Elliptic constructions of hyperkähler metrics

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Radu Aurelian Ionaş

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Radu Aurelian Ionaş

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation.

Prof. Martin Roček, Institute for Theoretical Physics

Prof. Peter van Nieuwenhuizen, Institute for Theoretical Physics

Prof. Thomas Weinacht, Department of Physics and Astronomy

Prof. Leon Takhtajan, Department of Mathematics

This dissertation is accepted by the Graduate School

Lawrence Martin
Dean of the Graduate School

Abstract of the Dissertation

Elliptic constructions of hyperkähler metrics

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In this dissertation we develop a twistor-theoretic method of constructing hyperkähler metrics from holomorphic functions and elliptic curves. We obtain, among other things, new results concerning the Atiyah-Hitchin manifold, asymptotically locally Euclidean spaces of type D_n and certain Swann bundles. For example, in the Atiyah-Hitchin case we derive in an explicit holomorphic coordinate basis closed-form formulas for the metric, the holomorphic symplectic form and all three Kähler potentials. The equation describing an asymptotically locally Euclidean space of type D_n is found to admit an algebraic formulation in terms of the group law on a Weierstrass cubic. This curve has the structure of a Cayley cubic for a pencil generated by two transversal plane conics, that is, it takes the form $Y^2 = \det(\mathcal{A} + X\mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are the defining 3×3 matrices of the conics. In this light, the equation can be interpreted as the closure condition for an elliptic billiard trajectory tangent to the conic $\mathcal{B}$ and bouncing into various conics of the pencil determined by the positions of the monopoles.

Părinţilor mei.

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0 Introduction

Lindström and Roček's generalized Legendre transform (GLT) approach to constructing hyperkähler metrics [1] emerged originally in the context of $\mathcal{N}=2$ supersymmetric sigma-models as the by-product of a superspace generalization of the Hodge duality between 0 and 2-form gauge fields in four dimensions that led eventually to the development of the concept of projective superspace [2, 3]. As it was later recognized, the natural mathematical setting of this approach is the theory of twistor spaces of hyperkähler manifolds [4, 5]. This is a classic generalization of Penrose's non-linear graviton construction [6] and forms an integral part of Salamon's theory of twistor spaces of quaternionic-Kähler manifolds [7]. The twistor space Z of a hyperkähler manifold $\mathcal{M}$ can be viewed as a holomorphic fibration $Z \to \mathbb{CP}^1$, with the fiber over a generic point $\zeta \in \mathbb{CP}^1 \simeq S^2$ being a copy of $\mathcal{M}$ endowed with only one complex structure, namely the one corresponding to ζ on the hyperkähler sphere of complex structures, and also with a symplectic form holomorphic with respect to this particular complex structure. In the twistor space picture, the GLT equations arise through a patching construction of a twisted holomorphic symplectic bundle over Z by means of twisted canonical transformations of type II. Remarkably, the metric information of the hyperkähler manifold is encoded in a single holomorphic function of one or several sections of $\mathcal{O}(2j)$ bundles over Z that satisfy a reality condition with respect to the real structure induced on Z by antipodal conjugation on the sphere of complex structures. The Penrose transform of this function is a real function F of the parameters of the $\mathcal{O}(2j)$ sections which satisfies a set of second order differential equations. A generalized Legendre-Fenchel transform of F yields both a Kähler potential of the hyperkähler manifold and a corresponding set of holomorphic coordinates. In this coordinate basis the components of the metric are given by the second derivatives of F and take the form of ratios of determinants of Hankel matrices. We observed, along the ideas of Penrose, that, formally, the structure of the real $\mathcal{O}(2j)$ sections is identical to that of quantum-mechanical wave functions describing the states of a particle with spin j in the so-called spin coherent representation [8]. These wave functions are sections of $\mathcal{O}(2j)$ bundles over the Bloch sphere, and so, locally, they are polynomials of degree j in the inhomogeneous coordinate on the sphere. Intuitively, such a state appears as a set of $2j$ elementary 'spins 1/2' with the origins at the center of the Bloch sphere, pointing out in the directions marked by a constellation of $2j$ dots on the surface of the sphere corresponding to the roots of the wave function polynomial.

A direction of applications for the GLT approach is the construction of Swann bundle metrics [9]. Swann bundles are hyperkähler varieties with an additional $\mathbb{H}^*$-action whose real component acts homothetically while the three purely imaginary components act isometrically and rotate the hyperkähler complex structures. The complex structures have furthermore the distinctive feature that they can be derived from the same Kähler potential, defined up to the addition of a constant. The importance of this class of hyperkähler manifolds stems from the fact that any quaternionic-Kähler manifold possesses a canonical Swann bundle from which it can be obtained through a quotient construction. They thus provide a holomorphic environment for the description of the generally non-holomorphic quaternionic-Kähler manifolds. In a joint work with Andrew Neitzke, we proved a very simple and generic criterion for a GLT construction to yield a hyperkähler manifold with a Swann bundle structure, generalizing previous results.

$\mathcal{O}(2)$-based GLT constructions of hyperkähler metrics have been extensively discussed in the literature [10, 4, 11]. Prominent examples include the asymptotically locally Euclidean (ALE) and asymptotically locally flat (ALF) metrics of type A_n (multi-Eguchi-Hanson and multi-Taub-NUT) and the 8-dimensional Swann bundle used by Calderbank and Pedersen to classify selfdual Einstein metrics with two commuting isometries [12] and by Anguelova, Roček and Vandoren to describe the geometry of the classical moduli space of the universal hypermultiplet in string theory compactifications [13]. Comparatively, the study of $\mathcal{O}(4)$-based constructions, expected to include among its applications the ALE and ALF metrics of type D_n as well as other interesting cases, is less developed, due to inherent difficulties [5, 14, 15, 16]. It is precisely this problem that we addresss in this dissertation, from two main directions: we took an algebraic-geometric approach combined with a search for Casimir invariants, suggested by the quantum-mechanical analogy. These two programs eventually converged into a unified picture and led to a generic and pervasive method. In what follows, we give a brief sketch of some of the results that we have obtained.

To each $\mathcal{O}(4)$ real section one associates canonically a quartic plane curve. Through a series of biholomorphic transformations this curve can be cast in Weierstrass normal form. The Weierstrass coefficients are real, $SO(3)$-invariant and have explicit homogeneous polynomial expressions in terms of the parameters of the $\mathcal{O}(4)$ section. This last characteristic makes this representation optimal for the GLT approach, where one needs to compute derivatives with respect to these parameters. On the Weierstrass curve, the antipodal conjugation property translates into a colinearity condition. The group law on the cubic thus comes into play. Moreover, the Weierstrass cubic turns out to be also a Cayley cubic for a pencil generated by two particular transversal plane conics, that is, it takes the form $Y^2 = \det(\mathcal{A} + X\mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are the defining 3×3 matrices of the conics. On the parameter space of real $\mathcal{O}(2j)$ sections there is an $SO(3)$ action induced by the automorphisms of the Riemann sphere that preserve the real structure of the section. The following table summarizes the number and type of invariants of this action associated to an $\mathcal{O}(2)$ and an $\mathcal{O}(4)$ section:

$\mathcal{O}(2)$ invariant	$\mathcal{O}(4)$ invariants	mixed invariants
r_1	r_2, r_2'	A, B
radial		angular

In particular, the two $\mathcal{O}(4)$ invariants are given essentially by the inverses of the Weierstrass periods of the $\mathcal{O}(4)$ curve. The elliptic nome and complementary nome take the forms $q = \exp(-\pi^2 r_2/r_2')$ and $q' = \exp(-r_2'/r_2)$ respectively, hence the two asymptotic regions $r_2 >> r_2'$ and $r_2 << r_2'$ can be analyzed perturbatively by performing series expansions in q respectively q'.

The application of these ideas to the Atiyah-Hitchin manifold [17] describing the moduli space of centered $SU(2)$ 2-monopoles leads to a host of new and interesting results. In this case, all metric information is contained in a single holomorphic function of one $\mathcal{O}(4)$ multiplet. The defining equation of the manifold in the GLT approach is simply $r_2 = \text{const}$. This leaves only one radial-type variable, r_2', to play the role of monopole separation distance. Furthermore, we were able to derive, in an explicit holomorphic coordinate basis, closed-form formulas for the metric, all three Kähler potentials, all three hyperkähler 2-forms as well as for the generating vector fields of the $SO(3)$ isometry.

There are other examples where the radius r_2 is not frozen as above. We discuss for instance an $\mathcal{O}(2)\oplus\mathcal{O}(4)$-based Swann bundle (conjectured to describe the nonperturbative universal hypermultiplet moduli space metric due to five-brane instantons [13] and possibly relevant for the classification of selfdual Einstein metrics with one toric isometry) whose hyperkähler potential yielded the following q'-expansion

$$K = 2\left(A - \frac{2}{3}\right)\frac{r_1^2}{r_2} - 6A\frac{r_1^2}{r_2'} + A\frac{r_1^2}{r_2'}\frac{144(5r_2^2 - 7r_2r_2' + 2r_2'^2)}{r_2^2}e^{-2r_2'/r_2} + \mathcal{O}(e^{-4r_2'/r_2}) \qquad (1)$$

Incidentally, in the limit when $r_2' >> r_2$, this metric approaches the previously mentioned Swann bundle metric of Calderbank and Pedersen, obtainable from a related $\mathcal{O}(2)\oplus\mathcal{O}(2)$-based GLT construction.

In the case of ALE spaces of type D_n, we found that the defining equation of the manifold admits an algebraic formulation in terms of the group law on the Weierstrass cubic associated to the $\mathcal{O}(4)$ multiplet, with a remarkable geometric quantization interpretation. Thus, the equation can be interpreted as the closure condition for an elliptic billiard trajectory tangent to the conic $\mathcal{B}$ and bouncing into various conics of the pencil determined by the positions of the monopoles. Poncelet's porism guarantees then that once a trajectory closes to a star polygon, any trajectory will close, regardless of the starting point and after the same number of steps. The elliptic billiards interpretation opens up a possible connection to integrable systems. Not long ago, Poncelet polygons have made an appearance in a different context, they were used by Hitchin to derive a special class of solutions to a certain Painlevé VI equation [18, 19]. Although otherwise quite disparate, these two circumstances have one thing in common, namely the presence of a dihedral symmetry. This suggests that we may have come across something of the kind of a universal pattern.

The contents of this dissertation are organized as follows: In section 1 we review the theory of twistors spaces of hyperkähler manifolds. In section 2 we review the $\mathcal{N} = 2$ projective superspace formalism and the scalar-tensor duality. In section 3 we outline the construction of hyperkähler metrics by means of the generalized Legendre transform. In section 4 we give a criterion for a hyperkähler manifold constructed by means of the generalized Legendre transform to have a Swann bundle structure. In section 5 we study the rotational properties of $\mathcal{O}(2j)$ multiplets using quantum mechanics, Penrose transforms and spherical geometry. In section 6 we outline the construction of ALE and ALF spaces of type A_n and in particular of the Taub-NUT space in this framework. In section 7 we review the construction of an 8-dimensional Swann bundle based on two $\mathcal{O}(2)$ multiplets. In section 8 we study in depth the $\mathcal{O}(4)$ multiplet and related elliptic curve. In section 9 we evaluate a set of contour integrals of fundamental practical importance for applications of the generalized Legendre transform construction involving $\mathcal{O}(4)$ multiplets. In section 10 we apply the ideas developed so far to the Atiyah-Hitchin metric and perform a series of explicit computations. Theta-function expressions for the coefficients of the metric in the $SO(3)$ coordinate basis are particularly suited, through their q and q'-series expansions, for calculating corrections virtually to any order in both the large and the small monopole separation limits of the metric. In section 11 we study the ALE spaces of type D_n. In section 12 we investigate an 8-dimensional Swann bundle related to the M-theory nonperturbative universal moduli space due to five-brane instantons and construct explicitly its hyperkähler potential. In section 13, along with reviewing some relevant aspects of the theory of elliptic functions and integrals, we develop the theory of (both complete and incomplete) Jacobi elliptic integrals of third kind within the frame of the Weierstrass formalism and obtain addition and differentiation formulas for them. The latter play a central role in our investigations. In section 14 we give a largely self-contained review of Poncelet's porism.

1 Twistor spaces of hyperkähler manifolds

1.1 The twistor construction

The theory of twistor spaces of hyperkähler manifolds has been introduced concurrently with the hyperkähler quotient, in [4]. A classic generalization of Penrose's non-linear graviton construction [6], it forms an integral part of Salamon's theory of twistor spaces of quaternionic-Kähler manifolds [7].

Let $\mathcal{M}$ be a hyperkähler manifold, with the standard triplet of complex structures J_1, J_2, J_3 forming a quaternionic algebra

$$J_1^2 = J_2^2 = J_3^2 = -I\,, \qquad J_1 J_2 = J_3\,, \qquad \text{a.s.o.} \tag{2}$$

where I is the identity endomorphism on the tangent bundle of $\mathcal{M}$. These generate in fact a 2-sphere's worth of complex structures on $\mathcal{M}$: for any unit vector $(x^1, x^2, x^3) \in \mathbb{R}^3$, the linear combination $x^1 J_1 + x^2 J_2 + x^3 J_3$ forms an integrable complex structure on $\mathcal{M}$, compatible with the metric and Levi-Civita connection. In particular, the quaternionic algebra properties of J_1, J_2, J_3 guarantee that $(x^1 J_1 + x^2 J_2 + x^3 J_3)^2 = -I$. Identifying the 2-sphere with the complex projective space $\mathbb{CP}^1$, one can describe it alternatively in terms of a set of two homogeneous coordinates, which we denote here by π^A, with $A \in \{1, 2\}$. The projective and the extrinsic descriptions of the Riemann sphere are related through the stereographic projection

$$x^i = \frac{\bar{\pi}_A (\sigma^i)^A{}_B \pi^B}{\bar{\pi}_C \pi^C} \tag{3}$$

where $i \in \{1, 2, 3\}$ is a Cartesian index, σ^i are the three standard 2×2 Pauli matrices and $\bar{\pi}_A$ denotes the complex conjugate of π^A. Using for instance a completeness property of the Pauli matrices one can verify that

$$\sum_{i=1}^{3} (x^i)^2 = 1 \tag{4}$$

For further reference, let us also record here two other direct consequences of equation (3), namely

$$\frac{\partial x^i}{\partial \pi^A} = \frac{\pi_A \bar{\pi}_B \bar{\pi}_D (\sigma^i)^{BD}}{(\bar{\pi}_C \pi^C)^2} \tag{5}$$

and

$$\epsilon_{ijk}\, x^j \frac{\partial x^k}{\partial \pi^A} = i \frac{\partial x^i}{\partial \pi^A} \tag{6}$$

We conventionally lower the $SL(2, \mathbb{C})$ indices by means of the two-dimensional ϵ-symbol and raise them by means of its inverse. For example, by definition, $\pi_A = \pi^B \epsilon_{BA}$ and so by way of consequence, $\pi^A = \epsilon^{AB} \pi_B$. Note incidentally that $(\sigma^i)^{AB} = (\sigma^i)^{BA}$ for all i.

So, to each point of homogeneous coordinates π^A on the Riemann sphere we can associate in $\operatorname{End} T(\mathcal{M})$ the complex structure

$$J(\pi, \bar{\pi}) = x^i J_i \tag{7}$$

with x^i given by (3). Based on the quaternionic relations (2), we have

$$J(\pi, \bar{\pi}) \frac{\partial J(\pi, \bar{\pi})}{\partial \pi^A} = \epsilon_{ijk}\, x^j \frac{\partial x^k}{\partial \pi^A} J_i - \frac{1}{2} \frac{\partial}{\partial \pi^A} (x^i x^i) I \tag{8}$$

Summation over repeated indices is assumed. Using then equations (4) and (6), we derive the key relation

$$[I + iJ(\pi,\bar{\pi})]\frac{\partial J(\pi,\bar{\pi})}{\partial \pi^A} = 0 \tag{9}$$

The twistor space Z is defined to be the direct product manifold $\mathcal{M} \times \mathbb{CP}^1$ endowed with the following complex structure

$$\mathcal{J} = J(\pi,\bar{\pi}) + i\frac{\partial}{\partial \pi^A} \otimes d\pi^A - i\frac{\partial}{\partial \bar{\pi}_A} \otimes d\bar{\pi}_A \tag{10}$$

This is an element of $\mathrm{End}\,[T(\mathcal{M}) \oplus T(\mathbb{CP}^1)] \simeq \mathrm{End}\,T(\mathcal{M}) \oplus \mathrm{End}\,T(\mathbb{CP}^1)$, with the first component given by the complex structure (7) and the second by the standard complex structure on $\mathbb{C}^2$ that descends on $\mathbb{CP}^1$. The twistor space can be viewed as a holomorphic fibration over $\mathbb{CP}^1$, with the fiber corresponding to the point of homogeneous coordinates π^A being a copy of the manifold $\mathcal{M}$ endowed with the complex structure $J(\pi,\bar{\pi})$. The holomorphic sections of this fibration are termed twistor lines.

To prove that this is indeed a complex structure, consider two arbitrary vector fields from $T(Z) \simeq T(\mathcal{M}) \oplus T(\mathbb{CP}^1)$

$$\mathcal{X} = X + X^A\frac{\partial}{\partial \pi^A} + \bar{X}_A\frac{\partial}{\partial \bar{\pi}_A} \qquad \text{and} \qquad \mathcal{Y} = Y + Y^A\frac{\partial}{\partial \pi^A} + \bar{Y}_A\frac{\partial}{\partial \bar{\pi}_A} \tag{11}$$

where $X, Y \in T(\mathcal{M})$. A direct calculation shows that the Nijenhuis tensor on Z corresponding to the almost complex structure (10) takes, when evaluated on $\mathcal{X}$ and $\mathcal{Y}$, the following form

$$\begin{aligned} N_{\mathcal{J}}(\mathcal{X},\mathcal{Y}) = {}& N_{J(\pi,\bar{\pi})}(X,Y) \\ & - iX^A[I + iJ(\pi,\bar{\pi})]\frac{\partial J(\pi,\bar{\pi})}{\partial \pi^A}Y + iY^A[I + iJ(\pi,\bar{\pi})]\frac{\partial J(\pi,\bar{\pi})}{\partial \pi^A}X \\ & + i\bar{X}_A\,[I - iJ(\pi,\bar{\pi})]\frac{\partial J(\pi,\bar{\pi})}{\partial \bar{\pi}_A}Y - i\bar{Y}_A\,[I - iJ(\pi,\bar{\pi})]\frac{\partial J(\pi,\bar{\pi})}{\partial \bar{\pi}_A}X \end{aligned} \tag{12}$$

The integrability of $J(\pi,\bar{\pi})$ on $\mathcal{M}$ together with equation (9) imply then that it vanishes. By the Newlander-Nirenberg theorem it follows that the complex structure (10) is indeed integrable and that Z is a complex manifold. As we shall see, the holomorphic structure on Z turns out to encode all the metric information of the hyperkähler manifold.

Let ω_1, ω_2, ω_3 be the Kähler 2-forms corresponding to the complex structures J_1, J_2, J_3 on $\mathcal{M}$ and define the 2-form-valued $\mathcal{O}(2)$ section

$$\omega(\pi) = -\frac{1}{2}\pi^A\pi^B(\sigma^i)_{AB}\,\omega_i \tag{13}$$

Let also g be the metric on $\mathcal{M}$. Then, for any two vector fields $X, Y \in T(\mathcal{M})$, we have

$$\omega(\pi)(X,Y) = -\frac{1}{2}\pi^A\pi^B(\sigma^i)_{AB}\,g(X, J_iY) = \frac{1}{2}\bar{\pi}_C\pi^C\pi_A\,g(X, \frac{\partial J(\pi,\bar{\pi})}{\partial \bar{\pi}_A}Y) \tag{14}$$

The second equality follows from the equation (5). Based on this result and the hermiticity of the metric we may write

$$\omega(\pi)([I + iJ(\pi,\bar{\pi})]X, Y) = \frac{1}{2}\bar{\pi}_C\pi^C\pi_A\,g(X, [I - iJ(\pi,\bar{\pi})]\frac{\partial J(\pi,\bar{\pi})}{\partial \bar{\pi}_A}Y) = 0 \tag{15}$$

The last equality follows from the equation (9). We have thus shown that $\omega(\pi)$ is a $(2,0)$-form on $\mathcal{M}$ with respect to the complex structure $J(\pi,\bar{\pi})$. Since all three Kähler 2-forms ω_i are closed, it follows that $\omega(\pi)$ is also closed and, moreover, holomorphic with respect to $J(\pi,\bar{\pi})$. In other words, for each point labelled by $\pi = (\pi^1, \pi^2)$ in $\mathbb{CP}^1$, $\omega(\pi)$ is a holomorphic symplectic form on the fiber of the projection $Z \to \mathbb{CP}^1$ above π.

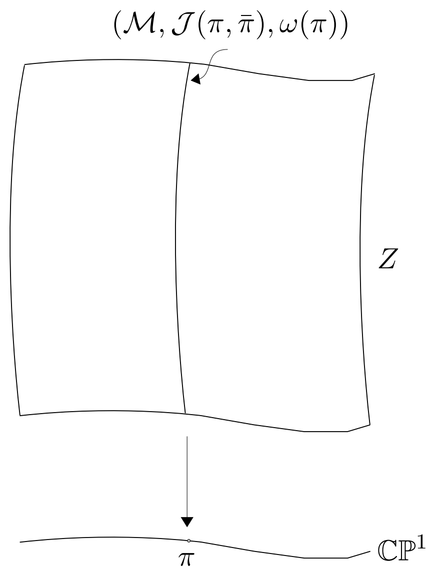


Figure 1. The twistor fibration

The antipodal conjugation on the sphere of hyperkähler complex structures induces on the twistor space Z the $\mathbb{Z}_2$-action

$$(m, \pi^A) \xrightarrow{\text{a. c.}} (m, \bar{\pi}^A) \tag{16}$$

for any $(m, \pi^A) \in \mathcal{M} \times \mathbb{CP}^1$. This action maps $\mathcal{J}$ into $-\mathcal{J}$ and therefore defines a *real structure* on Z. In particular, note that (16) takes $\omega(\pi)$ into its complex conjugate. This can be seen immediately by resorting to the hermiticity property $\overline{(\sigma^i)}_{AB} = (\sigma^i)^{BA}$.

1.2 The generalized Legendre transform

The theory of twistor spaces of hyperkähler manifolds provides a natural mathematical framework for the generalized Legendre transform approach to constructing hyperkähler manifolds [4]. This method emerged in the context of $\mathcal{N} = 2$ supersymmetric sigma models and a superspace generalization of the Hodge duality between 0 and 2-form gauge fields in four dimensions [1]. Originally it applied only to hyperkähler manifolds posessing a number of abelian isometries equal to their quaternionic dimensions, but later generalizations [2, 5] proved that this assumption was by no means necessary.

In the previous section we took a global approach and used homogeneous coordinates to describe the $\mathbb{CP}^1$ base of the twistor fibration. It is beneficial to take also a local point of view and describe bundles over $\mathbb{CP}^1$ and their sections in terms of patching by means of transition functions. Let U_1 and U_2 be the standard open charts on $\mathbb{CP}^1$, corresponding to the inhomogeneous coordinates $\zeta = \pi^2/\pi^1$ and $\tilde{\zeta} = \pi^1/\pi^2$, respectively. On $U_1 \cap U_2$ these are related by $\tilde{\zeta} = 1/\zeta$. Choosing the trivialization $\pi = (1, \zeta)$, the 2-form-valued

$\mathcal{O}(2)$ section (13) takes for example on U_1 the form

$$\omega(\zeta) = \omega_+ - \omega_3\zeta - \omega_-\zeta^2 \tag{17}$$

where $\omega_\pm = (\omega_1 \pm i\omega_2)/2$ are the hyperkähler holomorphic and anti-holomorphic 2-forms, respectively. The reality condition induced by antipodal conjugation reads in local coordinates

$$\overline{\omega\left(-\frac{1}{\bar{\zeta}}\right)} = -\left(\frac{1}{\zeta}\right)^2 \omega(\zeta) \tag{18}$$

Consider now the fiber of $Z \to \mathbb{CP}^1$ above $\zeta \in U_1$. As we have seen, $\omega(\zeta)$ is a holomorphic symplectic form with respect to the distinguished complex structure on the fiber. In particular, this means that one can always introduce a set of complex holomorphic Darboux coordinates $p = p(\zeta)$ and $q = q(\zeta)$, in terms of which $\omega(\zeta) = dq \wedge dp$. If the real dimension of the hyperkähler manifold is $4n$ then there are in fact n such pairs of Darboux coordinates and the symplectic form is a direct sum of such terms. To decongest the notation we omit here the indices differentiating among these, but they should be understood to exist. Similarly, one can introduce complex holomorphic Darboux coordinates $P = P(\tilde{\zeta})$ and $Q = Q(\tilde{\zeta})$ on the fiber above $\tilde{\zeta} \in U_2$. If these can furthermore be chosen such that

$$Q(\tilde{\zeta}) = \overline{q\left(-\frac{1}{\bar{\zeta}}\right)} \qquad \text{and} \qquad P(\tilde{\zeta}) = -\overline{p\left(-\frac{1}{\bar{\zeta}}\right)} \tag{19}$$

then the required reality property of ω with respect to antipodal conjugation holds automatically. Since ω is $\mathcal{O}(2)$-valued, on $U_1 \cap U_2$ we may write

$$\omega = dq \wedge dp = \zeta^2 dQ \wedge dP \tag{20}$$

The transition between the two sets of Darboux coordinates can be thus viewed as a twisted holomorphic symplectomorphism.

At this point we make a few restrictive assumptions. First of all, we assume that the q-coordinates are $\mathcal{O}(2j)$ sections over $\mathbb{CP}^1$ (not necessarily with the same value of j, when there are several of these) and that the p-coordinates are non-singular close to $\zeta = 0$. Accordingly, we have the following power expansions

$$q = \sum_{n=0}^{2j} q_n \zeta^n \qquad \text{and} \qquad p = \sum_{n=0}^{\infty} p_n \zeta^n \tag{21}$$

The twisted holomorphic symplectomorphism (20) can be derived from the twisted type II canonical transformation

$$Q = \frac{(-)^j}{\zeta^{2j}} \frac{\partial F_2(q,P)}{\partial P} \tag{22}$$

$$p = \frac{(-)^j}{\zeta^{2j-2}} \frac{\partial F_2(q,P)}{\partial q} \tag{23}$$

Our second assumption is that the generating function is of the form

$$F_2(q,P) = qP + (-)^j \zeta^{2j} G(q) \tag{24}$$

with G depending solely on q and perhaps on ζ but not on P. The factor in front of G is nonessential, it serves a formal purpose and it can be in principle absorbed in the definition of G. Equation (22) encodes in fact a reality condition for the $\mathcal{O}(2j)$ sections. Indeed, together with the first equation (19) it yields that

$$\overline{q\left(-\frac{1}{\bar{\zeta}}\right)} = (-)^j \left(\frac{1}{\zeta}\right)^{2j} q(\zeta) \tag{25}$$

In terms of the parameters, this is equivalent to $\bar{q}_n = (-)^{j-n} q_{2j-n}$. In particular, the middle parameter q_j is always real. Equation (23), on the other hand, yields a patching formula for the p-coordinates on $U_1 \cap U_2$. Integrating it on a closed contour around $\zeta = 0$ we obtain

$$\frac{\partial F}{\partial q_n} = p_{1-n} - (-)^{j-n} \bar{p}_{1-2j+n} \tag{26}$$

for $n = 1, \cdots, 2j$, where we have defined

$$F = \frac{1}{2\pi i} \oint d\zeta \, G(q(\zeta)) \tag{27}$$

The integration contour should be chosen such that the resulting F be purely imaginary. Explicitly, the system of equations (26) reads

$$\frac{\partial F}{\partial q_0} = p_1 \tag{28}$$

$$\frac{\partial F}{\partial q_1} = p_0 + (-)^j \bar{p}_{2-2j} \tag{29}$$

$$\frac{\partial F}{\partial q_2} = \cdots = \frac{\partial F}{\partial q_j} = 0 \tag{30}$$

The remaining equations are just the complex conjugates of these. Note that the second term in the r.h.s. of (29) vanishes for all values of j except for $j = 1$, when it is equal to $-\bar{p}_0$.

Substituting the power expansions (21) into the Darbox form of ω on U_1, we get

$$\omega(\zeta) = dq \wedge dp = dq_0 \wedge dp_0 + (dq_0 \wedge dp_1 + dq_1 \wedge dp_0)\zeta + \cdots \tag{31}$$

Comparison with (17) yields

$$\omega_+ = dq_0 \wedge dp_0 \tag{32}$$

$$\omega_3 = -(dq_0 \wedge dp_1 + dq_1 \wedge dp_0) \tag{33}$$

Observe now that in the limit when $\zeta \to 0$, the complex structure $J(\zeta, \bar{\zeta})$ on the twistor fiber above ζ converges to J_3, while the holomorphic symplectic structure $\omega(\zeta)$ converges to ω_+. Since p and q are holomorphic coordinates with respect to $J(\zeta, \bar{\zeta})$ we conclude that q_0 and p_0 are holomorphic with respect to J_3. Using that, we can write ω_3 as a total derivative

$$\omega_3 = d(p_1 dq_0 - q_1 dp_0) \tag{34}$$

On another hand, defining a real function K by the Legendre transform

$$iK(q_0, \bar{q}_0, p_0, \bar{p}_0) = F(q_0, \bar{q}_0, q_1, \bar{q}_1, \cdots) - (p_0 q_1 - \bar{p}_0 \bar{q}_1) \tag{35}$$

we get, by means of the equations (28) through (30), that

$$i\partial K = i\left(\frac{\partial K}{\partial q_0}dq_0 + \frac{\partial K}{\partial p_0}dp_0\right) = p_1 dq_0 - q_1 dp_0 \tag{36}$$

Plugging this in (34) yields

$$\omega_3 = i\bar{\partial}\partial K \tag{37}$$

which means that the function K thus defined is in fact a Kähler potential for the complex structure J_3. The equations (27) and (35) together with (29) and (30) form the basis of the generalized Legendre transform approach to constructing hyperkähler metrics.

2 Projective superspace

2.1 Definition and $\mathcal{N} = 2$ action

The analytic superspace formalism [2, 3] has proven useful in constructing field theories which realize $\mathcal{N} = 2$, $d = 4$ supersymmetry off-shell. The analytic superspace $\mathbb{A}$ is an extension of Minkowski space $\mathbb{R}^{3,1}$ by an auxiliary $\mathbb{CP}^1$ with four fermionic directions fibered over it. To describe this fibration we begin by fixing some notation for the larger space $\mathbb{R}^{3,1|8} \times \mathbb{CP}^1$. The derivatives along the eight fermionic directions are given by four Grassmann-odd operators $D_{A\alpha}$ and their conjugates $\bar{D}^A_{\dot{\alpha}}$; here $A \in \{1, 2\}$ is the R-symmetry index, and α is a complex spinor index. These obey

$$\{D_{A\alpha}, D_{B\beta}\} = 0 \qquad \{D_{A\alpha}, \bar{D}^B_{\dot{\alpha}}\} = i\delta_A{}^B \partial_{\alpha\dot{\alpha}} \tag{38}$$

Let π^A be homogeneous coordinates for $\mathbb{CP}^1$. Then a meromorphic function of the π^A which is homogeneous of degree zero descends to a meromorphic function on $\mathbb{CP}^1$. It will be useful in what follows also to consider functions which are homogeneous of nonzero degree k; by definition, these are meromorphic sections of the line bundle $\mathcal{O}(k)$ on $\mathbb{CP}^1$.

Now we are ready to define $\mathbb{A}$ as a quotient of $\mathbb{R}^{3,1|8} \times \mathbb{CP}^1$. Introduce the odd $\mathcal{O}(1)$-valued vector fields

$$\nabla = \pi^A D_A \qquad\qquad \bar{\nabla} = \pi_A \bar{D}^A \tag{39}$$

where $\pi_A = \pi^B \epsilon_{BA}$. From (38) it follows that

$$\{\nabla, \nabla\} = \{\nabla, \bar{\nabla}\} = \{\bar{\nabla}, \bar{\nabla}\} = 0 \tag{40}$$

In (39) and henceforward, we suppress the spectator spinor indices α and $\dot{\alpha}$. By definition, functions on $\mathbb{A}$ are functions on $\mathbb{R}^{3,1|8} \times \mathbb{CP}^1$ which are annihilated by ∇ and $\bar{\nabla}$. The anticommutation relations (40) play the role of integrability conditions.

There is a natural action of $SU(2)$ on $\mathbb{R}^{3,1|8} \times \mathbb{CP}^1$, under which D_A transform as a doublet, π^A in the dual, and $\bar{D}^A$ in the complex conjugate. Both ∇ and $\bar{\nabla}$ are invariant under this $SU(2)$, so it descends to an action on $\mathbb{A}$. Moreover, since the action is linear in terms of the π_A, $SU(2)$ acts not only on $\mathbb{A}$ but on any line bundle $\mathcal{O}(k)$ over $\mathbb{A}$.

