Gauge Covariant Actions From Strings

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Abstract of the Dissertation

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We derive Yang-Mills vertex operators for (super)string theory whose BRST invariance requires only the free gauge-covariant field equation and no gauge condition. The gauge transformation and conformal transformation of these vertex operators are studied. Standard conformal field theory methods yield the three-point vertices directly in gauge-invariant form and S-matrices in terms of free field strengths for vector states, which allows arbitrary gauge choices. As examples we give three and four-vector (super)string tree amplitudes in this form, and find the field theory actions that give the first three orders in the slope. Also, on the String Field Theory side, we construct the Zinn-Justin-Batalin-Vilkovisky action for tachyons and gauge bosons from Witten's 3-string vertex of the bosonic open string without gauge fixing. Through canonical transformations, we find the off-shell, local, gauge-covariant action up to 3-point terms, satisfying the usual field theory gauge transformations. Perturbatively, it can be extended to higher-point terms. It also gives a new gauge condition in field theory which corresponds to the Feynman-Siegel gauge on the world-sheet. Finally, we combine two partons as a vector state on a random lattice, which is another approach for strings quantization. In the ladder approximation, we find propagators of such states (after tuning the mass to vanish). We also construct some diagrams which are very similar to 3-string

vertices in string field theory for the first oscillator mode. Attaching 3 such lattice states to these vertices, we get Yang-Mills and cubic interactions up to 3-point as from bosonic string (field) theory. This gives another view of a gauge field as a bound state in a theory whose only fundamental fields are scalars.

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Chapter 1

Introduction

Strings were first introduced for hadrons and identified as bound states of gluons and quarks. It is reinterpreted as fundamental strings to include all four fundamental interactions. These string models can be quantized covariantly or in the light-cone formalism. The light-cone formalism will be free of ghosts, though not manifestly covariant. While in the covariant formalism, ghosts are introduced for the gauge condition. To have right conformal weight, those vertex operators for external states are not gauge-covariant and correspond to states with specifical gauges (for instance, the condition $k \cdot \epsilon = 0$ is imposed on the open-string vertex operator $\epsilon \cdot \partial X e^{ik\cdot X}$ for massless vectors) [1]. So they can't give YM-gauge-covariant amplitudes while the corresponding effective theory is supposed to be expressed directly in terms of field strengths. On the other hand, nonlinear sigma models directly give gauge invariant results, but only order-by-order in α', and thus not the complete scattering amplitude [2].

Another interesting feature of four-point amplitudes with four external gauge fields in both $D = 10$ superstrings and maximally supersymmetric gauge theories in $D \leq 10$ (and by supersymmetry, arbitrary external massless states) is that the kinematic factors are identical at the tree and one-loop level [3]. Because lower-point amplitudes vanish in these theories, the one-loop four-point amplitude consists of one-particle-irreducible graphs in the field theory case, and is thus expressed directly in terms of field strengths as a contribution to the effective action in a background-field gauge calculation, as the "non-field-strength" contributions (from non-spin couplings) exactly cancel [4]. On the other hand, tree graphs are never expressed in terms of field strengths, so the identity of these kinematic factors seems somewhat mysterious.

In general field theories, the fact that S-matrices always have external propagators amputated means that the generating functional for the S-matrix (as opposed to that for Green functions) can always be expressed in terms of fields rather than sources [5]. (Consider, e.g., the external vector states for the tree amplitude of an electron in an external electromagnetic field.) In fact, the external line factors of Feynman graphs are (asymptotic) fields, and satisfy their (free) wave equations and (linear) gauge conditions. However, the gauge conditions imposed on external states generally do not match those applied to internal ones, neither for propagators in loops ("quantum gauge") nor in attached trees ("background gauge"): Usually the latter two gauges are some variation of the Fermi-Feynman gauge, while the external states satisfy a Landau gauge, further restricted to some type of unitary gauge (lightcone or Coulomb) by the residual gauge invariance. An exception is when external polarizations are summed over in a cross section, a procedure that is often more cumbersome because cross sections involve double sums (i.e., over both amplitudes and their complex conjugates).

The consistency of this procedure follows from the fact that in general *three* independent gauges can be chosen in the calculation of an S-matrix element from Feynman diagrams, corresponding to three steps in the procedure: (1) First calculate the effective action, using the background field method. The gauge for the "quantum" fields, which appear inside the loops, is fixed but the background fields are not gauge-fixed. The resulting effective action, which depends only on the background fields, is thus gauge invariant, not merely BRST invariant (and in fact is not a functional of the ghosts). (2) Calculate the generating functional for the S-matrix from "tree" graphs of the effective action, treating the full effective action as "classical", fixing the gauge for the (background) fields of the effective action. The result can always be expressed as a functional of linearized, on-shell field strengths only, in a Lorentz and gauge covariant way. (3) Calculate a specific S-matrix element, choosing a (linear) unitary gauge condition for the external gauge fields, or expressing the external field strengths directly in terms of polarizations.

It is the second step that will be the focus of this article. We will also examine its analog in string and superstring theory. In that case, with the usual first-quantized methods, the effective action does not appear, so the procedure reduces to two steps: (1) Calculate the S-matrix in terms of field strengths by using *gauge-covariant vertex operators* [6]. (2) Same as step 3 of the field theory case. The main difference in the string case is that gauge invariance at the next-to-last step is automatic (although there is still some work to rearrange the result in terms of field strengths). The advantages of having the third gauge invariance are similar to those of the other gauge invariances, since the result (a) can be applied to different gauges (e.g., lightcone or Coulomb), depending on the application, (b) is generally simpler, since various terms of various derivatives of gauge fields can be combined into field strengths, (c) is more unique, simplifying comparison of different contributions, and (d) is manifestly Lorentz covariant.

Some of these advantages can also be obtained by instead using a twistor formalism ("spinor helicity" [7], "spacecone" [8], etc.), but that approach does not generalize conveniently to higher dimensions. In fact, the two methods are somewhat related in $D = 4$. As an example, consider the "maximally helicity violating" tree amplitudes of Yang-Mills theory [9]: In the usual twistor notation, these are written as

$$\mathcal{A} = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}, \qquad \langle kl\rangle = \lambda_k^\alpha \lambda_{l\alpha}$$

for an n-point amplitude with i and j labeling the lines whose helicites differ from the rest. The twistors themselves are "square roots" of the momenta,

$$p_\alpha{}^{\dot\alpha} = \lambda_\alpha \bar\lambda^{\dot\alpha}$$

so no residue of gauge invariance is visible, but manifestation of Lorentz invariance is possible because in $D = 4$ the little group is just U(1), as represented by helicity. On the other hand, a twistor can also be interpreted as the square root of (the selfdual f or anti-selfdual $\bar f$ part of) an antisymmetric tensor: In an appropriate normalization for external lines,

$$f_\alpha{}^\beta = \lambda_\alpha \lambda^\beta, \qquad \bar f_{\dot\alpha}{}^{\dot\beta} = \bar\lambda_{\dot\alpha}\bar\lambda^{\dot\beta}$$

as follows from Maxwell's equations. Thus the result can easily be expressed in terms of field strengths and the usual (helicity-independent) momentum invariants by completing the denominator of the amplitude to the square of its absolute value (thus making the usual pole structure obvious): In 2×2 matrix notation,

$$\mathcal{A} = \frac{tr(f_i f_j) tr(p_j \bar f_{j+1} ... \bar f_n \bar f_1 ... \bar f_{i-1} p_i^*) tr(p_i \bar f_{i+1} ... \bar f_{j-1} p_j^*)}{p_1 \cdot p_2 ... p_n \cdot p_1}$$

In string theory, the gauge-boson vertex operator $A(X) \cdot \partial X$, expanded in plane waves as $A(X) = \epsilon e^{ik\cdot X}$, is not gauge covariant, and requires the gauge condition $\partial \cdot A = 0$ for worldsheet conformal invariance. To relax this limitation and allow arbitrary gauge choices, we use the BRST operator and the integrated vertex operator to introduce gauge-covariant vertex operator with momentum-dependent conformal weights. In following two chapters, we derived gauge-covariant vertex operators for vectors in bosonic string and superstring [6]. Then standard conformal field theory methods yield the three-point vertices directly in gauge-invariant form: The result was the the gauge-unfixed

F^2 Yang-Mills action (and in the bosonic string, also an F^3 term). Also, the conformal symmetry of the amplitude is studied to show that the result follows from any 3-string vertex for string field theory [10]. What's more, we will use these gauge-covariant vertex operators to compute the gauge-invariant tree amplitude between 4 gauge bosons [11]. In particular, to our knowledge a complete, explicit expression for this amplitude (i.e., not simply as a functional derivative of some generating functional) in bosonic string theory has not appeared previously in the literature. Then we will reproduce the same amplitudes at order 1, α', and α'^2 from the appropriate F^2, F^3 (for the bosonic string), and F^4 terms in a field theory action.

Generalizing above formalism to off-shell, ie, String Field Theory (SFT) is not an easy work. A complete description of interacting strings and string fields was presented in the light-cone gauge [12] and generalized to the super case [13]. A covariant, gauge-invariant formulation of the bosonic open string field theory was given by Witten [14], based on the relation found between gauge transformations of the fields and first-quantized Becchi-Rouet-Stora-Tyutin transformations in the free action [15]. It was made more concrete by several groups: The explicit operator construction of the string field interaction was presented [16]; string field theory geometry was formulated by writing each term in the action as an expectation value in the 2D conformal field theory on the world surface [17]; the tensor constructions were analyzed from first principles [18]; etc. Because string field action is gauge invariant under the gauge transformations:

$$\delta\Psi = Q\Lambda + \Psi \star \lambda - \lambda \star \Psi \tag{1.1}$$

it is helpful to fix the gauge before computation the action. Usually, the external states are introduced by vertex operators with a gauge condition $b_0 = 0$, which is called the Feynman-Siegel gauge. Then the antifields in the string field expansion, which are associated with states that have a ghost zero-mode c_0, are taken to vanish. As we will show in chapter 4, the extra term in the gauge-covariant vertex operator will be canceled by Nakanishi-Lautrup field. So the action from the viewpoint of quantum field theory is gauge fixed, while it is not clear what kind of gauge condition is applied. We can only guess the action for these states (for example, the origin of the ϕA^2 term is not clear for the lack of gauge covariance) but are not able to write it down gauge invariantly. The simplest way to accomplish this is to find the Zinn-Justin-Batalin-Vilkovisky action [19, 20] with all antifields. The ZJBV formalism was first developed to deal with the renormalization of gauge theories, but follows naturally from any field theory action whose kinetic operator is expressed as the first-quantized BRST operator [21]. It allows the handling of very general gauge theories, including those with open or reducible symmetry algebras. The ZJBV action includes both the usual gauge-invariant action and the definition of the gauge (BRST) transformations. Here we will start from this ZJBV action for SFT and, through some canonical transformations (including field redefinitions and gauge transformations of both fields and antifields), get the explicit gauge-covariant action (and gauge transformations) for tachyons and massless vectors up to 3-point terms. We will show for the first time that it is just usual Yang-Mills coupled to scalars, plus F^3 and ϕF^2 interactions. These specific canonical transformations will tell us the gauge condition on the fields corresponding to Feynman-Siegel gauge on the world-sheet. Another advantage of this mechanism is that we pushed all nonlocal factors in 3-point interactions to higher-point interactions and make the 3-point interactions just the usual local YM form. But, as a price, there will be all possible higher-point interactions (nonrenormalizable in ordinary field theory).

Following this result, we are able to construct external states and vertices on the random lattice. It is known that, in nonrelativistic quantum mechanics, Regge behavior relates the angular momenta and energies of bound states [22]. In relativistic quantum field theory, the high-energy behavior of a scattering amplitude, $F(s,t) \sim \beta(s)t^{\alpha(s)}$ as $t \to \infty$ and $s < 0$, is also dominated by Regge poles, with trajectories $J = \alpha(s)$. Here the Bethe-Salpeter equation [23] takes the place of the Schrödinger equation, which can only be solved in certain approximations, such as the ladder approximation or a perturbative Feynman diagram analysis.

Experimental data confirms the existence of families of particles along trajectories $J = \alpha(s)$ which are *linear* as from the Veneziano model or string (field) theory. However, in many approximations of conventional field theory the trajectories rise for a while and then fall back towards negative values of J for increasing energy. Thus, only a few bound states are produced, as characteristic of a Higgs phase; instead, linearity and an infinite number of bound states are expected to arise as a consequence of confinement, perhaps due to some infrared catastrophe. However, such a catastrophe is absent in the usual calculations, which are always made for massive or off-shell states precisely in order to avoid infrared divergences.

Originally, strings were introduced for hadrons and later identified as bound states of "partons". Unfortunately, a suitable hadronic string theory serving that purpose hasn't been constructed. This led to the reinterpretation of the known strings as fundamental strings describing gluons and quarks, leptons, gravitons, etc. The target space is 26D for the bosonic theory and 10D for the super theory, which means compactification is necessary.

One nonperturbative approach to strings is quantization on a suitable random lattice representing the worldsheet [24]. It expresses the strings as bound states of underlying partons, and the lattices are identified with Feynman diagrams [25]. The two theories are "dual" to each other, and one is perturbative while the other is nonperturbative. The Feynman diagrams of the particles underlying this bosonic string were studied and linear Regge trajectories were reproduced in the ladder approximation [26]. This implies that the only fundamental fields are scalars and all others can be represented as composite fields. In chapter 5, the massless state are constructed as a bound state of partons, and two simple lattice interaction diagrams are introduced. In the ladder approximation, we find that such states have $1/p^2$ propagators. From the interaction diagrams, we found interactions of those bound states similar to the usual YM gauge field. The comparison of these 3-state vertices on the lattice with Witten's vertex on the continuous worldsheet shows all of them have the same symmetries, especially twist symmetry, which is absent in the CSV vertex. The twist symmetry restricted the gauge-fixed interaction to be proportional to the structure constants of the gauge group, or equivalently, the interaction term of the gauge condition must be proportional to the structure constants. That's the reason the Gervais-Neveu gauge can only be obtained from the CSV vertex. Anyway, we show here the possibility to bind the scalars on the lattice to get the massless vector state which behaves like the gauge field, i.e., the gauge field is no longer a fundamental particle but a composite state in the field theory. This also provided a new view of the 3-string coupling in Witten's bosonic open string field theory. Finally, chapter 6 gives some conclusions and discussions.

Chapter 2

Gauge-Covariant Vertex Operators in Bosonic Strings

In this chapter, a new gauge-covariant unintegrated vertex operator or massless vectors will be constructed for bosonic string. Then the gauge-invariant amplitude of 3 vectors will be calculated through this operator. Also, the gauge transformation and conformal transformation of the vertex and amplitude is studied. Moreover, the gauge-covariant S-matrices for 4 vectors are also computed which leading to new F^4 interactions in the effective action.

2.1 Bosonic vertex for massless vectors

There are two kinds of vertex operators In string theory – an integrated one $\oint W$ and an unintegrated one V. For a gauge vector

$$W = A(X) \cdot \partial X \tag{2.1}$$

or in Fourier expansion

$$W = \epsilon \cdot \partial X e^{ik\cdot X} \tag{2.2}$$

Then, using the BRST operator "Q", we can find the unintegrated operator V [27]:

$$[Q, \oint W\} = 0 \quad \Rightarrow \quad [Q, W\} = \partial V \quad \Rightarrow \quad [Q, V\} = 0 \tag{2.3}$$

Then, any amplitude can be computed from these two kinds of operators:

$$\mathcal{A} = \langle VVV \oint W \cdots \oint W \rangle \tag{2.4}$$

For the bosonic string, the BRST operator

$$Q = \oint \tfrac{1}{2\pi i} dz(-\tfrac{1}{4\alpha'} c\partial X \cdot \partial X + bc\partial c) \tag{2.5}$$

where b, c are world-sheet antighost and ghost. So, the commutator

$$[Q, \epsilon \cdot \partial X e^{ik\cdot X}] = \oint \tfrac{1}{2\pi i} dz'(-\tfrac{1}{4\alpha'} c\partial' X \cdot \partial' X)\ \epsilon \cdot \partial X e^{ik\cdot X} \tag{2.6}$$

For the open string, the propagator in the upper-half complex plane between X's is $-2\alpha'\eta^{\mu\nu} ln|z'-z|$, so the commutator above can be found through OPE (operator product expansion)

$$\begin{aligned}[Q, \epsilon \cdot \partial X e^{ik\cdot X}] &= \partial(c\epsilon \cdot \partial X)e^{ik\cdot X} + c(\epsilon \cdot \partial X)(ik \cdot \partial X)e^{ik\cdot X} \\ &\quad +\alpha'[(\epsilon \cdot \partial X)k^2 \partial c e^{ik\cdot X} - (ik \cdot \epsilon)\partial^2 c e^{ik\cdot X}]\end{aligned} \tag{2.7}$$

The first two terms come from the two ways to contract a single pair of $X^\mu(z')$ and $X^\nu(z)$, while the last two terms from the two ways to contract two pairs.

To find the new unintegrated vertex operator according to (2.3), this commutator has to be written as a total derivative. To accomplish it, we notice that external states should be on-shell and satisfied the *gauge-invariant* equation of motion of the free vector

$$\partial^\mu F_{\mu\nu} = 0 \qquad or \qquad k^2 \epsilon^\mu - k^\mu (k \cdot \epsilon) = 0 \tag{2.8}$$

Then the commutator in (2.6) can be interpreted as a total derivative:

$$[Q, \epsilon \cdot \partial X e^{ik\cdot X}] = \partial[c\epsilon \cdot \partial X e^{ik\cdot X} - i\alpha'(\partial c)(\epsilon \cdot k)e^{ik\cdot X}] \tag{2.9}$$

This is a BRST-invariant vertex operator for the gauge vector without gauge fixing:

$$V = c\epsilon \cdot \partial X e^{ik\cdot X} - i\alpha'(\partial c)(\epsilon \cdot k)e^{ik\cdot X} \tag{2.10}$$

$$or \quad V = cA \cdot \partial X - \alpha'(\partial c)(\partial \cdot A) \tag{2.11}$$

∂c is also the vertex operator for the Nakanishi-Lautrup field B [28]: In the Feynman-Siegel gauge $b_0 = 0$, $B = \partial \cdot A$, and the two ∂c terms cancel. We will study this more in this aspect later.

