactly marginal line κ.

7.2.1 Protected spectrum in $\mathcal{N}=2$ SCQCD

A generic long multiplet $\mathcal{A}^{\Delta}_{R,r(j,\bar{j})}$ of the $\mathcal{N}=2$ superconformal algebra is generated by the action of the 8 Poincaré supercharges $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ on a superconformal primary, which by definition is annihilated by all conformal supercharges $\mathcal{S}$. If some combination of the $\mathcal{Q}$'s also annihilates the primary, the corresponding multiplet is shorter and the conformal dimensions of all its members are protected against quantum corrections. We follow the conventions of [52] for the possible shortening conditions for the $\mathcal{N}=2$ superconformal algebra, see table H.1.

In table 7.1 we list all the generalized single-trace operators of $\mathcal{N}=2$ SCQCD made out of scalar fields, which obey any of the possible shortening conditions. Using the spin-chain Hamiltonian of section 7.1.1, we find that the only operators with zero anomalous dimension are the one listed in (7.24)[9]. The operators $\mathrm{Tr}\,\phi^{\ell}$ correspond to the vacuum of the spin chain, while the operators $\mathrm{Tr}\,T\,\phi^{\ell}$ correspond to the zero-momentum limit of the gapless excitation $T(p)$, eq. (7.13) . There is one more protected operator, which is "exceptional" in not belonging to an infinite sequence: $\mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}$. Its anomalous dimension is zero for gauge group $SU(N_c)$ but not for gauge group $U(N_c)$: the double-trace terms in the Lagrangian that arise from the removal of the $U(1)$ are crucial for the protection of this operator (see footnote at page 133).

[9] Together of course with their conjugates. Note that since in our conventions ϕ has $r=-1$, the multiplet $\bar{\mathcal{E}}_{-\ell(0,0)}$, $\ell>0$, is represented by $\mathrm{Tr}\phi^{\ell}$. The conjugate multiplet $\mathcal{E}_{\ell(0,0)}$ is represented by $\mathrm{Tr}\bar{\phi}^{\ell}$ and is of course also protected.

Scalar Multiplets	SCQCD operators	Protected
$\bar{\mathcal{B}}_{R,-\ell(0,0)}$	$\mathrm{Tr}[\phi^\ell \mathcal{M}_3^R]$	
$\bar{\mathcal{E}}_{-\ell(0,0)}$	$\mathrm{Tr}[\phi^\ell]$	Yes
$\hat{\mathcal{B}}_R$	$\mathrm{Tr}[\mathcal{M}_3^R]$	Yes for $R=1$
$\bar{\mathcal{C}}_{R,-\ell(0,0)}$	$\mathrm{Tr}[T\mathcal{M}_3^R\phi^\ell]$	
$\bar{\mathcal{C}}_{0,-\ell(0,0)}$	$\mathrm{Tr}[T\phi^\ell]$	Yes
$\hat{\mathcal{C}}_{R(0,0)}$	$\mathrm{Tr}[T\mathcal{M}_3^R]$	
$\hat{\mathcal{C}}_{0(0,0)}$	$\mathrm{Tr}[T]$	Yes
$\bar{\mathcal{D}}_{R(0,0)}$	$\mathrm{Tr}[\mathcal{M}_3^R\phi]$	

Table 7.1: $\mathcal{N}=2$ SCQCD protected operators at one loop

7.2.2 Protected spectrum in the orbifold theory

$\mathcal{N}=2$ SCQCD can be obtained as the $\check{g}_{YM}\to 0$ limit of a family of $\mathcal{N}=2$ superconformal field theories, which reduces for $g_{YM}=\check{g}_{YM}$ to the $\mathcal{N}=2$ $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM. In this section we find the protected spectrum of single-trace operators of the interpolating family. We start at the orbifold point, where the protected states are easy to determine, and follow their fate along the exactly marginal line towards $\mathcal{N}=2$ SCQCD.

At the orbifold point, operators fall into two classes: untwisted and twisted. In the untwisted sector, the protected states are well-known, since they are inherited from $\mathcal{N}=4$ SYM. The protected operators in the twisted sector are chiral with respect to $\mathcal{N}=1$ subalgebra and could be obtained by analyzing the chiral ring [145]. [10] Both the classes of operators can be rigorously checked to be protected by computing the superconformal index.[11] Using the index one can also argue that the protected multiplets found at the orbifold point *cannot* recombine into long multiplets as we vary $\check{g}$ [2], so in particular taking $\check{g}\to 0$ they must evolve into the protected multiplets of the theory

$$\{\mathcal{N}=2\ \mathrm{SCQCD}\ \oplus\ \text{decoupled}\ SU(N_{\hat{c}})\ \text{vector multiplet}\}\,. \tag{7.25}$$

[10]We confirm the spectrum in [2] up to one operator that was missed in the analysis of [145].

[11]The calculation for the orbifold was carried out already in [139], which we confirm up to a minor emendation in [2].

In section 7.2.3 we follow this evolution in detail. We find that the $SU(2)_L$-singlet protected states of the interpolating theory evolve into the list (7.24) of protected states of SCQCD, plus some extra states made purely from the decoupled vector multiplet. On the other hand, the interpolating theory has also many single-trace protected states with non-trivial $SU(2)_L$ spin, which are of course absent from the list (7.24): we see that in the limit $\check{g} \to 0$, a state with $SU(2)_L$ spin L can be interpreted as a "multiparticle state", obtained by linking together L short "open" spin chains of SCQCD and decoupled fields $\check{\phi}$. By this route we confirm that (7.24) is the correct and complete list of protected single-traces in the scalar sector for $\mathcal{N} = 2$ SCQCD. The results are also suggestive of a dual string theory interpretation: as $\check{g} \to 0$, single closed string states carrying $SU(2)_L$ quantum numbers disintegrate into multiple open strings. The above argument, however, doesn't imply that all the protected operators of SCQCD are obtained as degenerations of protected operators of the interpolating theory. Indeed, they aren't. In [2], we discuss an alternative mechanism that brings about more protected SCQCD operators from the decomposition of long multiplets of the interpolating theory as $\check{g} \to 0$.

In summary, the degeneracy of protected states is independent of the exactly marginal deformation that changes $\check{g}_{YM}$ and is thus the same for the orbifold theory and for the theory (7.25). At $\check{g}_{YM} = 0$ there is a symmetry enhancement, $SU(2)_L \times SU(N_{\check{c}}) \to U(N_f = 2N_c)$, and we can consistently truncate the spectrum of generalized single trace operators to singlets of the flavor group $U(N_f)$ – which in particular do not contain any of the decoupled states $\check{\phi}$. This is the flavor singlet spectrum of $\mathcal{N} = 2$ SCQCD that we have analyzed in the previous section.

7.2.3 Away from the orbifold point: matching with $\mathcal{N} = 2$ SCQCD

In the limit $\check{g} \to 0$, we must be able to match the protected states of the interpolating family with protected states of $\{\mathcal{N} = 2 \text{ SCQCD} \oplus \text{decoupled vector multiplet}\}$. For the purpose of this discussion, the protected states naturally splits into two sets: $SU(2)_L$ singlets and $SU(2)_L$ non-singlets. It is clear that all the (generalized) single-trace operators of $\mathcal{N} = 2$ SCQCD must

Multiplet	Orbifold operator $(R, \ell \geq 0,\ n \geq 2)$
$\hat{\mathcal{B}}_{R+1}$	$\mathrm{Tr}[(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R+1}]$
$\bar{\mathcal{E}}_{-(\ell+2)(0,0)}$	$\mathrm{Tr}[\phi^{\ell+2} + \check{\phi}^{\ell+2}]$
$\hat{\mathcal{C}}_{R(0,0)}$	$\mathrm{Tr}[\sum \mathcal{T}(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R}]$
$\bar{\mathcal{D}}_{R+1(0,0)}$	$\mathrm{Tr}[\sum(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R+1}(\phi^{+}\check{\phi})]$
$\bar{\mathcal{B}}_{R+1,-(\ell+2)(0,0)}$	$\mathrm{Tr}[\sum_i (Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R+1}\phi^i\check{\phi}^{\ell+2-i}]$
$\bar{\mathcal{C}}_{R,-(\ell+1)(0,0)}$	$\mathrm{Tr}[\sum_i \mathcal{T}(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R}\phi^i\check{\phi}^{\ell+1-i}]$
$\mathcal{A}^{\Delta=2R+\ell+2n}_{R,-\ell(0,0)}$	$\mathrm{Tr}[\sum_i \mathcal{T}^n(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R}\phi^i\check{\phi}^{\ell-i}]$

Table 7.2: Superconformal primary operators in the untwisted sector of the orbifold theory that descend from the $\frac{1}{2}$ BPS primary of $\mathcal{N} = 4$. The symbol $\sum$ indicates summation over all "symmetric traceless" permutations of the component fields allowed by the index structure.

Multiplet	Orbifold operator $(\ell \geq 0)$
$\hat{\mathcal{B}}_1$	$\mathrm{Tr}[(Q^{+\hat{+}}\bar{Q}^{+\hat{-}} - Q^{+\hat{-}}\bar{Q}^{+\hat{+}})] = \mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}$
$\bar{\mathcal{E}}_{-(\ell+2)(0,0)}$	$\mathrm{Tr}[\phi^{\ell+2} - \check{\phi}^{\ell+2}]$

Table 7.3: Superconformal primary operators in the twisted section of the orbifold theory.

arise from the $SU(2)_L$ singlets.

7.2.3.1 $\mathbf{SU(2)_L}$ singlets

They are:

(i) One $\hat{\mathcal{B}}_1$ multiplet, corresponding to the primary $\mathrm{Tr}[Q_{\hat{\mathcal{I}}\{\mathcal{I}}\bar{Q}^{\hat{\mathcal{I}}}_{\mathcal{J}\}}] = \mathrm{Tr}\,\boldsymbol{\mathcal{M}_3}$. Since this is the only operator with these quantum numbers, it cannot mix with anything and its form is independent of $\check{g}$.

(ii) Two $\bar{\mathcal{E}}_{-\ell(0,0)}$ multiplets for each $\ell \geq 2$, corresponding to the primaries $\mathrm{Tr}\,[\phi^\ell \pm \check{\phi}^\ell]$.

For each ℓ, there is a two-dimensional space of protected operators, and we may choose whichever basis is more convenient. For $g = \check{g}$, the natural basis vectors are the untwisted and twisted combinations (respectively even and odd under $\phi \leftrightarrow \check{\phi}$), while for $\check{g} = 0$ the natural basis vectors are $\mathrm{Tr}\,\phi^\ell$ (which is an operator of $\mathcal{N} = 2$ SCQCD) and $\mathrm{Tr}\,\check{\phi}^\ell$ (which belongs to the decoupled sector).

(iii) One $\hat{\mathcal{C}}_{0(0,0)}$ multiplet (the stress-tensor multiplet), corresponding to the primary $\mathrm{Tr}\,\mathcal{T} = \mathrm{Tr}\,[T + \check{\phi}\bar{\check{\phi}}]$. We have checked that this combination is an eigenstate with zero eigenvalue for all $\check{g}$.

For $\check{g} = 0$, we may trivially subtract out the decoupled piece $\mathrm{Tr}\,\check{\phi}\bar{\check{\phi}}$ and recover $\mathrm{Tr}\,T$, the stress-tensor multiplet of $\mathcal{N} = 2$ SCQCD.

(iv) One $\bar{\mathcal{C}}_{0,-\ell(0,0)}$ multiplet for each $\ell \geq 1$. In the limit $\check{g} \to 0$, we expect this multiplet to evolve to the $\mathrm{Tr}\,T\phi^\ell$ multiplet of $\mathcal{N} = 2$ SCQCD. Let us check this in detail.

The primary of $\bar{\mathcal{C}}_{0,-\ell(0,0)}$ has $R = 0$, $r = -\ell$ and $\Delta = \ell + 2$. The space of operators which classically have these quantum numbers is spanned by

$$\begin{aligned} |a\rangle &= \mathrm{Tr}[\check{\phi}^{\ell+1}\bar{\check{\phi}}], \quad |b_i\rangle \equiv \frac{1}{2}\mathrm{Tr}[\phi^i Q_{\mathcal{I}\hat{\mathcal{I}}}\check{\phi}^{\ell-i}\bar{Q}^{\hat{\mathcal{I}}\mathcal{I}}] \quad \text{for} \quad 0 \leq i \leq \ell \\ |c_\ell\rangle &\equiv \mathrm{Tr}[\phi^{\ell+1}\bar{\phi}] \end{aligned} \tag{7.26}$$

Diagonalizing the Hamiltonian in Fourier space, we find the protected operator to be

$$|\bar{\mathcal{C}}_{0,-\ell(0,0)}\rangle_\kappa = \kappa^\ell|a\rangle - \sum_{i=0}^{\ell}\kappa^{\ell-i}|b_i\rangle + |c_\ell\rangle \tag{7.27}$$

where $\kappa \equiv \check{g}/g$. In the limit $\kappa \to 0$,

$$|\bar{\mathcal{C}}_{0,-\ell(0,0)}\rangle_{\kappa\to 0} = \mathrm{Tr}[(\phi\bar{\phi} - \frac{1}{2}Q_{\mathcal{I}\hat{\mathcal{I}}}\bar{Q}^{\mathcal{I}\hat{\mathcal{I}}})\phi^\ell] = \mathrm{Tr}[T\phi^\ell]\,, \tag{7.28}$$

as claimed.

All in all, we see that this list reproduces the list (7.24) of one-loop protected scalar operators of $\mathcal{N}=2$ SCQCD, *plus* the extra states $\mathrm{Tr}\check{\phi}^\ell$ which decouple for $\check{g}=0$. This concludes the argument that that the operators (7.24) are protected at the full quantum level, and that they are the *complete* set of protected generalized single-trace primaries of $\mathcal{N}=2$ SCQCD.

7.2.3.2 $\mathbf{SU(2)_L}$ non-singlets

The basic protected primary of $\mathcal{N}=2$ SCQCD which is charged under $SU(2)_L$ is the $SU(2)_L$ triplet contained in the mesonic operator $\mathcal{O}^i_{\mathbf{3_R}\,j} = (\bar{Q}^i_a Q^a_j)_{\mathbf{3_R}}$. Indeed writing the $U(N_f = 2N_c)$ flavor indices i as $i = (\check{a},\hat{\mathcal{I}})$, with $\check{a} = 1,\ldots N_f/2 = N_c$ "half" flavor indices and $\mathcal{I} = \hat{\pm}$ $SU(2)_L$ indices, we can decompose

$$\mathcal{O}^i_{\mathbf{3_R}\,j} \to \mathcal{O}^{\check{a}}_{\mathbf{3_R}\mathbf{3_L}\,\check{b}}\,,\quad \mathcal{O}^{\check{a}}_{\mathbf{3_R}\mathbf{1_L}\,\check{b}}\,. \tag{7.29}$$

In particular we may consider the highest weight combination for both $SU(2)_L$ and $SU(2)_R$,

$$(\bar{Q}^{++\hat{+}}Q^{++\hat{+}})^{\check{a}}_{\check{b}}\,. \tag{7.30}$$

States with higher $SU(2)_L$ spin can be built by taking products of $\mathcal{O}_{\mathbf{3_R}\mathbf{3_L}}$ with $SU(2)_L$ and $SU(2)_R$ indices separately symmetrized – and this is the only way to obtain protected states of $\mathcal{N}=2$ SCQCD charged under $SU(2)_L$ which have finite conformal dimension in the Veneziano limit. It is then a priori clear that a protected primary of the interpolating theory with $SU(2)_L$ spin L must evolve as $\check{g}\to 0$ into a product of L copies of $(\bar{Q}^{++\hat{+}}Q^{++\hat{+}})$ and of as many additional decoupled scalars $\check{\phi}$ and $\bar{\check{\phi}}$ as needed to make up for the correct

$U(1)_r$ charge and conformal dimension. It is amusing to follow in more detail this evolution for the various multiplets:

(i) $\hat{\mathcal{B}}_R$ multiplet.

This is a trivial case, since for each R there is only one operator with the correct quantum numbers, namely

$$|\hat{\mathcal{B}}_R\rangle_\kappa \equiv \mathrm{Tr}[(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^R]\,, \tag{7.31}$$

for all g and $\check{g}$. We have checked that it is indeed an eigenstate of zero eigenvalue for all couplings.

(ii) $\bar{\mathcal{D}}_{R(0,0)}$ multiplet.

The primary of $\bar{\mathcal{D}}_{R(0,0)}$ has $SU(2)_R$ spin equal R, $U(1)_r$ charge $r = -1$ and $\Delta = 2R + 1$. The space of operators which classically have these quantum numbers is two-dimensional, spanned by $\mathrm{Tr}[(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^R\phi]$ and $\mathrm{Tr}[(\bar{Q}^{+\hat{+}}Q^{+\hat{+}})^R\check{\phi}]$. The spin chain Hamiltonian in this subspace reads

$$g^2 H_{\bar{\mathcal{D}}} = \begin{pmatrix} 4g^2 & -4g\check{g} \\ -4g\check{g} & 4\check{g}^2 \end{pmatrix} \tag{7.32}$$

The protected operator (eigenvector with zero eigenvalue) is

$$|\bar{\mathcal{D}}_{R(0,0)}\rangle_\kappa \equiv \mathrm{Tr}[\kappa(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^R\phi + (\bar{Q}^{+\hat{+}}Q^{+\hat{+}})^R\check{\phi}]\,. \tag{7.33}$$

For $\kappa = 0$, the protected operator is interpreted as a "multi-particle state" of R open chains of SCQCD and one decoupled scalar $\check{\phi}$. For example for $R = 2$, the operator will be broken into the following gauge-invariant pieces,

$$(\bar{Q}^{+\hat{+}}Q^{+\hat{+}})^{\check{a}}{}_{\check{b}}\,, \quad (\bar{Q}^{+\hat{+}}Q^{+\hat{+}})^{\check{b}}{}_{\check{c}} \quad \text{and} \quad \check{\phi}^{\check{c}}{}_{\check{a}}\,. \tag{7.34}$$

In the limit $\check{g} \to 0$, the "closed chain" of the interpolating theory effectively breaks into "open chains" of $\{\mathcal{N} = 2$ SCQCD $\oplus$ decoupled multiplet$\}$, with the rupture points at the contractions of the "half-flavor" indices $\check{a}$, $\check{b}$, $\check{c}$.

(iii) $\bar{\mathcal{B}}_{R,r(0,0)}$ multiplet.

Finding the protected multiplet for arbitrary coupling amounts to diagonalizing the spin-chain Hamiltonian of the interpolating theory in the space of operators with quantum numbers R, r and $\Delta = 2R - r$. The dimension of this space increases rapidly with R and r. Let us focus on two simple cases.

case 1*:* $R = 1$, $r \equiv -\ell < 0$

In this case, the space is $\ell + 1$ dimensional, spanned by

$$|\psi_i\rangle \equiv \mathrm{Tr}[\phi^i Q^{+\hat{+}} \check{\phi}^{\ell-i} \bar{Q}^{+\hat{+}}] \,, \quad i = 0, \ldots \ell \,. \tag{7.35}$$

The protected operator is found to be

$$|\bar{\mathcal{B}}_{1,-\ell(0,0)}\rangle_\kappa \equiv \sum_{i=0}^{\ell} \kappa^i |\psi_i\rangle \tag{7.36}$$

In our schematic notation of $\sum$, introduced earlier, the same operator would read

$$|\bar{\mathcal{B}}_{1,-\ell(0,0)}\rangle_\kappa = \mathrm{Tr}[\sum_i \kappa^i (Q^{+\hat{+}} \bar{Q}^{+\hat{+}}) \phi^i \check{\phi}^{\ell-i}] \,. \tag{7.37}$$

Note that at $\kappa = 0$, the $U(1)_r$ charge of the operator is all carried by the decoupled scalars $\check{\phi}$ – there are no ϕ. This is again consistent with the picture of the closed chain disintegrating into open pieces.

case 2: $r = -2$, $R = 2$

The relevant vector space is spanned by the operators

$$\begin{array}{lcllcl}
|0\rangle & = & \mathrm{Tr}[\phi\phi Q^{+\hat{+}} \bar{Q}^{+\hat{+}} Q^{+\hat{+}} \bar{Q}^{+\hat{+}}] & |\check{0}\rangle & = & \mathrm{Tr}[Q^{+\hat{+}} \check{\phi}\check{\phi} \bar{Q}^{+\hat{+}} Q^{+\hat{+}} \bar{Q}^{+\hat{+}}] \\
|1\rangle & = & \mathrm{Tr}[\phi Q^{+\hat{+}} \check{\phi} \bar{Q}^{+\hat{+}} Q^{+\hat{+}} \bar{Q}^{+\hat{+}}] & |\check{1}\rangle & = & \mathrm{Tr}[Q^{+\hat{+}} \check{\phi} \bar{Q}^{+\hat{+}} \phi Q^{+\hat{+}} \bar{Q}^{+\hat{+}}] \\
|2\rangle & = & \mathrm{Tr}[\phi Q^{+\hat{+}} \bar{Q}^{+\hat{+}} \phi Q^{+\hat{+}} \bar{Q}^{+\hat{+}}] & |\check{2}\rangle & = & \mathrm{Tr}[Q^{+\hat{+}} \check{\phi} \bar{Q}^{+\hat{+}} Q^{+\hat{+}} \check{\phi} \bar{Q}^{+\hat{+}}]
\end{array} \tag{7.38}$$

The Hamiltonian in this subspace is (the basis vectors are read in the sequence $|0\rangle$, $|\check{0}\rangle$, $|1\rangle$, ...)

$$g^2 H_{\bar{\mathcal{B}}_{2,-2(0,0)}} = \begin{pmatrix} 4g^2 & 0 & -2g\check{g} & -2g\check{g} & 0 & 0 \\ 0 & 4\check{g}^2 & -2g\check{g} & -2g\check{g} & 0 & 0 \\ -2g\check{g} & -2g\check{g} & 4g^2+4\check{g}^2 & 0 & -2g\check{g} & -2g\check{g} \\ -2g\check{g} & -2g\check{g} & 0 & 4g^2+4\check{g}^2 & -2g\check{g} & -2g\check{g} \\ 0 & 0 & -2g\check{g} & -2g\check{g} & 4g^2 & 0 \\ 0 & 0 & -2g\check{g} & -2g\check{g} & 0 & 4\check{g}^2 \end{pmatrix} \tag{7.39}$$

There is an eigenvector with zero eigenvalue for all κ, namely

$$\begin{aligned} |\bar{\mathcal{B}}_{2,-2(0,0)}\rangle_\kappa &\equiv \kappa^2|0\rangle + |\check{0}\rangle + \kappa|1\rangle + \kappa|\check{1}\rangle + \kappa^2|2\rangle + |\check{2}\rangle \\ &= \mathrm{Tr}[\sum_i \kappa^i (Q^{++\hat{+}}\bar{Q}^{++\hat{+}})^2 \phi^i \check{\phi}^{2-i}] \end{aligned}$$

As expected, for $\kappa = 0$ the operator contains $\check{\phi}$ and no ϕ.

Extrapolating from these cases, we make an educated guess for the form for general protected operator,

$$|\bar{\mathcal{B}}_{R,-\ell(0,0)}\rangle_\kappa = \mathrm{Tr}[\sum_i \kappa^i (Q^{++\hat{+}}\bar{Q}^{++\hat{+}})^R \phi^i \check{\phi}^{\ell-i}]\,. \tag{7.40}$$

In the limit $\kappa \to 0$, this operator breaks into R mesons $(\bar{Q}Q)^{\check{a}}_{\check{b}}$ of $\mathcal{N}=2$ SCQCD and ℓ decoupled scalars $\check{\phi}^{\check{a}}_{\check{b}}$.

(iv) $\hat{\mathcal{C}}_{R(0,0)}$ and $\bar{\mathcal{C}}_{R,-\ell(0,0)}$ multiplets.

We have not studied these cases in detail since they are technically quite involved. It is clear however that for $\check{g} \to 0$ the protected primaries must evolve into states of the schematic form

$$\mathrm{Tr}\left[\mathcal{O}^R_{\mathbf{3_R 3_L}} \check{\phi}^{\ell+n} \bar{\check{\phi}}^n\right]\,, \tag{7.41}$$

with $\ell = 0$, $n = 1$ for $\hat{\mathcal{C}}_{R(0,0)}$ and $n = 1$ for $\bar{\mathcal{C}}_{R,-\ell(0,0)}$.

7.3 Two-body scattering

In this section we study the scattering of two magnons in the spin chain for the interpolating SCFT. We take the chain to be infinite. Because of the index structure of the impurities, one of the asymptotic magnons must be a Q and the other a $\bar{Q}$, and their ordering is fixed – we can have a Q impurity always to the left of a $\bar{Q}$ impurity, or viceversa. The scattering is thus pure reflection. For the case of Q to the *left* of $\bar{Q}$, and suppressing momentarily the $SU(2)_L \times SU(2)_R$ quantum numbers, the asymptotic form of the eigenstates of the Hamiltonian is

$$\sum_{x_1 \ll x_2} \left(e^{ip_1 x_1 + ip_2 x_2} + S(p_2, p_1) e^{ip_2 x_1 + ip_1 x_2}\right) |\dots \phi Q(x_1) \check{\phi} \dots \check{\phi} \bar{Q}(x_2) \phi \dots\rangle \,. \tag{7.42}$$

This is the definition of the two-body S-matrix. In fact, thanks to the nearest-neighbor nature of the spin chain, if the impurities are not adjacent we are already in the "asymptotic" region, so $x_1 \ll x_2$ should be interpreted as $x_1 < x_2 - 1$. Similarly, for the case where Q to the *right* of $\bar{Q}$ the asymptotic form of the two-magnon state is

$$\sum_{x_1 \ll x_2} \left(e^{ip_1 x_1 + ip_2 x_2} + \check{S}(p_2, p_1) e^{ip_2 x_1 + ip_1 x_2}\right) |\dots \check{\phi} \bar{Q}(x_1) \phi \dots \phi Q(x_2) \check{\phi} \dots\rangle \,, \tag{7.43}$$

which defines $\check{S}$. The two-body S-matrices S and $\check{S}$ are related by exchanging $g \leftrightarrow \check{g}$,

$$S(p_1, p_2; g, \check{g}) = \check{S}(p_1, p_2; \check{g}, g) \,. \tag{7.44}$$

The total energy of a two-magnon state is just the sum of the energy of the individual magnons,

$$E(p_1, p_2; \kappa) = \left(2(1-\kappa)^2 + 8\kappa(\sin^2 \frac{p_1}{2})\right) + \left(2(1-\kappa)^2 + 8\kappa(\sin^2 \frac{p_2}{2})\right) \,. \tag{7.45}$$

Besides the continuum of states with real momenta p_1 and p_2, there can be bound and "anti-bound" states for special complex values of the momenta. A bound state occurs when

$$S(p_1, p_2) = \infty \,, \quad \text{with} \quad p_1 = \frac{P}{2} - iq \,, \quad p_2 = \frac{P}{2} + iq \,, \quad q > 0 \,. \tag{7.46}$$

Since $S(p_2, p_1) = 1/S(p_1, p_2) \to 0$, the asymptotic wave-function is

$$e^{iP\frac{x_1+x_2}{2} - q(x_2-x_1)}, \tag{7.47}$$

which is indeed normalizable (since $x_2 > x_1$ in our conventions). A bound state has smaller energy than any state in the two-particle continuum with the same total momentum P. An anti-bound state occurs when

$$S(p_1, p_2) = \infty\,, \quad \text{with} \quad p_1 = \frac{P}{2} - iq + \pi\,, \quad p_2 = \frac{P}{2} + iq - \pi\,, \quad q > 0\,. \tag{7.48}$$

The asymptotic wave-function is now

$$(-1)^{x_2-x_1} e^{iP\frac{x_1+x_2}{2} - q(x_2-x_1)}\,. \tag{7.49}$$

The energy of an anti-bound state is strictly bigger than the two-particle continuum. It is easy to see that (8.4) and (7.48) are the only allowed possibilities for complex p_1 and p_2 such that the total momentum and the total energy are real.

The analysis of two-body scattering proceeds independently in four different sectors, corresponding the choice of the triplet or singlet combinations for $SU(2)_L$ and $SU(2)_R$. In each sector, we will compute the S-matrix and look for the (anti)bound states associated to its poles.

7.3.1 $3_L \otimes 3_R$ Sector

In the $3_L \otimes 3_R$ sector, we write the general two-impurity state with Q to the left of $\bar{Q}$ as

$$|\Psi_{3\otimes 3}\rangle = \sum_{x_1 < x_2} \Psi_{3\otimes 3}(x_1, x_2) | \ldots \phi Q(x_1) \check{\phi} \ldots \check{\phi} \bar{Q}(x_2) \phi \ldots \rangle_{3\otimes 3}\,. \tag{7.50}$$

There is no mixing with states containing $\bar{\phi}$ and $\bar{\check{\phi}}$ since they have different $SU(2)_L \times SU(2)_R \times U(1)_r$ quantum numbers. Acting with the Hamiltonian, one finds:

- For $x_2 > x_1 + 1$,

$$g^2 H \cdot \Psi_{3\otimes 3}(x_1, x_2) = 4(g^2 + \check{g}^2)\Psi_{3\otimes 3}(x_1, x_2) - 2g\check{g}\Psi_{3\otimes 3}(x_1 + 1, x_2) \\ -2g\check{g}\Psi_{3\otimes 3}(x_1 - 1, x_2) - 2g\check{g}\Psi_{3\otimes 3}(x_1, x_2 + 1) - 2g\check{g}\Psi_{3\otimes 3}(x_1, x_2 - 1) \tag{7.51}$$

- For $x_2 = x_1 + 1$,

$$g^2 H \cdot \Psi_{3\otimes 3}(x_1, x_2) = 4g^2 \Psi_{3\otimes 3}(x_1, x_2) - 2g\check{g}\Psi_{3\otimes 3}(x_1 - 1, x_2) - 2g\check{g}\Psi_{3\otimes 3}(x_1, x_2 + 1)\,. \tag{7.52}$$

The plane wave states $e^{i(p_1 x_1 + p_2 x_2)}$ and $e^{i(p_1 x_2 + p_2 x_1)}$ are separately eigenstates for the "bulk" action of the Hamiltonian (7.51), with eigenvalue (7.45). The action of the Hamiltonian on the state with adjacent impurities, equ.(7.52), provides the boundary condition that fixes the exact eigenstate of asymptotic momenta p_1, p_2,

$$\Psi_{3\otimes 3}(x_1, x_2) = e^{i(p_1 x_1 + p_2 x_2)} + S_{3\otimes 3}(p_1, p_2) e^{i(p_1 x_2 + p_2 x_1)}\,, \tag{7.53}$$

where

$$S_{3\otimes 3}(p_1, p_2) = -\frac{1 + e^{ip_1 + ip_2} - 2\kappa e^{ip_1}}{1 + e^{ip_1 + ip_2} - 2\kappa e^{ip_2}}\,, \qquad \kappa \equiv \frac{\check{g}}{g}\,. \tag{7.54}$$

In this sector, the S-matrix coincides with the familiar S-matrix of the XXZ chain, with the identification $\Delta_{XXZ} = \kappa$. The pole of the S-matrix,

$$e^{ip_2} = \frac{1 + e^{i(p_1 + p_2)}}{2\kappa}\,, \tag{7.55}$$

is associated to a bound state. Writing $p_1 = P/2 - iq$, $p_2 = P/2 + iq$, we have

$$e^{-q} = \frac{\cos(\frac{P}{2})}{\kappa}\,. \tag{7.56}$$

The wave-function is normalizable provided $q > 0$, which implies $2 \arccos \kappa < |P| < \pi$. Substituting p_1 and p_2 into the expression for the total energy (7.45), we find that the dispersion relation of the bound state is simply

$$\left[Q\bar{Q}\right]^{bound}_{3_L\, 3_R}: \qquad E = 4\sin^2(\frac{P}{2})\,, \quad 2\arccos\kappa < |P| < \pi\,. \tag{7.57}$$

This dispersion relation is plotted as the dotted (orange) curve in the left column of figure 7.2. When the total momentum P is smaller than $2\arccos\kappa$ the bound state dissolves into the two-particle continuum. The bound state exists for the full range of P at the orbifold point $\kappa = 1$, but the allowed range of P shrinks as κ is decreased, and the bound state disappears in the SCQCD limit $\kappa \to 0$.

The S-matrix in the $3_L \otimes 3_R$ sector with Q to the *right* of $\bar{Q}$ is obtained by switching $g \leftrightarrow \check{g}$,

$$\check{S}_{3\otimes 3}(p_1, p_2; \kappa) = S_{3\otimes 3}(p_1, p_2; 1/\kappa) = -\frac{1 + e^{ip_1+ip_2} - \frac{2}{\kappa}e^{ip_1}}{1 + e^{ip_1+ip_2} - \frac{2}{\kappa}e^{ip_2}} . \tag{7.58}$$

Now the pole of the S-matrix is associated to a bound state with

$$e^{-q} = \kappa\cos(\frac{P}{2}) . \tag{7.59}$$

The bound state exists for all P in the whole range of $\kappa \in (0, 1]$. Its dispersion relation is

$$\left[\bar{Q}Q\right]^{bound}_{3_L\,3_R} : \qquad E = 4\kappa^2\sin^2(\frac{P}{2}) , \tag{7.60}$$

plotted as the dotted (orange) curve in the right column of figure 7.2. The existence of this bound state is consistent with our analysis of the protected spectrum in section 7.2.

7.3.2 $1_L \otimes 3_R$ Sector

The general two-body state with Q to the left of $\bar{Q}$ is

$$|\Psi_{1\otimes 3}\rangle = \sum_{x_1<x_2} \Psi_{1\otimes 3}(x_1, x_2)|\ldots\phi Q(x_1)\check{\phi}\ldots\check{\phi}\bar{Q}(x_2)\phi\ldots\rangle_{1\otimes 3} \tag{7.61}$$

The action of the Hamiltonian for $x_2 = x_1 + 1$ is now

$$g^2 H\cdot\Psi_{1\otimes 3}(x, x+1) = 8g^2\Psi_{1\otimes 3}(x, x+1) - 2g\check{g}\Psi_{1\otimes 3}(x-1, x+1) - 2g\check{g}\Psi_{1\otimes 3}(x, x+2) \tag{7.62}$$

Writing

$$\Psi_{1\otimes 3}(x_1, x_2) = e^{i(p_1x_1+p_2x_2)} + S_{1\otimes 3}(p_2, p_1)e^{i(p_1x_2+p_2x_1)} , \tag{7.63}$$

we find

$$S_{1\otimes 3}(p_1,p_2;\kappa) = -\frac{1+e^{ip_1+ip_2}-2(\kappa-\frac{1}{\kappa})e^{ip_1}}{1+e^{ip_1+ip_2}-2(\kappa-\frac{1}{\kappa})e^{ip_2}}\,, \tag{7.64}$$

which is again the S-matrix of the XXZ chain, now with $\Delta = \kappa - \frac{1}{\kappa}$. The S-matrix blows up for

$$e^{ip_2} = \frac{1+e^{i(p_1+p_2)}}{2(\kappa-\frac{1}{\kappa})}\,. \tag{7.65}$$

This pole is associated to an *anti*-bound state. Parametrizing $p_1 = P/2 - iq + \pi$, $p_2 = P/2 - iq - \pi$, the location of the pole is given by

$$e^{-q} = \frac{\cos(\frac{P}{2})}{\frac{1}{\kappa}-\kappa}\,. \tag{7.66}$$

Normalizability of the wave-function requires $q > 0$, which occurs for a restricted range of P for $\kappa_* < \kappa < 1$, and for the full range of P for $\kappa < \kappa_*$,

$$\begin{aligned} 2\arccos(\frac{1}{\kappa}-\kappa) < |P| < \pi \quad & \text{for } \frac{\sqrt{5}-1}{2} < \kappa < 1 \\ 0 < |P| < \pi \quad & \text{for } 0 < \kappa < \frac{\sqrt{5}-1}{2}\,. \end{aligned} \tag{7.67}$$

Substituting in $E(p_1,p_2;\kappa)$ we find the dispersion relation for the anti-bound state,

$$\big[Q\bar{Q}\big]^{antibound}_{1_L\,3_R}: \qquad E = \frac{4(2-\kappa^2)}{1-\kappa^2} - \frac{4\kappa^2}{1-\kappa^2}\sin^2\frac{P}{2}\,, \tag{7.68}$$

which is plotted as the solid (red) curve in the left column of figure 7.2. The anti-bound state is absent at the orbifold point $\kappa = 1$. For $\kappa \to 0$, $q \to +\infty$, so that the wave-function (7.49) localizes to two neighboring sites and in fact coincides with the dimeric excitation $\mathcal{M}_{\mathbf{3}} = (Q\bar{Q})_{\mathbf{3}}$ of $\mathcal{N} = 2$ SCQCD; in the limit we smoothly recover the $\mathcal{M}_{\mathbf{3}}$ dispersion relation $E(P) = 8$.

For $\bar{Q}Q$ scattering, we have

$$\check{S}_{1\otimes 3}(p_1,p_2;\kappa) = S_{1\otimes 3}(p_1,p_2;1/\kappa) = -\frac{1+e^{ip_1+ip_2}-2(\frac{1}{\kappa}-\kappa)e^{ip_1}}{1+e^{ip_1+ip_2}-2(\frac{1}{\kappa}-\kappa)e^{ip_2}}\,. \tag{7.69}$$

Now the pole corresponds to a bound state, indeed it occurs for $p_1 = P/2 - iq$, $p_2 = P/2 + iq$ with q and P related as in (7.66). Clearly the allowed range of

P is as in (7.67). We find the dispersion relation

$$[Q\bar{Q}]^{bound}_{1_L\,3_R}: \qquad E = \frac{4\kappa^2}{(1-\kappa^2)}(1 - 2\kappa^2 + \sin^2\frac{P}{2})\,, \tag{7.70}$$

which is plotted as the solid (red) curve in the right column of figure 7.2.

7.3.3 $3_L \otimes 1_R$ Sector

The scattering problem in the $3_L \otimes 1_R$ sector is solved by the same technique. We find

$$S_{3\otimes 1}(p_1, p_2) = \check{S}_{3\otimes 1}(p_1, p_2) = -1\,, \tag{7.71}$$

which coincides with the scattering matrix of free fermions, or with the $\Delta_{XXZ} \to \infty$ limit of the S-matrix for the XXZ chain. Clearly there are no (anti)bound states.