To contour-integrate over $\mathcal{C} \subset \mathbb{A}$ preserving this $SU(2)$ symmetry, we need to construct an invariant measure. First let us consider the bosonic directions of $\mathbb{CP}^1$. It is well known that $\mathbb{CP}^1$ does not possess any $SU(2)$-invariant holomorphic 1-form in the usual sense. However, it does have an $SU(2)$-invariant holomorphic 1-form valued in the line bundle $\mathcal{O}(2)$, namely

$$\mu_b = \pi_A d\pi^A. \tag{41}$$

To construct the fermionic measure, consider an additional pair of linear combinations

$$\Delta = \lambda^A D_A, \qquad \bar{\Delta} = \lambda_A \bar{D}^A \tag{42}$$

satisfying as before

$$\{\Delta, \Delta\} = \{\Delta, \bar{\Delta}\} = \{\bar{\Delta}, \bar{\Delta}\} = 0 \tag{43}$$

Choosing the linear coefficients such that $\lambda_A \pi^A = 2$ yields the following set of anticommutation relations with ∇ and $\bar{\nabla}$

$$\{\Delta, \nabla\} = \{\bar{\Delta}, \bar{\nabla}\} = 0 \tag{44}$$

$$\{\Delta_\alpha, \bar{\nabla}_{\dot{\alpha}}\} = 2i\partial_{\alpha\dot{\alpha}} \tag{45}$$

$$\{\nabla_\alpha, \bar{\Delta}_{\dot{\alpha}}\} = -2i\partial_{\alpha\dot{\alpha}} \tag{46}$$

Since Δ acts on the space of functions on $\mathbb{A}$, it should be considered only modulo shifts by a multiple of ∇. With this understanding, Δ is uniquely determined by (45); this is important because all other ingredients of (45) are $SU(2)$-invariant, so the unique Δ must also be $SU(2)$-invariant. Moreover, since $\bar{\nabla}$ is $\mathcal{O}(1)$-valued, we learn from (45) that Δ is $\mathcal{O}(-1)$-valued. The same type of argument shows that there is an $\mathcal{O}(-1)$-valued, $SU(2)$-invariant operator $\bar{\Delta}$ obeying (46). Writing $\Delta^2 = \epsilon^{\alpha\beta}\Delta_\alpha\Delta_\beta$, and likewise $\bar{\Delta}^2$, we get the $SU(2)$-invariant and $\mathcal{O}(-4)$-valued measure for integration over the fermions,

$$\mu_f = \Delta^2\bar{\Delta}^2. \tag{47}$$

Combining the measures for integration over bosons and fermions,

$$\mu = \mu_b \times \mu_f \tag{48}$$

is thus an $\mathcal{O}(-2)$-valued measure.

If $\hat{G}$ is a superfield on $\mathbb{A}$, $\mu\,\hat{G}$ can be integrated over some $\mathcal{C} \subset \mathbb{A}$ which projects to a closed contour in $\mathbb{CP}^1$ and is extended along all of the fermionic directions. Then

$$S = \int d^4x \oint_{\mathcal{C}} \mu\,\hat{G} \tag{49}$$

is a manifestly $\mathcal{N} = 2$ invariant action. This can be seen as follows: the infinitesimal $\mathcal{N} = 2$ supersymmetric variation of $\hat{G}$ is given by

$$\delta\hat{G} = (\epsilon^{\alpha A}Q_{\alpha A} + \bar{\epsilon}^{\dot{\alpha}}_A\bar{Q}^A_{\dot{\alpha}})\hat{G} \tag{50}$$

The supersymmetry generators $Q_{\alpha A}$ and $\bar{Q}^A_{\dot{\alpha}}$ differ from the corresponding supercovariant derivatives $D_{\alpha A}$ and $\bar{D}^A_{\dot{\alpha}}$ by total space-time derivative terms; consequently, under the action integral the former can be substituted with the latter, modulo boundary terms. The supercovariant derivatives, in their turn, can be replaced further with linear combinations of ∇ and Δ respectively $\bar{\nabla}$ and $\bar{\Delta}$. The terms containing ∇ and $\bar{\nabla}$ vanish when acting on $\hat{G}$, by the definition of $\mathbb{A}$, on which $\hat{G}$ is defined. The terms containing Δ and $\bar{\Delta}$ are annihilated by the fermionic measure due to the identities $\Delta^3 = \bar{\Delta}^3 = 0$. This proves the $\mathcal{N} = 2$ invariance of the action.

2.2 $\mathcal{O}(2j)$ multiplets

In what follows, we will assume that the superfield G is composite and, more specifically, that it is a function of analytic supermultiplets of a particular type, namely $\mathcal{O}(2j)$ multiplets [2]. These are $\mathcal{O}(2j)$ sections $\eta^{(2j)} = \eta_{A_1\cdots A_{2j}}\pi^{A_1}\cdots\pi^{A_{2j}}$ over $\mathbb{A}$ (we can define sections of $\mathcal{O}(k)$ over $\mathbb{A}$ by pulling back from $\mathbb{CP}^1$) which satisfy, additionally, a reality condition with respect to antipodal conjugation. If we define the local form $\eta^{(2j)}(\zeta)$, with $\zeta = \pi^2/\pi^1$ the standard inhomogeneous coordinate on $\mathbb{CP}^1$, by

$$\eta_{A_1\cdots A_{2j}}\pi^{A_1}\cdots\pi^{A_{2j}} = (\pi^1\pi^2)^j\eta^{(2j)}(\zeta) \tag{51}$$

then, in terms of it, the reality condition reads

$$\eta^{(2j)}(-\frac{1}{\bar{\zeta}}) = \overline{\eta^{(2j)}(\zeta)} \tag{52}$$

An $\mathcal{O}(2j)$ multiplet is thus generically of the form

$$\eta^{(2j)}(\zeta) = \frac{\bar{z}}{\zeta^j} + \frac{\bar{v}}{\zeta^{j-1}} + \frac{\bar{t}}{\zeta^{j-2}} + \cdots + x + (-)^j(\cdots + t\zeta^{j-2} - v\zeta^{j-1} + z\zeta^j) \tag{53}$$

The coefficients come in complex-conjugated pairs, except for the middle coefficient, x, which has to be real. The analytic superspace constraints

$$\nabla\eta^{(2j)} = \bar{\nabla}\eta^{(2j)} = 0 \tag{54}$$

yield the following set of $\mathcal{N}=2$ constraints for the components of the multiplet

$$D_{(A}\eta_{A_1\cdots A_{2j})} = 0 \tag{55}$$

$\mathcal{N}=2$ supersymmetry implies $\mathcal{N}=1$ supersymmetry. The reduced $\mathcal{N}=1$ superspace perspective offers some further insights. From an $\mathcal{N}=1$ superspace point of view only the the first two components of the multiplet, namely z and v, as well as their conjugates $\bar{z}$ and $\bar{v}$ are constrained. From the equations (55), we get

$$D_1\bar{z} = 0 \tag{56}$$

$$(D_1)^2\bar{v} = 0 \tag{57}$$

Thus $\bar{z}$ is an anti-chiral superfield and $\bar{v}$ is a complex linear superfield. The rest of the components of the multiplet are unconstrained in $\mathcal{N}=1$ superspace, in other words, they are auxiliary superfields.

Furthermore, by re-casting the operators (42) in the form

$$\Delta = \frac{1}{\pi^2}(2D_1 + \lambda^2\nabla) \qquad\qquad \bar{\Delta} = \frac{1}{\pi^1}(2\bar{D}_1 + \lambda^1\bar{\nabla}) \tag{58}$$

one can show that the $\mathcal{N}=2$ fermionic measure reduces to the standard $\mathcal{N}=1$ superspace measure as follows

$$\Delta^2\bar{\Delta}^2 = \left(\frac{4}{\pi^1\pi^2}\right)^2 (D_1)^2(\bar{D}_1)^2 + \text{trivial terms} \tag{59}$$

where by 'trivial terms' we mean either terms proportional to ∇ or $\bar{\nabla}$ which vanish when acting on superfields defined on $\mathbb{A}$ or terms proportional to space-time derivatives, which also vanish upon integration over space-time under appropriate boundary conditions. Note as well that, in terms of the inhomogeneous coordinate $\zeta = \pi^2/\pi^1$ on $\mathbb{CP}^1$, the bosonic measure (41) can be written as follows

$$\pi_A d\pi^A = \pi^1\pi^2\frac{d\zeta}{\zeta} \tag{60}$$

The manifestly $\mathcal{N}=1$ supersymmetric form of the $\mathcal{N}=2$ invariant action (49) is then

$$S = \int d^4x\,(D_1)^2(\bar{D}^1)^2\oint\frac{d\zeta}{\zeta}G(\eta^{(2j)}(\zeta)) \tag{61}$$

where, with respect to (49), we absorbed in the definition of G a factor of $16(\pi^1\pi^2)^{-1}$.

Now, suppose that G is an $\mathcal{O}(2)$-valued superfield. This implies in particular that under the rescaling $\eta^{(2j)} \longrightarrow \lambda^j \eta^{(2j)}$ G scales with weight one, and so, from (51), one has

$$G(\eta_{A_1 \cdots A_{2j}} \pi^{A_1} \cdots \pi^{A_{2j}}) = \pi^1 \pi^2 \, G(\eta^{(2j)}(\zeta)) \tag{62}$$

Then, based on the equations (59), (60) and (62), by the cancelation of the $\pi^1\pi^2$ factors, it follows that the corresponding action (61) can be written, modulo boundary terms, in the following manifestly $SU(2)$-invariant form

$$S = \frac{1}{16} \int d^4x \oint \pi_A d\pi^A \Delta^2 \bar{\Delta}^2 G(\eta_{A_1 \cdots A_{2j}} \pi^{A_1} \cdots \pi^{A_{2j}}) \tag{63}$$

The case of $\mathcal{O}(2)$ multiplets is in some sense special. Observe first that $G = \eta^{(2)}$ is $\mathcal{O}(2)$-valued but leads to a vanishing action. That is because $\eta^{(2)}$ multiplets have only three components - a chiral superfield, its complex conjugate and a real linear superfield - all of which are annihilated by the $\mathcal{N} = 1$ superspace measure in (61), as follows from (56) and (57). Alternatively, this can be viewed as an instance of a Cauchy theorem for projective superspace. Let U_1 and U_2 be the standard charts on the $\mathbb{CP}^1$ base of the projective superspace fibration, with $\zeta = \pi^2/\pi^1$ and $\tilde{\zeta} = \pi^1/\pi^2$ as corresponding inhomogeneous coordinates. One has, modulo trivial terms,

$$\begin{aligned} \mu\, \eta^{(2)} &\sim \quad 16(D_2)^2(\bar{D}^1)^2 d\zeta \, (\bar{z} + x\zeta - z\zeta^2) \quad \text{on } U_1 && (64) \\ &\sim -16(D_1)^2(\bar{D}^2)^2 d\tilde{\zeta} \, (\bar{z}\tilde{\zeta}^2 + x\tilde{\zeta} - z) \quad \text{on } U_2 && (65) \end{aligned}$$

Indeed, the two two components of the measure μ and the $\mathcal{O}(2)$ multiplet can be expressed in terms of the inhomogeneous coordinate on U_1 as follows

$$\pi_A d\pi^A = (\pi^1)^2 d\zeta \tag{66}$$

$$\Delta^2 \bar{\Delta}^2 = \left(\frac{2}{\pi^1}\right)^4 (D_2)^2(\bar{D}^1)^2 + \text{trivial terms} \tag{67}$$

$$\eta^{(2)} = \pi^A \pi^B \eta_{AB} = (\pi^1)^2(\bar{z} + x\zeta - z\zeta^2) \tag{68}$$

A similar result holds on U_2. Equations (64) and (65) follow immediately. Their significance is that $\mu\, \eta^{(2)}$ is a holomorphic 1-form on $\mathbb{CP}^1$ and so, by Cauchy's theorem, its closed-loop contour integrals must vanish.

Consider now instead $G = \eta^{(2)} \ln \eta^{(2)}$. Even though this is not an $\mathcal{O}(2)$ but rather an *affine* $\mathcal{O}(2)$ section, it nevertheless leads to a non-vanishing $SU(2)$-invariant action, as in this case the action can still be cast in the form (63). That this is so can be seen by writing

$$\begin{aligned} G(\eta_{AB}\pi^A\pi^B) &= (\eta_{AB}\pi^A\pi^B) \ln (\eta_{AB}\pi^A\pi^B) \\ &= \pi^1\pi^2[\eta^{(2)}(\zeta) \ln \eta^{(2)}(\zeta) + \eta^{(2)}(\zeta) \ln(\pi^1\pi^2)] \\ &= \pi^1\pi^2 \, G(\eta^{(2)}(\zeta)) + \pi^1\pi^2 \ln(\pi^1\pi^2)\, \eta^{(2)}(\zeta) \end{aligned} \tag{69}$$

and then noticing that the term linear in $\eta^{(2j)}(\zeta)$ vanishes under the measure μ, annihilated by its $\mathcal{N} = 1$ superspace component, just as above.

2.3 Scalar-tensor duality and the generalized Legendre transform

To describe the scalar-tensor duality [1], we start with the $\mathcal{N}=1$ form of the action[1]

$$S = \int d^4x \mathcal{D}^2\bar{\mathcal{D}}^2 F(z,\bar{z},v,\bar{v},t,\bar{t},\cdots,x) \tag{70}$$

with F-function given, as in (61), by

$$F(z,\bar{z},v,\bar{v},t,\bar{t},\cdots,x) = \oint \frac{d\zeta}{\zeta} G(\eta^{(2j)}(\zeta)) \tag{71}$$

and then relax the constraint on the linear superfield v by means of a chiral Lagrange multiplier u. This yields the first-order action

$$S_1 = \int d^4x \mathcal{D}^2\bar{\mathcal{D}}^2 [F(z,\bar{z},v,\bar{v},t,\bar{t},\cdots,x) - uv - \bar{u}\bar{v}] \tag{72}$$

Indeed, by varying[2] u and $\bar{u}$ and substituting the result back into S_1 we retrive the constraint (57) and the action S. If, on the other hand, we vary the linear and the auxiliary superfields we arrive at the dual action

$$S' = \int d^4x \mathcal{D}^2\bar{\mathcal{D}}^2 K(z,\bar{z},u,\bar{u}) \tag{73}$$

where

$$K(z,\bar{z},u,\bar{u}) = F(z,\bar{z},v,\bar{v},t,\bar{t},\cdots,x) - uv - \bar{u}\bar{v} \tag{74}$$

and

$$\frac{\partial F}{\partial v} = u \tag{75}$$

$$\frac{\partial F}{\partial t} = \cdots = \frac{\partial F}{\partial x} = 0 \tag{76}$$

If the multiplet is of $\mathcal{O}(2)$ type there are no auxiliary fields and $v = \bar{v} = x \in \mathbb{R}$. The equations (75) and (76) are replaced in this case by the single equation

$$\frac{\partial F}{\partial x} = u + \bar{u} \tag{77}$$

The equations (75)-(76) respectively (77) can be implicitly solved to express v, $\bar{v}$, t, $\bar{t}$, $\ldots$, x respectively x in terms of z, $\bar{z}$, u, $\bar{u}$. In particular, notice that when G is an $\mathcal{O}(2)$ section over $\mathbb{CP}^1$ and thus satisfies (62), the extremization conditions (76) can be collectively reformulated as the vanishing of the following Penrose transform

$$\oint \pi_C d\pi^C \pi^{B_1}\cdots\pi^{B_{2j-4}} \frac{\partial G}{\partial \eta^{(2j)}} (\eta_{A_1\cdots A_{2j}}\pi^{A_1}\cdots\pi^{A_{2j}}) = 0 \tag{78}$$

Clearly, for this, one has to have $j \geq 2$, otherwise these equations would not exist.

[1] For simplicity, we denote here the $\mathcal{N}=1$ supercovariant derivative D_1 by $\mathcal{D}$.

[2] The variation of the action with respect to a chiral superfield should respect the chiral constraint. $\bar{\mathcal{D}}u = 0$ is solved by $u = \bar{\mathcal{D}}^2\psi$, with ψ an unconstrained superfield. The chiral constraint is automatically preserved if one substitutes u with $\bar{\mathcal{D}}^2\psi$ and then varies the action with respect to ψ instead.

The actions S and S' are dual to each other in the sense that they can be derived from the same first-order action S_1. This duality is the superspace equivalent of the Hodge duality between 0 and 2-form gauge fields in 4-dimensional space. To see this, note that the constraint (57) can be solved by

$$v = \mathcal{D}^\alpha W_\alpha + \bar{\mathcal{D}}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} \tag{79}$$

with W_α a chiral spinor superfield. This form stays invariant under the infinitesimal transformations

$$\delta W_\alpha = \bar{\mathcal{D}}^2 \mathcal{D}_\alpha V \tag{80}$$

with V a vector superfield. The linear superfield v, the chiral spinor superfield W_α and the vector superfield V count among their components a 3-form, a 2-form and a 1-form, respectively. It is then natural to think of v as the 'field strength' of W_α and regard (57) as a Bianchi identity and (80) as a gauge transformation. This can be made more precise with the help of superforms [20].

K is a function of chiral superfields and their anti-chiral conjugates, and thus looks like a Kähler potential. More precisely, the action S' inherits the $\mathcal{N} = 2$ supersymmetry of S, so K *is* in fact one of the Kähler potentials of a hyperkähler manifold. This means that the set of equations (71), (74), (75) and (76) can essentially be used to construct hyperkähler metrics from holomorphic functions. One can arrive at these same results through purely geometrical means [4, 5]. In the geometric language the superfields are simply replaced by coordinates. This approach is known in the literature as the generalized Legendre transform construction and its natural mathematical setting is the theory of hyperkähler twistor spaces.

3 Hyperkähler metrics from the generalized Legendre transform

3.1 The generic case

Equations (75)-(76) can in principle be solved to give v, $\bar{v}$, t, $\bar{t}$, $\ldots$, x as implicit functions of z, $\bar{z}$, u, $\bar{u}$. Implicit differentiation returns

$$\frac{\partial a}{\partial z} = -F^{ab} F_{bz} \qquad\qquad \frac{\partial a}{\partial u} = F^{av} \tag{81}$$

where a and b run over the values v, $\bar{v}$, t, $\bar{t}$, $\ldots$, x and F^{ab} is by definition the inverse of the matrix of second derivatives F_{ab}. Summation over repeated indices is assumed. On another hand, taking the derivatives of (74) with respect to the holomorphic coordinates and imposing afterwards the generalized Legendre relations (75)-(76), one gets that

$$\frac{\partial K}{\partial z} = \frac{\partial F}{\partial z} \qquad \text{and} \qquad \frac{\partial K}{\partial u} = -v \tag{82}$$

Using further equations (81) to take the derivatives of (82) with respect to the anti-holomorphic variables this time, one obtains the metric components

$$\begin{pmatrix} K_{z\bar{z}} & K_{z\bar{u}} \\ K_{u\bar{z}} & K_{u\bar{u}} \end{pmatrix} = \begin{pmatrix} F_{z\bar{z}} - F_{za} F^{ab} F_{b\bar{z}} & F_{za} F^{a\bar{v}} \\ F^{va} F_{a\bar{z}} & -F^{v\bar{v}} \end{pmatrix} \tag{83}$$

In the GLT approach the three standard Kähler forms are given by

$$\omega^3 = -i\, \partial\bar{\partial} K \qquad\qquad \omega^+ = dz \wedge du \qquad\qquad \omega^- = \overline{\omega^+} \tag{84}$$

From the last two expressions in (84) and the fact that for any hyperkähler manifold, in a coordinate basis holomorphic with respect to the complex structure J_3, the components of its (2,0) and (0,2) Kähler forms satisfy

$$\omega^+{}_{\mu\rho}\, \omega^{-\rho\nu} = -\delta_\mu{}^\nu \qquad\qquad \omega^{-\rho\sigma} K_{\rho\bar{\mu}} K_{\sigma\bar{\nu}} = \omega^-{}_{\bar{\mu}\bar{\nu}} \tag{85}$$

it follows that the inverse metric relates in a direct way the metric itself, namely

$$\begin{pmatrix} K^{z\bar{z}} & K^{z\bar{u}} \\ K^{u\bar{z}} & K^{u\bar{u}} \end{pmatrix} = \begin{pmatrix} K_{u\bar{u}} & -K_{u\bar{z}} \\ -K_{z\bar{u}} & K_{z\bar{z}} \end{pmatrix} \tag{86}$$

On the other hand, one can attempt to invert directly (83) using the following elementary linear algebra result: if A and D are non-singular square matrices then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix} \tag{87}$$

By identifying the upper-left block of the inverse metric obtained in this way to the upper-left block of (86), one gets

$$F_{z\bar{z}} - F_{za} F^{ab} F_{b\bar{z}} + F_{za} F^{a\bar{v}} H_{\bar{v}v} F^{vb} F_{b\bar{z}} = -H_{\bar{v}v} \tag{88}$$

where $H_{\bar{v}v}$ denotes the matrix inverse of $F^{v\bar{v}}$. This allows then one to rewrite (83) in the more symmetric form

$$\begin{pmatrix} K_{z\bar{z}} & K_{z\bar{u}} \\ K_{u\bar{z}} & K_{u\bar{u}} \end{pmatrix} = \begin{pmatrix} \mathbb{I} & F_{za} F^{a\bar{v}} \\ 0 & -F^{v\bar{v}} \end{pmatrix} \begin{pmatrix} -H_{\bar{v}v} & 0 \\ 0 & -H_{\bar{v}v} \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ F^{va} F_{a\bar{z}} & -F^{v\bar{v}} \end{pmatrix} \tag{89}$$

or, equivalently,

$$ds^2 = -dzH_{\bar{v}v}d\bar{z} - (du - dzF_{za}F^{a\bar{v}}H_{\bar{v}v})F^{v\bar{v}}(d\bar{u} - H_{\bar{v}v}F^{va}F_{a\bar{z}}d\bar{z}) \tag{90}$$

From the form (89) of the metric it is immediately apparent that the following Monge-Ampère equation holds

$$\det K_{(z,u)} = 1 \tag{91}$$

This result was established by Lindström and Roček [1] for the case of several $\mathcal{O}(2)$ multiplets and subsequently by Cherkis [21] for the case of one $\mathcal{O}(2j)$ multiplet. Both proofs exploit the fact that the contour-integral form (71) implies that the function F satisfies the following set of second order differential equations

$$\begin{aligned}
F_{z\bar{z}} &= -F_{v\bar{v}} = F_{t\bar{t}} = \cdots = (-)^j F_{xx} \\
F_{z\bar{v}} &= -F_{v\bar{t}} = \cdots \\
F_{zt} &= F_{vv} \qquad \text{etc.} \\
F_{zv} &= F_{vz} \qquad \text{etc.}
\end{aligned} \tag{92}$$

We stress that the argument presented here can be adapted in a straightforward manner to generic combinations of multiplets.

In the case of a single $\mathcal{O}(2j)$ multiplet the components of the metric can be expressed as ratios of determinants of Hankel matrices, as follows from

$$F_{za}F^{a\bar{v}} = \frac{(-)^j}{\det F} \det\left(h_{a+b-1}\right)_{-(j-1)\leq a,b\leq +(j-1)} \tag{93}$$

$$-F^{v\bar{v}} = \frac{(-)^j}{\det F} \det\left(h_{a+b-1}\right)_{-(j-2)\leq a,b\leq +(j-1)} \tag{94}$$

$$\det F = \det\left(h_{a+b}\right)_{-(j-1)\leq a,b\leq +(j-1)} \tag{95}$$

where

$$h_k = \oint \frac{d\zeta}{\zeta}\zeta^{-k}\frac{\partial^2 G}{\partial \eta^2} \tag{96}$$

These relations can be derived by writing the elements of the matrix F^{ab} that occur in the l.h.s. in terms of the cofactors of its inverse matrix, F_{ab}, and, in the first case only, by using subsequently Laplace's determinant expansion formula. The resulting determinants can then be algebraically manipulated into the forms displayed above. This generalizes an observation made by Cherkis and Hitchin in [16].

3.2 $\mathcal{O}(2)$-based constructions

As noted already, $\mathcal{O}(2)$ multiplets are in some sense special and their presence require certain adjustments to the GLT formalism developed above. There are no extremization conditions of the type (76) associated with $\mathcal{O}(2)$ multiplets as they do not have enough components, or, in a field-theoretical language, there are no auxiliary fields to be integrated out. There is only one Legendre relation associated to each of them. Distinguishing with an index I between the various $\mathcal{O}(2)$ multiplets of the theory, these take the form

$$\frac{\partial F}{\partial x^I} = u_I + \bar{u}_I \tag{97}$$

The departure of these relations from the regular form (75) for higher-degree multiplets stems from the reality properties of $\mathcal{O}(2)$ multiplets which require that x^I be real. In the absence of $\bar{u}_I$ the regular form would simply be inconsistent.

Let us consider the class of F-functions constructed exclusively out of a number of n $\mathcal{O}(2)$ multiplets and review the particularities of the corresponding $4n$-real dimensional hyperkähler manifolds. This class of manifolds was studied in [4] and [11]. The metric formula (90) specializes in this case to

$$ds^2 = -dz^I F_{x^I x^J} d\bar{z}^J - (du_I - dz^K F_{z^K x^I}) F^{x^I x^J} (d\bar{u}_J - F_{x^J \bar{z}^L} d\bar{z}^L) \tag{98}$$

where $F^{x^I x^J}$ is the matrix inverse of $F_{x^I x^J}$, whereas the holomorphic $(2,0)$-form from (84) becomes

$$\omega^+ = dz^I \wedge du_I \tag{99}$$

The fact that the holomorphic coordinates u_I occur exclusively in the combination $u_I + \bar{u}_I$ implies that the metric is independent of the imaginary part of u_I, in other words it has n abelian holomorphic Killing vectors

$$\tilde{X}^I = i\left(\frac{\partial}{\partial u_I} - \frac{\partial}{\partial \bar{u}_I}\right) \tag{100}$$

Clearly, these Killing vectors preserve the $(2,0)$-form (99) and hence they are not only holomorphic but in fact tri-holomorphic.

In the holomorphic coordinate basis z^I, u_I the hyperkäler structure is manifest but the underlying $SO(3)$ structure deriving from the $\mathcal{O}(2)$ multiplets is obscure and so are the abelian isometries. We can make the symmetries manifest and obscure the holomorphic structure by switching to a set of real coordinates defined as follows

$$\vec{r}^I = (z^I + \bar{z}^I, -i(z^I - \bar{z}^I), x^I) \tag{101}$$

$$\psi_I = \operatorname{Im} u_I \tag{102}$$

In this coordinate basis the metric takes a generalized Gibbons-Hawking form,

$$ds^2 \sim \Phi_{IJ}\, d\vec{r}^I \cdot d\vec{r}^J + (\Phi^{-1})^{IJ} (d\psi_I + \vec{A}_{IK} \cdot d\vec{r}^K)(d\psi_J + \vec{A}_{JL} \cdot d\vec{r}^L) \tag{103}$$

where the tilde simbolizes 'equal, up to an overall factor 1/2' and $\vec{A}$, Φ are defined by

$$\vec{A}_{IJ} \cdot d\vec{r}^J = \frac{i}{2}(F_{x^I z^J} dz^J - F_{x^I \bar{z}^J} d\bar{z}^J) \tag{104}$$

$$\Phi_{IJ} = -\frac{1}{2} F_{x^I x^J} \tag{105}$$

In order to be able to derive this expression of the metric from the form (98) one needs to use, among other things, the fact that

$$F_{x^I z^J} = F_{x^J z^I} \tag{106}$$

a direct consequence of the contour-integral form (71) of F. By resorting to this relation again in conjunction with equations (104) and (105) one can prove moreover that the following generalized Bogomol'nyi equations hold

$$\vec{\nabla}_I \times \vec{A}_{KJ} = -\vec{\nabla}_I \Phi_{KJ} \qquad \text{and} \qquad \vec{\nabla}_I \Phi_{KJ} = \vec{\nabla}_J \Phi_{KI} \tag{107}$$

where $\vec{\nabla}_I = \partial / \partial \vec{r}^I$ are $\mathbb{R}^3$ gradient operators.

4 Swann bundles from the generalized Legendre transform

4.1 Conformal conditions

For simplicity, the G-potentials that we consider here depend on only one $\mathcal{O}(2j)$ multiplet. The subsequent discussion can be straightforwardly generalized to include several multiplets or combinations of multiplets with different values of j, including $j = 1$. This will amount, essentially, to introducing additional indices to differentiate among these.

The reality constraint (51) is preserved only by the $PSU(2)$ subgroup of the $PSL(2, \mathbb{C})$ group of automorphisms of $\mathbb{CP}^1$. The $PSU(2)$ automorphic transformations of $\mathbb{CP}^1$ induce $PSO(3)$ transformations on the parameter space of $\mathcal{O}(2j)$ sections, generated by the vector fields

$$L_- = v\frac{\partial}{\partial z} - 2j\bar{z}\frac{\partial}{\partial \bar{v}} + 2t\frac{\partial}{\partial v} - (2j-1)\bar{v}\frac{\partial}{\partial \bar{t}} + \cdots \tag{108}$$

$$L_+ = \bar{v}\frac{\partial}{\partial \bar{z}} - 2jz\frac{\partial}{\partial v} + 2\bar{t}\frac{\partial}{\partial \bar{v}} - (2j-1)v\frac{\partial}{\partial t} + \cdots \tag{109}$$

$$iL_3 = 2j\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) + 2(j-1)\left(v\frac{\partial}{\partial v} - \bar{v}\frac{\partial}{\partial \bar{v}}\right) + 2(j-2)\left(t\frac{\partial}{\partial t} - \bar{t}\frac{\partial}{\partial \bar{t}}\right) + \cdots \tag{110}$$

$$S = j\left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} + v\frac{\partial}{\partial v} + \bar{v}\frac{\partial}{\partial \bar{v}} + t\frac{\partial}{\partial t} + \bar{t}\frac{\partial}{\partial \bar{t}} + \cdots + x\frac{\partial}{\partial x}\right) \tag{111}$$

satisfying the Lie-bracket algebra

$$[L_i, L_j] = 2\epsilon_{ijk} L_k \qquad\qquad [L_i, S] = 0 \tag{112}$$

We define as usual L_1 and L_2 by $L_\pm = (L_1 \pm iL_2)/2$.

A natural question arises: what is the structure that is induced by this parameter-space $PSO(3)$ action on hyperkähler varieties constructed by means of the generalized Legendre transform method based on the corresponding $\mathcal{O}(2j)$ multiplet? We find that, provided that certain 'conformal conditions' are satisfied, the induced structure is a Swann bundle structure, *i.e.* an $\mathbb{H}^*$-action comprising one conformal homothetic and three isometric generators [9]. More precisely, we obtain the following criterion: provided that
(1) G does not depend explicitly on ζ under the contour integral

$$\oint d\zeta \left(\frac{dG}{d\zeta} - \frac{\partial G}{\partial \zeta}\right) = 0 \tag{113}$$

(2) G is homogeneous of degree 1 when η scales with weight j up to terms that vanish under contour integration

$$\oint \frac{d\zeta}{\zeta}\left(j\eta\frac{\partial G}{\partial \eta} - G\right) = 0 \tag{114}$$

then the hyperkähler variety constructed by means of the generalized Legendre transform method from this G-potential posseses a Swann bundle structure. This criterion generalizes an observation of [22], made in the case $j = 1$. The conditions (113) and (114) translate into the following set of differential equations for F

$$L_3(F) = 0 \tag{115}$$

$$S(F) = F \tag{116}$$

where the linear differential operators L_3 and S are defined in (110) and (111), respectively.

The first step in the proof is to determine the action that the transformations generated by (108)-(111) induce on the holomorphic coordinates z and u. For the z coordinate we get immediately

$$L_-(z) = -\frac{\partial K}{\partial u} \tag{117}$$

$$L_+(z) = -\frac{\partial K}{\partial \bar{u}} \tag{118}$$

$$iL_3(z) = 2jz \tag{119}$$

$$S(z) = jz \tag{120}$$

In deriving (117) and (118) we made use of the second equation (82). For the u coordinate, using equation (75), we obtain

$$L_-(u) = \quad 2j\bar{z}F_{\bar{z}z} + (2j-1)\bar{v}F_{\bar{v}z} + (2j-2)\bar{t}F_{\bar{t}z} + \cdots - 2tF_{tz} + vF_{vz} \tag{121}$$

$$L_+(u) = -2jzF_{zt} - (2j-1)vF_{vt} - (2j-2)\bar{t}F_{\bar{t}t} - \cdots - 2\bar{t}F_{\bar{t}t} - \bar{v}F_{\bar{v}t} \tag{122}$$

$$iL_3(u) = 2j(zF_{zv} - \bar{z}F_{\bar{z}v}) + 2(j-1)(vF_{vv} - \bar{v}F_{\bar{v}v}) + 2(j-2)(tF_{tv} - \bar{t}F_{\bar{t}v}) + \cdots \tag{123}$$

$$S(u) = j(zF_{zv} + \bar{z}F_{\bar{z}v} + vF_{vv} + \bar{v}F_{\bar{v}v} + tF_{tv} + \bar{t}F_{\bar{t}v} + \cdots + xF_{xv}) \tag{124}$$

In (121) and (122) we made use, additionally, of the differential equations (92). The remaining equations follow directly. So far, these results hold generically. If we now take into account the conformal constraints (115) and (116), the equations (121) through (124) become

$$L_-(u) = \frac{\partial K}{\partial z} \tag{125}$$

$$L_+(u) = \frac{\partial K}{\partial \bar{z}} \tag{126}$$

$$iL_3(u) = -2(j-1)u \tag{127}$$

$$S(u) = -(j-1)u \tag{128}$$

Equation (125) follows from acting with a z-derivative on equations (115) and (116), substracting the outcomes and then substituting the result into the r.h.s. of (121). Equation (126) follows in a similar manner, with the difference that one acts in this case with a t-derivative. Equations (127) and (128) follow from acting with a v-derivative on equations (115) respectively (116) and then substituting the results into the r.h.s. of equations (123) respectively (124). To arrive at the above form, we subsequently resorted to the equations (82).

Hence, we conclude that when the conformal conditions (115) and (116) hold, the $PSO(3)$ generators (108)-(111) induce on the hyperkähler manifold an action represented by the vector fields[3]

$$X_- = -\frac{\partial K}{\partial u}\frac{\partial}{\partial z} + \frac{\partial K}{\partial z}\frac{\partial}{\partial u} \tag{129}$$

[3]For later convenience, X corresponds here to $2S$ whereas $X_\pm$ and X_3 correspond to $L_\pm$ and L_3, respectively.

$$X_+ = -\frac{\partial K}{\partial \bar{u}}\frac{\partial}{\partial \bar{z}} + \frac{\partial K}{\partial \bar{z}}\frac{\partial}{\partial \bar{u}} \tag{130}$$

$$iX_3 = 2j\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) - 2(j-1)\left(u\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{u}}\right) \tag{131}$$

$$X \ = 2j\left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right) - 2(j-1)\left(u\frac{\partial}{\partial u} + \bar{u}\frac{\partial}{\partial \bar{u}}\right) \tag{132}$$

As before, instead of $X_\pm$ we sometimes alternatively use X_1 and X_2, defined by $X_\pm = (X_1 \pm iX_2)/2$.