To see this new operator is gauge invariant, make gauge transformations of the vectors

$$\delta A^\mu = \partial^\mu \lambda \quad \Rightarrow \quad \delta V = c(\partial^\mu \lambda)(\partial X_\mu) - \alpha'(\partial c)\partial_\mu(\partial^\mu \lambda) \tag{2.12}$$

This gauge variation δV can also be written as the commutator between Q and $\lambda(X)$

$$\begin{aligned} [Q,\lambda] &= \oint \tfrac{1}{2\pi i} dz'[-\tfrac{1}{4\alpha'} c\partial' X_(z')\cdot\partial' X(z')]\lambda[X(z)] \\ &= c(\partial^\mu\lambda)(\partial X_\mu) - \alpha'(\partial c)\partial_\mu(\partial^\mu\lambda) \\ &= \delta V \end{aligned} \tag{2.13}$$

It is easy to see the integrated operator is gauge invariant because the integral of a total derivative vanishes:

$$\delta\oint W = \oint \partial X^\mu \delta A_\mu = \oint \partial X^\mu \partial_\mu \lambda = \oint \partial\lambda = 0 \tag{2.14}$$

Then matrix elements are also gauge invariant:

$$\delta_{\lambda_1}\mathcal{A}_n = \langle \delta V_1 V_2 V_3 \oint W \cdots \oint W\rangle = \langle [Q,\lambda_1] V_2 V_3 \oint W\cdots\oint W\rangle = 0 \tag{2.15}$$

where the vacuum is BRST invariant, and similarly for the gauge transformations of V_2 and V_3. From above, in principle, we are able to compute gauge-invariant n-point amplitudes for vectors in string theory which can be comparable to effective actions in field theory. In the following subsection, we will calculate the simplest case: 3-point amplitudes for bosonic string.

2.2 Bosonic three-point amplitudes

The amplitude between three vectors (V_1, V_2, V_3) is :

$$\begin{aligned} \mathcal{A}_3 &= -\tfrac{ig_{YM}}{2\alpha'}\langle V_1 V_2 V_3\rangle \\ &= -\tfrac{ig_{YM}}{2\alpha'}\langle \prod_{i=1}^{3}[\underbrace{c(y_i)\epsilon_i\cdot\partial X(y_i)e^{ik_i\cdot X(y_i)}}_{G(y_i)} - \underbrace{i\alpha'\partial c(y_i)(\epsilon_i\cdot k_i)e^{ik_i\cdot X(y_i)}}_{H(y_i)}]\rangle \end{aligned} \tag{2.16}$$

where the α' is the Regge slope and g_{YM} the inaction parameter for Yang-Mills field.

Considering the lowest order of α' first, only 2 terms in (2.16) contribute:

$$\langle\prod_{i=1}^{3} G(y_i)\rangle \quad and \quad \langle \prod_{i\neq j} G(y_i)H(y_j)\rangle \tag{2.17}$$

Using the correlation between ghosts:

$$\langle c(y_1)c(y_2)c(y_3)\rangle = y_{12}y_{13}y_{23}$$
$$\langle \partial_{y_1} c(y_1)c(y_2)c(y_3)\rangle = \partial_{y_1}(y_{12}y_{13}y_{23}), \cdots \tag{2.18}$$

and the propagator between $X^\mu(z')$ and $X^\nu(z)$, the contracting rules of OPE, and the *gauge-invariant* equation of motion of the free vector (2.8), the amplitude (2.16) in the lowest order of α' is:

$$\begin{aligned}\mathcal{A}_3^{(1)} = \ & ig_{YM}(2\pi)^D\delta^D(\Sigma_i k_i)[(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot k_{12}) + (\epsilon_2\cdot\epsilon_3)(\epsilon_1\cdot k_{23}) \\ & +(\epsilon_3\cdot\epsilon_1)(\epsilon_2\cdot k_{31})]\end{aligned} \tag{2.19}$$

where $k_{ij} = k_i - k_j$. (To the lowest order in α', the factor in $\mathcal{A}_3$

$$|y_{12}|^{2\alpha' k_1\cdot k_2}|y_{13}|^{2\alpha' k_1\cdot k_3}|y_{23}|^{2\alpha' k_2\cdot k_3} \to 1 \tag{2.20}$$

when $\alpha' \to 0$).The effective action corresponds to this amplitude is the Yang-mills interaction without gauge fixing:

$$\frac{1}{g_{YM}^2}\int d^{26}x[-\tfrac{1}{4}Tr(F^{\mu\nu}F_{\mu\nu})] \tag{2.21}$$

To the second order in α', the technology is the same but calculation is tedious. We notice the contribution from expanding of the factors

$$|y_{12}|^{2\alpha' k_1\cdot k_2}|y_{13}|^{2\alpha' k_1\cdot k_3}|y_{23}|^{2\alpha' k_2\cdot k_3} \tag{2.22}$$

are just zero because

$$k_1\cdot k_2 = \tfrac{1}{2}(k_3^2 - k_1^2 - k_2^2), \cdots \qquad and$$
$$k_1^2[(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot k_{12}) + (\epsilon_2\cdot\epsilon_3)(\epsilon_1\cdot k_{23}) + (\epsilon_3\cdot\epsilon_1)(\epsilon_2\cdot k_{31})] = 0, \cdots \tag{2.23}$$

by using only the *gauge-covariant* equation of motion (2.8) and momentum conservation $k_1 + k_2 + k_3 = 0$. Then the amplitude in the second order of α' is only contributed by

$$\langle G(y_1)G(y_2)G(y_3)\rangle, \quad \langle G(y_1)G(y_2)H(y_3)\rangle, \cdots, \quad and \langle G(y_1)H(y_2)H(y_3)\rangle, \cdots \tag{2.24}$$

and H^3 contribution vanishes. We won't give the detail procedure and the result is:

$$\mathcal{A}_3^{(2)} \ = \ 2i\alpha' g_{YM}(2\pi)^{26}\delta^{26}(\sum_i k_i)\times$$

$$[(\epsilon_1 \cdot k_2)(\epsilon_2 \cdot k_3)(\epsilon_3 \cdot k_1) - (\epsilon_1 \cdot k_3)(\epsilon_2 \cdot k_1)(\epsilon_3 \cdot k_2)] \tag{2.25}$$

which corresponds to the F^3 interaction of the YM field:

$$\tfrac{-2i\alpha'}{3g_{YM}^2} Tr(F_\mu{}^\nu F_\nu{}^\omega F_\omega{}^\mu) \tag{2.26}$$

With the equation of motion (2.8), all higher terms besides $\mathcal{A}_3^{(1)}$ and $\mathcal{A}_3^{(2)}$ vanish. So the effective action we get here is complete.

Another simple example we give here is the 3-point amplitude between 1 gauge boson and 2 tachyons. Using the vertex operator for tachyons : $V_t = ce^{ik\cdot X}$, to the lowest order in α',

$$-ig_{YM}\langle V(y_1)V_t(y_2)V_t(y_3)\rangle = -ig_{YM}\epsilon_1 \cdot (k_2 - k_3)(2\pi)^{26}\delta^{26}(\Sigma_i k_i) \tag{2.27}$$

with $g_0 = (2\alpha')^{1/2} g_{YM}$ is the coupling constant for tachyons. This corresponds to the effective action

$$\tfrac{1}{g_{YM}^2}[-\tfrac{1}{2}Tr(D_\mu\phi D^\mu\phi)] \quad with \quad D_\mu\phi = \partial\phi - i[A_\mu, \phi] \tag{2.28}$$

in quantum field theory.

Beside the gauge invariance, we can observe that the amplitude constructed by vertices as (2.10) is independent of the conformal map from the world sheet to the complex plane. It can can be verified by checking the conformal transformation of the vertex operator:

$$\delta V = \oint \lambda(z')T(z')V(z) \tag{2.29}$$

The world-sheet energy momentum tensor $T(z')$ includes two parts: the X contribution T^m and ghost contributions T^g

$$T(z') = T^m(z') + T^g(z') \tag{2.30}$$

where

$$\begin{aligned} T^m(z) &= -\tfrac{1}{4\alpha'}\partial X_\mu \partial X^\mu \\ T^g(z) &= (\partial b)c - 2\partial(bc) \end{aligned} \tag{2.31}$$

Using the equation of motion (2.8),

$$\delta V = \lambda\partial V + \alpha' k^2(\partial\lambda)V \tag{2.32}$$

So the conformal weight of operator (2.10) is $\alpha' k^2$ instead of zero:

$$V'(z') = \left(\frac{dz'}{dz}\right)^{-\alpha' k^2} V(z) \tag{2.33}$$

In string field theory, an arbitrary 3-point vertex can be defined by

$$\langle V[h_1(0)]V[h_2(0)]V[h_3(0)]\rangle = \langle h_1[V(0)]h_2[V(0)]h_3[V(0)]\rangle \tag{2.34}$$

with h_i arbitrary maps of $z = 0$ into the upper complex plane. This differs from the previous expression only by terms with extra factors of k_i^2. By the previous argument (2.23), such terms vanish by the gauge-invariant equation of motion.

Till now, our calculation is up to 3-point interactions while there is also 4-point interactions in terms of F^2 and F^3 in field theory. It means at least the 4-point amplitudes in string are necessary to compare with the effect actions.

2.3 Four-point S-matrices

In this section, we will use this gauge-covariant vertex operator in (2.10) to compute the explicit expression of the gauge-invariant tree amplitude between 4 gauge bosons, which has not appeared previously in the literature. From the S-matrices, we will find the corresponding gauge-invariant effective actions in field theory.

For a 4-point amplitude, there are three unintegrated vertex and one integrated vertex:

$$\mathcal{A}_4 \;=\; \tfrac{g_{YM}^2}{2\alpha'^2}\langle V(y_1)\int dy_2 W(y_2)V(y_3)V(y_4)\rangle \tag{2.35}$$

Set $y_1 = 0, y_3 = 1, y_4 \to \infty$ and integrate y_2 from 0 to 1. When $y_4 \to \infty$, the factor appearing in $\mathcal{A}_4$

$$|y_{14}|^{2\alpha' k^1\cdot k^4}|y_{24}|^{2\alpha' k^2\cdot k^4}|y_{34}|^{2\alpha' k^3\cdot k^4} \to |y_4|^{-2\alpha' k^4\cdot k^4}. \tag{2.36}$$

with the momentum conservation $k^1 + k^2 + k^3 + k^4 = 0$. For the convenience, we also introduce the Mandelstam variables

$$s = -(k^1 + k^2)^2, \quad t = -(k^1 + k^4)^2, \quad u = -(k^1 + k^3)^2. \tag{2.37}$$

and the definitions

$$\overset{\circ}{F}{}^i_{\mu\nu} = k^i_{[\mu}\epsilon^i_{\nu]} = k^i_\mu\epsilon^i_\nu - k^i_\nu\epsilon^i_\mu$$

The integral in (2.35) gives Euler Beta function:

$$\int_0^1 dy\; y^a(1-y)^b = \tfrac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} \tag{2.38}$$

with the Gamma function

$$\Gamma(a) = \int_0^\infty dt\; t^{a-1}e^{-t}, \qquad \Gamma(a+1) = a\Gamma(a). \tag{2.39}$$

After using the gauge-invariant equation of motion of the free vector (2.8) to simplify the expression, the amplitude is

$$\alpha'^2(K_0 + \alpha' K_1 + \alpha'^2 stu K_2)\,\frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s - \alpha' t)} \tag{2.40}$$

where we have factored out the usual coupling constants and momentum conservation δ-function, as well as Chan-Paton factors for cyclic ordering. The kinematic factors are

$$\begin{aligned} K_0 &= (4\mathring{F}^{1\nu}_{\mu}\mathring{F}^{2\sigma}_{\nu}\mathring{F}^{3\rho}_{\sigma}\mathring{F}^{4\mu}_{\rho} - \mathring{F}^{1\nu}_{\mu}\mathring{F}^{2\mu}_{\nu}\mathring{F}^{3\rho}_{\sigma}\mathring{F}^{4\sigma}_{\rho}) + 2\; permutations \\ &\equiv t^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}\mathring{F}^1_{\mu\nu}\mathring{F}^2_{\rho\sigma}\mathring{F}^3_{\alpha\beta}\mathring{F}^4_{\gamma\delta}, \end{aligned} \tag{2.41}$$

$$K_1 = [4(\mathring{F}^{1\nu}_{\mu}\mathring{F}^{4\mu}_{\nu})(k^1-k^4)^\tau \mathring{F}^{2\sigma}_{\tau}\mathring{F}^{3\lambda}_{\sigma}k^4_\lambda + 8\mathring{F}^{1[\mu}_{\nu}\mathring{F}^{2\sigma]\nu}\mathring{F}^{3\rho}_{\sigma}k^4_\rho\mathring{F}^{4\tau}_{\mu}k^1_\tau] + 3\; permutations \tag{2.42}$$

$$K_2 = -2\left(\frac{\mathring{F}^{1\nu}_{\mu}\mathring{F}^{4\mu}_{\nu}\mathring{F}^{2\rho}_{\sigma}\mathring{F}^{3\sigma}_{\rho}}{t(1+\alpha' t)} + \frac{\mathring{F}^{1\nu}_{\mu}\mathring{F}^{2\mu}_{\nu}\mathring{F}^{3\rho}_{\sigma}\mathring{F}^{4\sigma}_{\rho}}{s(1+\alpha' s)} + \frac{\mathring{F}^{1\nu}_{\mu}\mathring{F}^{3\mu}_{\nu}\mathring{F}^{2\rho}_{\sigma}\mathring{F}^{4\sigma}_{\rho}}{u(1+\alpha' u)}\right) \tag{2.43}$$

Here, the permutations in K_0 are the order 1342 and 1423 which replace the cyclic order 1234 and the 3 permutations in K_1 are the replacing of 1234 by 2341, 3412 and 4123. Notice the K_1 term corresponds to the contribution from an F^3 term in the field theory action, and hence it is absent in the presence of supersymmetry. The K_2 term in (2.40) can be regarded as the contribution of tachyon poles in the s and t channels, while the apparent u pole is canceled by the Γ's. So it will be also absent in the corresponding superstring amplitude due to the GSO projection. (These amplitudes agree with earlier gauge fixed results obtained [29].)

Expanding above amplitude in orders of 1, α' and α'^2,

$$\frac{K_0 + \alpha' K_1}{st} + \alpha'^2(-\tfrac{\pi^2}{6}K_0 + uK_2'), \tag{2.44}$$

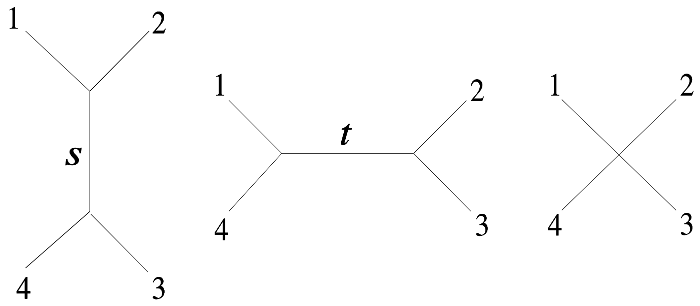

Figure 2.1: The s- and t-channel diagrams for 4 gauge bosons coupled by F^2 vertices.

$$K_2' = -2\left(\frac{\mathring{F}^{1\nu}_{\mu}\mathring{F}^{2\mu}_{\nu}\mathring{F}^{3\rho}_{\sigma}\mathring{F}^{4\sigma}_{\rho}}{s} + \frac{\mathring{F}^{1\nu}_{\mu}\mathring{F}^{4\mu}_{\nu}\mathring{F}^{2\rho}_{\sigma}\mathring{F}^{3\sigma}_{\rho}}{t} + \frac{\mathring{F}^{1\nu}_{\mu}\mathring{F}^{3\mu}_{\nu}\mathring{F}^{2\rho}_{\sigma}\mathring{F}^{4\sigma}_{\rho}}{u}\right) \tag{2.45}$$

Naively, it is hard to see what kind of effective action in field theory should correspond to the S-matrices in (2.44). From previous section 2.2, the effective action should include F^2 term (2.21) and F^3 term (2.26), which will contribute in 4-point amplitude. So, in next section, we will compute the S-matrices from these actions first.