7.3.4 $1_L \otimes 1_R$ Sector

The analysis for the $1_L \otimes 1_R$ sector is slightly more involved because a two-impurity state is not closed under the action of Hamiltonian: when Q and $\bar{Q}$ are next to each other they can transform into $\phi\bar{\phi}$. The general state for $Q\bar{Q}$ scattering in the singlet sector is

$$\begin{aligned} |\Psi_{1\otimes 1}\rangle \;&=\; \sum_{x_1<x_2} \Psi_{1\otimes 1}(x_1, x_2)|\ldots\phi Q(x_1)\check{\phi}\ldots\check{\phi}\bar{Q}(x_2)\phi\ldots\rangle_{1\otimes 1} \\ &\quad + \sum_{x} \Psi_{\bar{\phi}}(x)|\ldots\phi\bar{\phi}(x)\phi\ldots\rangle\,. \end{aligned} \tag{7.72}$$

The first term is an eigenstate for "bulk" action of the Hamiltonian ($x_2 > x_1 + 1$) with the usual eigenvalue $E(p_1, p_2; \kappa)$ of equ.(7.45). The action of the Hamiltonian for $x_2 = x_1 + 1$ is

$$\begin{aligned} &g^2 H \cdot \Psi_{1\otimes 1}(x, x+1) = 4(g^2 + \check{g}^2)\Psi_{1\otimes 1}(x, x+1) - 2g\check{g}\Psi_{1\otimes 1}(x-1, x+1) \\ &-2g\check{g}\Psi_{1\otimes 1}(x, x+2) + 2g^2\Psi_{\bar{\phi}}(x) + 2g^2\Psi_{\bar{\phi}}(x+1). \end{aligned} \tag{7.73}$$

Furthermore,

$$
\begin{aligned}
g^2 H \cdot \Psi_{\bar{\phi}}(x) &= 6g^2\Psi_{\bar{\phi}}(x) - g^2\Psi_{\bar{\phi}}(x+1) - g^2\Psi_{\bar{\phi}}(x-1) \\
&\quad +2g^2\Psi_{1\otimes 1}(x,x+1) + 2g^2\Psi_{1\otimes 1}(x-1,x)\,.
\end{aligned} \tag{7.74}
$$

We take the ansatz

$$
\Psi_{1\otimes 1}(x_1,x_2) = e^{i(p_1x_1+p_2x_2)} + S_{1\otimes 1}(p_2,p_1)e^{i(p_1x_2+p_2x_1)} \tag{7.75}
$$

$$
\Psi_{\bar{\phi}}(x) = S_{\bar{\phi}}(p_2,p_1)e^{i(p_1+p_2)x}\,. \tag{7.76}
$$

Note that $S_{1\otimes 1}(p_1,p_2)$ still has the interpretation of the scattering matrix of the magnons Q and $\bar{Q}$, which are the asymptotic excitations, while $\bar{\phi}$ is an "unstable" excitations created during the collision of Q and $\bar{Q}$. We find

$$
\begin{aligned}
S_{1\otimes 1}(p_1,p_2) &= -\left(\frac{1+e^{ip_1+ip_2}-2(\kappa-\frac{1}{\kappa})e^{ip_1}}{1+e^{ip_1+ip_2}-2(\kappa-\frac{1}{\kappa})e^{ip_2}}\right)\left(\frac{1+e^{ip_1+ip_2}-2\kappa e^{ip_1}}{1+e^{ip_1+ip_2}-2\kappa e^{ip_2}}\right)^{-1} \\
S_{\bar{\phi}}(p_1,p_2) &= \frac{4ie^{i(p_1+p_2)}(\sin p_1 - \sin p_2)}{(1+e^{ip_1+ip_2}-2\kappa e^{ip_1})(1+e^{ip_1+ip_2}-2(\kappa-\frac{1}{\kappa})e^{ip_2})}.
\end{aligned}
$$

$S_{1\otimes 1}$ is the product of two factors, and it admits two poles. The first factor coincides with $S_{1\otimes 3}$, so its pole is associated to an anti-bound state entirely analogous to the anti-bound state in the $1_L \otimes 3_R$ sector. The pole is located at $p_1 = P/2 - iq + \pi$, $p_2 = P/2 + iq - \pi$, with

$$
e^{-q} = \frac{\cos(P/2)}{\frac{1}{\kappa}-\kappa}\,. \tag{7.77}
$$

The dispersion relation is again

$$
\left[Q\bar{Q}\right]^{antibound}_{1_L\,1_R}: \qquad E = \frac{4(2-\kappa^2)}{1-\kappa^2} - \frac{4\kappa^2}{1-\kappa^2}\sin^2\frac{P}{2}\,, \tag{7.78}
$$

and the range of P for which the wave-function is normalizable is as in (7.67) – see the solid (red) curve in the left column of figure 7.2. It is interesting to analyze the explicit form of the wave-function in the $\kappa \to 0$ limit. The $Q\bar{Q}$ piece has the form

$$
\Psi_{1\otimes 1}(x_1,x_2) = (-1)^{x_2-x_1}e^{iP(\frac{x_1+x_2}{2})}e^{-q(x_2-x_1)}\,, \quad q\to\infty \tag{7.79}
$$

so only the $x_2 = x_1 + 1$ term survives in the limit, and we recover the dimeric impurity $\mathcal{M}_{\mathbf{1}}$ of SCQCD. A short calculation gives

$$\frac{\Psi_{\bar{\phi}}(x)}{\Psi(x, x+1)}|_{\kappa \to 0} = \frac{2}{(1 + e^{iP})} \, . \tag{7.80}$$

Comparison with (7.12) shows that that in the $\kappa \to 0$ limit the antibound state in the $Q\bar{Q}$ singlet sector becomes precisely the dimeric excitation $\widetilde{T}$ of SCQCD.

The pole in the second factor of $S_{1\otimes 1}$ corresponds instead to a bound state, with

$$e^q = \frac{\cos(P/2)}{\kappa} \, . \tag{7.81}$$

The dispersion relation and range of existence are

$$\left[Q\bar{Q}\right]^{bound}_{1_L \, 1_R} : \qquad E = 4 \sin^2 \frac{q}{2} \, , \quad 0 < |P| < 2 \arccos\kappa \, , \tag{7.82}$$

which are shown as the dashed (green) curve on the left column of figure 7.2. This bound state is absent at the orbifold point and comes into full existence (for any P) in the SCQCD limit $\kappa \to 0$. The natural guess is that in this limit it reduces to the gapless $T(p)$ magnon of SCQCD, and it does:

$$\frac{\Psi_{\bar{\phi}}(x)}{\Psi(x, x+1)}|_{\kappa \to 0} = -\frac{1 + e^{-iP}}{2} \, , \tag{7.83}$$

in agreement with (7.11).

The S-matrix in the $\bar{Q}Q$ channel is obtained as usual by $\kappa \to 1/\kappa$,

$$\begin{aligned} \check{S}_{1\otimes 1}(p_1, p_2; \kappa) &= -\left(\frac{1 + e^{ip_1 + ip_2} + 2(\kappa - \frac{1}{\kappa})e^{ip_1}}{1 + e^{ip_1 + ip_2} + 2(\kappa - \frac{1}{\kappa})e^{ip_2}}\right)\left(\frac{1 + e^{ip_1 + ip_2} - \frac{2}{\kappa}e^{ip_1}}{1 + e^{ip_1 + ip_2} - \frac{2}{\kappa}e^{ip_2}}\right)^{-1} \\ \check{S}_{\bar{\phi}}(p_1, p_2; \kappa) &= \frac{4ie^{i(p_1 + p_2)}(\sin p_1 - \sin p_2)}{(1 + e^{ip_1 + ip_2} - \frac{2}{\kappa}e^{ip_1})(1 + e^{ip_1 + ip_2} + 2(\kappa - \frac{1}{\kappa})e^{ip_2})} \, . \end{aligned}$$

The pole in the first factor of $\check{S}_{1\otimes 1}$ corresponds to a bound state, with

$$\left[\bar{Q}Q\right]^{bound}_{1_L \, 1_R} : \qquad E(P) = \frac{4\kappa^2}{1 - \kappa^2}\left(1 - 2\kappa^2 + \sin^2 \frac{P}{2}\right) \, , \tag{7.84}$$

with the range of existence given by (7.67). Finally, the pole in the second

	Pole of the S-matrix	Range of existence	Dispersion relation $E(P)$
$\mathcal{M}_{33}$	$e^{-q} = \cos(\frac{P}{2})/\kappa$	$2\arccos\kappa < \|P\| < \pi$	$4\sin^2(\frac{P}{2})$
T	$e^{q} = \cos(\frac{P}{2})/\kappa$	$0 < \|P\| < 2\arccos\kappa$	$4\sin^2(\frac{P}{2})$
$\widetilde{T}$ and $\mathcal{M}_3$	$e^{-q} = \cos(\frac{P}{2})/(\kappa - \frac{1}{\kappa})$	See equ.(7.67)	$\frac{4\kappa^2}{(1-\kappa^2)}(\frac{2}{\kappa^2} - 1 - \sin^2\frac{P}{2})$
$\check{\mathcal{M}}_{33}$	$e^{-q} = \kappa\cos(\frac{P}{2})$	$0 < \|P\| < \pi$	$4\kappa^2\sin^2(\frac{P}{2})$
$\check{T}$	$e^{q} = \kappa\cos(\frac{P}{2})$	No solution	
$\check{\widetilde{T}}$ and $\check{\mathcal{M}}_3$	$e^{-q} = \cos(\frac{P}{2})/(\frac{1}{\kappa} - \kappa)$	See equ.(7.67)	$\frac{4\kappa^2}{(1-\kappa^2)}(1 - 2\kappa^2 + \sin^2\frac{P}{2})$

Table 7.4: Dispersion relations and range of existence of the various (anti)bound states in two-body scattering. The first three entries correspond to the $Q\bar{Q}$ channel and the last three entries to the $\bar{Q}Q$ channel. The color-coding of the third entry is a reminder that these are *anti*-bound states with energy above the two-particle continuum.

factor of $\check{S}_{1\otimes 1}$ never corresponds to a normalizable solution.

7.3.5 Summary

The two-body scattering problem in the spin chain of the interpolating SCFT admits a rich spectrum of bound and anti-bound states. The results are summarized in table 7.4 and figure 7.2. The $Q\bar{Q}$ scattering channel (that is, the channel with Q to the left of $\bar{Q}$, and the ϕ vacuum on the outside) is the one relevant to make contact with $\mathcal{N} = 2$ SCQCD, which is obtained in the $\kappa \to 0$ limit. Remarkably, the magnon excitations of SCQCD are recovered as the smooth limits of the $Q\bar{Q}$ (anti)bound states: as $\kappa \to 0$ the wavefunctions of the (anti)bound states localize to two neighboring sites and reproduce the "dimeric" magnons $T(p)$, $\widetilde{T}(p)$ and $\mathcal{M}_{\mathbf{3}}(p)$ of SCQCD.

7.3.6 Left/right factorization of the two-body S-matrix

As is well-known, the magnon excitations of the $\mathcal{N} = 4$ SYM spin chain transform in the fundamental representation of $SU(2|2) \times SU(2|2)$, and their two-body S-matrix factorizes into the product of the S-matrices for the "left" and "right" $SU(2|2)$. The $\mathbb{Z}_2$ orbifold preserves this factorization. Remarkably, this left/right factorization persists even away from the orbifold point, for the full interpolation SCFT – or at least this is what happens at one-loop in the

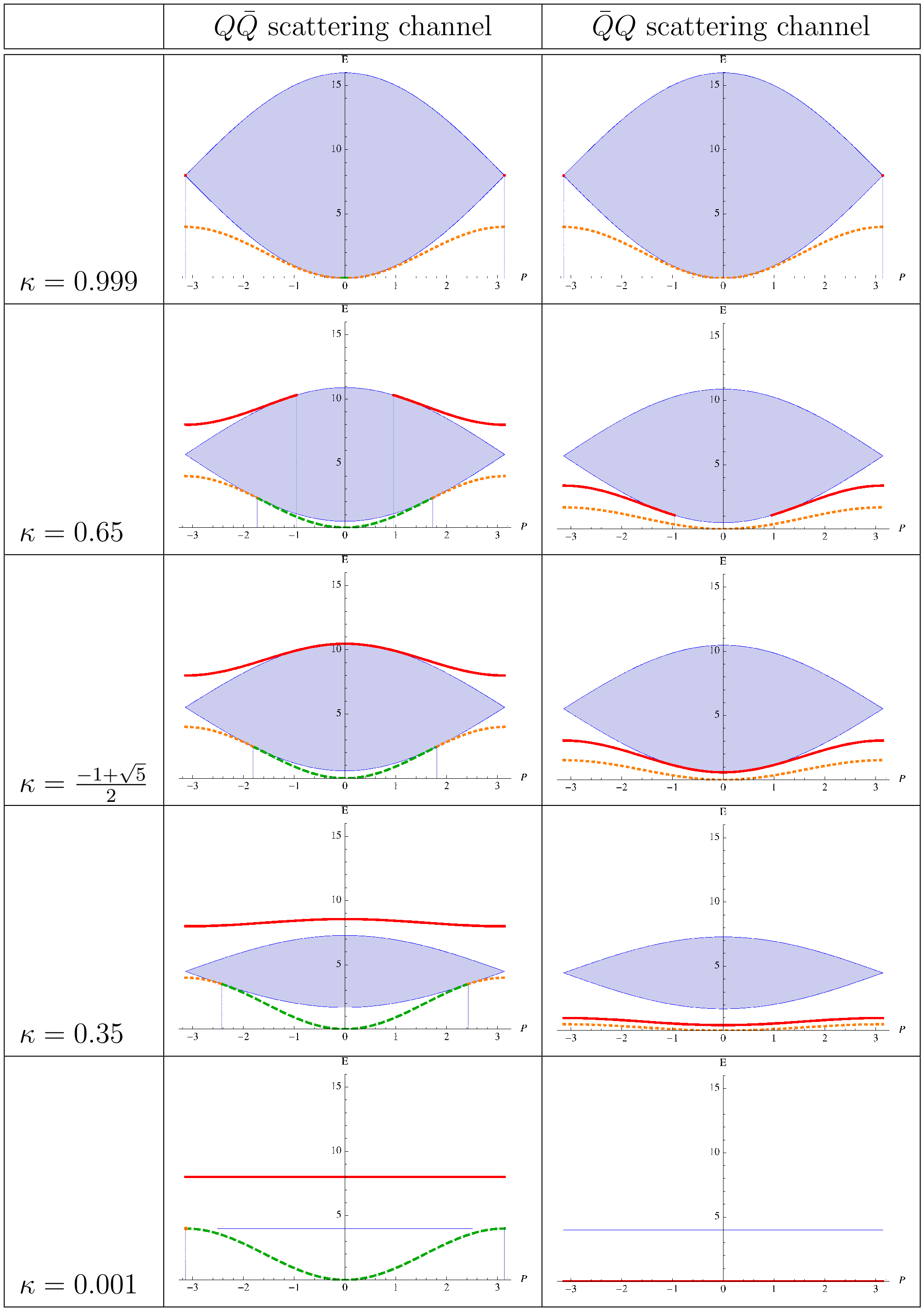


Figure 7.2: Plots of the dispersion relations of the (anti)bound states for different values of κ. The shaded region represents the two-particle continuum.

$L \otimes R$	$S(p_1, p_2, \kappa)$
$1 \otimes 1$	$-\mathcal{S}(p_1, p_2, \kappa - \frac{1}{\kappa})\mathcal{S}^{-1}(p_1, p_2, \kappa)$
$1 \otimes 3$	$\mathcal{S}(p_1, p_2, \kappa - \frac{1}{\kappa})$
$3 \otimes 1$	-1
$3 \otimes 3$	$\mathcal{S}(p_1, p_2, \kappa)$

Table 7.5: The S-matrix in the $Q\bar{Q}$ scattering channel.

$SU(2)_L$	$S_L(p_1, p_2; \kappa)$	$SU(2)_R$	$S_R(p_1, p_2; \kappa)$
1	$\mathcal{S}(p_1, p_2; \kappa - \frac{1}{\kappa})$	**1**	-1
3	$\mathcal{S}(p_1, p_2; \kappa)$	**3**	$\mathcal{S}(p_1, p_2; \kappa)$

Table 7.6: Definitions of the $SU(2)_L$ and $SU(2)_R$ S-matrices.

scalar sector. Our results for the S-matrix in the $Q\bar{Q}$ channel in the four different $SU(2)_L \times SU(2)_R$ sectors are summarized in table 7.5, where we have defined

$$\mathcal{S}(p_1, p_2, \kappa) \equiv -\frac{1 - 2\kappa e^{ip_1} + e^{i(p_1+p_2)}}{1 - 2\kappa e^{ip_2} + e^{i(p_1+p_2)}}\,, \tag{7.85}$$

i.e. the standard S-matrix of the XXZ chain, with $\Delta_{XXZ} = \kappa$. We see that we can write

$$S(p_1, p_2; \kappa) = \frac{S_L(p_1, p_2; \kappa) S_R(p_1, p_2; \kappa)}{S_{3\otimes 3}(p_1, p_2; \kappa)} \tag{7.86}$$

where S_L and S_R are defined in table 7.6.

In the analysis of the Yang-Baxter equation, it will be useful to write the S-matrices in both the $SU(2)_L$ and $SU(2)_R$ sectors using the identity ($\mathbb{I}$) and trace ($\mathbb{K}$) tensorial structures,

$$S_L(p_1, p_2; \kappa) = A_L(p_1, p_2; \kappa)\, \mathbb{I} + B_L(p_1, p_2; \kappa)\, \mathbb{K} \tag{7.87}$$

$$S_R(p_1, p_2; \kappa) = A_R(p_1, p_2; \kappa)\, \mathbb{I} + B_R(p_1, p_2; \kappa)\, \mathbb{K}\,. \tag{7.88}$$

Writing the indices explicitly,

$$(S_R)_{\mathcal{I}\mathcal{J}}^{\mathcal{M}\mathcal{N}} = A_R\, \delta_{\mathcal{I}}^{\mathcal{M}} \delta_{\mathcal{J}}^{\mathcal{N}} + B_R\, \epsilon_{\mathcal{I}\mathcal{J}} \epsilon^{\mathcal{M}\mathcal{N}}\,, \tag{7.89}$$

Recalling that eigenvalue of $\mathbb{K}$ on the triplet is zero while it is two on the singlet, we see that

$$A = S_{\mathbf{3}} \tag{7.90}$$
$$B = \frac{1}{2}(S_{\mathbf{1}} - S_{\mathbf{3}})\,. \tag{7.91}$$

The values of $S_{\mathbf{1}}$ and $S_{\mathbf{3}}$ in both the $SU(2)_L$ and $SU(2)_R$ sectors can be read off from table 7.6,

$$A_L(p_1, p_2, \kappa) = \mathcal{S}(p_1, p_2, \kappa) \tag{7.92}$$
$$B_L(p_1, p_2, \kappa) = \frac{1}{2}\left(\mathcal{S}(p_1, p_2, \kappa - \frac{1}{\kappa}) - \mathcal{S}(p_1, p_2, \kappa)\right) \tag{7.93}$$
$$A_R(p_1, p_2, \kappa) = \mathcal{S}(p_1, p_2, \kappa) \tag{7.94}$$
$$B_R(p_1, p_2, \kappa) = -\frac{1}{2}(1 + \mathcal{S}(p_1, p_2, \kappa))\,. \tag{7.95}$$

In complete analogy, in the $\bar{Q}Q$ channel we have the factorization

$$\check{S}(p_1, p_2; \kappa) = \frac{\check{S}_L(p_1, p_2; \kappa)\check{S}_R(p_1, p_2; \kappa)}{\check{S}_{3\otimes 3}(p_1, p_2; \kappa)}\,, \tag{7.96}$$

and we can write

$$\check{S}_L(p_1, p_2; \kappa) = \check{A}_L(p_1, p_2; \kappa)\,\mathbb{I} + \check{B}_L(p_1, p_2; \kappa)\,\mathbb{K} \tag{7.97}$$
$$\check{S}_R(p_1, p_2; \kappa) = \check{A}_R(p_1, p_2; \kappa)\,\mathbb{I} + \check{B}_R(p_1, p_2; \kappa)\,\mathbb{K}\,. \tag{7.98}$$

As always, each "checked" quantity is obtained from the corresponding unchecked one by sending $\kappa \to 1/\kappa$.

7.4 Yang-Baxter Equation

The one-loop spin chain of the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM is known to be integrable [172, 173]. A natural question is whether integrability persists for the $\check{g} \neq g$. We can explore the integrability of the spin chain for the interpolating SCFT by checking the Yang-Baxter equation for the two-body S-matrix. Integrability of the spin chain amounts to the existence of higher conserved quantities beyond the momentum and the Hamiltonian, which would imply exact factorization of many-body scattering into a sequence of two-body scat-

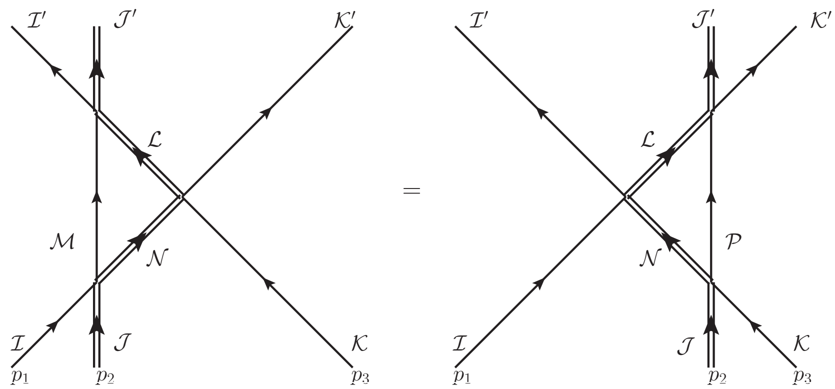


Figure 7.3: Yang-Baxter equation in each $SU(2)$ sector. Simple lines represent Q impurities, double lines $\bar{Q}$ impurities.

terings. For this to happen it is necessary that different ways to factorize three-body scattering into two-body scatterings should commute: the Yang-Baxter equation expresses this consistency condition.

The two-body S-matrix of our theory factorizes into the S-matrix for the $SU(2)_L$ sector times the S-matrix for the $SU(2)_R$ sector. The Yang-Baxter equation must be satisfied separately in each sector. Clearly this is a sufficient condition for the full Yang-Baxter equation to hold; it is also a necessary condition since we can always restrict the asymptotic states to one sector by setting their quantum numbers in the other sector to be highest weights. In each sector, the Yang-Baxter equation is represented by the diagram of figure 7.3, and reads explicitly

$$S_{\mathcal{I}\mathcal{J}}^{\mathcal{M}\mathcal{N}}(p_1,p_2)\check{S}_{\mathcal{N}\mathcal{K}}^{\mathcal{L}\mathcal{K}'}(p_1,p_3)S_{\mathcal{M}\mathcal{L}}^{\mathcal{I}'\mathcal{J}'}(p_2,p_3) = \check{S}_{\mathcal{L}\mathcal{P}}^{\mathcal{J}'\mathcal{K}'}(p_1,p_2)S_{\mathcal{I}\mathcal{N}}^{\mathcal{I}'\mathcal{L}}(p_1,p_3)\check{S}_{\mathcal{J}\mathcal{K}}^{\mathcal{N}\mathcal{P}}(p_2,p_3) \tag{7.99}$$

Using the decomposition introduced in the previous section, we can write the left-hand side as

$$\begin{aligned} & S_{\mathcal{I}\mathcal{J}}^{\mathcal{M}\mathcal{N}}(p_1,p_2)\check{S}_{\mathcal{N}\mathcal{K}}^{\mathcal{L}\mathcal{K}'}(p_1,p_3)S_{\mathcal{M}\mathcal{L}}^{\mathcal{I}'\mathcal{J}'}(p_2,p_3) \\ = \; & A\check{A}A\delta_{\mathcal{K}}^{\mathcal{K}'}\delta_{\mathcal{I}}^{\mathcal{I}'}\delta_{\mathcal{J}}^{\mathcal{J}'} + A\check{B}Bg_{\mathcal{J}\mathcal{K}}\delta_{\mathcal{I}}^{\mathcal{K}'}g^{\mathcal{I}'\mathcal{J}'} + B\check{B}Ag_{\mathcal{I}\mathcal{J}}\delta_{\mathcal{K}}^{\mathcal{I}'}g^{\mathcal{J}'\mathcal{K}'} \\ + \; & (A\check{A}B + B\check{A}A + 2B\check{A}B + B\check{B}B)\delta_{\mathcal{K}}^{\mathcal{K}'}g_{\mathcal{I}\mathcal{J}}g^{\mathcal{I}'\mathcal{J}'} + A\check{B}Ag_{\mathcal{J}\mathcal{K}}g^{\mathcal{J}'\mathcal{K}'}\delta_{\mathcal{I}}^{\mathcal{I}'} \end{aligned}$$

We have suppressed the momentum arguments with the convention that the first symbol in each term is a function of (p_1, p_2), the second is function of

(p_1, p_3) and the third (p_2, p_3). Similarly, for the right-hand side

$$\begin{aligned}
& \check{S}^{\mathcal{J}'\mathcal{K}'}_{\mathcal{L}\mathcal{P}}(p_1,p_2) S^{\mathcal{I}'\mathcal{L}}_{\mathcal{I}\mathcal{N}}(p_1,p_3) \check{S}^{\mathcal{N}\mathcal{P}}_{\mathcal{J}\mathcal{K}}(p_2,p_3) \\
= \; & \check{A}A\check{A}\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}'}_{\mathcal{J}}\delta^{\mathcal{K}'}_{\mathcal{K}} + \check{A}B\check{B}g^{\mathcal{I}'\mathcal{J}'}g_{\mathcal{J}\mathcal{K}}\delta^{\mathcal{K}'}_{\mathcal{I}} + \check{B}B\check{A}g^{\mathcal{J}'\mathcal{K}'}g_{\mathcal{I}\mathcal{J}}\delta^{\mathcal{I}'}_{\mathcal{K}} \\
+ \; & \check{A}B\check{A}g_{\mathcal{I}\mathcal{J}}g^{\mathcal{I}'\mathcal{J}'}\delta^{\mathcal{K}'}_{\mathcal{K}} + (\check{A}A\check{B} + \check{B}A\check{A} + 2\check{B}A\check{B} + \check{B}B\check{B})g^{\mathcal{J}'\mathcal{K}'}\delta^{\mathcal{I}'}_{\mathcal{I}}g_{\mathcal{J}\mathcal{K}}
\end{aligned}$$

Collecting the terms with the same index structure, the Yang-Baxter equation in each $SU(2)$ sector reduces to the following five equations:

$$A\check{A}A = \check{A}A\check{A} \tag{7.100}$$

$$A\check{B}B = \check{A}B\check{B} \tag{7.101}$$

$$B\check{B}A = \check{B}B\check{A} \tag{7.102}$$

$$2B\check{A}B + A\check{A}B + B\check{A}A + B\check{B}B = \check{A}B\check{A} \tag{7.103}$$

$$A\check{B}A = 2\check{B}A\check{B} + \check{A}A\check{B} + \check{B}A\check{A} + \check{B}B\check{B}. \tag{7.104}$$

At the orbifold point, $\kappa = 1/\kappa = 1$ and thus $A = \check{A}$, $B = \check{B}$: the first three equations are trivial; the forth and fifth become equivalent. In both the $SU(2)_L$ and $SU(2)_R$ sectors (which are in fact equivalent for $\kappa = 1$), the remaining equation is easily checked. Thus as expected, the Yang-Baxter equation is satisfied at the orbifold point. We then find that YB is *violated* as we move away from the orbifold point, for all $\kappa \in (0, 1)$, showing conclusively that the spin chain of the interpolating theory is *not* integrable for general κ. To our surprise however, YB holds again in the SCQCD limit $\kappa \to 0$! We take this as a good hint that planar $\mathcal{N} = 2$ SCQCD might be integrable, at least at one loop.

7.5 Discussion

Ordinarily, verification of the Yang-Baxter equation for the two-magnon S-matrix counts as strong evidence for integrability. In our case, however, for κ *strictly* zero, the elementary Q impurities "freeze", and only $Q\bar{Q}$ dimers can propagate on the chain. Correspondingly, the Q dispersion relation becomes

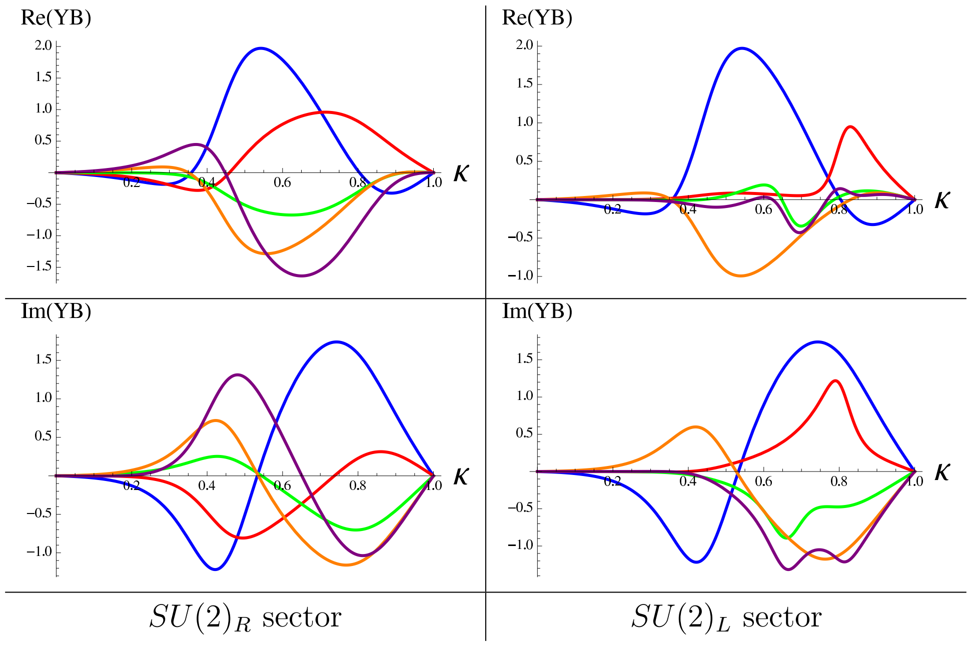


Figure 7.4: The figure shows the differences between the left and right-hand sides of the five Yang-Baxter equations, as a function of κ, for the specific choice of momenta $p_1 = 0.3$, $p_2 = 0.8$ and $p_3 = 1.4$. The blue, red, green, orange and purple curves show *(l.h.s)−(r.h.s)* for the the first to fifth equation.

momentum-independent,

$$E_Q(p;\kappa) = 2(1-\kappa)^2 + 8\kappa\,\sin^2(\frac{p}{2}) \underset{\kappa\to 0}{\longrightarrow} 2\,, \tag{7.105}$$

and the S-matrix also degenerates to a simple expression. Verification of YB strictly at $\kappa = 0$ may then appear like an accident due to this degenerate limit. What we find more significant, and non-trivial evidence for integrability, is that the integrable point $\kappa = 0$ is reached smoothly, with YB failing infinitesimally for infinitesimal κ – this is clear since the S-matrices are analytic (rational) functions of κ. This smooth behavior is illustrated in figure 7.4, where we plot the differences between the left and right hand sides of the five equations (7.100–7.104) (for some choice of the momenta).

An elegant way to conclusively prove integrability at $\kappa = 0$ would be to exhibit the algebraic Bethe ansatz for the SCQCD spin chain. The simplest guess for the R-matrix does not appear to work [186], but the issue needs further investigation.

Another approach being pursued [186] is to *assume* integrability to derive Bethe equations for the periodic chain, and then check whether their (numerical) solutions agree with the solutions obtained by direct diagonalization of the Hamiltonian. This is not entirely straightforward, because we cannot work strictly at $\kappa = 0$. The naive Bethe equations at $\kappa = 0$ have no interesting solutions for finite values of the Bethe roots – the non-trivial dynamics is hidden in Bethe roots with infinite imaginary parts (in the momentum variable). We saw this phenomenon in the evolution of the bound states as $\kappa \to 0$, where the individual magnon momenta behave as $i \log \kappa$. Taking the SCQCD limit $\kappa \to 0$ too early we lose information about the bound states. (It is conceivable that the failure of the (simplest) algebraic Bethe ansatz is also due to this order-of-limits issue.) Nevertheless, it makes sense to write Bethe equations for small κ, viewed as a regulator to be removed at the end of the calculation. We can also calculate the S-matrix of the bound states, by using the fusion procedure for infinitesimal κ, and check *their* YB equation in the SCQCD limit. The consistency of this approach should follow from the smoothness of the $\kappa \to 0$ limit.

A natural extension of our work is the calculation of one-loop dilation operator in the complete theory, including fermions and derivatives [187]. Let us briefly comment on the symmetry structure of the complete spin chain. As is well-known, the symmetry of the $\mathcal{N} = 4$ spin chain in the excitation picture is $PSU(2|2)_L \times PSU(2|2)_R \times \mathbb{R}$, where the central factor $\mathbb{R}$ is identified with the Hamiltonian. The $\mathbb{Z}_2$ orbifold projection preserves the $PSU(2|2)_R$ in the "right" sector (this is a subgroup of the $\mathcal{N} = 2$ superconformal group $SU(2,2|2)$), but breaks $PSU(2|2)_L$ to the bosonic subgroup $SU(2)_L \times SU(2)_\alpha$, where $SU(2)_\alpha$ denotes the left-handed Lorentz symmetry. At the orbifold point $\kappa = 1$, the breaking is only due to a global twist of the chain, while locally the symmetry is the same as in $\mathcal{N} = 4$, but for $\kappa \neq 1$ the symmetry is truly broken. All in all, the symmetry of the spin chain of the interpolating theory is $SU(2)_L \times SU(2)_\alpha \times PSU(2|2)_R \times \mathbb{R}$. In this chapter we have found that in the two-body S-matrix of Q impurities has a left $\times$ right factorization, and we expect this feature to persist for the full chain [187].

An obvious question is whether symmetry is sufficient to fix the form of the S-matrix, as it does to all loops in $\mathcal{N} = 4$ SYM (up to an overall scalar factor).

While unlikely for S_L, this is likely for S_R, which has a large supergroup symmetry. Note that the left sector is trivial in SCQCD limit (by construction, we restrict to flavor singlets, which are in particular $SU(2)_L$ singlets), so luckily the right-sector dynamics is both the most relevant and the most constrained. In fact, the symmetry in the right sector of the interpolating SCFT the same as in (either sector of) $\mathcal{N} = 4$ SYM. This raises a small puzzle. The S_R matrix of $\mathcal{N} = 4$ is uniquely fixed, up to an overall scalar factor, from the (centrally extended) $SU(2|2)_R$ symmetry [23]. But our results for S_R in the interpolating theory are definitely different (for $\kappa \neq 1$) from the $\mathcal{N} = 4$ results. This is clear already in the scalar sector studied in this chapter, by inspection of the S-matrix of the $Q_{\mathcal{I}\hat{+}}$ impurities. In the next chapter, we compute the S-matrix in the right-sector to all loops using centrally extended $SU(2|2)$ symmetry. In the process we answer the puzzle raised here.

Finally it would be very interesting to evaluate the two-body S-matrix at strong coupling, in the dual string sigma-model, and see whether it has the same κ dependence as the perturbative S-matrix. Failure of integrability for generic κ is not an issue here, since we would not be using in any way factorization of n-body scattering, but rather focus on the two-body S-matrix, which we expect to have a smooth interpolation from weak to strong coupling. The sigma-model at the orbifold point is well-known, and moving away from the orbifold point corresponds to changing the value of a theta angle β (the period of the NSNS B-field through the collapsed cycle of the orbifold). The orbifold point corresponds to $\beta = 1/2$, while the SCQCD limit corresponds to $\beta \to 0$. From the dual side, it is very natural to expect integrability precisely at the two extrema 0 and $1/2$, but not for generic values of the B-field. A toy model for this behavior is the $O(3)$ sigma-model in a magnetic field [188].[12]

One of our original motivations was to collect "bottom-up" clues about the string dual of $\mathcal{N} = 2$ SCQCD. While firm conclusions will have to wait a higher-oder (all order?) analysis, we can already see a qualitative agreement with the "top-down" approach of our previous chapter. We argued that $\mathcal{N} = 2$ SCQCD is dual to a non-critical string background, with seven geometric dimensions, containing both an AdS_5 and an S^1 factor. Rotation in S^1 corresponds to the $U(1)_r$ quantum number. In lightcone quantization of the

[12] Similar considerations apply to the ABJ model [177, 179, 180].

sigma-model, the lightcone coordinates would be obtained by combining this S^1 and the timelike direction of AdS_5. We then expect five bosonic gapless excitations, four associated to the transverse AdS coordinates and one to the seventh dimension. The vacuum of the lightcone sigma-model corresponds to chiral vacuum $\mathrm{Tr}\,\phi^\ell$ of the spin chain, while the four AdS excitations correspond to derivative impurities on the chain. In the scalar sector that we have studied in this chapter, *one* gapless excitation is then expected, the one corresponding to the seventh dimension: just what we found, the gapless magnon $T(p)$. As $\kappa \to 0$, the Q impurities, carriers of the $SU(2)_L \times SU(2)_R$ quantum numbers associated with the three extra dimensions (the transverse S^3, see [2] for details), become non-dynamical, and only their composite bound state $T(p)$ survives as a gapless mode. We interpret this phenomenon as the field theory counterpart of the transition from the critical to the non-critical background.

Chapter 8

Twisted Magnons

The spin chain associated to the planar dilation operator of $\mathcal{N} = 4$ super-Yang Mills [15, 166, 189] is strongly constrained by symmetry. While the structure of the Hamiltonian becomes unwieldy beyond one loop, and no closed form is yet in sight, the S-matrix of magnon excitations of the infinite chain is a relatively simple object [23, 167, 168]. Assuming integrability (for which there is by now strong evidence), the n-body S-matrix factorizes in terms of two-body S-matrices. In turn, the full matrix structure of the two-body S-matrix is fixed by Beisert's centrally extended $SU(2|2) \times SU(2|2)$ symmetry [23]. Finally, the overall phase is determined with the help of crossing symmetry and plausible physical assumptions [169, 190–192].

The centrally extended $SU(2|2)$ symmetry is a general feature of spin chains for $\mathcal{N} = 2$ $4d$ superconformal theories[1], indeed $SU(2|2)$ is a subgroup of the $\mathcal{N} = 2$ superconformal group $SU(2,2|2)$ preserved by the choice of the spin chain vacuum. In this chapter we explore the consequences of this symmetry in a class of $\mathcal{N} = 2$ SCFTs, the quiver theories related by exactly marginal deformations to $\mathcal{N} = 2$ orbifolds of $\mathcal{N} = 4$ super-Yang Mills.