Let us now observe that, together with equations (74) through (76) and (82), equations (115) and (116) imply

$$X_3(K) = 0 \tag{133}$$

$$X(K) = 2K \tag{134}$$

with the linear operators X_3 and X given above. In fact, we have

$$X_i(K) = 0 \tag{135}$$

for $i = 1, 2, 3$.

Given that the vector field X is holomorphic with respect to the manifest complex structure J_3 associated to the coordinates z and u and that, as can be easily seen from (131) and (132), $X_3 = -J_3 X$, we have

$$X^\mu K_{\mu\bar{\nu}} = \partial_{\bar{\nu}}(X^\mu \partial_\mu K) = \frac{1}{2}\partial_{\bar{\nu}}[X(K) + iX_3(K)] = \partial_{\bar{\nu}} K \tag{136}$$

The holomorphic indices μ, ν correspond to the holomorphic coordinate basis $\{z, u\}$. Thus X is a gradient, *i.e.*,

$$X^\mu = K^{\mu\bar{\nu}}\partial_{\bar{\nu}} K \tag{137}$$

Denoting with ∇ the Levi-Civita connection associated to the metric $K_{\mu\bar{\nu}}$, this implies that $\nabla_\nu X^\mu = \delta_\nu{}^\mu$. On the other hand, one also has $\nabla_{\bar{\nu}} X^\mu = 0$, an immediate consequence of the holomorphicity of X. Together, these two properties indicate that X is a homothetic conformal vector, *i.e.*,

$$\nabla X = \mathbb{I} \tag{138}$$

where $\mathbb{I}$ stands for the identity endomorphism on the tangent bundle of the manifold.

4.2 Swann bundles

Let $(\mathcal{M}, g, \vec{J})$ be a hyperkähler variety with a homothetic conformal Killing vector field X satisfying the condition (138). We want to show that such a variety is automatically endowed with a Swann bundle structure, *i.e.* with an $\mathbb{H}^*$-action consisting of one conformal homothetic and three isometric generators. Consider the three vector fields defined by

$$X_i = -J_i X \tag{139}$$

with $i = 1, 2, 3$. In the context of the generalized Legendre construction discussed above it can be shown that this definition yields precisely (modulo an obvious linear transformation) the vector fields (129)-(131). We will demonstrate that X together with X_1, X_2 and X_3 generate on $\mathcal{M}$ an $\mathbb{H}^*$-action with the required properties.

Given a system of coordinates $\{x^\alpha\}$ on $\mathcal{M}$, the the conformal homothetic condition (138) reads, in components,

$$\nabla_\beta X^\alpha = \delta^\alpha{}_\beta \tag{140}$$

On the other hand, equation (139) together with (140) imply that

$$\nabla_\beta X_i^\alpha = -J_i^\alpha{}_\beta \tag{141}$$

The Lie derivatives of the metric along these vector fields are evaluated as follows

$$(\mathcal{L}_X g)_{\alpha\beta} = X^\gamma(\nabla_\gamma g_{\alpha\beta}) + (\nabla_\alpha X^\gamma) g_{\gamma\beta} + (\nabla_\beta X^\gamma) g_{\alpha\gamma} = 2 g_{\alpha\beta} \tag{142}$$

$$(\mathcal{L}_{X_i} g)_{\alpha\beta} = X_i^\gamma(\nabla_\gamma g_{\alpha\beta}) + (\nabla_\alpha X_i^\gamma) g_{\gamma\beta} + (\nabla_\beta X_i^\gamma) g_{\alpha\gamma} = \omega_{i\,\alpha\beta} + \omega_{i\,\beta\alpha} = 0 \tag{143}$$

The partial derivatives may be replaced by covariant derivatives since the Levi-Civita connection is torsion-free. The second set of equalities follows from the equations (140) and (141) and the fact that the Levi-Civita connection preserves the metric, *i.e.* $\nabla g = 0$. In (143), the ω_i are the Kähler 2-forms corresponding to the complex structures J_i. We have thus shown that the action of X is conformal, whereas the actions of X_1, X_2 and X_3 are isometric. To evaluate their action on the three standard hyperkähler complex structures, we proceed similarly

$$(\mathcal{L}_X J_j)^\alpha{}_\beta = X^\gamma(\nabla_\gamma J_j^\alpha{}_\beta) - (\nabla_\gamma X^\alpha) J_j^\gamma{}_\beta + (\nabla_\beta X^\gamma) J_j^\alpha{}_\gamma = 0 \tag{144}$$

$$(\mathcal{L}_{X_i} J_j)^\alpha{}_\beta = X_i^\gamma(\nabla_\gamma J_j^\alpha{}_\beta) - (\nabla_\gamma X_i^\alpha) J_j^\gamma{}_\beta + (\nabla_\beta X_i^\gamma) J_j^\alpha{}_\gamma = [J_i, J_j]^\alpha{}_\beta = 2\epsilon_{ijk}\, J_k^\alpha{}_\beta \tag{145}$$

The second set of equalities follows again from the equations (140) and (141) as well as from the compatibility condition of the hyperkähler complex structures with the the metric connection, *i.e.* $\nabla J_j = 0$ for all j. In (145) we use additionally the the fact that the complex structures satisfy a quaternionic algebra. Thus, we get that the vector field X preserves the complex structures whereas the vector fields X_i rotate them into one another. Eventually, based on this last set of results, we have

$$[X\,, X_j] = -\mathcal{L}_X(J_j X) \; = -(\mathcal{L}_X J_j) X \; - \; J_j(\mathcal{L}_X X) \; = 0 \tag{146}$$

$$[X_i, X_j] = -\mathcal{L}_{X_i}(J_j X) = -(\mathcal{L}_{X_i} J_j) X - J_j(\mathcal{L}_{X_i} X) = 2\epsilon_{ijk}\, X_k \tag{147}$$

In our case, these commutation relations can verified by direct calculation, using the holomorphic coordinate basis forms (129)-(132) of the vector fields. A somehow subtle point: one needs to use, among other things, the fact that the hyperkähler holomorphic $(2,0)$-form takes in the generalized Legendre transform approach the Darboux form

$$\omega^+ = dz \wedge du \tag{148}$$

Based on this, one can derive a set of relations between the components of the metric and those of its inverse which plays a direct role in the calculation of the commutators.

A distinguishing feature of Swann bundles among hyperkähler varieties is the fact that there exists a scalar function $f : \mathcal{M} \to \mathbb{R}$, defined up to the addition of a constant, which is simultaneously a Kähler potential for J_1, J_2 and J_3. To see that, let us first observe that the homothetic Killing vector condition (140) implies, based on the compatibility of the Levi-Civita connection with the metric, that $\nabla_\beta X_\alpha = g_{\alpha\beta}$, where $X_\alpha = g_{\alpha\beta} X^\beta$ is the dual vector field with respect to the metric bilinear form. Exploiting further the symmetry of the metric and the torsion-free character of the connection, one derives the closure condition

$$\partial_\alpha X_\beta - \partial_\beta X_\alpha = 0 \tag{149}$$

This implies that, locally, the dual vector field can be expressed as a total derivative of a scalar function f, and so X must be a gradient vector field, *i.e.*,

$$X^\alpha = g^{\alpha\beta}\partial_\beta f \tag{150}$$

where, as usual, $g^{\alpha\beta}$ represents the inverse metric. Choose now an arbitrary complex structure J_k and define the complex-valued 1-form

$$\theta_k = (X^\alpha + iX_k^\alpha) g_{\alpha\beta}\, dx^\beta \tag{151}$$

On one hand, based on (150), we can write this as follows

$$\theta_k = \partial_\alpha f(\delta^\alpha{}_\beta + iJ_k{}^\alpha{}_\beta) dx^\beta = 2\,\bar{\partial}_{J_k} f \tag{152}$$

On the other hand, we have

$$d\theta_k = \nabla_\alpha \theta_{k\beta}\, dx^\alpha \wedge dx^\beta = 2i\,\omega_k \tag{153}$$

where ω_k is the Kähler 2-form corresponding to the complex structure J_k. Ordinary derivatives can be replaced with covariant derivatives for the same reason invoked above. The second equality in (153) follows by substituting (151) and then making use of (140) and (141). The exterior derivative decomposes into $d = \partial_{J_k} + \bar{\partial}_{J_k}$ and so, from (152) and (153) we infer that

$$\omega_k = -i\,\partial_{J_k}\bar{\partial}_{J_k} f \tag{154}$$

which means that f is a Kähler potential for ω_k. Since our choice of complex structure was arbitrary, it follows that the function f is equally a Kähler potential for J_1, J_2 and J_3.

One can further show that the function f is $SU(2)$-invariant. Comparing with (135), we conclude that the generalized Legendre transform construction yields, in the case of Swann bundles, not just a Kähler potential corresponding to the explicit complex structure but in fact the hyperkähler potential, *i.e.*,

$$K = f + \text{const.} \tag{155}$$

5 Spherical representations and invariants of $\mathcal{O}(2j)$ multiplets

A generic $\mathcal{O}(2j)$ multiplet can be written in either one of the following two equivalent forms

$$\eta^{(2j)}(\zeta) = \sum_{m=-j}^{j} \binom{2j}{j+m}^{1/2} \bar{\psi}^j_m \zeta^m \tag{156}$$

$$= \frac{\varrho}{\zeta^j} \prod_{l=1}^{j} \frac{(\zeta - \alpha_l)(\bar{\alpha}_l \zeta + 1)}{1 + |\alpha_l|^2} \tag{157}$$

The reality requirement

$$\eta^{(2j)}(-\frac{1}{\bar{\zeta}}) = \overline{\eta^{(2j)}(\zeta)} \tag{158}$$

translates into the condition $\psi^j_{-m} = (-)^m \bar{\psi}^j_m$ on the coefficients in the first line and the condition $\varrho \in \mathbb{R}$ as well as into the antipodal pairing of the roots in the second line.

Under the Möbius action of an element R of $SU(2)$

$$\zeta \quad \xrightarrow{R} \quad \frac{a\,\zeta + b}{-\bar{b}\,\zeta + \bar{a}} \tag{159}$$

with $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$, the roots of $\eta^{(2j)}$ transform in the same way, *i.e.*,

$$\alpha_l \quad \xrightarrow{R} \quad \frac{a\,\alpha_l + b}{-\bar{b}\,\alpha_l + \bar{a}} \tag{160}$$

whereas the scale factor ϱ remains inert. Thus, the root system consists of j antipodal pairs of points on the Riemann sphere that rotate together rigidly - a *constellation* in the language of [23, 24]. On the other hand, under (159) the coefficients ψ^j_m transform according to Wigner's D-function realization of the spin-j unitary irrep of $SO(3)$ - the double-cover of $SU(2)$,

$$\psi^j_m \quad \xrightarrow{R} \quad \sum_{m'=-j}^{j} D^j_{mm'}(\phi, \theta, \psi) \psi^j_{m'} \tag{161}$$

where the Euler angles are related to the Cayley-Klein parameters of R by

$$a = \cos\frac{\theta}{2}\, e^{\frac{i}{2}(\phi+\psi)} \qquad\qquad b = \sin\frac{\theta}{2}\, e^{\frac{i}{2}(\phi-\psi)} \tag{162}$$

5.1 Quantum spin coherent states

Polynomials of the type $\zeta^j \eta^{(2j)}(\zeta)$ occur in the context of Quantum Mechanics in the guise of (unnormalized) spin-j wave functions. In the form (156) they are known as being in the *spin coherent state representation* [8], whereas in the form (157) as being in Majorana's *stellar representation* [25]. For this latter reason we will refer to them in these notes as *Majorana polynomials.*

The quantum states of a particle with spin j are commonly described as linear superpositions of $2j+1$ spherical harmonics, *i.e.*,

$$|\psi\rangle = \sum_{m=-j}^{j} \psi_m^j \, |jm\rangle \tag{163}$$

The spherical harmonics are simultaneous eigenvalues of the Casimir operator J^2 and of the operator J_z corresponding to the projection of the angular momentum along a preferential axis and form an orthonormal basis in the Hilbert space of states that transforms according to the spin-j unitary irrep of $SU(2)$. Under such a transformation, the linear coefficients of (163) transform just as in (161). Corresponding to any $\zeta \in \mathbb{C} \cup \{\infty\}$ one defines in this basis a *spin coherent state* by [8]

$$|\zeta\rangle = \frac{1}{(1+|\zeta|^2)^j} \sum_{m=-j}^{j} \binom{2j}{j+m}^{1/2} \zeta^{j+m} |jm\rangle \tag{164}$$

The set of spin coherent states forms an overcomplete basis in the Hilbert space of states. The wave function $\psi(\zeta) = \langle\psi|\zeta\rangle$ is said to be the spin coherent state representation of $|\psi\rangle$, and is equal, up to a non-holomorphic normalization factor, to a Majorana polynomial holomorphic in ζ, *i.e.*

$$\langle\psi|\zeta\rangle = \frac{1}{(1+|\zeta|^2)^j} \zeta^j \eta^{(2j)}(\zeta) \tag{165}$$

with $\eta^{(2j)}(\zeta)$ expressed as in (156). In particular, the spin coherent state representation of a purely spin coherent state labeled by the complex number α takes the form

$$\langle\alpha|\zeta\rangle = \left[\frac{1+\bar{\alpha}\zeta}{\sqrt{(1+|\zeta|^2)(1+|\alpha|^2)}}\right]^{2j} \tag{166}$$

Corresponding to the factorization (157), the spin-j wave function (165) decomposes, up to a quantum-mechanically irrelevant phase factor, into a product of $2j$ spin-1/2 coherent wave functions

$$\langle\psi|\zeta\rangle \sim \varrho \prod_{l=1}^{j} \langle -\frac{1}{\bar{\alpha}_l} |\zeta\rangle_{1/2} \langle\alpha_l|\zeta\rangle_{1/2} \tag{167}$$

Similarly, the coherent wave function (166) can be written as

$$\langle\alpha|\zeta\rangle = [\langle\alpha|\zeta\rangle_{1/2}]^{2j} \tag{168}$$

A very intuitively appealing picture emerges: a quantum state with spin j appears to be described by a set of $2j$ elementary 'spins 1/2' with the origins at the center of a Bloch sphere, pointing out in the directions marked by a constellation of $2j$ dots on the surface of the sphere corresponding to the roots of the wave function polynomial. In particular, a spin state is real in the sense of (158) when all elementary spins come in oppositely oriented pairs and is coherent when all elementary spins point in the same direction. Clearly, in the spin coherent state representation the rotational structure is preserved manifestly and no preferential axis needs to be chosen.

These elementary spins correspond essentially to Penrose's notion of *principal spinors*, as defined *e.g.* in [26], see also [27]. Penrose frames the above result in the following language: any nonvanishing totally symmetric $\eta_{A_1\cdots A_{2j}}$ admits a canonical decomposition

$$\eta_{A_1\cdots A_{2j}} = \chi^{(1)}_{(A_1}\chi^{(2)}_{A_2}\cdots\chi^{(2j)}_{A_{2j})} \tag{169}$$

in terms of a set of $2j$ commutative spinors $\chi^{(k)}_A$, uniquely defined up to proportionality and reordering.

The properties of quantum spin-1/2 coherent states are especially fit for use in spherical geometry, and we will exploit this feature later on. The overlap between two spin-1/2 coherent states corresponding to $\alpha, \beta \in \mathbb{C}\cup\{\infty\} \simeq S^2$ is[4]

$$\langle\alpha|\beta\rangle = \frac{1+\bar{\alpha}\beta}{\sqrt{(1+|\alpha|^2)(1+|\beta|^2)}} \tag{170}$$

Note that this formula implies that the overlap between states corresponding to pairs of antipodally-conjugated points is zero. The norms

$$|\langle\alpha|\beta\rangle| = k_{\alpha\beta} \qquad\qquad |\langle -\frac{1}{\bar{\alpha}}|\beta\rangle| = k'_{\alpha\beta} \tag{171}$$

are related to the geodesic distance on the sphere between α and β, see equations (174) and (175) below. On the other hand, the phases of cyclic sequences of spin-1/2 coherent states have an area interpretation, namely,

$$\langle\alpha_1|\alpha_2\rangle\langle\alpha_2|\alpha_3\rangle\cdots\langle\alpha_{n-1}|\alpha_n\rangle\langle\alpha_n|\alpha_1\rangle = k_{\alpha_1\alpha_2}k_{\alpha_2\alpha_3}\cdots k_{\alpha_{n-1}\alpha_n}k_{\alpha_n\alpha_1}e^{iA_{\text{polygon}}/2} \tag{172}$$

where A_{polygon} is the area of the spherical polygon with vertices at the points $\alpha_1\,\cdots\,\alpha_n$. The factor 1/2 in front of the area makes the ambiguity in the choice of what one means by the 'inside' and the 'outside' of the polygon irrelevant. For later reference, let us also

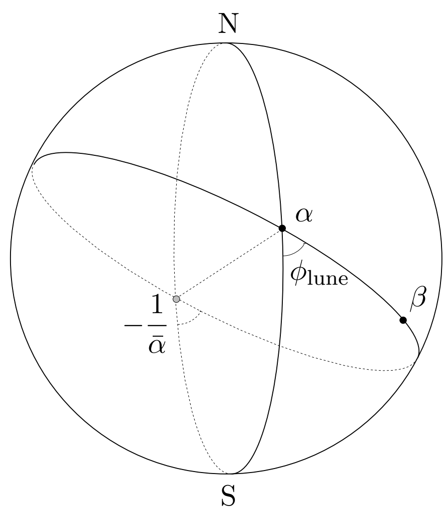


Figure 2.

[4]For simplicity, we drop here the index 1/2 from the notation of spin-1/2 coherent wave functions and will continue to do so throughout the remainder of these notes, unless we explicitly specify otherwise.

note that one can use equation (172) to show that

$$\langle -\frac{1}{\bar{\alpha}}|\beta\rangle\langle\beta|\alpha\rangle = k'_{\alpha\beta}k_{\alpha\beta}e^{iA_{\rm lune}/2} \tag{173}$$

where $A_{\rm lune}$ is the area of the *lune* cut on the sphere by the two geodesic circles that pass through β and α, respectively the South pole and α, equal to twice the dihedral angle $\phi_{\rm lune}$, see Figure 2.

5.2 Rotational invariants

For reasons to become clear later on, we are interested in constructing $SU(2)$ invariant quantities associated to one (pure-type invariants) or several (mixed-type invariants) multiplets. To this purpose we develop several different approaches that yield two basic classes of invariants: invariants that can be expressed explicitly and invariants that can be expressed only implicitly in terms of the Majorana coefficients. The former will turn out to be reducible in terms of the latter.

Our first approch is a natural off-shoot of the geometric picture detailed above. Consider a multiplet or a set of multiplets for which we want to compute invariants and the corresponding constellation of roots on the Riemann sphere endowed with the $SU(2)$-invariant metric of Fubini and Study. Given two such roots α and β from the same or from two different multiplets, the Fubini-Study distance between them is given by

$$\delta_{\alpha\beta} = 2\arccos k_{\alpha\beta} = 2\arcsin k'_{\alpha\beta} \tag{174}$$

with the chordal distance and radius expressed in terms of the roots as follows

$$k_{\alpha\beta} = \frac{|1+\bar{\alpha}\beta|}{\sqrt{(1+|\alpha|^2)(1+|\beta|^2)}} \qquad \text{and} \qquad k'_{\alpha\beta} = \frac{|\alpha-\beta|}{\sqrt{(1+|\alpha|^2)(1+|\beta|^2)}} \tag{175}$$

One has $k^2_{\alpha\beta}+k'^2_{\alpha\beta} = 1$ and thus $0 < k_{\alpha\beta}, k'_{\alpha\beta} < 1$. We can then use invariant Fubini-Study distances as building blocks to construct proper invariants by considering combinations of them subject to the additional condition that they are symmetric at the permutation of the roots of each of the multiplets involved.

A second approach involves constructing invariant Penrose-type transforms. It is based on the following result: let

$$\mathcal{I} = \oint_\Gamma \frac{d\zeta}{\zeta} G(\eta^{(2j)}(\zeta)) \tag{176}$$

be a contour integral, with G a meromorphic function possibly with branch cuts, depending on one or several multiplets denoted here collectively by $\eta^{(2j)}$ and Γ an integration contour that yields either a real or a purely imaginary $\mathcal{I}$, such that

(1) G does not depend explicitly on ζ other than through $\eta^{(2j)}(\zeta)$, and

(2) G scales, modulo terms that vanish under the contour integral, with weight -1 when each $\eta^{(2j)}(\zeta)$ is scaled with weight j.

Then, based on these conditions being satisfied, one can write $\mathcal{I}$ in the following manifestly $SU(2)$-invariant form

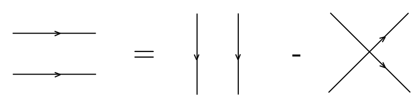
$$\mathcal{I} = \oint_\Gamma \pi_A d\pi^A G(\eta_{A_1\cdots A_{2j}}\pi^{A_1}\cdots\pi^{A_{2j}}) \tag{177}$$

Alternatively, one can use the spherical tensor properties of the coefficients ψ^j_m of the Majorana polynomials to construct spherical scalars by invariantly coupling two such tensors, three, a.s.o., *i.e.*,

$$\sum_m \bar{\psi}^j_m \psi^j_m \ , \qquad \sum_{m_1,m_2,m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \psi^{j_1}_{m_1} \psi^{j_2}_{m_2} \psi^{j_3}_{m_3} \ , \qquad \cdots \tag{178}$$

The coupling factors in the second expression are Wigner $3j$-symbols. The formulas become increasingly more complex with the number of angular momenta coupled.

A more uniform approach that leads to equivalent results is to form spherical scalars by completely contracting indices of various combinations of symmetric tensors $\eta_{A_1 \cdots A_{2j}}$ corresponding to the set of multiplets one is interested in computing invariants for. At first sight it may look like there exists an infinite number of such scalar configurations, but Penrose's canonical decomposition (169) implies that only a finite number of them are in fact independent. There is a nice way to depict these scalar combinations graphically by representing *e.g.* $\eta_{A_1 \cdots A_{2j}}$ as a vertex with $2j$ outgoing lines and $\eta^{A_1 \cdots A_{2j}}$ as a similar vertex but with $2j$ incoming lines. The resulting graphs can then be easily manipulated and related to each other by using diagrammatic identities such as



which expresses the ϵ-symbol identity $\epsilon^{AB}\epsilon_{CD} = \delta^A{}_C \delta^B{}_D - \delta^A{}_D \delta^B{}_C$. Graphs with legs starting and ending on the same vertex vanish, reflecting the fact that $\eta_{A_1 \cdots A_{2j}}$ is totally symmetric in its indices and hence yields zero when two of these are contracted with an ϵ-symbol. Reversing the orientation of a leg changes the sign of the graph.

5.3 Invariants of $\mathcal{O}(2)$ multiplets

A generic $\mathcal{O}(2)$ multiplet can be written locally in either one of the following two forms

$$\begin{aligned} \eta^{(2)}(\zeta) &= \frac{\bar{z}_1}{\zeta} + x_1 - z_1 \zeta \\ &= \frac{\sigma}{\zeta} \frac{(\zeta - \gamma)(\bar{\gamma}\zeta + 1)}{1 + |\gamma|^2} \end{aligned} \tag{179}$$

The coefficients can be expressesed in terms of the roots and scale factor σ explicitly. Conversely, in order to express the roots in terms of the coefficients one has to solve a quadratic equation.

To a real $\mathcal{O}(2)$ section one can associate only one independent invariant, namely σ. An invariant integral is

$$\oint_\Gamma \frac{d\zeta}{\zeta} \frac{1}{\eta^{(2)}(\zeta)} = \frac{2}{\sigma} \tag{180}$$

with the contour Γ depicted in Figure 3.

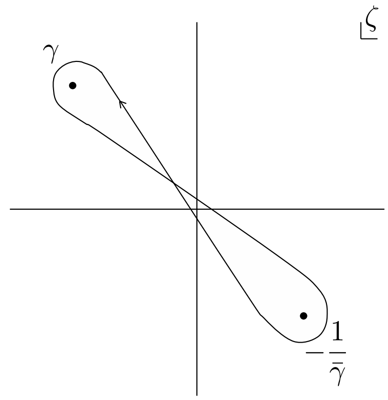
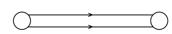


Figure 3. The contour Γ

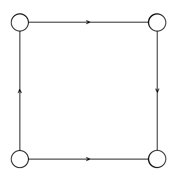
Alternatively, as suggested above, one can consider the spherical scalar

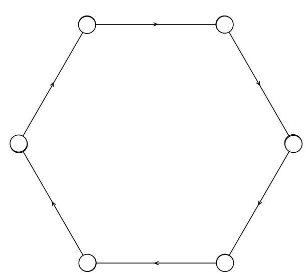
$$g_{\sigma^2} = -2\,\eta_{AB}\eta^{AB} = -2\times$$ 

The factor -2 has been inserted for convenience. A short calculation yields that

$$g_{\sigma^2} = 4|z_1|^2 + x_1^2 = \sigma^2 \tag{181}$$

As one can easily check, all other invariants can be deconstructed down to g_{σ^2}, *e.g.*,

 $= \frac{1}{8}g_{\sigma^2}^2$

 $= -\frac{1}{32}g_{\sigma^2}^3$

a.s.o. Polygons with $2k+1$ sides vanish identically. Polygons with $2k$ sides yield the k-th power of g_{σ^2}, times a numerical factor.

The coefficients z_1, x_1, $\bar z_1$ form an $SO(3)$ vector multiplet. This can be cast in an Euclidian basis by the linear transformation

$$z_1 = \frac{1}{2}(x+iy) \qquad x_1 = z \qquad \bar z_1 = \frac{1}{2}(x-iy) \tag{182}$$

We shall use the notation $\vec r_1$ for the $\mathbb{R}^3$ vector with components x, y, z. Clearly,

$$|\vec r_1| = \sqrt{x^2+y^2+z^2} = \sigma \tag{183}$$

i.e., σ represents the invariant length of the vector associated in this manner with the $\mathcal{O}(2)$ multiplet.

5.4 Invariants of $\mathcal{O}(4)$ multiplets

A generic $\mathcal{O}(4)$ multiplet can be written locally in either one of the following two forms

$$\begin{aligned}\eta^{(4)}(\zeta) &= \frac{\bar{z}_2}{\zeta^2} + \frac{\bar{v}_2}{\zeta} + x_2 - v_2\zeta + z_2\zeta^2 \\ &= \frac{\rho}{\zeta^2}\frac{(\zeta-\alpha)(\bar{\alpha}\zeta+1)}{1+|\alpha|^2}\frac{(\zeta-\beta)(\bar{\beta}\zeta+1)}{1+|\beta|^2}\end{aligned} \tag{184}$$

The coefficients can be expressed directly in terms of the roots, but conversely, expressing the roots explicitly in terms of the coefficients involves solving a quartic equation, an impractical approach.

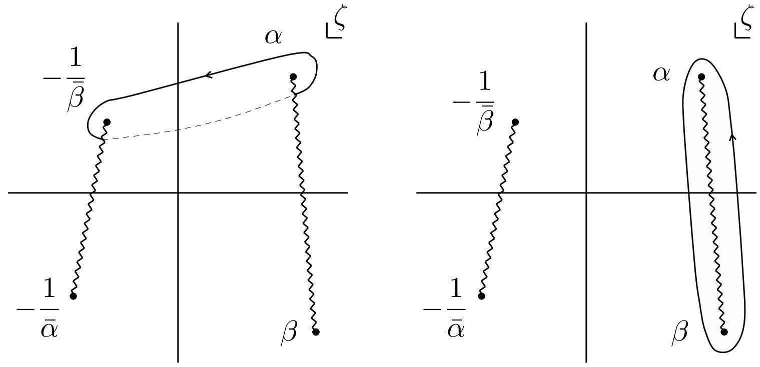


Figure 4. Integration contours Γ_a (left) and Γ_b (right).

An invariant integral with the required homogeneity property is

$$\mathcal{I}(\Gamma) = \oint_\Gamma \frac{d\zeta}{\zeta}\frac{1}{\sqrt{\eta^{(4)}(\zeta)}} \tag{185}$$

The two generators of the canonical homology basis for the closed contours Γ are depicted in Figure 4. They correspond to the a and b-cycles of the $\mathcal{O}(4)$ curve associated to the multiplet. The integrals over these two contours are precisely the period integrals of the $\mathcal{O}(4)$ curve. In section 8 we will show that they can be expressed in terms of the complete elliptic integrals of modulus $k_{\alpha\beta}$ respectively complementary modulus $k'_{\alpha\beta}$ as follows

$$\mathcal{I}(\Gamma_a) = \frac{2}{\sqrt{\rho}}K(k_{\alpha\beta}) \stackrel{\text{def}}{=} \frac{2}{r_2} \qquad\qquad \mathcal{I}(\Gamma_b) = \frac{2}{\sqrt{\rho}}iK(k'_{\alpha\beta}) \stackrel{\text{def}}{=} \frac{2\pi i}{r'_2} \tag{186}$$

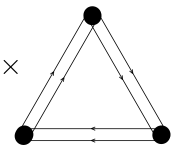
The second set of equalities are definitions inspired by and analogous to (180). The π-factor in the r.h.s. has been chosen for later convenience but is otherwise irrelevant. Since $0 < k_{\alpha\beta}, k'_{\alpha\beta} < 1$, the elliptic integrals are real and so

$$r_2, r'_2 > 0 \tag{187}$$

Based on this and their rotational invariance property, we shall refer to r_2 and r'_2 as '$\mathcal{O}(4)$ radii'.

We can also construct $\mathcal{O}(4)$ rotational invariants by completely contracting the indices of products of η_{ABCD} tensors. With the diagrammatic conventions introduced above, let for instance

$g_{\rho^2} = 2\times$ 

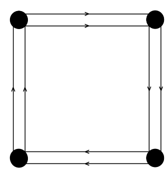
$g_{\rho^3} = \frac{8}{3}\times$ 

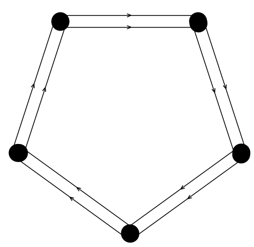
The multiplicative factors have been inserted for convenience. A straightforward calculation yields the following Majorana-coefficient expressions

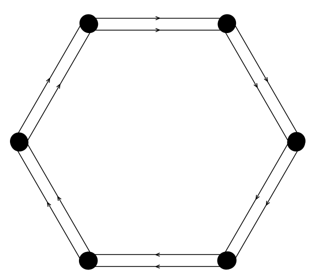
$$g_{\rho^2} = 4|z_2|^2 + |v_2|^2 + \frac{1}{3}x_2^2 \tag{188}$$

$$g_{\rho^3} = \frac{8}{3}|z_2|^2 x_2 - \frac{1}{3}|v_2|^2 x_2 - \frac{2}{27}x_2^3 - z_2\bar{v}_2^2 - \bar{z}_2 v_2^2 \tag{189}$$

g_{ρ^2} and g_{ρ^3} are essentially the only independent invariants that one can construct in this manner. All higher order spherical scalars break down ultimately into these two basic components. For example,

 $= \frac{1}{8}g_{\rho^2}^2$

 $= \frac{5}{32}g_{\rho^2}g_{\rho^3}$

 $= \frac{1}{64}(2g_{\rho^2}^3 + 3g_{\rho^3}^2)$

a.s.o.

And yet a third pair of invariants is provided by the scale ρ and the chordal distance $k_{\alpha\beta}$. Clearly though, these sets of pairs of invariants are not all independent. As a matter of fact, one can express g_{ρ^2} and g_{ρ^3} in terms of both ρ, $k_{\alpha\beta}$ and r_2, r_2'. For instance, by resorting to the Viète relations between the coefficients and the roots of (184), one can show that

$$g_{\rho^2} = -(e_1e_2 + e_2e_3 + e_3e_1) \tag{190}$$

$$g_{\rho^3} = e_1e_2e_3 \tag{191}$$

with

$$e_1 = -\frac{\rho}{3}(k_{\alpha\beta}^2 - 2) \qquad e_2 = \frac{\rho}{3}(2k_{\alpha\beta}^2 - 1) \qquad e_3 = -\frac{\rho}{3}(k_{\alpha\beta}^2 + 1) \tag{192}$$

On the other hand, as we shall see in section 8.5, g_{ρ^2} and g_{ρ^3} have theta-function representations which allow us to express them in terms of r_2 and r_2' in the form of infinite Lambert-type series.