2.4 Effective Actions

Obviously, the amplitude in $O(\alpha'^0)$ in (2.44) corresponds to the 4-point amplitude from 3 Feynman diagrams in Yang-Mills action (2.21), as shown in Fig. 2.1. To the order $O(\alpha')$, there are 5 Feynman diagrams involving the cubic interaction (2.26), as shown in Fig. 2.2. The summation of amplitudes from these 5 diagrams is

$$-2\alpha'(K_1' + 3\ permutations) \tag{2.46}$$

in which

$$\begin{aligned} K_1' \;=\; & \tfrac{p^a}{p^2}(\mathring{F}^{4b}_{a}\mathring{F}^{1\tau}_{b} - \mathring{F}^{1b}_{a}\mathring{F}^{4\tau}_{b})[2\epsilon^2_\tau(\epsilon^3\cdot k^2) - 2\epsilon^3_\tau(\epsilon^2\cdot k^3) + (k^3-k^2)_\tau(\epsilon^2\cdot\epsilon^3)] \\ & + \mathring{F}^{1b}_{a}\mathring{F}^{2c}_{b}(\epsilon^3_c\epsilon^{4a} - \epsilon^{3a}\epsilon^4_c) \end{aligned} \tag{2.47}$$

and $p = -k^1 - k^4$. The 3 permutations are the replacing of 1234 by 2341, 3412 and 4123 in K_1'. But it is not a explicitly gauge-covariant form as we

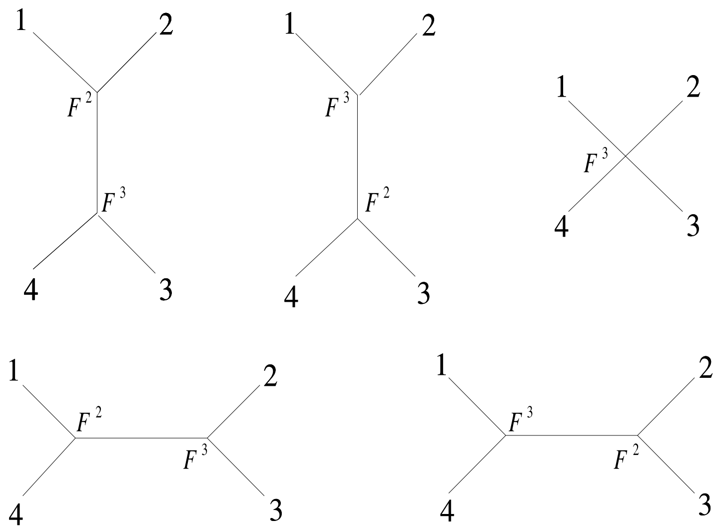

Figure 2.2: The s- and t-channel diagrams for 4 gauge bosons coupled by one F^2 vertex and one F^3 vertex.

expected. To make it explicit, apply the gauge transformation

$$\epsilon^i_\mu \to \epsilon^i_\mu - k^i_\mu \frac{\epsilon^i \cdot k^{i+1}}{k^i \cdot k^{i+1}} = -\frac{k^{i+1}_\nu \overset{\circ}{F}{}^i_{\mu\nu}}{k^i \cdot k^{i+1}}, \tag{2.48}$$

where $i+1 \to 1$ for $i = 4$. After using the Bianchi identity

$$k_{[\mu} \overset{\circ}{F}_{\nu\sigma]} = 0 \tag{2.49}$$

and the *gauge-invariant* equation of motion of the free vector (2.8), we get back the $O(\alpha')$ order term in the amplitude $\mathcal{A}_4$ in (2.44): $\frac{\alpha' K_1}{st}$.

It is not the whole story because there more terms in order $O(\alpha'^2)$ from the string amplitude (2.44) . It should be contributed by two Feynman diagrams, as shown in Fig. 2.3, and probably a higher-derivative gauge interaction F^4. The direct calculation for two Feynman diagrams in Fig. 2.3 gives a explicitly gauge-covariant amplitude

$$\alpha'^2 \left(\frac{s-u}{t} \overset{\circ}{F}{}^{1\nu}_{\mu} \overset{\circ}{F}{}^{4\mu}_{\nu} \overset{\circ}{F}{}^{2\rho}_{\sigma} \overset{\circ}{F}{}^{3\sigma}_{\rho} + \frac{t-u}{s} \overset{\circ}{F}{}^{1\nu}_{\mu} \overset{\circ}{F}{}^{2\mu}_{\nu} \overset{\circ}{F}{}^{3\rho}_{\sigma} \overset{\circ}{F}{}^{4\sigma}_{\rho} \right) \tag{2.50}$$

The difference between (2.50) and the $O(\alpha'^2)$ part of (2.44) corresponds to

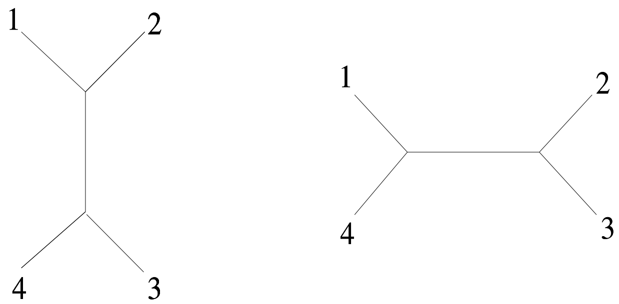

Figure 2.3: The s- and t-channel diagrams for 4 gauge bosons coupled only by the F^3 vertex.

higher-derivative interactions, i.e., the F^4 interactions in the effective theory. The difference is composed of two parts:

$$B_1 = -\tfrac{\pi^2}{6}\alpha'^2 K_0, \tag{2.51}$$

and

$$B_2 = \alpha'^2(\overset{\circ}{F}{}^{1\nu}_{\mu}\overset{\circ}{F}{}^{4\mu}_{\nu}\overset{\circ}{F}{}^{2\rho}_{\sigma}\overset{\circ}{F}{}^{3\sigma}_{\rho} + \overset{\circ}{F}{}^{1\nu}_{\mu}\overset{\circ}{F}{}^{2\mu}_{\nu}\overset{\circ}{F}{}^{3\rho}_{\sigma}\overset{\circ}{F}{}^{4\sigma}_{\rho} - 2\overset{\circ}{F}{}^{1\nu}_{\mu}\overset{\circ}{F}{}^{3\mu}_{\nu}\overset{\circ}{F}{}^{2\rho}_{\sigma}\overset{\circ}{F}{}^{4\sigma}_{\rho}). \tag{2.52}$$

Replace $\overset{\circ}{F}_{\mu\nu}$ by $-iF_{\mu\nu}$ and include a factor of $1/4$ for the cyclic identity (as well as the usual overall factor $1/g^2_{YM}$). So, the action from B_1 is

$$-\tfrac{\pi^2\alpha'^2}{4!g^2_{YM}}t^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}Tr(F_{\mu\nu}F_{\rho\sigma}F_{\alpha\beta}F_{\gamma\delta}) \tag{2.53}$$

and, from B_2, the action is

$$\tfrac{\alpha'^2}{2g^2_{YM}}Tr(F_{\mu\nu}F^{\nu\mu}F_{\rho\sigma}F^{\sigma\rho} - F_{\mu\nu}F_{\rho\sigma}F^{\nu\mu}F^{\sigma\rho}) \tag{2.54}$$

As we will see later, action (2.53) also exists in the superstring, while (2.54) is absent in superstring case. Totally, the low energy limit (2.44) of amplitude $\mathcal{A}_4$ in (2.40) corresponds to the effective action in field theory:

$$\begin{aligned} S = \tfrac{1}{g^2_{YM}}\int d^Dx\,[&-\tfrac{1}{4}Tr(F^{\mu\nu}F_{\mu\nu}) - \tfrac{2i\alpha'}{3}Tr(F_\mu{}^\nu F_\nu{}^\omega F_\omega{}^\mu) \\ &-\tfrac{\pi^2\alpha'^2}{4!}t^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}Tr(F_{\mu\nu}F_{\rho\sigma}F_{\alpha\beta}F_{\gamma\delta}) \\ &+\tfrac{\alpha'^2}{2}Tr(F_{\mu\nu}F^{\nu\mu}F_{\rho\sigma}F^{\sigma\rho} - F_{\mu\nu}F_{\rho\sigma}F^{\nu\mu}F^{\sigma\rho})]. \end{aligned} \tag{2.55}$$

As shown in this chapter, we have given a general construction for gauge-covariant vertex operators. It is applied it to the spin 1 vertex in the bosonic string, which leads to 3-point and 4-point gauge-covariant amplitudes. The corresponding field effective action is easier to deduce from these amplitudes for the covariance. This method allows direct calculation of gauge-invariant results, analogous to nonlinear sigma models, and can also be applied to string field theory. In the following chapter, we will applied a similar method to NSR superstring to find gauge-covariant vertex for vectors.

Chapter 3

Gauge-Covariant Vertex Operators in NSR Strings

In this chapter, the formalism for bosonic string in chapter 2 will be generalized to the Neveu-Schwarz string. To simplify the expression, the language of the "Big Picture" [27] will be introduced in this chapter. After the new vertex operator is constructed, the amplitude of 3 vectors will be calculated directly, with the properties of gauge and conformal transformation studied.

3.1 Neveu-Schwarz vertex for massless vector

For superstrings, it is more useful and convenient to interpret superconformal symmetry through a supermanifold: a world-sheet with one normal complex coordinate z and a anticommuting one θ. The superderivatives is defined on this supermanifold as: $D \equiv D_\theta = \partial_\theta + \theta\partial_z$ and $\overline{D} \equiv D_{\overline{\theta}} = \partial_{\overline{\theta}} + \overline{\theta}\partial_{\overline{z}}$. The action for NSR string in conformal gauge is constructed on the $(0,0)$ tenser $X^\mu(z,\theta,\widetilde{\theta}) = x^\mu(z) + i\theta\psi(z) + i\overline{\theta}\widetilde{\psi}(z) + \theta\overline{\theta}F^\mu$ and superfields of ghosts $C = c + \theta\gamma$ and $B = \beta + \theta b$ (c and b are anticommuting superconformal ghost and antighost, β and γ are commuting superconformal ghost and antighost):

$$S = \tfrac{1}{4\pi}\int d^2z d^2\theta (DX^\mu\overline{D}X_\mu + B\overline{D}C + \overline{B}D\overline{C}) \tag{3.1}$$

where, to simplify the calculation and expression, we replace x by $(\alpha'/2)^{1/2}x$ to absorb constants (We also write $\sqrt{\alpha'/2}\,\epsilon$ as ϵ and $\sqrt{\alpha'/2}\,k$ as k and restore these factors in the final resul). This action in conformal gauge gives the propagator between $X(z,\theta,\widetilde{\theta})$'s as

$$X^\mu(z',\theta')X^\nu(z,\theta) \sim -4ln|z' - z - \theta'\theta|\eta^{\mu\nu} \tag{3.2}$$

and ghosts

$$B(z_1,\theta_1)C(z_2,\theta_2) \sim \tfrac{\theta_1-\theta_2}{z_1-z_2}, \tag{3.3}$$

Similarly as in the case of bosonic open string, we start from the integrated vertex operator $\oint W = \oint \epsilon \cdot D_\theta X e^{ik\cdot X(Z)}$ with $Z = (z,\theta)$: the coordinate on the open string world-sheet. If the commutator between the BRST operator and the integrated vertex can be written as total derivative: $[Q,W\} = D_\theta V$, V will be the BRST invariant unintegrated operator: $[Q,V\} = 0$.

The BRST operator Q in the NSR string is

$$Q = \tfrac{1}{2\pi i}\oint dz'd\theta'(CT^x + \tfrac{1}{2}CT^g) \tag{3.4}$$

with the contribution of matter fields

$$T^x = -\tfrac{1}{4\alpha'}(D_{\theta'}X^\mu)(\partial' X_\mu) \tag{3.5}$$

and the contribution of ghosts

$$T^g = \tfrac{1}{2}(D_{\theta'}B)(D_{\theta'}C) - \tfrac{3}{2}B(\partial' C) - (\partial' B)C \tag{3.6}$$

Because W contains no ghosts, it is only necessary to compute

$$[Q,W\} = \tfrac{1}{2\pi i}\oint dz'd\theta'(-\tfrac{1}{8})[C(z',\theta')(D_{\theta'}X^\mu)(\partial' X_\mu)][\epsilon \cdot D_\theta X e^{ik\cdot X(Z)}] \tag{3.7}$$

Using

$$\oint dz'd\theta' \tfrac{1}{z'-z-\theta'\theta}[f(z') + \theta' g(z')] = D_\theta[f(z) + \theta g(z)] \tag{3.8}$$

$$\oint dz'd\theta' \tfrac{\theta'-\theta}{z'-z}[f(z') + \theta' g(z')] = f(z) + \theta g(z) \tag{3.9}$$

and gauge-covariant equation of motion (2.8), we find the BRST invariant vertex operator for the massless open string vector V is

$$\begin{aligned} V &= -D_\theta[C(\epsilon \cdot D_\theta X)e^{ik\cdot X(Z)}] + \tfrac{1}{2}(D_\theta C)(D_\theta X \cdot \epsilon)e^{ik\cdot X(Z)} \\ &\quad -2i(\epsilon \cdot k)(\partial C)e^{ik\cdot X(Z)} \\ &= G + H \end{aligned} \tag{3.10}$$

or equivalently

$$V = -D_\theta[CA^\mu(D_\theta X_\mu)] - 2(\partial C)(\partial_\mu A^\mu) + \tfrac{1}{2}(D_\theta C)A^\mu(D_\theta X_\mu), \tag{3.11}$$

where G represents the first two terms and H represents the last term.

The gauge transformation of the vertex operator (3.10) is

$$\begin{aligned} \delta V &= -D_\theta[C(\partial^\mu\lambda)(D_\theta X_\mu)] - 2(\partial C)[\partial_\mu(\partial^\mu\lambda)] + \tfrac{1}{2}(D_\theta C)(D_\theta X_\mu)(\partial^\mu\lambda) \\ &= -\tfrac{1}{2}(D_\theta C)(D_\theta X_\mu)\partial^\mu\lambda - 2(\partial C)\Box\lambda + C(\partial X_\mu)\partial^\mu\lambda \\ &\quad -C(D_\theta X_\mu)(D_\theta X_\nu)\partial^\nu\partial^\mu\lambda \end{aligned} \tag{3.12}$$

under $A \to A + \partial_\mu\lambda$. Obviously, the last term is zero because $D_\theta X_\mu$ and $D_\theta X_\nu$ anticommute. Then δV can be written as the commutator of the BRST operator and the function λ:

$$\begin{aligned} [Q, \lambda(X(z,\theta))\} &= \tfrac{1}{2\pi i}\oint dz' d\theta' \tfrac{1}{8} C(Z')(D_{\theta'}X^\mu)(\partial' X_\mu)\lambda(X(Z)) \\ &= -\tfrac{1}{2}D_\theta(CD_\theta X_\mu)\partial^\mu\lambda - 2(\partial C)\Box\lambda + \tfrac{1}{2}C(\partial X_\mu)\partial^\mu\lambda \end{aligned} \tag{3.13}$$

So, just as in the bosonic case, the amplitude $\langle VVV \oint W \cdots \oint W\rangle$ is gauge invariant

3.2 Three-point amplitude

The vertex operator given in previous section can be used to compute the 3-point amplitude between 3 gauge bosons. The difference between this amplitude and previous calculations is that this amplitude is gauge invariant which corresponds the gauge-invariant effective action in field theory directly.

Using the correlation function between superfields of ghosts

$$\begin{aligned} &\langle 0|C(z_1,\theta_1)C(z_2,\theta_2)C(z_3,\theta_3)|0\rangle \\ = \; &\theta_1\theta_2 z_3(z_1+z_2) + \theta_2\theta_3 z_1(z_2+z_3) + \theta_3\theta_1 z_2(z_3+z_1) \end{aligned}$$

and the propagator between X's as (3.2), and the gauge-covariant equation of motion (2.8), we find the 3-point amplitude for 3 gauge bosons, to the lowest order in α', as following

$$\begin{aligned} \mathcal{A}_3 &= \tfrac{2g_{YM}}{\alpha'^2}\langle V(z_1,\theta_1)V(z_2,\theta_2)V(z_3,\theta_3)\rangle \\ &= ig_{YM}(2\pi)^D\delta^D(\Sigma_i k_i)[(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot k_{12}) + (\epsilon_2\cdot\epsilon_3)(\epsilon_1\cdot k_{23}) \\ &\qquad + (\epsilon_3\cdot\epsilon_1)(\epsilon_2\cdot k_{31})] \end{aligned} \tag{3.14}$$

which corresponds to the 3-particle interaction in the YM theory.

The calculation becomes more complicate to second order in α'. Factors like

$$|y_{ij}-\theta_i\theta_j|^{2\alpha' k_i\cdot k_j}, \quad D_{\theta_k}|y_{ij}-\theta_i\theta_j|^{2\alpha' k_i\cdot k_j}, D_{\theta_k}D_{\theta_l}|y_{ij}-\theta_i\theta_j|^{2\alpha' k_i\cdot k_j}, \cdots \tag{3.15}$$

are involved in the amplitude $\mathcal{A}_3$. To lowest order in α', the first is just one, the rest zero. But the expansion of $2\alpha' k_i \cdot k_j$ contributes to the second order in α', as do

$$\begin{aligned}&\langle G(y_1)G(y_2)G(y_3)\rangle,\\ &\langle G(y_1)G(y_2)H(y_3)\rangle, \cdots,\\ &\langle G(y_1)H(y_2)H(y_3)\rangle, \cdots\end{aligned} \tag{3.16}$$

while

$$\langle H(y_1)H(y_2)H(y_3)\rangle \tag{3.17}$$

vanishes because of too many derivatives.

We calculate the expansion of $|y_{ij}-\theta_i\theta_j|^{2\alpha' k_i\cdot k_j}$ and its derivatives $D_{\theta_k}|y_{ij}-\theta_i\theta_j|^{2\alpha' k_i\cdot k_j}$, etc., using

$$\begin{aligned}&|y_{12}-\theta_1\theta_2|^a|y_{23}-\theta_2\theta_3|^b|y_{31}-\theta_3\theta_1|^c\\ =\ &[1-\tfrac{a}{y_{12}}\theta_1\theta_2-\tfrac{b}{y_{23}}\theta_2\theta_3-\tfrac{c}{y_{31}}\theta_3\theta_1]|y_{12}|^a|y_{23}|^b|y_{31}|^c\\ =\ &[D_{\theta_1}(|y_{12}|^a\theta_{12})][D_{\theta_2}(|y_{23}|^b\theta_{23})][D_{\theta_3}(|y_{31}|^c\theta_{31})]\end{aligned} \tag{3.18}$$

to the first order in a, b, c. Then

$$\begin{aligned}&D_{\theta_1}(|y_{12}-\theta_1\theta_2|^a|y_{23}-\theta_2\theta_3|^b|y_{31}-\theta_3\theta_1|^c)\\ =\ &(\partial_1|y_{12}|^a)|y_{23}|^b|y_{31}|^c\theta_{12}+|y_{12}|^a|y_{23}|^b(\partial_3|y_{31}|^c)\theta_{31}\\ &+\theta_1\theta_2\theta_3[(\partial_1|y_{12}|^a)\partial_3(|y_{23}|^b|y_{31}|^c)+\partial_2(|y_{12}|^a|y_{23}|^b)(\partial_3|y_{31}|^c)]\end{aligned} \tag{3.19}$$

$$D_{\theta_1}D_{\theta_2}(|y_{12}-\theta_1\theta_2|^a|y_{23}-\theta_2\theta_3|^b|y_{31}-\theta_3\theta_1|^c)=\tfrac{a}{y_{12}}+\tfrac{a}{y_{12}^2}\theta_1\theta_2, \cdots \tag{3.20}$$

Combining both contributions without showing all details here, the 3-point amplitude to the second order in α' is just zero as expected, which corresponds to the absence of F^3 terms in SYM. As mentioned in the bosonic case, using the gauge-covariant equation of motion (2.8), above amplitude (3.14) is complete and there is no more higher order terms in α'.