Unlike the case of $\mathcal{N} = 4$ SYM, only one copy of the $SU(2|2)$ supergroup is preserved, while the other is broken to its bosonic subgroup. We show how to fix the dispersion relations and two-body S-matrices of the magnons transforming under the surviving $SU(2|2)$ by a generalization of Beisert's approach. Since the $SU(2|2)$ representations are now "twisted", the generalization is not entirely trivial and leads to interesting functions of the exactly marginal cou-

[1] See also [193] for applications of $SU(2|2)$ to a class theories with 16 supercharges.

plings. At the orbifold point the magnons are gapless and the spin chain is integrable [172, 173] but as we perturb away from it, the magnons acquire a gap, and their two-body S-matrices do not satisfy the Yang-Baxter equation. So for general values of the couplings the theories are not integrable, and the complete magnon S-matrix cannot be deduced from the two-body S-matrix. Nevertheless the dispersion relations and two-body S-matrices are interesting pieces of information in their own right, and it is remarkable that one can obtain for them all-order expressions. At one-loop, we find agreement with the explicit perturbative calculations of [4, 187]. At strong 't Hooft coupling, one should be able to compare our field-theoretic results with a giant-magnon [194] calculation in the dual string theory, which is a deformation of the orbifold background $AdS_5 \times S^5/\Gamma$ [83, 144].

For ease of notation, in most of the chapter we focus on the simplest case, the $\mathcal{N} = 2$ superconformal quiver with $SU(N_c) \times SU(N_{\check{c}})$ gauge group,[2] which is in the moduli space of the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM. In section 8.1 we determine the dispersion relation of the bifundamental magnons and in section 8.2 their two-body S-matrix.

Following Berenstein et al. [195], in section 8.3 we re-derive the dispersion relations of the twisted magnons from a large N analysis of the quiver matrix model, obtained by quantizing the gauge theory on $S^3 \times \mathbb{R}$ and keeping the zero modes on S^3. It is not a priori obvious that this approach, which relies on an uncontrolled approximation, should give the same answer as the exact algebraic analysis, but it does. This viewpoint gives a simple geometric interpretation of dispersion relations, very suggestive of an emergent dual geometry.

The generalization to $\mathcal{N} = 2$ $\mathbb{Z}_k$ orbifolds is straightforward, and we indicate it in section 8.4.

In the rest of this introduction we describe the symmetry structure of the $\mathbb{Z}_2$-quiver spin chain, contrasting it with the $\mathcal{N} = 4$ chain. This will serve as an overview of our logic and to orient the reader through our notations.

The superconformal symmetry of $\mathcal{N} = 4$ SYM is $PSU(2, 2|4)$. It is broken to $PSU(2|2) \times PSU(2|2) \times \mathbb{R}$, where $\mathbb{R}$ is a central generator corresponding to

[2] The two gauge groups are identical, $N_c \equiv N_{\check{c}}$, but we find it useful to always denote with a "check" quantities associated to the second gauge group.

the spin chain Hamiltonian, by the choice of the BMN [196] vacuum $\mathrm{Tr}\,\Phi^J$. The magnon excitations on this vacuum are in the fundamental representation of the unbroken symmetry, and they are gapless because they are the Goldstone modes associated to the broken generators. The $PSU(2,2|4)$ symmetry generators are shown in table 8.1. The boxed generators, in the diagonal blocks, are preserved by the choice of the vacuum while the off-diagonal ones are broken and correspond to the magnons. The broken generators are labelled in terms of the corresponding magnons: the upper-right block contains the magnon creation operators and the lower-left block the magnon annihilation operators.

	$SU(2_{\dot\alpha})$	$SU(2_I)$	$SU(2_\alpha)$	$SU(2_{\hat I})$
$SU(2_{\dot\alpha})$	$\boxed{\mathcal{L}^{\dot\alpha}{}_{\dot\beta}}$	$\boxed{\mathcal{Q}^{\dot\alpha}{}_{J}}$	$D^{\dagger\dot\alpha}{}_{\beta}$	$\lambda^{\dagger\dot\alpha}{}_{\hat J}$
$SU(2_I)$	$\boxed{\mathcal{S}^{I}{}_{\dot\beta}}$	$\boxed{\mathcal{R}^{I}{}_{J}}$	$\lambda^{\dagger I}{}_{\beta}$	$\mathcal{X}^{\dagger I}{}_{\hat J}$
$SU(2_\alpha)$	$D^{\alpha}{}_{\dot\beta}$	$\lambda^{\alpha}{}_{J}$	$\boxed{\mathcal{L}^{\alpha}{}_{\beta}}$	$\boxed{\mathcal{Q}^{\alpha}{}_{\hat J}}$
$SU(2_{\hat I})$	$\lambda^{\hat I}{}_{\dot\beta}$	$\mathcal{X}^{\hat I}{}_{J}$	$\boxed{\mathcal{S}^{\hat I}{}_{\beta}}$	$\boxed{\mathcal{R}^{\hat I}{}_{\hat J}}$

Table 8.1: The $PSU(2,2|4)$ symmetry generators. The R-symmetry subgroup $SU(4)$ is represented as branched into $SU(2_I)\times SU(2_{\hat I})$. We have introduced the notation $SU(2_\alpha)$ for $SU(2)_\alpha$ etc.

A priori, the two-body magnon S-matrix, decomposed according to the $SU(2_\alpha|2_{\hat I})\times SU(2_{\dot\alpha}|2_I)$ quantum numbers, can take the schematic form

$$S_{SU(2_\alpha|2_{\hat I})\times SU(2_{\dot\alpha}|2_I)} = S_{SU(2_\alpha|2_{\hat I})}\otimes S_{SU(2_{\dot\alpha}|2_I)} + S'_{SU(2_\alpha|2_{\hat I})}\otimes S'_{SU(2_{\dot\alpha}|2_I)} + \dots \tag{8.1}$$

As it turns out, the $SU(2|2)$ S-matrix is unique up to an overall phase [23], so one has the useful factorization

$$S_{SU(2_\alpha|2_{\hat I})\times SU(2_{\dot\alpha}|2_I)} = S_{SU(2_\alpha|2_{\hat I})}\otimes S_{SU(2_{\dot\alpha}|2_I)}\,. \tag{8.2}$$

The $SU(2_\alpha|2_{\hat I})$ S-matrix describes the scattering of magnons in the highest weight state of $SU(2_{\dot\alpha}|2_I)$, and viceversa.

The $\mathbb{Z}_2$ projection of $\mathcal{N}=4$ SYM breaks $PSU(2_\alpha,2_{\dot\alpha}|4_{I\hat I})$ to $SU(2_\alpha,2_{\dot\alpha}|2_I)\times SU(2_{\hat I})$. At the orbifold point $g_{YM}=\check{g}_{YM}$ the breaking is only global (by boundary conditions on the periodic chain), but for general couplings the $PSU(2_\alpha, 2_{\dot\alpha}|4_{I\hat I})$ is truly lost. The symmetry preserved by the spin chain vacuum is $SU(2_{\dot\alpha}|2_I) \times SU(2_\alpha) \times SU(2_{\hat I})$. Table 8.2 lists the symmetry generators of the theory, with the broken generators identified as Goldstone modes. The Goldstone excitations (gapless magnons) are in the fundamental representation of $SU(2_\alpha) \times SU(2_{\dot\alpha}|2_I)$. The $\{\mathcal{X}^I_{\hat J}, \lambda^{\dot\alpha}_{\hat J}\}$ magnons, in the fundamental of $SU(2_{\hat I}) \times SU(2_{\dot\alpha}|2_I)$, are omitted in table 8.2 because they do not correspond to broken generators – indeed they have a gap for $g_{YM} \neq \check g_{YM}$. Their dynamics is the main focus of this chapter.

Here we are using the "orbifold" notation, where the fields are labeled as in $\mathcal{N} = 4$ SYM, and are $2N_c \times 2N_c$ matrices in color space (see equ.(8.19)). The state space of the spin chain consists of an twisted and and untwisted sector, distinguished by whether or not the twist operator τ (equ.(8.17)) is inserted on the chain. The two sectors mix for $g_{YM} \neq \check g_{YM}$. In particular the symmetry generators and the central charges acquire twisted components, see (8.22, 8.23).

	$SU(2_{\dot\alpha})$	$SU(2_I)$	$SU(2_\alpha)$	$SU(2_{\hat I})$
$SU(2_{\dot\alpha})$	$\boxed{\mathcal{L}^{\dot\alpha}_{\dot\beta}}$	$\boxed{\mathcal{Q}^{\dot\alpha}_J}$	$D^{\dagger\dot\alpha}_\beta$	
$SU(2_I)$	$\boxed{\mathcal{S}^I_{\dot\beta}}$	$\boxed{\mathcal{R}^I_J}$	$\lambda^{\dagger I}_\beta$	
$SU(2_\alpha)$	$D^\alpha_{\dot\beta}$	λ^α_J	$\boxed{\mathcal{L}^\alpha_\beta}$	
$SU(2_{\hat I})$				$\boxed{\mathcal{R}^{\hat I}_{\hat J}}$

Table 8.2: The generators of $SU(2,2|2) \times SU(2_{\hat I})$, the symmetry of the $\mathbb{Z}_2$ quiver. As before, the boxed generator are preserved by the choice of the spin-chain vacuum while the other correspond to Goldstone excitations.

The scattering of any two magnons (gapless or gapped) is given by a factorized two-body S-matrix,

$$S_{SU(2_\alpha)\times SU(2_{\hat I})\times SU(2_{\dot\alpha}|2_I)} = S_{SU(2_\alpha)\times SU(2_{\hat I})} \otimes S_{SU(2_{\dot\alpha}|2_I)}\,. \tag{8.3}$$

The $S_{SU(2_{\dot\alpha}|2_I)}$ S-matrix describes the scattering of magnons in the highest weight of $SU(2_\alpha)\times SU(2_{\hat I})$. It has both an untwisted and a twisted component, schematically

$$S_{SU(2_{\dot{\alpha}}|2_I)}\,|\mathcal{X}_1\mathcal{X}_2\rangle = \mathcal{S}^{\mathbb{I}}\,|\mathcal{X}_1\mathcal{X}_2\rangle + \mathcal{S}^{\tau}\,|\mathcal{X}_1\mathcal{X}_2\tau\rangle\,. \tag{8.4}$$

The centrally extended $SU(2|2)$ symmetry will fix both components uniquely, up to the usual phase ambiguity.

8.1 Magnon Dispersion Relations

8.1.1 Review: $\mathcal{N}=4$ magnons

The field content of $\mathcal{N}=4$ super Yang-Mills consists of the gauge field A_μ, four Weyl spinors λ^A_α and six real scalars X^i, where $A=1,\ldots 4$ and $i=1,\ldots 6$ are indices labelling fundamental and antisymmetric self-dual representation of the $SU(4_A)$ R-symmetry group respectively. Under $U(1)_r\times SU(2_I)_R\times SU(2_{\hat{I}})_L\subset SU(4_A)$, the scalars branch into one complex scalar Φ, charged under $U(1)_r$, and $SU(2_I)_R\times SU(2_{\hat{I}})_L$ bifundamental scalars $\mathcal{X}^{I\hat{I}}$, with zero $U(1)_r$ charge, satisfying the reality condition $\mathcal{X}^{I\hat{I}\dagger}=-\epsilon^{IJ}\epsilon^{\hat{I}\hat{J}}\mathcal{X}^{J\hat{J}}$. The fermions decompose as λ^I_α and $\lambda^{\hat{I}}_\alpha$. The $\mathcal{N}=2$ supersymmetry organizes $A_\mu,\lambda^I_\alpha,\Phi$ into a vector multiplet and $\mathcal{X}^{I\hat{I}},\lambda^{\hat{I}}_\alpha$ into a hypermultiplet.

For definiteness we focus on the "right-handed" magnons, in the fundamental of $SU(2_{\dot{\alpha}}|2_I)$ and in the highest-weight state of of $SU(2_\alpha|2_{\hat{I}})$,

$$\mathcal{X}^I_{\hat{+}}\equiv\mathcal{X}^I\,,\quad \lambda^{\dot{\alpha}}_{\hat{+}}\equiv\lambda^{\dot{\alpha}}\,. \tag{8.5}$$

Beisert determined the magnon dispersion relation from symmetry arguments, as we now review. The non-zero commutation relations of the $SU(2|2)$ generators are:

$$\begin{aligned}
[\mathcal{R}^I{}_J,\mathcal{J}^K] &= \delta^K_J\mathcal{J}^I-\frac{1}{2}\delta^I_J\mathcal{J}^K\\
[\mathcal{L}^{\dot{\alpha}}{}_{\dot{\beta}},\mathcal{J}^{\dot{\gamma}}] &= \delta^{\dot{\gamma}}_{\dot{\beta}}\mathcal{J}^{\dot{\alpha}}-\frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}\mathcal{J}^{\dot{\gamma}}\\
\{\mathcal{Q}^{\dot{\alpha}}{}_I,\mathcal{S}^J{}_{\dot{\beta}}\} &= \delta^J_I\mathcal{L}^{\dot{\alpha}}{}_{\dot{\beta}}+\delta^{\dot{\alpha}}_{\dot{\beta}}\mathcal{R}^J{}_I+\delta^J_I\delta^{\dot{\alpha}}_{\dot{\beta}}\mathcal{C}
\end{aligned}$$

where $\mathcal{J}$ represents any generator with the appropriate index. The central charge $\mathcal{C}$ is related to the scaling dimension as $\mathcal{C}=\frac{1}{2}(\Delta-|r|)$. The impurities

$(\mathcal{X}^{\mathcal{I}}, \lambda^{\dot{\alpha}})$ transform in the fundamental representation of $SU(2|2)$, and closure of the algebra fixes $\mathcal{C} = \frac{1}{2}$, corresponding to the canonical dimensions $\Delta = 1$ and $\Delta = \frac{3}{2}$ for $\mathcal{X}$ and λ. Consider now a magnon of momentum p,

$$\Psi(p) = \sum_{l=-\infty}^{\infty} e^{ipl} |\mathcal{X}(l)\rangle . \tag{8.6}$$

For $p \neq 0$, the state acquires a non-vanishing anomalous dimension, so $\mathcal{C} \neq \frac{1}{2}$, but the representation remains short, as there are no other degrees of freedom with which it could combine to become long. This is in conflict with the $SU(2|2)$ algebra. The resolution is to allow for a further central extension by momentum-dependent central charges $\mathcal{P}$ and $\mathcal{K}$,

$$\{\mathcal{Q}^{\dot{\alpha}}_{I}, \mathcal{Q}^{\dot{\beta}}_{J}\} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{IJ}\mathcal{P}, \quad \{\mathcal{S}^{I}_{\dot{\alpha}}, \mathcal{S}^{J}_{\dot{\beta}}\} = \epsilon^{IJ}\epsilon_{\dot{\alpha}\dot{\beta}}\mathcal{K} . \tag{8.7}$$

The most general action of the generators in the excitation picture is :

$$\begin{aligned}
\mathcal{Q}^{\dot{\alpha}}_{I}|\mathcal{X}^{J}\rangle &= a\delta^{J}_{I}|\lambda^{\dot{\alpha}}\rangle \\
\mathcal{Q}^{\dot{\alpha}}_{I}|\lambda^{\dot{\beta}}\rangle &= b\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{IJ}|\mathcal{X}^{J}\Phi^{+}\rangle \\
\mathcal{S}^{I}_{\dot{\alpha}}|\mathcal{X}^{J}\rangle &= c\epsilon^{IJ}\epsilon_{\dot{\alpha}\dot{\beta}}|\lambda^{\dot{\beta}}\Phi^{-}\rangle \\
\mathcal{S}^{I}_{\dot{\alpha}}|\lambda^{\dot{\beta}}\rangle &= d\delta^{\dot{\beta}}_{\dot{\alpha}}|\mathcal{X}^{I}\rangle ,
\end{aligned} \tag{8.8}$$

which implies

$$\mathcal{P}|\mathcal{X}\rangle = ab|\mathcal{X}\Phi^{+}\rangle \tag{8.9}$$

$$\mathcal{K}|\mathcal{X}\rangle = cd|\mathcal{X}\Phi^{-}\rangle . \tag{8.10}$$

$$\mathcal{C}|\mathcal{X}\rangle = \frac{1}{2}(ad + bc)|\mathcal{X}\rangle . \tag{8.11}$$

Closure of the algebra requires $ad - bc = 1$. We can then formally solve

$$\mathcal{C} = \frac{1}{2}\sqrt{1 + 4\mathcal{P}\mathcal{K}} . \tag{8.12}$$

For a quick heuristic derivation of the central charges, we can proceed as follows. The supersymmetry transformations of the fields appearing in the

Lagrangian,

$$\begin{aligned}
\mathcal{Q}^{\dot\alpha}_{I}\mathcal{X}^K &= \delta^K_I\lambda^{\dot\alpha} \\
\mathcal{Q}^{\dot\beta}_{J}\lambda^{\dot\alpha} &= \epsilon^{\dot\beta\dot\alpha}\frac{\partial W}{\partial \mathcal{X}^J} = \frac{g}{\sqrt{2}}\epsilon^{\dot\beta\dot\alpha}\epsilon_{JL}[\mathcal{X}^L,\Phi]
\end{aligned}$$

where $W = \frac{g}{\sqrt{2}}\mathrm{Tr}\ \mathcal{X}^{I\hat{I}}\Phi\mathcal{X}_{I\hat{I}}$ is the superpotential of $\mathcal{N}=4$ super Yang-Mills. The coupling g is the square root of the 't Hooft coupling, normalized as

$$g^2 = \frac{g^2_{YM}N_c}{8\pi^2}\,. \tag{8.13}$$

These susy transformations lead to the anticommutators

$$\begin{aligned}
\{\mathcal{Q}^{\dot\alpha}_{I},\mathcal{Q}^{\dot\beta}_{J}\}\mathcal{X}^K &= \frac{g}{\sqrt{2}}\epsilon^{\dot\alpha\dot\beta}\epsilon_{IJ}[\Phi,\mathcal{X}^K] \\
\{\mathcal{Q}^{\dot\alpha}_{I},\mathcal{Q}^{\dot\beta}_{J}\}\lambda^{\dot\gamma} &= \frac{g}{\sqrt{2}}\epsilon^{\dot\alpha\dot\beta}\epsilon_{IJ}[\Phi,\lambda^{\dot\gamma}]
\end{aligned}$$

Using the fact that momentum eigenstates satisfy

$$|\Phi^\pm\mathcal{X}\rangle = e^{\mp ip}|\mathcal{X}\Phi^\pm\rangle\,, \tag{8.14}$$

we can realize the susy transformation laws on the spin chain as

$$\{\mathcal{Q}^{\dot\alpha}_{I},\mathcal{Q}^{\dot\beta}_{J}\}|\mathcal{X}\rangle = \epsilon^{\dot\alpha\dot\beta}\epsilon_{IJ}\mathcal{P}|\mathcal{X}\rangle = \epsilon^{\dot\alpha\dot\beta}\epsilon_{IJ}\frac{g}{\sqrt{2}}(e^{-ip}-1)|\mathcal{X}\Phi^+\rangle\,, \tag{8.15}$$

implying $ab = \frac{g}{\sqrt{2}}(e^{-ip}-1)$. Similarly using $\{\mathcal{S},\mathcal{S}\}$, we can obtain $cd = \frac{g}{\sqrt{2}}(e^{ip}-1)$. Finally, from (8.12),

$$\Delta - |r| = 2\mathcal{C} = \sqrt{1+8g^2\sin^2\frac{p}{2}}\,. \tag{8.16}$$

This derivation[3] is only heuristic because of the assumption that the susy transformations in the excitation picture can be simply read off from the classical Lagrangian. In [23], Beisert used a purely algebraic method to determine the central charges, as we review in appendix O. The algebraic method confirms the form (8.16), but with g^2 a priori replaced by a renormalized coupling

[3]The first field-theoretic argument for the square-root form (8.16) was given in [197].

$\mathbf{g}^2 = g^2 + O(g^4)$. There is strong evidence that in $\mathcal{N} = 4$ SYM $\mathbf{g}^2 = g^2$. In the ABJM theory [178] one can run an identical argument, but the coupling *is* renormalized [176, 198, 199]. See [200, 201] for discussions of this issue.

8.1.2 The $\mathbb{Z}_2$ orbifold and its deformation

The $\mathbb{Z}_2$ orbifold theory is the well known quiver gauge theory living on the worldvolume of D3 branes probing $\mathbb{R}^2 \times \mathbb{R}^4/\mathbb{Z}_2$ singularity. It is obtained from $\mathcal{N} = 4$ super Yang-Mills by projecting onto the $\mathbb{Z}_2 \subset SU(2)_L$ invariant states. The $\mathbb{Z}_2$ action identifies $\mathcal{X}^{I\hat{I}} \to -\mathcal{X}^{I\hat{I}}$ while acting trivially on Φ. The supersymmetry is broken to $\mathcal{N} = 2$ as the supercharges with $SU(2)_L$ indices are projected out. The $SU(4)$ R symmetry group is broken to $SU(2)_R \times SO(3)_L \times U(1)_r$. $SU(2)_R \times U(1)_r$ is the R symmetry group of the $\mathcal{N} = 2$ theory while $SO(3)_L$ is a global symmetry. In color space, we start with $SU(2N_c)$ gauge group and declare the nontrivial element of the orbifold to be

$$\tau = \begin{pmatrix} \mathbb{I}_{N_c \times N_c} & 0 \\ 0 & -\mathbb{I}_{N_{\check{c}} \times N_{\check{c}}} \end{pmatrix}. \tag{8.17}$$

It acts on the fields of $\mathcal{N} = 4$ SYM as

$$A_\mu \to \tau A_\mu \tau, \quad \Phi \to \tau \Phi \tau, \quad \lambda^I \to \tau \lambda^I \tau, \quad \mathcal{X}^{I\hat{I}} \to -\tau \mathcal{X}^{I\hat{I}} \tau, \quad \lambda^{\hat{I}} \to -\tau \lambda^{\hat{I}} \tau. \tag{8.18}$$

The components that survive the projection are

$$\begin{aligned} A_\mu &= \begin{pmatrix} A_\mu & 0 \\ 0 & \check{A}_\mu \end{pmatrix}, \quad \Phi = \begin{pmatrix} \phi & 0 \\ 0 & \check{\phi} \end{pmatrix}, \quad \lambda^I = \begin{pmatrix} \lambda^I & 0 \\ 0 & \check{\lambda}^I \end{pmatrix}, \\ \mathcal{X}^{I\hat{I}} &= \begin{pmatrix} 0 & Q^{I\hat{I}} \\ \bar{Q}^{I\hat{I}} & 0 \end{pmatrix}, \quad \lambda^{\hat{I}} = \begin{pmatrix} 0 & \psi^{\hat{I}} \\ \tilde{\psi}^{\hat{I}} & 0 \end{pmatrix}. \end{aligned} \tag{8.19}$$

The orbifold theory has an untwisted sector of states, which descend by projection from $\mathcal{N}=4$, and a twisted sector of states, characterized by the presence of one insertion of the twist operator τ in the color trace. We refer to this presentation of the theory (in terms of $2N_c \times 2N_c$ matrices) as the "orbifold basis".

Equivalently, we can present the theory as an $\mathcal{N}=2$ quiver gauge theory with product gauge group $SU(N_c)\times SU(N_{\check{c}})$ and two bifundamental hypermultiplets: (A_μ, λ^I, ϕ) and $(\check{A}_\mu, \check{\lambda}, \check{\phi})$ are the two vector multiplets while $(Q^{I\hat{I}}, \psi^{\hat{I}})$ and $(\bar{Q}^{I\hat{I}}, \tilde{\psi}^{\hat{I}})$ are the two hypermultiplets transforming respectively in the $\mathbf{N}_c \times \overline{\mathbf{N}}_{\check{c}}$ and $\overline{\mathbf{N}}_c \times \mathbf{N}_{\check{c}}$ representations.

The two gauge couplings g and $\check{g}$ are exactly marginal. For $g \neq \check{g}$ the superpotential acquires a twisted term,

$$W = \frac{G}{\sqrt{2}}\mathrm{Tr}\,[\frac{1}{2}(\sqrt{\kappa}+\frac{1}{\sqrt{\kappa}}) + \tau\frac{1}{2}(\sqrt{\kappa}-\frac{1}{\sqrt{\kappa}})]\mathcal{X}^{I\hat{I}}\Phi\mathcal{X}_{I\hat{I}} \tag{8.20}$$

where

$$G \equiv \sqrt{g\check{g}}\,, \qquad \kappa \equiv \frac{\check{g}}{g}\,. \tag{8.21}$$

In the quiver language,

$$\begin{aligned} W &= \frac{g}{\sqrt{2}}\mathrm{Tr}\;\bar{Q}^{I\hat{I}}\phi Q_{I\hat{I}} + \frac{\check{g}}{\sqrt{2}}Q^{I\hat{I}}\check{\phi}\bar{Q}_{I\hat{I}} \\ &= \frac{G}{\sqrt{2}}(\mathrm{Tr}\;\frac{1}{\sqrt{\kappa}}\bar{Q}^{I\hat{I}}\phi Q_{I\hat{I}} + \sqrt{\kappa}Q^{I\hat{I}}\check{\phi}\bar{Q}_{I\hat{I}})\,. \end{aligned}$$

8.1.3 Twisted magnons

As we have explained in the introduction, the magnons of the $\mathbb{Z}_2$ theory fall into two classes: Goldstone magnons associated with the broken generators, carrying an α index, and magnons not associated with symmetries, carrying a $\hat{I}$ index. Both types are in the fundamental representation of $SU(2_{\dot{\alpha}}|2_I)$. The algebraic analysis for the Goldstone magnons is exactly as in $\mathcal{N}=4$ SYM, so they obey the same dispersion relation. On the other hand, the non-Goldstone magnons transform in a "twisted" representation of the $SU(2|2)$ superalgebra,

$$\begin{aligned}
\mathcal{Q}^{\dot\alpha}_I|\mathcal{X}^J\rangle &= a_0\delta^J_I|\lambda^{\dot\alpha}\rangle + a_1\delta^J_I|\tau\lambda^{\dot\alpha}\rangle \\
\mathcal{Q}^{\dot\alpha}_I|\lambda^{\dot\beta}\rangle &= b_0\epsilon^{\dot\alpha\dot\beta}\epsilon_{IJ}|\mathcal{X}^J\Phi^+\rangle + b_1\epsilon^{\dot\alpha\dot\beta}\epsilon_{IJ}|\tau\mathcal{X}^J\Phi^+\rangle \\
\mathcal{S}^I_{\dot\alpha}|\mathcal{X}^J\rangle &= c_0\epsilon^{IJ}\epsilon_{\dot\alpha\dot\beta}|\lambda^{\dot\beta}\Phi^-\rangle + c_1\epsilon^{IJ}\epsilon_{\dot\alpha\dot\beta}|\tau\lambda^{\dot\beta}\Phi^-\rangle \\
\mathcal{S}^I_{\dot\alpha}|\lambda^{\dot\beta}\rangle &= d_0\delta^{\dot\beta}_{\dot\alpha}|\mathcal{X}^I\rangle + d_1\delta^{\dot\beta}_{\dot\alpha}|\tau\mathcal{X}^I\rangle
\end{aligned} \tag{8.22}$$

One then finds for the central charges:

$$\begin{aligned}
\mathcal{P}|\mathcal{X}\rangle &= (a_0b_0 + a_1b_1)|\mathcal{X}\Phi^+\rangle + (a_0b_1 + a_1b_0)|\tau\mathcal{X}\Phi^+\rangle \\
\mathcal{K}|\mathcal{X}\rangle &= (c_0d_0 + c_1d_1)|\mathcal{X}\Phi^-\rangle + (c_0d_1 + c_1d_0)|\tau\mathcal{X}\Phi^-\rangle \\
\mathcal{C}|\mathcal{X}\rangle &= [\frac{1}{2}(a_0d_0 + b_0c_0) + \frac{1}{2}(a_1d_1 + b_1c_1)]|\mathcal{X}\rangle \\
&+ [\frac{1}{2}(a_0d_1 + b_0c_1) + \frac{1}{2}(a_1d_0 + b_1c_0)]|\tau\mathcal{X}\rangle\,.
\end{aligned} \tag{8.23}$$

Using the supersymmetry transformations following from the deformed superpotential (8.20), a little calculation gives

$$\begin{aligned}
a_0b_0 + a_1b_1 &= \frac{G}{\sqrt{2}}\frac{1}{2}(\frac{1}{\sqrt{\kappa}} + \sqrt{\kappa})(e^{-ip} - 1) \\
a_0b_1 + a_1b_0 &= \frac{G}{\sqrt{2}}\frac{1}{2}(\frac{1}{\sqrt{\kappa}} - \sqrt{\kappa})(e^{-ip} + 1) \\
c_0d_0 + c_1d_1 &= \frac{G}{\sqrt{2}}\frac{1}{2}(\frac{1}{\sqrt{\kappa}} + \sqrt{\kappa})(e^{ip} - 1) \\
c_0d_1 + c_1d_0 &= \frac{G}{\sqrt{2}}\frac{1}{2}(\frac{1}{\sqrt{\kappa}} - \sqrt{\kappa})(e^{ip} + 1)\,.
\end{aligned} \tag{8.24}$$

We can then read off the central charges

$$\begin{aligned}
C_0 &\equiv \frac{1}{2}(a_0d_0 + b_0c_0) + \frac{1}{2}(a_1d_1 + b_1c_1) \\
&= \frac{1}{2}\sqrt{1 + 8G^2\left(\sin^2\frac{p}{2} + \frac{1}{4}(\sqrt{\kappa} - \frac{1}{\sqrt{\kappa}})^2\right)} \\
C_1 &\equiv \frac{1}{2}(a_0d_1 + b_0c_1) + \frac{1}{2}(a_1d_0 + b_1c_0) = 0\,.
\end{aligned}$$

It is illuminating to repeat the exercise in the quiver basis, as it will give us the dispersion relation of the perhaps more "physical" bifundamental excitations that interpolate between the $\mathrm{Tr}\,\phi^J$ and $\mathrm{Tr}\check{\phi}^J$ vacua. In the quiver basis, the $(\mathcal{X},\, \lambda)$ doublet splits into two doublets, $(Q,\, \psi)$ and $(\bar{Q},\, \tilde{\psi})$. Let us call these two fundamental $SU(2|2)$ representations V and $\tilde{V}$. The action of the algebra $\mathcal{A} : V \to V$ and $\mathcal{A} : \tilde{V} \to \tilde{V}$ is given in table 8.3.

$$
\begin{array}{rclrcl}
\mathcal{Q}^{\dot\alpha}_I|Q^J\rangle &=& a\delta^J_I|\psi^{\dot\alpha}\rangle & \mathcal{Q}^{\dot\alpha}_I|\bar{Q}^J\rangle &=& \tilde{a}\delta^J_I|\tilde{\psi}^{\dot\alpha}\rangle \\
\mathcal{Q}^{\dot\alpha}_I|\psi^{\dot\beta}\rangle &=& b\epsilon^{\dot\alpha\dot\beta}\epsilon_{IJ}|Q^J\check{\phi}^+\rangle & \mathcal{Q}^{\dot\alpha}_I|\tilde{\psi}^{\dot\beta}\rangle &=& \tilde{b}\epsilon^{\dot\alpha\dot\beta}\epsilon_{IJ}|\bar{Q}^J\phi^+\rangle \\
\mathcal{S}^I_{\dot\alpha}|Q^J\rangle &=& c\epsilon^{IJ}\epsilon_{\dot\alpha\dot\beta}|\psi^{\dot\beta}\check{\phi}^-\rangle & \mathcal{S}^I_{\dot\alpha}|\bar{Q}^J\rangle &=& \tilde{c}\epsilon^{IJ}\epsilon_{\dot\alpha\dot\beta}|\tilde{\psi}^{\dot\beta}\phi^-\rangle \\
\mathcal{S}^I_{\dot\alpha}|\psi^{\dot\beta}\rangle &=& d\delta^{\dot\beta}_{\dot\alpha}|Q^I\rangle & \mathcal{S}^I_{\dot\alpha}|\tilde{\psi}^{\dot\beta}\rangle &=& \tilde{d}\delta^{\dot\beta}_{\dot\alpha}|\bar{Q}^I\rangle .
\end{array}
$$

Table 8.3: Representation of the magnons in the quiver basis.

The a, b, c, d coefficients in this basis are related to the coefficients in the orbifold basis as $a = a_0 + a_1$, $\tilde{a} = a_0 - a_1$ and so on. One easily finds

$$
\begin{array}{rclr}
ab &=& \frac{G}{\sqrt{2}}(\frac{e^{-ip}}{\sqrt{\kappa}} - \sqrt{\kappa}) \equiv P & \tilde{a}\tilde{b} = \frac{G}{\sqrt{2}}(e^{-ip}\sqrt{\kappa} - \frac{1}{\sqrt{\kappa}}) \equiv \tilde{P} \\
cd &=& \frac{G}{\sqrt{2}}(\frac{e^{+ip}}{\sqrt{\kappa}} - \sqrt{\kappa}) \equiv K & \tilde{c}\tilde{d} = \frac{G}{\sqrt{2}}(e^{+ip}\sqrt{\kappa} - \frac{1}{\sqrt{\kappa}}) \equiv \tilde{K}\,.
\end{array}
\tag{8.25}
$$

Finally the dispersion relations for (Q, ψ) and $(\bar{Q}.\tilde{\psi})$ are

$$
\Delta - |r| = 2C \;=\; \sqrt{1+4PK} = \sqrt{1 + 8G^2\left(\sin^2\frac{p}{2} + \frac{1}{4}(\sqrt{\kappa} - \frac{1}{\sqrt{\kappa}})^2\right)} \tag{8.26}
$$

$$
\tilde{\Delta} - |r| = 2\tilde{C} \;=\; \sqrt{1+4\check{P}\check{K}} = \sqrt{1 + 8G^2\left(\sin^2\frac{p}{2} + \frac{1}{4}(\sqrt{\kappa} - \frac{1}{\sqrt{\kappa}})^2\right)} \tag{8.27}
$$

Recall the definitions $G \equiv \sqrt{g\check{g}}$, $\kappa \equiv \check{g}/g$. As expected, the non-Goldstone magnons acquire a gap for $g \neq \check{g}$. The derivation of the dispersion relation just presented suffers from the same criticism as the derivation in the $\mathcal{N} = 4$ case: a priori we should allow for renormalization of the gauge couplings. A purely algebraic method for determining $\mathcal{P}$ and $\mathcal{K}$, along the lines of [23], is described in the appendix O, and confirms this expectation. From symmetry alone, one can only conclude that both dispersion relations take the form

$$2C = 2\check{C} = \sqrt{1 + 2(\mathbf{g} - \check{\mathbf{g}})^2 + 8\mathbf{g}\check{\mathbf{g}} \sin^2 \frac{p}{2}} \tag{8.28}$$

where $\mathbf{g}(g, \check{g}) = g + \dots$ and $\check{\mathbf{g}}(g, \check{g}) = \check{g} + \dots$ are a priori renormalized couplings. (Of course such renormalization is known to *not* occur at the orbifold point $g = \check{g}$.) This issue also affects the forthcoming expressions for the S-matrix: the couplings g and $\check{g}$ could in principle be replaced by $\mathbf{g}$ and $\check{\mathbf{g}}$. The expansion of (8.28) agrees at one-loop with the result of [4]. It will be interesting to test it at higher orders.

8.2 Two-body S-matrix

The scattering problem is formulated on the infinite spin chain. The scattering of two Goldstone magnons is uninteresting, since the matrix structure of their two-body S-matrix is exactly as in $\mathcal{N} = 4$ SYM. We will focus on the scattering of two "non-Goldstone" magnons, both in the highest weight of $SU(2_{\hat{I}})$. The scattering of a Goldstone and a non-Goldstone magnon is also non-trivial, and could be studied by the same methods.

In the quiver basis, because of the index structure of the impurities, one of the non-Goldstone magnons must be from the Q multiplet and the other from the $\bar{Q}$ multiplet. Their ordering is fixed, we can have Q type magnons always on left of $\bar{Q}$ type ones, or viceversa. The scattering is pure reflection. For the case of Q type magnon on the left of $\bar{Q}$ type magnon, the schematic asymptotic form of the two body wavefunction is

$$\sum_{x_1 \ll x_2} (e^{ip_1 x_1 + ip_2 x_2} + S(p_2, p_1) e^{ip_2 x_1 + ip_1 x_2}) | \dots \phi Q(x_1) \check{\phi} \dots \check{\phi} \bar{Q}(x_2) \phi \dots \rangle. \tag{8.29}$$

This is the definition of the two body S matrix $S(p_1, p_2)$. We dropped the $SU(2|2)$ indices of the excitations for clarity. Similarly, for the other case where Q is on the right side of $\bar{Q}$, the aymptotic form of the wavefunction is

$$\sum_{x_1 \ll x_2} (e^{ip_1 x_1 + ip_2 x_2} + \check{S}(p_2, p_1) e^{ip_2 x_1 + ip_1 x_2}) | \dots \check{\phi} \bar{Q}(x_1) \phi \dots \phi Q(x_2) \check{\phi} \dots \rangle \tag{8.30}$$

which defines $\check{S}$. The two-body S matrices S and $\check{S}$ are related by exchanging $g \leftrightarrow \check{g}$,

$$S(p_1, p_2; g, \check{g}) = \check{S}(p_1, p_2; \check{g}, g). \tag{8.31}$$

For this reason, without loss of generality, we restrict our analysis to finding $S(p_1, p_2)$.

8.2.1 Rapidity variables

Following Beisert, a preliminary step is to solve for the coefficients a, b, c, d and $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ appearing in the magnon representation (table 8.3) in terms of convenient rapidity variables.

For the representation coefficients of the Q multiplet, we write

$$a = \gamma, \quad b = -\frac{G}{\sqrt{2}}\frac{1}{\gamma x^+}(x^+\sqrt{\kappa} - \frac{x^-}{\sqrt{\kappa}}), \quad c = \frac{G}{\sqrt{2}}\frac{i\gamma'}{x^-}, \quad d = -\frac{i}{\gamma'}(\frac{x^+}{\sqrt{\kappa}} - x^-\sqrt{\kappa}) \tag{8.32}$$

The relative factor between γ and γ' corresponds to relative rescalings of the fields Q and ψ and affects the S matrix as an overall phase. We choose $\gamma = \gamma'$.