5.5 Mixed invariants of $\mathcal{O}(2)$ and $\mathcal{O}(4)$ multiplets

Consider now the combination of an $\mathcal{O}(2)$ with an $\mathcal{O}(4)$ multiplet. An invariant integral containing both is

$$\mathcal{I}(\Gamma) = \oint_\Gamma \frac{d\zeta}{\zeta} \frac{\eta^{(2)}(\zeta)}{\eta^{(4)}(\zeta)} \tag{193}$$

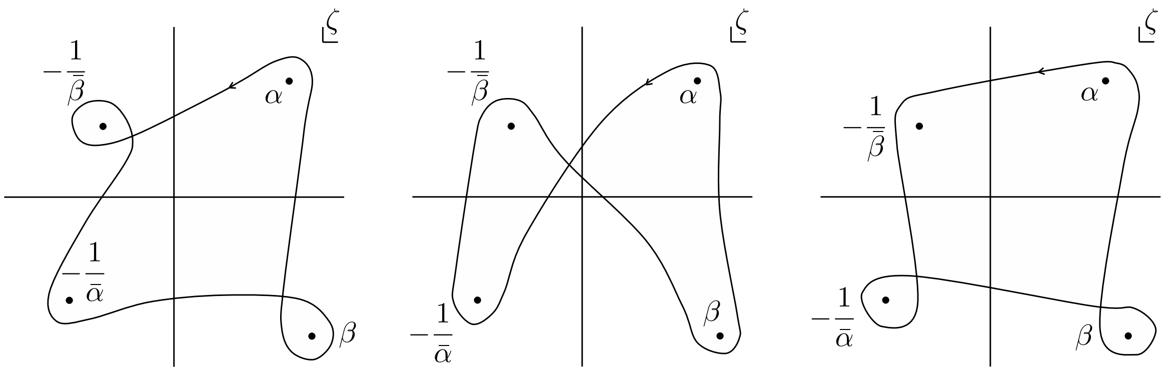


Figure 5. Integration contours Γ_0 (left), Γ_+ (middle) and Γ_- (right)

For the three independent contours depicted in Figure 5 we obtain

$$\mathcal{I}(\Gamma_0) = i\frac{\sigma}{\rho}\frac{Q_0}{k_{\alpha\beta}^2 k_{\alpha\beta}'^2} \qquad \mathcal{I}(\Gamma_+) = \frac{\sigma}{\rho}\frac{Q_+}{k_{\alpha\beta}^2} \qquad \mathcal{I}(\Gamma_-) = \frac{\sigma}{\rho}\frac{Q_-}{k_{\alpha\beta}'^2} \tag{194}$$

with

$$Q_\pm^2 = (\cos\delta_{\alpha\gamma} \pm \cos\delta_{\beta\gamma})^2 \tag{195}$$

and

$$Q_0^2 = \begin{vmatrix} 1 & \cos\delta_{\alpha\gamma} & \cos\delta_{\alpha\beta} \\ \cos\delta_{\alpha\gamma} & 1 & \cos\delta_{\beta\gamma} \\ \cos\delta_{\alpha\beta} & \cos\delta_{\beta\gamma} & 1 \end{vmatrix} = 36 \times (\mathrm{Vol}_{OABC})^2 \tag{196}$$

A, B and C are the points on the round sphere corresponding to the roots α, β and γ, O is the center of the sphere and Vol_{OABC} is the volume of the tetrahedron $OABC$. The vanishing of Q_0^2 is the necessary and sufficient condition for the three points A, B and C to lie on the same geodesic circle (*i.e.* to be colinear in the sense of projective geometry).

Alternatively, define the invariants

$$g_{\rho\sigma^2} = 4 \times \frac{1}{4!}\,\eta_{(AB}\,\eta_{CD)}\,\eta^{ABCD} \tag{197}$$

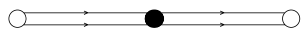
$$g_{\rho^2\sigma^2} = 24 \times \frac{1}{4!}\,\eta_{(AB}\,\eta_{CD)}\,\eta^{ABEF}\eta_{EF}{}^{CD} \tag{198}$$

The numerical factors are chosen for convenience. Explicitly, they take the nondescript forms

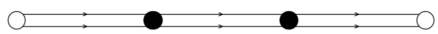
$$g_{\rho\sigma^2} = \frac{2}{3}x_2(x_1^2 - 2|z_1|^2) + 4z_2\bar{z}_1^2 + 4\bar{z}_2 z_1^2 + 2v_2\bar{z}_1 x_1 + 2\bar{v}_2 z_1 x_1 \tag{199}$$

$$\begin{aligned} g_{\rho^2\sigma^2} = {} & (8|z_2|^2 - |v_2|^2 - \frac{2}{3}x_2^2)(x_1^2 - 2|z_1|^2) - 12z_2\bar{v}_2\bar{z}_1 x_1 - 12\bar{z}_2 v_2 z_1 x_1 \\ & + 8z_2x_2\bar{z}_1^2 + 8\bar{z}_2x_2z_1^2 - 2v_2x_2\bar{z}_1x_1 - 2\bar{v}_2x_2z_1x_1 - 3v_2^2\bar{z}_1^2 - 3\bar{v}_2^2 z_1^2 \end{aligned} \tag{200}$$

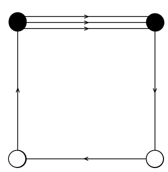
The diagram corresponding to the first spherical invariant is

$4\times$ 

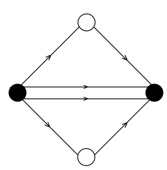
whereas the second one can be represented for instance by

$24\times$  $+\ 2g_{\rho^2}g_{\sigma^2}$

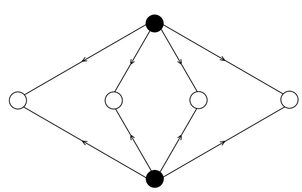
Any other diagram constructed from either one or both of these multiplets can be reduced to homogeneous rational polynomial expressions in terms of these basic invariants. For example,



$$= -\frac{1}{8}g_{\rho^2}g_{\sigma^2}$$



$$= \frac{1}{24}(g_{\rho^2\sigma^2} + g_{\rho^2}g_{\sigma^2})$$



$$= \frac{1}{96}(6g_{\rho\sigma^2}^2 + 2g_{\rho^2\sigma^2}g_{\sigma^2} - g_{\rho^2}g_{\sigma^2}^2)$$

a.s.o. Note that all combinations with an odd number of $\mathcal{O}(2)$ vertices vanish.

To relate the two types of mixed invariants that we have introduced so far, we start from the observation that $\eta^{(4)} - \lambda\,(\eta^{(2)})^2$ is a real (w.r.t. antipodal conjugation) $\mathcal{O}(4)$ multiplet for any real invariant coupling scale λ. In particular, one can construct the associated $\mathcal{O}(4)$ basic spherical invariants

$$g_2(\lambda) = g_{\rho^2} - 3g_{\rho\sigma^2}\left(\frac{\lambda}{3}\right) + 3g_{\sigma^2}^2\left(\frac{\lambda}{3}\right)^2 \tag{201}$$

$$g_3(\lambda) = g_{\rho^3} - g_{\rho^2\sigma^2}\left(\frac{\lambda}{3}\right) - 3g_{\rho\sigma^2}g_{\sigma^2}\left(\frac{\lambda}{3}\right)^2 + 2g_{\sigma^2}^3\left(\frac{\lambda}{3}\right)^3 \tag{202}$$

The following remarkable relations hold

$$g_3\left(\frac{3e_1}{\sigma^2}\right) = +\frac{3}{4}\rho^2 e_1 Q_-^2 \tag{203}$$

$$g_3\left(\frac{3e_2}{\sigma^2}\right) = -\frac{3}{4}\rho^2 e_2 Q_0^2 \tag{204}$$

$$g_3\left(\frac{3e_3}{\sigma^2}\right) = +\frac{3}{4}\rho^2 e_3 Q_+^2 \tag{205}$$

with the e_1, e_2 and e_3 given in (192). This can be verified by expressing everything in terms of the Majorana roots and scales. The simplification that occurs at these particular couplings is quite substantial in view of the fact that $g_{\rho\sigma^2}$ and $g_{\rho^2\sigma^2}$ alone contain 36 respectively 141 terms when expressed in terms of the roots. From any two of the equations (203) through (205) one obtains the relations

$$g_{\rho\sigma^2} = \rho\sigma^2(\cos\delta_{\alpha\gamma}\cos\delta_{\beta\gamma} - \frac{1}{3}\cos\delta_{\alpha\beta}) \tag{206}$$

$$g_{\rho^2\sigma^2} = g_{\rho^2}g_{\sigma^2} + \frac{1}{4}\rho^2\sigma^2(Q_0^2 - Q_+^2 - Q_-^2) \tag{207}$$

5.6 Rotational invariants as quantum amplitudes

Let $|\psi_{\eta^{(2)}}\rangle$ and $|\psi_{\eta^{(4)}}\rangle$ be the spin-1 respectively spin-2 quantum coherent states associated to $\eta^{(2)}$ and $\eta^{(4)}$ according the prescription of section 5.1. By taking tensor products of these elementary states one can construct composite states. For example, the tensor product $|\psi_{\eta^{(2)}}\otimes\psi_{\eta^{(2)}}\rangle$ has a spin-2 component equal to $|\psi_{(\eta^{(2)})^2}\rangle$, no spin-1 component, and a spin-0 component given by $-1/(2\sqrt{3})\sigma^2|00\rangle$. Hilbert scalar products of the quantum states formed in this way yield rotational invariants of the type discussed above:

$$\|\psi_{\eta^{(2)}}\|^2 = \frac{1}{2}g_{\sigma^2} \tag{208}$$

$$\|\psi_{\eta^{(4)}}\|^2 = \frac{1}{2}g_{\rho^2} \tag{209}$$

$$\langle\psi_{\eta^{(4)}}|\psi_{\eta^{(4)}}\otimes\psi_{\eta^{(4)}}\rangle = \frac{9}{4\sqrt{21}}g_{\rho^3} \tag{210}$$

$$\langle\psi_{\eta^{(2)}}|\psi_{\eta^{(2)}}\otimes\psi_{\eta^{(4)}}\rangle = -\frac{3}{4\sqrt{15}}g_{\rho\sigma^2} \tag{211}$$

$$\langle\psi_{\eta^{(4)}}|\psi_{\eta^{(2)}}\otimes\psi_{\eta^{(2)}}\rangle = \frac{1}{4}g_{\rho\sigma^2} \tag{212}$$

$$\langle\psi_{\eta^{(2)}}\otimes\psi_{\eta^{(2)}}|\psi_{\eta^{(4)}}\otimes\psi_{\eta^{(4)}}\rangle = \frac{1}{4\sqrt{21}}g_{\rho^2\sigma^2} - \frac{1}{4\sqrt{15}}g_{\rho^2}g_{\sigma^2} \tag{213}$$

This is because these scalar products lead to expressions of the type (178) when written in a spherical basis. Not all such scalar products lead to independent invariants. Some vanish, yielding orthogonality relations, *e.g.*,

$$\langle\psi_{\eta^{(2)}}|\psi_{\eta^{(4)}}\rangle = 0 \tag{214}$$

$$\langle\psi_{\eta^{(2)}}|\psi_{\eta^{(4)}}\otimes\psi_{\eta^{(4)}}\rangle = 0 \tag{215}$$

$$\langle\psi_{\eta^{(4)}}|\psi_{\eta^{(2)}}\otimes\psi_{\eta^{(4)}}\rangle = 0 \tag{216}$$

The representation of the invariants as quantum amplitudes can be put to use to derive various inequalities. For instance, from the positive-definiteness of the Hilbert space norm it follows that

$$g_{\sigma^2},\ g_{\rho^2} \geq 0 \tag{217}$$

This is consistent with the conclusion that one can derive in a more direct manner by examining the Majorana-coefficient expressions of g_{σ^2} and g_{ρ^2}. Other, less obvious relations follow by way of Cauchy-Schwarz inequalities on the Hilbert space. For example, from the equation (212) together with (208) and (209) one gets upper and lower bounds for $g_{\rho\sigma^2}$

$$-\sqrt{2} \leq \frac{g_{\rho\sigma^2}}{\sqrt{g_{\rho^2} g_{\sigma^2}}} \leq \sqrt{2} \tag{218}$$

whereas the equation (213) together with (208) and (209) yield upper and lower bounds for $g_{\rho^2\sigma^2}$

$$-\sqrt{\frac{7}{5}}(\sqrt{15}-1) \leq \frac{g_{\rho^2\sigma^2}}{g_{\rho^2} g_{\sigma^2}} \leq \sqrt{\frac{7}{5}}(\sqrt{15}+1) \tag{219}$$

We will henceforth refer to positive-definite invariants as in (217) as being of *radial* type and to doubly-bounded invariants as in (218) and (219) as being of *angular* type. To underline their angular character we will sometimes use instead of the mixed invariants $g_{\rho\sigma^2}$ and $g_{\rho^2\sigma^2}$ the equivalent pair of invariants

$$A = \frac{g_{\rho\sigma^2}}{\sqrt{3 g_{\rho^2} g_{\sigma^2}}} \qquad \text{and} \qquad B = -\frac{g_{\rho^2\sigma^2}}{3 g_{\rho^2} g_{\sigma^2}} \tag{220}$$

The numerical factors have been chosen for later convenience.

6 ALE and ALF metrics of type A_n

6.1 Multi-center metrics

In this section we survey the generalized Legendre transform construction of ALE and ALF spaces of type A_{n-1}, also known as multi-Eguchi-Hanson respectively multi-Taub-NUT spaces [28, 29, 30], [31, 32]. As observed for the first time in [4], their generating F-function is given by

$$F = -\frac{1}{2\pi i h}\oint_{\Gamma_0}\frac{d\zeta}{\zeta}\eta^2 + \sum_{l=1}^{n}\oint_{\Gamma_l}\frac{d\zeta}{\zeta}(\eta-\chi_l)\ln(\eta-\chi_l) \tag{221}$$

For $h \geq 0$ the resulting metrics are non-singular. The ALE metrics correspond to $h \to \infty$, the ALF metrics to $h = m^2 > 0$. The contour Γ_0 encloses the origin, whereas the n contours Γ_l are lemniscates enclosing the roots of $\eta - \chi_l$. The $\mathcal{O}(2)$ multiplet η takes the local form

$$\eta = \frac{\bar{z}}{\zeta} + x - z\zeta \tag{222}$$

Its parameters transform in the spin 1 representation of $SO(3)$ under the automorphic $SU(2)$ transformations of $\mathbb{CP}^1$. The usual vector representation is obtained through the linear transformation

$$z = \frac{1}{2}(x^1 + ix^2) \qquad\qquad x = x^3 \qquad\qquad \bar{z} = \frac{1}{2}(x^1 - ix^2) \tag{223}$$

Let $\vec{r}$ be the $\mathbb{R}^3$ vector with components x^1, x^2, x^3 and r its $SO(3)$-invariant length,

$$r = \sqrt{x^2 + 4|z|^2} \tag{224}$$

The n $\mathcal{O}(2)$ multiplets χ_l *parametrize* the positions of the monopoles. For each of them we introduce notations similar to the above, with an index l adjoined.

Evaluating the integrals in (221) explicitly, we obtain

$$F = -\frac{r^2 - 6|z|^2}{h} + 2\sum_{l=1}^{n}\left[|\vec{r}-\vec{r}_l|^2 - (x-x_l)\tanh^{-1}\frac{x-x_l}{|\vec{r}-\vec{r}_l|}\right] \tag{225}$$

Succesive derivations of (225) yield

$$F_x = -2\frac{x}{h} - 2\sum_{l=1}^{n}\tanh^{-1}\frac{x-x_l}{|\vec{r}-\vec{r}_l|} \tag{226}$$

$$F_{xx} = -\frac{2}{h} - \sum_{l=1}^{n}\frac{2}{|\vec{r}-\vec{r}_l|} \tag{227}$$

$$F_{zx} = \sum_{l=1}^{n}\frac{1}{z-z_l}\frac{x-x_l}{|\vec{r}-\vec{r}_l|} \tag{228}$$

The Kähler potential for this $\mathcal{O}(2)$-based theory is obtained through the Legendre transform

$$K = F - x(u+\bar{u}) \tag{229}$$

with

$$F_x = u + \bar{u} \tag{230}$$

We get

$$K = \frac{r^2 - 2|z|^2}{h} + 2\sum_{l=1}^{n}\left[|\vec{r} - \vec{r}_l| + x_l \tanh^{-1}\frac{x - x_l}{|\vec{r} - \vec{r}_l|}\right] \tag{231}$$

Equation (230), with F_x given in (226), allows one, in principle, to solve for x as a function of the holomorphic coordinates z, u and their complex conjugates (more precisely, as a function of $z\bar{z}$ and $u+\bar{u}$). Note that this equation is *transcendental* for finite h and *algebraic* for $h \to \infty$. In this latter case this is due to the addition theorem satisfied by $\tanh^{-1}$. But for practical purposes we do not need an explicit solution. Implicit differentiation yields

$$dx = \frac{1}{2\Phi}(A_z dz + A_{\bar{z}} d\bar{z} - du - d\bar{u}) \tag{232}$$

with

$$\Phi = \frac{1}{h} + \sum_{l=1}^{n}\frac{1}{|\vec{r} - \vec{r}_l|} \tag{233}$$

$$A_z = \sum_{l=1}^{n}\frac{1}{z - z_l}\frac{x - x_l}{|\vec{r} - \vec{r}_l|} \tag{234}$$

This allows us to calculate the derivatives of the Kähler potential (231) with respect to the holomorphic variables, leading to the metric

$$ds^2 = 2\Phi\, dz d\bar{z} + (2\Phi)^{-1}(du - A_z dz)(d\bar{u} - A_{\bar{z}} d\bar{z}) \tag{235}$$

As we have remarked above, u and $\bar{u}$ occur only in the combination $u + \bar{u}$. Shifts in the imaginary direction of u leave things unchanged. As a consequence, the metric (235) has an isometry generated by the vector field

$$\tilde{X} = i\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial \bar{u}}\right) \tag{236}$$

The generalized Legendre transform approach also prescribes the form of the hyperkäler holomorphic $(2,0)$-form in this particular set of holomorphic coordinates. Specifically,

$$\omega^+ = dz \wedge du \tag{237}$$

Using this and equation (235) it is straightforward to verify that the isometry (236) is tri-holomorphic.

In the holomorphic coordinate basis z, $\bar{z}$, u, $\bar{u}$ the hyperkäler structure is manifest but the underlying $SO(3)$ structure deriving from the $\mathcal{O}(2)$ multiplets is obscure, and so is the abelian isometry. We can make the symmetries manifest and obscure the holomorphic structure by switching to the real coordinates $\vec{r}$ and ψ, where

$$\psi = \mathrm{Im}\, u \tag{238}$$

coordinatizes the orbits of $\tilde{X}$. Equation (232) together with the equations (223) yield the Jacobian of the transformation. In the new coordinate basis, the metric (235) reads[5]

$$ds^2 \sim \Phi\, d\vec{r}^{\,2} + \Phi^{-1}(d\psi + \vec{A}\cdot d\vec{r})^2 \tag{239}$$

[5] In this and the next section the tilde sign means 'equal, up to an overall factor 1/2'.

with

$$\vec{A}\cdot d\vec{r} = \sum_{l=1}^{n} \frac{x^3 - x_l^3}{|\vec{r}-\vec{r}_l|} \frac{(x^2 - x_l^2)dx^1 - (x^1 - x_l^1)dx^2}{(x^1 - x_l^1)^2 + (x^2 - x_l^2)^2} \tag{240}$$

One can verify that

$$\vec{\nabla}\times\vec{A} = -\vec{\nabla}\Phi \tag{241}$$

6.2 The single-center case

The case $n = 1$ corresponds to the Taub-NUT metric when h is finite or to the flat-space metric when $h \to \infty$. Without loss of generality we can set $\vec{r}_1 = 0$ and thus, for instance, cast the generating F-function in the form

$$F = -\frac{1}{2\pi i h}\oint_{\Gamma_0} \frac{d\zeta}{\zeta}\eta^2 + \oint_{\Gamma} \frac{d\zeta}{\zeta}\eta \ln \eta \tag{242}$$

The Kähler potential (231) does not contain in this case any $\tan^{-1}$ terms, *i.e.*,

$$K = \frac{r^2 - 2|z|^2}{h} + 2r \tag{243}$$

and the harmonic potential (233) becomes simply

$$\Phi = \frac{1}{h} + \frac{1}{r} \tag{244}$$

A special feature of this case which does not survive for higher n is the presence of an $SO(3)$ isometry. The isometry transformations are induced by the obvious $SO(3)$ action on the parameter space of the $\mathcal{O}(2)$ multiplet. This determines their form almost completely, up to an ambiguity related to the presence of the tri-holomorphic isometry. In the holomorphic coordinate basis introduced above the isometry generators take the form

$$X_- = x\frac{\partial}{\partial z} + \frac{hr + 2|z|^2}{hz}\frac{\partial}{\partial u} + 2\frac{\bar{z}}{h}\frac{\partial}{\partial \bar{u}} \tag{245}$$

$$X_+ = x\frac{\partial}{\partial \bar{z}} + \frac{hr + 2|z|^2}{h\bar{z}}\frac{\partial}{\partial \bar{u}} + 2\frac{z}{h}\frac{\partial}{\partial u} \tag{246}$$

$$X_3 = -2i\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) \tag{247}$$

One can verify directly that they satisfy indeed the $SO(3)$ algebra

$$[X_i, X_j] = 2\epsilon_{ijk} X_k \tag{248}$$

and that, moreover, they rotate the hyperkäler 2-forms as one would expect from general considerations

$$\mathcal{L}_{X_i}\omega_j = 2\epsilon_{ijk}\omega_k \tag{249}$$

We use here the standard notation conventions $X_\pm = (X_1 \pm iX_2)/2$ and $\omega^\pm = (\omega_1 \pm i\omega_2)/2$. The action of each isometry generator is hamiltonian with respect to the corresponding

Kähler 2-form. The corresponding moment maps are

$$\mu_1 = 2r + \frac{r^2 - 2|z|^2 - 2\mathrm{Re}(z^2)}{h} \tag{250}$$

$$\mu_2 = 2r + \frac{r^2 - 2|z|^2 + 2\mathrm{Re}(z^2)}{h} \tag{251}$$

$$\mu_3 = 2r + \frac{4|z|^2}{h} \tag{252}$$

If we write the components of $\vec{r}$ in polar coordinates then the directional ϕ and θ angles associate naturally with the ψ coordinate to parametrize the $SO(3)$ group manifold. From (239) and (240) we obtain

$$ds^2_{n=1} \sim \Phi\,[dr^2 + r^2(\sigma_1^2 + \sigma_2^2)] + \Phi^{-1}\sigma_3^2 \tag{253}$$

with σ_1, σ_2, σ_3 the Cartan-Maurer left-invariant $SO(3)$ 1-forms.

Denoting $h = m^2$ and redefining the radial coordinate by $r = m(R-m)/2$, one obtains yet another common form of the Taub-NUT metric,

$$ds^2 \sim \frac{1}{4}\frac{R+m}{R-m}dR^2 + \frac{1}{4}(R+m)(R-m)(\sigma_1^2 + \sigma_2^2) + \frac{R-m}{R+m}\sigma_3^2 \tag{254}$$

The properties of the metrics discussed in this section have been extensively studied in the literature. We will not attempt to pursue them any further since we are primarily interested in the details of the construction method and in drawing from it general lessons that we can apply to more complex cases rather than in the particularities of the results.

7 An $\mathcal{O}(2) \oplus \mathcal{O}(2)$-based Swann bundle

In this section we review the GLT construction of the 8-dimensional Swann bundle with two abelian tri-holomorphic isometries generated by the F-function

$$F = \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \frac{(\eta_1^{(2)})^2}{\eta_2^{(2)}} \tag{255}$$

depending on two $\mathcal{O}(2)$ multiplets

$$\eta_I^{(2)} = \frac{\bar{z}_I}{\zeta} + x_I - z_I \zeta \tag{256}$$

($I = 1, 2$). The integration contour winds around the roots of $\eta_2^{(2)}$ in such a way that the integral yields a real outcome. That the resulting hyperkähler variety has a Swann bundle structure follows from the fact that F scales with weight one under the weight-one scaling of the two $\mathcal{O}(2)$ multiplets. This problem was considered by Calderbank and Pedersen in [12] and by Anguelova, Roček and Vandoren in [13]. We retrace here the basic steps of their construction and compute, additionally, the hyperkähler potential of the metric.

The residue theorem yields for the contour integral the expression

$$F = \frac{r_1^2 r_2^2 - (\vec{r}_1 \cdot \vec{r}_2)^2}{2 r_2 |z_2|^2} + \frac{r_2 (z_1 \bar{z}_2 - z_2 \bar{z}_1)^2}{|z_2|^4} \tag{257}$$

where $\vec{r}_1$ and $\vec{r}_2$, defined in (101), are the standard $\mathbb{R}^3$ vectors associated to $\eta_1^{(2)}$ and $\eta_2^{(2)}$. For the first derivatives of F with respect to x_1 and x_2 we get

$$x_1 \frac{\partial F}{\partial x_1} = -\frac{2 r_2 |z_1|^2}{|z_2|^2} + \frac{2(z_1 \bar{z}_2 - z_2 \bar{z}_1)^2}{r_2 |z_2|^2} + \frac{r_1^2 r_2^2 - (\vec{r}_1 \cdot \vec{r}_2)^2}{2 r_2 |z_2|^2} + \frac{2 r_1^2}{r_2} \tag{258}$$

$$x_2 \frac{\partial F}{\partial x_2} = \frac{2 r_2 |z_1|^2}{|z_2|^2} - \frac{2(z_1 \bar{z}_2 - z_2 \bar{z}_1)^2}{r_2 |z_2|^2} + \frac{r_2 (z_1 \bar{z}_2 - z_2 \bar{z}_1)^2}{|z_2|^4} - \frac{2(\vec{r}_1 \cdot \vec{r}_2)^2}{r_2^3} \tag{259}$$

A Legendre transform of F yields then the hyperkähler potential

$$K = -\frac{2(\vec{r}_1 \times \vec{r}_2)^2}{r_2^3} \tag{260}$$

The dependence on the holomorphic coordinates is implicit, the $SO(3)$ invariance, on the other hand, is manifest.

The metric follows from the second derivatives of F and it can be cast in the form of the generalized Gibbons-Hawking Ansatz (103), with the Higgs component given by

$$(\Phi_{IJ}) = \frac{2}{r_2} \begin{pmatrix} -1 & \dfrac{\vec{r}_1 \cdot \vec{r}_2}{r_2^2} \\ \dfrac{\vec{r}_1 \cdot \vec{r}_2}{r_2^2} & \dfrac{r_1^2 r_2^2 - 3(\vec{r}_1 \cdot \vec{r}_2)^2}{2 r_2^4} \end{pmatrix} \tag{261}$$

8 $\mathcal{O}(4)$ multiplets

8.1 Majorana normal form

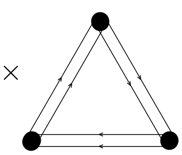
We will now focus our attention on the multiplets of $\mathcal{O}(4)$ type. These can be cast, generically, in either one of the following two local forms, to which we will henceforth refer as Majorana normal forms [25]

$$\begin{aligned}\eta^{(4)}(\zeta) &= \frac{\bar{z}}{\zeta^2} + \frac{\bar{v}}{\zeta} + x - v\zeta + z\zeta^2 \\ &= \frac{\rho}{\zeta^2}\frac{(\zeta-\alpha)(\bar{\alpha}\zeta+1)}{1+|\alpha|^2}\frac{(\zeta-\beta)(\bar{\beta}\zeta+1)}{1+|\beta|^2}\end{aligned} \tag{262}$$

The antipodal pairing of the roots in the second line is a consequence of the reality constraint. The coefficients of the $\eta^{(4)}$ multiplet can be expressed directly in terms of its roots through Viète's formulas, but conversely, expressing the roots explicitly in terms of the coefficients requires solving a quartic equation, an impractical approach.

The irreducible spherical invariants associated to $\eta^{(4)}$ are

$g_2 = 2\times$ 

$g_3 = \frac{8}{3}\times$ 

The numerical factors have been introduced for later convenience. All spherical invariants with diagrammatic representations having four or more vertices are reducible in the sense that they can be decomposed into homogeneous polynomial expressions in g_2 and g_3, with rational coefficients.

A direct calculation yields

$$g_2 = 4|z|^2 + |v|^2 + \frac{1}{3}x^2 \tag{263}$$

$$g_3 = \frac{8}{3}|z|^2 x - \frac{1}{3}|v|^2 x - \frac{2}{27}x^3 - z\bar{v}^2 - \bar{z}v^2 \tag{264}$$

If one substitutes further into these formulas the expressions for z, $\bar{z}$, v, $\bar{v}$ and x in terms of the roots and scale factor, one can arrange them in the form

$$g_2 = \frac{1}{3}\rho^2(1-k^2+k^4) \tag{265}$$

$$g_3 = \frac{1}{27}\rho^3(k^2-2)(2k^2-1)(k^2+1) \tag{266}$$

where $k = k_{\alpha\beta}$ is the Fubini-Study invariant defined in (175), the only such invariant for $j = 2$.

The fact that $\eta^{(4)}$ has scaling weight 2 suggests considering the associated quartic plane curve

$$\eta^2 = \zeta^2\eta^{(4)}(\zeta) \tag{267}$$

The projection $(\zeta, \eta) \longmapsto \zeta$ is a two-sheeted branched covering of the Riemann sphere, with the holomorphic elliptic involution $(\zeta, \eta) \longmapsto (\zeta, -\eta)$ interchanging the two sheets except at its fixed points, which are branching points for the covering. When these are all different, the curve is non-singular. The curve has, additionally, two anti-holomorphic involutions or real structures $(\zeta, \eta) \longmapsto (-1/\bar{\zeta}, \pm\bar{\eta}/\bar{\zeta}^2)$ induced by the antipodal map on the sphere, conjugated by the elliptic involution and preserving the set of branching points. This makes it a double-cover of the real projective plane, $\mathbb{RP}^2 \simeq \mathbb{CP}^1/\mathbb{Z}_2$, and thus a real algebraic curve of genus 1.

As an elliptic curve, it has an abelian differential form, *i.e.*, a globally defined holomorphic 1-form

$$\varpi = \frac{d\zeta}{2\zeta\sqrt{\eta^{(4)}(\zeta)}} \tag{268}$$

8.2 Legendre normal form

The equation of the curve can be cast into Legendre normal form by means of a birational transformation of ζ mapping three members of the root system α,$-1/\bar{\alpha}$, β,$-1/\bar{\beta}$ to 0, 1 and ∞, while also appropriately transforming η. As is well-known in the theory of elliptic curves, one can obtain in this way six possible moduli, representing points where the fourth root can be mapped. A very simple and elegant modulus, namely k^2, is returned by any one of the following four birational transformations: the one given in the form of the cross-ratio[6]

$$\nu = [\zeta, -\frac{1}{\bar{\alpha}},\ \alpha,\ \beta] \tag{269}$$

another one obtained by replacing in this relation α and β with their antipodal conjugates, as well as two others that can be obtained from these two by interchanging α and β. Note, incidentally, that antipodal conjugation translates in this new context into complex-conjugation. In all these four cases the abelian 1-form (268) transforms to

$$\varpi = \frac{d\nu}{2\sqrt{\rho\,\nu(\nu-1)(\nu-k^2)}} \tag{270}$$

and so equation (267) can be re-expressed in terms of ν and the coordinate $\mu = \eta\,\partial\nu/\partial\zeta$ as

$$\mu^2 = \rho\,\nu(\nu-1)(\nu-k^2) \tag{271}$$

The periods of ϖ are obtained by integrating it over the canonical cycles. By performing some standard changes of variables one can relate the period integrals to the familiar form of the complete Legendre elliptic integrals of the first kind. The period lattice takes thus the form

$$\Lambda = \frac{2}{\sqrt{\rho}}[\mathbb{Z}\cdot K(k) + \mathbb{Z}\cdot iK(k')] \tag{272}$$

Since $0 < k, k' < 1$, it follows that $K(k)$ and $K(k')$ are real and so Λ is an orthogonal lattice. The Fubini-Study invariants k and k' play in this context the role of modulus respectively complementary modulus of the elliptic integrals [5].

[6]The notation for cross-ratio that we employ here is $[z_1, z_2, z_3, z_4] = \dfrac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$.

8.3 Weierstrass normal form

At this point, let us note that equations (265) and (266) can be re-written in terms of

$$e_1 = -\frac{\rho}{3}(k^2 - 2) \qquad e_2 = \frac{\rho}{3}(2k^2 - 1) \qquad e_3 = -\frac{\rho}{3}(k^2 + 1) \tag{273}$$

satisfying

$$e_1 + e_2 + e_3 = 0 \tag{274}$$

as follows

$$g_2 = -(e_1e_2 + e_2e_3 + e_3e_1) \tag{275}$$

$$g_3 = e_1e_2e_3 \tag{276}$$

Equations (273) can be reverted to yield

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} \qquad \text{and} \qquad \rho = e_1 - e_3 \tag{277}$$

Recognizing that equations (274), (275) and (276) are Viète-type formulas, it is then immediately apparent that e_1, e_2 and e_3 have to be the three roots of the cubic equation $X^3 - g_2X - g_3 = 0$. This suggests that one should try to cast the equation of the elliptic curve associated to $\eta^{(4)}$ into the Weierstrass normal form

$$Y^2 = X^3 - g_2X - g_3 \tag{278}$$

This is indeed possible and is accomplished in practice by performing a further birational transformation of the complex variable ν in the Legendre normal form (271) of the equation

$$\nu = \frac{X - e_3}{e_1 - e_3} \tag{279}$$

while also substituting $\mu = Y/\rho$. Under the transformation (279) the abelian 1-form (270) becomes

$$\varpi = \frac{dX}{2\sqrt{X^3 - g_2X - g_3}} \tag{280}$$

The corresponding period lattice is conventionally written in the form

$$\Lambda = \mathbb{Z} \cdot 2\omega + \mathbb{Z} \cdot 2\omega' \tag{281}$$

with the Weierstrass half-periods given by

$$\omega = \frac{1}{\sqrt{\rho}}K(k) \qquad \text{and} \qquad \omega' = \frac{1}{\sqrt{\rho}}iK(k') \tag{282}$$

The discriminant of the Weierstrass cubic (288)

$$\Delta = 4g_2^3 - 27g_3^2 = \rho^6k^4k'^4 = [(e_1 - e_2)(e_2 - e_3)(e_3 - e_1)]^2 \tag{283}$$

is strictly positive as long as the elliptic modulus k is not 0 or 1, just as expected, given the fact that the roots e_1, e_2 and e_3 are all real.