This amplitude here is independent of the anticommuting coordinate θ, as expected. It is also independent of z, as in the bosonic case. It is the result of conformal invariance of the amplitude, which we are going to prove in the following part. The conformal transformation of vertex V in (3.10) is

$$\delta V = \frac{1}{2\pi i}\oint dz' d\theta' \lambda (T^x + T^g)(z', \theta') V(z, \theta) \tag{3.21}$$

with T^x and T^g defined in eqs. (3.5-3.6). Using the OPE of superfields of X, B and C,

$$\delta V = (\alpha' k^2)(\partial \lambda) V + \tfrac{1}{2}(D_\theta \lambda)(D_\theta V) + \lambda(\partial V) \tag{3.22}$$

Under the infinitesimal superconformal transformation $\lambda = 2\theta\eta$ with $\eta(z)$ anticommuting,

$$\delta V = (2\alpha' k^2)(\theta \partial \eta) V + \eta Q_\theta V \tag{3.23}$$

where $Q_\theta = \partial_\theta - \theta \partial_z$. So the vertex operator in (3.10) has the weight $\alpha' k^2$ and transforms as

$$V'(z', \theta') = (D_\theta \theta')^{-2\alpha' k^2} V(z, \theta) \tag{3.24}$$

which leads to the conformal transformation of the n-point amplitude as

$$\begin{aligned} & \langle V'(z_1', \theta_1') V'(z_2', \theta_2') V'(z_3', \theta_3') \oint W \cdots \oint W \rangle \\ = \; & (D_{\theta_1}\theta_1')^{-2\alpha' k_1^2} (D_{\theta_2}\theta_2')^{-2\alpha' k_2^2} (D_{\theta_3}\theta_3')^{-2\alpha' k_3^2} \\ & \times \langle V(z_1, \theta_1) V(z_2, \theta_2) V(z_3, \theta_3) \oint W \cdots \oint W \rangle \end{aligned} \tag{3.25}$$

Using equation of motion $\Box F = 0$, the expansion of $\alpha' k_i^2$ gives nothing than just one if the amplitude is proportional to the product of field of strengthes. In the case of 3-point amplitude (3.14), this implies the conformal invariance:

$$\langle V'(z_1', \theta_1') V'(z_2', \theta_2') V'(z_3', \theta_3') \rangle = \langle V(z_1, \theta_1) V(z_2, \theta_2) V(z_3, \theta_3) \rangle \tag{3.26}$$

3.3 4-point S-matrices in NS section

In this section, as the case bosonic string, the new vertex operator (3.10) is used to compute the S-matrices for 4 massless gauge bosons, which will be explicitly gauge-covariant. The corresponding effective action for gauge bosons can be deduced from this amplitude directly.

The 4-point amplitude in the superstring is

$$\mathcal{A}_4^{NSR} = -\frac{2 g_{YM}^2}{\alpha'^2} \langle V(Z_1) \int dz_2 d\theta_2 W(Z_2) V(Z_3) V(Z_4) \rangle, \tag{3.27}$$

with $z_1 = 0, z_3 = 1, z_4 \to \infty$ and integrating z_2 from 0 to 1. To make the calculation simpler, we will first set θ_1, θ_3, and θ_4 to zero, because the superconformal weight of unintegrated vertices ensure the superconformal invariance of the amplitude. So, as we will see, the S-matrices will be independent from the choice of $\theta_1 = \theta_3 = \theta_4$. The vertex operator (3.10) can also be written as

$$\begin{aligned} V &= -\tfrac{1}{2}(D_\theta C)(\epsilon \cdot D_\theta X)e^{ik\cdot X(Z)} + CD_\theta[(D_\theta X \cdot \epsilon)e^{ik\cdot X(Z)}] \\ &\quad -2i(\epsilon \cdot k)(\partial C)e^{ik\cdot X(Z)}. \end{aligned} \tag{3.28}$$

Using the anticommutation relation between C, D_θ, and $\int d\theta$, move C, $D_\theta C$, and ∂C to the left side of $\mathcal{A}_4^{NSR}$. To make the calculation simpler, we first set θ_1, θ_3, and θ_4 to zero. Then we only have to compute the terms independent of θ_1, θ_3, and θ_4, which is from the parts with two $D_\theta C$'s and one C or ∂C:

$$\mathcal{A}_4^{NSR}(\theta_1 = 0, \theta_3 = 0, \theta_4 = 0) = \alpha'^2 K_0 \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s - \alpha' t)} \tag{3.29}$$

where K_0 is defined by (2.41).

Because operator V has the weight $\alpha' k^2$, through a conformal transformation $\theta_1 = 0 \to \theta_1'$, $\theta_2 = 0 \to \theta_2'$ and $\theta_4 = 0 \to \theta_4'$, the amplitude (3.29) in transforms as

$$\begin{aligned} &\langle V'(z_1', \theta_1') \oint W V'(z_3', \theta_3') V'(z_4', \theta_4')\rangle \\ = \;& (D_{\theta_1}\theta_1')^{-2\alpha' k_1^2}(D_{\theta_3}\theta_3')^{-2\alpha' k_3^2}(D_{\theta_4}\theta_4')^{-2\alpha' k_4^2}\langle V(z_1,\theta_1)\oint W V(z_3,\theta_3)V(z_4,\theta_4)\rangle \\ = \;& [(D_{\theta_1}\theta_1')^{-2\alpha' k_1^2}(D_{\theta_3}\theta_3')^{-2\alpha' k_3^2}(D_{\theta_4}\theta_4')^{-2\alpha' k_4^2}]|_{\theta_1=\theta_3=\theta_4=0}\mathcal{A}_4^{NSR}(0,0,0) \end{aligned} \tag{3.30}$$

Using the equation of motion (2.8),

$$\mathcal{A}_4^{NSR}(\theta_1', \theta_2', \theta_3') = \mathcal{A}_4^{NSR}(0,0,0)$$

for the amplitude in (3.29) is in form of $\overset{\circ}{F}_{\mu\nu}$. So (3.29) is exactly the 4-point tree amplitude for arbitrary values of parameters θ_1, θ_3 and θ_4.

What we notice first here is that, Since there is no tachyon in the superstring, the amplitude doesn't give the terms associated with tachyon poles in (2.40) , which is not surprising. Second, expanding the function $\frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)}$, the leading term corresponds to the quadratic Yang-Mills action (2.21) as in the bosonic case. For the lacking of F^3 interaction in field side, there is no terms of $O(\alpha')$ order and the $O(\alpha'^2)$ terms represent the higher-derivative F^4 action directly. As in the bosonic case, replacing $\overset{\circ}{F}_{\mu\nu}$ by $-iF_{\mu\nu}$, the complete effective action corresponding to (3.29) is

$$S = \frac{1}{g_{YM}^2} \int d^D x \; [-\tfrac{1}{4} Tr(F^{\mu\nu} F_{\mu\nu}) - \tfrac{\pi^2 \alpha'^2}{4!} t^{\mu\nu\rho\sigma\alpha\beta\gamma\delta} Tr(F_{\mu\nu} F_{\rho\sigma} F_{\alpha\beta} F_{\gamma\delta})]. \quad (3.31)$$

These actions agree with those obtained from non-gauge-covariant amplitudes [30].

Chapter 4

Feynman-Siegel gauge

In previous Chapter 2, we pointed the relation between the vertex operator in (2.10) to the external state in string field theory (SFT) (Witten's theory here). In this chapter, we will study more details. We start from constructing the Zinn-Justin-Batalin-Vilkovisky action for tachyons and gauge bosons from Witten's 3-string vertex of the bosonic open string without gauge fixing. Through canonical transformations, we find the off-shell, local, gauge-covariant action up to 3-point terms, satisfying the usual field theory gauge transformations. Perturbatively, it can be extended to higher-point terms. It also gives the relation between a new gauge condition in field theory and the Feynman-Siegel gauge on the world-sheet.

4.1 Witten's 3-string vertex

In string field theory, the 3-string interaction can be interpreted as

$$\langle h_1[\vartheta_A]h_2[\vartheta_b]h_3[\vartheta_c]\rangle = \langle V_{123}(|A\rangle_1 \otimes |B\rangle_2 \otimes |C\rangle_3) \tag{4.1}$$

where ϑ_i is the vertex operator for each external state and $h_i(z)$ is the conformal mapping from each string state to the complex plane. In Witten's bosonic open string field theory, strings interact by identifying the right half of each string with the left half of the next one. The conformal mapping for this interactive world-sheet geometry can be expressed as

$$h_1(z) = e^{i\frac{2\pi}{3}}h(z), \quad h_2 = h(z), \quad h_3 = e^{-i\frac{2\pi}{3}}h(z) \tag{4.2}$$

where

$$h(z) = (\tfrac{1-iz}{1+iz})^{\frac{2}{3}} \tag{4.3}$$

Then the action is

$$S = \langle V_2|\Psi, Q\Psi\rangle + \tfrac{g}{3}\langle V_3|\Psi,\Psi,\Psi\rangle \tag{4.4}$$

where Q is the usual string theory BRST operator. Using the string oscillation modes α_n of the matter sector and b_n, c_n of the ghost sector, the two-string "vertex" is

$$\begin{aligned} \langle V_2| &= \delta^D(p_1+p_2)\left(\langle 0;p_1|\otimes\langle 0;p_2|\right)\left(c_0^{(1)}+c_0^{(2)}\right) \\ &\quad\times\exp\left(\sum_{n=1}^{\infty}(-1)^{n+1}[\alpha_n^{(1)}\alpha_n^{(2)}+c_n^{(1)}b_n^{(2)}+c_n^{(2)}b_n^{(1)}]\right) \end{aligned} \tag{4.5}$$

and the 3-string vertex associated with the three-string overlap can be written as

$$\begin{aligned} \langle V_3| &= \mathcal{N}\delta^D(p_1+p_2+p_3)(\langle 0|c_{-1}c_0)^{(3)})(\langle 0|c_{-1}c_0)^{(2)})(\langle 0|c_{-1}c_0)^{(1)}) \\ &\quad\times\exp\Big(\sum_{r,s=1}^{3}\sum_{n,m\geq 1}\frac{1}{2}\alpha_m^{(r)}N_{mn}^{rs}\alpha_n^{(s)}+p^{(r)}N_{0m}^{rs}\alpha_m^{(s)}+\frac{1}{2}N_{00}\sum_{r=1}^{3}(p^{(r)})^2\Big) \\ &\quad\times\exp\Big(\sum_{r,s=1}^{3}\sum_{\substack{m\geq 0\\ n\geq 1}}b_m^{(r)}X_{mn}^{rs}nc_n^{(s)}\Big) \end{aligned} \tag{4.6}$$

with the normalization factor $\mathcal{N} = 3^{9/2}/2^6$ [31]. Because we will focus on the fields and antifields up to oscillation modes 1, the only relevant Neumann coefficients are

$$\begin{gathered} N_{11}^{11} = N_{11}^{22} = N_{11}^{33} = -\tfrac{5}{27} \\ N_{11}^{12} = N_{11}^{23} = N_{11}^{31} = \tfrac{16}{27} \\ N_{01}^{12} = -N_{01}^{13} = N_{01}^{23} = -N_{01}^{21} = N_{01}^{31} = -N_{01}^{32} = -\tfrac{2\sqrt{3}}{9} \\ N_{00}^{11} = N_{00}^{22} = N_{00}^{33} = -\tfrac{1}{2}\ln(27/16) \\ N_{01}^{11} = N_{01}^{22} = N_{01}^{33} = 0 \end{gathered} \tag{4.7}$$

for the matter sector and

$$\begin{gathered} X_{11}^{11} = X_{11}^{22} = X_{11}^{33} = -\tfrac{11}{27} \\ X_{11}^{12} = X_{11}^{23} = X_{11}^{31} = X_{11}^{21} = X_{11}^{32} = X_{11}^{13} = -\tfrac{8}{27} \\ X_{01}^{12} = -X_{01}^{13} = X_{01}^{23} = -X_{01}^{21} = X_{01}^{31} = -X_{01}^{32} = -\tfrac{4\sqrt{3}}{9} \\ X_{01}^{11} = X_{01}^{22} = X_{01}^{33} = 0 \end{gathered} \tag{4.8}$$

for the ghost sector.

Usually, the three-string interactions are calculated in the Feynman-Siegel gauge

$$b_0|\Psi\rangle = 0 \tag{4.9}$$

Then what we get is the gauge-fixed action, and the gauge condition for this action was never clear. Also we will get some ϕA^2 interactions whose origin was not obvious due to the lack of gauge covariance. In the next section, we will construct the ZJBV action from string field theory to study the gauge condition from the aspect of field theory.

4.2 ZJBV

In the usual Hamiltonian formalism for a phase space $(q,\ p)$, the Poisson bracket, which is useful for studying symmetry properties and relates to the commutator of the quantum theory, can be defined. In gauge field theory, there is a similar interpretation where the fields (including ghosts) correspond to q and the antifields (with opposite statistics) to p. In the YM case (including scalars), $\phi, A_\mu, C, \widetilde{C}$ are fields and $\phi^*, A^*_\mu, C^*, \widetilde{C}^*$ are antifields. As a generalization of the Poisson bracket, the "antibracket" $(f(\Phi), g(\Phi)) = f \circ g$ is introduced [19]:

$$\circ = \int dx(-1)^I \left(\frac{\overleftarrow{\delta}}{\delta\phi^*_I}\frac{\delta}{\phi^I} + \frac{\overleftarrow{\delta}}{\delta\phi^I}\frac{\delta}{\phi^*_I} \right) \tag{4.10}$$

It has the following useful properties:

$$\begin{gathered} (f,ga) = (f,g)a, \quad (af,g) = a(f,g) \\ (f,g) = -(-1)^{(f+1)(g+1)}(g,f) \\ (f,gh) = (f,g)h + (-1)^{(f+1)g}g(f,h) \\ (-1)^{(f+1)(h+1)}(f,(g,h)) + cyc. = 0 \end{gathered} \tag{4.11}$$

The existence of a bracket with these properties allows the definition of a Lie derivative, $\mathcal{L}_A B \equiv (A,B)$ and a unitary transformation

$$S' = e^{\mathcal{L}_G}S = S + \mathcal{L}_G S + \tfrac{1}{2!}\mathcal{L}_G\mathcal{L}_G S + \cdots \tag{4.12}$$

For the example we are going to discuss, the antibrackets for fields and antifields are:

$$(A^*_\mu, A_\nu) = \eta_{\mu\nu}, \quad (\phi^*,\phi) = 1, \quad (C, C^*) = 1, \quad (\widetilde{C}, \widetilde{C}^*) = 1 \tag{4.13}$$

The general Lagrangian path integral for BRST quantization is

$$\mathcal{A} = \int D\psi^I e^{-iS_{gf}}, \quad S_{gf} = e^{\mathcal{L}_\Lambda} S_{ZJBV}| \tag{4.14}$$

where S_{gf} is evaluated at all antifields $\psi^* = 0$. Expanding the ZJBV action in antifields, using ψ_m and ψ_{nm} to indicate all minimal and non-minimal fields,

$$S_{ZJBV} = S_{gi} + (Q\psi_m)\psi_m^* + \psi_{nm}^*\psi_{nm}^*, \tag{4.15}$$

then

$$S_{gf} = e^{\mathcal{L}_\Lambda} S_{ZJBV}| = S_{gi} + (\delta\Lambda/\delta\psi_m)(Q\psi_m) + (\delta\Lambda/\delta\psi_{nm})^2 \tag{4.16}$$

where S_{gi} and Λ depend only on coordinates ψ^I. Also, the BRST transformations can be written as $\delta_Q\psi^I = (S_{ZJBV}, \psi^I)$. Gauge independence requires

$$(-1)^I \tfrac{\delta^2 S_{ZJBV}}{\delta\psi_I^*\delta\psi^I} + i\tfrac{1}{2}(S_{ZJBV}, S_{ZJBV}) = 0 \tag{4.17}$$

which is called the "quantum master equation". It is the approach to BRST of Zinn-Justin, Batalin, and Vilkovisky (ZJBV).

To see the equivalence of the ZJBV combination of the gauge-invariant action with the BRST operator to ordinary BRST, here is an example, pure Yang-Mills theory. The ZJBV action in YM can be written as

$$S_{ZJBV} = -F_{\mu\nu}F^{\mu\nu} - 2(\widetilde{C}^*)^2 - 2i[\nabla_\mu, C]A^{*\mu} + C^2C^*$$

We have the usual BRST transformations of fields from $Q\psi = (S, \psi)$:

$$QA_\mu = -2i[\nabla_\mu, C], \quad QC = -C^2, \quad Q\widetilde{C} = 4\widetilde{C}^*, \quad Q\widetilde{C}^* = 0 \tag{4.18}$$

Taking

$$\Lambda = tr \int \tfrac{1}{4}\widetilde{C}f(A), \tag{4.19}$$

we find the usual gauge fixed action

$$S_{gf} = S_{gi} - \tfrac{1}{4}f(A)^2 - \tfrac{i}{2}\widetilde{C}\tfrac{\partial f}{\partial A} \cdot [\nabla, C] \tag{4.20}$$

as from the usual BRST formalism.