For the $\bar{Q}$ coefficients, we write

$$\tilde{a} = \tilde{\gamma}, \quad \tilde{b} = -\frac{G}{\sqrt{2}}\frac{1}{\tilde{\gamma}\tilde{x}^+}(\frac{\tilde{x}^+}{\sqrt{\kappa}} - \tilde{x}^-\sqrt{\kappa}), \quad \tilde{c} = \frac{G}{\sqrt{2}}\frac{i\tilde{\gamma}}{\tilde{x}^-}, \quad \tilde{d} = -\frac{i}{\tilde{\gamma}}(\tilde{x}^+\sqrt{\kappa} - \frac{\tilde{x}^-}{\sqrt{\kappa}}). \tag{8.33}$$

Both pairs of rapidity variables obey $\frac{x^+}{x^-} = \frac{\tilde{x}^+}{\tilde{x}^-} = e^{ip}$. For hermitian representations we have to choose

$$|\gamma| = |i(x^-\sqrt{\kappa} - \frac{x^+}{\sqrt{\kappa}})|^{1/2}, \qquad |\tilde{\gamma}| = |i(\frac{\tilde{x}^-}{\sqrt{\kappa}} - \tilde{x}^+\sqrt{\kappa})|^{1/2}. \tag{8.34}$$

The closure of the algebra requires $ad - bc = 1$ and $\tilde{a}\tilde{d} - \tilde{b}\tilde{c} = 1$ *i.e.*

$$\begin{aligned}
\frac{x^+}{\sqrt{\kappa}} - x^-\sqrt{\kappa} + \frac{G^2}{2}(\frac{1}{x^+\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{x^-}) &= i \\
\tilde{x}^+\sqrt{\kappa} - \frac{\tilde{x}^-}{\sqrt{\kappa}} + \frac{G^2}{2}(\frac{\sqrt{\kappa}}{\tilde{x}^+} - \frac{1}{\tilde{x}^-\sqrt{\kappa}}) &= i.
\end{aligned}$$

The central charges are then

$$\begin{aligned}\mathcal{C} &= \frac{1}{2}+i\frac{G^2}{2}(\frac{1}{x^+\sqrt{\kappa}}-\frac{\sqrt{\kappa}}{x^-}) = -i\frac{x^+}{\sqrt{\kappa}}+ix^-\sqrt{\kappa}-\frac{1}{2} \\ \tilde{\mathcal{C}} &= \frac{1}{2}+i\frac{G^2}{2}(\frac{\sqrt{\kappa}}{\tilde{x}^+}-\frac{1}{\tilde{x}^-\sqrt{\kappa}}) = -i\tilde{x}^+\sqrt{\kappa}+i\frac{\tilde{x}^-}{\sqrt{\kappa}}-\frac{1}{2}.\end{aligned}$$

Although the expressions for the central charges (=anomalous dimensions) of Q and $\bar{Q}$ look different in terms of rapidity variables x and $\tilde{x}$, they are in fact equal (by construction) as functions of the momenta.

8.2.2 The S-matrix

The S-matrix S is an operator

$$S: \quad V\otimes\tilde{V}\to V\otimes\tilde{V} \tag{8.35}$$

and similarly

$$\check{S}: \quad \tilde{V}\otimes V\to\tilde{V}\otimes V\,. \tag{8.36}$$

The $SU(2|2)$ algebra acts on $V\otimes\tilde{V}$ as follows,

$$\mathcal{A}(v\times\tilde{v}) = (\mathcal{A}v)\times\tilde{v}+(-1)^{F_{\mathcal{A}}F_v}v\times(\mathcal{A}\tilde{v})\,, \tag{8.37}$$

where $\mathcal{A}$ is an element of the algebra, $v,\tilde{v}$ vectors in V and $\tilde{V}$, and F the fermion number. To guarantee the $SU(2|2)$ symmetry of the S-matrix we simply need to impose the matrix equation $[\mathcal{A},S]=0$. This is sufficient to determine S up to an overall phase.

Following [23], we parametrize the S-matrix as

$$\begin{aligned}S|Q_1^I\bar{Q}_2^J\rangle &= A|Q_2^{\{I}\bar{Q}_1^{J\}}\rangle+B|Q_2^{[I}\bar{Q}_1^{J]}\rangle+\frac{1}{2}C\epsilon^{IJ}\epsilon_{\dot{\alpha}\dot{\beta}}|\psi_2^{\dot{\alpha}}\tilde{\psi}_1^{\dot{\beta}}\phi^-\rangle \\ S|\psi_1^{\dot{\alpha}}\tilde{\psi}_2^{\dot{\beta}}\rangle &= D|\psi_2^{\{\dot{\alpha}}\tilde{\psi}_1^{\dot{\beta}\}}\rangle+E|\psi_2^{[\dot{\alpha}}\tilde{\psi}_1^{\dot{\beta}]}\rangle+\frac{1}{2}F\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{IJ}|Q_2^I\bar{Q}_1^J\phi^+\rangle \\ S|Q_1^I\tilde{\psi}_2^{\dot{\beta}}\rangle &= G|\psi_2^{\dot{\beta}}\bar{Q}_1^I\rangle+H|Q_2^I\tilde{\psi}_1^{\dot{\beta}}\rangle \\ S|\psi_1^{\dot{\alpha}}\bar{Q}_2^J\rangle &= K|\psi_2^{\dot{\alpha}}\bar{Q}_1^J\rangle+L|Q_2^J\tilde{\psi}_1^{\dot{\alpha}}\rangle.\end{aligned} \tag{8.38}$$

The linear constraints obeyed by the S-matrix are listed in equ.(P.9). Below we give the solution for the components A, B, C, G, H, K, L. The solution for B, D and E involve lengthier expressions – they can be readily obtained from equ.(P.9) with `Mathematica`'s help.

$$\begin{aligned}
A &= \frac{\tilde{x}_1^- x_2^-}{x_1^- \tilde{x}_2^-}\left(\frac{\tilde{x}_2^+ - x_1^-}{x_2^- - \tilde{x}_1^+}\right) \\
B &= \tilde{x}_1^- x_2^- [\tilde{x}_1^+ x_2^+ \kappa(2x_2^- x_1^+ \tilde{x}_2^+ - \tilde{x}_1^+ x_2^+ (x_1^- + \tilde{x}_2^+)) \\
&\quad + \tilde{x}_1^- (2\tilde{x}_1^+ x_2^+ (x_1^- x_2^+ + \tilde{x}_2^+ (x_2^+ - x_1^-)) \\
&\quad + x_2^- (-2x_1^+ \tilde{x}_2^+ x_2^+ + \kappa \tilde{x}_1^+ (2x_1^+ \tilde{x}_2^+ - x_2^+ (x_1^- + \tilde{x}_2^+))))]/ \\
&\quad \kappa \tilde{x}_1^+ x_2^+ x_1^- \tilde{x}_2^- (x_2^- - \tilde{x}_1^+)(\tilde{x}_1^- x_2^- - \tilde{x}_1^+ x_2^+) \\
C &= 2\sqrt{2}\tilde{\gamma}_1 \gamma_2 \tilde{x}_1^- x_2^- (\tilde{x}_1^+ x_2^+ (x_1^- + \tilde{x}_2^+) - x_1^+ \tilde{x}_2^+ (x_2^- + \tilde{x}_1^+))/ \\
&\quad \kappa G x_1^- \tilde{x}_2^- (x_2^- - \tilde{x}_1^+)(\tilde{x}_1^- x_2^- - \tilde{x}_1^+ x_2^+) \\
G &= \frac{\gamma_2}{\tilde{\gamma}_2}\frac{\tilde{x}_1^- x_2^- \tilde{x}_2^+}{x_1^- \tilde{x}_2^- x_2^+}\left(\frac{x_2^+ - x_1^+}{x_2^- - \tilde{x}_1^+}\right) \\
H &= \frac{\tilde{\gamma}_1 \tilde{x}_1^- x_2^- \tilde{x}_2^+}{\tilde{\gamma}_2 x_1^- \tilde{x}_2^- x_2^+ \tilde{x}_1^+}\left(\frac{\tilde{x}_1^+ x_2^+ - x_2^- x_1^+}{x_2^- - \tilde{x}_1^+}\right) \\
K &= \frac{\gamma_2 \tilde{x}_1^- x_2^-}{\gamma_1 x_1^- \tilde{x}_2^- x_2^+}\left(\frac{x_1^+ \tilde{x}_2^+ - x_1^- x_2^+}{x_2^- - \tilde{x}_1^+}\right) \\
L &= \frac{\tilde{\gamma}_1}{\gamma_1}\frac{\tilde{x}_1^- x_2^-}{x_1^- \tilde{x}_2^- \tilde{x}_1^+ x_2^+}\left(\frac{x_2^- x_1^+ \tilde{x}_2^+ - x_1^- \tilde{x}_1^+ x_2^+}{x_2^- - \tilde{x}_1^+}\right)
\end{aligned} \tag{8.39}$$

The Yang-Baxter equation fails to hold for $g \neq \check{g}$, as already observed in the one-loop result of [4].

One-loop limit

At one-loop, going back to the momentum representation, the S-matrix simplifies to

$$\begin{aligned}
A &= E = -\frac{1 + e^{ip_1 + ip_2} - 2\kappa e^{ip_2}}{1 + e^{ip_1 + ip_2} - 2\kappa e^{ip_1}} \\
B &= D = -1 \\
C &= F = 0 \\
G &= L = -\frac{\kappa(e^{ip_1} - e^{ip_2})}{1 + e^{ip_1 + ip_2} - 2\kappa e^{ip_1}} \\
H &= K = -\frac{1 + e^{ip_1 + ip_2} - \kappa(e^{ip_1} + e^{ip_2})}{1 + e^{ip_1 + ip_2} - 2\kappa e^{ip_1}}
\end{aligned} \tag{8.40}$$

The S-matrix $\check{S}$ for $\bar{Q}Q$ scattering is given by sending $\kappa \to \frac{1}{\kappa}$ in the above expressions.

The bosonic and fermionic impurities do not mix at one-loop. The $Q\,\bar{Q}$ S-matrix agrees with the explicit perturbative calculation of [4]. The fermion S-matrix has also been successfully checked against one-loop perturbation theory [187].

All-loops at $\kappa = 0$

For $\kappa = 0$, the all-loop S matrix at $\kappa = 0$ in the $Q\bar{Q}$ channel is rather trivial,

$$\begin{aligned} A &= E = -1 \\ B &= D = -1 \\ C &= F = 0 \\ G &= L = 0 \\ H &= K = -1\,. \end{aligned} \tag{8.41}$$

This is intuitively clear: the Q and $\bar{Q}$ impurities are separated by adjoint fields in the "checked" vector multiplet, which decouples in the limit $\kappa \to 0$.

On the other hand, in the $\bar{Q}Q$ scattering sector the scattering retains a a non-trivial dependence on the coupling (now the impurities are separated by the interacting fields of the "unchecked" vector multiplet),

$$\begin{aligned}
&\check{A} = -e^{i(p_2-p_1)} && \check{D} = -1 \\
&\check{B} = -e^{i(p_2-p_1)}(\cos(p_1-p_2) - i\tfrac{\sin(p_1-p_2)}{\sqrt{1+2g^2}}) && \check{E} = -(\cos(p_1-p_2) + i\tfrac{\sin(p_1-p_2)}{\sqrt{1+2g^2}}) \\
&\check{C} = -ie^{ip_2}\sqrt{2}g\tfrac{\sin(p_1-p_2)}{\sqrt{1+2g^2}} && \check{F} = -ie^{-ip_1}\sqrt{2}g\tfrac{\sin(p_1-p_2)}{\sqrt{1+2g^2}} \\
&\check{G} = \tfrac{1}{2}(1-e^{i(p_2-p_1)}) && \check{L} = \tfrac{1}{2}(1-e^{i(p_2-p_1)}) \\
&\check{H} = -\tfrac{1}{2}(1+e^{i(p_2-p_1)}) && \check{K} = -\tfrac{1}{2}(1+e^{i(p_2-p_1)})\,.
\end{aligned}$$

The limit $\kappa \to 0$ is interesting because the $\mathbb{Z}_2$ quiver theory reduces to $\mathcal{N} = 2$ superconformal QCD (plus the decoupled "checked" vector multiplet).

We refer to [2, 4] for detailed discussions. For $\kappa = 0$ the global symmetry $SU(2_{\hat{I}})$ combines with the second gauge group $SU(N_{\check{c}})$ and there is a symmetry enhancement to the flavor group $U(N_f = 2N_c)$.

An important question is whether the flavor-singlet sector of the SCQQD spin-chain is integrable. We may now look forward to shed new light on this question using the above all-loop results. Unfortunately, flavor singlets are in particular $SU(2_{\hat{I}})$ singlets, and the methods of this chapter only allow us to consider scattering of $SU(2_{\hat{I}})$ triplets. So our results have no direct bearing on the question of integrability of the $\mathcal{N} = 2$ SQCD spin-chain. With this caveat, we may nevertheless go ahead and check whether the Yang-Baxter equation holds at $\kappa = 0$ for $SU(2_{\hat{I}})$ triplets. It doesn't.[4]

8.3 Emergent Magnons

In [195], following [202], Berenstein et al. reproduced the all-loop magnon dispersion relation in $\mathcal{N} = 4$ SYM using a simple matrix quantum mechanics. The matrix quantum mechanics is obtained by truncating to the lowest modes of $SU(N_c)$ $\mathcal{N} = 4$ SYM on S^3. The ground state is obtained by minimizing the potential energy, which leads to a model of commuting hermitian matrices. The matrix eigenvalues are localized on a five-sphere of radius[5] $\frac{1}{\sqrt{2}}$, which is naturally identified with the S^5 in the dual background. This gives a simple picture for emergent geometry. Each point in the emergent geometry corresponds to an eigenvalue and is labelled by an $SU(N_c)$ index. In [203, 204] the same exercise for orbifolds of $\mathcal{N} = 4$ SYM shows that the ground state of the matrix model is localized on the orbifolded S^5.

The excitations of the vacuum obtained by turning on off-diagonal modes of the matrix model are interpreted as string bits. They are bilocal in the emergent geometry because they are labelled by two $SU(N_c)$ indices and are visualized as string bits stretching between two points (see figure 8.1). An off-diagonal excitation of momentum p is peaked at the configuration where

[4] In [4], it was found that in the scalar sector, at one-loop, the YB equation holds as $\kappa \to 0$ both for $SU(2_{\hat{I}})$ triplets and $SU(2_{\hat{I}})$ singlets. Only the result for singlets is relevant to the integrability question.

[5] Our normalization for the fields are related to the normalization in [195] as $\phi_{here} = \phi_{there}/\sqrt{N_c}$.

the corresponding string bit subtends an angle p at the center. The expectation value of their energy precisely reproduces the exact magnon dispersion relation [195]. A very similar picture for the magnons was obtained in [194]

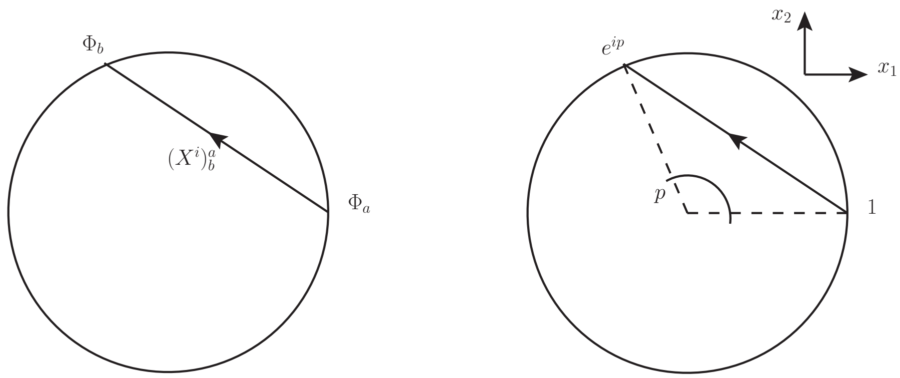


Figure 8.1: The left figure shows the string bit corresponding to the off-diagonal excitation $(X^i)^a_b$. The right figure shows the configuration where the wavefunction of a magnon with momentum p is peaked.

on the dual string side. Moreover, the x_1 and x_2 components of the vector $\vec{M}$ associated with the magnon were identified with the central charges of the $SU(2|2)$ algebra [194]

$$M_1 = \frac{1}{2}(K+P)\,, \quad M_2 = \frac{1}{2i}(K-P)\,. \tag{8.42}$$

8.3.1 Emergent magnons for the $\mathbb{Z}_2$ quiver

Following [195], we truncate the $\mathbb{Z}_2$ quiver theory to its lowest bosonic modes on S^3, which gives us the matrix quantum mechanis

$$\begin{aligned} S &= N_c \int dt\, \mathrm{Tr}\, \frac{1}{2}\Big((D_t\phi)^2 + (D_t\check{\phi})^2 + (D_t Q^{I\hat{I}})^2 - \phi^2 - \check{\phi}^2 - (Q^{I\hat{I}})^2\Big) \\ &- g^2\left([\phi,\bar{\phi}]^2 + \sqrt{2}Q^{I\hat{I}}\bar{Q}_{I\hat{I}}(\phi\bar{\phi}+\bar{\phi}\phi) + Q^{I\hat{I}}\bar{Q}_{J\hat{I}}Q^{J\hat{J}}\bar{Q}_{I\hat{J}} - \frac{1}{2}Q^{I\hat{I}}\bar{Q}_{I\hat{I}}Q^{J\hat{J}}\bar{Q}_{J\hat{J}}\right) \\ &- \check{g}^2\left([\check{\phi},\bar{\check{\phi}}]^2 + \sqrt{2}\bar{Q}_{I\hat{I}}Q^{I\hat{I}}(\check{\phi}\bar{\check{\phi}}+\bar{\check{\phi}}\check{\phi}) + \bar{Q}_{J\hat{I}}Q^{I\hat{I}}\bar{Q}_{I\hat{J}}Q^{J\hat{J}} - \frac{1}{2}\bar{Q}_{I\hat{I}}Q^{I\hat{I}}\bar{Q}_{J\hat{J}}Q^{J\hat{J}}\right) \\ &+ \sqrt{g\check{g}}\left(4Q^{I\hat{I}}\check{\phi}\bar{Q}_{I\hat{I}}\bar{\phi} + h.c.\right) + \frac{1}{N_c}(\text{double}-\ \text{trace}). \end{aligned} \tag{8.43}$$

The mass terms arise due to the conformal couplings of the scalars to curvature of S^3. The eigenvalue distribution of the ground state is same as that of the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM. We now excite the off-diagonal mode $(Q^{I\hat{I}})^a_{\check{b}}$. The linearized theory describing this excitation is the harmonic oscillator,

$$\begin{aligned} H &= \frac{1}{2}(\Pi_{I\hat{I}})^a_{\check{b}}(\Pi^{I\hat{I}})^{\check{b}}_a + \frac{1}{2}\omega_{a\check{b}}(Q^{I\hat{I}})^a_{\check{b}}(\bar{Q}_{I\hat{I}})^{\check{b}}_a \\ \omega_{a\check{b}} &= 1 + 4|g\phi_a - \check{g}\check{\phi}_{\check{b}}|^2 . \end{aligned}$$

Note the difference in the frequency compared to the $\mathcal{N} = 4$ case, where $\omega_{ab} = 1 + 4g^2|\phi_a - \phi_b|^2$. This motivates the effective picture of figure 8.2.

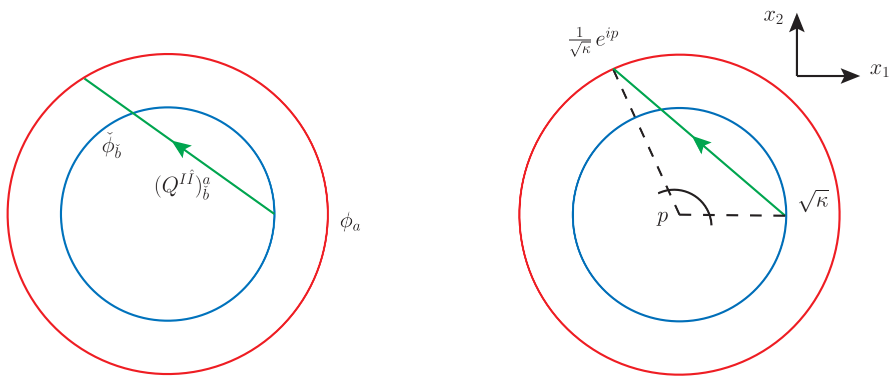


Figure 8.2: The figure on the left shows the string bit in the $\mathbb{Z}_2$ quiver theory. On the right, the wavefunction of the bifundamental magnon $Q^{I\hat{I}}$ with momentum p.

The circle spanned by the eigenvalues of Φ has split into two circles, one spanned by the eigenvalues of ϕ and the other by eigenvalues of $\check{\phi}$. The radii of the two circles are taken to be $\frac{1}{\sqrt{\kappa}}\frac{G}{\sqrt{2}}$ and $\sqrt{\kappa}\frac{G}{\sqrt{2}}$ respectively, by normalizing the tension of the string bit to unity. The string bit corresponding to a bifundamental excitation stretches from one circle to the other. A magnon of momentum p again localizes on the configuration where the string bit subtends an angle p at the center. Using (8.42) we learn

$$P = x_1 - ix_2 = \frac{G}{\sqrt{2}}(e^{-ip}\frac{1}{\sqrt{\kappa}} - \sqrt{\kappa}) = K^* , \tag{8.44}$$

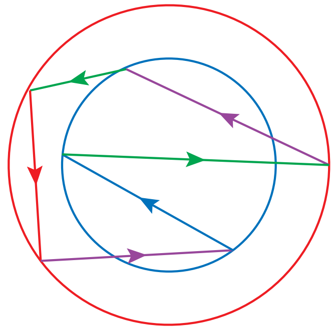


Figure 8.3: A state of the spin chain with six magnons.

so the energy of the magnon is

$$\Delta - |r| = \sqrt{1 + 8G^2 \left(\sin^2 \frac{p}{2} + \frac{1}{4}(\sqrt{\kappa} - \frac{1}{\sqrt{\kappa}})^2 \right)}. \tag{8.45}$$

The central charges agree precisely with the from obtained earlier from the algebraic method.[6]

It is clear that the adjoint excitations λ and D ($\check{\lambda}$ and $\check{D}$) are string bits that stretch between two points of ϕ circle ($\check{\phi}$ circle). Their dispersion relation coincides with the $\mathcal{N} = 4$ SYM dispersion relation, as clear from the picture. A generic state of the spin chain is shown in figure 8.3.

At strong 't Hooft coupling, Hofman and Maldacena [194] obtained the dual description of an $\mathcal{N} = 4$ magnon as a semiclassical strings rotating on the $S^2 \subset S^5$. In LLM coordinates this "giant magnon" has precisely the shape of figure 8.1. The energy of the string was matched with the strong coupling limit of the exact magnon dispersion relation. (See also [205] for a sigma-model derivation of the $SU(2|2)$ central charges.) The $\mathbb{Z}_2$ quiver theory is dual to the $AdS_5 \times S^5/\mathbb{Z}_2$ background. The ratio of the gauge couplings is related the period of the NSNS B-field through the collapsed two-cycle. It must be possible to reproduce the effective picture of figure 8.2 and the associated dispersion relation by studying the giant magnon solution in this background. This problem is under investigation [206].

[6]Of course, as before, there is no guarantee that the couplings do not get renormalized. This caveat is all the more obvious in this approach, since integrating out massive modes would generically lead to such a renormalization.

8.3.2 Bound states

In addition to the elementary magnons with real momenta, the spectrum of the theory also contains bound states at some special complex values of the momenta. A two-magnon bound state occurs at the pole of the two-body S-matrix,

$$S(p_1,p_2)=\infty \qquad \text{with} \quad p_1=\frac{P}{2}-iq, \quad p_2=\frac{P}{2}+iq, \quad q>0\,. \tag{8.46}$$

Since $S(p_2,p_1)=1/S(p_1,p_2)\to 0$, the asymptotic wavefunction becomes

$$e^{iP\frac{x_1+x_2}{2}-q|x_2-x_1|}. \tag{8.47}$$

A bound state has smaller energy than any state in the two particle continuum with the same total mometum P. The exact dispersion relation of the bound states in $\mathcal{N}=4$ SYM was found in [207] and their S-matrix in [208]. The two-body S-matrix in the present case allows us to determine the bound state dispersion relation. Finding their S-matrix, however, would requires the four-body magnon S-matrix, which we cannot determine in the absence of integrability.

Let us first analyze the bound state of Q^+ (on the left of the chain) and $\bar{Q}^+$ (on the right). Their scattering matrix given in equ.(8.39),

$$A(p_1,p_2)=S^0_{12}\frac{\tilde{x}_2^- x_1^-}{x_2^-\tilde{x}_1^-}(\frac{\tilde{x}_1^+-x_2^-}{x_1^- - \tilde{x}_2^+})\,, \tag{8.48}$$

where S^0_{12} is the overall dressing factor which is not determined by symmetries. Clearly there is a pole is at $x_1^-=\tilde{x}_2^+$. We assume that this pole is not cancelled by a zero of the dressing factor. Following [209], we define the bound state rapidity variables as

$$X^+\equiv x_1^+, \qquad X^-\equiv \tilde{x}_2^-\,. \tag{8.49}$$

Remarkably, at the pole they obey the relations

$$\begin{aligned}\frac{X^+}{X^-} &= e^{iP}\\ X^+-X^-+\frac{G^2}{2}(\frac{1}{X^+}-\frac{1}{X^-}) &= 2i\sqrt{\kappa}.\end{aligned}$$

The bound state dispersion relation can also be expressed completely in terms of $X^{\pm}$,

$$\begin{aligned} C_{Q\bar{Q}} &= C_1 + \tilde{C}_2 = 1 + i\frac{G^2}{2\sqrt{\kappa}}(\frac{1}{X^+} - \frac{1}{X^-}) \\ &= \frac{1}{2}\sqrt{4 + 8g^2 \sin^2 \frac{p}{2}}. \end{aligned} \tag{8.50}$$

This dispersion is exactly the same as the one of the two-magnon bound states in $\mathcal{N} = 4$ SYM. Thus the $Q\bar{Q}$ bound state can be elegantly represented as a string bit of "weight two" stretching between two points of the outer circle. The analogous exercise for the $\bar{Q}Q$ bound state gives the dispersion relation

$$C_{\bar{Q}Q} = \frac{1}{2}\sqrt{4 + 8\check{g}^2 \sin^2 \frac{p}{2}}\,. \tag{8.51}$$

This bound state is represented as a weight-two string bit stretching between two points of the inner circle.

As we vary the momentum P of the bound state the pole iq moves on the positive imaginary axis. For certain values of P where q approaches zero, the bound state is only marginally stable. This phenomenon does not occur in $\mathcal{N} = 4$ SYM, the bound states of $\mathcal{N} = 4$ are stable for all values of P but this is not the case for the $\mathbb{Z}_2$ quiver theory. The marginal stability condition $q = 0$ gives respectively for the $Q\bar{Q}$ and $\bar{Q}Q$ bound states

$$\kappa = \cos\frac{P}{2} \qquad \text{and} \qquad \frac{1}{\kappa} = \cos\frac{P}{2} \tag{8.52}$$

In the latter case, there is no solution which means that $\bar{Q}Q$ bound state is stable for all values of the momenta. On the other hand, the $Q\bar{Q}$ bound state on the other hand can decay at $P = 2\arccos\kappa$. These conclusions exactly match with results obtained at one loop in [4].

Geometrically, there is simple way of understanding the boud state decay, see figure 8.4. As the bound state string bit stretching in the outer circle (which means it is a $Q\bar{Q}$ bound state) touches the inner circle, its energy becomes manifestly equal to the sum of the energies of the constituents. Vanishing of the binding energy allows the $Q\bar{Q}$ state to decay. Simple trigonometry reveals the threshold momentum $P = 2\arccos\kappa$ at this point. From this picture it is

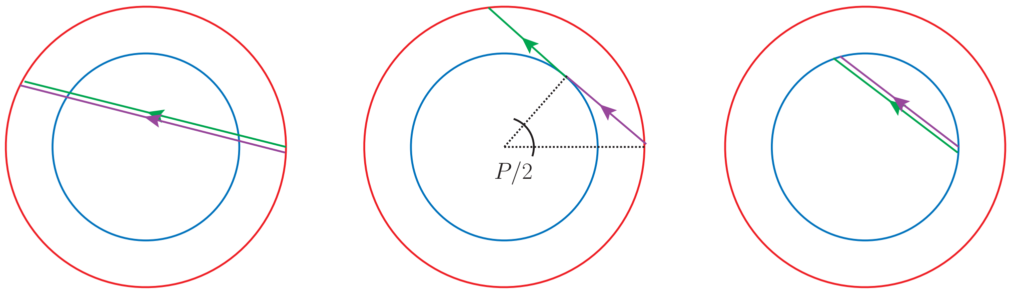


Figure 8.4: The figure on the left represents a $Q\bar{Q}$ bound state at generic momenta. In the middle is the marginally stable $Q\bar{Q}$ bound state. From the figure one can easily see that $P = 2 \arccos \kappa$ since the ratio of the radii of the two circles is κ. On the right is a $\bar{Q}Q$ bound state, which is stable for all values of momenta.

also immediate to see that the $\bar{Q}Q$ bound state is stable for all values of the momenta.

As we move around in the parameter space of the quiver gauge theory, at certain codimension one "walls", the bound states of the elementary magnons decay. It would be interesting to understand bound state decay as a wall-crossing phenomenon in the dual sigma model.

8.4 Generalization to $\mathbb{Z}_k$ orbifolds

The analysis presented for the $\mathbb{Z}_2$ quiver can be extended to a general ADE $\mathcal{N}=2$ orbifold of $\mathcal{N}=4$ SYM. In this section we indicate the generalization for the (marginally deformed) $\mathbb{Z}_k$ orbifolds. The quiver gauge theory describing such an orbifold is shown in figure 8.5.

The superpotential at a generic point in the parameter space is

$$W = \frac{1}{\sqrt{2}} \sum_i g_{(i)} \left(\mathrm{Tr} Q^I_{(i-1,i)} \phi_{(i)} \bar{Q}_{I(i,i-1)} + \mathrm{Tr} \bar{Q}_{I(i+1,i)} \phi_{(i)} Q^I_{(i,i+1)} \right) . \qquad (8.53)$$

We impose the periodicity condition $i + k \sim i$ on the indices.

To compute the $SU(2|2)$ central charges for the representation of the

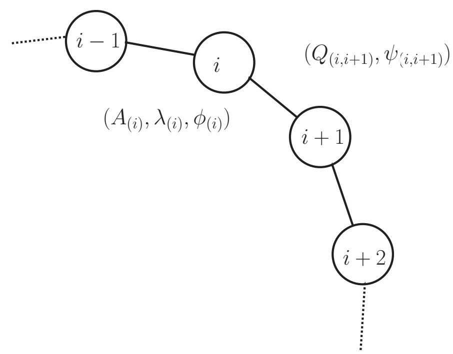


Figure 8.5: The quiver diagram for $\mathcal{N} = 2$ $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM. It is a circular necklace with k nodes, four of which are shown. A vector multiplet (A, λ, ϕ) is associated to each node and a hypermultiplet (Q^I, ψ) is associated to each edge.

$Q^I_{(i,i+1)}$ magnon we evaluate the anticommutator of two supersymmetries,

$$\{Q^{\dot{\alpha}}_I, Q^{\dot{\beta}}_J\} Q^K_{(i,i+1)} = \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon_{IJ} \left(\frac{g_{(i)}}{\sqrt{2}} \phi_{(i)} Q^K_{(i,i+1)} - \frac{g_{(i+1)}}{\sqrt{2}} Q^K_{(i,i+1)} \phi_{(i+1)} \right) \tag{8.54}$$

which, on the spin chain, leads to

$$\begin{aligned} \{Q^{\dot{\alpha}}_I, Q^{\dot{\beta}}_J\} |Q^K_{(i,i+1)}\rangle &= \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon_{IJ} \frac{1}{\sqrt{2}} (g_{(i)} e^{-ip} - g_{(i+1)}) |Q^K_{(i,i+1)} \phi^+\rangle \\ \Rightarrow \mathcal{P} &= \frac{1}{\sqrt{2}} (g_{(i)} e^{-ip} - g_{(i+1)}) = \mathcal{K}^*. \end{aligned}$$

Interchanging $g_{(i)} \leftrightarrow g_{(i+1)}$ gives us the central charges of the $\bar{Q}_{(i+1,i)}$ representation. In both cases we get the dispersion relation

$$\Delta - |r| = 2\mathcal{C} = \sqrt{1 + 8G^2_{(i,i+1)} \left(\sin^2 \frac{p}{2} + \frac{1}{4} \left(\sqrt{\kappa_{(i,i+1)}} - \frac{1}{\sqrt{\kappa_{(i,i+1)}}} \right)^2 \right)}. \tag{8.55}$$

Here we have defined

$$G_{(i,i+1)} = \sqrt{g_{(i)} g_{(i+1)}} \qquad \text{and} \qquad \kappa_{(i,i+1)} = \frac{g_{(i+1)}}{g_{(i)}}. \tag{8.56}$$

The dispersion relation of the adjoint magnons $\lambda_{(i)}$ and $D_{(i)}$ works the same

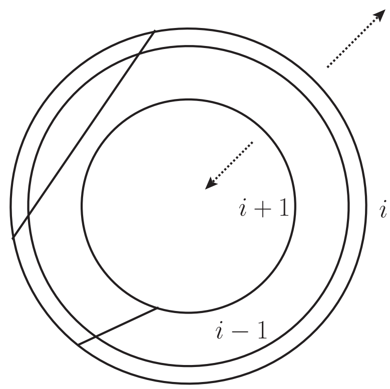


Figure 8.6: The emergent picture describing $\mathbb{Z}_k$ orbifold. Only the circles corresponding to $i-1, i, i+1$ gauge node are shown. We have also shown two magnons, one in the adjoint of $SU(N)_i$ and the other in the bifundamental of $SU(N)_i \times SU(N)_{i+1}$.

way as $\mathcal{N}=4$ and is equal to

$$\Delta - |r| = 2\mathcal{C} = \sqrt{1 + 8g_{(i)}^2 \sin^2 \frac{p}{2}}. \tag{8.57}$$

The picture presented in section 8.3 also generalizes to $\mathbb{Z}_k$ orbifolds, see figure 8.6. It consists of k concentric circles which are labelled by i, corresponding to the gauge group $SU(N_c)_i$. The radius of i-th circle is $\frac{g_{(i)}}{\sqrt{2}}$. The magnons in the adjoint of the i-th node are represented by string bits that stretch between the i-th circle, while the $SU(N)_i \times SU(N)_{i+1}$ bifundamental magnons correspond to string bits stretching from i-th to $i+1$-th circle. The dispersion relations of both adjoint and bifundamental magnons is summarized by the simple formula

$$\Delta - |r| = \sqrt{1 + 4\ell^2} \tag{8.58}$$

where ℓ is the length of the corresponding string bit. The two-body S-matrix is also fixed by the centrally extended $SU(2|2)$ symmetry, and can be obtained by straightforward extension of our analysis of the $\mathbb{Z}_2$ case.

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Appendix A

S-duality for $\mathcal{N} = 4$ $SO(2n+1)/Sp(n)$ SYM

In this Appendix we compute the superconformal indices for $\mathcal{N} = 4$ SYM with gauge groups $SO(2n+1)$ and $Sp(n)$. Since the SO and Sp theories are related by S-duality, their indices are expected to agree. These are in fact the only non-trivial $\mathcal{N} = 4$ cases from the viewpoint of index calculations. Indeed the index depends on the adjoint representation of the group: the A, D, E, F and G cases are manifestly self-dual, and the only interesting duality is $B \leftrightarrow C$.

The characters of the adjoint representations of for $Sp(n)$ and $SO(2n+1)$ are

$$\begin{aligned} \chi_{Sp(n)}(\{z_i\}) : &\quad \textstyle\sum_{1\le i<j\le n} \left(z_i z_j + z_i z_j^{-1} + z_j z_i^{-1} + z_i^{-1} z_j^{-1}\right) + \sum_{i=1}^{n} (z_i^2 + z_i^{-2}) + n, \\ \chi_{SO(2n+1)}(\{z_i\}) : &\quad \textstyle\sum_{1\le i<j\le n} \left(z_i z_j + z_i z_j^{-1} + z_j z_i^{-1} + z_i^{-1} z_j^{-1}\right) + \sum_{i=1}^{n} (z_i + z_i^{-1}) + n. \end{aligned} \tag{A.1}$$

Their Haar measures are

$$Sp(n) \quad : \quad \int_{Sp(n)} d\mu(z) f(z) = \frac{(-)^n}{2^n\, n!} \oint_{\mathbb{T}_n} \prod_{j=1}^{n} \frac{dz_j}{2\pi i z_j} \prod_{j=1}^{n} (z_j - z_j^{-1})^2 \Delta(\mathbf{z} + \mathbf{z}^{-1})^2 f(z), \tag{A.2}$$

$$SO(2n+1) \quad : \quad \int_{So(2n+1)} d\mu(z) f(z) = \frac{(-)^n}{2^n\, n!} \oint_{\mathbb{T}_n} \prod_{j=1}^{n} \frac{dz_j}{2\pi i z_j} \prod_{j=1}^{n} (z_j^{1/2} - z_j^{-1/2})^2 \Delta(\mathbf{z} + \mathbf{z}^{-1})^2 f(z),$$

where $\mathbb{T}_n$ is an n-dimensional torus with unit radii and $\Delta(\mathbf{x})$ the van der Monde determinant

$$\Delta(\mathbf{x}) = \prod_{i<j} (x_i - x_j)\,. \tag{A.3}$$

The single letter partition function is in both cases equal to [19]

$$f(t,y) = \frac{3t^2 - 3t^4 - t^3(y+y^{-1}) + 2t^6}{(1-t^3\,y)(1-t^3y^{-1})}\,, \tag{A.4}$$

where for simplicity we have omitted the chemical potentials of the R-charges – we will restore them in the end. Using the identities (2.24),

$$e^{\sum_k \frac{f_k}{k}\chi_{Sp(n)}(\{z_i^k\})} = \Gamma^{3n}(t^2;p,q)(p;p)^n(q;q)^n \prod_{i<j} \frac{z_j^2}{(1-z_iz_j)^2(1-z_i^{-1}z_j)^2} \frac{1}{\Gamma(z_i^{\pm1}z_j^{\pm1};p,q)}$$
$$\prod_j \frac{-z_j^2}{(1-z_j^2)^2}\frac{1}{\Gamma(z_j^{\pm2};p,q)} \prod_{i<j}\Gamma(t^2z_i^{\pm1}z_j^{\pm1};p,q)^3 \prod_j \Gamma(t^2z_i^{\pm2};p,q)^3. \tag{A.5}$$

Recall the definition (2.26) of the product $(x;y)$. Further, using

$$\prod_{i<j}(1-z_iz_j)(1-z_i/z_j)(1-z_j/z_i)(1-1/(z_iz_j)) = \Delta(\mathbf{z}+\mathbf{z}^{-1})^2, \tag{A.6}$$
$$\prod_j(1-z_j^2)(1-1/z_j^2) = (-1)^n\prod_j(z_j-1/z_j)^2\,,$$

we obtain

$$\int_{Sp(n)} d\mu(z)\, e^{\sum_k \frac{1}{k}f_k\chi_{Sp(n)}(\{z_i\})} = \tag{A.7}$$
$$\frac{\Gamma^{3n}(t^2;p,q)}{2^n\,n!}(p;p)^n(q;q)^n \oint \prod_j \frac{dz_j}{2\pi i z_j}\prod_{i<j}\frac{\Gamma(t^2z_i^{\pm1}z_j^{\pm1};p,q)^3}{\Gamma(z_i^{\pm1}z_j^{\pm1};p,q)}\prod_j\frac{\Gamma(t^2z_j^{\pm2};p,q)^3}{\Gamma(z_j^{\pm2};p,q)}.$$

In complete analogy we obtain for the $SO(2n+1)$ gauge group

$$\int_{SO(2n+1)} d\mu(z)\, e^{\sum_k \frac{1}{k}f_k\chi_{So(2n+1)}} = \tag{A.8}$$
$$\frac{\Gamma^{3n}(t^2;p,q)}{2^n\,n!}(p;p)^n(q;q)^n \oint \prod_j \frac{dz_j}{2\pi i z_j}\prod_{i<j}\frac{\Gamma(t^2z_i^{\pm1}z_j^{\pm1};p,q)^3}{\Gamma(z_i^{\pm1}z_j^{\pm1};p,q)}\prod_j\frac{\Gamma(t^2z_j^{\pm1};p,q)^3}{\Gamma(z_j^{\pm1};p,q)}.$$

S-duality predicts that the integrals (A.7) and (A.8) must agree. For $Sp(1)\cong SO(3)$ this is trivially checked by a change of variable: in the $SO(3)$ integral make the substitution $z\to y=\sqrt{z}$. The case of $Sp(2)\cong SO(5)$ is also trivial (as it should be). Define $\hat{z}_1=\sqrt{z_1z_2}$ and $\hat{z}_2=\sqrt{z_1/z_2}$. Then in (A.8) the first product is exchanged with the second with a doubled power of the z argument and we obtain (A.7). We have checked for the first few orders in a series expansion in t that (A.7) (A.8) also agree for higher rank groups. We do not have an analytic proof of this statement.