8.4 The $\mathcal{O}(4)$ Weierstrass cubic is a Cayley cubic

If we define in place of the Majorana coefficients the related real variables

$$x_\pm = \frac{x \pm 6|z|}{3} \qquad v_+ = \mathrm{Im}\frac{v}{\sqrt{z}} \qquad v_- = \mathrm{Re}\frac{v}{\sqrt{z}} \tag{284}$$

then in terms of these, the expressions (263) and (264) can be rewritten as follows

$$g_2 = x_+^2 + x_+x_- + x_-^2 + \frac{1}{4}(x_+ - x_-)(v_-^2 + v_+^2) \tag{285}$$

$$g_3 = -(x_+ + x_-)x_+x_- - \frac{1}{4}(x_+ - x_-)(x_+v_-^2 + x_-v_+^2) \tag{286}$$

This form of the Weierstrass coefficients facilitates two key observations. First, we note that the four points with (X, Y)-coordinates

$$\begin{array}{cc} (x_-, v_-(x_+ - x_-)/2) & (x_+, iv_+(x_+ - x_-)/2) \\ (x_-, v_-(x_- - x_+)/2) & (x_+, iv_+(x_- - x_+)/2) \end{array} \tag{287}$$

are points on the $\mathcal{O}(4)$ curve in the Weierstrass representation, *i.e.* they satisfy the equation

$$Y^2 = X^3 - g_2X - g_3 \tag{288}$$

This can be checked by direct substitution. The pairs of points on each column in (287) are conjugated under the elliptic involution. The pairs of points along the two diagonals are conjugated under the $\mathbb{Z}_2$ action

$$\begin{array}{ccc} x_- & \longleftrightarrow & x_+ \\ v_- & \longleftrightarrow & iv_+ \end{array} \tag{289}$$

Clearly, this action leaves the coefficients g_2 and g_3 invariant.

Secondly, we note that we can write the Weierstrass cubic as a determinant, *i.e.*,

$$X^3 - g_2X - g_3 = \begin{vmatrix} X - x_+ & \sqrt{|z|}v_+ & 0 \\ \sqrt{|z|}v_+ & X + x_+ + x_- & \sqrt{|z|}v_- \\ 0 & \sqrt{|z|}v_- & X - x_- \end{vmatrix} \tag{290}$$

We give this fact the following interpretation: *The Weierstrass cubic curve (288) associated to the $\mathcal{O}(4)$ multiplet is a Cayley cubic, i.e.,*

$$Y^2 = \det(\mathcal{A} + X\mathcal{B}) \tag{291}$$

for the pencil generated by the two plane conics with defining real-valued matrices

$$\mathcal{A} = \begin{pmatrix} -x_+ & \sqrt{|z|}v_+ & 0 \\ \sqrt{|z|}v_+ & x_+ + x_- & \sqrt{|z|}v_- \\ 0 & \sqrt{|z|}v_- & -x_- \end{pmatrix} \qquad \text{and} \qquad \mathcal{B} = \mathbb{I}_{3\times3} \tag{292}$$

Let us make one more remark. In the Legendre normal form of the curve the ν-coordinates corresponding to the points (287) on the Weierstrass cubic are

$$\nu_{\pm} = \frac{x_{\pm} - e_3}{e_1 - e_3} \tag{293}$$

Using equations (273), (175) and the relations between the coefficients and the roots in (262), one finds that

$$\begin{aligned}
\nu_{\pm} &= \frac{(1 \pm |\alpha\beta|)^2}{(1+|\alpha|^2)(1+|\beta|^2)} \\
1 - \nu_{\pm} &= \frac{(|\alpha| \mp |\beta|)^2}{(1+|\alpha|^2)(1+|\beta|^2)} \\
k^2 - \nu_{\pm} &= \frac{(\sqrt{\alpha\bar{\beta}} \mp \sqrt{\bar{\alpha}\beta})^2}{(1+|\alpha|^2)(1+|\beta|^2)}
\end{aligned} \tag{294}$$

A quick inspection of these relations yields the inequalities

$$0 < \nu_- < k^2 < \nu_+ < 1 \tag{295}$$

On the Weierstrass side, together with the obvious ordering of the Weierstrass roots, they imply that

$$e_3 < x_- < e_2 < x_+ < e_1 < e_0 = \infty \tag{296}$$

8.5 Radial $\mathcal{O}(4)$ invariants

In the limit when an $\mathcal{O}(4)$ multiplet degenerates to the square of an $\mathcal{O}(2)$ multiplet, *i.e.* when $\eta^{(4)} \longrightarrow (\eta^{(2)})^2$, one has $\rho \longrightarrow \sigma^2$. We saw during the discussion of $\mathcal{O}(2)$ invariants in section 5.3 that the $SO(3)$ invariant $\sigma = r_1 > 0$ can in effect be interpreted as a radius. This suggests that, in the (generic, non-degenerate) $\mathcal{O}(4)$ case, positively-defined $SO(3)$-invariant quantities which are proportional to $\sqrt{\rho}$ could be thought of as some sort of radii, too. There are essentially only two independent quantities satisfying these requirements, namely

$$r = \frac{1}{2\omega} > 0 \qquad \text{and} \qquad r' = \frac{i\pi}{2\omega'} > 0 \tag{297}$$

The disparity in the normalization factors serves a subsequent formal purpose but is otherwise irrelevant.

The $SO(3)$ invariants g_2 and g_3 can be expressed entirely in terms of r and r', and in this sense they form two sets of equivalent invariants. Indeed, it is well-known in the theory of elliptic functions that the Weierstrass coefficients admit the following double power series representation in terms of the periods

$$g_2 = 15 \sum_{m,m'=-\infty}^{\infty}{}' \frac{1}{(2m\omega + 2m'\omega')^4} \tag{298}$$

$$g_3 = 35 \sum_{m,m'=-\infty}^{\infty}{}' \frac{1}{(2m\omega + 2m'\omega')^6} \tag{299}$$

where the prime sum symbol signifies that the term with $(m, m') = (0, 0)$ must be omitted. Alternatively, each of these double series can be recast as a Lambert-type q-series

$$g_2 = \frac{1}{3}\left(\frac{\pi}{2\omega}\right)^4 \left(1 + 240 \sum_{n=1}^{\infty} n^3 \frac{q^{2n}}{1 - q^{2n}}\right) \tag{300}$$

$$g_3 = \frac{2}{27}\left(\frac{\pi}{2\omega}\right)^6 \left(1 - 504 \sum_{n=1}^{\infty} n^5 \frac{q^{2n}}{1 - q^{2n}}\right) \tag{301}$$

where $q = \exp(i\pi\tau)$ is the elliptic *nome* and $\tau = \omega'/\omega$ is the elliptic modulus. Since the Weierstrass coefficients are invariant under the modular transformation $\tau' = -1/\tau$, see the discussion in section 13.5, their q'-series expansions are formally identical, but with q replaced by q' and ω by ω'. In terms of the radii,

$$q = e^{-\pi^2 r/r'} \qquad \text{and} \qquad q' = e^{-r'/r} \tag{302}$$

The fact that $r, r' > 0$ implies that $0 < q, q' < 1$, which in turn guarantees convergence. The two asymptotic regions $r >> r'$ and $r << r'$ can be analyzed perturbatively by performing expansions in q respectively q'.

8.6 The Jacobian picture

One can check that, for any $\zeta \in \mathbb{C} \cup \{\infty\}$, one has

$$(\bar{X}_\zeta - e_2)(X_{-1/\bar{\zeta}} - e_2) = (e_1 - e_2)(e_3 - e_2) \tag{303}$$

We use the notation X_ζ for the image of ζ through the birational transformations (269) and (279). Equation (303) implies that it is possible to choose the ambiguous signs of the Y-coordinates of the curve points with X-coordinates $\bar{X}_\zeta$ and $X_{-1/\bar{\zeta}}$ such that

$$\begin{vmatrix} 1 & \bar{X}_\zeta & \bar{Y}_\zeta \\ 1 & X_{-1/\bar{\zeta}} & Y_{-1/\bar{\zeta}} \\ 1 & e_2 & 0 \end{vmatrix} = 0 \tag{304}$$

which is just the *colinearity condition* for the three points $(\bar{X}_\zeta, \bar{Y}_\zeta)$, $(X_{-1/\bar{\zeta}}, Y_{-1/\bar{\zeta}})$ and $(e_2, 0)$. Moreover, this allows one to choose the corresponding points (through the Abel-Jacobi map) on the Jacobian variety such that

$$\bar{u}_\zeta + u_{-1/\bar{\zeta}} = \omega_2 \tag{305}$$

This relation expresses the action of the antipodal conjugation-induced real structure on the Jacobian of the curve.

By a straightforward calculation one can show that

$$\mathrm{cn}(\sqrt{\rho}\, u_\zeta) = \sqrt{\frac{\alpha - \beta}{1 + \bar{\alpha}\beta} \frac{1 + \bar{\alpha}\zeta}{\alpha - \zeta}} \tag{306}$$

$$\mathrm{dn}(\sqrt{\rho}\, u_\zeta) = \sqrt{\frac{\alpha - \beta}{1 + \bar{\beta}\beta} \frac{1 + \bar{\beta}\zeta}{\alpha - \zeta}} \tag{307}$$

where cn and dn are the usual Jacobi elliptic functions. A similar expression holds for $\mathrm{sn}(\sqrt{\rho}\, u_\zeta)$. These formulas are unsatisfactory for a number of reasons, chief among them being the fact that the roots α and β do not appear on the same footing. We clearly need a different perspective. The crucial observation is contained in the following result

$$\mathrm{cn}[\sqrt{\rho}\,(u_\zeta \pm \bar{u}_\zeta)] = \frac{k'_{\alpha\beta}}{k_{\alpha\beta}} \frac{k_{\alpha\zeta}k'_{\alpha\zeta} \mp k_{\beta\zeta}k'_{\beta\zeta}}{k'^2_{\alpha\zeta} - k'^2_{\beta\zeta}} = \frac{\tan\dfrac{\delta_{\alpha\beta}}{2}}{\tan\dfrac{\delta_{\alpha\zeta} \pm \delta_{\beta\zeta}}{2}} \tag{308}$$

$$\mathrm{dn}[\sqrt{\rho}\,(u_\zeta \pm \bar{u}_\zeta)] = k'_{\alpha\beta} \frac{k_{\alpha\zeta}k'_{\beta\zeta} \mp k_{\beta\zeta}k'_{\alpha\zeta}}{k^2_{\alpha\zeta} - k^2_{\beta\zeta}} = \frac{\sin\dfrac{\delta_{\alpha\beta}}{2}}{\sin\dfrac{\delta_{\beta\zeta} \pm \delta_{\alpha\zeta}}{2}} \tag{309}$$

Incidentally, note that the these are the same type of trigonometric ratios that appear in the Napier and Delambre analogies of spherical trigonometry. The first equalites in (308) and (309) follow from applying the addition formulas (523) for the Jacobi elliptic functions cn and dn. We use that

$$\mathrm{sn}(\sqrt{\rho}\, u) = \sqrt{\frac{e_1 - e_3}{X - e_3}} \qquad \mathrm{cn}(\sqrt{\rho}\, u) = \sqrt{\frac{X - e_1}{X - e_3}} \qquad \mathrm{dn}(\sqrt{\rho}\, u) = \sqrt{\frac{X - e_2}{X - e_3}} \tag{310}$$

with $X = \wp(u)$, as in (447). Despite the simple form of the outcome, the calculation is quite entangled and laborious if approached frontally. We managed to simplify and streamline it significantly by resorting to the spin coherent state techniques developed in section 5.1. First, observe that we have the following cross-ratio expressions

$$\frac{X_\zeta - e_1}{e_3 - e_1} = [\beta, -\frac{1}{\bar{\alpha}},\ \alpha,\ \zeta] \tag{311}$$

$$\frac{X_\zeta - e_2}{e_3 - e_2} = [\beta, -\frac{1}{\bar{\beta}},\ \alpha,\ \zeta] \tag{312}$$

$$\frac{X_\zeta - e_3}{e_1 - e_3} = [\zeta, -\frac{1}{\bar{\alpha}},\ \alpha,\ \beta] \tag{313}$$

The relation (313) is just equation (269) taken together with (279); the preceding two follow from this one. The second observation is that cross-ratios can be expressed in terms of spin-1/2 coherent states as follows

$$[z_1, z_2, z_3, z_4] = \frac{\langle -\frac{1}{\bar{z}_1}|z_3\rangle\langle -\frac{1}{\bar{z}_2}|z_4\rangle}{\langle -\frac{1}{\bar{z}_1}|z_4\rangle\langle -\frac{1}{\bar{z}_2}|z_3\rangle} \tag{314}$$

Together, these relations allow one to cast the cn and dn addition formulas entirely in terms of spin-1/2 coherent states. The k and k' expressions emerge from the coherent state picture by means of the norm relations (171). The second equalities in (308) and (309) follow by using further the relations (174) and some trigonometry.

For any $\zeta \in \mathbb{C} \cup \{\infty\}$, let us define

$$u^\pm_\zeta = u_\zeta \pm u_{-1/\bar{\zeta}} \tag{315}$$

i.e. the 'real' and 'imaginary' parts of u_ζ with respect to the real structure induced by the antipodal conjugation on the sphere. Based on the equations (305), (308), (309) and the half-period addition formula $\mathrm{sn}[v \pm (K+iK')] = \pm\, \mathrm{dn}v \div k\, \mathrm{cn}v$ we obtain

$$\mathrm{sn}(\sqrt{\rho}\, u_\zeta^\pm) = \sec\frac{\delta_{\alpha\zeta} \mp \delta_{\beta\zeta}}{2} \tag{316}$$

If we resort instead to the addition formula $\mathrm{sn}[v \pm K] = \pm\, \mathrm{cn}v \div \mathrm{dn}v$, we obtain

$$\mathrm{sn}[\sqrt{\rho}\,(u_\zeta^\pm - \omega')] = \frac{\cos\dfrac{\delta_{\alpha\zeta} \mp \delta_{\beta\zeta}}{2}}{\cos\dfrac{\delta_{\alpha\beta}}{2}} \tag{317}$$

We use here the conventional notations $K = K(k)$ and $K' = K(k')$ for the complete elliptic integrals of the first kind of complementary moduli. Remember that we work on the S^2 sphere with the antipodal points identified. This means essentially that we always consider points α, β and ζ which are on the *same hemisphere* of S^2. They determine a spherical triangle with vertices at α, β, ζ and sides $\delta_{\alpha\zeta}$, $\delta_{\beta\zeta}$, $\delta_{\alpha\beta}$, which, for this reason, has the following properties: (1) $\delta_{\alpha\zeta}$, $\delta_{\beta\zeta}$, $\delta_{\alpha\beta} \in [0,\pi]$, meaning the triangle is convex, which further implies that the usual triangle inequalities hold, *i.e.* $\delta_{\alpha\zeta} + \delta_{\beta\zeta} \geq \delta_{\alpha\beta}$, *etc.* and (2) $\delta_{\alpha\zeta} + \delta_{\beta\zeta} + \delta_{\alpha\beta} \leq 2\pi$. Based on these inequalities being satisfied one determines that both equations (316) (*i.e.* with both sets of signs considered) and the equation (317) with the upper set of signs are ≥ 1, whereas the equation (317) with the lower set of signs is ≤ 1 and ≥ -1. It seems then natural to set this latter equation equal to the sine of an angle, let us call it $\sin D_\zeta$. In the light of (517) we find it convenient to write this *definition* in the form

$$\frac{\sin\dfrac{\pi - \delta_{\alpha\zeta} - \delta_{\beta\zeta}}{2}}{\sin D_\zeta} = k \tag{318}$$

Inverting the lower equation (317) on a fundamental domain yields

$$u_\zeta^- = \frac{1}{\sqrt{\rho}} F(\sin D_\zeta, k) + \omega' \tag{319}$$

with $F(\cdot, k)$ an incomplete Legendre elliptic integral of the first kind. This gives us a very explicit expression for u_ζ^-, with a clearly resolved complex structure: the first term in the r.h.s. of (319) is real, the second one is a purely imaginary constant shift.

We end this section with yet another important observation. We found that it is possible to choose the ambiguous signs of Y_0, Y_∞ and $y_\pm$ corresponding on the Weierstrass curve (288) to X_0, X_∞ and $x_\pm$, such that

$$\begin{vmatrix} 1 & X_\infty & Y_\infty \\ 1 & X_0 & \pm Y_0 \\ 1 & x_\pm & -y_\pm \end{vmatrix} = 0 \tag{320}$$

This can be verified for instance by expressing everything in terms of the roots α, β, their complex conjugates and the scale ρ, by means of the equations (279), (269), (273) and (175). The equation (320) is a colinearity condition. By comparing the corresponding equation on the Jacobian to equation (315) with $\zeta = \infty$, we infer immediately that

$$\wp(u_\infty^\pm) = x_\pm \tag{321}$$

i.e., the four points $\pm u_\infty^+$ and $\pm u_\infty^-$ from the Jacobian are mapped by the inverse Abel-Jacobi map to the four points (287) on the Weierstrass curve, with the X-coordinates equal to x_+ respectively x_-.

8.7 Euler-angle parametrization

The rotational transformation properties of the Majorana coefficients suggest an alternative parametrization of the $\mathcal{O}(4)$ multiplet in which the rotational structure appears more explicitly. Consider the following *Ansatz*, inspired by (161)

$$
\begin{aligned}
z &= \sqrt{1}\sum_{m=-2}^{2} D^{(2)}_{-2m}(\phi,\theta,\psi)\chi^2_m \\
v &= \sqrt{4}\sum_{m=-2}^{2} D^{(2)}_{-1m}(\phi,\theta,\psi)\chi^2_m \\
x &= \sqrt{6}\sum_{m=-2}^{2} D^{(2)}_{0m}(\phi,\theta,\psi)\chi^2_m
\end{aligned}
\tag{322}
$$

Three of the new parameters will be the Euler angles ϕ, θ and ψ. Wigner's rotation matrices ensure the right transformation properties. This leaves two rotation-invariant parameters on which the 5-component χ^2_m can depend. Observe now that due to the rotational invariance of g_2 and g_3 one can replace in equations (263) and (264) z, v and x by $\sqrt{1}\,\chi^2_{-2}$, $\sqrt{4}\,\chi^2_{-1}$ and $\sqrt{6}\,\chi^2_0$, respectively. Passing then to a form similar to (285) and (286), $x_\pm$ gets replaced by $e_\pm = (\sqrt{6}\chi^2_0 \pm 6|\chi^2_{-2}|)/3$, and it is clear that if we put $\chi^2_{-1} = 0 = \chi^2_{+1}$ and compare the remainder with (275) and (276) we can identify e_+ and e_- with any two of e_1, e_2 and e_3. Taking for instance $e_+ = e_1$ and $e_- = e_3$, it follows that $\chi^2_0 = -\sqrt{6}\,e_2/4$ and $|\chi^2_{-2}| = (e_1 - e_3)/4$. A choice consistent with these constraints would be, *e.g.*,

$$
\chi^2_{m=-2\cdots+2} = \frac{1}{4}\begin{pmatrix} e_1 - e_3 \\ 0 \\ -\sqrt{6}\,e_2 \\ 0 \\ e_1 - e_3 \end{pmatrix}
\tag{323}
$$

It depends, as required, on two rotation-independent parameters, since e_1, e_2 and e_3 satisfy (274). The parametrization of the $\mathcal{O}(4)$ multiplet given by (322) and (323) corresponds essentially to the one used in [5].

9 $\mathcal{O}(4)$ elliptic integrals

Generalized Legendre transform constructions based on $\mathcal{O}(4)$ multiplets oftentimes involve evaluating contour integrals of the type

$$\mathcal{I}_m = \int_\Gamma \frac{d\zeta}{\zeta} \frac{\zeta^m}{2\sqrt{\eta^{(4)}}} \tag{324}$$

with Γ an integration contour which may be either open or closed, depending on context, and m an integer taking values from -2 to 2. In fact, it suffices to consider only $m = 0, 1, 2$, since the integrals corresponding to m and $-m$ are complex conjugated to each other, modulo a shift. More precisely,

$$\mathcal{I}_{-m} = (-)^m \bar{\mathcal{I}}_m \pm 2\pi i \frac{m\bar{\beta}^{m-1}}{\sqrt{z}} \tag{325}$$

This can be seen by changing in (324) the integration variable ζ to $-1/\bar{\zeta}$ and then deforming the resulting contour back to the original one; in the process, one picks up a residue, which accounts for the shift term. Shifts are usually discarded by means of a doubling trick: we can always choose two contours, one which gives a $+$ and one which gives a $-$ in (325); by summing the two contributions up, the residue terms will mutually cancel.

By 'evaluating' these contour integrals we mean of course reducing them to standard elliptic integrals. For various reasons, we are particularly interested in obtaining as explicit a dependence on the Majorana coefficients of $\eta^{(4)}$ as possible. As it turns out, the Weierstrass framework is best suited to this purpose. Hence the first step of our approach is to transform the integrals from what we refer to as the Majorana picture to the Weierstrass picture by means of the birational transformation that results from the two succesive transformations (269) and (279), and which, with the help of the notation that we introduced at the begining of section 8.6, can be conveniently written in the form

$$\zeta = \beta \frac{X - X_0}{X - X_\infty} \tag{326}$$

The abelian differential that plays the role of integration measure transforms, according to (268) and (280), as follows

$$\frac{d\zeta}{\zeta} \frac{1}{2\sqrt{\eta^{(4)}}} = \frac{dX}{2Y} \tag{327}$$

Once an integral is expressed completely in terms of Weierstrass variables, we follow the standard procedure in evaluating elliptic integrals, see *e.g.* [33]: we expand the rational coefficient of the measure (327) into partial fractions centered on X_∞ and then use formulas (444) through (446) to express each resulting term in terms of Weierstrass elliptic functions. That is of course not possible to do directly for the $\mathcal{I}_2$ integral, as the partial fraction expansion yields in that case a term proportional to $Y_\infty^2/(X - X_\infty)^2$. One handles this by noticing that

$$\left(\frac{Y_\infty}{X - X_\infty}\right)^2 = \frac{1}{2}\left(X - X_\infty - \frac{3X_\infty^2 - g_2}{X - X_\infty}\right) - Y \frac{d}{dX}\left(\frac{Y}{X - X_\infty}\right) \tag{328}$$

The last term in (328) leads eventually to a total derivative which can be easily integrated. The other ones lead directly to elliptic integrals of the three kinds, just as in the other cases.

The outcome at this stage can be simplified by using that

$$X_\infty = \frac{x}{3} - \beta v + 2\beta^2 z \tag{329}$$

$$\frac{3X_\infty^2 - g_2}{Y_\infty} = \frac{v}{\sqrt{z}} - 4\beta\sqrt{z} \tag{330}$$

$$\frac{X_\infty - X_0}{Y_\infty} = -\frac{1}{\beta\sqrt{z}} \tag{331}$$

and that

$$2\zeta(u_\infty) = \zeta(u_\infty^+) + \zeta(u_\infty^-) + 2\beta\sqrt{z} \tag{332}$$

The first three identities can be verified by expressing everything in terms the Majorana roots and scale. The last one follows by applying succesively the doubling formula and then the addition theorem for the Weierstrass ζ-function. Note that (315) implies that $2u_\infty = u_\infty^+ + u_\infty^-$.

In the end, we obtain

$$\mathcal{I}_0 = u + \mathcal{C} \tag{333}$$

$$\mathcal{I}_1 = -\frac{1}{2\sqrt{z}}\left[\ln\frac{\sigma(u-u_\infty)}{\sigma(u+u_\infty)} + [\zeta(u_\infty^+) + \zeta(u_\infty^-)]u\right] + \mathcal{C} \tag{334}$$

$$\begin{aligned}\mathcal{I}_2 = -\frac{1}{4z}&\left\{\zeta(u-u_\infty) + \zeta(u+u_\infty) + (x_+ + x_-)u\right.\\ &\left.+\frac{v}{\sqrt{z}}\left[\ln\frac{\sigma(u-u_\infty)}{\sigma(u+u_\infty)} + [\zeta(u_\infty^+) + \zeta(u_\infty^-)]u\right]\right\} + \mathcal{C}\end{aligned} \tag{335}$$

where u is related to X as in equation (447). The corresponding complete integrals, obtained by integrating over the contours Γ_i with $i = 1, 2, 3$ defined in the paragraph preceding equations (448) through (450), are

$$\mathcal{I}_0^{(i)} = 2\omega_i \tag{336}$$

$$\mathcal{I}_1^{(i)} = \frac{1}{\sqrt{z}}[\pi_i(x_+) + \pi_i(x_-)] \tag{337}$$

$$\mathcal{I}_2^{(i)} = -\frac{1}{2z}\left[2\eta_i + (x_+ + x_-)\omega_i - \frac{v}{\sqrt{z}}[\pi_i(x_+) + \pi_i(x_-)]\right] \tag{338}$$

To derive (336) - (338) from (333) - (335) we made use of the σ-function monodromy property as well as of a version of the ζ-function addition theorem.

10 The Atiyah-Hitchin metric

10.1 The hyperkähler structure

The F-function that yields the Atiyah-Hitchin metric through the generalized Legendre transform construction of [4] is given, according to [5], by

$$F = F_2 + F_1 = \frac{1}{2\pi i h}\oint_{\Gamma_0}\frac{d\zeta}{\zeta}\eta^{(4)} - \oint_{\Gamma}\frac{d\zeta}{\zeta}\sqrt{\eta^{(4)}} \tag{339}$$

Γ_0 is an integration contour around $\zeta = 0$ whereas Γ is a contour that winds once around the branch-cut between the roots α and $-1/\bar{\beta}$; h is a constant coupling scale.

The first integral can be evaluated by means of a straightforward application of Cauchy's integral formula. One gets

$$F_2 = \frac{x}{h} \tag{340}$$

The evaluation of the second integral presents a more challenging problem. Observe that it satisfies the homogeneity property

$$F_1 = 2\bar{z}\frac{\partial F_1}{\partial \bar{z}} + 2\bar{v}\frac{\partial F_1}{\partial \bar{v}} + 2x\frac{\partial F_1}{\partial x} + 2v\frac{\partial F_1}{\partial v} + 2z\frac{\partial F_1}{\partial z} \tag{341}$$

and that each derivative is equal to a complete integral of the type (324), which we have already evaluated. More precisely,

$$\frac{\partial F_1}{\partial z} = -\mathcal{I}_2^{(1)} \qquad \frac{\partial F_1}{\partial v} = \mathcal{I}_1^{(1)} \qquad \frac{\partial F_1}{\partial x} = -\mathcal{I}_0^{(1)} \tag{342}$$

with $\mathcal{I}^{(1)}_{m=0,1,2}$ given in (336) - (338). To obtain a real F_1 we should consider a combination of contours around the branch-cuts between the roots α and $-1/\bar{\beta}$ as well as between β and $-1/\bar{\alpha}$ in such a way as to cancel the residue terms in (325). Henceforth, we will automatically assume that this is done, and we will just ignore them. With this assumption, equations (341), (342) and (336) - (338) yield

$$F_1 = 4\eta - 4(x_+ + x_-)\omega + 2v_-\pi(x_-) + 2iv_+\pi(x_+) \tag{343}$$

From the equations (340) and (342) one also readily obtains the derivatives

$$\frac{\partial F}{\partial v} = \frac{\pi(x_-) + \pi(x_+)}{\sqrt{z}} \tag{344}$$

$$\frac{\partial F}{\partial x} = \frac{1}{h} - 2\omega \tag{345}$$

We are now prepared to impose the generalized Legendre relations, which in this case read

$$\frac{\partial F}{\partial v} = u \tag{346}$$

$$\frac{\partial F}{\partial x} = 0 \tag{347}$$

Equation (347) defines implicitly the Atiyah-Hitchin manifold as a codimension 1 subspace in the five real-dimensional space of moduli of $\eta^{(4)}$ sections. It takes the remarkably simple form

$$2\omega = 1/h \tag{348}$$

Equation (346), on the other hand, serves to introduce a second holomorphic coordinate u (the other one being z). Note that due to the inequalities (296), $\pi(x_-)$ is real whereas $\pi(x_+)$ is purely imaginary. Then equations (344) and (346) imply that $\pi(x_-) = \mathrm{Re}\, u\sqrt{z}$ and $\pi(x_+) = i\, \mathrm{Im}\, u\sqrt{z}$, and so

$$2v_-\pi(x_-) + 2iv_+\pi(x_+) = uv + \overline{uv} \tag{349}$$

It follows that the Jacobi terms in F, in conjunction with the Legendre relation that introduces the second holomorphic coordinate, play a pivotal role in canceling the quadratic terms in the Legendre transform that yields the Kähler potential. This constitutes a generic mechanism for $\mathcal{O}(4)$ multiplets. Putting together equations (343), (340) and (349) and making also use of equation (348), one obtains quite effortlessly a formula for the Kähler potential corresponding to the complex structure to which the holomorphic coordinates z and u are associated, *i.e.*,

$$K = 4\eta - (x_+ + x_-)\omega \tag{350}$$

A Kähler potential for the Atiyah-Hitchin metric has also been derived by Olivier [34], who followed a different, symmetry-based approach. It is straightforward to check that the rotation-invariant term of (350), namely 4η, coincides, up to a constant factor, with the corresponding part of Olivier's potential.

We found *a posteriori* that the metric takes a simpler form if one uses instead of u and z a new pair of holomorphic variables defined as follows

$$U = u\sqrt{z} \qquad Z = 2\sqrt{z} \tag{351}$$

This transformation is a holomorphic symplectomorphism, it leaves the hyperkähler holomorphic (2,0)-form invariant,

$$\omega^+ = dZ \wedge dU \tag{352}$$

Taking also into account the reality properties of $\pi(x_\pm)$, equation (346) is then equivalent to

$$U = \pi(x_-) + \pi(x_+) \qquad \bar{U} = \pi(x_-) - \pi(x_+) \tag{353}$$

To derive the metric one can follow two equivalent paths: one is to use the general formula (90) giving the metric in terms of the second derivatives of F without resorting explicitly to holomorphic variables, the other is to use the Kähler potential (350) and the manifest holomorphic structure. In the first approach, the second derivatives of F follow from derivating once more F_v and F_x, given by equations (344) and (345). The absence of F_z from this list can be compensated by the relations (92) between the second derivatives. Regarding ω and $\pi(x_\pm)$ as functions of g_2, g_3 and (in the latter case only) $x_\pm$, one makes use of the differentiation formulas (486) and (488) from section 13.3 in combination with the explicit forms (263) and (264) of g_2 and g_3. In the second approach, one employs equations (348) and (353), regarding now $\pi(x_\pm)$ as a function of ω, η and $x_\pm$. From the equations (353) together with $|Z|^2 = x_+ - x_-$, by making use of the differentiation relation (494) one can obtain the partial derivatives of $x_\pm$ and η with respect to the holomorphic variables U and Z,

$$d\eta = \frac{(A_+ - A_-)dU + (A_+ + A_-)d\bar{U} + 2A_+A_-(\bar{Z}dZ + Zd\bar{Z})}{2(A_-B_+ - A_+B_-)} \tag{354}$$

$$dx_\pm = \frac{(B_+ - B_-)dU + (B_+ + B_-)d\bar{U} + 2A_\mp B_\pm(\bar{Z}dZ + Zd\bar{Z})}{2(A_-B_+ - A_+B_-)} \tag{355}$$

where, with $y_+ = iv_+(x_+ - x_-)/2$ and $y_- = v_-(x_- - x_+)/2$ as in (287) and V given in (495),

$$\begin{aligned} A_\pm &= \frac{\eta + x_\pm \omega}{2y_\pm} \\ B_\pm &= \frac{V\omega + x_\pm}{y_\pm} \end{aligned} \tag{356}$$

The components of the metric can now be computed directly from the Kähler potential (350). Following either path, one obtains

$$\begin{aligned} K_{Z\bar{Z}} &= \quad \frac{1}{\mathcal{Q}|Z|^2}\mathcal{K}_4 \\ K_{U\bar{Z}} &= -\frac{1}{2\mathcal{Q}\bar{Z}}(v_-\mathcal{K}_{3+} + iv_+\mathcal{K}_{3-}) \\ K_{Z\bar{U}} &= -\frac{1}{2\mathcal{Q}Z}(v_-\mathcal{K}_{3+} - iv_+\mathcal{K}_{3-}) \\ K_{U\bar{U}} &= \quad \frac{1}{\mathcal{Q}|Z|^2}\mathcal{K}_2 \end{aligned} \tag{357}$$

where

$$\mathcal{Q} = (\eta + e_1\omega)(\eta + e_2\omega)(\eta + e_3\omega) \tag{358}$$

and

$$\begin{aligned} \mathcal{K}_2 &= (g_2 - 3x_+x_-)\eta^2 - [6g_3 + 2(x_+ + x_-)g_2]\omega\eta + [g_2^2 + 3(x_+ + x_-)g_3 + x_+x_-g_2]\omega^2 \\ \mathcal{K}_3 &= \eta^3 + 3x_\pm\omega\eta^2 + g_2\,\omega^2\eta - (2g_3 + x_\pm g_2)\omega^3 \\ \mathcal{K}_4 &= \eta^4 + 2(x_+ + x_-)\omega\eta^3 + (g_2 + 3x_+x_-)\omega^2\eta^2 - 2g_3\omega^3\eta - [(x_+ + x_-)g_3 + x_+x_-g_2]\omega^4 \end{aligned} \tag{359}$$

One verifies that the following Monge-Ampère equation holds

$$\det K_{(Z,U)} = 1 \tag{360}$$

Putting all this together, the metric eventually reads

$$ds^2 = \phi\, dU d\bar{U} + \phi^{-1}(dZ + \mathcal{A})\left(d\bar{Z} + \bar{\mathcal{A}}\right) \tag{361}$$

with

$$\phi = \frac{\mathcal{Q}}{\mathcal{K}_4}|Z|^2 \qquad \mathcal{A} = -\frac{v_-\mathcal{K}_{3+} + iv_+\mathcal{K}_{3-}}{2\mathcal{K}_4} Z dU \tag{362}$$

It has three isometries induced by the $SO(3)$ structure inherent to the $\eta^{(4)}$ sections out of which it has been constructed, generated by the three vector fields

$$X_3 = -2i\left(Z\frac{\partial}{\partial Z} - \bar{Z}\frac{\partial}{\partial \bar{Z}}\right) \tag{363}$$

$$X_- = (v_- + iv_+)\frac{\partial}{\partial Z} + \frac{(\eta + x_-\omega) + (\eta + x_+\omega)}{Z}\frac{\partial}{\partial U} + \frac{(\eta + x_-\omega) - (\eta + x_+\omega)}{Z}\frac{\partial}{\partial \bar{U}} \tag{364}$$

$$X_+ = \overline{X_-} \tag{365}$$

They preserve the metric, form an $SO(3)$ algebra,[7]

$$[X_i, X_j] = 2\epsilon_{ijk} X_k \tag{366}$$

and rotate the hyperkähler structure

$$\mathcal{L}_{X_i}\omega_j = 2\epsilon_{ijk}\,\omega_k \tag{367}$$

It is clear from this last property that the action of each vector field X_i is hamiltonian with respect to the corresponding Kähler 2-form ω_i and thus, provided that the manifold is simply connected, they all must have associated moment maps. We calculated these and obtained

$$\mu_1 = 4\eta - (x_+ + x_-)\omega + \mathrm{Re}Z^2\omega \tag{368}$$

$$\mu_2 = 4\eta - (x_+ + x_-)\omega - \mathrm{Re}Z^2\omega \tag{369}$$

$$\mu_3 = 4\eta + 2(x_+ + x_-)\omega \tag{370}$$

Hitchin [4, 35] shows that if on a simply-connected hyperkähler manifold there exists a vector field X_i which acts on the hyperkähler structure as in (367) then the corresponding moment map is a Kähler potential for any complex structure orthogonal[8] to the one preserved by X_i. In explicit agreement with this theorem, the moment maps (368) and (369) belong manifestly to the Kähler class of (350).