4.3 The gauge covariant action

In this section, we will use Witten's 3-string vertex to get the interactions for tachyons and vectors without the Feynman-Siegel gauge. The action will be in the ZJBV formalism including fields and antifields. From this ZJBV action, through some canonical transformations, we can get the gauge invariant action back. Observing the forms of these transformations, we will be able to tell which gauge condition in field theory corresponds to the Feynman-Siegel gauge in Witten's string field theory.

First, let see if we can generalize the on-shell bosonic vertex to off-shell case. In string field theory, the general external state (without $b_0 = 0$) is

$$\begin{aligned} |\psi\rangle &= (C + \phi c_1 + A \cdot a_{-1}c_1 + \widetilde{C}c_{-1}c_1 + \widetilde{C}^* c_0 \\ &\quad + \phi^* c_0 c_1 + A^* \cdot a_{-1}c_0 c_1 + C^* c_{-1}c_0 c_1 + \cdots)|0, k\rangle \end{aligned} \tag{4.21}$$

Now we try to generalize the on-shell vertex in (2.10)

$$V = cA \cdot \partial X - \alpha'(\partial c)(\partial \cdot A) \tag{4.22}$$

to off-shell. The first term corresponds to $A \cdot a_{-1}c_1$ in (4.21). For the second term, (∂c) is $\frac{i}{2\alpha'}c_0$ in string field theory. Then

$$-\alpha'(\partial c)(\partial \cdot A) \to -\tfrac{i}{2}(\partial \cdot A)c_0 \tag{4.23}$$

Thus, the external state is modified to

$$\begin{aligned} |\psi\rangle &= [C + \phi c_1 + A \cdot a_{-1}c_1 + \widetilde{C}c_{-1}c_1 + (\widetilde{C}^* - \tfrac{i}{2}\partial \cdot A)c_0 \\ &\quad + \phi^* c_0 c_1 + A^* \cdot a_{-1}c_0 c_1 + C^* c_{-1}c_0 c_1 + \cdots]|0, k\rangle \end{aligned} \tag{4.24}$$

It seems different but once the Feynman-Siegel gauge $b_0 = 0$ is imposed, $\widetilde{C}^* = \frac{i}{2}\partial \cdot A$ and all other antifield vanish. It gets back to usual external state under Feynman-Siegel gauge. So, in following, we will start from most general state (4.21) to get ZJBV action which will help us to get the gauge covariant action in field theory.

Apply (4.21) to string field action (4.4), it gives the free terms and 3-point interactions for tachyons, YM gauge bosons, ghosts, antighosts, and their antifields. The free part is

$$\begin{aligned} S_2^{ZJBV} &= \langle V_2 | \Psi, Q\Psi \rangle \\ &= -\tfrac{1}{2}\phi(\Box + 2)\phi - \tfrac{1}{2}A_\mu \Box A^\mu + \widetilde{C}\Box C - 2i(\partial_\mu C)A^{*\mu} \\ &\quad -2(\widetilde{C}^*)^2 - 2i(\partial \cdot A)\widetilde{C}^* \end{aligned} \tag{4.25}$$

and the interaction part is

$$S_3^{ZJBV} = \tfrac{g}{3}\langle V_3|\Psi,\Psi,\Psi\rangle = S_3^{(0)} + S_3^{(1)} + S_3^{(2)} + S_3^{(3)} \tag{4.26}$$

where (to lowest order in Regge slope for those nonlocal factors $e^{\frac{1}{2}N_{00}^{rr}(P_i^2+m_i^2)}$; we will discuss them in the next section)

$$\begin{aligned}
S_3^{(0)} &= \tfrac{1}{3}\phi^3 + \phi A^2 + (\widetilde{C}^*)^2\phi - \tfrac{1}{2}[\widetilde{C}, C]\phi + \tfrac{1}{2}\{\widetilde{C}, C\}\widetilde{C}^* \\
&\quad +\{C,\phi^*\}\phi + C^2C^* + [\phi^*, C]\widetilde{C}^* + [A_\mu, C]A^{*\mu} && (4.27)\\
S_3^{(1)} &= \tfrac{i}{2}\partial_\mu\phi[A^\mu,\phi] + \tfrac{i}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)[A^\mu, A^\nu] \\
&\quad +\tfrac{i}{2}[\widetilde{C}^*, \partial_\mu\widetilde{C}^*]A^\mu + \tfrac{i}{4}\widetilde{C}[A_\mu, \partial^\mu C] - \tfrac{i}{4}\partial^\mu\widetilde{C}[A_\mu, C] \\
&\quad +\tfrac{i}{2}\phi^*(\{\partial_\mu C, A^\mu\} + \partial_\mu\{C, A^\mu\}) \\
&\quad +\tfrac{i}{2}A_\mu^*([C, \partial^\mu\widetilde{C}^*] - [\partial^\mu C, \widetilde{C}^*]) \\
&\quad +\tfrac{i}{2}A_\mu^*(\{C, \partial^\mu\phi] - [\partial^\mu C, \phi]) && (4.28)\\
S_3^{(2)} &= \phi(\partial_\mu A_\nu)(\partial^\nu A^\mu) + \tfrac{1}{2}\phi\{\partial_\mu(\partial\cdot A), A^\mu\} + \tfrac{1}{4}\phi(\partial\cdot A)^2 \\
&\quad +(\tfrac{1}{2}[\partial_\nu C, \partial_\mu A^\nu] - \tfrac{1}{2}[\partial_\mu\partial_\nu C, A^\nu] \\
&\qquad +\tfrac{1}{4}[C, \partial_\mu(\partial\cdot A)] - \tfrac{1}{4}[\partial_\mu C, (\partial\cdot A)])A^{*\mu} && (4.29)\\
S_3^{(3)} &= \tfrac{i}{6}(\partial^\mu\partial^\nu A_\lambda)[A_\mu, \partial^\lambda A_\nu] + \tfrac{i}{24}\partial^\mu(\partial\cdot A)[\partial^\nu A_\mu, A_\nu] && (4.30)
\end{aligned}$$

This gives the gauge fixed action after setting antifields to zero. Before setting them to zero, it is related to the usual ZJBV action by a canonical (with respect to the antibracket) transformation. Since such transformations can mix fields and antifields, the transformation itself (followed by setting antifields to zero) is one way to define the gauge-fixing procedure in this formalism. So, one way to find the gauge invariant action is to undo this transformation.

Another way is to take this action with antifields, drop all fields with nonvanishing ghost number, and then eliminate the remaining zero-ghost-number antifields (Nakanishi-Lautrup fields) by their equations of motion. However, the resulting action is kind of messy and has unusual gauge transformations.

The advantage of working with the entire ZJBV action is that it contains both the gauge invariant action and the gauge (BRST) transformations. Furthermore, canonical transformations perform field redefinitions (including antifield redefinitions that define the gauge fixing) in a way that preserves the (anti)bracket (as in ordinary quantum mechanics). Thus, we look for canonical transformations that produce the standard form for gauge transformations of the fields, a well as eliminate terms in the action that could normally be ignored "on shell".

Notice there are antifield-independent terms from gauge fixing in the ZJBV action of (4.25) and (4.26). So we have to find transformations to "undo" the gauge fixing. For example, the gauge transformation generated by $-\frac{i}{2}(\partial\cdot A)\widetilde{C}$ will cancel the gauge fixing term $\widetilde{C}\Box C$ because $(-\frac{i}{2}(\partial\cdot A)\widetilde{C}, -2i(\partial_\mu C)A^{*\mu}) = -\widetilde{C}\Box C$. Also notice that the ZJBV actions of (4.25) and (4.26) don't give the usual gauge transformations (from terms linear in antifields), so we also look for transformations to give them the usual form. For instance, the term $[\phi^*, C]\widetilde{C}^*$ will give unusual contributions for gauge transformations of ϕ and $\widetilde{C}$, but it can be canceled through the field redefinition generated by $\frac{1}{4}[\phi^*, C]\widetilde{C}$. We also look for terms that generate field redefinitions that cancel cubic antifield-independent terms that are proportional to the linearized field equations. For example, $\frac{1}{2}A^2\phi^*$ will generate the counter term $-\phi A^2 - \frac{1}{2}(\Box\phi)A^2$, which converts ϕA^2 into $-\frac{1}{2}(\Box\phi)A^2$, which will be part of the covariant interaction $\phi F_{\mu\nu}F^{\mu\nu}$.

The calculation is straightforward, but to find the complete transformation we need more steps, because some transformations applied to cancel terms we don't want will have byproducts to be canceled by further transformations. The complete transformation is given as follows: First, make the transformation generated by

$$G_g = -\tfrac{i}{2}(\partial\cdot A)\widetilde{C} + \tfrac{1}{16}\widetilde{C}[\Box A_\mu + \partial_\mu(\partial\cdot A), A^\mu] + \tfrac{1}{8}\widetilde{C}^2 C + \tfrac{1}{16}(\partial_\mu\widetilde{C})^2 C - \tfrac{i}{8}\widetilde{C}\{\partial\cdot A, \phi\} \tag{4.31}$$

to "undo" the gauge fixing. It is independent of antifields, and so can be identified with gauge fixing. Then we make the transformation

$$\begin{aligned} G_0 \;=\;& \tfrac{1}{4}[\phi^*, C]\widetilde{C} + \tfrac{1}{2}A^2\phi^* + \tfrac{1}{4}\{C,\phi\}C^* + \tfrac{1}{8}\{\phi, \widetilde{C}^*\}\widetilde{C} - \tfrac{1}{2}\{\phi, A^*_\mu\}A^\mu \\ & -\tfrac{i}{4}A^*_\nu[\partial^\nu A^\mu, A_\mu] + \tfrac{i}{8}\{C, A^*_\mu\}(\partial^\mu\widetilde{C}) - \tfrac{i}{8}[\widetilde{C}^*, A_\mu](\partial^\mu\widetilde{C}) \end{aligned} \tag{4.32}$$

This generator is linear in antifields, and so can be identified with a field redefinition. (However, there is some subtlety in that the Nakanishi-Lautrup fields in this form of ZJBV appear as antifields $\widetilde{C}^*$.) As the result of the above transformations, the action (up to 3-point terms and lowest order in Regge slope) can be written as

$$\begin{aligned} S \;=\;& S_2 + S_3 \\ =\;& \tfrac{1}{2}[\nabla_\mu, \phi][\nabla^\mu, \phi] - \phi^2 - F_{\mu\nu}F^{\mu\nu} - 2i[\nabla_\mu, C]A^{*\mu} - 2(\widetilde{C}^*)^2 + \{C, \phi^*\}\phi \\ & + C^2C^* + \tfrac{1}{3}\phi^3 + 2\phi F_{\mu\nu}F^{\mu\nu} - \tfrac{4}{3}F_\mu{}^\nu F_\nu{}^\lambda F_\lambda{}^\mu \end{aligned} \tag{4.33}$$

with $\nabla_\mu = \partial_\mu + \frac{i}{2}A_\mu$. Now it is explicitly gauge covariant (to this order) even off-shell! Thus the F^3 interaction appears explicitly (which was done only on shell before), and a new gauge invariant interaction term ϕF^2 is found. Furthermore, the YM gauge condition corresponding to the world-sheet Feynman-Siegel gauge is now known: The usual gauge-fixing function $\partial \cdot A$ of the Fermi-Feynman gauge is modified to

$$\partial\cdot A+\tfrac{i}{8}[\Box A_\mu+\partial_\mu(\partial\cdot A), A^\mu]+\tfrac{1}{4}\{\partial\cdot A,\phi\}+\tfrac{i}{8}\{\widetilde{C},C\}-\tfrac{i}{16}\{\Box\widetilde{C},C\}-\tfrac{i}{16}\{\partial_\mu\widetilde{C},\partial^\mu C\} \tag{4.34}$$

The additional gauge fixing terms simplify the F^3 and ϕF^2 interactions, and make the gauge fixed action symmetric in ghosts and antighosts [32].

4.4 High orders of Regge slope

This is not the end of the story, because we only made the action manifestly gauge invariant to lowest order in the Regge slope expanded from the nonlocal factors. Remember, in the 3-string vertex in (4.6), the Neumann coefficients $\frac{1}{2}N^{rr}_{00} = -\lambda$ will contribute nonlocal factors to interactions. That means the full interaction will have the form of replacing each (anti)field ψ_i in (4.26) by $e^{-\lambda(p_i^2+m_i^2)}\psi_i$. But the above canonical transformations can be performed in the same way except that the (anti)fields ψ_i in G_g and G_0 are replaced by $e^{\lambda(\Box_i-m_i^2)}\psi_i$. Then we will get the full action as in (4.33) while attaching the factor $e^{\lambda(\Box_i-m_i^2)}$ to each (anti)field ψ_i in the interaction part:

$$S_2^{full} = -\tfrac{1}{2}\phi(\Box+2)\phi + \tfrac{1}{4}\partial_{[\mu}A_{\nu]}\partial^{[\mu}A^{\nu]} - 2i(\partial_\mu C)A^{*\mu} - 2(\widetilde{C}^*)^2 \tag{4.35}$$

$$\begin{aligned} S_3^{full}(\lambda) \;=\;& \tfrac{i}{2}\partial_\mu\hat\phi[\hat A^\mu,\hat\phi] + \tfrac{i}{4}\widehat{F}_{\mu\nu}[\hat A^\mu,\hat A^\nu] + \tfrac{1}{3}\hat\phi^3 + [\hat A_\mu,\hat C]\hat A^{*\mu} + \{\hat C,\hat\phi^*\}\hat\phi \\ & +\hat C^2\hat C^* + 2\hat\phi\hat F_{\mu\nu}\hat F^{\mu\nu} - \tfrac{4}{3}\hat F_\mu^{\ \nu}\hat F_\nu^{\ \lambda}\hat F_\lambda^{\ \mu} \end{aligned} \tag{4.36}$$

where

$$\hat F_{\mu\nu} = \partial_{[\mu}\hat A_{\nu]} \tag{4.37}$$

and

$$\begin{aligned} &\hat\phi = e^{\lambda(\Box+2)}\phi, \quad \hat\phi^* = e^{\lambda(\Box+2)}\phi^*, \quad \hat A = e^{\lambda\Box}A, \quad \hat A^* = e^{\lambda\Box}A^* \\ &\hat C = e^{\lambda\Box}C, \quad \hat C^* = e^{\lambda\Box}C^*, \quad \hat{\widetilde{C}} = e^{\lambda\Box}\widetilde{C}, \quad \hat{\widetilde{C}}^* = e^{\lambda\Box}\widetilde{C}^* \end{aligned} \tag{4.38}$$

We now perform more field redefinitions to push these nonlocal factors into higher-point interactions and restore the usual gauge invariant action up to 3-point terms. Let's first expand the exponential factor $e^{\lambda(\Box_i-m_i^2)}$ to the first order. Then there are extra terms like $\phi^2[\lambda(\Box+2)\phi]$ from $\frac{1}{3}\phi^3$ to be absorbed. The naive guess is making the field redefinition through $G = \lambda\phi^2\phi^*$, which will give a counter term through the antibracket:

$$\delta S_3 = (G, S_2) = (\lambda\phi^2\phi^*, -\tfrac{1}{2}\phi(\Box + 2)\phi) = -\lambda\phi^2[(\Box + 2)\phi] \tag{4.39}$$

where we use S_2 to represent the free part and S_3 the interaction part in (4.33) (to lowest order in Regge slope).

Fortunately, it turns out this is almost the right guess. To the first order in Regge slope, the redefinition should come through

$$G = \lambda(\phi^*, S_3)\phi^* + \lambda(A^*_\mu, S_3)A^*_\mu + \lambda(C^*, S_3)(-\tfrac{i}{2})(\partial\cdot A) + \lambda(\partial\cdot A, S_3)(\tfrac{i}{2}C^*) \tag{4.40}$$

Then

$$\begin{aligned} (\lambda(\phi^*, S_3)\phi^*, S_2) &= \lambda(\phi^*, S_3)(\phi^*, S_2) + \lambda\phi^*((\phi^*, S_3), S_2) \\ &= -\lambda(\phi^*, S_3)(\Box + 2)\phi + \lambda\phi^*(S_2, (\phi^*, S_3)) \end{aligned} \tag{4.41}$$

Using the properties of antibrackets in (4.11) and the gauge invariant condition $(S_3, S_2) = 0$,

$$\begin{aligned} &-(S_2, (\phi^*, S_3)) + (\phi^*, (S_3, S_2)) + (S_3, (S_2, \phi^*)) = 0 \\ \Rightarrow\ &(S_2, (\phi^*, S_3)) = (S_3, (S_2, \phi^*)) = (S_3, (\Box + 2)\phi) \\ &= (-[C, \phi]\phi^*, (\Box + 2)\phi) = -[C, \phi](\Box + 2) \end{aligned} \tag{4.42}$$

Thus (4.41) gives

$$(\lambda(\phi^*, S_3)\phi^*, S_2) = -\lambda(\phi^*, S_3)(\Box + 2)\phi - \lambda\{C, (\Box + 2)\phi^*\}\phi \tag{4.43}$$

which will cancel the additional terms from the first-order expansions of $e^{\lambda(\Box - m^2)}$ for ϕ's and ϕ^*'s in the 3-point interactions. Similar calculations show that G in (4.40) does cancel all additional terms from the first-order expansions of $e^{\lambda(\Box_i - m_i^2)}$ for all (anti)fields: $\phi, \phi^*, A_\mu, A^*_\mu, C, C^*, \widetilde{C}, \widetilde{C}^*$ in S_3^{full}.