Given an orthonormal basis e_i of $\mathbb{R}^n$ the root system of C_n ($Sp(n)$) consists of vectors of the form $X(C_n) = \{\pm 2e_i,\, \pm e_i \pm e_j, i < j\}$. The root system of B_n ($SO(2n+1)$) on the other hand consists of vectors of the form $X(B_n) = \{\pm e_i,\, \pm e_i \pm e_j, i < j\}$. These two systems are dual to one other. The integrands in (A.7) and (A.8) are given by

$$\prod_{\alpha \in X} \frac{\Gamma(t^2\, e^{\alpha}; p, q)^3}{\Gamma(e^{\alpha}; p, q)}\,, \tag{A.9}$$

where X is the corresponding root system and we formally identify $z_i = e^{e_i}$. In this language it is easy to understand why the integrals (A.8) with $SO(3)/SO(5)$, (A.7) with $Sp(1)/Sp(2)$ are equal to one other. In these cases the two root systems are linear transformations of one other, *i.e.* rescaling and in the case of $Sp(2)/SO(5)$ also rotation. For higher n the relation is more complicated. For example for $n = 3$ the $SO(7)$ lattice is a cube and the $Sp(3)$ lattice is an octahedron.

Finally, let us indicate how the expressions for the indices are modified by adding the chemical potentials for the R-symmetry charges [19]. The only differences are in the numerators of (A.7,A.8), which become

$$\begin{aligned}
Sp(n):\qquad & \prod_{i<j} \Gamma(t^2 v\, z_i^{\pm 1} z_j^{\pm 1}; p, q)\Gamma(\frac{t^2}{w}\, z_i^{\pm 1} z_j^{\pm 1}; p, q)\Gamma(\frac{wt^2}{v}\, z_i^{\pm 1} z_j^{\pm 1}; p, q) \\
& \prod_{j} \Gamma(t^2 v\, z_j^{\pm 2}; p, q)\Gamma(\frac{t^2}{w}\, z_j^{\pm 2}; p, q)\Gamma(\frac{wt^2}{v}\, z_j^{\pm 2}; p, q), \\
SO(2n+1):\qquad & \prod_{i<j} \Gamma(t^2 v\, z_i^{\pm 1} z_j^{\pm 1}; p, q)\Gamma(\frac{t^2}{w}\, z_i^{\pm 1} z_j^{\pm 1}; p, q)\Gamma(\frac{wt^2}{v}\, z_i^{\pm 1} z_j^{\pm 1}; p, q) \\
& \prod_{j} \Gamma(t^2 v\, z_j^{\pm 1}; p, q)\Gamma(\frac{t^2}{w}\, z_j^{\pm 1}; p, q)\Gamma(\frac{wt^2}{v}\, z_j^{\pm 1}; p, q),
\end{aligned} \tag{A.10}$$

and in the prefactor of the integrals,

$$\Gamma^{3n}(t^2; p, q) \quad \to \quad \Gamma^{n}(t^2\, v; p, q)\, \Gamma^{n}(\frac{t^2}{w}; p, q)\, \Gamma^{n}(\frac{w\, t^2}{v}; p, q)\,. \tag{A.11}$$

Appendix B

TQFT Algebra for $v = t$

For $v = t$ we can rewrite the algebra of the topological quantum field theory (2.9) in a more elegant way, removing the delta-functions by making use of identities obeyed by elliptic Beta integrals. This does not appear to be a preferred limit physically, except for the fact that the contribution to the index of the chiral superfield in the $\mathcal{N} = 2$ vector multiplet vanishes, see (4.3). Our manipulations will be slightly formal since the limit $v = t$ of the formulae we will use is somewhat singular. We start by quoting the important identity

$$E^{(m=0)}(t_1, \ldots, t_6) \quad = \quad \kappa \oint \frac{dz}{z} \frac{\prod_{k=1}^{6} \Gamma\left(t_k\, z^{\pm 1}; p, q\right)}{\Gamma\left(z^{\pm 2}; p, q\right)} = \prod_{1 \le j < k \le 6} \Gamma\left(t_j\, t_k; p, q\right). \tag{B.1}$$

with $\prod_{k=1}^{6} t_k = pq$. This is a vast generalization to elliptic Gamma functions of that seminal object in string theory, the classic Beta integral of Euler,

$$B(\alpha, \beta) = \int_0^1 dt\, t^{\alpha-1}(1-t)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,, \tag{B.2}$$

which is recovered as a special limit, see *e.g.* [46]. Applying (B.1) we have

$$\kappa \oint \frac{dz}{z} \frac{\Gamma\left(\tau\sqrt{\nu}\, a^{\pm 1} b^{\pm 1} z^{\pm 1}\right)\, \Gamma\left(\frac{\tau}{\nu}\, z^{\pm 1} y^{\pm 1}\right)}{\Gamma\left(z^{\pm 2}\right)} = \tag{B.3}$$
$$\Gamma\left(\frac{\tau^2}{\sqrt{\nu}}\, a^{\pm 1} b^{\pm 1} y^{\pm 1}\right)\, \Gamma\left(\tau^2 \nu\, a^{\pm 2}\right) \Gamma\left(\tau^2 \nu\, b^{\pm 2}\right)\, \Gamma\left(\frac{\tau^2}{\nu^2}\right)\, \Gamma\left(\tau^2\, \nu\right)^2\,.$$

For brevity we have omitted the p and q parameters in the Gamma functions. We assume $pq = \tau^6$. For these values of p and q, $\Gamma(\tau^3 z^{\pm1}) = 1$. Now if we take $\nu = \tau$,

$$\kappa \oint \frac{dz}{z} \frac{\Gamma\left(\tau^{3/2}\, a^{\pm1} b^{\pm1} z^{\pm1}\right)\, \Gamma\left(z^{\pm1} y^{\pm1}\right)}{\Gamma\left(z^{\pm2}\right)} = \Gamma\left(\tau^{3/2}\, a^{\pm1} b^{\pm1} y^{\pm1}\right)\, \Gamma\left(1\right) . \quad \text{(B.4)}$$

Strictly speaking the elliptic Beta integral formula (B.1) holds when $|t_k| < 1$ for all $k = 1 \ldots 6$. For $\nu = \tau$ some of the t_ks in (B.3) saturate this bound. The elliptic Beta integral (B.3) is proportional to $\Gamma(\frac{\tau^2}{\nu^2}; p, q) \to \Gamma(1; p, q)$. Since the elliptic Gamma function has a simple pole when its argument approaches $z = 1$ (see (2.20)), (B.3) diverges in the limit. We will proceed by keeping formal factors of $\Gamma(1)$ in all the expressions. Thanks to (B.4), the expression

$$\frac{\Gamma(z^{\pm1} y^{\pm1})}{\Gamma(z^{\pm2})\Gamma(1)} \equiv \delta_y^z \quad \text{(B.5)}$$

acts as a formal identity operator. All factors of $\Gamma(1)$ will cancel in the final expression for the index.

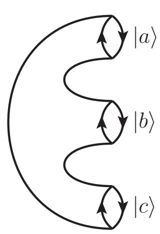
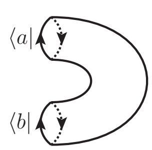
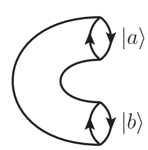
Symbol	Surface	Value	Symbol	Surface	Value
C_{abc}		$\Gamma(t^{\frac{3}{2}} a^{\pm1} b^{\pm1} c^{\pm1})$	V^a		$\frac{1}{\Gamma(1)^2} \frac{\Gamma(t^{\pm\frac{3}{2}} a^{\pm1})}{\Gamma(a^{\pm2})}$
η^{ab}		$\frac{1}{\Gamma(1)} \frac{\Gamma(a^{\pm1} b^{\pm1})}{\Gamma(a^{\pm2}, b^{\pm2})}$	η_{ab}		$\frac{1}{\Gamma(1)} \Gamma\left(a^{\pm1} b^{\pm1}\right)$

Table B.1: The basic building blocks of the topological algebra in the $v = t$ case.

For $t = v$ we can write the building blocks of the topological algebra in the form summarized in Table B.1. Contraction of indices is defined as

$$A^{..a..}\, B_{..a..} \to \kappa \oint \frac{d\,a}{a}\, A^{..a..}\, B_{..a..}. \tag{B.6}$$

We now proceed to perform a few sample calculations and consistency checks. We can raise an index of the structure constants to obtain

$$C_{abe}\, \eta^{ec} = \frac{\kappa}{\Gamma(1)} \oint \frac{d\,e}{e} \Gamma(t^{\frac{3}{2}} a^{\pm 1} b^{\pm 1} e^{\pm 1}) \frac{\Gamma(e^{\pm 1} c^{\pm 1})}{\Gamma(e^{\pm 2}, c^{\pm 2})} = \frac{\Gamma(t^{\frac{3}{2}} a^{\pm 1} b^{\pm 1} c^{\pm 1})}{\Gamma(c^{\pm 2})} = C_{ab}{}^{c} \tag{B.7}$$

In particular we see that the index (2.28) is finite and is simply given by $C_{ab}{}^{c}\, C_{cde}$. The "vacuum state" $|V\rangle \equiv V^a|a\rangle$ satisfies by definition (see *e.g.* [54]) $C_{abc}\, V^c = \eta_{ab}$, as illustrated in Figure B.1. This determines V^a to be the expression in Table B.1,

$$C_{abc}\, V^c = \frac{\kappa}{\Gamma(1)^2} \oint \frac{dz}{z} \Gamma(t^{\frac{3}{2}} a^{\pm 1} b^{\pm 1} z^{\pm 1})\, \frac{\Gamma(t^{\pm \frac{3}{2}} z^{\pm 1})}{\Gamma(z^{\pm 2})} = \frac{1}{\Gamma(1)} \Gamma(a^{\pm 1} b^{\pm 1}) = \eta_{ab} \tag{B.8}$$

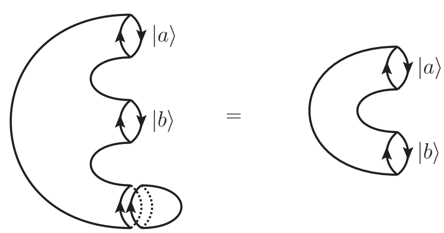


Figure B.1: Constructing the metric by capping off the trivalent vertex.

Further, we can check that η_{ab} and η^{ab} in Table B.1 are one the inverse of the other,

$$\eta^{ae}\, \eta_{ec} = \frac{\kappa}{\Gamma(1)^2} \oint \frac{d\,e}{e} \frac{\Gamma(a^{\pm 1} e^{\pm 1})}{\Gamma(a^{\pm 2}, e^{\pm 2})} \Gamma\left(e^{\pm 1} c^{\pm 1}\right) = \frac{1}{\Gamma(1)} \frac{\Gamma(a^{\pm 1} c^{\pm 1})}{\Gamma(a^{\pm 2})} = \delta^a_c\, . \tag{B.9}$$

As a consistency check one can verify in examples that δ^a_b is indeed an identity.

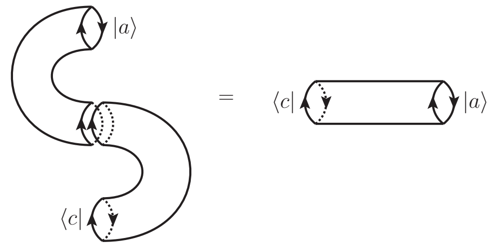


Figure B.2: Topological interpretation of the property $\eta^{ce}\,\eta_{ea} = \delta^c_a$.

For instance

$$\delta^z_a\,C_{zbc} = \frac{\kappa}{\Gamma(1)} \oint \frac{dz}{z}\,\frac{\Gamma(a^{\pm1}z^{\pm1})}{\Gamma(z^{\pm2})}\,\Gamma(t^{\frac{3}{2}}z^{\pm1}b^{\pm1}c^{\pm1}) = \Gamma(t^{\frac{3}{2}}a^{\pm1}b^{\pm1}c^{\pm1}) = C_{abc}, \qquad \text{(B.10)}$$

as illustrated in Figure B.3. For completeness we can also compute the sphere

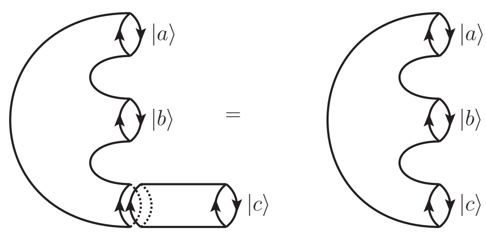


Figure B.3: The consistency requirement $\delta^z_c\,C_{abz} = C_{abc}$.

and the torus partition functions. (These partition functions do not appear in any index computation of a 4d superconformal theory so their physical interpretation is unclear.)

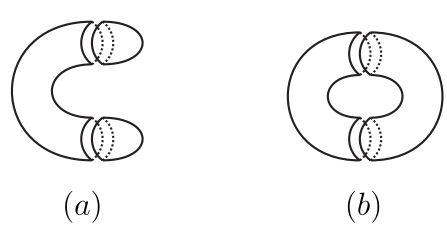


Figure B.4: The sphere **(a)** and the torus **(b)** partition functions.

The sphere partition function is given by

$$
\begin{aligned}
V^c\,V^e\,\eta_{ce} &= \frac{\kappa^2}{\Gamma(1)^5}\oint\frac{de}{e}\oint\frac{dc}{c}\frac{\Gamma\left(c^{\pm1}e^{\pm1}\right)\Gamma\left(t^{\pm3/2}\,c^{\pm1}\right)\,\Gamma\left(t^{\pm3/2}\,e^{\pm1}\right)}{\Gamma\left(c^{\pm2}\right)\Gamma\left(e^{\pm2}\right)} = \\
&= \frac{\kappa}{\Gamma(1)^4}\oint\frac{de}{e}\frac{\Gamma\left(t^{\pm3/2}\,e^{\pm1}\right)^2}{\Gamma\left(e^{\pm2}\right)} = \Gamma(t^{-3})\,\frac{1}{\Gamma(1)}.
\end{aligned}
\tag{B.11}
$$

The torus partition function is given by

$$
\eta_{ab}\eta^{ab} = \frac{\kappa}{\Gamma(1)}\oint\frac{d\,a}{a}\frac{\Gamma(a^{\pm1}a^{\pm1})}{\Gamma(a^{\pm2})} = \kappa\,\Gamma(1)\oint\frac{d\,a}{a} = 2\,\pi\,i\,\kappa\,\Gamma(1). \tag{B.12}
$$

Since $\Gamma(1)=\infty$ the sphere partition function vanishes and the torus partition function diverges.

Appendix C

t expansion in the weakly-coupled frame

We expand the index (3.10) in t as

$$\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}} = \sum_{k=0}^{\infty} b_k\, t^k. \tag{C.1}$$

The first few orders are

$$\begin{aligned}
b_0 &= 1, \\
b_1 &= b_2 = b_3 = 0, \\
b_4 &= \frac{1}{v}\chi^{SU(6)}_{\mathbf{35},adj} + \frac{1}{v} + v^2, \\
b_5 &= -v(y + \frac{1}{y}), \\
b_6 &= \frac{1}{v^{3/2}}\chi^{SU(6)}_{\mathbf{20}}((\frac{a}{b})^{3/2} + (\frac{b}{a})^{3/2}) - \chi^{SU(6)}_{\mathbf{35},adj} + v^3 - 1, \\
b_7 &= \frac{1}{v}(y + \frac{1}{y})\chi^{SU(6)}_{\mathbf{35},adj} + \frac{2}{v}(y + \frac{1}{y}), \\
b_8 &= \frac{1}{v^2}\chi^{SU(6)}_{sym^2\mathbf{35}} + v\chi^{SU(6)}_{\mathbf{35},adj} - \frac{1}{\sqrt{v}}\chi^{SU(6)}_{\mathbf{20}}((\frac{a}{b})^{3/2} + (\frac{b}{a})^{3/2}) + v^4 - v(y + \frac{1}{y})^2 + 2v, \\
b_9 &= -2(y + \frac{1}{y})\chi^{SU(6)}_{\mathbf{35},adj} + \frac{1}{v^{3/2}}(y + \frac{1}{y})\chi^{SU(6)}_{\mathbf{20}}((\frac{a}{b})^{3/2} + (\frac{b}{a})^{3/2}) - 2(y + \frac{1}{y}).
\end{aligned} \tag{C.2}$$

In the above equation we decomposed $SU(6) \supset SU(3)_z \otimes SU(3)_{y^{-1}} \otimes U(1)$. The branching of **35** and **20** of $SU(6)$ is given by (see [210]),

$$\begin{aligned} \mathbf{35} &= (\mathbf{1},\mathbf{1})_0 + (\mathbf{8},\mathbf{1})_0 + (\mathbf{1},\mathbf{8})_0 + (\bar{\mathbf{3}},\mathbf{3})_2 + (\mathbf{3},\bar{\mathbf{3}})_{-2}\,, \\ \mathbf{20} &= (\mathbf{1},\mathbf{1})_3 + (\mathbf{1},\mathbf{1})_{-3} + (\bar{\mathbf{3}},\mathbf{3})_{-1} + (\mathbf{3},\bar{\mathbf{3}})_1\,. \end{aligned} \tag{C.3}$$

For example, the character of the adjoint is

$$\begin{aligned} \chi^{SU(6)}_{\mathbf{35},adj} &= \left[(a\,b)^{1/2}\,(z_1+z_2+z_3) + (a\,b)^{-1/2}\,(\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3})\right] \times \\ &\quad \times \left[(a\,b)^{-1/2}\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right) + (a\,b)^{1/2}\,(y_1+y_2+y_3)\right] - 1\,. \end{aligned} \tag{C.4}$$

We conclude that the $U(1)$ charge in $SU(6)$ can be identified as $(a\,b)^{-1/2}$.

Appendix D

Inversion theorem

In this appendix we quote the inversion theorem [55], which we use in section 3.1.3 to obtain the index of the E_6 theory. Define

$$\delta(z,\, w; T) \equiv \frac{\Gamma(T\, z^{\pm 1}\, w^{\pm 1};\, p, q)}{\Gamma(T^2,\, z^{\pm 2};\, p, q)}. \tag{D.1}$$

If T, p and q are such that

$$|\max(p,\, q)| < |T| < 1\,, \tag{D.2}$$

then the following theorem holds true. For fixed w on the unit circle we define a contour C_w (see figure D.1) in the annulus $\mathbb{A} = \{|T| - \epsilon < |z| < |T|^{-1} + \epsilon\}$ with small but finite $\epsilon \in \mathbb{R}^+$, such that the points $T^{-1} w^{\pm 1}$ are in its interior and $C_w = C_w^{-1}$ (i.e. an inverse of the point in the interior of C_w is in the exterior of C_w). Let $f(z) = f(z^{-1})$ be a holomorphic function in $\mathbb{A}$. Then for $|T| < |x| < |T|^{-1}$,

$$\hat{f}(w) = \kappa \oint_{C_w} \frac{dz}{2\pi i\, z}\, \delta(z, w; , T^{-1})\, f(z) \to f(x) = \kappa \oint_{\mathbb{T}} \frac{dw}{2\pi i\, w}\, \delta(w, x; , T)\, \hat{f}(w)\,. \tag{D.3}$$

Our expression for the index in the strongly-coupled frame (3.20) is of the form of the right hand side of (D.3). Thus, to use the inversion theorem to

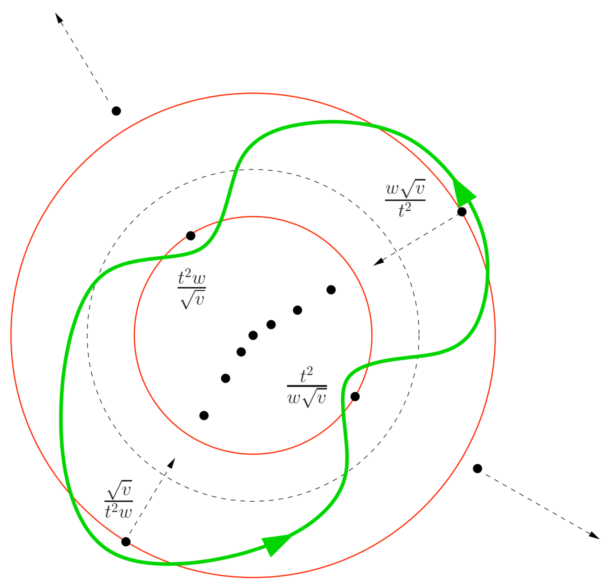


Figure D.1: The integration contour C_w (green). The dashed (black) circle is the unit circle $\mathbb{T}$. Black dots are poles of $\Gamma\left(\frac{\sqrt{v}}{t^2}\, w^{\pm 1}\, z^{\pm 1}\right)$. There are four sequences of poles: two sequences starting at $\frac{\sqrt{v}}{t^2} w^{\pm 1}$ and converging to $z = 0$, and two sequences starting at $\frac{t^2}{\sqrt{v}} w^{\pm 1}$ and converging to $z = \infty$. The contour encloses the two former sequences.

obtain the index of E_6 theory we *assume* that this index can be written as

$$\Gamma(t^2 v\, w^{\pm 2})\, C^{(E_6)}\left((w,\, r), \mathbf{y}, \mathbf{z}\right) = \kappa \oint_{C_w} \frac{ds}{2\pi i\, s} \frac{\Gamma(\frac{\sqrt{v}}{t^2} w^{\pm 1}\, s^{\pm 1})}{\Gamma(\frac{v}{t^4},\; s^{\pm 2})} F\left(s, r; \mathbf{y}, \mathbf{z}\right) \quad \text{(D.4)}$$

for some function F. The theorem (D.3) then implies that $F\left(s, r; \mathbf{y}, \mathbf{z}\right) = \hat{\mathcal{I}}\left(s, r; \mathbf{y}, \mathbf{z}\right)$ with $\mathcal{I}\left(s, r; \mathbf{y}, \mathbf{z}\right)$ given in (3.20).

Appendix E

The Coulomb and Higgs branch operators of E_6 SCFT

We collect here a few facts about the Coulomb and the Higgs branches of E_6 SCFT, following the analysis of [62]. Argyres-Seiberg duality can be used to determine the quantum numbers of protected operators of E_6 theory if their dual operators in the dual $SU(3)$ theory are known. The Coulomb branch operator u of the E_6 theory (the operator whose vev parametrized the Coulomb branch) is identified as $\mathrm{Tr}\,\phi^3$ in the $SU(3)$ theory. Since ϕ has quantum numbers $(E, j_1, j_2, R, r) = (1, 0, 0, 0, -1)$, u should have quantum numbers $(3, 0, 0, 0, -3)$ and contribute to the superconformal index as $t^6 v^3$.

The operator $\mathbb{X}$ whose vev parametrized the Higgs branch transforms in the adjoint representation of E_6. Under the $SU(2) \otimes SU(6)$ subgroup of E_6 it decomposes as

$$X^i_j, \quad Y^{[ijk]}_\alpha, \quad Z_{\alpha\beta}\,, \tag{E.1}$$

where $i, j, k = 1, \ldots, 6$ are the $SU(6)$ indices, and $\alpha, \beta = 1, 2$ are the $SU(2)$ indices. At the same time, the $SU(2)$ gauge theory provides the quarks q_α, $\tilde{q}_\alpha$ and the F-term constraint

$$Z_{\alpha\beta} + q_{(\alpha}\tilde{q}_{\beta)} = 0\,. \tag{E.2}$$

Thus the gauge-invariant operators are

$$(q\tilde{q}), \quad X^i_j, \quad (Y^{ijk}q), \quad (Y_{ijk}\tilde{q})\,. \tag{E.3}$$

On the $SU(3)$ side, the Higgs branch is parameterized by gauge invariant operators

$$M^i_j = Q^i_a \tilde{Q}^a_j, \quad B^{ijk} = \epsilon^{abc} Q^i_a Q^j_b Q^k_c, \quad \tilde{B}_{ijk} = \epsilon_{abc} \tilde{Q}^a_i \tilde{Q}^b_j \tilde{Q}^c_k \,, \tag{E.4}$$

where Q^i_a and $\tilde{Q}^a_i$ are the squark fields, $i = 1, \ldots, 6$ are flavor indices, and $a = 1, 2, 3$ the color indices.

The duality of the two sides suggests the following identification

$$\mathrm{Tr} M \leftrightarrow (q\tilde{q}), \qquad \hat{M}^i_j \leftrightarrow X^i_j, \tag{E.5}$$

$$B^{ijk} \leftrightarrow (Y^{ijk} q), \qquad \tilde{B}_{ijk} \leftrightarrow (Y_{ijk}\tilde{q}) \tag{E.6}$$

where $\hat{M}^i_j$ is the traceless part of M^i_j. Since the quantum numbers of Q are $(1, 0, 0, 1/2, 0)$, the quantum numbers of $\mathbb{X}$ should be $(2, 0, 0, 1, 0)$, and contribute to the index as t^4/v.

Appendix F

Identities from S-duality

In this appendix we summarize identities of integrals of elliptic Gamma functions implied by S-duality of the $SU(3)$ quiver theories.

Generalization of [49]

We define

$$\mathcal{I}^{(n)}\left(a\,,\mathbf{z}_{SU(n)};b\,,\mathbf{y}_{SU(n)}\right) \equiv \frac{2^{n-1}}{n!}\kappa^{n-1}\Gamma(t^2 v)^{n-1}\times \tag{F.1}$$
$$\oint_{\mathbb{T}^{n-1}}\prod_{i=1}^{n-1}\frac{dx_i}{2\pi i\,x_i}\frac{\prod_{i=1}^{n}\prod_{j=1}^{n}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{a z_i}{x_j}\right)^{\pm 1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\left(b\,y_i\,x_j\right)^{\pm 1}\right)\prod_{i\neq j}\Gamma\left(t^2 v\,\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}\Bigg|_{\prod_{j=1}^{n}x_j=1}$$

The claim is that

$$\mathcal{I}^{(n)}\left(a\,,\mathbf{z}_{SU(n)};b\,,\mathbf{y}_{SU(n)}\right)=\mathcal{I}^{(n)}\left(b\,,\mathbf{z}_{SU(n)};a\,,\mathbf{y}_{SU(n)}\right)\,. \tag{F.2}$$

For $SU(2)$ this identity was proven in [49], and for $SU(3)$ we have performed perturbative checks. The usual S-duality of $N_f=2n$ $SU(n)$ theories implies that this identity should be true for any n. Note that for $t=v$ this is a special case of identities discussed in [56].

E_6 Integral

We define

$$C^{(E_6)}\left((w,\,r),\mathbf{y},\mathbf{z}\right) \equiv \frac{2\,\kappa^3\Gamma(t^2v)^2}{3\,\Gamma(t^2v\,w^{\pm2})}\oint_{C_w}\frac{ds}{2\pi i\,s}\,\frac{\Gamma(\frac{\sqrt{v}}{t^2}w^{\pm1}\,s^{\pm1})}{\Gamma(\frac{v}{t^4},\;s^{\pm2})}\times$$

$$\times\oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i\,x_i}\frac{\prod_{i=1}^{3}\prod_{j=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{s^{\frac{1}{3}}\,z_i}{x_j\,r}\right)^{\pm1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{s^{-\frac{1}{3}}\,y_i\,x_j}{r}\right)^{\pm1}\right)\prod_{i\neq j}\Gamma\left(t^2v\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}$$

This integral has manifest symmetry under $SU(2)_w\otimes SU(6)$, where the $SU(6)$ has been decomposed as $SU(3)_{\mathbf{z}}\otimes SU(3)_{\mathbf{y}^{-1}}\otimes U(1)_r$. The identification with the index of the E_6 SCFT implies that there must be a symmetry enhancement $SU(2)_w\otimes SU(6)\to E_6$. Two properties that are sufficient to guarantee E_6 covariance are: first,

$$C^{(E_6)}\left((w,\,r),\mathbf{y},\mathbf{z}\right) = C^{(E_6)}\left(\left(\frac{w^{1/2}}{r^{3/2}},\,\frac{1}{w^{1/2}\,r^{1/2}}\right),\mathbf{y},\mathbf{z}\right)\,, \qquad \text{(F.3)}$$

which is the statement that (w,r) combine into a character of $SU(3)$ (which we shall denote by $\mathbf{w}$); second,

$$C^{(E_6)}(\mathbf{w},\mathbf{y},\mathbf{z}) = C^{(E_6)}(\mathbf{y},\mathbf{w},\mathbf{z})\,. \qquad \text{(F.4)}$$

We presented perturbative evidence for the full E_6 symmetry in the text.

S-dualities of $SU(3)$ quivers

Define

$$\mathcal{I}_{3333}\left(\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{s}\right) \equiv \oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i x_i}\prod_{i\neq j}\frac{\Gamma\left(t^2vx_i/x_j\right)}{\Gamma\left(x_i/x_j\right)}C^{(E_6)}\left(\mathbf{y},\mathbf{z},\mathbf{x}\right)C^{(E_6)}(\mathbf{u},\mathbf{s},\mathbf{x}^{-1})\,, \qquad \text{(F.5)}$$

$$\mathcal{I}_{3331}\left(\mathbf{y},\mathbf{z},\mathbf{u},a\right) \equiv \oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i x_i}\prod_{i\neq j}\frac{\Gamma\left(t^2vx_i/x_j\right)}{\Gamma\left(x_i/x_j\right)}C^{(E_6)}\left(\mathbf{y},\mathbf{z},\mathbf{x}\right)\prod_{i,j=1}^{3}\Gamma(\frac{t^2}{\sqrt{v}}(a\,x_i^{-1}\,u_j)^{\pm}).$$

The S-dualities of the $SU(3)$ quivers imply

$$\begin{aligned}\mathcal{I}_{3333}\left(\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{s}\right) &= \mathcal{I}_{3333}\left(\mathbf{y},\mathbf{u},\mathbf{z},\mathbf{s}\right)\ , \\ \mathcal{I}_{3331}\left(\mathbf{y},\mathbf{z},\mathbf{u},a\right) &= \mathcal{I}_{3331}\left(\mathbf{y},\mathbf{u},\mathbf{z},a\right)\ .\end{aligned} \tag{F.6}$$

Appendix G

Refinement of $3d$ partition function

The superconformal index defined in section 5.1 is a function of fugacities t, y and v. In order to recover the matrix model of Kapustin et al. [20, 21] in section 5.2 we simply fixed the $v \to t$ and $y \to 1$. In this appendix we refine the $3d$ partition function by keeping track of all the fugacities in the index. It is convenient to define the chemical potentials

$$v = e^{-\beta(1/3+u)}, \qquad y = e^{-\beta\eta}. \tag{G.1}$$

The index, in terms of β, u and η becomes

$$\mathcal{I} = \mathrm{Tr}(-1)^F e^{-\beta[\frac{2}{3}(E+j_2)-\frac{1}{3}(r+R)-(r+R)u+2j_1\eta]}. \tag{G.2}$$

Let us compute the partition function of the hypermultiplet after turning on only u.

$$\begin{aligned}
\mathcal{I}^{hyp} &= \prod_i \Gamma\left(\frac{t^2}{\sqrt{v}} a_i; t^3 y, t^3 y^{-1}\right) = \prod_i \prod_{n\geqslant 1} \left(\frac{[n+\frac{1}{2}+\frac{u}{2}+i\alpha_i]_q}{[n-\frac{1}{2}-\frac{u}{2}-i\alpha_i]_q}\right)^n \\
&\xrightarrow{q\to 1} \prod_i \left[\cosh\pi\left(\alpha_i - i\frac{u}{2}\right)\right]^{-\frac{1}{2}} \\
\mathcal{I}^{vector} &= \prod_{i<j} \frac{1}{1-q^{-i(\alpha_i-\alpha_j)}} \frac{1}{1-q^{i(\alpha_i-\alpha_j)}} \frac{\Gamma(q^{1+u\pm i(\alpha_i-\alpha_j)}; q, q)}{\Gamma(q^{\pm i(\alpha_i-\alpha_j)}; q, q)} \\
&\xrightarrow{q\to 1} \prod_{i<j} \left(\frac{\sinh\pi(\alpha_i-\alpha_j)}{\pi(\alpha_i-\alpha_j)}\right)^2 \left(\frac{\cosh\pi(\mp(\alpha_i-\alpha_j)+i/2)}{\cosh\pi(\mp(\alpha_i-\alpha_j)+i(u+1/2))}\right)^{1/2} .
\end{aligned} \tag{G.3}$$

Both partition functions reduce to the ones in section 5.2 as we set u to zero.

Now we restore $y = q^{-\beta\eta}$ to produce the more refined $3d$ partition function. The chemical potential η has a nice physical interpretation as the $U(1)\times U(1)$ isometry preserving squashing deformation of the S^3. The partition function of $3d$ gauge theories on this squashed background was computed in [78].

The contribution due to the hypermultiplet with η deformation turned on is

$$\begin{aligned}
\mathcal{I}^{hyp} &= \prod_i \Gamma(\frac{t^2}{\sqrt{v}} a_i; t^3 y, t^3/y) \\
&\xrightarrow{y\to q^{-\beta\eta}} \prod_i \Gamma(q^{1/2-u/2-i\alpha_i}; q^{1+\eta}, q^{1-\eta}) \\
&= \prod_i \prod_{j,k\geqslant 0} \frac{1-q^{3/2+u/2+i\alpha_i} q^{(1+\eta)j} q^{(1-\eta)k}}{1-q^{1/2-u/2-i\alpha_i} q^{(1+\eta)j} q^{(1-\eta)k}}.
\end{aligned} \tag{G.4}$$

Using the regularized infinite product representation of Barnes' double-Gamma function

$$\Gamma_2(x|\epsilon_1,\epsilon_2) \propto \prod_{m,n\geqslant 0} (x + m\epsilon_1 + n\epsilon_2)^{-1}, \tag{G.5}$$

the partition function of hyper-multiplet can be written in a compact way

$$\begin{aligned}\mathcal{I}^{hyper} &\to \prod_i \frac{\Gamma_2(1/2 - u/2 - i\alpha_i|1+\eta, 1-\eta)}{\Gamma_2(3/2 + u/2 + i\alpha_i|1+\eta, 1-\eta)} \\ &= \prod_i \frac{\Gamma_2(\frac{Q}{2}(1/2 - u/2) - i\hat{\alpha}_i|b, b^{-1})}{\Gamma_2(\frac{Q}{2}(3/2 + u/2) + i\hat{\alpha}_i|b, b^{-1})},\end{aligned} \tag{G.6}$$

where we have defined[1]

$$\hat{\alpha}_i = \frac{\alpha_i}{\sqrt{1-\eta^2}}, \qquad b = \sqrt{\frac{1-\eta}{1+\eta}}, \qquad Q = b + b^{-1}. \tag{G.7}$$

With this change of variables it is easy to see that for $u = 0$, our result is in agreement with [78]. The partition function of the vector multiplet:

$$\mathcal{I}^{vector} \to \prod_{i<j} \frac{1}{1 - q^{-i(\alpha_i - \alpha_j)}} \frac{1}{1 - q^{i(\alpha_i - \alpha_j)}} \frac{\Gamma(q^{1+u\pm i(\alpha_i-\alpha_j)}; q^{1+\eta}, q^{1-\eta})}{\Gamma(q^{\pm i(\alpha_i-\alpha_j)}; q^{1+\eta}, q^{1-\eta})}. \tag{G.8}$$

reduces to

$$\begin{aligned}\mathcal{I}^{vector} =& \prod_{i<j} \frac{(1-\eta^2)\sinh\frac{\pi(\alpha_i-\alpha_j)}{1+\eta}\sinh\frac{\pi(\alpha_i-\alpha_j)}{1-\eta}}{\pi^2(\alpha_i-\alpha_j)^2} \frac{\Gamma_2(1+u \pm i(\alpha_i-\alpha_j)|1+\eta, 1-\eta)}{\Gamma_2(1-u \pm i(\alpha_i-\alpha_j)|1+\eta, 1-\eta)} \\ =& \prod_{i<j} \frac{\sinh \pi b(\hat{\alpha}_i - \hat{\alpha}_j)\sinh \pi b^{-1}(\hat{\alpha}_i - \hat{\alpha}_j)}{\pi^2(\hat{\alpha}_i - \hat{\alpha}_j)^2} \frac{\Gamma_2(\frac{Q}{2}(1+u) \pm i(\hat{\alpha}_i - \hat{\alpha}_j)|b, b^{-1})}{\Gamma_2(\frac{Q}{2}(1-u) \pm i(\hat{\alpha}_i - \hat{\alpha}_j)|b, b^{-1})}.\end{aligned} \tag{G.9}$$

Again, we find a precise agreement with the partition function of the vector multiplet on squashed S^3.

[1]We thank Davide Gaiotto for pointing out this change of variables.