In the derivation of the relations (366) through (370) we found the following partial results useful

$$Z\frac{\partial K}{\partial Z} = \bar{Z}\frac{\partial K}{\partial \bar{Z}} = \frac{\mu_3}{2} \tag{371}$$

$$Z\frac{\partial \mu_3}{\partial Z} = \bar{Z}\frac{\partial \mu_3}{\partial \bar{Z}} = \frac{2\mathcal{K}_4}{\mathcal{Q}} \tag{372}$$

$$\frac{\partial K}{\partial U} = -(v_- + iv_+) \tag{373}$$

$$\frac{\partial \mu_3}{\partial U} = -\frac{v_-\mathcal{K}_{3+} + iv_+\mathcal{K}_{3-}}{\mathcal{Q}} \tag{374}$$

10.2 The $SO(3)$ structure

The connection between the complex holomorphic basis form (361) of the metric and the well-known form of Atiyah and Hitchin [17], in which the (hyper-)complex structure is obscure but the non-triholomorphic $SO(3)$ isometry is manifest, emerges through a change of variables. Specifically, one needs to switch from the complex holomorphic coordinate basis Z, $\bar{Z}$, U, $\bar{U}$ to the coordinate basis given by the $SO(3)$ angles ϕ, θ, ψ and the elliptic *nome* q. The Majorana coefficients z, $\bar{z}$, v, $\bar{v}$ and x will serve as intermediate variables. The Jacobian matrix of this transformation is computed as follows: the partial derivatives of Z and $\bar{Z}$ with respect to the Majorana coefficients are quite trivially computed from

[7] We use the standard conventions $X_\pm = (X_1 \pm iX_2)/2$ and $\omega^\pm = (\omega_1 \pm i\omega_2)/2$.

[8] A hyperkähler manifold with standard complex structures I, J, K has in fact a 2-sphere worth of complex structures compatible with the metric and the Levi-Civita connection, given by $xI + yJ + zK$ for all unit vectors (x, y, z) from $\mathbb{R}^3$. One refers to two complex structures as being orthogonal when the corresponding unit vectors are orthogonal with respect to the scalar product on $\mathbb{R}^3$.

the second relation (351); the partial derivatives of U and $\bar{U}$ on the other hand can be obtained from (353), regarding $\pi(x_\pm)$ as functions of g_2, g_3 and $x_\pm$. In this way one gets, *e.g.*,

$$\frac{\partial U}{\partial z} = \frac{\bar{Z}}{4Z}(M\eta^* + N\omega^*) \tag{375}$$

with η^* and ω^* defined in (489) and

$$\begin{aligned} M &= v_-(5x_+ - x_-) - iv_+(5x_- - x_+) \\ N &= v_-(8g_2 - 3x_+x_- - 9x_+^2) - iv_+(8g_2 - 3x_+x_- - 9x_-^2) \end{aligned} \tag{376}$$

For the remaining derivatives, a detour through the relations (92) allows for simpler calculations. Specifically,

$$\begin{aligned} \frac{\partial U}{\partial x} &= \frac{Z}{2}F_{xv} = \frac{Z}{2}F_{vx} = -Z\frac{\partial \omega}{\partial v} \\ \frac{\partial U}{\partial \bar{v}} &= \frac{Z}{2}F_{\bar{v}v} = -\frac{Z}{2}F_{xx} = Z\frac{\partial \omega}{\partial x} \\ \frac{\partial U}{\partial v} &= \frac{Z}{2}F_{vv} = \frac{Z}{2}F_{zx} = -Z\frac{\partial \omega}{\partial z} \\ \frac{\partial U}{\partial \bar{z}} &= \frac{Z}{2}F_{\bar{z}v} = -\frac{Z}{2}F_{\bar{v}x} = Z\frac{\partial \omega}{\partial \bar{v}} \end{aligned} \tag{377}$$

Explicit forms along the lines of (375)-(376) can be easily obtained by means of the formula (486) in combination with the equations (263) and (264). It remains now to compute the derivatives of the Majorana coefficients with respect to ϕ, θ, ψ and q. To do this, we use the equations (322) and (323) together with (497) to express the Majorana coefficients in terms of ϕ, θ, ψ, q and ω. The half-period ω is fixed by the equation (348). The derivatives may then be computed with the help of, among other things, the formulas (498), which gives the partial derivatives of the Weierstrass roots with respect to the *nome*.

In the new coordinate basis the metric takes the form[9]

$$ds^2 \sim (abc)^2\frac{dq^2}{q^2} + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2 \tag{378}$$

with σ_1, σ_2 and σ_3 the (left-)invariant Cartan-Maurer 1-forms of $SO(3)$ and a, b, c determined by

$$\begin{aligned} 2bc &= -\left(\frac{2}{\pi}\right)^2\omega(\eta + e_1\omega) \\ 2ab &= -\left(\frac{2}{\pi}\right)^2\omega(\eta + e_2\omega) \\ 2ca &= -\left(\frac{2}{\pi}\right)^2\omega(\eta + e_3\omega) \end{aligned} \tag{379}$$

Comparison with the theta-function representation formula (500) from section 13.4 yields eventually for the metric coefficients the expressions

$$2bc = \frac{\vartheta_2''(0,q)}{\vartheta_2(0,q)}$$

[9] An overall numerical factor is omitted.

$$
\begin{aligned}
2ab &= \frac{\vartheta_3''(0,q)}{\vartheta_3(0,q)} \\
2ca &= \frac{\vartheta_4''(0,q)}{\vartheta_4(0,q)}
\end{aligned}
\tag{380}
$$

A change of variable from the *nome* q to the complementary *nome* q' through the modular transformation (505) gives the following alternative expression for the metric

$$
ds^2 \sim \left(\frac{\ln q'}{\pi}\right)^2 \left[(ABC)^2 \frac{dq'^2}{q'^2} + A^2\sigma_1^2 + B^2\sigma_2^2 + C^2\sigma_3^2\right] \tag{381}
$$

with the coefficients A, B, C determined this time by

$$
\begin{aligned}
2BC &= \frac{2}{\ln q'} - \frac{\vartheta_4''(0,q')}{\vartheta_4(0,q')} \\
2AB &= \frac{2}{\ln q'} - \frac{\vartheta_3''(0,q')}{\vartheta_3(0,q')} \\
2CA &= \frac{2}{\ln q'} - \frac{\vartheta_2''(0,q')}{\vartheta_2(0,q')}
\end{aligned}
\tag{382}
$$

One can check that these two expressions correspond precisely to the metric of Atiyah and Hitchin, up to an irrelevant numerical scale factor.

10.3 Large and small monopole separation limits

In terms of the radial invariants introduced in (297), the generalized Legendre transform equation (348) reads

$$
r = h = \text{const.} \tag{383}
$$

The obvious interpretation is that a constant distance scale is thus set into the problem. The remaining radius, r', takes the meaning of monopole separation distance when one views the Atiyah-Hitchin manifold as the moduli space of centered two-monopoles. As discussed in section 8.5, the elliptic *nome* and complementary *nome* are given by

$$
q = e^{-\pi^2 h/r'} \qquad \text{and} \qquad q' = e^{-r'/h} \tag{384}
$$

The small monopole separation limit $r' << h$ thus corresponds to $q \to 0$, the large monopole separation limit $r' >> h$ to $q' \to 0$. These asymptotic regions can therefore be probed by series-expanding in q respectively q' the theta-function expressions (380) respectively (382) of the coefficients of the metric. The infinite series representation formulas (503) make this a straighforward task.

A q'-series expansion gives

$$
\begin{aligned}
a^2 &= \frac{r'(r'-2h)}{2h^2} - \frac{4r'^2(r'-2h)}{h^3}e^{-r'/h} + \frac{4r'^2(4r'^2-8hr'+h^2)}{h^4}e^{-2r'/h} + \cdots \\
b^2 &= \frac{2r'}{r'-2h} - \frac{16r'^2(2r'^2-6hr'+5h^2)}{h^2(r'-2h)^2}e^{-2r'/h} + \cdots \\
c^2 &= \frac{r'(r'-2h)}{2h^2} + \frac{4r'^2(r'-2h)}{h^3}e^{-r'/h} + \frac{4r'^2(4r'^2-8hr'+h^2)}{h^4}e^{-2r'/h} + \cdots
\end{aligned}
\tag{385}
$$

Retaining only the non-exponential terms yields the asymptotic form of the Atiyah-Hitchin metric

$$ds_\infty^2 = h^{-1}\left[\left(\frac{1}{2h}-\frac{1}{r'}\right)(dr'^2 + r'^2\sigma_1^2 + r'^2\sigma_3^2) + \left(\frac{1}{2h}-\frac{1}{r'}\right)^{-1}\sigma_2^2\right] \tag{386}$$

This is a Euclidean Taub-NUT metric with negative mass parameter. It has a singularity at $r' = 2h$, far away from the asymptotic region, and thus harmless.

On the other hand, expanding in the *nome* q, we get

$$\begin{aligned} a^2 &= 32\pi^2(e^{-2\pi^2 h/r'} - 4e^{-4\pi^2 h/r'} + \cdots) \\ b^2 &= \frac{\pi^2}{2}(1 - 4e^{-\pi^2 h/r'} + 16e^{-2\pi^2 h/r'} + \cdots) \\ c^2 &= \frac{\pi^2}{2}(1 + 4e^{-\pi^2 h/r'} + 16e^{-2\pi^2 h/r'} + \cdots) \end{aligned} \tag{387}$$

Truncating to order q^2 and changing the radial variable to $R = 4\,e^{-\pi^2 h/r'}$, we obtain the small monopole separation limit of the metric

$$ds_0^2 = \frac{\pi^2}{2}[dR^2 + 4R^2\sigma_1^2 + (1 - R + R^2)\sigma_2^2 + (1 + R + R^2)\sigma_3^2] \tag{388}$$

These limits of the Atiyah-Hitchin metric and the first few exponential corrections to them have been studied by Gibbons and Manton [36], to which we refer for more details. Note that, unlike in [36], here we do not need to choose the gauge $f = -b/r'$ instead of the more symmetric $f = abc$ for the radial diagonal component of the metric. A set of expressions closely related to the form (380) of the metric coefficients has been obtained in [37] through solving a Halphen system of differential equations, see also [38]. The equations (380) and (382) together with the series expansions (503) allow for a straightforward computation of the corrections to both the large and the small separation limit of the metric virtually to any order.

11 ALE manifolds of type D_n

The F-function that generates the asymptotically locally Euclidean (ALE) D_n metric through the GLT construction of [4] is given, according to [14, 15, 16], by

$$F = \oint_{\Gamma} \frac{d\zeta}{\zeta} \sqrt{\eta^{(4)}} - \sum_{l=1}^{n} \sum_{+,-} \frac{1}{2\pi i} \oint_{\Gamma_l} \frac{d\zeta}{\zeta} [\sqrt{\eta^{(4)}} \pm \chi_l^{(2)}] \ln[\sqrt{\eta^{(4)}} \pm \chi_l^{(2)}] \tag{389}$$

The parameters of the $\mathcal{O}(2)$-multiplets $\chi_l^{(2)}$, which transform as the components of a vector at rotations, do not coordinatize the ALE space but rather specify the positions of the monopoles. The contour Γ winds around the canonical 2-cycles of $\sqrt{\eta^{(4)}}$. The n contours Γ_l surround the roots a_l, $-1/\bar{a}_l$, b_l, $-1/\bar{b}_l$ of the deformed $\mathcal{O}(4)$ multiplets $\eta^{(4)} - (\chi_l^{(2)})^2$ in the way depicted schematically in Figure 6.

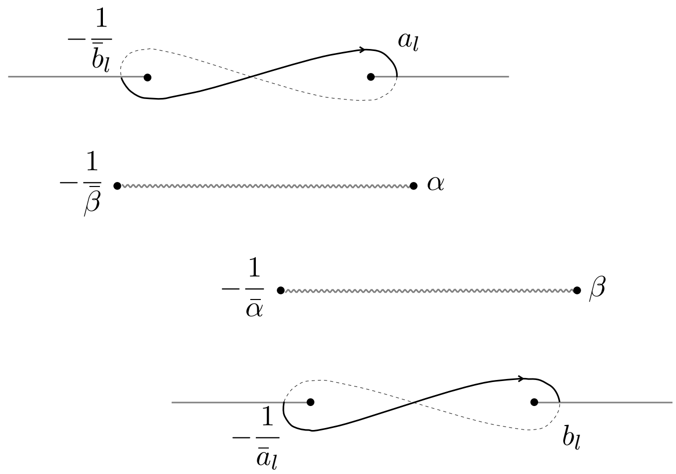


Figure 6. The two components of the contour Γ_l.

The roots a_l, $-1/\bar{a}_l$, b_l, $-1/\bar{b}_l$ are obtained by solving for ζ the equation

$$\eta^{(4)}(\zeta) = \eta_l^{(2)}(\zeta)^2 \tag{390}$$

This is an equation on $\mathbb{RP}^2$, the 2-sphere with antipodal points identified. We can get some insight into it by using spin-1/2 coherent wave-functions. In terms of these, the equation can be re-written as follows

$$\rho\,\langle -\frac{1}{\bar{\zeta}}|\alpha\rangle\langle\alpha|\zeta\rangle\langle -\frac{1}{\bar{\zeta}}|\beta\rangle\langle\beta|\zeta\rangle = [\sigma_l\langle -\frac{1}{\bar{\zeta}}|\gamma_l\rangle\langle\gamma_l|\zeta\rangle]^2 \tag{391}$$

Then, based on the equations (171) and (173), by taking the norm and, separately, comparing the phase factors on the two sides, one obtains

$$\rho\,\sin\delta_{\alpha\zeta}\sin\delta_{\beta\zeta} = (\sigma_l \sin\delta_{\gamma_l\zeta})^2 \tag{392}$$

$$\phi_{\widehat{\gamma_l\zeta\alpha}} + \phi_{\widehat{\gamma_l\zeta\beta}} = 2\pi k \qquad (k \in \mathbb{Z}) \tag{393}$$

where $\phi_{\widehat{\gamma_l\zeta\alpha}}$ is the (oriented) angle formed by the two geodesic circles that pass through γ_l respectively α and intersect at ζ; $\phi_{\widehat{\gamma_l\zeta\beta}}$ is defined similarly. Equation (393) means geometrically that γ_l sits on the geodesic circle that bisects the angle formed by the two geodesic circles that pass through α respectively β and intersect at ζ. Unfortunately we do not yet possess a satisfactory understanding of the geometric picture behind these equations. But notice that if we think of them not as equations for ζ but for γ_l, or, in other words, if we formulate the problem in this way: given ζ fixed (α and β are assumed fixed in either case), find γ_i that leads to it, then a simple geometric picture emerges. In this case, equation (392) can be easily solved to yield $\delta_{\gamma_l\zeta}$. Clearly, since we assume that $\delta_{\alpha\zeta}$, $\delta_{\beta\zeta}$, $\delta_{\gamma_l\zeta} \in [0,\pi]$, one can have either no solution or two solutions (two supplementary angles), counting multiplicities. Notice that if $\sigma_l^2 \geq \rho$ then one always has two solutions. So let us assume there are two solutions. Arrange the sphere such that ζ and $-1/\bar{\zeta}$ lie on the North-South axis. Then the locus of γ_l corresponding to a given pair of solutions for $\delta_{\gamma_l\zeta}$ is given by two circles parallel to the equator. But, as we stated above, the locus of solutions of equation (393) is the geodesic circle that bisects the spherical angle $\widehat{\alpha\zeta\beta}$ - a meridian, in our picture. The solutions for γ_l lie at the intersection of the pair of paralel circles with this meridian. Note that these solutions come in antipodally-conjugated pairs, as objects that descend on $\mathbb{RP}^2$ should.

Denoting with F_χ the sum of χ-deformed terms in (389), then by commuting the derivatives with the integrals one obtains

$$\begin{aligned}\frac{\partial F_\chi}{\partial x} &= -\sum_{l=1}^{n} \frac{1}{2\pi i} \oint_{\Gamma_l} \frac{d\zeta}{\zeta} \frac{1}{2\sqrt{\eta^{(4)}}} \ln[\eta^{(4)} - (\chi_l^{(2)})^2] \\ &= -\sum_{l=1}^{n} \int_{-1/\bar{b}_l}^{a_l} + \int_{-1/\bar{a}_l}^{b_l} \frac{d\zeta}{\zeta} \frac{1}{2\sqrt{\eta^{(4)}}}\end{aligned} \tag{394}$$

and

$$\begin{aligned}\frac{\partial F_\chi}{\partial v} &= \sum_{l=1}^{n} \frac{1}{2\pi i} \oint_{\Gamma_l} \frac{d\zeta}{\zeta} \frac{\zeta}{2\sqrt{\eta^{(4)}}} \ln[\eta^{(4)} - (\chi_l^{(2)})^2] \\ &= \sum_{l=1}^{n} \int_{-1/\bar{b}_l}^{a_l} + \int_{-1/\bar{a}_l}^{b_l} \frac{d\zeta}{\zeta} \frac{\zeta}{2\sqrt{\eta^{(4)}}}\end{aligned} \tag{395}$$

The logarithm can be dropped out of the integral at the expense of turning closed contours into open contours. We thus arrive at *incomplete* elliptic integrals of the type (324), with $m = 0, 1$. The first integral in (389) appears also in the Atiyah-Hitchin case and leads to *complete* elliptic integrals of the same type. Using the fundamental results of section 9 we derive in a straightforward manner the following formulas

$$\frac{\partial F}{\partial x} = 2m\omega + 2m'\omega' - \sum_{l=1}^{n} (u_{a_l}^- + u_{b_l}^-) \tag{396}$$

and

$$\begin{aligned}\frac{\partial F}{\partial v} &= \frac{1}{2\sqrt{z}} \ln \frac{\sigma(2m\omega - u_\infty)\sigma(2m'\omega' - u_\infty)}{\sigma(2m\omega + u_\infty)\sigma(2m'\omega' + u_\infty)} \prod_{l=1}^{n} \prod_{\zeta = a_l, b_l} \frac{\sigma(u_\zeta + u_\infty)\sigma(u_{-1/\bar{\zeta}} - u_\infty)}{\sigma(u_\zeta - u_\infty)\sigma(u_{-1/\bar{\zeta}} + u_\infty)} \\ &+ \frac{1}{2\sqrt{z}} [\zeta(u_\infty^+) + \zeta(u_\infty^-)] \frac{\partial F}{\partial x}\end{aligned} \tag{397}$$

with $m, m' \in \mathbb{Z}$. Observe that equation (396) determines the winding number m' if we require that F be real. In this case, since x is real, the whole equation has to be real. From (319) it is clear that the imaginary parts of both $u^-_{a_l}$ and $u^-_{b_l}$ are equal to ω'. To cancel them, one needs $m' = n$.

Since the n multiplets $\chi^{(2)}_l$ are spectators, the Legendre relations read

$$\frac{\partial F}{\partial v} = u \tag{398}$$

$$\frac{\partial F}{\partial x} = 0 \tag{399}$$

Together with equation (397) they imply

$$e^{2u\sqrt{z}} = \frac{\sigma(2m\omega - u_\infty)\sigma(2m'\omega' - u_\infty)}{\sigma(2m\omega + u_\infty)\sigma(2m'\omega' + u_\infty)} \prod_{l=1}^{n} \prod_{\zeta = a_l, b_l} \frac{\sigma(u_\zeta + u_\infty)\sigma(u_{-1/\bar{\zeta}} - u_\infty)}{\sigma(u_\zeta - u_\infty)\sigma(u_{-1/\bar{\zeta}} + u_\infty)} \tag{400}$$

The expression on the r.h.s. is a meromorphic elliptic function in u_∞, with zeros at $2m\omega$, $2m'\omega'$, $-u_{a_l}$, $u_{-1/\bar{a}_l}$, $-u_{b_l}$, $u_{-1/\bar{b}_l}$ for all values of l, and poles at the mirror points, of opposite sign.

On the other hand, equation (399) together with the expression (396) imply

$$\sum_{l=1}^{n} [F(\sin D_{a_l}, k) + F(\sin D_{b_l}, k)] = \mathbb{Z}_+ \cdot 2K(k) \tag{401}$$

where the angles D_{a_l} and D_{b_l} are defined by

$$\frac{\sin \dfrac{\pi - \delta_{\alpha a_l} - \delta_{\beta a_l}}{2}}{\sin D_{a_l}} = \frac{\sin \dfrac{\pi - \delta_{\alpha b_l} - \delta_{\beta b_l}}{2}}{\sin D_{b_l}} = k \qquad \text{for all } l = 1, \cdots, n \tag{402}$$

We write these relations in this form to make the resemblance to (516) transparent. In the light of the discussion in section 14 and, in particular, of the equation (546), the generalized Legendre relation (401) has a startling interpretation: *it represents the closure condition for a Poncelet star polygon!*

12 An $\mathcal{O}(2)\oplus\mathcal{O}(4)$-based Swann bundle

12.1 The hyperkähler potential

In [13] it was conjectured, based on a symmetry argument, that the nonperturbative universal hypermultiplet moduli space metric due to five-brane instantons is a certain deformation of the hyperkäler manifold that is generated, through the GLT, by the F-function

$$F=\oint_{\Gamma}\frac{d\zeta}{\zeta}\frac{(\eta^{(2)})^2}{\sqrt{\eta^{(4)}}} \tag{403}$$

The contour Γ around the branch-cuts of $\sqrt{\eta^{(4)}}$ is chosen in such a way that the outcome of the contour integration is real. Since F scales with weight 1 under the scaling transformation

$$\eta^{(2)}\longrightarrow\lambda\eta^{(2)}\qquad\qquad\eta^{(4)}\longrightarrow\lambda^2\eta^{(4)} \tag{404}$$

the resulting 8-dimensional hyperkähler variety will have a Swann bundle structure. Swann bundles possess a so-called hyperkähler potential, a function defined up to the addition of a constant which is simultaneously a Kähler potential for each complex structure compatible with the hyperkähler structure. For Swann bundles, the GLT construction produces the hyperkähler potential. In our case, the GLT relations read

$$K=F-u_2v_2-\bar{u}_2\bar{v}_2-(u_1+\bar{u}_1)x_1 \tag{405}$$

with

$$\frac{\partial F}{\partial x_1}=u_1+\bar{u}_1 \tag{406}$$

$$\frac{\partial F}{\partial v_2}=u_2 \tag{407}$$

$$\frac{\partial F}{\partial x_2}=0 \tag{408}$$

The holomorphic coordinates are z_1, u_1, z_2, u_2. We differentiate by means of an index 1 or 2 between quantities related to the $\mathcal{O}(2)$ and the $\mathcal{O}(4)$ multiplet respectively, and use in general the notations established in sections 5.3, 5.4 and 5.5.

To evaluate F, observe that we can write

$$2F=z_1^2F_{z_1z_1}+2z_1x_1F_{z_1x_1}+(x_1^2-2|z_1|^2)F_{x_1x_1}+2\bar{z}_1x_1F_{\bar{z}_1x_1}+\bar{z}_1^2F_{\bar{z}_1\bar{z}_1} \tag{409}$$

and further express the double derivatives of F in terms of the complete versions of the primary integrals of section 9 as follows

$$F_{z_1z_1}=4\mathcal{I}_2^{(1)}\qquad\qquad F_{z_1x_1}=-4\mathcal{I}_1^{(1)}\qquad\qquad F_{x_1x_1}=4\mathcal{I}_0^{(1)} \tag{410}$$

From equations (336) through (338) we thus obtain for F the expression

$$\begin{aligned}F&=4(z_{1+}^2-z_{1-}^2)\,\eta+4(x_1^2+x_-z_{1+}^2-x_+z_{1-}^2)\,\omega\\&+2\,\mathrm{Re}\left[\frac{z_1}{\sqrt{z_2}}\left(\frac{v_2z_1}{z_2}-4x_1\right)\right]\pi(x_-)+2i\,\mathrm{Im}\left[\frac{z_1}{\sqrt{z_2}}\left(\frac{v_2z_1}{z_2}-4x_1\right)\right]\pi(x_+)\end{aligned} \tag{411}$$

where we define

$$z_{1+}=\mathrm{Im}\frac{z_1}{\sqrt{z_2}}\qquad\qquad z_{1-}=\mathrm{Re}\frac{z_1}{\sqrt{z_2}} \tag{412}$$

Then, similarly to the Atiyah-Hitchin case, by means of the elliptic differentiation formulas developed in section 13.3, we compute the following derivatives of (411)

$$\frac{\partial F}{\partial x_1} = 8[x_1\omega - z_{1-}\pi(x_-) - iz_{1+}\pi(x_+)] \tag{413}$$

$$\frac{\partial F}{\partial v_2} = \frac{2M\eta^* + 2N\omega^* + (z_{1-} + iz_{1+})^2[\pi(x_-) + \pi(x_+)]}{\sqrt{z_2}} \tag{414}$$

$$\frac{\partial F}{\partial x_2} = -2(g_{\rho\sigma^2}\,\eta^* - g_{\rho^2\sigma^2}\,\omega^*) \tag{415}$$

where η^* and ω^* are defined in (489) and $M = M_- + iM_+$, $N = N_- + iN_+$, with

$$M_\pm = \frac{(2g_{\rho^2}x_\pm + 3g_{\rho^3})[g_{\sigma^2} - 3(x_+ - x_-)z_{1\pm}^2] - 3x_\pm^2 g_{\rho\sigma^2} - x_\pm g_{\rho^2\sigma^2}}{3(x_+ - x_-)v_\pm}$$

$$N_\pm = \frac{-(9g_{\rho^3}x_\pm + 2g_{\rho^2}^2)[g_{\sigma^2} - 3(x_+ - x_-)z_{1\pm}^2] + (3g_{\rho^2}x_\pm + 9g_{\rho^3})g_{\rho\sigma^2} + (3x_\pm^2 - 2g_{\rho^2})g_{\rho^2\sigma^2}}{3(x_+ - x_-)v_\pm}$$

The imaginary parts iM_+ and iN_+ of the coefficients M and N are conjugates of the corresponding real parts M_- and N_- under the $\mathbb{Z}_2$ action given by (289) together with

$$z_{1-} \longleftrightarrow iz_{1+} \tag{416}$$

The r.h.s. of equation (415) is manifestly $SO(3)$-invariant. That this should be so can be argued independently, without resorting to direct calculation, as follows: commuting the derivative with the integral, one obtains the integral representation

$$\frac{\partial F}{\partial x_2} = -\frac{1}{2}\oint_\Gamma \frac{d\zeta}{\zeta}\frac{(\eta^{(2)})^2}{(\eta^{(4)})^{3/2}} \tag{417}$$

Under the scaling transformation (404) this integral transforms with weight -1. According to the discussion following equation (176), it should then result in a $SO(3)$-invariant quantity. The equation (415) also provides us with a good opportunity to advertize the superiority of the Weierstrass approach. Had we expressed the multiplets in terms of the Majorana roots and evaluated the derivative of F with respect to x_2 within the Legendre frame we would have obtained an expression with 709 terms!

Note the structural similarity between the Jacobi terms in equation (411) and equation (343). The same mechanism as in the Atiyah-Hitchin case gives us now the hyperkähler potential: the Jacobi terms cancel against the quadratic terms in the Legendre transform when the Legendre relations (406) and (407) are used. The resulting Kähler potential is

$$K = -\frac{4}{3}(g_{\rho^2\sigma^2} + 4g_{\rho^2}g_{\sigma^2})\eta^* + 4(g_{\rho^2}g_{\rho\sigma^2} + 6g_{\rho^3}g_{\sigma^2})\omega^* - 4(x_+ + x_-)(g_{\rho\sigma^2}\,\eta^* - g_{\rho^2\sigma^2}\omega^*) \tag{418}$$

On the other hand, the GLT relation (408) reads

$$g_{\rho\sigma^2}\eta^* = g_{\rho^2\sigma^2}\omega^* \tag{419}$$

Upon resorting to it, the $(x_+ + x_-)$-dependent terms in (418) drop out and the resulting hyperkähler potential takes the remarkably compact manifestly $SO(3)$-invariant form

$$K = -\frac{4}{3}(g_{\rho^2\sigma^2} + 4g_{\rho^2}g_{\sigma^2})\eta^* + 4(g_{\rho^2}g_{\rho\sigma^2} + 6g_{\rho^3}g_{\sigma^2})\omega^* \tag{420}$$

That the hyperkähler potential must be invariant under $SO(3)$ transformations can be argued on general grounds, and this provides an additional validation for our result.

The hyperkähler holomorphic $(2,0)$-form takes the Darboux form

$$\omega^{+} = dz_1 \wedge du_1 + dz_2 \wedge du_2 \tag{421}$$

Just as in the Atiyah-Hitchin case it is worthwhile to perform the following holomorphic symplectomorphism

$$U_2 = u_2\sqrt{z_2} \qquad\qquad Z_2 = 2\sqrt{z_2} \tag{422}$$

In the new holomorphic coordinate basis,

$$\omega^{+} = dz_1 \wedge du_1 + dZ_2 \wedge dU_2 \tag{423}$$

and the conformal homothetic Killing vector field reads

$$X = 2\left(z_1\frac{\partial}{\partial z_1} + \bar{z}_1\frac{\partial}{\partial \bar{z}_1} + Z_2\frac{\partial}{\partial Z_2} + \bar{Z}_2\frac{\partial}{\partial \bar{Z}_2}\right) \tag{424}$$

One can check explicitly that the Kähler potential K is an eigenfunction of X, *i.e.*,

$$X(K) = 2K \tag{425}$$

Let us also recall that, besides the Swann bundle structure, the variety has an additional abelian tri-holomorphic isometry that is due to the presence of the $\mathcal{O}(2)$ multiplet. This is generated by the Killing vector field

$$\tilde{X} = i\left(\frac{\partial}{\partial u_1} - \frac{\partial}{\partial \bar{u}_1}\right) \tag{426}$$

One can go further and compute the metric explicitly in this holomorphic coordinate basis. For that, one needs to compute the second derivatives of F with respect to the Majorana coefficients of the two multiplets. In principle, this is a straightforward task, since all necessary tools have already been developed in the preceding sections. Unfortunately we have not been able to cast the result in a presentable compact form. A reasonable guess is that, nevertheless, such a form is very likely to exist, perhaps in a coordinate basis better adapted to the many symmetries of the problem than our own.

12.2 Asymptotic expansions

The single $\mathcal{O}(2)$ invariant $\sigma = r_1$ is of radial type. On the $\mathcal{O}(4)$ side, there are two invariants of radial type, namely r_2 and r_2' defined in (297). Other $\mathcal{O}(4)$ invariants such as g_{ρ^2}, g_{ρ^3} and η have theta-function representations which allow one to express them in terms of r_2 and r_2' in the form of infinite Lambert-type series. By constrast, the mixed invariants are essentially of angular type. The following table summarizes the various radial and angular invariants associated to an $\mathcal{O}(2) \oplus \mathcal{O}(4)$ system of multiplets

$\mathcal{O}(2)$ invariant	$\mathcal{O}(4)$ invariants	mixed invariants
r_1	r_2, r_2'	A, B
radial		angular

We want to investigate the behavior of the GLT equation (419) and the hyperkähler potential (420) in and around the asymptotic limits $r_2 >> r_2'$ and $r_2 << r_2'$. This is facilitated in a decisive manner by their manifest $SO(3)$ invariance. In practice, the two asymptotic regions are probed by expanding the $\mathcal{O}(4)$ invariants in the *nome* q respectively the complementary *nome* q', since

$$q = e^{-\pi^2 r_2/r_2'} \qquad \text{and} \qquad q' = e^{-r_2'/r_2} \tag{427}$$

Specifically, the q-series expansions for g_{ρ^2} and g_{ρ^3} are given by (298) and (299), while for η we have (see *e.g.* [39])

$$\eta = \frac{\pi^2}{12\omega}\left(1 - 24\sum_{n=1}^{\infty} n\frac{q^{2n}}{1-q^{2n}}\right) \tag{428}$$

To obtain the q'-series expansions for the Weierstrass coefficients one can use the fact that they are invariant unde the modular transformation $\tau' = -1/\tau$ and so the equations (298) and (299) still hold if one replaces ω and q with ω' and q'. Similarly, the equation (428) still holds if one replaces η, ω and q by η', ω' and q'. This yields a q'-series expansion for η'. Furthermore, η' is related to η by means of the Legendre identity

$$\begin{vmatrix} \omega' & \omega \\ \eta' & \eta \end{vmatrix} = i\frac{\pi}{2} \tag{429}$$

which then allows us to write down a q'-series expansion for the latter. Clearly, one can perform these expansions virtually to any order.