Basically, we can do it order by order, and here is the field redefinition for all orders:

$$\begin{aligned} G \;=\;& (\phi^*, \int_0^\lambda d\alpha S_3^{full}(\alpha))\phi^* + (A^*_\mu, \int_0^\lambda d\alpha S_3^{full}(\alpha))(A^*)^\mu \\ &+(C^*, \int_0^\lambda d\alpha S_3^{full}(\alpha))(-\tfrac{i}{2})(\partial\cdot A) \\ &+(\partial\cdot A, \int_0^\lambda d\alpha S_3^{full}(\alpha))(\tfrac{i}{2}C^*) \end{aligned} \tag{4.44}$$

The integral is easy to perform:

$$\int_0^\lambda d\alpha S_3^{full}(\alpha) = \tfrac{1}{(\Box_1-m_1^2)+(\Box_2-m_2^2)+(\Box_3-m_3^2)}(e^{\lambda(\Box_1-m_1^2)+\lambda(\Box_2-m_2^2)+\lambda(\Box_3-m_3^2)} - 1)S_3 \tag{4.45}$$

where the indices $1, 2, 3$ indicate the three fields in each term of S_3. The proof is very similar to the first-order case and we won't bother to give the details here.

Then we will have N-point interactions for any big N just from a 3-string interaction in SFT. This is because in the above calculation we only accounted for corrections up to 3-point, while the full transformed action should be

$$e^{\mathcal{L}_G} S = S + (G, S) + \tfrac{1}{2!}(G, (G, S)) + \cdots \tag{4.46}$$

Essentially, we can perform this mechanism perturbatively in higher-point interactions. We have not studied whether the nonlocal interactions can be eliminated at any finite order of perturbation, or whether this procedure is consistent nonperturbatively.

Chapter 5

Random lattice

In chapter 4, we clarified the Yang-Mills gauge condition corresponding to Feynman-Siegel gauge on the world-sheet. In this chapter, we will construct similar interactions as the Witten's string field theory on a random lattice. We combine two partons on a random lattice as a vector state. In the ladder approximation, we find that such states have $1/p^2$ propagators (after tuning the mass to vanish). We also construct some diagrams which are very similar to 3-string vertices in string field theory for the first oscillator mode. Attaching 3 such lattice states to these vertices, we get Yang-Mills and F^3 interactions up to 3-point as from bosonic string (field) theory. This gives another view of a gauge field as a bound state in a theory whose only fundamental fields are scalars.

5.1 Regge theory

As suggested by Regge, Regge poles might be relevant to the analysis of high-energy scattering. Many results about poles' locations and properties were obtained on the basis of analyticity assumptions, mostly in ϕ^3 theory [33]. A simpler consideration is to examine the high-energy behavior of scattering amplitudes directly by summing suitable sets of Feynman diagrams [34].

The two-particle elastic scattering amplitude $A(s,t)$ for an appropriate set of Feynman diagrams (e.g., ladders) can be of the form:

$$A(s,t) = \int d^4 k_i \prod_a \tfrac{1}{p_a^2+m^2} \sim \int d^4 k_i \int_0^\infty \prod_a d\beta_a e^{-\beta_a(p_a^2+m^2)/2} \tag{5.1}$$

where k_i are independent loop momenta, β_a are Schwinger parameters to exponentiate the propagators and the Mandelstam variables are

$$s = -(q_1 + q_2)^2 = -(q_3 + q_4)^2, \quad t = -(q_1 - q_3)^2 \tag{5.2}$$

As we will see, the only difference between an ordinary field theory and lattice string theory is the integration over the parameters β_a. In the lattice string case reviewed in the next section, they are fixed at $\beta_a = \alpha'$.

Integrating out Gaussian loop momenta,

$$A(s,t) \sim \int_0^\infty \prod_a d\beta_a \tfrac{N(\beta)}{[C(\beta)]^2} e^{-g(\beta)t - d(s,\beta)} \tag{5.3}$$

When $t \to \infty$, it is dominated by the region near $g(\beta) = 0$. So to make the coefficient of t vanish, one can set those β's to zero everywhere except in $g(\beta)$, which shortcircuits the diagram to eliminate the t dependence. Then the integration can be carried out to obtain the asymptotic behavior as $t \to 0$. For ladder graphs, the ladder with n rungs has an expression of the form:

$$A_n(s,t) \sim g^2 \tfrac{1}{t} [g^2 K(s) \ln t]^{n-1} \tag{5.4}$$

where $K(s)$ is just a self-energy diagram evaluated from a bubble in 2 fewer dimensions. So the asymptotic behavior comes from the sum of ladder diagrams:

$$\sum A_n(s,t)/(n-1)! = g^2 t^{\alpha(s)}, \qquad \alpha(s) = -1 + g^2 K(s) \tag{5.5}$$

which is the result associated with the Regge trajectory.

5.2 Bosonic lattice string review

The main difference between the lattice and continuum approaches to the string is that a lattice requires a scale, while conformal invariance of the continuum string includes scale invariance. To break the conformal invariance of the worldsheet, a term proportional to the area (the simplest scale-*variant* and coordinate-*invariant* property of the worldsheet) with coefficient (cosmological constant) μ is added to the string action. Furthermore, to describe the string interaction, the string coupling constant, which is counted by the integral of the worldsheet curvature R, should be included. So totally, the action is

$$S = \oint \frac{d^2\sigma}{2\pi} \sqrt{-g} \left[\tfrac{1}{\alpha'} g^{mn} \tfrac{1}{2} (\partial_m X \cdot \partial_n X) + \mu + (\ln \kappa) \tfrac{1}{2} R \right] \tag{5.6}$$

On the random lattice, this action can be written as

$$S_1 = \tfrac{1}{\widetilde{\alpha}'}\sum_{\langle ij\rangle}\tfrac{1}{2}(x_i - x_j)^2 + \mu\sum_i 1 + \ln\kappa\left(\sum_i 1 - \sum_{\langle ij\rangle} + \sum_J 1\right) \tag{5.7}$$

where j are vertices, $\langle ij\rangle$ the links (edges), and J the plaquets (faces, planar loops) of the lattice. The functional integration over the worldsheet metric in usual string theory is repalced by a sum over Feynman diagrams. The positions of vertices are integrated (except external vertices; alternatively, external states will be introduced to calculate the full amplitudes, as shown in later sections):

$$A = \sum\int\prod dx\; e^{-S_1} = \sum e^{-\mu\sum_i 1}\int dx\;\prod_{ij} e^{-\frac{1}{2\widetilde{\alpha}'}(x_i - x_j)^2} \tag{5.8}$$

Now, by identifying the lattice with a position-space Feynman diagram, we can find the underlying field theory as follows: Vertices of the lattice correspond to those of Feynman diagram and links to propagators; the 1/N expansion is associated with the faces of the worldsheet polyhedra with $U(N)$ indices. Thus, the area term (counting the number of vertices) in the lattice action (5.7) gives the coupling constant factor for each vertex in the field action, and the worldsheet curvature term gives the string coupling 1/N of the topological expansion [35]. Explicitly, the action of an n-point-interaction scalar-field action is

$$S_2 = N\; tr\int\frac{d^Dx}{(2\pi\widetilde{\alpha}')^{D/2}}\left(\tfrac{1}{2}\phi e^{-\widetilde{\alpha}'\Box/2}\phi - G\tfrac{1}{n}\phi^n\right) \tag{5.9}$$

with

$$G = e^{-\mu}, \qquad \tfrac{1}{N} = \kappa \tag{5.10}$$

The interaction ϕ^n can be chosen arbitrarily; restrictions may come from consistency of the worldsheet continuum limit [36]. In this paper, we will focus on the minimum coupled lattice, ϕ^3 theory, but the calculation for a ϕ^4 interaction is pretty much the same.

5.3 Ladder graphs and Regge trajectories

In this section we review the ladder graphs responsible for a Regge trajectory $\alpha(s)$, and compare with those done in the early days of Regge theory. Since somewhat similar procedures will be used in following sections, we give details

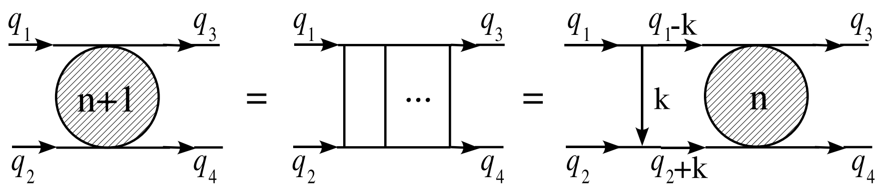

Figure 5.1: Ladder diagrams

in this section.

Consider 4-point functions in the parton theory with Gaussian propagators and cubic interaction ϕ^3 with coupling constant λ. The amplitude is evaluated by solving the Bethe-Salpeter equation in the ladder approximation with two incoming particles of momenta q_1 and q_2 and two outgoing particles of momenta q_3 and q_4, as depicted in Fig. 5.1.

The two-particle propagator Δ satisfies the Bethe-Salpeter equation in D dimentions

$$\Delta = 1 + e^{-H}\Delta \tag{5.11}$$

where e^{-H} sticks an extra rung on the sum of ladders (as in Fig. 1). Explicitly, it can be written as

$$e^{-H} = (\text{rung propagator}) \times (\text{two “side” propagators}) \tag{5.12}$$

with integration over either loop momentum or positions of vertices. The propagator is given by

$$\Delta = \frac{1}{1-e^{-H}} = \sum (e^{-H})^n \tag{5.13}$$

Here, we will replace integrals with operator expressions as in usual string theory. Thus, adding the two sides followed by adding the rung in (5.12) is performed by the operator

$$e^{-H} = e^{-(x_1-x_2)^2/2} e^{-(p_1^2+p_2^2)/2} \tag{5.14}$$

where the p's and x's are now the operators for the two particles. Separating p's and x's into average and relative coordinates,

$$p_{1,2} = P \pm p, \quad x_{1,2} = \tfrac{1}{2}X \pm \tfrac{1}{2}x \tag{5.15}$$

(5.14) is then

$$e^{-H} = e^{-x^2/2}e^{-P^2+p^2} = e^{-x^2/2}e^{-p^2}e^{s/4} \tag{5.16}$$

where

$$P^2 = -\tfrac{1}{4}(q_1+q_2)^2 = -\tfrac{1}{4}(q_3+q_4)^2 = -s/4$$

By a similarity transformation, we can put half of one exponential on each side,

$$e^{-H} \quad \to \quad e^{s/4}e^{-x^2/4}e^{-p^2}e^{-x^2/4} \quad or \quad e^{s/4}e^{-p^2/2}e^{-x^2/2}e^{-p^2/2} \tag{5.17}$$

To write H as a manifestly Hermitian expression, we apply the Baker-Campbell-Haussdorf theorem to combine the exponentials into a single one. Because the exponents, $\frac{1}{2}x^2$, $\frac{1}{2}p^2$, satisfy the commutation relations of raising and lowering operators and the Baker-Campbell-Haussdorf theorem requires only commutators, we can use the representation

$$\tfrac{1}{2}x^2 \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tfrac{1}{2}p^2 \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad i\tfrac{1}{2}\{x,p\} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.18}$$

So, in general,

$$\begin{aligned} & e^{-\alpha p^2/2}e^{-\beta x^2/2}e^{-\alpha p^2/2} \quad \to \quad e^{-\left(\begin{smallmatrix} 0 & 0 \\ \alpha & 0 \end{smallmatrix}\right)}e^{-\left(\begin{smallmatrix} 0 & \beta \\ 0 & 0 \end{smallmatrix}\right)}e^{-\left(\begin{smallmatrix} 0 & 0 \\ \alpha & 0 \end{smallmatrix}\right)} \\ = & \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix}\begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} = \begin{pmatrix} 1+\alpha\beta & -\beta \\ -\alpha(2+\alpha\beta) & 1+\alpha\beta \end{pmatrix} \\ = & e^{-\left(\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right)} = cosh(\sqrt{ab}) - \frac{sinh(\sqrt{ab})}{\sqrt{ab}}\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \end{aligned} \tag{5.19}$$

Then, H in (5.17) becomes the hamiltonian of a harmonic oscillator

$$\begin{aligned} H &= -\tfrac{1}{4}s - ln\lambda^2 + \omega(m\omega\tfrac{1}{2}x^2 + \tfrac{1}{m\omega}\tfrac{1}{2}p^2) \\ &= -\tfrac{1}{4}s - ln\lambda^2 + \tfrac{\omega}{2}D + \omega a^\dagger \cdot a \end{aligned} \tag{5.20}$$

with λ restored. If we work in coordinate space, as in the following sections,

$$\alpha = \tfrac{1}{2}, \beta = 2 \quad \Rightarrow \quad \omega = ln(2+\sqrt{3}), \quad m\omega = \frac{\sqrt{3}}{2} \tag{5.21}$$

we can find the Regge trajectory from the spectrum of this harmonic oscillator. The harmonic oscillators (a D-vector) can be interpreted as the oscillators in the usual string theory (but only one such vector) as follows: The positions of the two partons in the Bethe-Salpeter equation are two adjacent points on the random lattice, and the relative coordinate represents the first order derivative of $x(\sigma)$ corresponding to the first oscillator. (A similar model was considered in [37].)

Taking $(D/2)\omega$ as the ground-state energy and integer excitation J as the (maximum) spin of the D oscillators (acting with J vector oscillators on the vacuum), the "energy" of the harmonic oscillator Hamiltonian $m\omega^2\frac{1}{2}x^2 + (1/m)\frac{1}{2}p^2$ can be identified as $(J + D/2)\omega$. Since the Bethe-Salpeter equation corresponds to perturbatively solving a Schrödinger equation with free Hamiltonian 1, potential e^{-H} and vanishing total energy $e^{-H} - 1 = 0$, it gives $H = 2\pi in$,

$$2\pi in = -\tfrac{1}{4}s - ln(\lambda^2) + \omega(J + \tfrac{1}{2}D) \tag{5.22}$$

So we have the trajectory $J = \alpha(s)$

$$\alpha(s) = -\tfrac{1}{2}D + \frac{1}{\omega}[\tfrac{1}{4}s + ln(\lambda^2) + 2\pi in] \tag{5.23}$$

The real part of (5.23)

$$\alpha(s) = -\tfrac{1}{2}D + \frac{1}{\omega}[\tfrac{1}{4}s + ln(\lambda^2)] \tag{5.24}$$

is linear with positive slope. The real pole gave us the asymptotic behavior, while complex poles do not affect the Regge trajectory, as shown in [26]. We require the vertical intercept of this Regge trajectory, which is given by $s = 0$, to be $J = 1$, so the corresponding spin-one particle is massless. Thus $\alpha(0) = 1$ gives

$$e^{-\omega(D+2)}\lambda^4 = 1 \tag{5.25}$$

(In the usual continuum approach, this constraint, as well as $D = 26$, are found perturbatively, but in the lattice approach they would be nonperturbative, so we impose them by hand.)

There are several ways to interpret the group theory of this state: (1) We can examine only color-singlet states (the partons are N by N matrices of U(N) color); then we should take the color trace of this vector, which would make it Abelian. (2) If we examine color-nonsinglets, the vector is in the adjoint representation, and so represents a Reggeized bound-state gauge field of color, and thus not a true string state. (3) If we introduce a second type of scalar parton which is in the fundamental representation of both color and a second, "flavor" symmetry, we can consider ladders where these scalar "quarks" run along the outside, giving an open string instead of a closed one [35]. Then the vector is the gauge field of this flavor symmetry. It is really only in this last case that string theory implies the state is massless.

5.4 External vertex operator for gauge field

Now we are ready to introduce the ground state and first excited state for the harmonic oscillator in ladders (5.20):

$$e^{ik\cdot(x_i+x_j)/2}|0\rangle \quad and \quad \epsilon\cdot(x_i-x_j)e^{ik\cdot(x_i+x_j)/2}|0\rangle$$

They are the very same as the vertex operators $e^{ik\cdot X}$ and $\epsilon\cdot\partial X e^{ik\cdot X}$ in the bosonic string, except latticized.

Defining

$$x = x_i - x_j = \sqrt{\tfrac{1}{2m\omega}}(a+a^\dagger)$$

where $a, a^\dagger$ are creation and annihilation operators of the harmonic oscillator in ladders, the first excitation can also be written as

$$\sqrt{\tfrac{1}{2m\omega}}\epsilon\cdot a^\dagger|0,k\rangle \tag{5.26}$$

with

$$|0,k\rangle = e^{ik\cdot(x_i+x_j)/2}|0\rangle \tag{5.27}$$

As in usual string or string field theory, this first excited state should be a massless state and the propagator should have a massless pole. To check it, let's consider the amplitude for one incoming and one outgoing state with momenta k and k' respectively. In the ladder approximation as reviewed in the last section, we have to evaluate the amplitude depicted in Fig. 5.1:

$$\mathcal{A} = -\tfrac{1}{2m\omega}\langle 0,k|(\epsilon_1\cdot a)\Delta(\epsilon_2\cdot a^\dagger)|0,k'\rangle \tag{5.28}$$

with the definitions $k = q_1+q_2$ and $k' = q_3+q_4$. The calculation is pretty similar to the previous section. The two-particle propagator $\Delta = \frac{1}{1-e^{-H}}$ satisfying the Bethe-Salpeter equation as in (5.11) and H is expressed by annihilation (creation) operators as in (5.20).

In such ladder approximations, the propagator should be written as a summation of all ladders

$$\Delta = \tfrac{1}{1-e^{-H}} = \sum(e^{-H})^n \tag{5.29}$$

Thus, using the commutator $[a_\mu, a^\dagger_\nu] = \delta_{\mu,\nu}$ and integrating out X's, the amplitude in (5.28) is

$$\begin{aligned}
\mathcal{A} &= -\tfrac{1}{2m\omega}\langle 0,k|(\epsilon_1\cdot a)\tfrac{1}{1-e^{-H}}(\epsilon_2\cdot a^\dagger)|0,k'\rangle \\
&= -\tfrac{1}{2m\omega}\tfrac{\epsilon_1\cdot\epsilon_2}{1-e^{s/4}\lambda^2 e^{-\omega(1+D/2)}}\delta^D(k+k') \qquad (5.30)
\end{aligned}$$

As given in (5.25), the real Regge trajectory $\alpha(0)=1$ gives $e^{-\omega(D+2)}\lambda^4=1$. Then (5.30)

$$\begin{aligned}
\mathcal{A} &= -\tfrac{1}{2m\omega}\tfrac{\epsilon_1\cdot\epsilon_2}{1-e^{-k^2/4}}\delta^D(k+k') \qquad (5.31)\\
&= -\tfrac{2}{m\omega}\tfrac{1}{k^2}\epsilon_1\cdot\epsilon_2 \quad , \quad k^2\to 0 \qquad (5.32)
\end{aligned}$$

has a massless pole.