Appendix H

Shortening Conditions of the $\mathcal{N} = 2$ Superconformal Algebra

A generic long multiplet $\mathcal{A}^{\Delta}_{R,r(j,\bar{j})}$ of the $\mathcal{N} = 2$ superconformal algebra is generated by the action of the 8 Poincaré supercharges $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ on a superconformal primary, which by definition is annihilated by all conformal supercharges $\mathcal{S}$. If some combination of the $\mathcal{Q}$'s also annihilates the primary, the corresponding multiplet is shorter and the conformal dimensions of all its members are protected against quantum corrections. A comprehensive list of the possible shortening conditions for the $\mathcal{N} = 2$ superconformal algebra was given in [52] . Their findings are summarized in Table H.1. We take a moment to explain the notation.[1] The state $|R, r\rangle^{h.w.}_{(j,\bar{j})}$ is the highest weight state with $SU(2)_R$ spin $R > 0$, $U(1)_r$ charge r, which can have either sign, and Lorentz quantum numbers $(j, \bar{j})$. The multiplet built on this state is denoted as $\mathcal{X}_{R,r(j,\bar{j})}$, where the letter $\mathcal{X}$ characterizes the shortening condition. The left column of Table H.1 labels the condition. A superscript on the label corresponds to the index $\mathcal{I} = 1, 2$ of the supercharge that kills the primary: or example $\mathcal{B}^1$ refers to $\mathcal{Q}^1_\alpha$. Similarly a "bar" on the label refers to the conjugate condition: for example $\bar{\mathcal{B}}^2$ corresponds to $\bar{Q}_{2\,\dot{\alpha}}$ annihilating the state; this would result in the short anti-chiral multiplet $\bar{\mathcal{B}}_{R,r(j,0)}$, obeying $\Delta = 2R - r$. Note that conjugation reverses the signs of r, j and $\bar{j}$ in the expression of the conformal dimension. We refer to [52] for more details.

[1] We follow the conventions of [52], except that we have introduced the labels $\mathcal{D}$, $\mathcal{F}$, $\hat{\mathcal{F}}$ and $\mathcal{G}$ to denote some shortening conditions that were left nameless in [52].

Shortening Conditions				Multiplet
$\mathcal{B}_1$	$\mathcal{Q}^1_\alpha \vert R, r\rangle^{h.w.} = 0$	$j = 0$	$\Delta = 2R + r$	$\mathcal{B}_{R,r(0,\bar{j})}$
$\bar{\mathcal{B}}_2$	$\bar{\mathcal{Q}}_{2\dot{\alpha}} \vert R, r\rangle^{h.w.} = 0$	$\bar{j} = 0$	$\Delta = 2R - r$	$\bar{\mathcal{B}}_{R,r(j,0)}$
$\mathcal{E}$	$\mathcal{B}_1 \cap \mathcal{B}_2$	$R = 0$	$\Delta = r$	$\mathcal{E}_{r(0,\bar{j})}$
$\bar{\mathcal{E}}$	$\bar{\mathcal{B}}_1 \cap \bar{\mathcal{B}}_2$	$R = 0$	$\Delta = -r$	$\bar{\mathcal{E}}_{r(j,0)}$
$\hat{\mathcal{B}}$	$\mathcal{B}_1 \cap \bar{B}_2$	$r = 0,\ j, \bar{j} = 0$	$\Delta = 2R$	$\hat{\mathcal{B}}_R$
$\mathcal{C}_1$	$\epsilon^{\alpha\beta}\mathcal{Q}^1_\beta \vert R, r\rangle^{h.w.}_\alpha = 0$ $(\mathcal{Q}^1)^2 \vert R, r\rangle^{h.w.} = 0$ for $j = 0$		$\Delta = 2 + 2j + 2R + r$ $\Delta = 2 + 2R + r$	$\mathcal{C}_{R,r(j,\bar{j})}$ $\mathcal{C}_{R,r(0,\bar{j})}$
$\bar{\mathcal{C}}_2$	$\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\mathcal{Q}}_{2\dot{\beta}} \vert R, r\rangle^{h.w.}_{\dot{\alpha}} = 0$ $(\bar{\mathcal{Q}}_2)^2 \vert R, r\rangle^{h.w.} = 0$ for $\bar{j} = 0$		$\Delta = 2 + 2\bar{j} + 2R - r$ $\Delta = 2 + 2R - r$	$\bar{\mathcal{C}}_{R,r(j,\bar{j})}$ $\bar{\mathcal{C}}_{R,r(j,0)}$
$\mathcal{F}$	$\mathcal{C}_1 \cap \mathcal{C}_2$	$R = 0$	$\Delta = 2 + 2j + r$	$\mathcal{C}_{0,r(j,\bar{j})}$
$\bar{\mathcal{F}}$	$\bar{\mathcal{C}}_1 \cap \bar{\mathcal{C}}_2$	$R = 0$	$\Delta = 2 + 2\bar{j} - r$	$\bar{\mathcal{C}}_{0,r(j,\bar{j})}$
$\hat{\mathcal{C}}$	$\mathcal{C}_1 \cap \bar{\mathcal{C}}_2$	$r = \bar{j} - j$	$\Delta = 2 + 2R + j + \bar{j}$	$\hat{\mathcal{C}}_{R(j,\bar{j})}$
$\hat{\mathcal{F}}$	$\mathcal{C}_1 \cap \mathcal{C}_2 \cap \bar{\mathcal{C}}_1 \cap \bar{\mathcal{C}}_2$	$R = 0, r = \bar{j} - j$	$\Delta = 2 + j + \bar{j}$	$\hat{\mathcal{C}}_{0(j,\bar{j})}$
$\mathcal{D}$	$\mathcal{B}_1 \cap \bar{\mathcal{C}}_2$	$r = \bar{j} + 1$	$\Delta = 1 + 2R + \bar{j}$	$\mathcal{D}_{R(0,\bar{j})}$
$\bar{\mathcal{D}}$	$\bar{\mathcal{B}}_2 \cap \mathcal{C}_1$	$-r = j + 1$	$\Delta = 1 + 2R + j$	$\bar{\mathcal{D}}_{R(j,0)}$
$\mathcal{G}$	$\mathcal{E} \cap \bar{\mathcal{C}}_2$	$r = \bar{j} + 1, R = 0$	$\Delta = r = 1 + \bar{j}$	$\mathcal{D}_{0(0,\bar{j})}$
$\bar{\mathcal{G}}$	$\bar{\mathcal{E}} \cap \mathcal{C}_1$	$-r = j + 1, R = 0$	$\Delta = -r = 1 + j$	$\bar{\mathcal{D}}_{0(j,0)}$

Table H.1: Shortening conditions and short multiplets for the $\mathcal{N} = 2$ superconformal algebra.

Appendix I

$\mathcal{N} = 1$ Chiral Ring

An important subset of the protected operators of a supersymmetry theory are the operators in the chiral ring. Chiral operators, by definition, are annihilated by the supercharge of one chirality, $\bar{\mathcal{Q}}^{\dot{\alpha}}$, and thus obey a $\mathcal{B}$-type shortening condition. (If the theory has extended supersymmetry we focus on an $\mathcal{N} = 1$ subalgebra.) The product of two chiral operators is again chiral. Chiral operators are normally considered modulo $\bar{\mathcal{Q}}^{\dot{\alpha}}$-exact operators. The chiral cohomology classes can be specified by a set of generators and relations, which are easy to determine at weak (infinitesimal but non-zero) coupling. At higher orders the relations may get corrected, but the basic counting of chiral states is not expected to change [19, 211].

Let us first consider the case of pure $\mathcal{N} = 2$ SYM with gauge group $SU(N_c)$. Under an $\mathcal{N} = 1$ subalgebra the field content is decomposed as a chiral superfield Φ and a vector superfield W_α, both in the adjoint representation of the gauge group.. A generic chiral operator of the theory in the adjoint representation of the gauge group obeys

$$[W_\alpha, \mathcal{O}\} = \left[\bar{\mathcal{Q}}^{\dot{\alpha}}, D_{\alpha\dot{\alpha}}\mathcal{O}\right\} \ . \tag{I.1}$$

Substituting $\mathcal{O} = \Phi$ and $\mathcal{O} = W_\beta$ we see that, modulo $\bar{\mathcal{Q}}$ exact terms, W_α (anti-)commutes with Φ and W_β respectively. Using these relations we can narrow down the single-trace chiral operators to

$$\operatorname{Tr}\Phi^{k+2}, \qquad \operatorname{Tr}\Phi^{k+1}W_\alpha, \qquad \operatorname{Tr}\Phi^k\epsilon^{\alpha\beta}W_\alpha W_\beta\,, \qquad \text{for } k \geq 0\,. \tag{I.2}$$

We have listed one representative from each cohomology class. For finite N_c the operators are further related by trace relations. In the large N_c limit of $N = 2$ supersymmetric Yang Mills, (I.2) is the complete and unconstrained list of single-trace chiral operators. Taking products we generate the whole chiral ring. In $\mathcal{N} = 2$ language the chiral operators are assembled in a single supermultiplet for each k, the multiplet with primary $\mathrm{Tr}\,\phi^{k+2}$.

To obtain $\mathcal{N} = 2$ SCQCD we add N_f fundamental hypermultiplets, equivalent to N_f fundamental chiral multiplets $\mathfrak{Q}$ and N_f antifundamental chiral multiplets $\tilde{\mathfrak{Q}}$, with the $\mathcal{N} = 2$ invariant superpotential $\tilde{\mathfrak{Q}}\Phi\mathfrak{Q}$. There are no chiral operators containing both W_α and $\mathfrak{Q}$ because $W_\alpha\mathfrak{Q}$ is $\bar{\mathcal{Q}}$ exact. Generally, in a theory with superpotential, further relations are imposed by the equations of motion

$$\partial_A W(A_i) = \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}A \quad \Rightarrow \quad \partial_A W(A_i)_{c.r.} = 0\,, \tag{I.3}$$

where $\{A_i\}$ is the set of chiral superfields. The subscript $c.r.$ denotes that the relation is valid in the chiral ring. In our case this implies that operators containing both Φ and $\mathfrak{Q}$ are constrained by the equations of motion

$$\Phi\mathfrak{Q} = 0, \quad \tilde{\mathfrak{Q}}\Phi = 0 \quad \text{and} \quad \mathfrak{Q}^a{}_i\tilde{\mathfrak{Q}}^i{}_b - \frac{1}{N_c}\delta^a_b\mathfrak{Q}^c{}_i\tilde{\mathfrak{Q}}^i{}_c = 0\,. \tag{I.4}$$

These relations set to zero all generalized single-trace operators[1] containing $\mathfrak{Q}$, except for $\mathrm{Tr}\,\mathfrak{Q}\tilde{\mathfrak{Q}}$. When expressed in $SU(2)_R$ covariant fashion, this operator corresponds to the $\mathcal{N} = 2$ superconformal primary $\mathrm{Tr}\mathcal{M}_{\mathbf{3}}$. Note that for gauge group $U(N_c)$ instead of $SU(N_c)$ the third relation gets modified to $\mathfrak{Q}^a{}_i\tilde{\mathfrak{Q}}^i{}_b = 0$ implying that even $\mathrm{Tr}\,\mathfrak{Q}\tilde{\mathfrak{Q}}$ is absent from the chiral ring. (For $U(N_c)$ we would have to also *add* the operator $\mathrm{Tr}\,\Phi$ to the list (I.2)). All in all, consideration of the chiral ring for $\mathcal{N} = 2$ SCQCD has led to identify the following protected $\mathcal{N} = 2$ superconformal primaries:

$$\mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}\,, \quad \mathrm{Tr}\,\phi^{\ell+2}\,, \quad \ell \geq 0\,. \tag{I.5}$$

Note that the multiplets $\{\mathrm{Tr}\,T\phi^\ell\}$, as well as the extra exotic protected states discussed in section 6.5.4, are not part of the chiral ring.

[1] In the flavor non-singlet sector they also allow for $\mathfrak{Q}^a_i\tilde{\mathfrak{Q}}^j_a$.

It is straightforward to repeat this exercise for the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM. In $\mathcal{N}=1$ language the field content of the orbifold theory consists of vector multiplets (Φ, W_α) and $(\check{\Phi}, \check{W}_\alpha)$, in the adjoint representation of $SU(N_c)$ and $SU(N_{\check{c}})$ respectively. They are coupled to bifundamental chiral multiplets $(\mathfrak{Q}_{\hat{\mathcal{I}}}, \tilde{\mathfrak{Q}}^{\hat{\mathcal{J}}})$ through the superpotential $\tilde{\mathfrak{Q}}^{\hat{\mathcal{I}}}\Phi\mathfrak{Q}_{\hat{\mathcal{I}}} + \mathfrak{Q}_{\hat{\mathcal{I}}}\check{\Phi}\tilde{\mathfrak{Q}}^{\hat{\mathcal{I}}}$. Here $\hat{\mathcal{I}}, \hat{\mathcal{J}}$ are $SU(2)_L$ indices. At large N_c, the chiral ring of the orbifold is generated by the operators (I.2), by a second copy of (I.2) with $\Phi, W_\alpha \to \check{\Phi}, \check{W}_\alpha$ corresponding to the two vector multiplets, and by single-trace operators involving the fields from hypermultiplets. The latter obey following constraints due to the superpotential:

$$\tilde{\mathfrak{Q}}^{\hat{\mathcal{I}}}\Phi = -\check{\Phi}\tilde{\mathfrak{Q}}^{\hat{\mathcal{I}}}, \qquad \Phi\mathfrak{Q}_{\hat{\mathcal{I}}} = -\mathfrak{Q}_{\hat{\mathcal{I}}}\check{\Phi} \tag{I.6}$$
$$\mathfrak{Q}^{a}_{\hat{\mathcal{I}}\,\check{a}}\tilde{\mathfrak{Q}}^{\hat{\mathcal{I}}\check{a}}{}_{b} - \frac{1}{N_c}\delta^a_b\mathfrak{Q}^{c}_{\hat{\mathcal{I}}\,\check{a}}\tilde{\mathfrak{Q}}^{\hat{\mathcal{I}}\check{a}}{}_{c} = 0, \qquad \tilde{\mathfrak{Q}}^{\hat{\mathcal{I}}\check{a}}{}_{a}\mathfrak{Q}^{a}_{\hat{\mathcal{I}}\,\check{b}} - \frac{1}{N_{\check{c}}}\delta^{\check{a}}_{\check{b}}\tilde{\mathfrak{Q}}^{\hat{\mathcal{I}}\check{c}}{}_{a}\mathfrak{Q}^{a}_{\hat{\mathcal{I}}\,\check{c}} = 0$$

Using the first two equivalence relations we could always choose a class representative that doesn't contain any $\check{\Phi}$. Then the relations in the second line allow for highest $SU(2)_L$ spin chiral operators of schematic form $\mathrm{Tr}\,(\mathfrak{Q}\tilde{\mathfrak{Q}})^{\ell+1}_{\mathbf{3_L}}\Phi^k$. This operator is in the untwisted sector as it is invariant under quantum $\mathbb{Z}_2$ symmetry of the orbifold upto $\bar{\mathcal{Q}}^{\dot{\alpha}}$ exact terms. As before, the chiral ring of the $SU(N_c)$ theory (as opposed to $U(N_c)$), also contains the "exceptional" operator $\mathrm{Tr}\,(\mathfrak{Q}\tilde{\mathfrak{Q}})_{\mathbf{1_L}}$, which belongs to the twisted sector. Assembling these $\mathcal{N}=1$ chiral multiplets into full $\mathcal{N}=2$ multiplets, we find the following list of $\mathcal{N}=2$ superconformal primaries:

$$\mathrm{Tr}\,(\phi^{k+2} + \check{\phi}^{k+2})\,, \qquad \mathrm{Tr}\,(\mathcal{M}^{\ell+1}_{\mathbf{3_R 3_L}}\,\phi^k)\,, \tag{I.7}$$
$$\mathrm{Tr}\,(\phi^{k+2} - \check{\phi}^{k+2})\,, \qquad \mathrm{Tr}\,\mathcal{M}_{\mathbf{3_R 1_L}}\,, \qquad \text{for } k \geq 0,\ \ell \geq 0\,. \tag{I.8}$$

The primaries in the first line belong to the untwisted sector and the primaries in the second line belong to the twisted sector. We know from inheritance from $\mathcal{N}=4$ SYM that in the untwisted sector there are additional protected operators (see section 6.4.1.1). On the other hand, in the twisted sector this is plausibly the complete list, as confirmed by the calculation of the superconformal index in appendix J.

As we move away from the orbifold point by taking $\check{g} \neq g$, the calculation of the chiral ring is almost unchanged, we only need to perform the substitutions

$\check{\Phi}, \check{W}_\alpha \to \kappa\check{\Phi}, \kappa\check{W}_\alpha$, with $\kappa \equiv \check{g}/g$ that take into account the deformation of the superpotential. The quantum numbers of the chiral operators remain unchanged.

Appendix J

The Index of Some Short multiplets

In this appendix we calculate the index of various short multiplets. A first goal is to determine the index of the set $\{\ \hat{\mathcal{B}}_1,\ \mathcal{E}_{\ell(0,0)},\ \ell \geq 2\ \}$ (the multiplets found by the analysis of the chiral ring in the twisted sector of the orbifold), and show that it agrees with (6.64). A second goal is to calculate $\mathcal{I}_{naive}$, the index of the "naive" protected spectrum (7.24) of $\mathcal{N}=2$ SCQCD.

J.1 $\mathcal{E}_{\ell(0,0)}$ multiplet

The chiral multiplet $\mathcal{E}_{\ell(0,0)}$ [52] is defined to be the multiplet that descends from the operator with $R=0$, that is annihilated by both $\mathcal{Q}^1$ and $\mathcal{Q}^2$. The shortening condition is $\Delta=\ell$. We have arranged the operator content of the multiplet in the array below. We represent the action of the supercharge $\mathcal{Q}$ to the left and $\bar{\mathcal{Q}}$ to the right. As $\mathcal{E}_{\ell(0,0)}$ is annihilated by $\mathcal{Q}$s, it only extends to the right.

$$
\begin{array}{c|ccccc}
\Delta & & & & & \\
\ell & \underline{0_{(0,0)}} & & & & \\
\ell+\frac{1}{2} & & \underline{\frac{1}{2}_{(0,\frac{1}{2})}} & & & \\
\ell+1 & & & 0_{(0,1)}, \underline{1_{(0,0)}} & & \\
\ell+\frac{3}{2} & & & & \frac{1}{2}_{(0,\frac{1}{2})} & \\
\ell+2 & & & & & 0_{(0,0)} \\
\hline
r & \ell & \ell-\frac{1}{2} & \ell-1 & \ell-\frac{3}{2} & \ell-2
\end{array}
\tag{J.1}
$$

This multiplet contributes only to the left index $\mathcal{I}^{\mathtt{L}}$. The operators with $\delta^{\mathtt{L}} = 0$ are underlined and their contribution to the index is listed in table J.1.

Δ	$R_{(j,\bar{j})}$	$\mathcal{I}^{\mathtt{L}}(t,y,v)$
ℓ	$0_{(0,0)}$	$t^{2\ell}v^{\ell}$
$\ell+\frac{1}{2}$	$\frac{1}{2}_{(0,\frac{1}{2})}$	$-t^{2\ell+1}v^{\ell-1}\left(y+\frac{1}{y}\right)$
$\ell+1$	$1_{(0,0)}$	$t^{2\ell+2}v^{\ell-2}$

Table J.1: Operators with $\delta^{\mathtt{L}} = 0$ in $\mathcal{E}_{\ell(0,0)}$

For $\ell > 1$, we sum the contribution of the operators from the above table and divide it by the contribution $(1-t^3y)\,(1-t^3y^{-1})$ from the derivatives,

$$
\begin{aligned}
\sum_{\ell=2}^{\infty}\mathcal{I}^{\mathtt{L}}_{\mathcal{E}_{\ell(0,0)}} &= \frac{1}{(1-t^3y)\,(1-t^3y^{-1})}\sum_{\ell=2}^{\infty}t^{2\ell}v^{\ell}(1-t^1v^{-1}(y+y^{-1})+t^2v^{-2}) \\
&= \frac{t^4v^2(1-\frac{t}{vy})(1-\frac{ty}{v})}{(1-t^2v)\,(1-t^3y)\,(1-t^3y^{-1})}
\end{aligned}
$$

The conjugate multiplet $\bar{\mathcal{E}}_{-\ell(0,0)}$ contributes exactly the same but to $\mathcal{I}^{\mathtt{R}}$.

J.2 $\hat{\mathcal{B}}_1$ multiplet

Next we consider the nonchiral multiplet $\hat{\mathcal{B}}_1$ [52], with the shortenning condition that the highest weight state is anihilated by $\mathcal{Q}^2, \bar{\mathcal{Q}}_1$. This shortening condition requires $r = 0$, $j = \bar{j} = 0$ and $\Delta = 2$ for the highest weight state.

$$
\begin{array}{c|ccccc}
\Delta & & & & & \\
2 & & & \underline{1_{(0,0)}} & & \\
\frac{5}{2} & & \underline{\frac{1}{2}_{\left(\frac{1}{2},0\right)}} & & \frac{1}{2}_{\left(0,\frac{1}{2}\right)} & \\
3 & 0_{(0,0)} & & 0_{\left(\frac{1}{2},\frac{1}{2}\right)} & & 0_{(0,0)} \\
\frac{7}{2} & & & & & \\
4 & & & -0_{(0,0)} & & \\
\hline
r & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & -1
\end{array}
\tag{J.2}
$$

The operator $-0_{(0,0)}$ at $\Delta = 4$ stands for an equation of motion – the negative sign in front of it means that its contribution to the index (partition function in general) has to be subtracted. We have underlined the operators with $\delta^{\mathrm{L}} = 0$ and their contribution to $\mathcal{I}^{\mathrm{L}}$ is listed in table J.2.

Δ	$R_{(j,\bar{j})}$	$\mathcal{I}^{\mathrm{L}}(t,y,v)$
2	$1_{(0,0)}$	$\frac{t^4}{v}$
$\frac{5}{2}$	$\frac{1}{2}_{\left(\frac{1}{2},0\right)}$	$-t^6$

Table J.2: Operators with $\delta^{\mathrm{L}} = 0$ in $\mathcal{B}_1$

Summing the individual contributions and dividing with the contribution from the derivatives, we get the index for this multiplet as,

$$
\mathcal{I}^{\mathrm{L}}_{\mathcal{B}_1} = \frac{t^4\,(1 - t^2 v)}{v\,(1 - t^3 y)\,(1 - t^3 y^{-1})}\,. \tag{J.3}
$$

J.3 $\hat{\mathcal{C}}_{0(0,0)}$ multiplet

The stress tensor, supercurrents and R-symmetry currents of the $\mathcal{N}=2$ theory are part of this multiplet. Its shortening condition $\hat{\mathcal{C}}$ is explained in table H.1. The operator content of this multiplet is displayed in the array below.

$$
\begin{array}{c|ccccc}
\Delta & & & & & \\
2 & & & 0_{(0,0)} & & \\
\frac{5}{2} & & \underline{\frac{1}{2}_{(\frac{1}{2},0)}} & & \frac{1}{2}_{(0,\frac{1}{2})} & \\
3 & \underline{0_{(1,0)}} & & \underline{1_{(\frac{1}{2},\frac{1}{2})}},\ 0_{(\frac{1}{2},\frac{1}{2})} & & 0_{(0,1)} \\
\frac{7}{2} & & \underline{\frac{1}{2}_{(1,\frac{1}{2})}} & & \frac{1}{2}_{(\frac{1}{2},1)} & \\
4 & & & 0_{(1,1)} & & \\
 & & & -0_{(0,0)},\ -1_{(0,0)} & & \\
\frac{9}{2} & & -\frac{1}{2}_{(\frac{1}{2},0)} & & -\frac{1}{2}_{(0,\frac{1}{2})} & \\
10 & & & -0_{(\frac{1}{2},\frac{1}{2})} & & \\
\hline
r & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & -1
\end{array}
\tag{J.4}
$$

The operators with negative signs stand for equations of motion as before. We have underlined the operators with $\delta^{\mathrm{L}}=0$ and their contribution is listed in the table below. Summing the contributions, we get the left index of this multiplet to be

$$
\mathcal{I}^{\mathrm{L}}_{\hat{\mathcal{C}}_{(0,0)}} = -t^6(1-vt^2)(1-\frac{t}{v}(y+\frac{1}{y}))\,. \tag{J.5}
$$

Δ	$R_{(j,\bar{j})}$	$\mathcal{I}^{\mathtt{L}}(t,y,v)$
$\frac{5}{2}$	$\frac{1}{2}_{(\frac{1}{2},0)}$	$-t^6$
3	$0_{(1,0)}$	$t^8 v$
3	$1_{(\frac{1}{2},\frac{1}{2})}$	$\frac{t^7}{v}(y+\frac{1}{y})$
$\frac{7}{2}$	$\frac{1}{2}_{(1,\frac{1}{2})}$	$-t^9(y+\frac{1}{y})$

Table J.3: Operators with $\delta^{\mathtt{L}}=0$ in $\hat{\mathcal{C}}_{0(0,0)}$

Being a nonchiral multipet, it contributes the same to the right index as well.

J.4 $\mathcal{C}_{\ell(0,0)}$ multiplet, $\ell \geq 1$

This multiplet obeys the shortening condition $\mathcal{F}=\mathcal{C}_1 \cap \mathcal{C}_2$. The operator content of $\mathcal{C}_{\ell(0,0)}$ is displayed below.

Δ \ r	$\ell+1$	$\ell+\frac{1}{2}$	ℓ	$\ell-\frac{1}{2}$	$\ell-1$	$\ell-\frac{3}{2}$	$\ell-2$
$\ell+2$			$0_{(0,0)}$				
$\ell+\frac{5}{2}$		$\underline{\frac{1}{2}_{(\frac{1}{2},0)}}$		$\frac{1}{2}_{(0,\frac{1}{2})}$			
$\ell+3$	$\underline{0_{(1,0)}}$		$\underline{1_{(\frac{1}{2},\frac{1}{2})}}$, $0_{(\frac{1}{2},\frac{1}{2})}$		$0_{(0,1)}$, $1_{(0,0)}$		
$\ell+\frac{7}{2}$		$\underline{\frac{1}{2}_{(1,\frac{1}{2})}}$		$\frac{1}{2}_{(\frac{1}{2},1)}$, $\frac{1}{2}_{(\frac{1}{2},0)}$, $\underline{\frac{3}{2}_{(\frac{1}{2},0)}}$		$\frac{1}{2}_{(0,\frac{1}{2})}$	
$\ell+4$			$0_{(1,1)}$, $\underline{1_{(1,0)}}$		$0_{(\frac{1}{2},\frac{1}{2})}$, $1_{(\frac{1}{2},\frac{1}{2})}$		$0_{(0,0)}$
$\ell+\frac{9}{2}$				$\frac{1}{2}_{(1,\frac{1}{2})}$		$\frac{1}{2}_{(\frac{1}{2},0)}$	
$\ell+5$					$0_{(1,0)}$		

The operators with $\delta^{\mathtt{L}}=0$ are underlined as usual. Table J.4 lists their contribution to $\mathcal{I}^{\mathtt{L}}$. Summing the contribution to the left index from $\mathcal{C}_{\ell(0,0)}$ with $\ell \geq 1$ we get,

$$\sum_{\ell=1}^{\infty} \mathcal{I}^{\mathtt{L}}_{\mathcal{C}_{\ell(0,0)}} = -t^8 v(1-vt^2)(1-\frac{t}{v}(y+\frac{1}{y})) - \frac{t^{10}}{v}\,. \tag{J.6}$$

Δ	$R_{(j,\bar{j})}$	$\mathcal{I}^{\mathrm{L}}(t,y,v)$
$\ell+\frac{5}{2}$	$\frac{1}{2}_{(\frac{1}{2},0)}$	$-t^{6+2\ell}v^{\ell}$
$\ell+3$	$0_{(1,0)}$	$t^{8+2\ell}v^{\ell+1}$
$\ell+3$	$1_{(\frac{1}{2},\frac{1}{2})}$	$t^{7+2\ell}v^{\ell-1}(y+\frac{1}{y})$
$\ell+\frac{7}{2}$	$\frac{1}{2}_{(1,\frac{1}{2})}$	$-t^{9+2\ell}v^{\ell}(y+\frac{1}{y})$
$\ell+\frac{7}{2}$	$\frac{3}{2}_{(\frac{1}{2},0)}$	$-t^{8+2\ell}v^{\ell-2}$
$\ell+4$	$1_{(1,0)}$	$t^{10+2\ell}v^{\ell-1}$

Table J.4: Operators with $\delta^{\mathrm{L}}=0$ in $\mathcal{C}_{\ell(0,0)}$

J.5 The $\mathcal{I}_{twist}$ of the orbifold and $\mathcal{I}_{naive}$ of SCQCD

The protected operators in the twisted sector of the orbifold are listed in Table 7.3. The conjugates, which contribute to $\mathcal{I}^{\mathrm{L}}$, are of the type:

$$\hat{\mathcal{B}}_1, \qquad \mathcal{E}_{\ell(0,0)} \qquad\qquad \text{for} \quad \ell \geq 2\,. \tag{J.7}$$

So we get,

$$\mathcal{I}_{twist} = \mathcal{I}_{\hat{\mathcal{B}}_1} + \sum_{\ell=2}^{\infty} \mathcal{I}_{\mathcal{E}_{\ell(0,0)}} \tag{J.8}$$

$$= \frac{t^4\,(1-t^2v)}{v\,(1-t^3y)\,(1-t^3y^{-1})} + \frac{t^4v^2(1-\frac{t}{vy})(1-\frac{ty}{v})}{(1-t^2v)\,(1-t^3y)\,(1-t^3y^{-1})} \tag{J.9}$$

$$= \frac{t^2v}{1-t^2v} - \frac{t^3y}{1-t^3y} - \frac{t^3y^{-1}}{1-t^3y^{-1}} - f_V(t,y,v)\,. \tag{J.10}$$

This precisely matches with (6.64), confirming the protected operators in the twisted sector of the orbifold. Let us now compute the $\mathcal{I}_{naive}$ of SCQCD that follows from the preliminary list 7.24 of protected operators. Their conjugates, which contribute to $\mathcal{I}^{\mathrm{L}}$, are of the type:

$$\hat{\mathcal{B}}_1, \qquad \mathcal{E}_{\ell+2(0,0)}, \qquad \hat{\mathcal{C}}_{0,0}, \qquad \mathcal{C}_{\ell+1(0,0)} \qquad\qquad \text{for} \quad \ell \geq 0\,. \tag{J.11}$$

The $\mathcal{I}_{naive}$ then is

$$\begin{aligned}
\mathcal{I}_{naive} &= \mathcal{I}_{\hat{\mathcal{B}}_1} + \sum_{\ell=2}^{\infty} \mathcal{I}_{\mathcal{E}_{\ell(0,0)}} + \mathcal{I}_{\hat{\mathcal{C}}_{0,0}} + \sum_{\ell=1}^{\infty} \mathcal{I}_{\mathcal{C}_{\ell(0,0)}} && \text{(J.12)} \\
&= \frac{-t^6(1-\frac{t}{v}(y+\frac{1}{y})) - \frac{t^{10}}{v} + \frac{t^4 v^2 (1-\frac{t}{vy})(1-\frac{ty}{v})}{1-t^2 v} + \frac{t^4}{v}(1-t^2 v)}{(1-t^3 y)(1-\frac{t^3}{y})} && \text{(J.13)}
\end{aligned}$$

Appendix K

KK Reduction of the $6d$ Tensor Multiplet on $AdS_5 \times S^1$

In this appendix we discuss the Kaluza-Klein reduction of the $6d$ tensor multiplet on $AdS_5 \times S^1$, and its matching with the twisted spectrum of the orbifold theory.

The tensor multiplet of maximal chiral supersymmetry in six dimensions (we will refer to it as (2,0) susy) has the following field content

$$B^{-}_{\mu\nu}\,, \quad \lambda^{\mathfrak{J}}_{\alpha}\,, \quad \Phi^{[\mathfrak{J}\mathfrak{K}]}\,. \tag{K.1}$$

The indices $\mathfrak{J}, \mathfrak{K}$ are the $USp(4)$ indices which is the R-symmetry group of the chiral supergravity. The spinors $\lambda^{\mathfrak{J}}_{\alpha}$ are in the **4** (complex) representation of $USp(4)$ and the scalars $\Phi^{[\mathfrak{J}\mathfrak{K}]}$ in the **5** (real) representation. The $\lambda^{\mathfrak{J}}_{\alpha}$ are Weyl, symplectic Majorana spinors. The symplectic Majorana condition is a psuedo-reality condition, $\bar{\lambda}_{\mathfrak{J}} = \Omega_{\mathfrak{J}\mathfrak{K}}\lambda^{\mathfrak{K}}$, where Ω is the symplectic form.

Consider now the background $AdS_5 \times S^1$. The natural embedding of the $SU(2)_R \times U(1)_r$ R-symmetry of the $\mathcal{N} = 4$ AdS_5 superalgebra (or equivalently of the $\mathcal{N}=2$ $4d$ superconformal algebra) into $USp(4)$ is

$$\left(\begin{array}{c|c} SU(2)_R \times U(1)_r & \\ \hline & SU(2)_R \times U(1)_r^* \end{array}\right) \tag{K.2}$$

The five scalars decompose as

$$\begin{aligned} \Phi^{[\mathfrak{J}\mathfrak{K}]} \quad &\longrightarrow \quad \Phi^i + \ \Phi \ + \ \bar{\Phi} \\ \mathbf{5} \quad &\longrightarrow \quad \mathbf{3}_0 + \mathbf{1}_{-\mathbf{1}} + \mathbf{1}_{+\mathbf{1}}\,, \end{aligned} \tag{K.3}$$

where the subscripts denote $U(1)_r$ charges. The spinors decompose as two (conjugate) $SU(2)_R$ doublets, with opposite $U(1)_r$ charges $r = \pm\frac{1}{2}$.

We are interested in the Kaluza-Klein reduction of the tensor multiplet on the S^1. We borrow the results of [145] (see also [212]), where all the KK modes with non-zero momentum were matched with the multiplets $\{\bar{\mathcal{E}}_{2+\ell(0,0)}\, \ell \geq 0\}$, corresponding to the twisted primaries $\{\mathrm{Tr}\phi^{2+\ell} - \mathrm{Tr}\check{\phi}^{\ell+2}\,\}$ of the orbifold theory. We will add the zero modes to the analysis of [145].

Let us indeed start with the zero modes on S^1. The bosonic zero modes comprise the following AdS_5 fields [145]: a complex scalar Φ, with $m^2 = -3$ (in AdS units)[1]; a triplet of scalars Φ^i, with $m^2 = -4$; a massless two form $B_{\hat{m}\hat{n}}$, *or* equivalently a massless gauge field $A_{\hat{m}}$. The massless two-form $B_{\hat{m}\hat{n}}$ arises from the $6d$ anti-selfdual two-form $B^-_{\mu\nu}$ when both indices are taken to be along AdS_5, while the gauge field $A_{\hat{m}}$ arises from $B^-_{\mu\nu}$ when one index is taken to be along AdS_5 and the other along S^1. Because of the anti-selfduality of $B^-_{\mu\nu}$, the two possibilities are not independent: $B_{\hat{m}\hat{n}}$ and $A_{\hat{m}}$ are dual to each other as $5d$ fields, and we must pick one *or* the other. This ambiguity translates into two alternative ways to fit the zero modes into supermultiplets of the $\mathcal{N}=2$ $4d$ superconformal algebra. Let us look at them in turn:

- Choosing $B_{\hat{m}\hat{n}}$.

  The massless two-form $B_{\hat{m}\hat{n}}$ is dual to a boundary two-form operator F'_{mn}

[1] The complex scalar Φ corresponds to the $k=-1$ real scalar in Family 2 and the $k=1$ real scalar in Family 3 of [145]. We have just relabeled them as $n=0$ modes.

of dimension $\Delta = 2$. We claim that the full supermultiplet of boundary operators is $\{\phi', \lambda'^{\mathcal{I}}_\alpha, F'_{mn}\, D'_i\}$, which is the the familiar off-shell $\mathcal{N} = 2$ vector multiplet (or $\mathcal{N} = 2$ "supersingleton" multiplet). Here ϕ' is a complex scalar with $r = \pm 1$ and $\Delta = 1$, dual to the bulk scalar Φ of $m^2 = -3$. The mass of Φ is in the range that allows both the Δ_+ and the Δ_- quantization schemes [213, 214], and supersymmetry forces the choice of $\Delta_- = 2 - \sqrt{m^2+4} = 1$. Since ϕ' saturates the unitarity bound, it must be a free scalar field. We recognize F'_{mn} as the Maxwell field strength and D'_i, $i = 1, 2, 3$, which form $SU(2)_R$ triplet with $\Delta = 2$ and are dual to the bulk fields Φ^i, as the auxiliary fields. Finally $\lambda'^{\mathcal{I}}_\alpha$ are the free fermionic fields with $\Delta = \frac{3}{2}$. The AdS/CFT relation for spin $\frac{1}{2}$ fields is usually quoted as $\Delta = 2 + |m|$, but this is evidently a case where we must pick instead $\Delta_- = 2 - |m|$, with $m = \frac{1}{2}$. We are not aware of an explicit discussion of the $\Delta_\pm$ quantization ambiguity for spinors, but it must be there because of supersymmetry. (Incidentally, similar issues arise in the familiar IIB on $AdS_5 \times S^5$ background if one looks at the zero modes, which can be organized in the $\mathcal{N} = 4$ supersingleton multiplet. Again both the scalars in the $\mathbf{6}$ of $SU(4)$ *and* the spinors in the $\mathbf{4}$ must be quantized in the Δ_- scheme.)

- Choosing $A_{\hat{\mu}}$.

  The boundary dual to $A_{\hat{m}}$ is a conserved current J_m ($\Delta = 3$). In this case we claim that supersymmetry forces the usual Δ_+ quantization scheme for Φ and $\lambda^{\mathfrak{I}}_\alpha$. It is easy to check that the zero modes can be precisely organized into the $\hat{\mathcal{B}}_1$ multiplet (summarized in (J.2)).

The two possibilities have a nice physical interpretation. The first alternative corresponds to keeping the $U(1)$ degree of freedom in the twisted sector (this is the "relative" $U(1)$ in the product gauge recall the discussion after equ.(8.19)) – in other terms we should identify $\phi' = \mathrm{Tr}(\phi - \hat{\phi})$. The second possibility corresponds instead to *removing* the relative $U(1)$. Then clearly the multiplet built on $\mathrm{Tr}(\phi - \hat{\phi})$ is lost, but as we have emphasized in section 6.4.1.2 and appendix B, an *additional* protected multiplet appears, the $\hat{\mathcal{B}}_1$ multiplet built on the primary $\mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}$. The AdS/CFT dictionary handles this subtle ambiguity in a very elegant way. For our purposes, the second alternative is the

Field Theory			Gravity	
Operator	$U(1)_r$	Δ	Mass	Field
$\mathrm{Tr}[\bar{\phi}^{n+1}] - \mathrm{Tr}[\bar{\check{\phi}}^{n+1}]$	$n+1$	$n+1$	$(n+1)(n-3)$	$\bar{\Phi}$
$\mathrm{Tr}[F\bar{\phi}^{n}] - \mathrm{Tr}[\check{F}\bar{\check{\phi}}^{n}]$	n	$n+2$	n^2	$B_{\hat{m}\hat{n}}$
$\mathrm{Tr}[\lambda\lambda\bar{\phi}^{n-1}] - \mathrm{Tr}[\check{\lambda}\check{\lambda}\bar{\check{\phi}}^{n-1}]$	n	$n+2$	n^2-4	Φ^i
$\mathrm{Tr}[F^2\bar{\phi}^{n-1}] - \mathrm{Tr}[\check{F}^2\bar{\check{\phi}}^{n-1}]$	$n-1$	$n+3$	$(n-1)(n+3)$	Φ

Table K.1: Matching of the positive KK modes ($n \geq 1$). The negative KK modes ($n \leq -1$) correspond to the conjugate operators.

relevant one, since we must remove the relative $U(1)$ in order to have a truly conformal field theory.