The q'-series expansion of equation (419) yields

$$\frac{B}{A} = 1 - \frac{288(3r_2 - r_2')}{r_2}e^{-2r_2'/r_2} + \frac{6912(39r_2^2 - 26r_2r_2' + 5r_2'^2)}{r_2^2}e^{-4r_2'/r_2} + \cdots \tag{430}$$

The limit $q' \to 0$ corresponds to the pinching of the b-cycle of the torus associated to the $\mathcal{O}(4)$ multiplet. In terms of the roots of $\eta^{(4)}$ this limit corresponds to $\alpha \to \beta$, while the $\mathcal{O}(4)$ multiplet degenerates into the square of an $\mathcal{O}(2)$ multiplet. Putting $\alpha = \beta$ in equation (206) and letting $\delta = \delta_{\alpha\gamma} = \delta_{\beta\gamma}$ be the Fubini-Study distance on the Riemann sphere between the confounding limit point and the $\eta^{(2)}$ root γ, we get

$$A = \cos^2\delta - \frac{1}{3} \tag{431}$$

Doing the same in equation (207) we obtain that $B = A$, in agreement with the zero-order term in the expansion (430).

On the other hand, solving equation (419) for $g_{\rho^2\sigma^2}$, substituting the result in the formula (420) for the hyperkähler potential and then performing a q'-series expansion, we get

$$\begin{aligned} K = {} & 2\left(A - \frac{2}{3}\right)\frac{r_1^2}{r_2} - 6A\frac{r_1^2}{r_2'} \\ & + A\frac{r_1^2}{r_2'}\frac{144(5r_2^2 - 7r_2r_2' + 2r_2'^2)}{r_2^2}e^{-2r_2'/r_2} \\ & - A\frac{r_1^2}{r_2'}\frac{432(285r_2^3 - 678r_2^2r_2' + 416r_2r_2'^2 - 80r_2'^3)}{r_2^3}e^{-4r_2'/r_2} + \cdots \end{aligned} \tag{432}$$

When $r_2 << r_2'$, the dominating contribution comes from the non-exponential term. This, in turn, contains a leading and a sub-leading part. Using the zero-order result (431), the leading part of the hyperkähler potential can be cast in the form

$$K_0 = 2\left(A - \frac{2}{3}\right)\frac{r_1^2}{r_2} = -\frac{2r_1^2 \sin^2\delta}{r_2} \tag{433}$$

Observe that this coincides precisely with the hyperkähler potential (260) of the $\mathcal{O}(2) \oplus \mathcal{O}(2)$ model discussed in section 7!

Let us now look at the other asymptotic region. The q-series expansion of equation (419) yields

$$\frac{B}{A} = \frac{7}{5} - \frac{504}{5}e^{-2\pi^2 r_2/r_2'} + \frac{101808}{5}e^{-4\pi^2 r_2/r_2'} + \cdots \tag{434}$$

while the expansion of the hyperkähler potential gives

$$K = \frac{2}{5}\left(A - \frac{10}{3}\right)\frac{r_1^2}{r_2} + A\frac{r_1^2}{r_2}\frac{216}{5}e^{-2\pi^2 r_2/r_2'} - A\frac{r_1^2}{r_2}\frac{14832}{5}e^{-4\pi^2 r_2/r_2'} + \cdots \tag{435}$$

In terms of the roots of $\eta^{(4)}$ the limit $q \to 0$ corresponds to $\alpha \to -1/\bar{\beta}$, while the $\mathcal{O}(4)$ multiplet degenerates into *minus* the square of an $\mathcal{O}(2)$ multiplet. Putting $\alpha = -1/\bar{\beta}$ and using that $\delta_{-1/\bar{\beta}\gamma} = \pi - \delta_{\beta\gamma}$ in the equations (206) and (207), we get that $B = -A$, which seems to be in contradiction to the leading term of (434). The resolution of this paradox comes from realizing that while $B = -A$ is a purely zero-order result, no corrections whatsoever being taken into account during its derivation, the leading term in (434) is fundamentally a first-order result in q^2. Indeed, we have the q-series expansions

$$\eta^*\Delta = 112\pi\left(\frac{\pi}{2\omega}\right)^7[q^2 + 66q^4 + \cdots] \tag{436}$$

$$\omega^*\Delta = -80\pi\left(\frac{\pi}{2\omega}\right)^5[q^2 + 18q^4 + \cdots] \tag{437}$$

There are no zero-order terms to begin with. The leading term in (434) follows from substituting these expansions in equation (419) and truncating consistently to first-order in q^2. This is to be contrasted with the situation at the other asymptotic region, where we have the q'-series expansions

$$\eta^*\Delta = -\frac{2i}{3}\left(\frac{\pi}{2\omega'}\right)^7[1 - 168(\ln q' + 3)q'^2 + \cdots] \tag{438}$$

$$\omega^*\Delta = -\frac{2i}{3}\left(\frac{\pi}{2\omega'}\right)^5[1 + 120(\ln q' + 2)q'^2 + \cdots] \tag{439}$$

which do have zero-order terms in q'^2 and where the resulting leading term of (430) is of truly zero-order nature.

13 Elliptic functions and integrals

13.1 Legendre and Weierstrass elliptic integrals

The incomplete elliptic integrals of first, second and third kind in Legendre normal form are

$$F(z,k) = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{440}$$

$$E(z,k) = \int_0^z \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt \tag{441}$$

$$\Pi(z,\nu,k) = \int_0^z \frac{1}{1-\nu t^2}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{442}$$

In place of Legendre's integral of third kind, Jacobi introduced a modified version, namely

$$\Pi_{\mathcal{J}}(z,\nu,k) = \sqrt{\frac{(\nu-1)(\nu-k^2)}{\nu}}[\Pi(z,\nu,k) - F(z,k)] \tag{443}$$

Jacobi's integral enjoys a great many formal advantages over Legendre's, for a detailed discussion in the Legendre formalism frame see *e.g.* [40]. The parameters z, k and ν are termed *amplitude*, *modulus* and *characteristic*, respectively. The corresponding complete integrals are obtained by putting the amplitude equal to 1: $K(k) = F(1,k)$, $E(k) = E(1,k)$, $\Pi(\nu,k) = \Pi(1,\nu,k)$ and $\Pi_{\mathcal{J}}(\nu,K(k)) = \Pi_{\mathcal{J}}(1,\nu,k)$ (we will justify later on this last notation). $K(k)$ arises from integrating the elliptic abelian differential form around one of the canonical cycles of the Legendre elliptic curve. The integral around the other canonical cycle yields $iK(k')$, with the *complementary modulus* k' satisfying $k^2 + k'^2 = 1$. The simplified notations $K(k) = K$, $K(k') = K'$, *etc.* are sometimes used.

In the Weierstrass theory the role of the incomplete elliptic integrals is played by

$$\int \frac{dX}{2Y} = u + \mathcal{C} \tag{444}$$

$$-\int X\frac{dX}{2Y} = \zeta(u) + \mathcal{C} \tag{445}$$

$$-\int \frac{Y_0}{X-X_0}\frac{dX}{2Y} = \frac{1}{2}\ln\frac{\sigma(u+u_0)}{\sigma(u-u_0)} - u\,\zeta(u_0) + \mathcal{C} \tag{446}$$

where $\mathcal{C}$ is an indefinite integration constant, (X,Y) and (X_0,Y_0) are points on the Weierstrass curve $Y^2 = X^3 - g_2X - g_3$, u and u_0 are the corresponding points on the Jacobian variety, and $\sigma(u)$, $\zeta(u)$ are the Weierstrass sigma respectively zeta pseudo-elliptic functions. The expressions on the r.h.s. are obtained by substituting X and Y with the corresponding Weierstrass elliptic functions, *i.e.*,

$$X = \wp(u;4g_2,4g_3) \qquad 2Y = \wp'(u;4g_2,4g_3) \tag{447}$$

The derivation of the first two expressions is fairly straightforward and standard. The derivation of the third one requires the use of a variant of the addition theorem of the Weierstrass ζ-function.

The corresponding complete integrals are obtained by integrating in the complex X-plane along the closed countours Γ_1, surrounding the roots e_2 and e_3, Γ_2, surrounding the roots e_3 and e_2 and Γ_3, surrounding the roots e_2 and e_1, or, more precisely, on the Jacobian, from $u = \omega_2$ to $-\omega_3$, from $u = \omega_3$ to $-\omega_2$ and from $u = \omega_2$ to $-\omega_1$, respectively. We get

$$\oint_{\Gamma_i} \frac{dX}{2Y} = 2\omega_i \tag{448}$$

$$-\oint_{\Gamma_i} X\frac{dX}{2Y} = 2\eta_i \tag{449}$$

$$-\oint_{\Gamma_i} \frac{Y_0}{X - X_0}\frac{dX}{2Y} = 2\begin{vmatrix} u_0 & \omega_i \\ \zeta(u_0) & \zeta(\omega_i) \end{vmatrix} \stackrel{\text{def}}{=} 2\,\pi_i(X_0) \tag{450}$$

where u_0 is the image of (X_0, Y_0) through the Abel-Jacobi map and $i = 1, 2, 3$. Equation (450) follows by way of the monodromy property of the Weierstrass σ-function in the r.h.s. of (446). The notation $\pi_i(X_0)$ is not quite rigorous, a more appropriate one would be for instance $\pi_i(X_0, Y_0)$ or $\pi_i(u_0)$. Nonetheless, for simplicity reasons as well as for other practical reasons soon to become clear, we use it in this form, but with the implicit *caveat* that it conceals a sign ambiguity. Clearly, only two out of three integrals of each set of integrals are independent, as $\omega_1+\omega_2+\omega_3 = 0$, $\eta_1+\eta_2+\eta_3 = 0$ and $\pi_1(X)+\pi_2(X)+\pi_3(X) = 0$. In line with the usual notation conventions $\omega_1 = \omega$, $\omega_3 = \omega'$, $\eta_1 = \eta$, $\eta_3 = \eta'$ we also denote $\pi_1(X) = \pi(X)$ and $\pi_3(X) = \pi'(X)$. The integrals ω and ω' respectively η and η' are termed half-periods and half-pseudo-periods because, for any $m, m' \in \mathbb{Z}$

$$\wp(u + 2m\omega + 2m'\omega') = \wp(u) \tag{451}$$

$$\zeta(u + 2m\omega + 2m'\omega') = \zeta(u) + 2m\eta + 2m'\eta' \tag{452}$$

Based on Legendre's identity

$$\begin{vmatrix} \omega' & \omega \\ \zeta(\omega') & \zeta(\omega) \end{vmatrix} = i\frac{\pi}{2} \tag{453}$$

one determines that

$$u \longrightarrow u + 2m\omega + 2m'\omega' \qquad \Longrightarrow \qquad \begin{aligned} \pi(X) &\longrightarrow \pi(X) + i\pi m' \\ \pi'(X) &\longrightarrow \pi'(X) - i\pi m \end{aligned} \tag{454}$$

This means that $\pi(X)$ and $\pi'(X)$ are *not* elliptic functions. Legendre's identity can be also used to show that

$$\begin{vmatrix} \omega' & \omega \\ \pi'(X) & \pi(X) \end{vmatrix} = i\frac{\pi}{2}u \tag{455}$$

for any $X = \wp(u)$.

The connection between the complete integrals in the Weierstrass and the Jacobi theories is given by the following formulas

$$K = \sqrt{\rho}\,\omega \tag{456}$$

$$E = \frac{\eta + e_1\omega}{\sqrt{\rho}} \tag{457}$$

$$\Pi_{\mathcal{J}}(\nu, K) = \pi(X) + i\frac{\pi}{2} \tag{458}$$

where X is related to ν as in equation (279). Similar relations hold for the corresponding primed quantities. Equations (457) and (458) can be proved by performing the following changes of integration variable in (441) and (443): $t^2 = \tilde{\nu}/k^2$ respectively $t^2 = 1/\tilde{\nu}$, with $\tilde{\nu} = (X - e_3)/(e_1 - e_3)$. Note that had we defined $\pi(X)$ as an integral from $X = \infty$ to e_1 instead of from $X = e_3$ to e_2 then it it would have been precisely equal to Jacobi's integral $\Pi_{\mathcal{J}}(\nu, K)$.

13.2 Jacobi's elliptic integral of third kind in Weierstrass form

Let us get a bit more specific than in (446) and define the elliptic integral of third kind in Weierstrass form by

$$\pi(X_1, X_2) = -\int_{\infty}^{X_1} \frac{Y_2}{X - X_2} \frac{dX}{2Y} \tag{459}$$

In terms of Weierstrass elliptic functions,

$$\pi(X_1, X_2) = \frac{1}{2} \ln \frac{\sigma(u_2 + u_1)}{\sigma(u_2 - u_1)} - u_1 \zeta(u_2) \tag{460}$$

Alternatively, based on the representation formula for the Weierstrass σ-function in terms of Jacobi theta functions, one obtains the remarkably similar theta-function formula

$$\pi(X_1, X_2) = \frac{1}{2} \ln \frac{\vartheta_1(v_2 + v_1, q)}{\vartheta_1(v_2 - v_1, q)} - v_1 \frac{\vartheta_1'(v_2, q)}{\vartheta_1(v_2, q)} \tag{461}$$

with

$$v = \frac{\pi}{2} \frac{u}{\omega} \tag{462}$$

and the standard definition for q. A more rigorous notation for the integral (459) would be $\pi(u_1, u_2)$. Note for instance that, since $\sigma(u)$ and $\zeta(u)$ are odd functions, then so is $\pi(u_1, u_2)$ in either argument. Also, under lattice shifts, one has

$$u_2 \longrightarrow u_2 + 2\omega_i \quad \Longrightarrow \quad \pi(X_1, X_2) \longrightarrow \pi(X_1, X_2) \tag{463}$$

$$u_1 \longrightarrow u_1 + 2\omega_i \quad \Longrightarrow \quad \pi(X_1, X_2) \longrightarrow \pi(X_1, X_2) + 2\pi_i(X_2) \tag{464}$$

The notation $\pi(X_1, X_2)$ is completely obscure if not misleading with respect to properties such as (464). The careful reader is urged to use throughout $\pi(u_1, u_2)$ instead of $\pi(X_1, X_2)$, wherever the latter occurs in the text.

The following interchange of amplitude and parameter formula holds

$$\pi(X_1, X_2) - \pi(X_2, X_1) = \begin{vmatrix} u_1 & u_2 \\ \zeta(u_1) & \zeta(u_2) \end{vmatrix} + \mathbb{Z} \cdot i\frac{\pi}{2} \tag{465}$$

Jacobi's integrals satisfy an addition theorem. To see that, consider a set of $n+1$ points (X_1, Y_1), $\cdots$, (X_n, Y_n) and (X, Y) on the Weierstrass curve together with the corresponding points u_1, $\cdots$, u_n and u on its Jacobian and note that from (460) one has

$$\sum_{i=1}^{n} \pi(X_i, X) = \frac{1}{2} \ln \frac{F(-u, u_1, \cdots, u_n)}{F(\ \ u, u_1, \cdots, u_n)} - \zeta(u) \sum_{i=1}^{n} u_i \tag{466}$$

with

$$F(v, v_1, \cdots v_n) = \prod_{i=1}^{n} \frac{\sigma(v - v_i)}{\sigma(v)\sigma(v_i)} \tag{467}$$

Now, provided that

$$\sum_{i=1}^{n} u_i = 0 \tag{468}$$

one has

$$F(u, u_1, \cdots, u_n) = \frac{1}{(n-1)!} \frac{\Delta_{(n)}(u, u_2, \cdots, u_n)}{\Delta_{(n-1)}(u_2, \cdots, u_n)} \tag{469}$$

with

$$\Delta_{(n)}(u_1, \cdots, u_n) = \begin{vmatrix} 1 & \wp(u_1) & \wp'(u_1) & \cdots & \wp^{(n-2)}(u_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(u_n) & \wp'(u_n) & \cdots & \wp^{(n-2)}(u_n) \end{vmatrix} \tag{470}$$

The proof of equation (469) goes as follows [41]: for fixed u_1, $\cdots$, u_n, both

$$F(u, u_1, \cdots, u_n)\Delta_{(n-1)}(u_2, \cdots, u_n) \qquad \text{and} \qquad \Delta_{(n)}(u, u_2, \cdots, u_n)$$

are meromorphic functions in u with a pole of order n at $u = 0$ and simple poles at $u = u_1$, $\cdots$, u_n. Their ratio is therefore a first order elliptic function, and hence a constant in u. To compute this constant one uses that $\sigma(u) = u + \cdots$, $\wp(u) = 1/u^2 + \cdots$ and then compares the Laurent series of the two functions in a neighborhood of $u = 0$.

In this case, equation (466) becomes

$$\sum_{i=1}^{n} \pi(X_i, X) = \frac{1}{2} \ln \frac{\Delta_{(n)}(-u, u_2, \cdots, u_n)}{\Delta_{(n)}(u, u_2, \cdots, u_n)} \tag{471}$$

As the derivatives of $\wp(u)$ are elliptic functions and so belong to the polynomial ring $\mathbb{C}[\wp, \wp']$, then, based on the fact that adding a multiple of a column to another leaves a determinant unchanged, one can show that (470) is proportional to

$$\Xi_{(n)}(X_1, Y_1, \cdots, X_n, Y_n) = \begin{vmatrix} 1 & X_1 & Y_1 & X_1^2 & Y_1 X_1 & X_1^3 & \cdots \\ \vdots & & & & & \vdots & \\ 1 & X_n & Y_n & X_n^2 & Y_n X_n & X_n^3 & \cdots \end{vmatrix} \tag{472}$$

Using this in equation (471), one obtains the addition theorem for the Jacobi elliptic integrals of third kind

$$\sum_{i=1}^{n} \pi(X_i, X) = \frac{1}{2} \ln \frac{\Xi_{(n)}(X, -Y, X_2, Y_2, \cdots, X_n, Y_n)}{\Xi_{(n)}(X, Y, X_2, Y_2, \cdots, X_n, Y_n)} \tag{473}$$

Note that (469) implies also the addition formula for the Weierstrass $\wp$-function

$$\Delta_{(n)}(u_1, \cdots, u_n) = 0 \tag{474}$$

or, equivalently,

$$\Xi_{(n)}(X_1, Y_1, \cdots, X_n, Y_n) = 0 \tag{475}$$

Putting the upper limit in (459) equal to e_1 yields the complete Jacobi integral of third kind, *i.e.*,

$$\pi(e_1, X) = \pi(X) + \mathbb{Z} \cdot i\frac{\pi}{2} \tag{476}$$

The ambiguity arises essentially because we define here $\pi(X)$ as an integral from $X = e_3$ to e_2 instead of from $X = \infty$ to e_1. In the process of deforming the second path into the first one picks a residue from the pole at X.

The complete integral has the following Weierstrass elliptic-function representation

$$\pi(X) = \begin{vmatrix} u & \omega \\ \zeta(u) & \zeta(\omega) \end{vmatrix} \tag{477}$$

and the theta-function representation

$$\pi(X) = -\frac{\pi}{2}\frac{\vartheta_1'(v, q)}{\vartheta_1(v, q)} \tag{478}$$

with v given in terms of u as in (462). This last relation is just the well-known theta-function representation formula for the Weierstrass ζ-function re-expressed in terms of $\pi(X)$.

An addition theorem holds also for the complete Jacobi integrals of third kind. Consider three points u_1, u_2, u_3 on the Jacobian, satisfying

$$u_1 + u_2 + u_3 = 0 \tag{479}$$

The corresponding points on on the Weierstrass curve are collinear

$$\begin{vmatrix} 1 & X_1 & Y_1 \\ 1 & X_2 & Y_2 \\ 1 & X_3 & Y_3 \end{vmatrix} = 0 \tag{480}$$

The addition theorem for the Weierstrass ζ-function together with the equation (477) imply then that

$$\pi(X_1) + \pi(X_2) + \pi(X_3) = \omega\frac{Y_1 - Y_2}{X_1 - X_2} = \omega\frac{Y_2 - Y_3}{X_2 - X_3} = \omega\frac{Y_3 - Y_1}{X_3 - X_1} \tag{481}$$

The ratios on the r.h.s. are equal (apart from the ω factor) to the slope of the line determined by the three colinear points under consideration. Interesting particular cases of this addition formula are obtained by putting *e.g.* $(X_3, Y_3) = (e_i, 0)$ with $i = 1, 2, 3$, in which case $\pi(X_3)$ takes the values 0, $-i\pi/2$ and $+i\pi/2$, respectively. Such formulas have appeared in the XIX century mathematical literature in the guise of identities relating pairs of Legendre elliptic integrals of the third kind with characteristics ν satisfying special relations, see *e.g.* [40].

13.3 Differentiation formulas

Legendre's complete elliptic integrals have the well-known differentiation formulas

$$dK(k) \quad = \quad \frac{E(k) - k'^2 K(k)}{2k^2 k'^2} dk^2 \tag{482}$$

$$dE(k) = \frac{E(k) - K(k)}{2k^2} dk^2 \tag{483}$$

$$\begin{aligned} d\Pi(\nu,k) &= \frac{E(k) - k'^2\Pi(\nu,k)}{2(k^2-\nu)k'^2} dk^2 \\ &+ \frac{(\nu-k^2)K(k) - \nu E(k) - (\nu^2-k^2)\Pi(\nu,k)}{2\nu(\nu-1)(\nu-k^2)} d\nu \end{aligned} \tag{484}$$

Based on these, we derive the following differentiation formula for Jacobi's version of the complete elliptic integral of the third kind

$$d\Pi_{\mathcal{J}}(\nu,K) = -\frac{2\nu(\nu-1)dK + [(\nu-1)K+E]d\nu}{2\sqrt{\nu(\nu-1)(\nu-k^2)}} \tag{485}$$

This justifies the parametrization by ν and K. Comparing with (484), one can see that whereas the derivatives of Legendre's integral of third kind with respect to its parameters depend on the integral itself, the derivatives of Jacobi's do not.

On the Weierstrass side one obtains, by means of the relations (456) through (458), as well as the equations (482), (483) and (485), the following set of differentiation formulas

$$d\omega = -\frac{2g_2^2\omega - 9g_3\eta}{2\Delta} dg_2 + 3\frac{3g_3\omega - 2g_2\eta}{2\Delta} dg_3 \tag{486}$$

$$d\eta = -g_2\frac{3g_3\omega - 2g_2\eta}{2\Delta} dg_2 + \frac{2g_2^2\omega - 9g_3\eta}{2\Delta} dg_3 \tag{487}$$

and

$$\begin{aligned} d\pi(X) = &\frac{\eta + X\omega}{2Y} dX \\ &+ \frac{(g_2X + 3g_3)(3g_3\omega - 2g_2\eta) - X^2(2g_2^2\omega - 9g_3\eta)}{2Y\Delta} dg_2 \\ &+ \frac{(3X^2 - 2g_2)(3g_3\omega - 2g_2\eta)\ -\ X(2g_2^2\omega - 9g_3\eta)}{2Y\Delta} dg_3 \end{aligned} \tag{488}$$

Note that the derivatives of $\pi(X)$ have again the remarkable feature that they do not depend on $\pi(X)$ itself, and are, moreover, elliptic functions. Occasionally we may use the shorthand notations

$$\eta^* = \frac{2g_2^2\omega - 9g_3\eta}{\Delta} \qquad \text{and} \qquad \omega^* = \frac{3g_3\omega - 2g_2\eta}{\Delta} \tag{489}$$

Incidentally, note that equations (486) and (487) constitute the first step in a recurrent series. For any integer $k \geq 0$ one has

$$d\omega_k = \frac{1}{2}(-\eta_{k+1}dg_2 + [2+(-1)^k]\omega_{k+1}dg_3) \tag{490}$$

$$d\eta_k = \frac{1}{2}(-g_2\omega_{k+1}dg_2 + [2-(-1)^k]\eta_{k+1}dg_3) \tag{491}$$

where

$$\eta_0 = \eta \qquad \omega_0 = \omega \tag{492}$$

and

$$\begin{pmatrix} \omega_{k+1} \\ \eta_{k+1} \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} [2-(-1)^k]\,3g_3 & 2g_2 \\ 2g_2^2 & [2+(-1)^k]\,3g_3 \end{pmatrix}\begin{pmatrix} 6k+1 & 0 \\ 0 & 6k-1 \end{pmatrix}\begin{pmatrix} \omega_k \\ \eta_k \end{pmatrix} \tag{493}$$

This is proved by induction.

Observe that equations (486) and (487) can be reverted to express dg_2 and dg_3 in terms of $d\eta$ and $d\omega$, which amounts to regarding g_2 and g_3 as functions of η and ω and not the other way around. From this perspective, $\pi(X)$ is then a function of X, η and ω. Its partial derivatives with respect to this alternative set of variables, as results from substituting the reverted equations (486)-(487) into equation (488), are given by

$$d\pi(X) = \frac{\eta + X\omega}{2Y} dX - \frac{V\omega + X}{Y} d\eta - \frac{V\eta - X^2}{Y} d\omega \tag{494}$$

where

$$V = \frac{2g_2\eta - 3g_3\omega}{3\eta^2 - g_2\omega^2} \tag{495}$$

The Weierstrass roots e_1, e_2 and e_3 can be regarded as functions of the modular coefficients g_2 and g_3. As one can easily check by means of equations (275) and (276), their partial derivatives with respect to these variables are given by

$$de_1 = \frac{e_1 dg_2 + dg_3}{(e_1 - e_2)(e_1 - e_3)} \tag{496}$$

and permutations thereof.

The Weierstrass roots can equally be regarded as functions of the period ω and the *nome* q. This dependence can be made in some sense explicit by means of the Jacobi theta-function representation formulas, which follow immediately from

$$\begin{aligned}
e_1 - e_2 &= \left(\frac{\pi}{2\omega}\right)^2 \vartheta_4^4(0, q) \\
e_1 - e_3 &= \left(\frac{\pi}{2\omega}\right)^2 \vartheta_3^4(0, q) \\
e_2 - e_3 &= \left(\frac{\pi}{2\omega}\right)^2 \vartheta_2^4(0, q)
\end{aligned} \tag{497}$$

The partial derivatives of the roots with respect to ω are trivial to compute. The partial derivatives with respect to q can be computed by using the fact that theta functions satisfy the 1-dimensional heat equation to exchange each derivative with respect to the elliptic modulus on a theta constant with a double derivative with respect the first argument of the corresponding theta function, evaluated at zero. By resorting to the equations (497) and the theta-function representation formulas (500), one can re-write the resulting theta-function expressions in terms of the roots. In the end, one obtains

$$de_1 = -2\, e_1 \frac{d\omega}{\omega} + \frac{1}{3}\left(\frac{2}{\pi}\right)^2 \omega \left[(e_1 - e_2)(\eta + e_3\omega) + (e_1 - e_3)(\eta + e_2\omega)\right] \frac{dq}{q} \tag{498}$$

and the permutations thereof.

13.4 Theta-function representation formulas

Differentiating equation (478) with respect to v and using the fact that $X = \wp(u) = -\zeta'(u)$, we get

$$\omega(\eta + X\omega) = -\left(\frac{\pi}{2}\right)^2 \frac{\partial^2}{\partial v^2} \ln \vartheta_1(v, q) \tag{499}$$

Setting then in turns $X = e_1$, e_2, e_3 corresponding to which $v = \pi/2$, $-(1+\tau)\pi/2$, $\tau\pi/2$ respectively, and then making use of the well-known pseudo-periodicity properties of the theta functions yields

$$\omega(\eta + e_i\,\omega) = -\left(\frac{\pi}{2}\right)^2 \frac{\vartheta''_{i+1}(0,q)}{\vartheta_{i+1}(0,q)} \tag{500}$$

with $i = 1, 2, 3$. Derivatives with respect to the first argument of the theta functions are denoted, conventionally, with primes. Equations (499) and (500) can be interpreted as theta-function representation formulas for X and the Weierstrass roots, respectively. For η one has [39]

$$\eta = -\frac{\pi^2}{12\omega}\frac{\vartheta_1'''(0,q)}{\vartheta_1'(0,q)} \tag{501}$$

The logarithmic derivatives of Jacobi's theta functions have the following Fourier series expansions (see *e.g.* [42])

$$\begin{aligned}
\frac{\vartheta_1'(v,q)}{\vartheta_1(v,q)} &= \cot v + 4\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin(2nv)\\
\frac{\vartheta_2'(v,q)}{\vartheta_2(v,q)} &= -\tan v + 4\sum_{n=1}^{\infty}(-)^n\frac{q^{2n}}{1-q^{2n}}\sin(2nv)\\
\frac{\vartheta_3'(v,q)}{\vartheta_3(v,q)} &= 4\sum_{n=1}^{\infty}(-)^n\frac{q^{n}}{1-q^{2n}}\sin(2nv)\\
\frac{\vartheta_4'(v,q)}{\vartheta_4(v,q)} &= 4\sum_{n=1}^{\infty}\frac{q^{n}}{1-q^{2n}}\sin(2nv)
\end{aligned} \tag{502}$$

By differentiating the last three relations with respect to v and then setting $v = 0$ we obtain the q-series representations

$$\begin{aligned}
\frac{\vartheta_2''(0,q)}{\vartheta_2(0,q)} &= -1 + 8\sum_{n=1}^{\infty}(-)^n n\frac{q^{2n}}{1-q^{2n}}\\
\frac{\vartheta_3''(0,q)}{\vartheta_3(0,q)} &= 8\sum_{n=1}^{\infty}(-)^n n\frac{q^{n}}{1-q^{2n}}\\
\frac{\vartheta_4''(0,q)}{\vartheta_4(0,q)} &= 8\sum_{n=1}^{\infty} n\frac{q^{n}}{1-q^{2n}}
\end{aligned} \tag{503}$$

The q-series expansion relevant for η is [39]

$$\frac{\vartheta_1'''(0,q)}{\vartheta_1'(0,q)} = -1 + 24\sum_{n=1}^{\infty} n\frac{q^{2n}}{1-q^{2n}} \tag{504}$$

13.5 Modular transformations and q'-series expansions

The Jacobi theta-functions of *nome* $q = \exp(i\pi\tau)$ and those of complementary *nome* $q' = \exp(i\pi\tau')$ with

$$\tau' = -\frac{1}{\tau} \tag{505}$$

are related as follows (see *e.g.* [39])

$$i\vartheta_1(v,q) = \sqrt{-i\tau'}e^{i\tau' v^2/\pi}\vartheta_1(\tau' v,q') \tag{506}$$

$$\vartheta_2(v,q) = \sqrt{-i\tau'}e^{i\tau' v^2/\pi}\vartheta_4(\tau' v,q') \tag{507}$$

$$\vartheta_3(v,q) = \sqrt{-i\tau'}e^{i\tau' v^2/\pi}\vartheta_3(\tau' v,q') \tag{508}$$

$$\vartheta_4(v,q) = \sqrt{-i\tau'}e^{i\tau' v^2/\pi}\vartheta_2(\tau' v,q') \tag{509}$$

This set of relations allows one to derive the modular transformation properties of any theta-function-dependent quantity. For instance, the Weierstrass coefficients g_2 and g_3 admit the following theta-function representation formulas

$$g_2 = \frac{1}{6}\left(\frac{\pi}{2\omega}\right)^4[\vartheta_2(0,q)^8 + \vartheta_3(0,q)^8 + \vartheta_4(0,q)^8] \tag{510}$$

$$g_3 = \frac{1}{27}\left(\frac{\pi}{2\omega}\right)^6[\vartheta_2(0,q)^4 + \vartheta_3(0,q)^4][\vartheta_3(0,q)^4 + \vartheta_4(0,q)^4][\vartheta_4(0,q)^4 - \vartheta_2(0,q)^4] \tag{511}$$

Using the equations (506)-(509) with v set to 0 one can easily show that they remain invariant under the modular transformation (505), *i.e.*,

$$g_2 = \frac{1}{6}\left(\frac{\pi}{2\omega'}\right)^4[\vartheta_2(0,q')^8 + \vartheta_3(0,q')^8 + \vartheta_4(0,q')^8] \tag{512}$$

$$g_3 = \frac{1}{27}\left(\frac{\pi}{2\omega'}\right)^6[\vartheta_2(0,q')^4 + \vartheta_3(0,q')^4][\vartheta_3(0,q')^4 + \vartheta_4(0,q')^4][\vartheta_4(0,q')^4 - \vartheta_2(0,q')^4] \tag{513}$$

On the other hand, quantities such as η and $\pi(X)$ transform non-trivially under this modular transformation. Their theta-function representations are given by equations (501) and (478). The corresponding primed quantities have similar representations

$$\eta' = -\frac{\pi^2}{12\omega'}\frac{\vartheta_1'''(0,q')}{\vartheta_1'(0,q')} \tag{514}$$

$$\pi'(X) = -\frac{\pi}{2}\frac{\vartheta_1'(\tau' v,q')}{\vartheta_1(\tau' v,q')} \tag{515}$$

and q'-expansion formulas given by the equations (504) and (502) with q replaced by q' and v by $\tau' v$. Using the equations (506)-(509) one can show then that the modular transformation rules for η and $\pi(X)$ under (505) are given precisely by the Legendre relations (453) respectively (455). Their q'-series expansions follow immediately from these.

13.6 Incomplete elliptic integrals of first kind and spherical triangles

Theorem (Legendre, [43]) *The equality*

$$F(\sin A,k) + F(\sin B,k) + F(\sin C,k) = 2K(k) \tag{516}$$

holds if and only if the amplitudes A, B and C form the angles of a spherical triangle, the lengths of the sides of which can be determined from

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = k \tag{517}$$

where $k \in [0,1]$ is the elliptic modulus.[10]

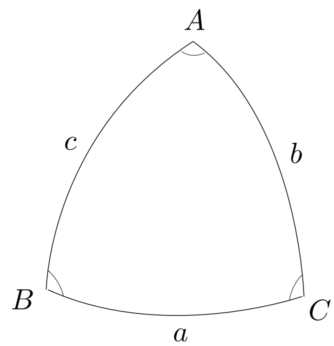


Figure 7.

Legendre's addition theorem is equivalent to the addition theorems for Jacobi's elliptic functions in the real domain. The theory of Jacobi elliptic functions emerged historically from the study of the problem of inverting incomplete Legendre elliptic integrals of the first kind, *i.e.*,

$$\mathrm{sn}(F(u,k),k) = u \tag{518}$$

An excellent reference for this topic is Cayley's treatise on elliptic functions [40].