This result can also be seen from the ladder integration if we rewrite (5.28) as

$$\begin{aligned}
\mathcal{A} &= -\tfrac{1}{2m\omega}\sum_n\langle 0,k|(\epsilon_1\cdot a)(e^{-H})^n(\epsilon_2\cdot a^\dagger)|0,k'\rangle \\
&= -\tfrac{1}{2m\omega}\sum_n A_n \qquad (5.33)
\end{aligned}$$

Here A_n is the amplitude for a single ladder with n loops (including external loops)

$$A_n=\int(\prod_{i=0}^n d^Dx_i d^Dy_i)\langle 0,k|(\epsilon_1\cdot a)|x_0,y_0\rangle[\prod_{i=1}^n(e^{-H_i})]\langle x_n,y_n|(\epsilon_2\cdot a^\dagger)|0,k'\rangle \qquad (5.34)$$

with

$$\begin{aligned}
\langle 0,k|(\epsilon_1\cdot a)|x_0,y_0\rangle &= \epsilon_1\cdot(x_0-y_0)e^{-\frac{m\omega}{2}(x_0-y_0)^2}e^{ik\cdot\frac{(x_0+y_0)}{2}} \\
\langle x_n,y_n|(\epsilon_2\cdot a^\dagger)|0,k'\rangle &= \epsilon_2\cdot(x_n-y_n)e^{-\frac{m\omega}{2}(x_n-y_n)^2}e^{ik'\cdot\frac{(x_n+y_n)}{2}} \qquad (5.35)
\end{aligned}$$

according to the definition of the ground state of the harmonic oscillator, and

$$\begin{aligned}
H_i &= \langle x_{i-1},y_{i-1}|H|x_i,y_i\rangle \\
&= \ln(\lambda^{-2})+\tfrac{1}{4}s+\tfrac{1}{4}(x_{i-1}-y_{i-1})^2+\tfrac{1}{2}(x_{i-1}-x_i)^2 \\
&\quad +\tfrac{1}{2}(y_{i-1}-y_i)^2+\tfrac{1}{4}(x_i-y_i)^2 \qquad (5.36)
\end{aligned}$$

Doing the Gaussian integrals for x_i's and y_i's, (5.34) becomes

$$A_n = -\tfrac{\epsilon_1 \cdot \epsilon_2}{2m\omega}[\lambda^2 e^{-k^2/4} e^{-\omega(1+D/2)}]^n \delta^D(k+k') \tag{5.37}$$

which gives the same massless pole as in (5.30).

This massless pole means the first excited state (5.26) is a massless state, and has the same propagator as the YM gauge field in Feynman gauge. We will discuss in the following sections how this generalizes to interactions.

5.5 The 3-string vertex

To get the 3-point gauge interaction in YM fields, we have to find a way to join three states. One analogue is Witten's open string field theory, in which stings interact by identifying the right half of each string with the left half of the next one. On the lattice we need to sum over an infinite number of diagrams representing this situation, each giving a result very similar to the 3-vector vertex in string field theory. Here we will give the 2 simplest examples to show that they give the same interaction as the usual YM field.

Similarly to string field theory (SFT), the 3-state interaction can be written as

$$\mathcal{A}_3^{(0)} = (\langle 0|_1 \otimes \langle 0|_2 \otimes \langle 0|_3)|V_1 V_2 V_3 \widehat{F}|0\rangle \tag{5.38}$$

Using the definitions

$$\begin{aligned} x_1 + x_2 = X \quad &, \quad x_1 - x_2 = x; \\ y_1 + y_2 = Y \quad &, \quad y_1 - y_2 = y; \\ z_1 + z_2 = Z \quad &, \quad z_1 - z_2 = z; \end{aligned} \tag{5.39}$$

$$V_1 = \epsilon_1 \cdot x e^{ik\cdot X/2}, \quad V_2 = \epsilon_2 \cdot y e^{ik\cdot Y/2} \quad \textit{and} \quad V_3 = \epsilon_3 \cdot z e^{ik\cdot Y/2}$$

are the external vertex operators for massless fields as considered in the previous section. In the operator formulation,

$$x = \sqrt{\tfrac{1}{2m\omega}}(a+a^\dagger), \quad y = \sqrt{\tfrac{1}{2m\omega}}(b+b^\dagger), \quad z = \sqrt{\tfrac{1}{2m\omega}}(c+c^\dagger)$$

where $a(a^\dagger)$, $b(b^\dagger)$, $c(c^\dagger)$ are annihilation (creation) operators for three independent harmonic oscillators in the ladder approximation.

The simplest figure for the 3-string lattice vertex is shown in Fig. 5.2. Then the 3-state vertex can be constructed with annihilation (creation) operators of

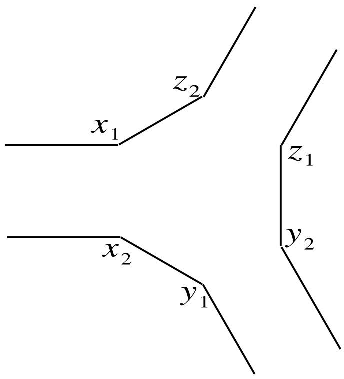

Figure 5.2: The interaction lattice of order λ^0 with the vertex given by $\widehat{F}$.

ladders after integrating out X, Y, Z

$$\begin{aligned}
\widehat{F} &= \int d^D X d^D Y d^D Z e^{-\frac{1}{2}[x_1-z_2)^2+(y_1-x_2)^2+(z_1-y_2)^2]+\frac{i}{2}(k_1\cdot X+k_2\cdot Y+k_3\cdot Z)} \\
&= e^{-\frac{1}{6}(x+y+z)^2-\frac{i}{6}(x\cdot k_{23}+y\cdot k_{31}+z\cdot k_{12})} \qquad (5.40)
\end{aligned}$$

where x, y, z can be expressed with annihilation (creation) operators $a(a^\dagger)$, $b(b^\dagger)$,$c(c^\dagger)$ and $k_{ij} = k_i - k_j$.

Thus the amplitude in (5.38) can be evaluated using commutators of 3 annihilation (creation) operators and the Baker-Campbell-Haussdorf theorem. First we write

$$-\tfrac{1}{6}(x+y+z)^2 = -\tfrac{1}{6}\tfrac{1}{2m\omega}[(a^\dagger+b^\dagger+c^\dagger)^2+(a+b+c)^2+\{a+b+c, a^\dagger+b^\dagger+c^\dagger\}] \quad (5.41)$$

It is noticed that the ingredients of exponents satisfy the commutation relations of raising and lowering operators of SU(1,1). We use the representation

$$\begin{aligned}
&\tfrac{i}{6}(a+b+c)^2 \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tfrac{i}{6}(a^\dagger+b^\dagger+c^\dagger)^2 \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
&-\tfrac{1}{6}\{a+b+c, a^\dagger+b^\dagger+c^\dagger\} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (5.42)
\end{aligned}$$

to calculate the commutators when we apply the Baker-Campbell-Haussdorf theorem. Then

$$
\begin{aligned}
e^{-\frac{1}{6}(x+y+z)^2} &= e^{\begin{pmatrix} \frac{1}{2m\omega} & \frac{i}{2m\omega} \\ \frac{1}{2m\omega} & \frac{-1}{2m\omega} \end{pmatrix}} = \begin{pmatrix} 1+\frac{1}{2m\omega} & \frac{i}{2m\omega} \\ \frac{i}{2m\omega} & 1-\frac{1}{2m\omega} \end{pmatrix} \\
&= e^{\left(\begin{smallmatrix} 0 & 0 \\ \alpha & 0 \end{smallmatrix}\right)} e^{-\left(\begin{smallmatrix} \beta & 0 \\ 0 & -\beta \end{smallmatrix}\right)} e^{-\left(\begin{smallmatrix} 0 & \alpha \\ 0 & 0 \end{smallmatrix}\right)} \\
&= \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} e^{\beta} & 0 \\ 0 & e^{-\beta} \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\beta} & \alpha e^{\beta} \\ \alpha e^{\beta} & \alpha^2 e^{\beta} + e^{-\beta} \end{pmatrix}
\end{aligned}
\tag{5.43}
$$

which gives

$$\alpha = \tfrac{i}{1+2m\omega}, \quad \beta = \ln(1 + \tfrac{1}{2m\omega}) \tag{5.44}$$

Thus,

$$
\begin{aligned}
e^{-\frac{1}{6}(x+y+z)^2}|0\rangle &= e^{\alpha\frac{i}{6}(a^\dagger+b^\dagger+c^\dagger)^2} e^{-\frac{\beta}{6}\{a+b+c,a^\dagger+b^\dagger+c^\dagger\}} e^{\alpha\frac{i}{6}(a+b+c)^2}|0\rangle \\
&= e^{\alpha\frac{i}{6}(a^\dagger+b^\dagger+c^\dagger)^2} e^{-\frac{D}{2}\beta}|0\rangle
\end{aligned}
\tag{5.45}
$$

for D-dimensional spacetime. Finally, using the Baker-Campbell-Haussdorf theorem again to write

$$e^{-\frac{i}{6\sqrt{2m\omega}}(a+a^\dagger)\cdot k_{23}} = e^{-\frac{i}{6\sqrt{2m\omega}}a^\dagger\cdot k_{23}} e^{-\frac{i}{6\sqrt{2m\omega}}a\cdot k_{23}} e^{\frac{1}{2}(-\frac{i}{6\sqrt{2m\omega}})^2 k_{23}^2}, \quad etc., \tag{5.46}$$

we find the 3-state interaction for the above massless state as

$$
\begin{aligned}
\mathcal{A}_3^{(0)} &= (\tfrac{1}{2m\omega})^{3/2} e^{-\frac{D}{2}\beta} \langle 0|(\epsilon_1\cdot a)(\epsilon_2\cdot b)(\epsilon_3\cdot c) \\
&\qquad \times e^{-\frac{i}{6\sqrt{2m\omega}}[(a+a^\dagger)\cdot k_{23}+(b+b^\dagger)\cdot k_{31}+(c+c^\dagger)\cdot k_{12}]} e^{\alpha\frac{i}{6}(a^\dagger+b^\dagger+c^\dagger)^2}|0\rangle \\
&= \kappa\Big\{\frac{-\frac{i}{6\sqrt{2m\omega}}\alpha}{3} i[(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot k_{12}) + permutations] \\
&\qquad + (-\tfrac{i}{6\sqrt{2m\omega}})^3(\epsilon_1\cdot k_{23})(\epsilon_2\cdot k_{31})(\epsilon_3\cdot k_{12})\Big\}
\end{aligned}
\tag{5.47}
$$

where

$$
\begin{aligned}
\kappa &= (\tfrac{1}{2m\omega})^{3/2} e^{-\frac{D}{2}\beta} e^{\frac{1}{2}(-\frac{i}{6\sqrt{2m\omega}})^2(k_{23}^2+k_{31}^2+k_{12}^2)} \\
&= (\tfrac{1}{2m\omega})^{3/2} e^{-\frac{D}{2}\beta} e^{-\frac{1}{48m\omega}(k_1^2+k_2^2+k_3^2)}
\end{aligned}
\tag{5.48}
$$

The result is the very same as the usual YM and F^3 3-point interactions as obtained from bosonic open string field theory in Feynman-Siegel gauge, except for the different ratio between the coefficients of the F^2 term and the F^3 term. Also, (5.47) has nonlocal coupling factors κ as in bosonic open string field theory.

Also, instead of using the operators of harmonic oscillators, direct Gaussian integration gives exactly same result as above for the diagram in Fig. 5.2. (5.38) can be written as

$$\mathcal{A}_3^{(0)} = \int d^D x d^D y d^D z \langle 0|V_1 V_2 V_3|x,y,z\rangle\langle x,y,z|\widehat{F}|0\rangle \tag{5.49}$$

Substituting (5.35) into it and integrating out all x_1, x_2, y_1, y_2, z_1 and z_2, we get the same 3-point vertex for gauge bosons as (5.47).

Another 3-string lattice vertex is shown in Fig. 5.3, which is order λ^4 in the lattice coupling. Then the 3-point amplitude is the same as in (5.38) with

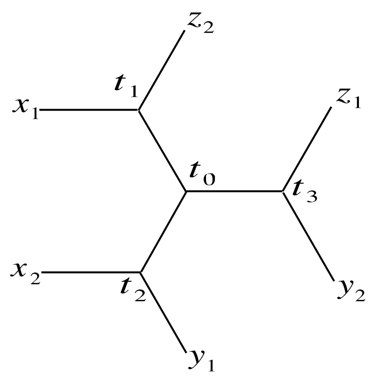

Figure 5.3: The interaction lattice of order λ^4 with the vertex given by $\widehat{G}$.

different 3-state vertex

$$\begin{aligned}\widehat{G} &= \int d^D X d^D Y d^D Z d^D t_0 d^D t_1 d^D t_2 d^D t_3 e^{-\frac{1}{2}[(t_1-t_0)^2+(t_2-t_0)^2+(t_3-t_0)^2]} \\ &\quad \times e^{-\frac{1}{2}[(x_1-t_1)^2+(x_2-t_2)^2+(y_1-t_2)^2+(y_2-t_3)^2+(z_1-t_3)^2+(z_2-t_1)^2]} e^{\frac{i}{2}(k_1\cdot X+k_2\cdot Y+k_3\cdot Z)} \\ &= e^{-\frac{1}{10}(x^2+y^2+z^2)-\frac{1}{20}(x+y+z)^2-\frac{i}{10}(x\cdot k_{23}+y\cdot k_{31}+z\cdot k_{12})}\end{aligned} \tag{5.50}$$

With the same external massless state of section 5.4, the 3-state interaction is

$$\mathcal{A}_3^{(1)} = \lambda^4(\langle 0|_1 \otimes \langle 0|_2 \otimes \langle 0|_3)|V_1 V_2 V_3 \widehat{G}|0\rangle \tag{5.51}$$

In the operator formalism, the computation of $\mathcal{A}_3^{(1)}$ is a little trickier. Introduce three new variables through an orthogonal rotation:

$$
\begin{aligned}
x' &= \tfrac{1}{\sqrt{3}}(x+y+z) \\
y' &= \tfrac{1}{\sqrt{2}}(x-y) \\
z' &= \tfrac{1}{\sqrt{6}}(x+y-2z)
\end{aligned}
\tag{5.52}
$$

Thus

$$
e^{-\frac{1}{10}(x^2+y^2+z^2)-\frac{1}{20}(x+y+z)^2} = e^{-\frac{1}{4}x'^2-\frac{1}{10}(y'^2+z'^2)} \tag{5.53}
$$

Also, 3 pairs of new annihilation (creation) operators

$$
\begin{aligned}
a' &= \tfrac{1}{\sqrt{3}}(a+b+c) \quad , \quad a'^\dagger = \tfrac{1}{\sqrt{3}}(a^\dagger+b^\dagger+c^\dagger) \\
b' &= \tfrac{1}{\sqrt{2}}(a-b) \quad , \quad b'^\dagger = \tfrac{1}{\sqrt{2}}(a^\dagger-b^\dagger) \\
c' &= \tfrac{1}{\sqrt{6}}(a+b-2c) \quad , \quad c'^\dagger = \tfrac{1}{\sqrt{6}}(a^\dagger+b^\dagger-2c^\dagger)
\end{aligned}
\tag{5.54}
$$

are introduced, which are independent of each other because $[a', b'^\dagger] = [a', c'^\dagger] = 0$, etc. Then

$$
\begin{aligned}
e^{-\frac{1}{4}x'^2} &= e^{-\frac{1}{4}\frac{1}{2m\omega}(a'^2+a'^{\dagger 2}+\{a',a'^\dagger\})} \\
e^{-\frac{1}{10}y'^2} &= e^{-\frac{1}{10}\frac{1}{2m\omega}(b'^2+b'^{\dagger 2}+\{b',b'^\dagger\})} \\
e^{-\frac{1}{10}x'^2} &= e^{-\frac{1}{10}\frac{1}{2m\omega}(c'^2+c'^{\dagger 2}+\{c',c'^\dagger\})}
\end{aligned}
\tag{5.55}
$$

The ingredients of each exponent satisfy the commutation relations of raising and lowering operators for SU(1,1), We use the same representation as in (5.42):

$$
\begin{aligned}
\tfrac{i}{2}(a')^2 \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tfrac{i}{2}(a'^\dagger)^2 \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
-\tfrac{1}{2}\{a', a'^\dagger\} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}
\tag{5.56}
$$

and, with the same procedure, find

$$
e^{-\frac{1}{4}x'^2} = e^{-\frac{1}{4}\frac{1}{2m\omega}(a'^2+a'^{\dagger 2}+\{a',a'^\dagger\})} = e^{\alpha_1\frac{i}{2}a'^{\dagger 2}} e^{-\alpha_2\frac{1}{2}\{a',a'^\dagger\}} e^{\alpha_1\frac{i}{2}a'^2} \tag{5.57}
$$

Similarly,

$$
e^{-\frac{1}{10}y'^2} = e^{-\frac{1}{10}\frac{1}{2m\omega}(b'^2+b'^{\dagger 2}+\{b',b'^\dagger\})} = e^{\beta_1\frac{i}{2}b'^{\dagger 2}} e^{-\beta_2\frac{1}{2}\{b',b'^\dagger\}} e^{\beta_1\frac{i}{2}b'^2} \tag{5.58}
$$

$$e^{-\frac{1}{10}z'^2} = e^{-\frac{1}{10}\frac{1}{2m\omega}(c'^2+c'^{\dagger 2}+\{c',c'^\dagger\})} = e^{\beta_1\frac{i}{2}c'^{\dagger 2}}e^{-\beta_2\frac{1}{2}\{c',c'^\dagger\}}e^{\beta_1\frac{i}{2}c'^2} \tag{5.59}$$

Here $\alpha_1, \alpha_2, \beta_1, \beta_2$ are defined as:

$$\begin{aligned} \alpha_1 &= \tfrac{i}{1+2(2m\omega)} \quad , \quad \alpha_2 = \ln[1+\tfrac{1}{2(2m\omega)}] \\ \beta_1 &= \tfrac{i}{1+5(2m\omega)} \quad , \quad \beta_2 = \ln[1+\tfrac{1}{5(2m\omega)}] \end{aligned} \tag{5.60}$$

Obviously, annihilation operators a', b', c' also annihilate the vacuum $|0\rangle$ and

$$\begin{aligned} e^{-\frac{1}{10}(x^2+y^2+z^2)-\frac{1}{20}(x+y+z)^2}|0\rangle &= Ce^{i\frac{\alpha_1}{6}(a+b+c)^2+i\frac{\beta_1}{3}(a^{\dagger 2}+b^{\dagger 2}+c^{\dagger 2}-a^\dagger\cdot b^\dagger-b^\dagger\cdot c^\dagger-c^\dagger\cdot a^\dagger}|0\rangle \\ C &= e^{-\frac{\alpha_2}{2}D-\frac{\beta_2}{2}D-\frac{\beta_2}{2}D} \end{aligned} \tag{5.61}$$

Finally, using the Baker-Campbell-Haussdorf theorem directly, up to a constant,

$$\begin{aligned} &\mathcal{A}_3^{(1)} = \lambda^4\langle 0|V_1V_2V_3\widehat{G}|0\rangle \\ \propto\ &\kappa'\{\tfrac{4+40m\omega}{1+4m\omega}[(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot k_{12}) + \mathit{permutations}] + (\epsilon_1\cdot k_{23})(\epsilon_2\cdot k_{31})(\epsilon_3\cdot k_{12})\} \end{aligned} \tag{5.62}$$

with

$$\kappa' = i\lambda^4 e^{-\frac{3}{40}\frac{1}{1+10m\omega}(k_1^2+k_2^2+k_3^2)} \tag{5.63}$$

The exponents of k_i^2's in κ' will vanish if it is on-shell but will make coupling factors nonlocal off-shell. Again, this result can be obtained by Gaussian integration directly, as in (5.49). We won't go through the details.