The matching of the higher Kaluza-Klein modes was discussed in [145], we summarize the results in Table K.1.

Appendix L

The Cigar Background and 7d Gauged Sugra

This appendix collects some facts about the non-critical string theory obtained in the double-scaling limit of two colliding NS branes [116, 117], namely IIB on $\mathbb{R}^{5,1} \times SL(2)_2/U(1)$. We start by reviewing well-known results, see *e.g.* [114, 115, 115–117, 123–125], and then make a new claim about a space-time "effective action" description. We are going to argue that the "lighest" delta-function normalizable modes in the continuum are described by a $7d$ maximally supersymmetric supergravity with non-standard gauging, recently constructed in [160, 161].

L.1 Preliminaries and Worldsheet Symmetries

A class of "non-critical" supersymmetric string backgrounds can defined in the RNS formalism by taking the tensor product of $\mathbb{R}^{d-1,1}$ with the Kazama Suzuki supercoset $SL_2(\mathbb{R})_k/U(1)$. The $\mathbb{R}^{d-1,1}$ part is described as usual by d free bosons X^μ and d free fermions ψ^μ. The coset $SL_2(\mathbb{R})_k/U(1)$ has a sigma-model description with target space the "cigar" background (setting $\alpha' = 2$)

$$ds^2 = d\rho^2 + \tanh^2(\frac{Q\rho}{2})d\theta^2 \qquad \rho \geq 0 \quad \theta \sim \theta + \frac{4\pi}{Q} \tag{L.1}$$

with vanishing B field and dilaton varying as

$$\Phi = -\ln\cosh(\frac{Q\rho}{2})\,. \tag{L.2}$$

The level k of the coset is related to the parameter Q as $k = 2/Q^2$. The central charge is

$$c_{cig} = 3 + \frac{6}{k} = 3 + 3Q^2\,. \tag{L.3}$$

Adding the usual superconformal ghost system $\{b\,, c\,, \beta\,, \gamma\}$ of central charge -15 and requiring cancellation of the total conformal anomaly, one finds $Q = \sqrt{\frac{1}{2}(8-d)}$. In the asymptotic region $\rho \to \infty$ the cigar becomes a cylinder of radius $\frac{2}{Q}$, with the dilaton varying linearly with ρ, and the theory is thus a free CFT. We will soon restrict to the $d = 6$ case, implying $c_{cig} = 6$, $Q = 1$ and $k = 2$.

For generic level k the Kazama-Susuki coset $SL(2)_k/U(1)$ has $(2,2)$ supersymmetry. In the asymptotic linear-dilaton region the holomorphic currents of $\mathcal{N} = 2$ susy take the form

$$\begin{aligned} T_{\rm cig} &= -\frac{1}{2}(\partial\rho)^2 - \frac{1}{2}(\partial\theta)^2 - \frac{1}{2}(\psi_\rho\partial\psi_\rho + \psi_\theta\partial\psi_\theta) - \frac{1}{2}Q\partial^2\rho && \text{(L.4)} \\ J_{\rm cig} &= -i\psi_\rho\psi_\theta + iQ\partial\theta \equiv i\partial H + iQ\partial\theta \equiv i\partial\phi && \text{(L.5)} \\ G^\pm_{\rm cig} &= \frac{i}{2}(\psi_\rho \pm i\psi_\theta)\partial(\rho \mp i\theta) + \frac{i}{2}Q\partial(\psi_\rho \pm i\psi_\theta)\,, && \text{(L.6)} \end{aligned}$$

with analogous expressions for the anti-holomorphic currents. For $k = 2$, which is the case of interest for us, worldsheet supersymmetry is enhanced to $(4,4)$. This is the generic enhancement of worldsheet susy from $\mathcal{N} = 2$ to $\mathcal{N} = 4$ that takes place when $c = 6$. Indeed for this value of the central charge the currents $J^i_{\rm cig} = \{e^{\pm\int J_{\rm cig}}\,, J_{\rm cig}\}$, $i = \pm, 3$, generate a left-moving $SU(2)$ current algebra, the R subalgebra of the left-moving $\mathcal{N} = 4$ worldsheet superconformal algebra. The two extra odd currents $\hat{G}^\pm_{\rm cig}$ are generated in the OPE of $G^\pm_{\rm cig}$ with $J^i_{\rm cig}$. Similarly for the right-movers. In the full cigar background the wordsheet superconformal currents have more complicated expressions but the theory still has exact $(2,2)$ susy, enhanced to $(4,4)$ for $k = 2$.

In the free linear dilaton theory, $i\partial\theta$ and $i\partial H$ defined in (L.5) are separately holomorphic, but only their linear combination $J_{\rm cig}$ is holomorphic in the full cigar background. This reflects the non-conservation of winding around the cigar (strings can unwrap at the tip). Momentum P^θ around the cigar is still conserved, and there is a corresponding Noether current with both holomorphic and anti-holomorphic components, which asymptotically takes the form $\frac{1}{Q}(i\partial\theta\,, i\bar{\partial}\theta)$. For $k=2$, the field θ is asymptotically at the free fermion radius. Thus in the linear dilaton theory the left-moving susy $U(1)$ generated by $(i\partial\theta\,,\psi_\theta)$ is enhanced to a left-moving $SU(2)_2$ current algebra, which can be represented by three free fermions ψ_i, with $\psi_3 \equiv \psi_\theta$ and $\psi_\pm \equiv e^{\pm i\theta}$. To avoid confusions with other $SU(2)$ symmetries will refer to this algebra as $SU(2)_{\psi_i}$. Similarly in the right-moving sector we have the analogous $SU(2)_{\tilde{\psi}_i}$. In the full cigar background the $SU(2)_{\psi_i}$ and $SU(2)_{\tilde{\psi}_i}$ current algebras are *not* symmetries, and only a *global* diagonal $SU(2)$ survives, whose Cartan generator is the momentum P^θ. This is interpreted as the $SU(2)_R$ *spacetime* R-symmetry.

L.2 Cigar Vertex Operators

To characterize the primary vertex operators of the cigar it is sufficient to give their asymptotic form in the linear-dilaton region. While the exact expressions are more complicated, their quantum numbers (including conformal dimensions) remain the same and can thus be evaluated in the asymptotic region. Splitting the vertex operators in left-moving and right-moving parts, we have the asymptotic left-moving expressions

$$\begin{aligned} V^{NS}_{j,m} &= e^{iQm\theta} e^{Qj\rho} \\ V^{R}_{j,m} &= e^{\pm\frac{i}{2}\phi} e^{iQm\theta} e^{Qj\rho} \end{aligned} \tag{L.7}$$

and the asymptotic anti-holomorphic expressions

$$\begin{aligned} \tilde{V}^{NS}_{j,\tilde{m}} &= e^{-iQ\tilde{m}\bar{\theta}} e^{Qj\bar{\rho}} \\ \tilde{V}^{R}_{j,\tilde{m}} &= e^{\pm\frac{i}{2}\tilde{\phi}} e^{-iQ\tilde{m}\bar{\theta}} e^{Qj\bar{\rho}}\,. \end{aligned} \tag{L.8}$$

Left-moving and right-moving terms can be glued together provided they have the same value of the quantum number j. We will sometimes re-express j in terms of p, the momentum in the radial direction, as

$$j = -\frac{Q}{2} + ip\,. \tag{L.9}$$

The quantum numbers m and $\tilde{m}$ are related to the integer winding w and the integer momentum n in the angular direction of the cylinder as

$$m = \frac{1}{2}(n + wk) \qquad \tilde{m} = -\frac{1}{2}(n - wk)\,. \tag{L.10}$$

Recall however that winding is not a conserved quantum number in the cigar background. Conformal dimensions of the primary operators (L.7,L.8) are

$$\begin{aligned}
\Delta^{NS}_{j,m} &= \frac{m^2 - j(j+1)}{k} \\
\bar{\Delta}^{NS}_{j,\tilde{m}} &= \frac{\tilde{m}^2 - j(j+1)}{k} \\
\Delta^{R\pm}_{j,m} &= \frac{1}{8} + \frac{(m \pm \frac{1}{2})^2 - j(j+1)}{k} \\
\bar{\Delta}^{R\pm}_{j,\tilde{m}} &= \frac{1}{8} + \frac{(\tilde{m} \mp \frac{1}{2})^2 - j(j+1)}{k}
\end{aligned}$$

.

L.3 Spacetime Supersymmetry

From now on we restrict to the case of interest, $d = 6$. The RNS vertex operators for $\mathbb{R}^{5,1}$ are familiar. To describe the Ramond sector, we bosonize the fermions in the usual fashion,

$$\begin{aligned}
\pm\psi_0 + \psi_1 &= e^{\pm\phi_0} \\
\psi_2 \pm i\psi_3 &= e^{\pm i\phi_1} \\
\psi_4 \pm i\psi_5 &= e^{\pm i\phi_2}
\end{aligned}$$

Spinors of $\mathbb{R}^{5,1}$ are then written

$$V_\alpha = e^{\frac{1}{2}(\epsilon_0\phi_0 + i\epsilon_1\phi_1 + i\epsilon_2\phi_2)} \tag{L.11}$$

with $\epsilon_a = \pm 1$. With these notations at hand, the BRST invariant vertex operators for the spacetime supercharges for the IIB theory read

$$\begin{aligned} S_\alpha &= e^{-\varphi/2} e^{+\frac{i}{2}\phi} V_\alpha^+ \qquad \bar{S}_\alpha = e^{-\varphi/2} e^{-\frac{i}{2}\phi} V_\alpha^+ \\ \tilde{S}_\alpha &= e^{-\tilde{\varphi}/2} e^{+\frac{i}{2}\tilde{\phi}} \tilde{V}_\alpha^+ \qquad \bar{\tilde{S}}_\alpha = e^{-\tilde{\varphi}/2} e^{-\frac{i}{2}\tilde{\phi}} \tilde{V}_\alpha^+ \end{aligned}$$

where φ is the usual chiral boson arising in the bosonization of the $\beta\gamma$ system. We use a bar to denote conjugation, and a tilde to distinguish the right-movers. By V_α^+ we mean the positive chirality spinor, *i.e.* we impose $\epsilon_0\epsilon_1\epsilon_2 = 1$. Choosing the same chirality in the left and right-moving sectors is the statement of the type IIB GSO projection. The supercharges obey the supersymmetry algebra

$$\{S_\alpha\,, \bar{S}_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu \qquad \{\tilde{S}_\alpha\,, \bar{\tilde{S}}_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu\,, \tag{L.12}$$

where P_μ is the momentum in $\mathbb{R}^{5,1}$. Thus the theory has $(2,0)$ supersymmetry in the six Minkowski directions. Note that

$$[P^\theta, S_\alpha\,(\tilde{S}_\alpha)] = \frac{1}{2} S_\alpha\,(\tilde{S}_\alpha)\,, \quad [P^\theta, \tilde{S}_\alpha\,(\bar{\tilde{S}}_\alpha)] = -\frac{1}{2} S_\alpha\,(\bar{\tilde{S}}_\alpha)\,, \tag{L.13}$$

confirming the interpretation of P^θ as a spacetime R-symmetry.

Physical vertex operators are constrained to be local with the spacetime supercharges. Locality implies the GSO condition

$$\begin{aligned} m + F_L &\in 2\mathbb{Z} + 1 \qquad (NS) \\ m + F_L &\in 2\mathbb{Z} \qquad\quad\;\; (R) \end{aligned}$$

where F_L is the left-moving worldsheet fermion number. The analogous condition holds for the right-movers. In the asymptotic region we may fermionize the field θ into $\psi^\pm$. Then the quantum number m, instead of denoting left-moving momentum in the θ direction, gets re-interpreted as $\psi^\pm$ fermion number. Denoting by $F_L' = F_L + m$ the new total left-moving fermion number, the GSO projection becomes simply

$$\begin{aligned} F'_L &\in 2\mathbb{Z}+1 \qquad (NS) \\ F'_L &\in 2\mathbb{Z} \qquad\qquad (R) \end{aligned}$$

and analogously for the right-movers.

L.4 Spectrum: generalities

The physical spectrum of the theory comprises:

(i) A discrete set of truly normalizable states, localized at the tip of the cigar.
$(j < -Q/2)$

(ii) A continuum of delta-function normalizable states, corresponding to incoming and outgoing waves in the ρ direction.
$(j = -Q/2 + i\mathbb{R},\ i.e.\ p \in \mathbb{R})$

(iii) Non-normalizable vertex operators, supported in the asymptotic large ρ region.
$(j > -Q/2)$

States of type (i) live in $\mathbb{R}^{5,1}$ at $\rho \sim 0$ and they fill in a massless tensor multiplet of the 6d $(2,0)$ supersymmetry. More precisely they are:

NSNS: four scalars, in the $\mathbf{3} + \mathbf{1}$ of $SU(2)_R$;

RR: one scalar and one anti-selfdual antisymmetric tensor, both $SU(2)_R$ singlets;

RNS: one left-handed Weyl spinor, which can be thought of an $SU(2)_R$ doublet of left-handed Majorana-Weyl spinors;

NSR: same as RNS.

See [215] for a detailed analysis.

In the rest of this appendix we will focus on the states of type (ii). These are the states relevant for the determination of a spacetime "effective action" for the non-critical string. Recall that our philosophy is to use the $\mathbb{R}^{5,1}\times$ cigar background as an intermediate step towards the AdS background dual to $\mathcal{N} = 2$ SCQCD. Both backgrounds should arise as solutions of the same non-critical string field theory. We would like to use the cigar background, for which we have a solvable worldsheet CFT, to derive an "effective action" description. The "effective action" is expected to be background independent and should admit both the cigar background and the AdS background as different classical solutions. We will restrict to the lowest level in a "Kaluza-Klein expansion" on the cigar circle (to be defined more precisely below). The states will then propagate in seven dimensions, $\mathbb{R}^{5,1}$ times the radial direction ρ. Because of the linear dilaton, they obey massive field equations in 7d, but they are in another sense "massless" – they are closely related to the massless states of the critical IIB 10d theory and possess the gauge invariances expected for massless 7d fields. We should emphasize from the outset that the linear dilaton varies with a string-scale gradient, so there is no real separation of scales between the "massless" level that we are keeping and the higher levels. This is why we are using "effective action" in quotation marks. Nevertheless the distinction between the lowest level obeying massless gauge-invariances and the higher genuinely massive levels is a meaningful one, and we still expect such an "effective action" to contain useful information. Remarkably, we will see that it is a $7d$ gauged supergravity with non-standard gauging.

Finally we should mention the operators of type (iii). They have an interesting holographic interpretation as "off-shell" observables of little string theory, which "lives" on the $\mathbb{R}^{5,1}$ boundary at $\rho = \infty$. However we are not interested in the cigar background per se and we are after a different incarnation of holography, so it is not immediately clear what the significance of these operators is for our story. In analogy with $c = 1$ non-critical string, our non-critical superstring background is expected to possess a rich spectrum of "discrete states", with Liouville dressing of type (iii). A closely related phenomenon is the existence of a chiral ring, which has been demonstrated in [216] (see also [217]). This infinite tower of discrete states may be related to the exotic extra protected states of $\mathcal{N} = 2$ SCQCD.

L.5 Delta-function normalizable states: the lowest mass level

We are now going to exhibit in detail the physical states of type (ii) at the lowest mass level. We first organize the states according their symmetries in the asymptotic linear dilaton region, and later discuss the symmetry breaking induced by the cigar interaction. The asymptotic cylinder is at free-fermion radius, and we wish to work covariantly in the enhanced $SU(2)_{\psi_i} \times SU(2)_{\tilde{\psi}_i}$ symmetry.

After fermionizing θ into $\psi^{\pm}$, we have in total ten worldsheet fermions: ψ_μ, $\mu = 0, \dots 5$ associated with $\mathbb{R}^{5,1}$, ψ_ρ associated to the radial direction and ψ_i, $i = 3, \pm$ associated to the stringy circle. It is then clear from outset that the lowest mass level of our theory will be formally similar to the massless spectrum of 10d *critical* IIB string theory, but of course the states will propagate only in the seven dimensions $x_{\hat{\mu}} = (x_\mu, \rho)$.

L.5.1 NS sector

In the left-moving NS sector the lowest states are the three 7d scalars

$$V_i^{\mathrm{NS}} = \psi_i e^{-\varphi} e^{j\rho} e^{ik\cdot X}\,, \tag{L.14}$$

in a triplet of $SU(2)_{\psi_i}$, and the 7d vector

$$V_{\hat{\mu}}^{\mathrm{NS}} = \psi_{\hat{\mu}} e^{-\varphi} e^{j\rho} e^{ik\cdot X}\,, \tag{L.15}$$

where $\hat{\mu} = \mu\,,\rho$. The mass-shell condition $L_0 = 1$ gives, for both the scalar and the vector,

$$\frac{1}{2}k^2 - \frac{1}{2}j(j+1) = 0\,, \tag{L.16}$$

which using $j = -1/2 + ip$ we may write as

$$-k^2 - p^2 = k_0^2 - \vec{k}^2 - p^2 = \frac{1}{4}\,. \tag{L.17}$$

Because of the linear dilaton, the wave equations appear to be "massive" with $m^2 = \frac{1}{4}$. Introducing a polarization vector $e^{\hat{\mu}} = (e_\mu\, e_\rho)$, the superconformal invariance condition $G_{\frac{1}{2}} e^{\hat{\mu}} V_{\hat{\mu}}^{\mathrm{NS}} = 0$ gives a modified transversality equation for the vector[1]

$$k \cdot e - \sqrt{-1}(j+1)e_\rho = 0\,. \tag{L.18}$$

A short calculation shows that the polarization

$$e = k \quad \text{and} \quad e_\rho = -\sqrt{-1}\, j \tag{L.19}$$

corresponds to a null state. Thus despite the mass term in the wave equation, $V_{\hat{\mu}}^{\mathrm{NS}}$ the 7-2 = 5 physical degrees of freedom of a massless 7d vector.

The theory is super-Poincaré invariant in $\mathbb{R}^{5,1}$, and we may label the states in terms of 6d quantum numbers. In assigning 6d Lorentz quantum numbers, we may focus for convenience on the states with radial momentum $p = \frac{1}{2}$, which obey a massless 6d wave-equation (see L.17). We can then label them according to the 6d little group $SO(4) = SU(2) \times SU(2)$. It must kept in mind that this is just a notational device, since the states are really part of a 7d continuum with arbitrary real p. We use the notation $|j_1\,, j_2\rangle^{2I+1}$ for a state with spins (j_1, j_2) under the 6d little group, and in the $2I+1$-dimensional representation of $SU(2)_{\psi_i}$. All in all, in this 6d notation we may summarize the lowest NS states as

$$|\frac{1}{2}, \frac{1}{2}\rangle^1 \oplus |0,0\rangle^1 \oplus |0,0\rangle^3\,. \tag{L.20}$$

L.5.2 R sector

The construction of vertex operators in the Ramond sector proceeds just as in to the familiar critical (10d) case, except of course that momenta are only seven-dimensional,

$$V^R = e^{-\varphi/2} e^{\frac{i}{2}(\epsilon_0\phi_0 + \epsilon_1\phi_1 + \epsilon_2\phi_2\epsilon_\theta\theta + \epsilon_H H)} e^{j\rho} e^{ip\cdot X}\,, \quad \epsilon_0\epsilon_1\epsilon_2\epsilon_\theta\epsilon_H = 1\,, \tag{L.21}$$

which we may write as

$$\Psi_\alpha(p_\mu) e^{\pm\frac{i}{2}(\theta+H)}\, e^{j\rho}\,, \qquad \Psi_{\dot\alpha}(p_\mu) e^{\pm\frac{i}{2}(\theta-H)}\, e^{j\rho}\,. \tag{L.22}$$

[1] Apologies for the $\sqrt{-1}$, but here the symbol i would look confusing next to the momentum j.

Here Ψ_α and $\Psi_{\dot\alpha}$ are 6d pseudo-real (Majorana-Weyl) spinors, respectively left-handed and right-handed. Choosing the 7d momentum as $p = \frac{1}{2}$ the spinors obey a massless 6d wave equation, but as above we should keep in mind that they are really part of 7d continuum. For each chirality we have an $SU(2)$ doublet of 6d Majorana-Weyl spinors (equivalently, one complex Weyl spinor) so in "massless 6d notation" we write the spectrum as

$$|\frac{1}{2},0\rangle^2 \oplus |0,\frac{1}{2}\rangle^2\,. \tag{L.23}$$

In 7d the wave-equation looks "massive", but the counting of degrees of freedom is again the one for massless states.

L.5.3 Gluing

Table L.1–L.4 show the result of gluing the left- and right-moving sectors. In the first column of each table we list the $(m,\tilde m)$ quantum numbers, recall (L.10). In the second and third columns the Lorentz quantum numbers are specified in the the 6d "massless" notation, that is we label states by their spins (j_1,j_2) of the little group $SO(4) = SU(2)_1 \times SU(2)_2$. The superscripts $2I+1$ and $2\tilde I+1$ in the second column denote the dimensions of the representations under $SU(2)_{\psi_i}$ and $SU(2)_{\tilde\psi_i}$, respectively (the superscript is omitted for singlets). Finally the superscript $2R+1$ in the third column denotes the dimension of the $SU(2)_R$ representation, with $SU(2)_R$ defined as the *diagonal* combination of $SU(2)_{\psi_i}$ and $SU(2)_{\tilde\psi_i}$ which is preserved by the cigar interaction.

It is interesting to organize the spectrum according to massless supermultiplets of $6d$ supersymmetry (again, we may pretend that the states are massless in $6d$ by focussing on the value $p=\frac{1}{2}$ of the momentum along ρ). Massless supermultiplets are constructed by taking the direct product of a primary $|j_1,j_2\rangle^{2R+1}$ with a set $\mathcal{R}$ of raising operators. For $(2,0)$ susy in six dimensions,

$$\mathcal{R} = (1,0) + 2(\frac{1}{2},0)^2 + (0,0)^3 + 2(0,0) \tag{L.24}$$

For example the graviton multiplet is obtained acting with $\mathcal{R}$ on the primary $|0,1\rangle$, while the tensor multiplet is obtained starting with the primary $|0,0\rangle$.

$(\{m\},\{\tilde{m}\})$	$\vert j_1,j_2\rangle^{2I+1}\otimes\vert j_1,j_2\rangle^{2\tilde{I}+1}$	Decomposition: $\vert j_1,j_2\rangle^{2R+1}$	$6d$ Fields
$(\{0\},\{0\})$	$\vert\frac{1}{2},\frac{1}{2}\rangle\otimes\vert\frac{1}{2},\frac{1}{2}\rangle$	$\vert 1,1\rangle\oplus\vert 1,0\rangle\oplus\vert 0,1\rangle\oplus\vert 0,0\rangle$	$G_{\mu\nu},B_{\mu\nu},\phi$
	$\vert\frac{1}{2},\frac{1}{2}\rangle\otimes\vert 0,0\rangle$	$\vert\frac{1}{2},\frac{1}{2}\rangle$	V_μ
	$\vert 0,0\rangle\otimes\vert\frac{1}{2},\frac{1}{2}\rangle$	$\vert\frac{1}{2},\frac{1}{2}\rangle$	$\tilde{V}_\mu$
	$\vert 0,0\rangle\otimes\vert 0,0\rangle$	$\vert 0,0\rangle$	ρ
$(\{\pm1,0\},\{0\})$	$\vert 0,0\rangle^3\otimes\vert\frac{1}{2},\frac{1}{2}\rangle$	$\vert\frac{1}{2},\frac{1}{2}\rangle^3$	$\tilde{V}^3_\mu$
	$\vert 0,0\rangle^3\otimes\vert 0,0\rangle$	$\vert 0,0\rangle^3$	ρ^3
$(\{0\},\{\pm1,0\})$	$\vert\frac{1}{2},\frac{1}{2}\rangle\otimes\vert 0,0\rangle^3$	$\vert\frac{1}{2},\frac{1}{2}\rangle^3$	V^3_μ
	$\vert 0,0\rangle\otimes\vert 0,0\rangle^3$	$\vert 0,0\rangle^3$	$\tilde{\rho}^3$
$(\{\pm1,0\},\{\pm1,0\})$	$\vert 0,0\rangle^3\otimes\vert 0,0\rangle^3$	$\vert 0,0\rangle^5\oplus\vert 0,0\rangle^3\oplus\vert 0,0\rangle$	T^5,T^3,T

Table L.1: Field Content in NSNS sector.

$(\{m\},\{\tilde{m}\})$	$\vert j_1,j_2\rangle^{2I+1}\otimes\vert j_1,j_2\rangle^{2\tilde{I}+1}$	Decomposition: $\vert j_1,j_2\rangle^{2R+1}$	$6d$ Fields
$(\{0\},\{0\})$	$\vert\frac{1}{2},0\rangle^2\otimes\vert\frac{1}{2},0\rangle^2$	$\vert 1,0\rangle^3\oplus\vert 1,0\rangle\oplus\vert 0,0\rangle^3\oplus\vert 0,0\rangle$	$A^{3+}_{\mu\nu},A^+_{\mu\nu},A^3,A$
$(\{\pm1\},\{0\})$	$\vert 0,\frac{1}{2}\rangle^2\otimes\vert\frac{1}{2},0\rangle^2$	$\vert\frac{1}{2},\frac{1}{2}\rangle^3\oplus\vert\frac{1}{2},\frac{1}{2}\rangle$	A^3_μ,A_μ
$(\{0\},\{\pm1\})$	$\vert\frac{1}{2},0\rangle^2\otimes\vert 0,\frac{1}{2}\rangle^2$	$\vert\frac{1}{2},\frac{1}{2}\rangle^3\oplus\vert\frac{1}{2},\frac{1}{2}\rangle$	$\tilde{A}^3_\mu,\tilde{A}_\mu$
$(\{\pm1\},\{\pm1\})$	$\vert 0,\frac{1}{2}\rangle^2\otimes\vert 0,\frac{1}{2}\rangle^2$	$\vert 0,1\rangle^3\oplus\vert 0,1\rangle\oplus\vert 0,0\rangle^3\oplus\vert 0,0\rangle$	$A^{3-}_{\mu\nu},A^-_{\mu\nu},A'^3,A'$

Table L.2: Field Content in RR sector

$(\{m\},\{\tilde{m}\})$	$\vert j_1,j_2\rangle^{2I+1}\otimes\vert j_1,j_2\rangle^{2\tilde{I}+1}$	Decomposition: $\vert j_1,j_2\rangle^{2R+1}$	$6d$ Fields
$(\{0\},\{0\})$	$\vert\frac{1}{2},\frac{1}{2}\rangle\otimes\vert\frac{1}{2},0\rangle^2$	$\vert 1,\frac{1}{2}\rangle^2\oplus\vert 0,\frac{1}{2}\rangle^2$	$\Psi^2_{\mu\dot{\alpha}},\Psi^2_{\dot{\alpha}}$
	$\vert 0,0\rangle\otimes\vert\frac{1}{2},0\rangle^2$	$\vert\frac{1}{2},0\rangle^2$	Ψ^2_α
$(\{\pm1,0\},\{0\})$	$\vert 0,0\rangle^3\otimes\vert\frac{1}{2},0\rangle^2$	$\vert\frac{1}{2},0\rangle^4\oplus\vert\frac{1}{2},0\rangle^2$	$\Psi^4_\alpha,\Psi^2_\alpha$
$(\{0\},\{\pm1\})$	$\vert\frac{1}{2},\frac{1}{2}\rangle\otimes\vert 0,\frac{1}{2}\rangle^2$	$\vert\frac{1}{2},1\rangle^2\oplus\vert\frac{1}{2},0\rangle^2$	$\Psi^2_{\mu\alpha},\Psi^2_\alpha$
	$\vert 0,0\rangle\otimes\vert 0,\frac{1}{2}\rangle^2$	$\vert 0,\frac{1}{2}\rangle^2$	$\Psi^2_{\dot{\alpha}}$
$(\{\pm1,0\},\{\pm1\})$	$\vert 0,0\rangle^3\otimes\vert 0,\frac{1}{2}\rangle^2$	$\vert 0,\frac{1}{2}\rangle^4\oplus\vert 0,\frac{1}{2}\rangle^2$	$\Psi^4_{\dot{\alpha}},\Psi^2_{\dot{\alpha}}$

Table L.3: Field Content in NSR sector

$(\{m\},\{\tilde{m}\})$	$\vert j_1,j_2\rangle^{2I+1}\otimes\vert j_1,j_2\rangle^{2\tilde{I}+1}$	Decomposition: $\vert j_1,j_2\rangle^{2R+1}$	$6d$ Fields
$(\{0\},\{0\})$	$\vert\frac{1}{2},0\rangle^2\otimes\vert\frac{1}{2},\frac{1}{2}\rangle$	$\vert 1,\frac{1}{2}\rangle^2\oplus\vert 0,\frac{1}{2}\rangle^2$	$\Psi^2_{\mu\dot{\alpha}},\Psi^2_{\dot{\alpha}}$
	$\vert\frac{1}{2},0\rangle^2\otimes\vert 0,0\rangle$	$\vert\frac{1}{2},0\rangle^2$	Ψ^2_{α}
$(\{\pm1\},\{0\})$	$\vert 0,\frac{1}{2}\rangle^2\otimes\vert\frac{1}{2},\frac{1}{2}\rangle$	$\vert\frac{1}{2},1\rangle^2\oplus\vert\frac{1}{2},0\rangle^2$	$\Psi^2_{\mu\alpha},\Psi^2_{\alpha}$
$(\{0\},\{\pm1,0\})$	$\vert 0,\frac{1}{2}\rangle^2\otimes\vert 0,0\rangle$	$\vert 0,\frac{1}{2}\rangle^2$	$\Psi^2_{\dot{\alpha}}$
	$\vert\frac{1}{2},0\rangle^2\otimes\vert 0,0\rangle^3$	$\vert\frac{1}{2},0\rangle^4\oplus\vert\frac{1}{2},0\rangle^2$	$\Psi^4_{\alpha},\Psi^2_{\alpha}$
$(\{\pm1\},\{\pm1,0\})$	$\vert 0,\frac{1}{2}\rangle^2\otimes\vert 0,0\rangle^3$	$\vert 0,\frac{1}{2}\rangle^4\oplus\vert 0,\frac{1}{2}\rangle^2$	$\Psi^4_{\dot{\alpha}},\Psi^2_{\dot{\alpha}}$

Table L.4: Field Content in RNS sector

The complete field content of (the lowest level of) the cigar theory is obtained by action of $\mathcal{R}$ on the set of primaries,

$$|0,1\rangle+2|0,\frac{1}{2}\rangle^2+|0,0\rangle^3+2|0,0\rangle \tag{L.25}$$

Comparison with (L.24) suggests us that there are two other hidden supercharges at work, of opposite chirality, namely $(0,2)$, which relate the primaries of all the $(2,0)$ supermultiplets. In other words, we might conclude that we have obtained the maximally supersymmetric non-chiral $(2,2)$ supergravity in six dimensions. This is correct as the counting of states with $7d$ momentum $p=\frac{1}{2}$ goes, but the right-handed supersymmetries are broken by interactions. Nevertheless this is a useful hint: we should regard the effective theory for the lowest level as a spontaneously broken version of a maximally supersymmetric theory. And since the $7d$ momentum can be arbitrary, the candidate theory before symmetry breaking is maximally supersymmetry *seven*-dimensional supergravity.

L.6 Maximal 7d Supergravity with $SO(4)$ Gauging

To pursue this hint, in Table L.5 we have organized the lowest level of the linear-dilaton theory (before turning on the cigar interaction) according to $7d$ quantum numbers. The little group in $7d$ is $SO(5)\cong USp(4)$ and we label $USp(4)$ representations by their dimension. In the linear dilaton theory the full

Sector	$\vert USp(4)\rangle^{2I+1} \otimes \vert USp(4)\rangle^{2\tilde{I}+1}$	Decomposition: $\vert USp(4)\rangle^{(2I+1,2\tilde{I}+1)}$	$7d$ Fields
NSNS	$\vert 5\rangle \otimes \vert 5\rangle$	$\vert 14\rangle \oplus \vert 10\rangle \oplus \vert 1\rangle$	$G_{\hat{\mu}\hat{\nu}},\ B_{\hat{\mu}\hat{\nu}},\ \phi$
	$\vert 5\rangle \otimes \vert 1\rangle^3$	$\vert 5\rangle^{(3,1)\oplus(1,3)}$	$V_{\hat{\mu}}^{(3,1)\oplus(1,3)}$
	$\vert 1\rangle^3 \otimes \vert 5\rangle$		
	$\vert 1\rangle^3 \otimes \vert 1\rangle^3$	$\vert 1\rangle^{(3,3)}$	$T^{(3,3)}$
RR	$\vert 4\rangle^2 \otimes \vert 4\rangle^2$	$\vert 10\rangle^{(2,2)} \oplus \vert 5\rangle^{(2,2)} \oplus \vert 1\rangle^{(2,2)}$	$C_{\hat{\mu}\hat{\nu}}^{(2,2)},\ C_{\hat{\mu}}^{(2,2)},\ C^{(2,2)}$
RNS	$\vert 4\rangle^2 \otimes \vert 5\rangle$	$\vert 16\rangle^{(2,1)\oplus(1,2)} \oplus \vert 4\rangle^{(2,1)\oplus(1,2)}$	$\Psi_{\hat{\mu}}^{(2,1)\oplus(1,2)},\ \Psi^{(2,1)\oplus(1,2}$
NSR	$\vert 5\rangle \otimes \vert 4\rangle^2$		
	$\vert 4\rangle^2 \otimes \vert 1\rangle^3$	$\vert 4\rangle^{(2,3)\oplus(3,2)}$	$\Psi^{(2,3)\oplus(3,2)}$
	$\vert 1\rangle^3 \otimes \vert 4\rangle^2$		

Table L.5: Seven-dimensional labeling of the spectrum of the linear-dilaton theory

$SU(2)_{\psi_i} \otimes SU(2)_{\tilde{\psi}_i} \cong SO(4)$ is unbroken and we label states with superscripts $(2I+1, 2\tilde{I}+1)$ indicating the representation dimensions of the two $SU(2)$s. Remarkably, the resulting spectrum is precisely the field content of maximal $7d$ supergravity with $SO(4)$ gauging, a theory that has been fully constructed only quite recently [160, 161]. The massless vector $V_{\hat{\mu}}^{(3,1)+(1,3)}$ are the $SO(4)$ gauge fields. On the other hand the vectors $C_{\hat{\mu}}^{4}$ are eaten by the two forms $C_{\hat{\mu}\hat{\nu}}^{4}$, which become massive through a vectorial Higgs mechanism [160, 161].

Recall that the standard gauging of maximal $7d$ sugra is of the full $SO(5)$ R-symmetry – this is the famous supergravity that arises by consistent truncation of $11d$ supergravity compactified on S^4 and that admits a maximally supersymmetric AdS_7 vacuum. By contrast, the scalar potential of the $SO(4)$ theory does not allow for a stationary solution, but only for a domain wall solution [160, 161], that is, our linear-dilaton background. A closely related interpretation of the $SO(4)$ gauged supergravity was given in [218] (before its explicit construction!) as the effective $7d$ supergravity arising from a "warped compactification" of IIB supergravity on the near-horizon NS5 brane background $\mathbb{R}^{5,1}\times$ linear dilaton $\times S^3$.

The cigar background is obtained by further turning on a "tachyon" perturbation, a profile for the NSNS scalar fields $T^{(3,3)}$ that decays for large ρ and acts as a wall for $\rho \sim 0$. Note that the scalars are in the symmetric traceless tensor of $SO(4)$, and choosing a vev for them breaks $SO(4) \to SO(3) \cong SU(2)_R$, the diagonal combination of $SU(2)_{\psi_i} \times SU(2)_{\tilde{\psi}_i}$, as expected. In the IIA set-up of colliding NS5 branes, this breaking corresponds to choosing an angular direction in the transverse S^3 to the coincident NS5 brane – the direction along which the branes are separated (we called it τ in Figure 6.4). Under the preserved diagonal $SU(2)_R$, the nine NSNS scalars $T^{(3,3)}$ decompose as $\mathbf{5} + \mathbf{3} + \mathbf{1}$. The $\mathbf{1}$ and the $\mathbf{3}$ are associated to moduli, corresponding respectively (in the T-dual picture) to the radial and angular separations of the two NS5 branes; together with an extra $SU(2)_R$-singlet scalar from the RR sector they comprise the five scalars of the $6d$ tensor multiplet localized at the tip of the cigar.

In the application of the $SO(4)$-gauged $7d$ supergravity to our problem of finding the dual $\mathcal{N} = 2$ SCQCD, we are not interested in turning on a background for the NSNS scalars, but rather for the RR fields corresponding to N_c D3 branes and N_f D5 branes. D3 branes are magnetically charged unde the RR one-form $C_\mu^{(2,2)}$ and D5 branes are magnetically charged under the RR zero-form $C^{(2,2)}$. As the superscripts indicate both of the RR one-form and zero-form transform as vectors of $SO(4)$. It is possible to choose a common direction in $SO(4)$ space for both forms, so that again we break $SO(4) \to SO(3) \cong SU(2)_R$. This is again consistent with the IIA Hanany-Witten picture. Separating the NS5 branes in breaks $SO(4)$ to $SO(3)$, and it is clear that both the compact and the non-compact $D4$-branes are extended in the same direction along which the NS5 branes are separated, so that their fluxes are oriented coherently in $SO(4)$ space. The surviving $SO(3) \cong SU(2)_R$ is interpreted as the $SU(2)_R$ R-symmetry of the $\mathcal{N} = 2$ gauge theory.

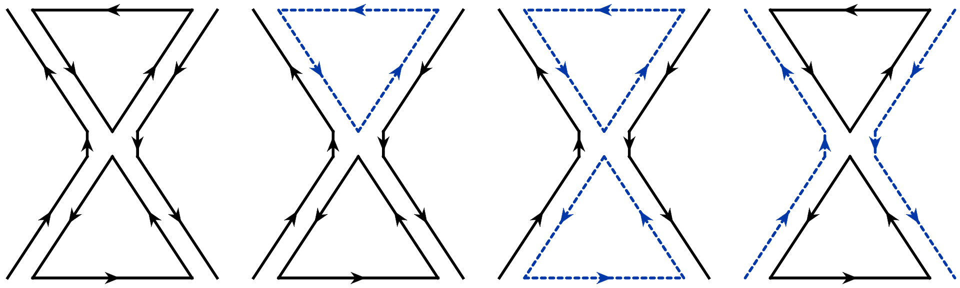
Appendix M

Simplified computation of the one-loop dilation operator

In this appendix we determine the one-loop spin-chain Hamiltonian by a simple shortcut. The interactions contributing to $H_{k,k+1}$ at one loop are listed schematically in figure 7.1. The first and second interactions (self-energy and gluon exchange) in figure 7.1 are proportional to the identity operator in $V_k \otimes V_{k+1}$, while the non-trivial tensorial structures are contributed only by the third diagram (quartic interaction). The idea is to evaluate explicitly the third diagram, and to fix the terms proportional to the identity by requiring that the anomalous dimensions of a few protected operators vanish.