The list of properties of Jacobi's elliptic functions $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$ and $\mathrm{dn}(u,k)$ includes

- trigonometric-like relations

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1 \qquad\qquad \mathrm{dn}^2 u + k^2 \mathrm{sn}^2 u = 1 \tag{519}$$

- reflection symmetry

$$\begin{aligned} \mathrm{sn}(-u) &= -\mathrm{sn}\, u \\ \mathrm{cn}(-u) &= +\mathrm{cn}\, u \\ \mathrm{dn}(-u) &= +\mathrm{dn}\, u \end{aligned} \tag{520}$$

- double-periodicity

$$\begin{aligned} \mathrm{sn}(v + 2mK + 2m'iK') &= (-)^{m} \quad\;\; \mathrm{sn}\, v \\ \mathrm{cn}(v + 2mK + 2m'iK') &= (-)^{m+m'} \mathrm{cn}\, v \\ \mathrm{dn}(v + 2mK + 2m'iK') &= (-)^{m'} \quad\; \mathrm{dn}\, v \end{aligned} \tag{521}$$

- addition theorems

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}\, u\, \mathrm{cn}\, v\, \mathrm{dn}\, v + \mathrm{sn}\, v\, \mathrm{cn}\, u\, \mathrm{dn}\, u}{1 - k^2 \mathrm{sn}^2 u\, \mathrm{sn}^2 v} \tag{522}$$

$$\mathrm{cn}(u+v) = \frac{\mathrm{cn}\, u\, \mathrm{cn}\, v - \mathrm{sn}\, u\, \mathrm{dn}\, u\, \mathrm{sn}\, v\, \mathrm{dn}\, v}{1 - k^2 \mathrm{sn}^2 u\, \mathrm{sn}^2 v} \tag{523}$$

$$\mathrm{dn}(u+v) = \frac{\mathrm{dn}\, u\, \mathrm{dn}\, v - k^2 \mathrm{sn}\, u\, \mathrm{cn}\, u\, \mathrm{sn}\, v\, \mathrm{cn}\, v}{1 - k^2 \mathrm{sn}^2 u\, \mathrm{sn}^2 v}$$

[10] This addition theorem appears in Legendre's treatise in a slightly different form and states that if the sum of two elliptic integrals equals a third one then their amplitudes must form the *sides* of a spherical triangle. Their asymmetric occurence in the elliptic formula forces then one to impose certain sign choice prescriptions. The form presented here, manifestly symmetric, avoids this formal inconvenience.

To prove Legendre's addition theorem, one should take first to the other side of the equality in (516) one of the incomplete elliptic integrals, for instance $F(\sin C, k)$, and then act in turns on the equation thus obtained with Jacobi's elliptic functions cn and dn of modulus k. This allows one to employ on the l.h.s. the addition formulas (523) and on the r.h.s. the properties (520) and (521) of Jacobi's elliptic functions, and then use the equations

$$\begin{aligned}\operatorname{sn} F(\sin A, k) &= \sin A \\ \operatorname{cn} F(\sin A, k) &= \cos A \\ \operatorname{dn} F(\sin A, k) &= \cos a\end{aligned}$$

and similarly for $F(\sin B, k)$. The remaining k^2's will be substituted by $\sin a \sin b \div \sin A \sin B$, in accordance with equations (517). Eventually, one should arrive after some algebraic manipulations at the cosine theorems of spherical trigonometry. Conversely, it is also possible to derive Jacobi's addition theorems starting from Legendre's [40]. The two are thus equivalent in the real domain.

14 Poncelet's closure theorem

14.1 Poncelet's porism

We will be concerned in these notes with the following theorem, due to Poncelet:

Theorem (Poncelet's porism[11]) *Given two plane conics $\mathcal{B}$ and $\mathcal{C}$, with $\mathcal{C}$ lying inside $\mathcal{B}$, if there exists a (star) polygon inscribed in $\mathcal{B}$ and circumscribed about $\mathcal{C}$ then there exist an infinity of such polygons.*

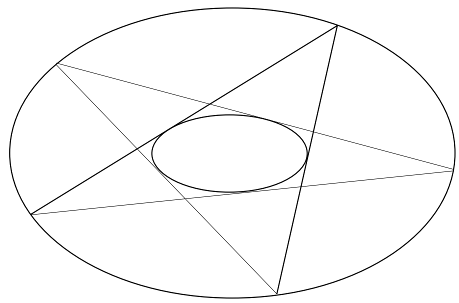


Figure 8.

Equivalently, consider a point P_0 lying on the conic $\mathcal{B}$ and from it draw a tangent L_1 to the conic $\mathcal{C}$, which will intersect again $\mathcal{B}$ at a point P_1. Repeat this construction starting this time from P_1, a.s.o. This yields a series of pairs of tangents to $\mathcal{C}$ and points on $\mathcal{B}$, (P_1, L_1), (P_2, L_2), $\cdots$. If, after a finite number of steps, one arrives back at P_0, then Poncelet's porism states that this will happen regardless of which starting point P_0 one chooses.

For original papers, reviews and related material, see [45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 18, 19].

14.2 The projective geometry of plane conics

A point in the projective plane $\mathbb{P}^2$ is specified by its homogeneous coordinates, $x = [x_1 : x_2 : x_3]$. The equation of a projective line in $\mathbb{P}^2$ that passes through the point x is

$$(y, x) = \sum_{i=1}^{3} y_i x_i = 0 \tag{524}$$

A line is specified by its coefficients, the tangential coordinates $y^* = (y_1 : y_2 : y_3)$, which can be thought of as being the homogeneous coordinates of a point in the dual projective plane $\mathbb{P}^{2*}$. One can similarly argue that a line in $\mathbb{P}^{2*}$ corresponds to a point in $\mathbb{P}^2$. The symmetry of the equation (524) at the interchange of x and y results in an ambiguity of interpretation of what one means by 'points' and 'lines' which lays at the heart of the principle of duality of projective geometry. For example, a fundamental theorem of projective geometry states that through any two distinct points in a projective plane there passes exactly one line. Applying it to the dual projective plane yields the dual

[11] "A proposition affirming the possibility of finding such conditions as will render a certain problem indeterminate, or capable of innumerable solutions" [44]

theorem: any two distinct lines in a projective plane intersect exactly once. The duality correspondence preserves incidence relationships.

Projective conics in $\mathbb{P}^2$ are described by means of quadratic equations

$$(x, Qx) = \sum_{i,j=1}^{3} x_i Q_{ij} x_j = 0 \tag{525}$$

where Q is a symmetric 3×3 matrix. Such a projective variety is a smooth submanifold of $\mathbb{P}^2$ and thus a Riemann surface if and only if the matrix Q is non-singular.

Given any smooth conic $\mathcal{C}$, let P_0 be a point on $\mathcal{C}$ and L_0 be a line that does not contain P_0, see Figure 9. By Bézout's theorem, any line in the projective plane intersects a smooth conic exactly twice, counting multiplicities. Then any line that passes through P_0 will intersect the conic at one other point which is in one-to-one correspondence with the point at which the line intersects L_0. This stereographic projection-like construction establishes a biholomorphic mapping $\mathcal{C} \longrightarrow L_0 \simeq \mathbb{P}^1$, *i.e.*, it provides a *rational parametrization* of the conic.

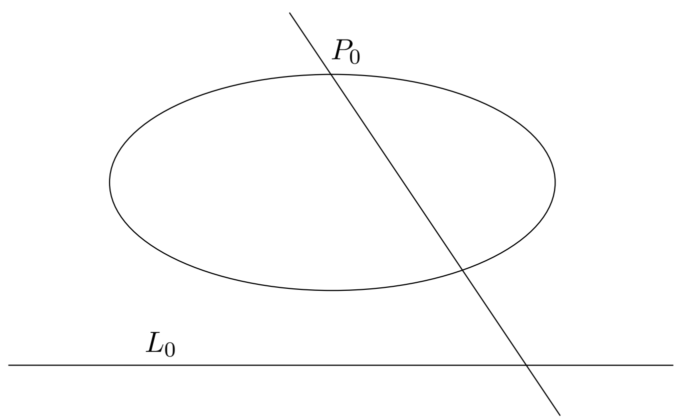


Figure 9.

The equation of the tangent line to a smooth conic $\mathcal{C}$ at a point $x \in \mathcal{C}$ is

$$0 = \frac{1}{2} \sum_{i=1}^{3} y_i \frac{\partial}{\partial x^i} (x, Qx) = (Qx, y) \tag{526}$$

The tangent is thus the subspace of $\mathbb{P}^2$ orthogonal to x with respect to the symmetric bilinear form associated to $\mathcal{C}$. The dual coordinates of the tangent line are $x^* = Qx$. They satisfy

$$(x^*, Q^{-1} x^*) = 0 \tag{527}$$

i.e., they are points on the dual conic $\mathcal{C}^*$ defined by the inverse matrix Q^{-1}. So, the envelope of tangents to a conic is also a conic. Points are dual to lines, conics are self-dual.

14.3 The construction of Griffiths and Harris

Consider two smooth conics defined by the matrices[12] $\mathcal{A}$ and $\mathcal{B}$, with no common components, so that they intersect transversally at four points in general position. The set of

[12] In the following, we will refer to a conic using the symbol of its defining matrix.

plane conics that contain these four points, *i.e.*, the *pencil* of conics generated by $\mathcal{A}$ and $\mathcal{B}$, is given by the one-parameter family

$$\mathcal{C}_X = \mathcal{A} + X\mathcal{B} \tag{528}$$

with $X \in \mathbb{C} \cup \{\infty\} \simeq \mathbb{P}^1$. In particular, $\mathcal{C}_0 = \mathcal{A}$ and $\mathcal{C}_\infty = \mathcal{B}$. Among the conics in the pencil there are three singular ones, consisting of the three pairs of lines obtained by joining in all possible ways pairs of the four intersection points.

Fix a conic $\mathcal{C}_{X_0}$ from the pencil, non-singular and different from $\mathcal{B}$. In order to address the Poncelet problem, Griffiths and Harris [50] construct the incidence correspondence[13]

$$\Sigma = \{(P, L) \in \mathcal{B} \times \mathcal{C}^*_{X_0} \mid P \in L\} \tag{529}$$

i.e., the set of pairs of points P on $\mathcal{B}$ and tangents L to $\mathcal{C}_{X_0}$ subject to the incidence condition that L passes through P. As both conics $\mathcal{B}$ and $\mathcal{C}^*_{X_0}$ can be rationally parametrized, $\Sigma \subset \mathcal{B} \times \mathcal{C}^*_{X_0} \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The transversality of the intersection $\mathcal{B} \cap \mathcal{C}_{X_0} = \mathcal{B} \cap \mathcal{A}$ insures that Σ is a non-singular variety.

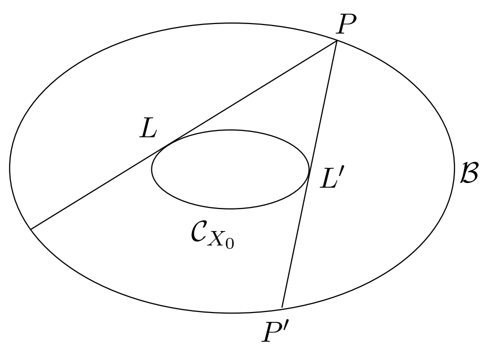


Figure 10.

Given a point and a smooth conic, there exist exactly two lines, counting multiplicities, that are tangent to the conic and intersect each other at the given point, with the two tangents being confounded if and only if the point belongs to the conic. This is dual to the statement that a line intersects a smooth conic exactly twice, counting multiplicities, with the intersection points coinciding if and only if the line is tangent to the conic. As a consequence, the variety Σ has two natural involutive automorphisms, namely

$$i_1(P, L') = (P', L') \tag{530}$$

$$i_2(P, L\ ) = (P\ , L') \tag{531}$$

for notations see Figure 10. The fixed points of i_1 are the four points of $\mathcal{B}^* \cap \mathcal{C}^*_{X_0}$, *i.e.*, the four common tangents of $\mathcal{B}$ and $\mathcal{C}_{X_0}$, whereas the fixed points of i_2 are the four points of $\mathcal{B} \cap \mathcal{C}_{X_0} = \mathcal{B} \cap \mathcal{A}$. The relevance to the Poncelet problem becomes transparent when we observe that the action of the composed automorphism $j = i_1 \circ i_2$, namely,

$$j(P, L) = (P', L') \tag{532}$$

offers a realization of the basic step in the geometric construction of Poncelet polygons.

[13]The construction that we present here is in fact dual to that of Griffiths and Harris.

The projection

$$\begin{tikzcd} \Sigma \arrow[d, "2:1"] & (P,L) \arrow[d, mapsto] \\ \mathcal{B} & P \end{tikzcd} \tag{533}$$

exhibits Σ as a branched double-cover of $\mathcal{B} \simeq \mathbb{P}^1$. The action of i_2 interchanges the sheets of the double-cover, the branching points being the fixed points of i_2. The Riemann-Hurwitz formula tells us then that Σ has genus 1, *i.e.*, Σ is an elliptic curve.

To cast Σ in a more explicit form, Griffiths and Harris, following the ideas of Cayley [45], use an ingenious rational parametrization construction for $\mathcal{B}$. Choose one of the four intersection points, and take the tangent to an arbitrary conic $\mathcal{C}_X$ from the pencil through this point, see Figure 11. The tangent intersects $\mathcal{B}$ at one more point, which we then label P_X. Together with the tangent to $\mathcal{B}$ through one of the other intersection points, this gives a rational parametrization of $\mathcal{B}$ by the complex parameter X.

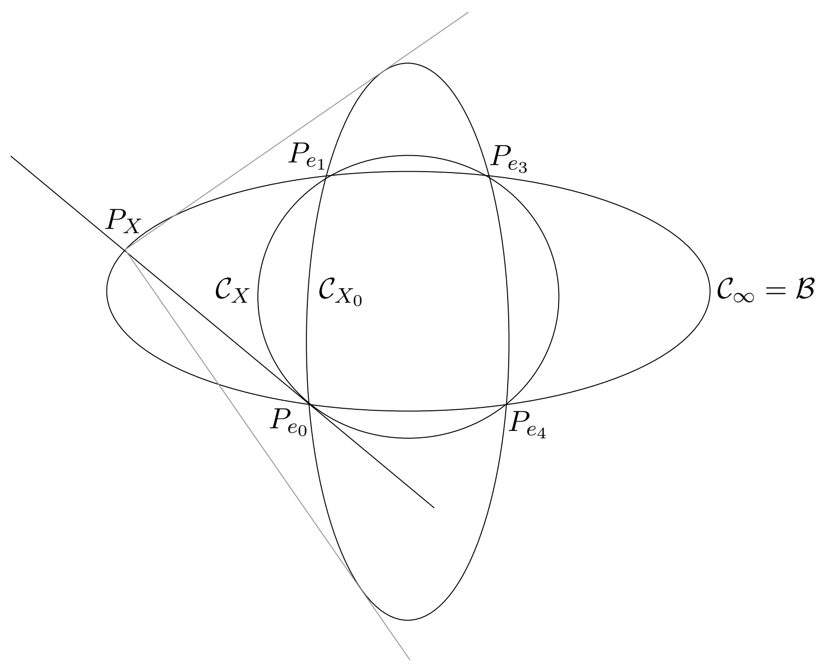


Figure 11.

The chosen point of intersection itself is the limit case given by the tangent to $\mathcal{C}_\infty = \mathcal{B}$, so it corresponds to $P_{e_0=\infty}$. Let P_{e_1}, P_{e_2}, P_{e_3} be the other three intersection points. The conics $\mathcal{C}_{e_1}$, $\mathcal{C}_{e_2}$, $\mathcal{C}_{e_3}$ that parametrize them are the three singular conics of the pencil. Indeed, a conic corresponding to one of the points P_{e_i} with $i = 1, 2$ or 3 has to simultaneously satisfy the following two properties: 1) the line joining P_{e_0} and P_{e_i} is tangent to it and 2) the two, by assumption disjoint, intersection points P_{e_0} and P_{e_i} belong to it, as they belong to all conics in the pencil. But these two requirements cannot be satisfied at once unless the conic is degenerate. This occurs when the defining matrix is singular, *i.e.*, when $X = e_i$ is a solution of $\det(\mathcal{A} + X\mathcal{B}) = 0$.

A point $P_X \in \mathcal{B}$ together with a choice of tangent to $\mathcal{C}_{X_0}$ define a point on the double cover Σ. Branching occurs when the tangents through P_X coincide, and this cannot happen unless $P_X \in \mathcal{C}_{X_0}$, in which case $P_X \in \mathcal{C}_{X_0} \cap \mathcal{B} = \mathcal{A} \cap \mathcal{B}$, and so, by the argument above, X has to be a solution of $\det(\mathcal{A} + X\mathcal{B}) = 0$. From these considerations if follows that the elliptic curve Σ is isomorphic to Cayley's cubic[14]

$$Y^2 = \det(\mathcal{A} + X\mathcal{B}) \tag{534}$$

the isomorphism between them being given by

$$\begin{aligned} (P_X, L\ ) \quad &\longleftrightarrow \quad (X, +Y) \\ (P_X, L') \quad &\longleftrightarrow \quad (X, -Y) \end{aligned} \tag{535}$$

As an elliptic curve, Σ posesses an abelian differential, *i.e.*, a globally holomorphic 1-form

$$\varpi = \frac{dX}{Y} \tag{536}$$

with associated period lattice $X = \mathbb{Z} \cdot 2\omega + \mathbb{Z} \cdot 2\omega'$. The fundamental periods 2ω and $2\omega'$, chosen such that $\operatorname{Im} \omega'/\omega > 0$, are the integrals of ϖ over the a and b-cycles of the torus Σ. One can exploit ϖ to give an alternative description of Σ by means of the Abel-Jacobi map, an analytic isomorphism between Σ and its Jacobian variety, $\mathbb{C}/\Lambda$,

$$\begin{aligned} \Sigma &\xrightarrow{\quad \simeq \quad} \mathbb{C}/\Lambda \\ (X, Y) &\longmapsto \int_\Gamma \varpi \end{aligned} \tag{537}$$

The integral, taken on a path Γ on Σ based at an arbitrary fixed point, is independent of the path modulo integer multiples of the periods, that is to say, it defines an equivalence class on $\mathbb{C}/\Lambda$.

The automorphisms of Σ are carried over by the Abel-Jacobi map to $\mathbb{C}/\Lambda$, and for simplicity we will denote the corresponding automorphisms of $\mathbb{C}/\Lambda$ by the same letters. Any automorphism i of $\mathbb{C}/\Lambda$ is induced by an automorphism of its universal cover $\mathbb{C}$, $\tilde{\imath}(u) = au + b$, for any $u \in \mathbb{C}$. Then i is involutive, *i.e.*, $i^2(u) = u \bmod \Lambda$ if and only if $a^2 = 1$ and $(a+1)b = 0 \bmod \Lambda$. In the case when $a = +1$, one easily argues that i has no fixed points unless it is the trivial automorphism of Σ, in which case all points of Σ are fixed points. Since we want i_1 and i_2 to have no more and no less than four fixed points each, this cannot be the case. So $a = -1$ for both, that is, $i_1(u) = -u + b_1 \bmod \Lambda$ and $i_2(u) = -u + b_2 \bmod \Lambda$. Moreover, one can always redefine u by a shift to put $b_2 = 0$. Eventually, renaming $b_1 = u_0$, we have

$$i_1(u) = -u + u_0 \bmod \Lambda \tag{538}$$

$$i_2(u) = -u \bmod \Lambda \tag{539}$$

and so

$$j(u) = u + u_0 \bmod \Lambda \tag{540}$$

Note that on $\mathbb{C}/\Lambda$ i_1 has the four fixed points $u_0/2 \bmod \Lambda/2$ and i_2 the four fixed points $0 \bmod \Lambda/2$.

[14]For this reason, in what follows we will denote Griffiths and Harris's incidence correspondence and Cayley's cubic curve by the same symbol, Σ.

On the other hand, the inverse of the Abel-Jacobi map

$$\begin{tikzcd}
\mathbb{P}^2 \supset \Sigma & \mathbb{C}/\Lambda \arrow[l, "\simeq"'] \\
(X,Y) = (\wp(u), \wp'(u)) & u \arrow[l, mapsto]
\end{tikzcd} \tag{541}$$

gives $X = \wp(u)$, $Y = \wp'(u)$ and thus $\varpi = du$, with $\wp(u)$ and its derivative $\wp'(u)$ elliptic functions of order 2 respectively 3, doubly-periodic with period lattice Λ, meromorphic on $\mathbb{C}$. Based on (539) and on the fact that the induced action of i_2 on the cubic curve interchanges $(X, +Y)$ with $(X, -Y)$, one can argue that $\wp(u)$ is even and $\wp'(u)$ is odd. The map

$$\begin{tikzcd}
\mathbb{C}/\Lambda \arrow[d, "2:1"] & u \bmod \Lambda \arrow[d, mapsto] \\
\mathbb{P}^1 & \wp(u)
\end{tikzcd} \tag{542}$$

is a branched double-covering of $\mathbb{P}^1$ by $\mathbb{C}/\Lambda$, with $\pm u \bmod \Lambda$ mapped to $\wp(u) = \wp(-u)$ and the four fixed points of i_2 on $\mathbb{C}/\Lambda$, *i.e.*, $0 \bmod \Lambda/2$, mapped to the branching points e_0, e_1, e_2, e_3. We will assume that $\wp(0) = e_0$.

The various correspondences are summarized in the diagram

$$\begin{tikzcd}
(X,Y) = (\wp(u), \wp'(u)) \arrow[ddd, mapsto] & & & u \bmod \Lambda \arrow[lll, mapsto] \arrow[ddd, mapsto] \\
& \Sigma \arrow[r, "\simeq", "\text{Abel-Jacobi}"'] \arrow[d, "2:1"'] & \mathbb{C}/\Lambda \arrow[d, "2:1"] & \\
& \mathcal{B} \arrow[r, "\simeq", "\text{rational parametrization}"'] & \mathbb{P}^1 & \\
P_X \arrow[rrr, mapsto] & & & X = \wp(u)
\end{tikzcd} \tag{543}$$

A line that passes through the point $P_{e_0} \in \mathcal{B}$ and is tangent to the fixed conic $\mathcal{C}_{X_0}$ will, by the above choice of rational parametrization for $\mathcal{B}$, intersect $\mathcal{B}$ again at the point parametrized by X_0, *i.e.*, P_{X_0}. From $e_0 = \wp(0)$ together with (538) we then obtain the interpretation of the u_0-shift of j, namely

$$\wp(u_0) = X_0 \tag{544}$$

One has then the following (see also Figure 12)

Corollary (Cayley) *The tangents from a point $P_{\wp(u)} \in \mathcal{B}$ to a non-singular conic $\mathcal{C}_{\wp(u_0)}$ from the pencil generated by the conics $\mathcal{A}$ and $\mathcal{B}$ will intersect again $\mathcal{B}$ at the points $P_{\wp(u \pm u_0)}$.*

As observed above, the basic step in the construction of Poncelet polygons corresponds on the Jacobian variety of Σ to the action of the automorphism j. The Poncelet problem can be reformulated in the following terms: the polygon closes after n steps if j^n has fixed points on Σ. From $j^n(u) = u + nu_0$ it follows that the necessary and sufficient condition that j^n has a fixed point is

$$nu_0 = 0 \bmod \Lambda \tag{545}$$

The elements of the Jacobian form an abelian group with respect to addition modulo lattice shifts. The condition (545) means that u_0 is a cyclic element of this group of order

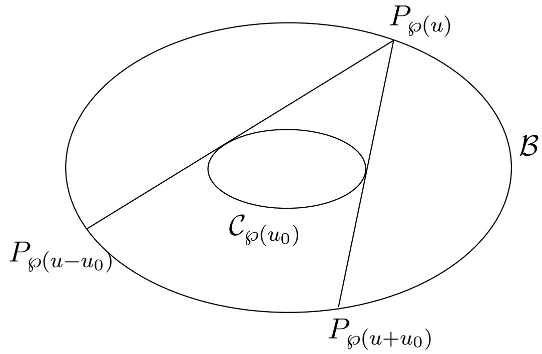


Figure 12.

n. This condition is clearly independent of the point $u \in \mathbb{C}/\Lambda \longleftrightarrow (P, L) \in \Sigma$, and this proves the porism.

The Poncelet problem can be generalized in the following way: consider a conic $\mathcal{B}$ and a series of conics $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\cdots$ from the pencil generated by $\mathcal{B}$ and another transversal conic, $\mathcal{A}$. Take a point P_0 on $\mathcal{B}$ and draw a tangent L_1 to the conic $\mathcal{C}_1$ which intersects again $\mathcal{B}$ at the point P_1. From P_1 draw a tangent L_2 to the conic $\mathcal{C}_2$, a.s.o. Dually, this reads as follows: take a point L_1 on $\mathcal{C}_1^*$ and draw a tangent P_1 to $\mathcal{B}^*$ which intersects $\mathcal{C}_2^*$ at a point L_2. From L_2 draw a tangent to $\mathcal{B}^*$ that intersects $\mathcal{C}_3^*$ at the point L_3, a.s.o. In this case one has not one but a series of automorphisms of type j, one for each conic $\mathcal{C}_i$. The above arguments can be easily extended to give the condition for this construction to close after n steps: one has to have

$$u_1 + \cdots + u_n = 0 \bmod \Lambda \tag{546}$$

where u_1, $\cdots$, u_n are such that $\mathcal{C}_i = \mathcal{C}_{\wp(u_i)}$. Again, since this condition is independent on the starting point, it follows that a generalized Poncelet porism holds, too.

References

[1] U. Lindström and M. Roček, "Scalar-tensor duality and $\mathcal{N}=1$, $\mathcal{N}=2$ non-linear sigma models," *Nucl. Phys.* **B222** (1983) 285–308.

[2] U. Lindström and M. Roček, "New hyperkähler metrics and new supermultiplets," *Commun. Math. Phys.* **115** (1988) 21.

[3] U. Lindström and M. Roček, "$\mathcal{N}=2$ Super Yang-Mills theory in projective superspace," *Commun. Math. Phys.* **128** (1990) 191.

[4] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček, "Hyperkähler metrics and supersymmetry," *Commun. Math. Phys.* **108** (1987) 535.

[5] I. T. Ivanov and M. Roček, "Supersymmetric sigma models, twistors, and the Atiyah- Hitchin metric," *Commun. Math. Phys.* **182** (1996) 291–302, `hep-th/9512075`.

[6] R. Penrose, "Nonlinear gravitons and curved twistor theory," *General Relativity and Gravitation* **7** (1976), no. 1, 31–52.

[7] S. Salamon, "Quaternionic Kähler manifolds," *Invent. Math.* **67** (1982), no. 1, 143–171.

[8] J. Radcliffe, "Some properties of coherent spin states," *J. Phys. A: Gen. Phys.* **4** (1971) 313–323. Reprinted in J. R. Klauder, B.-S. Skagerstam, *Coherent states - Applications in physics and mathematical physics*, World Scientific (1985).

[9] A. Swann, "Hyper-Kähler and quaternionic Kähler geometry," *Math. Ann.* **289** (1991), no. 3, 421–450.

[10] A. Karlhede, U. Lindstrom, and M. Rocek, "Selfinteracting tensor multiplets in n=2 superspace," *Phys. Lett.* **B147** (1984) 297.

[11] H. Pedersen and Y. S. Poon, "Hyper-Kähler metrics and a generalization of the Bogomolny equations," *Comm. Math. Phys.* **117** (1988), no. 4, 569–580.

[12] D. M. J. Calderbank and H. Pedersen, "Self-dual Einstein metrics with torus symmetry," `math/0105263`.

[13] L. Anguelova, M. Roček, and S. Vandoren, "Quantum corrections to the universal hypermultiplet and superspace," *Phys. Rev.* **D70** (2004) 066001, `hep-th/0402132`.

[14] G. Chalmers, M. Roček, and S. Wiles, "Degeneration of ALF D(n) metrics," *JHEP* **01** (1999) 009, `hep-th/9812212`.

[15] S. A. Cherkis and A. Kapustin, "D(k) gravitational instantons and Nahm equations," *Adv. Theor. Math. Phys.* **2** (1999) 1287–1306, `hep-th/9803112`.

[16] S. A. Cherkis and N. J. Hitchin, "Gravitational instantons of type D(k)," *Commun. Math. Phys.* **260** (2005) 299–317, `hep-th/0310084`.

[17] M. F. Atiyah and N. J. Hitchin, *The geometry and dynamics of magnetic monopoles.* Princeton University Press, Princeton, NJ, 1988.

[18] N. Hitchin, "A lecture on the octahedron," *Bull. London Math. Soc.* **35** (2003), no. 5, 577–600.

[19] N. J. Hitchin, "Poncelet polygons and the Painlevé equations," in *Geometry and analysis (Bombay, 1992)*, pp. 151–185. Tata Inst. Fund. Res., Bombay, 1995.

[20] S. J. Gates, M. T. Grisaru, M. Roček, and W. Siegel, "Superspace, or one thousand and one lessons in supersymmetry," *Front. Phys.* **58** (1983) 1–548, `hep-th/0108200`.

[21] S. A. Cherkis, unpublished.

[22] B. de Wit, M. Roček, and S. Vandoren, "Hypermultiplets, hyperkähler cones and quaternion-Kähler geometry," *JHEP* **02** (2001) 039, `hep-th/0101161`.

[23] H. Bacry, "Orbits of the rotation group on spin states," *J. Math. Phys.* **15** (1974) 1686–1688.

[24] H. Bacry, A. Grossmann, and J. Zak, "Geometry of generalized coherent states," in *Group theoretical methods in physics (Fourth Internat. Colloq., Nijmegen, 1975)*, pp. 249–268. Lecture Notes in Phys., Vol. 50. Springer, Berlin, 1976.

[25] E. Majorana, "Oriented atoms in a variable magnetic field," *Nuovo Cim.* **9** (1932) 43–50.

[26] R. Penrose and W. Rindler, *Spinors and space-time*, vol. I & II. Cambridge University Press, Cambridge, 1984, 1986.

[27] R. Penrose, *Shadows of the mind.* Oxford University Press, Oxford, 1994.

[28] T. Eguchi and A. J. Hanson, "Asymptotically flat self-dual solutions to Euclidean gravity," *Phys. Lett.* **B74** (1978) 249.

[29] T. Eguchi and A. J. Hanson, "Self-dual solutions to Euclidean gravity," *Ann. Phys.* **120** (1979) 82.

[30] T. Eguchi and A. J. Hanson, "Gravitational instantons," *Gen. Rel. Grav.* **11** (1979) 315–320.

[31] G. W. Gibbons and S. W. Hawking, "Gravitational multi-instantons," *Phys. Lett.* **B78** (1978) 430.

[32] G. W. Gibbons and S. W. Hawking, "Classification of gravitational instanton symmetries," *Commun. Math. Phys.* **66** (1979) 291–310.

[33] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions*, vol. II. Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981. Based on notes left by Harry Bateman.

[34] D. Olivier, "Complex coordinates and Kähler potential for the Atiyah-Hitchin metric," *Gen. Rel. Grav.* **23** (1991) 1349–1362.

[35] N. Hitchin, "Integrable systems in Riemannian geometry," in *Surveys in differential geometry: integrable systems*, Surv. Differ. Geom., IV, pp. 21–81. Int. Press, Boston, MA, 1998.

[36] G. W. Gibbons and N. S. Manton, "The moduli space metric for well-separated BPS monopoles," *Phys. Lett.* **B356** (1995) 32–38, `hep-th/9506052`.

[37] A. Hanany and B. Pioline, "(Anti-)instantons and the Atiyah-Hitchin manifold," *JHEP* **07** (2000) 001, `hep-th/0005160`.

[38] L. A. Takhtajan, "A simple example of modular forms as tau-functions for integrable equations," *Theor. Math. Phys.* **93** (1992) 1308–1317.

[39] N. I. Akhiezer, *Elements of the theory of elliptic functions*, vol. 79 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1990.

[40] A. Cayley, *An elementary treatise on elliptic functions*. 2nd ed. Dover Publications, Inc., New York, 1961. A reprinting of the 2nd edition, George Bell and Sons, London, 1895.

[41] L. Gavrilov and A. M. Perelomov, "On the explicit solutions of the elliptic Calogero system," *J. Math. Phys.* **40** (1999), no. 12, 6339–6352.

[42] `http://functions.wolfram.com`.

[43] A.-M. Legendre, *Traité des fonctions elliptiques*, vol. 1. 1825. p. 20.

[44] J. Playfair *Trans. Roy. Soc. Edinburgh* **3** (1792) 156.

[45] A. Cayley *Phil. Mag.* **V** (1853) 281–284.

[46] A. Cayley, "Note on the porism of the in-and-circumscribed polygon," *Phil. Mag.* **VI** (1853) 99–102.

[47] A. Cayley *Phil. Mag.* **VI** (1853) 103–105.

[48] A. Cayley, "On the porism of the in-and-circumscribed polygon," *Phil. Trans. R. Soc. London* **CLI** (1861) 225–239.

[49] C. Jacobi, *Gesammelte Werke*, vol. 1. 1881.

[50] P. Griffiths and J. Harris, "A Poncelet theorem in space," *Comment. Math. Helv.* **52** (1977), no. 2, 145–160.

[51] P. Griffiths and J. Harris, "On Cayley's explicit solution to Poncelet's porism," *Enseign. Math. (2)* **24** (1978), no. 1-2, 31–40.

[52] H. J. M. Bos, C. Kers, F. Oort, and D. W. Raven, "Poncelet's closure theorem," *Exposition. Math.* **5** (1987), no. 4, 289–364.

[53] S.-J. Chang and R. Friedberg, "Elliptical billiards and Poncelet's theorem," *J. Math. Phys.* **29** (1988), no. 7, 1537–1550.

[54] S.-J. Chang, B. Crespi, and K. J. Shi, "Elliptical billiard systems and the full Poncelet's theorem in n dimensions," *J. Math. Phys.* **34** (1993), no. 6, 2242–2256.

[55] B. Crespi, S.-J. Chang, and K. J. Shi, "Elliptical billiards and hyperelliptic functions," *J. Math. Phys.* **34** (1993), no. 6, 2257–2289.