It is easy to notice that both vertices $\widehat{F}$ in (5.40) and $\widehat{G}$ in (5.50) give some gauge-fixed interactions for the massless state constructed from partons, as in Witten's bosonic open string field theory, which will be discussed in the next section. Their comparison will be interesting because it will give another view of string field theory, from the lattice.

5.6 Comparison to string field theory

In this section, we will compare the two 3-state vertices mentioned in the last section and the 3-state coupling from them with those in SFT. As we will notice, if all oscillator modes but the zeroth and first are truncated, the structure of 3-state vertices $\widehat{F}$ in (5.40) and $\widehat{G}$ in (5.50) seem similar to the 3-string vertex from Witten's interaction in SFT, except for different coefficients.

In above sections, the scale of the lattice was set to 1, which leads to the slope of the Regge trajectory $\frac{1}{4\omega}$. So before comparing with string field theory, we have to restore the scale of the lattice to match the slope with the Regge slope from usual string theory (or string field theory).

We use the lattice actions (5.7), with the lattice scale $\tilde{\alpha}'$. The calculations in previous sections are unchanged except for rescaling the momenta by

$$k_i \to \sqrt{\tilde{\alpha}'} k_i \tag{5.64}$$

and renormalizing the lattice coupling by

$$\lambda \to \lambda' \tag{5.65}$$

Then the real Regge trajectory is

$$\alpha(s) = -\tfrac{1}{2}D + \frac{1}{\omega}[\tfrac{\tilde{\alpha}'}{4}s + ln(\lambda'^2)] \tag{5.66}$$

with the slope $\frac{\tilde{\alpha}'}{4\omega}$. Setting it to be the same as the Regge slope from string theory, which is α', we need the lattice scale

$$\tilde{\alpha}' = 4\omega\alpha'$$

The intercept condition will be the same as (5.25) but replacing λ by λ'.

It is easy to see the propagator (5.28) for the gauge boson in the lattice string still has a massless pole. Also, the lattice rescaling did nothing to either 3-string vertex but change the scale of momenta and so change the ratio between coefficients of F^2 terms and F^3 terms in 3-point amplitudes.

In string field theory, the general 3-string interaction can be interpreted as

$$\langle h_1[\vartheta_A]h_2[\vartheta_b]h_3[\vartheta_c]\rangle = (\langle A|_1 \otimes \langle B|_2 \otimes \langle C|_3)|V_{123}\rangle \tag{5.67}$$

where ϑ_i is the vertex operator for each external state and $h_i(z)$ is the conformal mapping from each string strip to the complex plane [17]. In Witten's theory, the strings couple by overlapping the right half of each string with the left half of the next [14]. Because there is only one oscillator mode in our ladder approximation for the lattice, here only the zeroth and first level oscillator modes will be considered in SFT aspect. After truncating oscillator modes and ignoring ghost contributions (there is no worldsheet gauge fixing on the lattice), the 3-string vertex in the oscillator approach is [16]:

$$|V_{123}\rangle = \mathcal{N}\delta^D(k_1+k_2+k_3)\exp\Big(-\tfrac{1}{2}\sum_{I,J=1}^{3}[a^I_{-1}N^{IJ}_{-1,-1}a^J_{-1} + 2a^I_{-1}N^{IJ}_{-1,0}p^J$$

$$+p^I N_{00}^{IJ} p^J]\Big)|0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \quad (5.68)$$

The Neumann coefficients N_{mn}^{IJ} depend on the choice of the conformal mappings. It was shown that the different conformal mappings correspond to different formulations of string field theory that are equivalent to each other. The most widely used open string field theory is Witten's theory. The action in Witten's open string field theory is

$$S = +\tfrac{1}{2}\langle V_2|\Psi, Q\Psi\rangle + \tfrac{g}{3}\langle V_3|\Psi,\Psi,\Psi\rangle\,. \quad (5.69)$$

in which the first part gives the free term and the second part gives the interactions. The Neumann coefficients read

$$N_{-1,-1}^{11} = N_{-1,-1}^{22} = N_{-1,-1}^{33} = \tfrac{5}{27} \quad (5.70)$$

$$N_{-1,-1}^{12} = N_{-1,-1}^{23} = N_{-1,-1}^{31} = -\tfrac{16}{27} \quad (5.71)$$

$$N_{-1,0}^{12} = -N_{-1,0}^{13} = N_{-1,0}^{23} = -N_{-1,0}^{21} = N_{-1,0}^{31} = -N_{-1,0}^{32} = \tfrac{2\sqrt{6\alpha'}}{9} \quad (5.72)$$

$$N_{00}^{11} = N_{00}^{22} = N_{00}^{33} = \alpha' \ln(27/16) \quad (5.73)$$

and zero for others [31]. With the Feynman-Siegel gauge $b_0 = 0$, for tachyon and massless states and up to 3-point interactions, we will get the gauge-fixed action from the gauge-invariant action

$$S = \tfrac{1}{2}[\nabla_\mu,\phi][\nabla^\mu,\phi] - \phi^2 - F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{3}\phi^3 + 2\phi F_{\mu\nu}F^{\mu\nu} - \tfrac{4}{3}F_\mu^\nu F_\nu^\lambda F_\lambda^\mu \quad (5.74)$$

with a particular gauge as we discussed in chapter 4. Here we focus on only the massless state and set

$$\langle\psi_i| = \langle 0, k_i|_I A(k_i)\cdot a_1^I \quad (5.75)$$

It gives the 3-point gauge interactions

$$\begin{aligned}\mathcal{A}_3 &= \tfrac{g}{3}\langle V_3|\Psi,\Psi,\Psi\rangle \\ &\propto ige^{-\frac{1}{2}N_{00}^{11}(k_1^2+k_2^2+k_3^2)}\{[(A_1\cdot A_2)(A_3\cdot k_{12}) + permutations] \\ &\qquad + \tfrac{\alpha'}{2}(A_1\cdot k_{23})(A_2\cdot k_{31})(A_3\cdot k_{12})\}\end{aligned} \quad (5.76)$$

after α' is restored.

From the previous section, in the oscillator approach, the 3-state vertices in (5.40) or (5.50) give the same form as (5.68) except for some different Neumann coefficients.

For (5.40),

$$N^{11}_{-1,-1} = N^{22}_{-1,-1} = N^{33}_{-1,-1} = \tfrac{1}{3}\tfrac{1}{1+2m\omega} \tag{5.77}$$

$$N^{12}_{-1,-1} = N^{23}_{-1,-1} = N^{31}_{-1,-1} = \tfrac{1}{3}\tfrac{1}{1+2m\omega} \tag{5.78}$$

$$N^{12}_{-1,0} = -N^{13}_{-1,0} = N^{23}_{-1,0} = -N^{21}_{-1,0} = N^{31}_{-1,0} = -N^{32}_{-1,0} = \tfrac{\sqrt{\tilde{\alpha}'}}{3\sqrt{2m}} \tag{5.79}$$

$$N^{11}_{00} = N^{22}_{00} = N^{33}_{00} = \tfrac{\tilde{\alpha}'}{6m} \tag{5.80}$$

and zero for others. This gives a 3-point interaction for massless bosons as in (5.47) with the ratio of F^3 and F^2 coefficients

$$\tilde{\alpha}'\tfrac{1+2m\omega}{6m} = \alpha'\tfrac{1+2m\omega}{3m}\omega \tag{5.81}$$

instead of the ratio $\alpha'/2$ from Witten's vettex.

For (5.50),

$$N^{11}_{-1,-1} = N^{22}_{-1,-1} = N^{33}_{-1,-1} = \tfrac{1}{3}\tfrac{1}{1+4m\omega} + \tfrac{2}{3}\tfrac{1}{1+10m\omega} \tag{5.82}$$

$$N^{12}_{-1,-1} = N^{23}_{-1,-1} = N^{31}_{-1,-1} = \tfrac{1}{3}[\tfrac{1}{1+4m\omega} - \tfrac{1}{1+10m\omega}] \tag{5.83}$$

$$N^{12}_{-1,0} = -N^{13}_{-1,0} = N^{23}_{-1,0} = -N^{21}_{-1,0} = N^{31}_{-1,0} = -N^{32}_{-1,0} = \sqrt{\tilde{\alpha}'}\tfrac{\sqrt{2m\omega}}{1+10m\omega} \tag{5.84}$$

$$N^{11}_{00} = N^{22}_{00} = N^{33}_{00} = \tfrac{3}{5}\tilde{\alpha}'\tfrac{\omega}{1+10m\omega} \tag{5.85}$$

and zero for others. Again, the 3-point interaction for the massless state is the same as in (5.62) with a ratio of F^3 and F^2 coefficients of

$$\tilde{\alpha}'\tfrac{1+4m\omega}{1+10m\omega}\omega = 4\alpha'\tfrac{1+4m\omega}{1+10m\omega}\omega^2 \tag{5.86}$$

As Witten's theory in Feynman-Siegel gauge, both (5.40) and (5.50) give gauge-fixed 3-point interactions with nonlocal $e^{\tau\Box}$ factors. The mismatch of F^2 and F^3 coefficients may be due to the fact we only considered the two simplest interacting lattice diagrams. In principle, all interaction diagrams should be summed, which may give the same interaction as from usual string theory (on-shell) or Witten's string field theory (off-shell). But it does show that the massless state given in the beginning of section 5.4 can have the same interactions as the usual YM field. So this will be an interesting start to view the YM gauge field as the bound state of an underlying scalar field instead of as a fundamental field.

Another similarity between vertices (5.40) or (5.50) and Witten's vertex is that they all have the same symmetries. First, there is a cyclic symmetry under $I \to J, J \to K, K \to I$, which corresponds to cyclic symmetry of each interaction diagram. Second, there is a symmetry for Neumann coefficients under $I \leftrightarrow J, m \leftrightarrow n$. Finally, there is a twist symmetry under $N^{IJ}_{nm} = (-1)^{m+n} N^{JI}_{nm}$ associated with twisting of the lattices (strings). It is nontrivial and restricts the group structure of the gauge-fixed action. For the case here, we only consider the first excited state $|\psi_i\rangle$ in (5.75), which is a twist-odd state under the twist operator Ω: $\Omega|\psi_i\rangle = -|\psi_i\rangle$. Then the twist invariance requires the gauge-fixed interaction to be proportional to the structure constants f^{abc} because

$$\Omega\langle\Psi_1, \Psi_2 * \Psi_3\rangle = \langle(\Omega\Psi_1), (\Omega\Psi_3) * (\Omega\Psi_2)\rangle = -\langle\Psi_1, \Psi_3 * \Psi_2\rangle \tag{5.87}$$

as shown in Fig 5.4. There, diagram I gives the term $\propto Tr(T_a T_b T_c)$ while diagram II gives the term $\propto -Tr(T_c T_b T_a)$ and their sum gives an interaction term $\propto f^{abc}$. Because the gauge-invariant YM action can always be written as

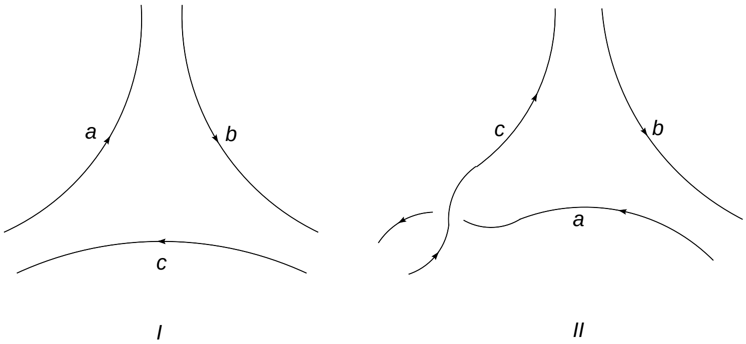

Figure 5.4: The twist symmetry in Witten's vertex: $I \propto Tr(T_a T_b T_c)$; $II \propto -Tr(T_c T_b T_a)$

a function of structure constants, the gauge condition in this case should also be expressed in terms of f^{abc}, which excludes the Gervais-Neveu gauge. These symmetries apply not only to massless states but also to general states (but with the usual extra sign factors in the twist). Obviously, both Fig. 5.2 and Fig. 5.3 are similar to the diagram of joining three open strings in Witten's theory except they are on a discrete lattice while Witten's vertex is on a continuous worldsheet.

Another 3-string vertex in SFT we will mention here is the CSV vertex, which is equivalent to Witten's vertex on-shell. Here we only review the coefficients for zero-modes and first excited modes:

$$N^{11}_{-1,-1} = N^{22}_{-1,-1} = N^{33}_{-1,-1} = 0 \tag{5.88}$$

$$N^{12}_{-1,-1} = N^{23}_{-1,-1} = N^{31}_{-1,-1} = 1 \tag{5.89}$$

$$N^{12}_{-1,0} = N^{23}_{-1,0} = N^{31}_{-1,0} = \sqrt{2\alpha'} \tag{5.90}$$

$$N^{21}_{-1,0} = N^{32}_{-1,0} = N^{13}_{-1,0} = 0 \tag{5.91}$$

and all N^{IJ}_{00} vanish. Comparing to the above vertices, the CSV vertex lacks twist symmetry. So, as has been shown previously, it corresponds to the well-known Gervais-Neveu gauge without nonlocal coupling factors.

Chapter 6

Conclusion and Discussion

In this thesis, we have given a general construction for gauge-covariant vertex operators, and applied it to the YM vertex in the string and superstring in 3-point amplitudes. This method allows direct calculation of gauge-invariant results, analogous to nonlinear sigma models. We also computed the gauge-invariant tree amplitude between 4 gauge bosons, whose explicit expression in bosonic string theory has not appeared previously in the literature. The results reproduce the same amplitudes at order 1, α', and α'^2 from the appropriate F^2, F^3 (for the bosonic string), and F^4 terms in a field theory action.

On the bosonic string field theory side, as the Feynman-Siegel gauge is imposed, the action is a Yang-Mills gauge fixed one. We computed the ZJBV action for Witten's open string field theory for tachyons and massless vectors, including all ghosts and antifields. We find after some canonical transformations that the action up to 3-point terms is just the usual Yang-Mills one plus ϕF^2 and F^3 interactions , which is explicitly gauge invariant now. The gauge condition in field theory which corresponds to the Feynman-Siegel gauge on the world-sheet is also known. Furthermore, there are no nonlocal interactions in the action up to 3-point terms. (A higher-point analysis would require analyzing the massive fields, since redefinitions of massive fields appearing in propagators, in 4-point and higher diagrams, will produce new local terms for massless fields on external lines.) We pushed these nonlocal factors in 3-point interactions to higher-point interactions. It may be possible that all such explicit factors can be eliminated in the complete action, so that all "nonlocality" can be attributed to the presence of higher-spin fields.

At the end, we followed another approach of string quantization: replace the world-sheet by the random lattice, where the random nature reflects the arbitrariness of the world-sheet metric. From the bosonic lattice string, we constructed the massless state as a bound state of partons, and two simple lattice interaction diagrams. Using such interaction diagrams, we found interactions of those bound states similar to the usual YM gauge field. The comparison of these 3-state vertices on the lattice with Witten's vertex on the continuous worldsheet shows all of them have the same symmetries, especially twist symmetry, which is absent in the CSV vertex. The twist symmetry restricted the gauge-fixed interaction to be proportional to the structure constants of the gauge group, or equivalently, the interaction term of the gauge condition must be proportional to the structure constants. That's the reason the Gervais-Neveu gauge can only be obtained from the CSV vertex. Anyway, we show here the possibility to bind the scalars on the lattice to get the massless vector state which behaves like the gauge field, i.e., the gauge field is no longer a fundamental particle but a composite state in the field theory. This also provided a new view of the 3-string coupling in Witten's bosonic open string field theory.

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