M.1 SCQCD

Let us recall our notations. The indices $\mathfrak{p}, \mathfrak{q} = \pm$ label the $U(1)_r$ charges of ϕ and $\bar{\phi}$, in other terms we define $\phi^- \equiv \phi$, $\phi^+ \equiv \bar{\phi}$, and $g_{\mathfrak{pq}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The



(a) (b) (c) (d)

Figure M.1: The color/flavor structure of the quartic vertex. The solid black line represents the flow of the color index while the dotted blue line show the flow of the flavor index. Diagram (a) shows the ϕ^4 interaction vertex, whose contribution is proportional to N_c as compared to the tree level. In (b) the $Q^2\phi^2$ interaction vertex has a factor of N_f/N_c compared to (a) because of the presence of one flavor loop. The Q^4 vertex in (c) has an additional factor of $(N_f/N_c)^2$ compared to (a) due to the presence of two flavor loops. Diagram (d), however, does not carry any additional N_f/N_c factors.

elements of the Hamiltonian due to quartic vertices are:

$$\langle \phi_{\mathfrak{p}'}\phi_{\mathfrak{q}'}|H|\phi^{\mathfrak{p}}\phi^{\mathfrak{q}}\rangle_{\phi^4} = \delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathfrak{q}}_{\mathfrak{q}'} + g^{\mathfrak{p}\mathfrak{q}}g_{\mathfrak{p}'\mathfrak{q}'} - 2\delta^{\mathfrak{p}}_{\mathfrak{q}'}\delta^{\mathfrak{q}}_{\mathfrak{p}'} \tag{M.1}$$

$$\langle \phi_{\mathfrak{p}'}\phi_{\mathfrak{q}'}|H|Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}}\rangle_{Q^2\phi^2} = \sqrt{\frac{N_f}{N_c}}g_{\mathfrak{p}'\mathfrak{q}'}\delta^{\mathcal{J}}_{\mathcal{I}} \tag{M.2}$$

$$\langle \bar{Q}^{\mathcal{I}'}Q_{\mathcal{J}'}|H|Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}}\rangle_{Q^4} = \frac{N_f}{N_c}(2\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'} - \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'}) \tag{M.3}$$

$$\langle Q_{\mathcal{J}'}\bar{Q}^{\mathcal{I}'}|H|\bar{Q}^{\mathcal{J}}Q_{\mathcal{I}}\rangle_{Q^4} = 2\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'} - \delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'} \tag{M.4}$$

The factors of $\frac{N_f}{N_c}$ are explained in figure M.1. Figures M.1a,M.1b,M.1c,M.1d correspond to equations (M.1,M.2,M.3,M.4) respectively. This fixes the Hamiltonian up to the terms proportional to the identity,

$$H_{k,k+1} =$$

$$\begin{array}{c c}
 & \begin{array}{cccc} \phi^{\mathfrak{p}}\phi^{\mathfrak{q}} & Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}} & \bar{Q}^{\mathcal{K}}Q_{\mathcal{L}} & Q_{\mathcal{I}}\phi^{\mathfrak{p}} \end{array} \\
\begin{array}{c} \phi_{\mathfrak{p}'}\phi_{\mathfrak{q}'} \\ \bar{Q}^{\mathcal{I}'}Q_{\mathcal{J}'} \\ Q_{\mathcal{K}'}\bar{Q}^{\mathcal{L}'} \\ \bar{Q}^{\mathcal{I}'}\phi_{\mathfrak{p}'} \end{array} &
\begin{pmatrix}
\alpha\delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathfrak{q}}_{\mathfrak{q}'} + g^{\mathfrak{p}\mathfrak{q}}g_{\mathfrak{p}'\mathfrak{q}'} - 2\delta^{\mathfrak{p}}_{\mathfrak{q}'}\delta^{\mathfrak{q}}_{\mathfrak{p}'} & \sqrt{\frac{N_f}{N_c}}g_{\mathfrak{p}'\mathfrak{q}'}\delta^{\mathcal{J}}_{\mathcal{I}} & 0 & 0 \\
\sqrt{\frac{N_f}{N_c}}g^{\mathfrak{p}\mathfrak{q}}\delta^{\mathcal{I}'}_{\mathcal{J}'} & \beta\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'} - \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'}\frac{N_f}{N_c} & 0 & 0 \\
0 & 0 & \gamma\delta^{\mathcal{K}}_{\mathcal{K}'}\delta^{\mathcal{L}'}_{\mathcal{L}} + 2\delta^{\mathcal{K}}_{\mathcal{L}}\delta^{\mathcal{L}'}_{\mathcal{K}'} & 0 \\
0 & 0 & 0 & \eta\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathfrak{p}}_{\mathfrak{p}'}
\end{pmatrix}
\end{array}$$

We can now find the coefficients α, β, γ and η from knowledge of the protected spectrum. Vanishing of the anomalous dimension of $\mathrm{Tr}\phi^k$ gives $\alpha = 2$. Another protected multiplet is the multiplet containing the stress-energy tensor. Its superconformal primary, called $\mathrm{Tr}\, T$, has $R, r = 0$ and $\Delta = 2$. Hence, it is a linear combination of $\mathrm{Tr}[Q_{\mathcal{I}}\bar{Q}^{\mathcal{I}}]$ and $\mathrm{Tr}[\phi\bar{\phi}]$. The restriction of the Hamiltonian to this subspace is

$$H = \begin{array}{c c} & \begin{array}{cc} \mathrm{Tr}[\phi\bar{\phi}] & \mathrm{Tr}[\mathcal{M}_{\mathbf{1}}] \end{array} \\ \begin{array}{c} \mathrm{Tr}[\phi\bar{\phi}] \\ \mathrm{Tr}[\mathcal{M}_{\mathbf{1}}] \end{array} & \begin{pmatrix} 4 & 2\sqrt{\frac{2N_f}{N}} \\ 2\sqrt{\frac{2N_f}{N}} & (\beta+\gamma) - 2(\frac{N_f}{N_c} - 2) \end{pmatrix} \end{array} \tag{M.5}$$

This matrix must have a zero at the superconformal point $N_f = 2N_c$, yielding $\beta + \gamma = 4$. Finally, the fact that $\mathrm{Tr}\, T\phi$ is also a protected operator gives the relation $\beta + 2\eta = 8$. We started with four coefficients α, β, γ, η and imposed three relations. The undetermined degrees of freedom corresponds to the "gauge" freedom of adding to the nearest neighbor Hamiltonian terms that vanish upon evaluating the full H on a closed chain. We may solve the constraints by writing

$$\alpha = 2\,, \qquad \beta = 4 + \frac{1}{2}(1+\xi)\,, \qquad \gamma = -\frac{1}{2}(1+\xi)\,, \qquad \eta = \frac{1}{4}(7-\xi)\,, \tag{M.6}$$

where ξ is the arbitrary gauge parameter. The resulting Hamiltonian is in perfect agreement (for $N_f = 2N_c$) with the answer (7.4) obtained by the slightly lengthier route of explicit evaluating all relevant one-loop diagrams. All in all, this confirms our understanding of the protected spectrum.

M.2 Interpolating SCFT

We can repeat the same exercise for the interpolating SCFT. The quartic vertices give

$$\langle \phi_{\mathfrak{p}'}\phi_{\mathfrak{q}'}|\phi^{\mathfrak{p}}\phi^{\mathfrak{q}}\rangle_{\phi^4} = \delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathfrak{q}}_{\mathfrak{q}'} + g^{\mathfrak{p}\mathfrak{q}}g_{\mathfrak{p}'\mathfrak{q}'} - 2\delta^{\mathfrak{p}}_{\mathfrak{q}'}\delta^{\mathfrak{q}}_{\mathfrak{p}'} \tag{M.7}$$

$$\langle \check\phi_{\mathfrak{p}'}\check\phi_{\mathfrak{q}'}|\check\phi^{\mathfrak{p}}\check\phi^{\mathfrak{q}}\rangle_{\check\phi^4} = \kappa^2(\delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathfrak{q}}_{\mathfrak{q}'} + g^{\mathfrak{p}\mathfrak{q}}g_{\mathfrak{p}'\mathfrak{q}'} - 2\delta^{\mathfrak{p}}_{\mathfrak{q}'}\delta^{\mathfrak{q}}_{\mathfrak{p}'}) \tag{M.8}$$

$$\begin{aligned}\langle \bar Q^{\hat{\mathcal{L}}\mathcal{L}}Q_{\mathcal{K}\hat{\mathcal{K}}}|Q_{\mathcal{I}\hat{\mathcal{I}}}\bar Q^{\hat{\mathcal{J}}\mathcal{J}}\rangle_{Q^4} &= 2\,\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\delta^{\mathcal{J}}_{\mathcal{K}}\delta^{\hat{\mathcal{L}}}_{\hat{\mathcal{K}}}\delta^{\mathcal{L}}_{\mathcal{I}} - \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\delta^{\mathcal{L}}_{\mathcal{K}}\delta^{\hat{\mathcal{L}}}_{\hat{\mathcal{K}}}\\ &+ \kappa^2(2\,\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{K}}}\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\hat{\mathcal{L}}}_{\hat{\mathcal{I}}}\delta^{\mathcal{L}}_{\mathcal{K}} - \delta^{\mathcal{L}}_{\mathcal{I}}\delta^{\hat{\mathcal{L}}}_{\hat{\mathcal{I}}}\delta^{\mathcal{J}}_{\mathcal{K}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{K}}})\end{aligned} \tag{M.9}$$

$$\begin{aligned}\langle Q_{\mathcal{I}\hat{\mathcal{I}}}\bar Q^{\hat{\mathcal{J}}\mathcal{J}}|\bar Q^{\hat{\mathcal{L}}\mathcal{L}}Q_{\mathcal{K}\hat{\mathcal{K}}}\rangle_{Q^4} &= 2\,\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\delta^{\mathcal{J}}_{\mathcal{K}}\delta^{\hat{\mathcal{L}}}_{\hat{\mathcal{K}}}\delta^{\mathcal{L}}_{\mathcal{I}} - \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\delta^{\mathcal{L}}_{\mathcal{K}}\delta^{\hat{\mathcal{L}}}_{\hat{\mathcal{K}}}\\ &+ \kappa^2(2\,\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{K}}}\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\hat{\mathcal{L}}}_{\hat{\mathcal{I}}}\delta^{\mathcal{L}}_{\mathcal{K}} - \delta^{\mathcal{L}}_{\mathcal{I}}\delta^{\hat{\mathcal{L}}}_{\hat{\mathcal{I}}}\delta^{\mathcal{J}}_{\mathcal{K}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{K}}})\end{aligned} \tag{M.10}$$

$$\langle \phi_{\mathfrak{p}'}\phi_{\mathfrak{q}'}|Q_{\mathcal{I}\hat{\mathcal{I}}}\bar Q^{\hat{\mathcal{J}}\mathcal{J}}\rangle_{Q^2\phi^2} = g_{\mathfrak{p}'\mathfrak{q}'}\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}} \tag{M.11}$$

$$\langle \check\phi_{\mathfrak{p}'}\check\phi_{\mathfrak{q}'}|\bar Q^{\hat{\mathcal{J}}\mathcal{J}}Q_{\mathcal{I}\hat{\mathcal{I}}}\rangle_{Q^2\check\phi^2} = \kappa^2 g_{\mathfrak{p}'\mathfrak{q}'}\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}} \tag{M.12}$$

$$\langle \bar Q^{\hat{\mathcal{J}}\mathcal{J}}\check\phi_{\mathfrak{q}}|\phi^{\mathfrak{p}}Q_{\mathcal{I}\hat{\mathcal{I}}}\rangle_{\phi Q\check\phi\bar Q} = -2\kappa\delta^{\mathfrak{p}}_{\mathfrak{q}}\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}} \tag{M.13}$$

$$\langle \phi^{\mathfrak{p}}\bar Q^{\hat{\mathcal{J}}\mathcal{J}}|Q_{\mathcal{I}\hat{\mathcal{I}}}\check\phi_{\mathfrak{q}}\rangle_{\phi Q\check\phi\bar Q} = -2\kappa\delta^{\mathfrak{p}}_{\mathfrak{q}}\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}} \tag{M.14}$$

The first four elements can have additional identity pieces. They are easily determined by imposing the symmetry under $g \leftrightarrow \check g$, $Q \leftrightarrow \bar Q$ and $\phi \leftrightarrow \check\phi$ and by requiring the Hamiltonian to reduce to that of SCQCD in the limit $\kappa \to 0$. The one loop Hamiltonian (7.17) is precisely reproduced by this method.

Appendix N

The Hamiltonian for SCQCD in the Dimer Picture

In this appendix we rewrite the Hamiltonian for SCQCD as acting on adjoint fields and dimers $Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}}$, regarded as basic objects. We define the singlet combination $\mathcal{M} = \frac{1}{\sqrt{2}}\mathcal{M}_{\mathcal{I}}{}^{\mathcal{J}}\delta^{\mathcal{I}}_{\mathcal{J}}$ and the triplet $\mathcal{M}^i = \frac{1}{\sqrt{2}}\mathcal{M}_{\mathcal{I}}{}^{\mathcal{J}}(\sigma^i)^{\mathcal{I}}_{\mathcal{J}}$, where σ^i are three Pauli matrices. These can be rewritten in an $SO(4)$ notation as $\mathcal{M}^m = \frac{1}{\sqrt{2}}\mathcal{M}_{\mathcal{I}}{}^{\mathcal{J}}(\sigma^m)^{\mathcal{I}}_{\mathcal{J}}$, where $m = 0, \ldots, 3$ and $\sigma^0 \equiv \mathbb{I}_{2\times 2}$.

Consider the action of H on following sequence in the spin chain,

$$\begin{array}{ccccccc} \phi^{\mathfrak{p}} & & Q_{\mathcal{I}} & & \bar{Q}^{\mathcal{J}} & & \phi^{\mathfrak{q}} \\ & \frac{1}{2}(3+\frac{\xi}{2}) & & (5-\frac{\xi}{2})\mathbb{I}_{QQ} - 2\mathbb{K}_{QQ} & & \frac{1}{2}(3+\frac{\xi}{2}) & \\ & \downarrow & & \downarrow & & \downarrow & \\ \phi_{\mathfrak{p}'} & & \bar{Q}^{\mathcal{I}'} & & Q_{\mathcal{J}'} & & \phi_{\mathfrak{q}'} \end{array} \tag{N.1}$$

In the new picture, where $\mathcal{M}$ is regarded as a basic impurity, the middle term $(5-\frac{\xi}{2})\mathbb{I}_{QQ} - 2\mathbb{K}_{QQ}$ is the "self energy" of $\mathcal{M}$, and we split it evenly between the $\phi\mathcal{M}$ and $\mathcal{M}\phi$ matrix elements. So we write

$$\begin{aligned} \langle \ldots \phi_{\mathfrak{p}'}\bar{\mathcal{M}}^{\mathcal{I}'}{}_{\mathcal{J}'} \ldots |H| \ldots \phi^{\mathfrak{p}}\mathcal{M}_{\mathcal{I}}{}^{\mathcal{J}} \ldots\rangle &= [\frac{1}{2}(3+\frac{\xi}{2}) + \frac{1}{2}(5-\frac{\xi}{2})]\delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'} - \delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'} \\ &= (4\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'} - \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'})\delta^{\mathfrak{p}}_{\mathfrak{p}'} \\ \langle \ldots \phi_{\mathfrak{p}'}\bar{\mathcal{M}}^{m'} \ldots |H| \ldots \phi^{\mathfrak{p}}\mathcal{M}^{m} \ldots\rangle &= \delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{mm'}(4 - 2\delta^{m0})\,. \end{aligned}$$

Similarly, to find the action of H on two neighboring $\mathcal{M}$s, we consider the sequence

$$
\begin{array}{ccccccc}
Q_{\mathcal{I}} & & \bar{Q}^{\mathcal{J}} & & Q_{\mathcal{K}} & & \bar{Q}^{\mathcal{L}} \\
& (5-\frac{\xi}{2})\mathbb{I}_{QQ}-2\mathbb{K}_{QQ} & & (\frac{\xi}{2}-1)\mathbb{I}_{QQ}+2\mathbb{K}_{QQ} & & (5-\frac{\xi}{2})\mathbb{I}_{QQ}-2\mathbb{K}_{QQ} & \\
& \downarrow & & \downarrow & & \downarrow & \\
\bar{Q}^{\mathcal{I}'} & & Q_{\mathcal{J}'} & & \bar{Q}^{\mathcal{K}'} & & Q_{\mathcal{L}'}
\end{array}
\tag{N.2}
$$

This gives

$$
\begin{aligned}
\langle \ldots \bar{\mathcal{M}}^{m'} \bar{\mathcal{M}}^{n'} \ldots | H | \ldots \mathcal{M}^{m} \mathcal{M}^{n} \ldots \rangle \quad = \quad & \delta^{mm'} \delta^{nn'} (13 - 4\delta^{m0} - 4\delta^{n0}) \\
& + \delta^{mn} \delta^{m'n'} - \delta^{mn'} \delta^{nm'} + i\epsilon^{mnn'm'} \,.
\end{aligned}
$$

Appendix O

Algebraic constraints on the central charges

O.1 $\mathcal{N} = 4$ super Yang-Mills

Let us review the logic used in [23] to constrain the central elements $\mathcal{P}$ and $\mathcal{K}$. The action of $\mathcal{P}$ on a state with K $\mathcal{X}$-excitations with momenta $p_1, \ldots p_K$ is

$$\mathcal{P}|\mathcal{X}_1\mathcal{X}_2\ldots\mathcal{X}_K\rangle = \sum_{k=1}^{K} a_k b_k \prod_{l=k+1}^{K} e^{-ip_l}|\mathcal{X}_1\mathcal{X}_2\ldots\mathcal{X}_K\Phi^+\rangle \tag{O.1}$$

On a physical state like the one above, the central charge must vanish. Since in the $\mathcal{N} = 4$ case all the $\mathcal{X}$-excitations belong to the same (fundamental) representation of $SU(2|2)$, the central charge only depends upon the momentum and not on the type of excitation, and the only possibility is for the sum in (O.1) to telescope to zero on physical states,

$$a_i b_i = \alpha(e^{-ip_i} - 1) \equiv P \tag{O.2}$$

with α being an undetermined constant. Here we use the fact that the total momentum of a physical state is zero. A similar exercise for $\mathcal{K}$ gives

$$c_i d_i = \beta(e^{ip_i} - 1) \equiv K\,. \tag{O.3}$$

On a single-particle state,

$$\mathcal{P}|\mathcal{X}\rangle = \alpha(e^{ip}-1)|\mathcal{X}\Phi^+\rangle, \qquad \mathcal{K}|\mathcal{X}\rangle = \beta(e^{-ip}-1)|\mathcal{X}\Phi^-\rangle\,. \tag{O.4}$$

The hermiticity condition translates into $\alpha = \beta^*$. Finally

$$C = \frac{1}{2}\sqrt{1+4PK} = \frac{1}{2}\sqrt{1+16\alpha\beta\sin^2\frac{p}{2}}\,. \tag{O.5}$$

Comparing with the one loop dispersion relation one finds $\alpha\beta = \frac{g^2}{2} + O(g^4) \equiv \frac{\mathbf{g}^2}{2}$.

O.2 $\mathbb{Z}_2$ quiver

A physical state is constructed by having alternating Q and $\bar{Q}$ type impurities on a periodic spin chain. The central charge should vanish on such a state. To determine the central charges $\mathcal{P}$ and $\mathcal{K}$ as functions of magnon mometum, we follow same steps as before. The action of $\mathcal{P}$ and $\mathcal{K}$ is

$$\begin{aligned}
&\mathcal{P}|Q_1\bar{Q}_2\ldots Q_{K-1}\bar{Q}_K\rangle\\
&= (a_1b_1(e^{-ip_2}\ldots e^{-ip_K}) + \tilde{a}_2\tilde{b}_2(e^{-ip_3}\ldots e^{-ip_K}) + \ldots + \tilde{a}_K\tilde{b}_K)|Q_1\bar{Q}_2\ldots Q_{K-1}\bar{Q}_K\phi^+\rangle\\
&\mathcal{K}|Q_1\bar{Q}_2\ldots Q_{K-1}\bar{Q}_K\rangle\\
&= (c_1d_1(e^{ip_2}\ldots e^{ip_K}) + \tilde{c}_2\tilde{d}_2(e^{ip_3}\ldots e^{ip_K}) + \ldots + \tilde{c}_K\tilde{d}_K)|Q_1\bar{Q}_2\ldots Q_{K-1}\bar{Q}_K\phi^-\rangle.
\end{aligned}$$

As before, let us define $P_i \equiv a_ib_i, K_i \equiv c_id_i$ and $\tilde{P}_i \equiv \tilde{a}_i\tilde{b}_i, \tilde{K}_i \equiv \tilde{c}_i\tilde{d}_i$. Now we impose

1. Physical state condition:
   $\mathcal{P}$ and $\mathcal{K}$ should vanish when the total momentum of the state is zero.

2. BPS condition:
   A BPS state of the interpolating theory is obtained from a BPS state of the orbifold by the substitution (in the one-loop approximation) $\check{\phi} \to \kappa\check{\phi}$, $\kappa \equiv \check{g}/g$ (see the last paragraph of appendix B in [2]). At higher orders we may have a renormalized substitution $\check{\phi} \to \kappa'\check{\phi}$, $k' \equiv \check{\mathbf{g}}/\mathbf{g}$ with $\mathbf{g}(g,\check{g})$ and $\check{\mathbf{g}}(g,\check{g})$ renormalized couplings. This means $Q(\bar{Q})$ moving with momentum $i\ln\kappa'$ $(-i\ln\kappa')$ is chiral and we expect that P_iK_i $(\tilde{P}_i\tilde{K}_i)$ should vanish on that state.

3. Hermiticity:
   $K = P^*$ and $\tilde{K} = \tilde{P}^*$.

From these condition it follows that

$$P = \alpha(e^{-ip}\frac{1}{\sqrt{\kappa'}} - \sqrt{\kappa'}), \qquad K = \alpha^*(e^{ip}\frac{1}{\sqrt{\kappa'}} - \sqrt{\kappa'}),$$
$$\tilde{P} = \alpha(e^{-ip}\sqrt{\kappa'} - \frac{1}{\sqrt{\kappa'}}), \qquad \tilde{K} = \alpha^*(e^{ip}\sqrt{\kappa'} - \frac{1}{\sqrt{\kappa'}}).$$

($\{P, K\} \leftrightarrow \{\tilde{P}, \tilde{K}\}$ is of course also a solution since the conditions above make no intrinsic distinction between the Q and $\bar{Q}$ impurities.) We then have

$$C = \frac{1}{2}\sqrt{1+4PK} = \frac{1}{2}\sqrt{1+16|\alpha|^2\Big(\sin^2\frac{p}{2} + \frac{1}{4}(\sqrt{\kappa'} - \frac{1}{\sqrt{\kappa'}})\Big)^2} \quad (O.6)$$
$$\tilde{C} = \frac{1}{2}\sqrt{1+4\tilde{P}\tilde{K}} = \frac{1}{2}\sqrt{1+16|\alpha|^2\Big(\sin^2\frac{p}{2} + \frac{1}{4}(\sqrt{\kappa'} - \frac{1}{\sqrt{\kappa'}})\Big)^2}. \quad (O.7)$$

Comparing with the one-loop dispersion relation [4] one finds $|\alpha|^2 \equiv \frac{\mathbf{g}\check{\mathbf{g}}}{2} = \frac{g\check{g}}{2} + \ldots$. All in all,

$$C = \tilde{C} = \sqrt{1 + 2(\mathbf{g} - \check{\mathbf{g}})^2 + 8\mathbf{g}\check{\mathbf{g}}\sin^2\frac{p}{2}}\,. \quad (O.8)$$

Appendix P

Solving for the S-matrix

$SU(2|1)$ subsector: Determining A, K, G, H, L

We first consider the $SU(2_{\dot\alpha}|1_I)$ subsector, which is closed under scattering. Consider the scattering of two bosonic magnons Q^+ and $\bar{Q}^+$. Requiring invariance under the supercharge $\mathcal{Q}^{\dot\alpha}_+$ we find

$$\begin{aligned}
\mathcal{Q}^{\dot\alpha}_+ S_{12}|Q^+_1\bar{Q}^+_2\rangle &= \mathcal{Q}^{\dot\alpha}_+ A_{12}|Q^+_2\bar{Q}^+_1\rangle \\
&= A_{12}a_2|\psi^{\dot\alpha}_2\bar{Q}^+_1\rangle + A_{12}\tilde{a}_1|Q^+_2\tilde{\psi}^{\dot\alpha}_1\rangle \\
S_{12}\mathcal{Q}^{\dot\alpha}_+|Q^+_1\bar{Q}^+_2\rangle &= S_{12}(a_1|\psi^{\dot\alpha}_1\bar{Q}^+_2\rangle + \tilde{a}_2|Q^+_1\tilde{\psi}^{\dot\alpha}_2\rangle) \\
&= (a_1K_{12} + \tilde{a}_2G_{12})|\psi^{\dot\alpha}_2\bar{Q}^+_1\rangle + (a_1L_{12} + \tilde{a}_2H_{12})|Q^+_2\tilde{\psi}^{\dot\alpha}_1\rangle) \\
[\mathcal{Q}^{\dot\alpha}_+, S] = 0 &\Rightarrow \\
A_{12} &= \frac{a_1}{a_2}K_{12} + \frac{\tilde{a}_2}{a_2}G_{12} \\
A_{12} &= \frac{a_1}{\tilde{a}_1}L_{12} + \frac{\tilde{a}_2}{\tilde{a}_1}H_{12}.
\end{aligned}$$

More constraints are obtained by imposing invariance under conformal supersymmetries $\mathcal{S}$. In this subsector it is sufficient to focus on $\mathcal{S}^-_{\dot\alpha}$,

$$\begin{aligned}
\mathcal{S}^-_{\dot\alpha}S_{12}|Q_1^+\bar{Q}_2^+\rangle &= A_{12}(-c_2\epsilon_{\dot\alpha\dot\beta}|\psi_2^{\dot\beta}\check{\phi}^-\bar{Q}_1^+\rangle - \tilde{c}_1\epsilon_{\dot\alpha\dot\beta}|Q_2^+\tilde{\psi}_1^{\dot\beta}\phi^-\rangle) \\
&= A_{12}(-c_2\epsilon_{\dot\alpha\dot\beta}\frac{x_2^-}{x_2^+}|\phi^-\psi_2^{\dot\beta}\bar{Q}_1^+\rangle - \tilde{c}_1\epsilon_{\dot\alpha\dot\beta}\frac{x_2^-\tilde{x}_1^-}{x_2^+\tilde{x}_1^+}|\phi^-Q_2^+\tilde{\psi}_1^{\dot\beta}\rangle) \\
S_{12}\mathcal{S}^+_{\dot\alpha}|Q_1^+\bar{Q}_2^+\rangle &= S_{12}(-c_1\epsilon_{\dot\alpha\dot\beta}\frac{x_1^-}{x_1^+}|\phi^-\psi_1^{\dot\beta}\bar{Q}_2^+\rangle - \tilde{c}_2\epsilon_{\dot\alpha\dot\beta}\frac{\tilde{x}_2^-x_1^-}{\tilde{x}_2^+x_1^+}|\phi^-Q_1^+\tilde{\psi}_2^{\dot\beta}\rangle) \\
&= -\epsilon_{\dot\alpha\dot\beta}(c_1\frac{x_1^-}{x_1^+}K_{12} + \tilde{c}_2\frac{\tilde{x}_2^-x_1^-}{\tilde{x}_2^+x_1^+}G_{12})|\phi^-\psi_2^{\dot\beta}\bar{Q}_1^+\rangle \\
&- \epsilon_{\dot\alpha\dot\beta}(c_1\frac{x_1^-}{x_1^+}L_{12} + \tilde{c}_2\frac{\tilde{x}_2^-x_1^-}{\tilde{x}_2^+x_1^+}H_{12})|\phi^-Q_1^+\tilde{\psi}_2^{\dot\beta}\rangle\,.
\end{aligned}$$

This gives another pair of constraints on the coefficients,

$$A_{12} = \frac{c_1}{c_2}\frac{x_2^+}{x_2^-}\frac{x_1^-}{x_1^+}K_{12} + \frac{\tilde{c}_2}{c_2}\frac{x_2^+}{x_2^-}\frac{\tilde{x}_2^-x_1^-}{\tilde{x}_2^+x_1^+}G_{12} \tag{P.1}$$

$$A_{12} = \frac{c_1}{\tilde{c}_1}\frac{x_2^+}{x_2^-}\frac{\tilde{x}_1^+}{\tilde{x}_1^-}\frac{x_1^-}{x_1^+}L_{12} + \frac{\tilde{c}_2}{\tilde{c}_1}\frac{x_2^+}{x_2^-}\frac{\tilde{x}_1^+}{\tilde{x}_1^-}\frac{\tilde{x}_2^-x_1^-}{\tilde{x}_2^+x_1^+}H_{12} \tag{P.2}$$

Bosonic singlet: Determining B, C

To evaluate the B and C matrix elements, we have to study the scattering of two bosons of opposite spins. Requiring $[\mathcal{Q}_+^+, S] = 0$ is sufficient to determine them. From

$$\begin{aligned}
\mathcal{Q}_+^+S_{12}|Q_1^+\bar{Q}_2^-\rangle &= \mathcal{Q}_+^+[(\frac{1}{2}A_{12}+\frac{1}{2}B_{12})|Q_2^+\bar{Q}_1^-\rangle + (\frac{1}{2}A_{12}-\frac{1}{2}B_{12})|Q_2^-\bar{Q}_1^+\rangle \\
&+ \frac{1}{2}C_{12}(|\psi_2^+\tilde{\psi}_1^-\phi^-\rangle - |\psi_2^-\tilde{\psi}_1^+\phi^-\rangle)] \\
&= a_2(\frac{1}{2}A_{12}+\frac{1}{2}B_{12})|\psi_2^+\bar{Q}_1^-\rangle + \tilde{a}_1(\frac{1}{2}A_{12}-\frac{1}{2}B_{12})|Q_2^-\tilde{\psi}_1^+\rangle \\
&- \tilde{b}_1\frac{1}{2}C_{12}|\psi_2^+\bar{Q}_1^-\phi^+\phi^-\rangle - b_2\frac{1}{2}C_{12}|Q_2^-\check{\phi}^+\tilde{\psi}_1^+\phi^-\rangle \\
&= a_2(\frac{1}{2}A_{12}+\frac{1}{2}B_{12})|\psi_2^+\bar{Q}_1^-\rangle + \tilde{a}_1(\frac{1}{2}A_{12}-\frac{1}{2}B_{12})|Q_2^-\tilde{\psi}_1^+\rangle \\
&- \tilde{b}_1\frac{1}{2}C_{12}|\psi_2^+\bar{Q}_1^-\rangle - b_2\frac{1}{2}C_{12}\frac{\tilde{x}_1^-}{\tilde{x}_1^+}|Q_2^-\tilde{\psi}_1^+\rangle \\
S_{12}\mathcal{Q}_+^+|Q_1^+\bar{Q}_2^-\rangle &= S_{12}a_1|\psi_1^+\bar{Q}_2^-\rangle \\
&= a_1[K_{12}|\psi_2^+\bar{Q}_1^-\rangle + L_{12}|Q_2^-\tilde{\psi}_1^+\rangle]
\end{aligned}$$

we find

$$a_2\frac{A_{12}+B_{12}}{2}-\tilde{b}_1\frac{C_{12}}{2} = a_1K_{12} \tag{P.3}$$

$$\tilde{a}_1\frac{A_{12}-B_{12}}{2}-b_2\frac{\tilde{x}_1^-}{\tilde{x}_1^+}\frac{C_{12}}{2} = a_1L_{12}\,. \tag{P.4}$$

We now turn to the scattering of fermions.

$SU(1|2)$ Subsector: Determining D

As before, we first focus on the $SU(1_{\dot{\alpha}}|2_I)$ sector and consider the scattering of two fermions in the triplet of $SU(2)_{\dot{\alpha}}$. This sector will enable us to determine D. We look at the condition $[\mathcal{S}_+^I, S]=0$. From

$$\begin{aligned}
\mathcal{S}_+^I S_{12}|\psi_1^+\tilde{\psi}_2^+\rangle &= \mathcal{S}_+^I D_{12}|\psi_2^+\tilde{\psi}_1^+\rangle \\
&= D_{12}d_2|Q_2^I\tilde{\psi}_1^+\rangle - D_{12}\tilde{d}_1|\psi_2^+\bar{Q}_1^I\rangle \\
S_{12}\mathcal{S}_+^I|\psi_1^+\psi_2^+\rangle &= S_{12}(d_1|Q_1^I\tilde{\psi}_2^+\rangle - \tilde{d}_2|\psi_1^+\bar{Q}_2^I\rangle) \\
&= (d_1H_{12}-\tilde{d}_2L_{12})|Q_2^I\tilde{\psi}_1^+\rangle + (d_1G_{12}-\tilde{d}_2K_{12})|\psi_2^+\bar{Q}_1^I\rangle
\end{aligned}$$

we find

$$D_{12} = \frac{d_1}{d_2}H_{12}-\frac{\tilde{d}_2}{d_2}L_{12} \tag{P.5}$$

$$D_{12} = -\frac{d_1}{\tilde{d}_1}G_{12}+\frac{\tilde{d}_2}{\tilde{d}_1}K_{12}. \tag{P.6}$$

A consistent solution needs to satisfy both equations.

Fermionic singlet: Determining E, F

To determine the remaining coefficients E and F, we scatter two fermions of opposite spins. It is sufficient to require $[\mathcal{S}_+^+, S] = 0$. From

$$
\begin{aligned}
\mathcal{S}_+^+ S_{12}|\psi_1^+\tilde{\psi}_2^-\rangle &= \mathcal{S}_+^+[(\frac{1}{2}D_{12} + \frac{1}{2}E_{12})|\psi_2^+\tilde{\psi}_1^-\rangle + (\frac{1}{2}D_{12} - \frac{1}{2}E_{12})|\psi_2^-\tilde{\psi}_1^+\rangle \\
&+ \frac{1}{2}F_{12}(|Q_2^+\bar{Q}_1^-\phi^+\rangle - |Q_2^-\bar{Q}_1^+\phi^+\rangle)] \\
&= d_2(\frac{1}{2}D_{12} + \frac{1}{2}E_{12})|Q_2^+\tilde{\psi}_1^-\rangle - \tilde{d}_1(\frac{1}{2}D_{12} - \frac{1}{2}E_{12})|\psi_2^-\bar{Q}_1^+\rangle \\
&+ \frac{1}{2}F_{12}(\tilde{c}_1|Q_2^+\tilde{\psi}_1^-\phi^-\phi^+\rangle - c_2|\psi_2^-\check{\phi}^-\bar{Q}_1^+\phi^+\rangle) \\
&= \frac{1}{2}(d_2 D_{12} + d_2 E_{12} + \tilde{c}_1 F_{12})|Q_2^+\tilde{\psi}_1^-\rangle \\
&+ \frac{1}{2}(-\tilde{d}_1 D_{12} + \tilde{d}_1 E_{12} - c_2\frac{\tilde{x}_1^+}{\tilde{x}_1^-}F_{12})|\psi_2^-\bar{Q}_1^+\rangle \\
S_{12}\mathcal{S}_+^+|\psi_1^+\tilde{\psi}_2^-\rangle &= S_{12}d_1|Q_1^+\tilde{\psi}_2^-\rangle \\
&= d_1(G_{12}|\psi_2^-\bar{Q}_1^+\rangle + H_{12}|Q_2^+\tilde{\psi}_1^-\rangle)
\end{aligned}
$$

we find

$$
\begin{aligned}
d_2\frac{D_{12} + E_{12}}{2} + \tilde{c}_1\frac{F_{12}}{2} &= d_1 H_{12} && \text{(P.7)} \\
-\tilde{d}_1\frac{D_{12} - E_{12}}{2} - c_2\frac{\tilde{x}_1^+}{\tilde{x}_1^-}\frac{F_{12}}{2} &= d_1 G_{12}. && \text{(P.8)}
\end{aligned}
$$

In summary, a sufficient set of linear equations that determine all the coefficients is:

$$
\begin{aligned}
A_{12} &= \frac{a_1}{a_2}K_{12} + \frac{\tilde{a}_2}{a_2}G_{12} \\
A_{12} &= \frac{a_1}{\tilde{a}_1}L_{12} + \frac{\tilde{a}_2}{\tilde{a}_1}H_{12}. \\
A_{12} &= \frac{c_1}{c_2}\frac{x_2^+}{x_2^-}\frac{x_1^-}{x_1^+}K_{12} + \frac{\tilde{c}_2}{c_2}\frac{x_2^+}{x_2^-}\frac{\tilde{x}_2^- x_1^-}{\tilde{x}_2^+ x_1^+}G_{12} \\
A_{12} &= \frac{c_1}{\tilde{c}_1}\frac{x_2^+}{x_2^-}\frac{\tilde{x}_1^+}{\tilde{x}_1^-}\frac{x_1^-}{x_1^+}L_{12} + \frac{\tilde{c}_2}{\tilde{c}_1}\frac{x_2^+}{x_2^-}\frac{\tilde{x}_1^+}{\tilde{x}_1^-}\frac{\tilde{x}_2^- x_1^-}{\tilde{x}_2^+ x_1^+}H_{12} \\
a_1 K_{12} &= \frac{1}{2}a_2(A_{12}+B_{12}) - \frac{1}{2}\tilde{b}_1 C_{12} \\
a_1 L_{12} &= \frac{1}{2}\tilde{a}_1(A_{12}-B_{12}) - \frac{1}{2}b_2\frac{\tilde{x}_1^-}{\tilde{x}_1^+}C_{12} \\
D_{12} &= \frac{d_1}{d_2}H_{12} - \frac{\tilde{d}_2}{d_2}L_{12} \\
D_{12} &= -\frac{d_1}{\tilde{d}_1}G_{12} + \frac{\tilde{d}_2}{\tilde{d}_1}K_{12} \\
d_1 H_{12} &= \frac{1}{2}d_2(D_{12}+E_{12}) + \frac{1}{2}\tilde{c}_1 F_{12} \\
d_1 G_{12} &= -\frac{1}{2}\tilde{d}_1(D_{12}-E_{12}) - \frac{1}{2}c_2\frac{\tilde{x}_1^+}{\tilde{x}_1^-}F_{12}\,.
\end{aligned} \tag{P.9}
$$